Normal Modes of Scalar Fields in BTZ Black Hole Spacetime

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Abstract

Study of BTZ black hole spacetime becomes important and interesting recently. We obtained all the normal models and eigenfunctions for the scalar fields in (2+1) BTZ black hole spacetime. We impose Dirichlet boundary condition at infinity and Dirichlet or Neumann boundary condition at horizon. We studied the effect by negative cosmological constant and the effects by rotation extensively.

1 Scalar field in BTZ spacetime

This section is preparation of definitions and notations in the following sections. For the negative cosmological constant ($\Lambda = -1/\ell^2$) in (2+1) dimension, the Black Hole metric is obtained by Banados, Teitelboim and Zanelli (BTZ) [1]:

$$\begin{aligned} ds^2 &= g_{tt}dt^2 + g_{\phi\phi}d\phi^2 + 2g_{t\phi}dtd\phi + g_{rr}dr^2 \\ g_{tt} &= M - \frac{r^2}{\ell^2}\ , \ \ g_{t\phi} = -\frac{J}{2}\ , \ \ g_{\phi\phi} = r^2\ , \ \ g_{rr} = \left(-M + \frac{J^2}{4r^2} + \frac{r^2}{\ell^2}\right)^{-1}\ , \end{aligned}$$

where M and J are mass and angular momentum of black hole respectively. Outside and inside horizon are defined by:

$$r_\pm^2 = \frac{M\ell^2}{2}\left(1 \pm \sqrt{1 - \frac{J^2}{M^2\ell^2}}\right)\ . \tag{1}$$

The action of complex scalar field $\Phi(x)$ of mass μ is

$$I_{\text{scalar}} = -\int dtdrd\phi(-g)^{1/2}(g^{\mu\nu}\partial_\mu\Phi^*(x)\partial_\nu\Phi(x) + \frac{\mu}{\ell^2}\Phi^*(x)\Phi(x))\ . \tag{2}$$

The scalar field is written in the form of separation of variables $\Phi = \mathrm{e}^{-\mathrm{i}\omega t + \mathrm{i}m\phi}R(r)$ with frequency ω and azimuthal angular momentum m. Then the equation for radial wave function $R(z)$ is obtained :

$$\left(g_{rr}\left(\omega - \frac{J}{2r^2}m\right)^2 - \frac{m^2}{r^2} + \frac{1}{r}\partial_r\frac{r}{g_{rr}}\partial_r - \frac{\mu}{\ell^2}\right)R(r) = 0\ . \tag{3}$$

Introducing variable $z = (r^2 - r_+^2)/(r^2 - r_-^2)$ and function $F(z) = z^{i\alpha}(1-z)^{-\beta}R(z)$, the Hypergeometric differential equation is obtained :

$$z(1-z)\frac{\mathrm{d}^2F}{\mathrm{d}z^2} + (c - (1+a+b)z)\frac{\mathrm{d}F}{\mathrm{d}z} - abF = 0\ . \tag{4}$$

The parameters a, b, c are defined :

$$a = \beta - i\frac{\ell^2}{2(r_+ + r_-)}\left(\omega + \frac{m}{\ell}\right)\ , \ b = \beta - i\frac{\ell^2}{2(r_+ - r_-)}\left(\omega - \frac{m}{\ell}\right)\ , \ c = 1 - 2i\alpha\ , \tag{5}$$

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$$\alpha = \frac{\ell^2 r_+}{2(r_+^2 - r_-^2)}(\omega - \Omega_{\mathsf{H}} m) \ , \quad \beta = \frac{1-(1+\mu)^{1/2}}{2} \ , \tag{6}$$

where $\Omega_{\mathsf{H}} = J/2r_+^2$ is angular velocity at horizon. General solution of Hypergeometric differential equation is expressed by linear combination of two independent solutions at horizon or infinity.

2 Eigenstate problem of scalar field

We set up Dirichlet boundary condition of eigenstate problem for normal modes at infinity because BTZ solution is asymptotic AdS spacetime:

$$R_\infty = \frac{z^{-i\alpha}(1-z)^\beta (1-z)^{c-a-b}}{\Gamma(c-a-b+1)} F(c-a, c-b, c-a-b+1; 1-z) \ . \tag{7}$$

