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A SEMICLASSICAL MONOPOLE CONFIGURATION FOR ELECTROMAGNETISM*

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ABSTRACT

A semiclassical monopole configuration of generalized electromagnetic charge and its associated electromagnetic field that satisfies Maxwell's equations and has a symmetry under the subgroup of the conformal group isomorphic to O(4) is explicitly constructed. This configuration has no singularities, carries quanta of two different kinds of angular momenta, and is called a vorton. It also carries topological charge. It is shown that the rotation associated with the angular momenta leads to a minimum energy for the configuration. Setting the energy to this minimum and quantizing the angular momenta of rotation yields a quantization condition for the magnitude of the electromagnetic charge. The smallest allowable nonzero electromagnetic charge carried by a configuration which has a nonzero topological charge equals 25.8e.

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I. INTRODUCTION

Nearly half a century ago it was pointed out[1] that magnetic monopoles may exist, and if so, they could furnish an explanation for the universal value of electric charge associated with elementary particles. Recently it was shown[2] that certain gauge transformations in compact, non-Abelian groups will yield field configurations having the character of finite energy monopole solutions to classical field equations. Since that time, there has been an enormous amount of work[3] on monopole solutions and their associated topological charge.

The purpose of this paper is to give an explicit construction of a configuration of electromagnetic source and field that satisfies (generalized) Maxwell's equations and that at the same time manifests a symmetry with respect to transformations generated by the operators of a certain subgroup of the conformal group. The generalization of Maxwell's equations which is used here is one which treats the electric and magnetic parts of the field tensor on an equal footing,[4] and the straightforward extension of such an approach to include both electric and magnetic charge.[5,6,7] It is seen that this configuration is a finite energy monopole solution without singularities which carries topological charge. As will be shown below, the topological charge is associated with the symmetry which is specifically built into the configuration.

II. THE CONFORMAL CONNECTION

It has been known for a long time that Maxwell's equations in Minkowski space are invariant under the operators of the conformal group,[8] a group with 15 generators. These include the ten generators of the Poincaré group, $M_{\mu\nu}$ and P_μ, as well as five additional ones: K_μ, the generator of special conformal transformations, and D, the generator of dilatations. The conformal group in Minkowski space is isomorphic to the group O(4,2).[9]

In an analysis of O(4,2), it has been shown[10] that there is a system of six operators L_i and X_i (where i = 1,2,3) which obey commutation relations isomorphic to those of O(4). [The global properties of this subgroup, however, differ from those of O(4); see Appendix C.] Invariance under the transformations generated by these operators is explicitly built into the configurations constructed in this paper. In terms of the generators of the conformal group,

$$L_i = M_{jk} \qquad\qquad i, j, k \text{ cyclic} ,$$

and

$$X_i = (K_i - P_i)/2 \quad , \tag{1}$$

Since $[L_i, X_i] = 0$, no summation over i, one can simultaneously diagonalize L_3 and X_3. The transformations associated with L_3, the usual angular momentum operator, are rotations about the z-axis, while those associated with X_3, are shown to be toroidal "rotations," also associated with the z-axis. The eigenvalues associated with L_3 and X_3 have been shown to be either both integer or both half-integer.[10] Clearly, these eigenvalues respectively denote the projections of ordinary angular momentum and toroidal angular momentum on the z-axis. In keeping with the semiclassical outlook of this paper, only integral eigenvalues are considered here.

While the nature of the rotation generated by L_3 is familiar, and may be measured by the azimuthal angle ϕ, the rotation generated by X_3 is not so well known. It is easy to generate this latter rotation or mapping by using the isomorphism of the conformal group in Minkowski space to the group O(4,2) in a 6 dimensional pseudo-Euclidean space with coordinates[11]

$$\eta_A = \kappa\left[x_\mu, \frac{1+x^2}{2}, \frac{1-x^2}{2}\right] , \tag{2}$$

where $\mu = 0,1,2,3$ and $A = 0,1,2,3,5,6$. The metric in x_μ space is diag [1,-1,-1,-1] while that in η_A space is diag [1,-1,-1,-1,-1, 1]. By an arbitrary choice of κ, all of Minkowski space maps onto the light cone in conformal space (i.e., onto a hypersurface satisfying $\eta_A \eta^A = 0$, where the usual summation over A is taken).

The coordinates η_1, η_2, η_3, and η_5 are the ones which exhibit the O(4) (sub)symmetry. The operators L_i generate rotations in the $\eta_j - \eta_k$ plane, i, j, k, = 1, 2, or 3 and cyclic, while the operators X_i generate them in the $\eta_i - \eta_5$ plane. Using an angle ϕ to describe a rotation in the $\eta_1 - \eta_2$ plane and an angle χ to describe one in the $\eta_3 - \eta_5$ plane, one writes

$$\begin{aligned} \eta_1' &= \eta_1 \cos\phi + \eta_2 \sin\phi , \\ \eta_2' &= -\eta_1 \sin\phi + \eta_2 \cos\phi , \end{aligned} \tag{3}$$

$$\eta_3' = \eta_3 \cos\chi + \eta_5 \sin\chi ,$$

and

$$\eta_5' = -\eta_3 \sin\chi + \eta_5 \cos\chi . \tag{4}$$

The transformations (3) and (4) are clearly independent and leave invariant the condition $\eta_A \eta^A = 0$.

To study the effect of the rotation in 3-space associated with the angle χ we set $\kappa = 1$ and choose $\eta_0 = 0$ (i.e., $x_0 = 0$). This latter requirement ensures that the points of 3-space under the action of X_3 remain in the same time frame, as is appropriate for what we shall see are static configurations. These steps taken, one can easily demonstrate that the rotation induced by X_3, moves the points of 3-space along circular paths which are identical to those of constant σ and ϕ in a toroidal coordinate system,[12] (σ,ψ,ϕ). The toroidal angle ψ in this case (is opposite in sense to χ and) measures displacement around the toroidal generating circles (labeled by σ), while ϕ measures the usual azimuthal angle in the x-y plane. The O(4) symmetry of the configurations of this model, then, is an invariance with respect to arbitrary rotations of the electromagnetic charge distributions through the two angles ϕ and ψ. Finite rotations are to be constructed as an integral of infinitesimal rotations. In the course of these (infinitesimal) rotations, the charge should be viewed as a (compressible) fluid, moving along with their local coordinate points as though entrained by them.

The angle ϕ going from zero to 2π effects one full rotation about z-axis, mapping the points of 3-space back upon themselves. Likewise ψ going from zero to 2π effects one full "rotation," also mapping 3-space back upon itself. Such mappings are members of homotopy classes[13] associated with the group under consideration, in this case the "O(4)" subgroup of the conformal group. If we perform both rotations at the same time (ϕ from 0 to 2π and ψ from 0 to 2π), we again effect a mapping of all points in 3-space back onto themselves. This mapping has a topology like that of a Möbius strip (except that it has a full twist rather than the usual half-twist). Hence, intuition suggests that it is not a member of the homotopy class containing the identity operation. This would mean that solutions to Maxwell's equations that are characterized by nonzero eigenvalues m_ϕ and m_ψ would exhibit topological charge. In Appendix C it is shown that for the configurations constructed in this paper, this is indeed the case.

Since we have indicated the O(4) symmetry in which we are interested here is related to the toroidal coordinate system in Euclidean 3-space, it would be natural to study solutions of the wave equation in a toroidal coordinate system. Unfortunately, the wave equation, $\Box\psi = 0$ (and hence also Maxwell's equations), does not separate in toroidal coordinates,[14] and general solutions to the wave equation in this coordinate system have never been explicitly found. Consequently, the approach of this paper is to use the well established electrodynamical relations between sources and fields, to construct monopole solutions which satisfy Maxwell's equations and which at the same time manifest the above described O(4) (sub)symmetry of the conformal group.

The toroidal aspect that is so constructed into these solutions bears some resemblance to a smoke ring. However, rather than a localized vortex ring, the "motion" generated by X_3 is spread throughout all of 3-space (as is the rotational motion associated with L_3). Because the vortex ring is an essential feature of the monopole configurations constructed in this paper and because they are subject to a quantum condition, we shall refer to them as "vortons."

III. GENERAL VORTON STRUCTURE

The basis of the vorton as a monopole configuration is an electromagnetic charge density distribution q centered at the origin of 3-space, normalized such that the total charge

$$Q = \iiint q \, dV \quad . \tag{5}$$

Since we are employing the generalized form of Maxwell's equations, Q can be either electric or magnetic charge. In fact, by the symmetry of these equations, the vorton can in general carry both electric charge and magnetic charge given by $Q\sin\Theta$ and $Q\cos\Theta$ respectively, where Θ is the dyality angle[15] employed by Han and Biedenharn.[7] For simplicity in the following calculations, we shall set $\Theta = \frac{\pi}{2}$, yielding an electric vorton configuration. But we shall keep in mind that since electromagnetic theory is invariant with respect to this angle, Θ will be a free parameter of the vorton configuration.

This charge distribution is assumed to be in a state of "double rotation," by which is meant that it behaves as though it is moving through equal increments of ϕ in equal increments of time, and through equal increments of ψ in equal increments of time. The vorton configuration is, therefore, characterized by "angular velocities" ω_ϕ and ω_ψ. While the ϕ motion is just like that of a rigid body rotating about the z-axis in 3-space, the ψ rotation is rigid only in the six dimensional pseudo-Euclidean space. But as we shall see, in spite of the difference in these two kinds of rotations in 3-space, the integral quantities derived from them are equal--a manifestation of the underlying O(4) symmetry.

Each type of (charge) rotation will be associated with an electromagnetic dipole-like field (magnetic for $\Theta = \pm\pi/2$, electric for $\Theta = 0$ or π), proportional to ω_ϕ and ω_ψ respectively. Since we assume that ω_ϕ and ω_ψ do not vary with time and the charge distribution is constructed to be O(4) symmetric, these fields do not vary with time. In fact, as a result of the symmetry built into the vorton configuration, none of the vortonic quantities vary with time, including the charge density itself; the (static) dipole-like fields are the only external evidence of the postulated motion of the charge. Therefore, the vorton configurations constructed in this paper are actually static solutions, and for this class of solutions we have no need to consider time as an independent variable. Consequently, we shall view the motion of the charge as "internal" in nature and the quantities ω_ϕ and ω_ψ as parameters rather than representing physical velocities.

This view enables us to ignore the possible difficulty that there are regions of the vorton charge distribution to which are attributed internal velocities exceeding that of light. We recall that (phase) velocities exceeding that of light are common in certain microwave field configurations, but that such velocities are not regarded to be in contradiction with relativistic theory as long as they do not entail superluminal group velocities. On this point, we note that since the fields and charge distribution of the vorton configuration do not change with time, the relevant group velocities are zero.

IV. DETAILED VORTON STRUCTURE

In order to attain an object exhibiting the desired O(4) symmetry, the charge density must be distributed such that there are equal increments of charge contained within equal increments of ϕ and of ψ. Assuming further, that q is constant over a sphere of radius a, where a sets the scale of a toroidal coordinate system, it is shown in Appendix A that

$$q = \frac{4a^3Q}{\pi^2(a^2 + r^2)^3} \qquad (6)$$

which is spherically symmetric. It follows that the electric displacement vector has only a radial component. Using Gauss's law one obtains in Gaussian units:

$$D(r) = \frac{4a^3Q}{\pi r^2}\left[\frac{-r}{(a^2 + r^2)^2} + \frac{r}{2a^2(a^2 + r^2)} + \frac{\tan^{-1} r/a}{2a^3}\right] . \qquad (7)$$

The last term in the square bracket is dominant for $r \gg a$ and we see that the distant field goes like Q/r^2, as expected for a monopole field. For $r \to 0$, $q \to \frac{4Q}{\pi^2 a^3}$ and Gauss's law yields $D \to \frac{16Qr}{3\pi a^3}$, a linear dependence upon r which agrees with the lowest order nonzero term of the power series expansion of Eq. (7).