This solution is also expressed near horizon as outgoing wave and incoming wave to black hole as:

$$R_\infty \ = \ \frac{\Gamma(1-c)}{\Gamma(1-a)\Gamma(1-b)} R_{r_+,\mathsf{in}} + \frac{\Gamma(c-1)}{\Gamma(c-a)\Gamma(c-b)} R_{r_+,\mathsf{out}} \ .$$

where ingoing wave is expressed: $R_{r_+,\mathsf{in}} = z^{-i\alpha}(1-z)^\beta F(a,b,c;z)$ and outgoing wave is $R_{r_+,\mathsf{out}} = z^{i\alpha}(1-z)^\beta F(1+b-c, 1+a-c, 2-c; z)$. Eigenvalue equations for normal modes are obtained for each boundary condition:

(i) Eigenvalue equation for Dirichlet boundary condition:

$$(\omega - \Omega_{\mathsf{H}} m) r_{*,\mathsf{H}} + \gamma_0(\omega) = -\pi \left(n + \frac{1}{2} \right) \quad \text{for} \quad n = 0, 1, 2, \cdots \ , \tag{8}$$

(ii) Eigenvalue equation for Neumann boundary condition:

$$(\omega - \Omega_{\mathsf{H}} m) r_{*,\mathsf{H}} + \gamma_0(\omega) = -\pi n \quad \text{for} \quad n = 0, 1, 2, \cdots \ , \tag{9}$$

where the phase function $\gamma_0(\omega)$ is defined :

$$\gamma_0(\omega) = \frac{\ell^2 r_+ (\omega - \Omega_{\mathsf{H}} m)}{2(r_+^2 - r_-^2)} \log \left(\frac{4r_+^2}{r_+^2 - r_-^2} \right) + \arg \left(\frac{\Gamma(c-1)}{\Gamma(c-a)\Gamma(c-b)} \right) \ . \tag{10}$$

The number n labels each quantum state and the horizon expressed by tortoise coordinate with regularlization parameter ϵ is: $r_{*,\mathsf{H}} \simeq \ell^2 r_+ \log(\epsilon/2r_+)/2(r_+^2 - r_-^2)$.

Square of absolute value of eigenfunction at horizon with respect to frequency ω is shown in Figure 1, where the zeros correspond to normal frequencies ω for Dirichlet boundary condition. In this report, only Dirichlet boundary condition is considered. Square of absolute values of eigenfunction with respect to radial coordinate r for lowest two state ($n = 0, 1$) are shown in Fig. 2. The sets of eigenvalue (ω, m) for each fixed n in case without rotation ($J = 0$) form convex curves due to the negative cosmological constant effects shown in left hand side of Fig. 3. Curves become flat for the case that ℓ is set to larger value 2ℓ and M to $4M$ shown in the right hand side of Fig. 3. The rotation effects to the eigenvalue $(\omega,\ m)$ are shown in Fig. 4. Eigenvalues $(\omega,\ m)$ rotate corresponding to the angler velocity Ω_{H} for the cases of $J = 0.2$ (Fig. 4) compared with the cases of $J = 0$ (Fig. 3).

3 Summary

(1) The set of eigenvalues (ω, m) forms curved lines in (ω, m) plane for fixed n. The coefficient to ω is invariant but the coefficient to m decrease as $1/\lambda$ in parameters a, b and c under the scalar transformation : $\ell \to \lambda\ell, \quad M \to \lambda^2 M, \quad (0 \le \lambda)$. Therefore the slope of eigenvalue becomes more flat for more small cosmological constant ($\ell \to \lambda\ell$).

(2) Eigenvalues (ω, m) rotate corresponding to the angular velocity Ω_{H}. Then the allowed region of $0 < \omega$ for $J = 0$ becomes $0 < \omega - \Omega_{\mathsf{H}} m$ for $J \neq 0$.

The eignvalues for normal modes relate the super-radiant problem [2, 3] and statistical mechanics of scalar fields around BTZ black hole [4].