In Appendix B the components of the magnetic field intensity vector associated with ω_ϕ were found:

$$H_r = \frac{8a^3Q\omega_\phi \cos\theta}{3\pi c r^3}\left\{\ \right\}_r \qquad (8)$$

where

$$\left\{\ \right\}_r = \frac{a^2 r}{(a^2 + r^2)^2} - \frac{5r}{2(a^2 + r^2)} + \frac{3\tan^{-1} r/a}{2a} + \frac{r^3}{(a^2 + r^2)^2} , \qquad (9)$$

and

$$H_\theta = \frac{8a^3 Q\omega_\phi \sin\theta}{3\pi c r^3} \left\{ \ \right\}_\theta \quad , \tag{10}$$

where

$$\left\{ \ \right\}_\theta = \frac{a^2 r}{2(a^2 + r^2)^2} - \frac{5r}{4(a^2 + r^2)} + \frac{3 \tan^{-1} r/a}{4a} - \frac{r^3}{(a^2 + r^2)^2} \quad . \tag{11}$$

In both expressions, the distant field is given by the arc tangent term in the curly brackets. The distant field, then, goes like r^{-3} and is the same as that of a dipole of moment[17] $Qa^2\omega_\phi$. Thus, the effective radius of the rotating vortonic charge distribution is $\sqrt{2}a$, which is just the result one obtains by calculating $\langle\rho^2\rangle = \frac{1}{Q}\int_0^\infty q\rho^2 r^2 dr = 2a^2$, where ρ is the usual cylindrical coordinate.

Also derived in Appendix B is the component of the magnetic field intensity vector associated with ω_ψ. By symmetry, one sees that this $\vec{H}$ has only a ϕ component:

$$H_\phi = \frac{-4Q\omega_\psi a^2 r \sin\theta}{\pi c(a^2 + r^2)^2} \tag{12}$$

Again, we see that the distant field goes like r^{-3}, as is characteristic of a dipole.

It is already evident that the vorton configuration has features which characterize objects now known as solitons.[18] It is therefore appropriate to investigate the topological charge of the vorton. One indicator of topological charge is a nonzero value for the Hopf invariant or Hopf charge.[19] In Appendix C, it is explicitly shown how to derive the relevant components of the vortonic electromagnetic field from a specified triplet of scalar fields, φ_i, from which is then obtained the value of the Hopf invariant Q_H, a mapping index associated with the φ_i.

$$Q_H = Cm'_\phi m'_\psi \quad , \tag{13}$$

where C is the largest common factor in m_ϕ and m_ψ; i.e., m'_ϕ and m'_ψ are relatively prime. Thus, the vorton configuration carries a topological charge if and only if both types of angular momenta are present, in agreement with the intuition already drawn from the topology of the mappings of 3-space induced by the operators L_3 and X_3. It is also relevant to note that there is an infinite set of possible values of Q_H for the vorton, indicating that Q_H will be additive modulo infinity. (Recall that in 't Hooft's SO(3) model,[2] there was only one class of mappings in addition to that associated with the identity, and thus the associated topological charge for that model was additive modulo 2.)

V. ENERGY AND ACTION INTEGRALS AND THE QUANTIZATION OF VORTONIC CHARGE

Since the energy density of an electromagnetic field is $(\vec{E} \cdot \vec{D} + \vec{H} \cdot \vec{B})/8\pi$, the energy W_m associated with the monopole field is proportional to Q^2:

$$W_m = K_m Q^2 \quad . \tag{14}$$

K_m is evaluated in Appendix D;

$$K_m = \frac{5}{4\pi a} \quad , \tag{15}$$

inversely proportional to the size of the vorton, as would be expected from dimensional analysis. Similarly, one can write for the energy of the vorton configuration associated with the dipole-like components of its electromagnetic field the equation

$$W_d = K_d Q^2 (a_\phi \beta_\phi^2 + a_\psi \beta_\psi^2) \tag{16}$$

where the dimensionless quantities

$$\beta_\phi \equiv \frac{a\omega_\phi}{c}$$

$$\beta_\psi \equiv \frac{a\omega_\psi}{c} \quad . \tag{17}$$

In Appendix D it is shown that we can set

$$K_d \equiv \frac{1}{6\pi a} \tag{18}$$

and take

$$a_\phi = a_\psi = 1 \quad . \tag{19}$$

Equation (19) is a manifestation of the O(4) symmetry built into the vorton configuration. Again we see that the vorton energy is inversely proportional to the vorton size.

We now observe that the variables ϕ and ψ are cyclic, and we may apply the Bohr-Sommerfeld quantum condition to the action (integrals) associated with these variables, quantizing them in units of Planck's constant h. These action integrals may be used to define associated angular momenta, which are thereby equivalently quantized in units of $\hbar$, and which yield (semi-classical) equations of the form

$$Q^2 \beta_\phi = b_\phi m_\phi \hbar \quad , \tag{20}$$

and

$$Q^2 \beta_\psi = b_\psi m_\psi \hbar \quad , \tag{21}$$

where m_ϕ and m_ψ are integers respectively denoting the (quantized) projections of the ϕ and ψ angular momenta on the z-axis. In Appendix E the angular momenta associated with these rotations are defined in terms of action integrals and are then evaluated. Quantizing them according to the Bohr-Sommerfeld quantum condition, and using Eqs. (20) and (21) yields

$$b_\phi = b_\psi = 3\pi c \equiv b \quad . \tag{22}$$

Again we see that the O(4) symmetry obtains.

Now β_ϕ and β_ψ are related by Eqs. (20) and (21) which we use to define the quantity

$$\overline{\beta} = \frac{\beta_\phi}{m_\phi} = \frac{\beta_\psi}{m_\psi} \tag{23}$$

obtaining the relation

$$Q^2 = b\hbar/\overline{\beta} \quad . \tag{24}$$

Using Eq. (24) in Eq. (14) yields

$$W_m = K'_m/\overline{\beta} \tag{25}$$

where

$$K'_m \equiv K_m b\hbar \quad . \tag{26}$$

Similarly, Eq. (16) becomes

$$W_d = K'_d \overline{\beta} \quad , \tag{27}$$

where

$$K'_d \equiv K_d b\hbar\,(m_\phi^2 + m_\psi^2) \quad . \tag{28}$$

One can now see that the total vorton energy,

$$W_T = W_m + W_d = K'_m \overline{\beta}^{-1} + K'_d \overline{\beta} \tag{29}$$

will have a minimum with respect to $\overline{\beta}$ when

$$\overline{\beta}^2 = K'_m/K'_d \quad . \tag{30}$$

At this minimum

$$W_m = W_d = \frac{W_T}{2} = \sqrt{K'_m K'_d} \quad ; \tag{31}$$

the monopole and dipole-like contributions are equal. It is straightforward to reduce this minimum W_T to the form

$$W_T = \Big[|m_\phi\ \omega_\phi| + |m_\psi\ \omega_\psi|\Big]\hbar \quad . \tag{32}$$