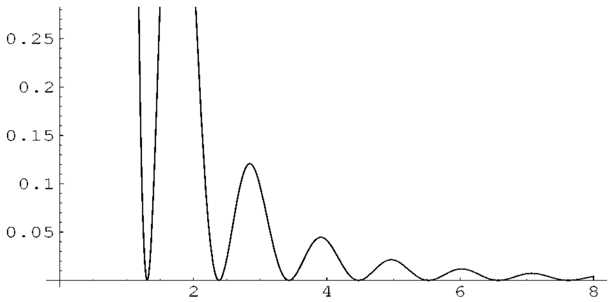

Figure 1: Square of absolute value of eigenfunction with frequency

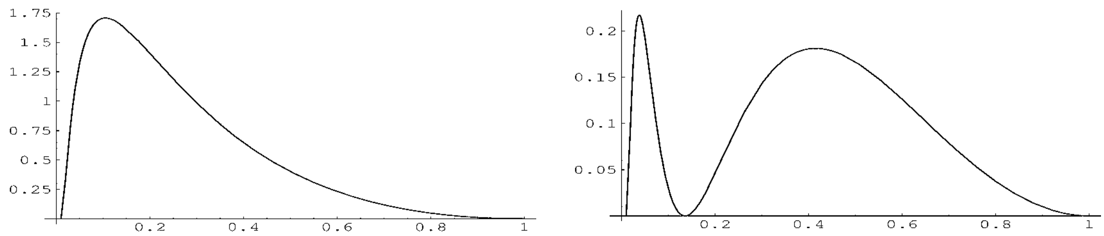

Figure 2: Square of absolute value of eigenfunctions with radial coordinate for n=0,1

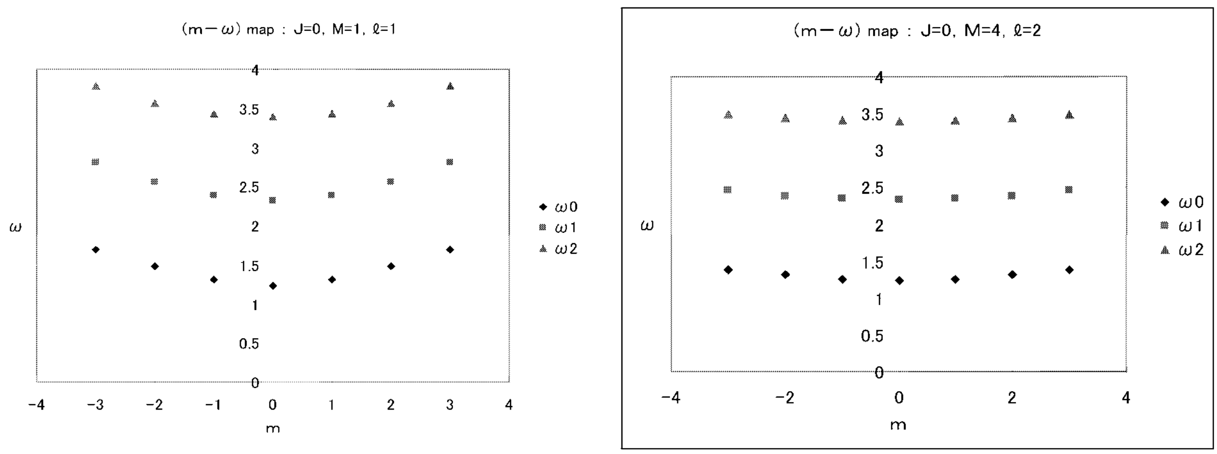

Figure 3: Eigenvalue map in (m,ω) plane without rotation

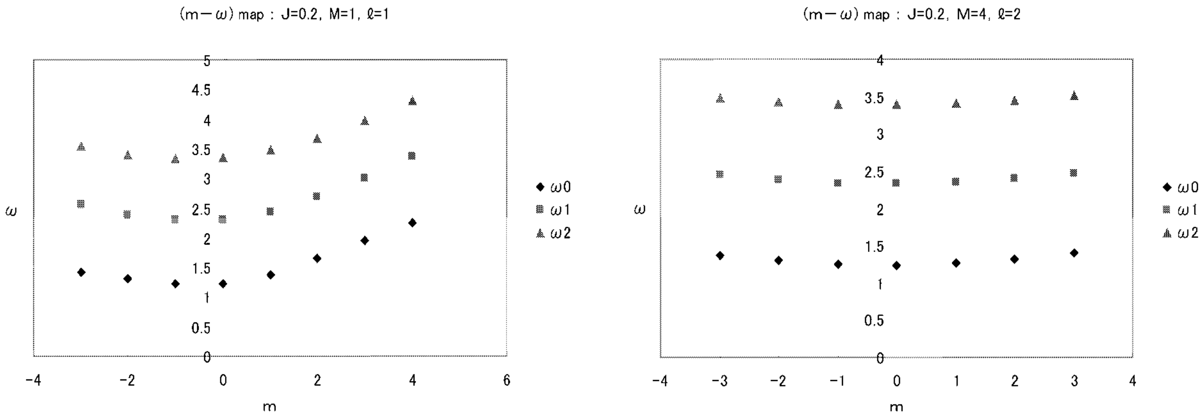

Figure 4: Eigenvalue map in (m, ω) plane with rotation

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