When $\overline{\beta}$ is such that W_T is minimized, we have a (semi-classical) quantization condition on the vorton charge:

$$Q^2 = b\hbar\sqrt{\frac{K_d}{K_m}}\sqrt{m_\phi^2 + m_\psi^2} \quad . \tag{33}$$

Putting the values for the constants into Eq. (33) yields

$$Q^2 = \pi\sqrt{\frac{6}{5}}\sqrt{m_\phi^2 + m_\psi^2}\ \hbar c \quad . \tag{34}$$

We see that the allowable values for the quantized electromagnetic charge depend upon the value of the topological charge, as specified by m_ϕ and m_ψ. If vortons would exist as physical entities, one presumes that those with the smallest nonzero topological charge would be the most important. The smallest nonzero topological charge, which obtains when $|m_\phi|=|m_\psi|=1$, yields

$$Q^2 = 2\pi \sqrt{\frac{3}{5}}\hbar c \quad , \tag{35}$$

which, using $e^2/\hbar c = \alpha \cong 1/137$, is equivalent to

$$Q \cong 25.8e \quad . \tag{36}$$

VI. SUMMARY AND DISCUSSION

In order to construct our monopole solution, called a vorton, we have postulated a generalized electromagnetic charge distribution having a certain O(4) symmetry and in a state of "double rotation." It is seen that the simultaneous presence of these two kinds of rotation leads to a topological charge associated with the vorton. Due to the symmetry of the configuration none of the electromagnetic quantities vary with time and the configuration is, in fact, static. Further, the vorton fields are seen to have no singularities. Thus the vorton has features which characterize objects now known as solitons.[18] It is also worthwhile to remark that since its electromagnetic and topological quantities obey Maxwell's equations, Lorentz invariance of the vorton configuration is assured.

An energy minimum with respect to a rotational parameter $\bar{\beta}$ was shown to exist. Setting the energy to this minimum and quantizing the angular momenta associated with the rotation of the charge distribution led to a quantization condition upon the total electromagnetic charge. Thus, the allowable values of the magnitude of the electromagnetic charge are seen to depend upon the topological charge of the vorton as specified by quantum numbers m_ϕ and m_ψ. This electromagnetic charge is, by employing an invariance pointed out by Rainich, a combination of electric and magnetic charge specified by the dyality angle Θ, a free parameter of the configuration.

The electromagnetic charge (magnitude) of the vorton with the smallest nonzero topological charge was shown to be 25.8e, different from 68.5e or 137e anticipated from the analyses of Dirac or Schwinger.[20] Of course, one could seek values of m_ϕ and m_ψ which would bring a calculated value of Q into better agreement with these anticipated charge strengths, but such an approach lacks a serious logical or physical foundation, and was not undertaken. Rather, one expects that quantum mechanical effects, not accounted for in this semi-classical analysis, might cause the value of Q to differ from 25.8e; a factor of about five would be required to yield e/α.

One cannot resist the speculation that vortons might be more than just an interesting mathematical construction. In fact, there is historical precedent for supposing that vortices might play a key role in the structure of matter. Lord Kelvin once proposed that atoms might be vortex rings.[21] But, if vortons should exist as physical particles, they would be quite different from presently known particles.

Indeed, since vortons find their basis in a subgroup of the conformal group, they would not be included in the celebrated particle classification scheme of Wigner,[22] which is based upon the representations of the Poincaré group, a different subgroup of the conformal group.

The total vorton energy was derived and found to be finite and inversely proportional to the size of the vorton; there is no fixed vorton "mass." This result is not a surprise since it is often remarked that mass is not a conformally invariant quantity.[23] Presumably, then, vortons could come with different energies (just as photons come with different energies), and the energy content of a given vorton would depend upon the conditions prevailing in its production process.

Once a vorton of some particular size and energy would be produced, it is unclear what its time evolution might be. In hydrodynamics, in the absence of viscous forces, vortices have been shown to persist indefinitely.[24] So it is, perhaps, not unjustified to suppose that vortons might persist long enough to enable experimental observation. On the other hand, a subsequent expansion of the vorton size could take place rendering difficult its observation as a localized particle, relieving a possible conflict with the negative results of monopole searches.[25]

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APPENDIX A

Derivation of the Vortonic Charge Distribution and the Monopole Component of Its Field

In the construction of the vorton solution, it is postulated that, centered at the origin of 3-space, there is a generalized electromagnetic charge density $q(\sigma, \psi, \phi)$, where σ, ψ, and ϕ are toroidal coordinates.[12] As mentioned in the text, to simplify the calculations q will be assumed to consist of electric charge (dyality angle $\Theta = \pi/2$) only.

Since we are seeking a solution which exhibits the O(4) symmetry described in Sec. II, there must be equal increments of charge contained in equal increments of ϕ and of ψ. Thus, in general we expect that

$$q(\sigma, \psi, \phi) \equiv q(\sigma, \psi) = \frac{K(\sigma)}{h_\sigma h_\psi h_\phi} = \frac{K(\sigma)(\cosh\sigma - \cos\psi)^3}{a^3 \sinh\sigma} \qquad \text{(A-1)}$$

where a gives the scale of the toroidal coordinate system, h_σ, h_ψ, and h_ϕ are the usual coordinate scaling factors (proportional to a) and $K(\sigma)$ is at our disposal to effect the desired symmetry of q. To this end, we shall choose $K(\sigma)$ such that q is uniform on the surface of the sphere of radius a. This choice is the only one which will achieve a distribution which is symmetric with respect to rotations about the x-axis and the y-axis (i.e., under L_1 and L_2) as well as the z-axis. On this sphere $\psi = \pm\pi/2$, $\cos\psi = 0$, and we obtain

$$q(\sigma, \pm\pi/2) = \frac{K(\sigma)\ \cosh^3\sigma}{a^3\ \sinh\sigma} = K_0/a^3 \quad . \tag{A-2}$$

Thus,

$$K(\sigma) = K_0 \frac{\sinh\sigma}{\cosh^3\sigma} \quad , \tag{A-3}$$

which yields

$$q(\sigma,\psi) = \frac{K_0(\cosh\sigma - \cos\psi)^3}{a^3\ \cosh^3\sigma} \quad . \tag{A-4}$$

Since the radius

$$r = a\left(\frac{\cosh\sigma + \cos\psi}{\cosh\sigma - \cos\psi}\right)^{1/2} \quad , \tag{A-5}$$

q may be simplified to

$$q = \frac{4a^3Q}{\pi^2(a^2 + r^2)^3} \quad , \tag{A-6}$$

where K_0 was evaluated by setting $\iiint q\ dV = Q$. Thus we see that the charge distribution is indeed spherically symmetric, being a function of radius alone.

The charge density of Eq. (A-6) may be used with Gauss's law to obtain the radial component of the electric displacement vector associated with this monopole configuration. That is, in Gaussian units,

$$D(r) = \frac{4\pi}{r^2}\int_0^r q(r')r'^2dr' \tag{A-7}$$

which yields[26]

$$D(r) = \frac{4a^3Q}{\pi r^2}\left[\frac{-r}{(a^2 + r^2)^2} + \frac{r}{2a^2(a^2 + r^2)} + \frac{\tan^{-1} r/a}{2a^3}\right] \quad . \tag{A-8}$$

APPENDIX B

The Fields Associated with the Rotation of Vortonic Charge

In the construction of this monopole solution, it is assumed that the vortonic charge distribution can take on rotations consistent with the subgroup of the conformal group of Minkowsky space-time isomorphic to O(4). Since L_3 and X_3 can be simultaneously diagonalized, we consider only rotations associated with the z-axis. As described in Sec. III, these rotations are characterized by the parameters ω_ϕ for the ordinary rotation and ω_ψ for the toroidal rotation.

In order to find the fields proportional to ω_ϕ, one can use the calculation of the (magnetic) field components at the point (r,θ,ϕ) generated by a circular loop carrying a current ΔI around the z-axis at distance r', and latitude angle θ' from the origin:[27]

$$\Delta H_r = \frac{4\pi\Delta I \sin\theta'}{2ac} \sum_{n=1}^{\infty} \left(\frac{r}{r'}\right)^{n-1} P_n^1(\cos\theta')\, P_n(\cos\theta) \qquad \text{(B-1)}$$

and

$$\Delta H_\theta = \frac{4\pi\Delta I \sin\theta'}{-2ac} \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{r}{r'}\right)^{n-1} P_n^1(\cos\theta')\, P_n^1(\cos\theta) \quad , \qquad \text{(B-2)}$$

where $r' > r$. The formulae for $r' < r$ may be obtained by the substitutions $(r'/r)^{n+2}$ for $(r/r')^{n-1}$ in Eq. (B-1) and $-n(r'/r)^{n+2}/(n+1)$ for $(r/r')^{n-1}$ in Eq. (B-2). The factor $4\pi/c$ replaces the permeability μ in the formulae of Smythe to obtain the expressions in Gaussian units.

Since the ΔI here represents an increment of the total current in question, we set

$$\Delta I = q\, dr' r' d\theta' \omega_\phi\, \rho' \tag{B-3}$$

where q is given by Eq. (A-6) and $\rho' = r' \sin\theta'$. By noting that $\sin\theta' = P_1^1(\xi')$ where $\xi' = \cos\theta'$, we can integrate over the ϕ' dependence[28] and set the value of n in the summations:

$$\int_{-1}^{1} d\xi'\, P_1^1(\xi') \sum_{n=1}^{\infty} P_n^1(\xi') = \frac{2}{2n+1} \frac{(n+1)!}{(n-1)!} \delta_{n1} = \frac{4}{3} \quad . \tag{B-4}$$

It is now straightforward to obtain

$$H_\theta = \frac{16a^3\, Q\omega_\phi \sin\theta}{3\pi c} \left[\mathscr{I}_I(r) + \mathscr{I}_{II}(r) \right] \quad , \tag{B-5}$$

where

$$\mathscr{I}_I(r) \equiv \frac{1}{r^3} \int_o^r \frac{r'^4 dr'}{(a^2 + r'^2)^3} \tag{B-6}$$

and

$$\mathscr{I}_{II}(r) \equiv -2 \int_r^\infty \frac{r' dr'}{(a^2 + r^2)^3} \tag{B-7}$$

The forms of $\mathscr{I}_I$ and $\mathscr{I}_{II}$ are tabulated:[29]

$$\mathscr{I}_I = \frac{1}{r^3} \left[\frac{a^2 r}{4(a^2 + r^2)^2} - \frac{5r}{8(a^2 + r^2)} + \frac{3}{8a} \tan^{-1} \frac{r}{a} \right] \tag{B-8}$$

and

$$\mathscr{I}_{II} = - \frac{1}{2(a^2 + r^2)^2} \quad . \tag{B-9}$$

In similar fashion one obtains

$$H_r = \frac{16a^3 Q\omega_\phi \cos\theta}{3\pi c}\left[2\mathcal{I}_I - \mathcal{I}_{II}\right] \tag{B-10}$$

While one can derive the H_ϕ field which is proportional to ω_ψ by a similar calculation, it is much simpler to exploit the symmetry of the problem, and use the relationship

$$\oint \vec{H}\cdot d\vec{\ell} = \frac{4\pi}{c} \oint\!\!\oint \vec{i}\cdot d\vec{s} \quad . \tag{B-11}$$

The line integral on the left-hand side can be taken around a cap of constant ψ while the surface integral on the right-hand side is taken over the surface of that cap.

Since, by the symmetry in this problem, there is only a ϕ component of $\vec{H}$, the left-hand side becomes $2\pi H_\phi \rho$, where ρ is the cap radius. The current density vector is normal to the cap and given by $qh_\psi \omega_\psi$, where $h_\psi = a \sinh\sigma/(\cosh\sigma - \cos\psi)$. The surface integral may then be written as

$$\oint\!\!\oint \vec{i}\cdot d\vec{s} = \int_0^{2\pi} d\phi \int_0^{\sigma_0} h_\phi\, q h_\psi \omega_\psi h_\sigma d\sigma \quad , \tag{B-12}$$

which reduces to

$$\frac{Q\omega_\psi}{\pi}\int_0^{\sigma_0} \frac{d(\cosh\sigma)}{\cosh^3\sigma} \quad , \tag{B-13}$$

an elementary integral.

Noting the directions of the unit vectors 1_ψ and 1_ϕ, these results combine to yield

$$H_\phi = \frac{-Q\omega_\psi}{\pi a c}\,\frac{\sinh\sigma(\cosh\sigma - \cos\psi)}{\cosh^3\sigma} \quad , \tag{B-14}$$

expressed in toroidal coordinates, and

$$H_\phi = \frac{-4Q\omega_\psi \, a^2 r \sin\theta}{\pi c(a^2 + r^2)^2} \quad , \qquad \text{(B-15)}$$

expressed in spherical coordinates.

APPENDIX C

The Determination of the Hopf Charge for the Vorton Configuration

In order to determine the Hopf invariant[19] or charge for the vorton configuration, we must first record some results of homotopy theory. Since our intent here is in the application of these results rather than their derivation, the interested reader should consult appropriate references[30] if more mathematical detail is desired.

The Hopf invariant is defined in terms of a mapping function from the sphere S^{2n-1} onto the sphere S^n, n even. It is appropriate to set n = 2 for our analysis of the "static" vorton configuration in Minkowski space. Our mapping, then, will be between S^3 and S^2. By adding the point at infinity, the three spatial coordinates of Minkowski space are compactified to form an S^3. The vector $\vec{r}$ (defined in the vorton rest frame) identifies the points on S^3. The desired mapping is performed by a triplet of scalar functions φ_i, where i = 1,2,3, which satisfy the normalization condition.

$$\sum_{i=1}^{3} \varphi_i^2(\vec{r},t) = R^2 \quad . \tag{C-1}$$

The φ_i specifies the points on S^2, a sphere of radius R. While it is common to use the unit sphere in φ - space, the Hopf index is invariant with respect to R, and the flexibility introduced by R is useful in relating the configuration of the Hopf mapping functions to that of the vorton.

One then introduces a "Hopf" field tensor function

$$f_{\mu\nu} \equiv \varepsilon_{ijk}\, \varphi_i\, \partial_\mu\, \varphi_j\, \partial_\nu\, \varphi_k \quad , \tag{C-2}$$

where $\mu,\nu = 0,1,2,3$; $\frac{\partial}{\partial x^\mu}$ with $x^\mu = (ct, x^i)$; $i,j,k = 1,2,3$; and ε_{ijk} is the totally antisymmetric tensor with $\varepsilon_{123} = 1$. Clearly $f_{\mu\nu} = -f_{\nu\mu}$. Since we shall be interested in relating (the "six-vector") $f_{\mu\nu}$ to the Maxwell $F_{\mu\nu}$, we shall denote its components $f_{0i} = e_i$ and $f_{ij} = h_k$; i,j,k cyclic. For completeness, we introduce two more vector fields, $\vec{d} = \varepsilon\vec{e} = \vec{e}$ and $\vec{b} = \mu\,\vec{h} = \vec{h}$, using $\varepsilon = 1$ and $\mu = 1$, which is the equivalent of Gaussian units for the Hopf fields.

From Eq. (C-1), one can show[31] that the dual of $f_{\mu\nu}$ is conserved. That is,

$$\partial_\mu(\tfrac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\, f_{\rho\sigma}) = 0 \quad , \tag{C-3}$$

where $\varepsilon^{\mu\nu\rho\sigma}$ is the totally antisymmetric tensor, with $\varepsilon^{0123} = 1$. One can also form the four-vector

$$\partial_\mu\, f^{\mu\nu} \equiv \frac{4\pi}{c}\, j^\nu \quad , \tag{C-4}$$

where by analogy to electromagnetism, c denotes the velocity of light. Using Eq. (C-2) in Eq. (C-4), it is easy to show that j^ν is conserved. That is,

$$\partial_\nu\, j^\nu = 0 \quad . \tag{C-5}$$

Eq. (C-3) is equivalent to

$$\vec{\nabla}\cdot\vec{b} = 0 \tag{C-6}$$

and

$$\vec{\nabla} \times \vec{e} + \frac{1}{c}\frac{\partial\vec{b}}{\partial t} = 0 \quad , \tag{C-7}$$

while Eq. (C-4) is equivalent to

$$\vec{\nabla}\cdot\vec{d} = 4\pi\rho \tag{C-8}$$

$$\vec{\nabla} \times \vec{h} - \frac{1}{c}\frac{\partial\vec{d}}{\partial t} = \frac{4\pi}{c}\vec{j} \quad , \tag{C-9}$$

if, as is conventional, we define $j^{\nu} = (c\rho, \vec{j})$. Eqs. (C-6) through (C-9) show that the components of $f_{\mu\nu}$ obey the familiar Maxwell's equations. This fact is crucial in relating $f_{\mu\nu}$ to $F_{\mu\nu}$.

We now have enough background to introduce a construction for the Hopf index. Consider an arbitrary fixed point φ_i^0 on our S^2. Then

$$\varphi_i(\vec{r}) = \varphi_i^0 \tag{C-10}$$

is the equation of a closed curve $\mathcal{Z}^0$ on S^3. (We have dropped the time dependence as unimportant here; since time development is a smooth process, it cannot alter the homotopy class of the functions.[32]) Taking Σ^0 as a two dimensional closed connected surface on S^3 having $\mathcal{Z}^0$ as a border, then $\varphi_i(\vec{r})$ maps Σ^0 onto the whole S^2 sphere, covering it an integral number of times. This integer is the Hopf index or charge Q_H. It can be shown that Q_H is independent of the chosen point φ_i^0.

Integral expressions have also been found which yield Q_H. To write these down one introduces, as is permitted by Eq. (C-3), a vector (potential) a_{μ} such that

$$f_{\mu\nu} = \partial_\mu a_\nu - \partial_\mu a_\mu \quad . \tag{C-11}$$

One of the equivalent expressions for the Hopf invariant Q_H is given by an integral over 3-space:[31]

$$Q_H = \frac{1}{4\pi R^6} \iiint \vec{a} \cdot (\vec{\nabla} \times \vec{a})\, dV \tag{C-12}$$

or

$$Q_H = \frac{1}{4\pi R^6} \iiint \vec{a} \cdot \vec{h}\, dV \tag{C-13}$$

where, as is conventional,

$$\vec{h} = \vec{\nabla} \times \vec{a} \quad . \tag{C-14}$$

The factor R^{-6} accounts for the radius R of our S^2. It is straightforward to demonstrate (using integration by parts) that Q_H, as given by Eq. (C-13), is invariant under the gauge transformation $\vec{a} \to \vec{a} \times \vec{\nabla} \Lambda$, where Λ is an arbitrary function.

It is important to note that Eq. (C-13) shows that the Hopf invariant is determined by the "magnetic" part of the tensor function $f_{\mu\nu}$, that is, $\vec{h}$. (In fact, for a static configuration $\vec{e} = 0$.) Equivalently, from Eq. (C-9) the Hopf invariant may be viewed as residing in the configuration of the current $\vec{j}$. Thus in our investigation of the Hopf charge of the vorton, we need only concern ourselves with its dipole-like fields. Since for convenience of calculation, $\Theta = \pi/2$ was chosen (an "electric" vorton), we shall therefore be interested in the $\vec{H}$ fields. But because Maxwell's equations, which govern both $f_{\mu\nu}$ and $F_{\mu\nu}$, are invariant with respect to Θ, the results obtained will be valid for any Θ.

In order to specify the functions φ_i which will yield the correct F_{ij}, the magnetic part of $F_{\mu\nu}$, we first define a quasi-toroidal coordinate system $(\bar{\sigma},\bar{\psi},\bar{\phi})$ based upon the vorton $\vec{H}$ fields as follows: $0 \le \bar{\sigma} \le \infty$ with sheets (of revolution about the z-axis) of constant $\bar{\sigma}$ lying parallel to the local $\vec{H}^{(\phi)}$ fields, the superscript denoting that the field is due to the ϕ rotation, i.e., proportional to ω_ϕ. As in the usual toroidal coordinate system, we choose $\bar{\sigma} = 0$ on the z-axis and $\bar{\sigma} = \infty$ on a ring of radius $\bar{a}$ in the x = y = 0 plane. $\bar{a} \cong 1.825a$ is the root of $\{\}_\theta = 0$, where $\{\}_\theta$ is given by Eq. (11). $0 \le \bar{\psi} \le 2\pi$. where the sheets of constant $\bar{\psi}$ are constructed to be orthogonal to the sheets of constant $\bar{\sigma}$. $\bar{\phi} = \phi$, the usual azimuthal coordinate.

We now consider the functions

$$\varphi_1 = R \cos S(\bar{\sigma}) \quad ,$$

$$\varphi_2 = R \sin S(\bar{\sigma}) \cos (m_\psi \bar{\psi} + m_\phi \bar{\phi}) \quad , \text{ and}$$

$$\varphi_3 = R \sin S(\bar{\sigma}) \sin (m_\psi \bar{\psi} + m_\phi \bar{\phi}) \quad , \qquad \text{(C-15)}$$

where $S(\bar{\sigma})$ is an arbitrary function to be specified later. $S(\bar{\sigma})$ is an angle with boundaries $S(0) = \pi$ and $S(\infty) = 0$. These φ_i clearly satisfy Eq. (C-1).

Using these φ_i in Eq. (C-2) to calculate the (magnetic) components f_{ij} in the system $(\bar{\sigma},\bar{\psi},\bar{\phi})$, which we denote by $\bar{h}_k$, (with a bar to avoid confusion with the usual coordinate scaling factors $h_{\bar{\sigma}}$, $h_{\bar{\psi}}$, and $h_{\bar{\phi}}$) we find that

$$f_{12} = \bar{h}_{\bar{\phi}} = \frac{m_\psi \sin S \, S'}{h_{\bar{\sigma}} \, h_{\bar{\psi}}} \quad , \qquad \text{(C-16)}$$

$$f_{23} = \bar{h}_{\bar{\sigma}} = 0 \quad , \quad \text{and}$$

$$f_{31} = \bar{h}_{\bar{\psi}} = \frac{m_\phi \sin S \, S'}{h_{\bar{\phi}} \, h_{\bar{\sigma}}} \quad , \qquad \text{(C-17)}$$

where $S' = dS/d\bar{\sigma}$.

We note that since $\bar{h}_{\bar{\sigma}} = 0$, the fields $\bar{h}_{\bar{\psi}}$ are parallel to the vorton fields $\vec{H}^{(\phi)}$, which are due to the rotations proportional to m_ϕ. Clearly, the function $S(\bar{\sigma})$ can be selected to make $\bar{h}_{\bar{\psi}}$ everywhere proportional to $\vec{H}^{(\phi)}$ and then R to make

$$\bar{h}_{\bar{\psi}} \, 1_{\bar{\psi}} = \vec{H}^{(\phi)} \quad . \qquad \text{(C-18)}$$

With this selection, assuming that the Hopf currents $\vec{j}$ are due to a uniform rotation of (Hopf) charge density parametrized by ω_ϕ, we see that, because $f_{\mu\nu}$ and j^ν obey Maxwell's equations, the Hopf currents and charge densities must be equal to those of the vorton. Thus the Hopf charge distribution is distributed according to Eq. (6). [What we refer to here is a moving charge density associated with the $\vec{j}$ and not the integrand of Eq. (C-13), which is also sometimes referred to as a Hopf charge density. As in an electric wire, this moving charge density is evidently electrically neutralized by an equal and opposite charge density at rest.] We now can invoke the conformal invariance of Maxwell's equations in both systems to assert that the Hopf field

$$\bar{h}_{\bar{\phi}} \, 1_{\bar{\phi}} = \vec{H}^{(\psi)} = H_\phi \, 1_\phi \qquad \text{(C-19)}$$

(which is proportional to ω_ψ). Thus the (relevant) fields of the vorton can be derived from a specified triplet of scalar functions φ_i. As a consequence we attribute to the vorton the Q_H associated with these φ_i.

While we could now use Eq. (C-12) with the vorton fields (and vector potential) in place of the Hopf fields to compute the Q_H for the vorton, it is simpler and more revealing to examine directly the mapping properites of the φ_i given by Eqs. (C-15). The choice of a φ_i^0 selects a $\bar{\sigma}^0$ and specifies the sum

$$\bar{\Phi}^0 = m_\psi \bar{\psi} + m_\phi \bar{\phi} \quad . \tag{C-20}$$

The closed curve Z^0 in S^3, then, will lie on a (quasi)toroidal surface of constant $\bar{\sigma}$, spiraling around it in a path determined by m_ψ and m_ϕ. If we extract C, the product of the common factors in m_ψ and m_ϕ, such that

$$m_\psi = Cm'_\psi$$

and

$$m_\phi = Cm'_\phi \quad , \tag{C-21}$$

where m'_ψ and m'_ϕ are relative primes, then the curve Z^0 will in one circuit wind m'_ψ times around the z-axis and m'_ϕ times around the ring of radius $\bar{a}$. The surface Σ^0 can be taken in such a way to include m'_ϕ cuts, with $\bar{\phi}$ constant, in the $\bar{\sigma}^0$ torus or doughnut, and m'_ψ (central) webs with $\bar{\psi}$ constant, across the hole in that doughnut. (The remaining portions of Σ^0 lying on $\bar{\sigma} = \bar{\sigma}^0$ do not contribute to the mapping multiplicity.)

The cuts in the doughnut govern the mapping on S^2 for $\bar{\sigma} > \bar{\sigma}^0$ while the webs govern it for $\bar{\sigma} < \bar{\sigma}^0$. Since there are m'_ϕ cuts with a mapping multiplicity m_ψ and m'_ψ webs with a mapping multiplicity m_ϕ, we see that

$$Q_H = m'_\phi m_\psi = m_\phi m'_\psi = C m'_\phi m'_\psi \quad ; \qquad \text{(C-22)}$$

the Hopf charge is nonzero if and only if m_ϕ and $m_\psi \neq 0$. The sign of Q_H is determined by the relative sense of m_ψ and m_ϕ.

That quanta of both types of angular momenta must be present in order for $Q_H \neq 0$ can also be deduced from the symmetry of the vorton and the form of Eq. (C-13). The integrand simplifies to the sum of two terms which are of the form $\vec{A}^{(\phi)} \cdot \vec{H}^{(\psi)} + \vec{A}^{(\psi)} \cdot \vec{H}^{(\phi)}$. We note in passing that a simple loop of current would be characterized by $m_\phi \neq 0$ and $m_\psi = 0$ and hence would have a null topological charge. This result can be deduced from the fact that (in Coulomb gauge) such a configuration has $\vec{A} \cdot \vec{H} = 0$ everywhere.

Finally, it is of interest to remark that while the six operators L_i and X_i form a subgroup of the conformal group locally isomorphic to O(4), the global properties of this subgroup are different from those of O(4). O(4) is quadruply connected,[13] meaning it has only four distinct homotopic classes of mappings, while Q_H, which enumerates the distinct homotopic classes for the vorton, can take on an infinite set of values.

APPENDIX D

The Energy in the Vortonic Electromagnetic Fields

The energy associated with the vortonic electromagnetic fields may be found by integrating the energy density

$$u = \frac{1}{8\pi}\left(\vec{D}\cdot\vec{E} + \vec{B}\cdot\vec{H}\right) \tag{D-1}$$

over all space.

Gaussian units are such that $\vec{D} = \vec{E}$ and $\vec{B} = \vec{H}$ (in free space). Using these free space relationships in the integral

$$W = \iiint u \, dV \tag{D-2}$$

the energy associated with the monopole field reduces to

$$W_m = \frac{2a^6Q^2}{\pi^2}\int_0^\infty \left[\frac{-2r}{(a^2 + r^2)^2} + \frac{r}{a^2(a^2 + r^2)} + \frac{1}{a^3}\tan^{-1}\frac{r}{a}\right]^2 \frac{dr}{r^2} \quad . \tag{D-3}$$

The resultant integrals are either tabulated[33] or convertible to tabulated forms. The $(\tan^{-1} r/a)^2$ term is converted to tabulated form[34] by the substitution $\tan^{-1}\frac{r}{a} = \eta$, to be integrated over the range $0 \le \eta \le \pi/2$. A further substitution $\sin^{-1}X = \eta$, yields tabulated forms,[35] to be integrated over $0 < X < 1$, for the terms linear in $\tan^{-1} r/a$. Collecting all terms yields

$$W_m = \frac{5Q^2}{4\pi a} \quad . \tag{D-4}$$

The calculation of the energy associated with the ϕ rotation may be simplified by using the expressions of the fields given in Eqs. (B-5) and (B-10) directly in Eq. (D-2). Thus we obtain

$$W_d^{(\phi)} = \frac{1}{2}\left(\frac{16a^3Q\omega_\phi}{3\pi c}\right)^2 \int_0^\infty r^2 dr \left[2\mathscr{I}_I^2(r) + \mathscr{I}_{II}^2(r)\right] \quad . \tag{D-5}$$

where $\mathscr{I}_I(r)$ and $\mathscr{I}_{II}(r)$ are given by Eqs. (B-8) and (B-9), respectively. While the second term can be done employing a tabulated form,[36]

$$\int_0^\infty r^2 dr\ \mathscr{I}_{II}^2(r) = \frac{\pi}{128a^5} \quad ., \tag{D-6}$$

the first is more conveniently done by integrating by parts. We use $\int udv = uv - \int vdu$ where $u = \mathscr{I}_I^2(r)$ and $dv = r^2dr$, which means $du = 2\mathscr{I}_I(r)\frac{d\mathscr{I}_I}{dr}\,dr$ and $v = r^3/3$. We see that the term $uv|_0^\infty = 0$. Evaluation of the $\int vdu$ term and some manipulation yields

$$\int_0^\infty r^2 dr\, \mathscr{I}_I^2(r) = \frac{2}{3}\int_0^\infty \frac{rdr}{(a^2+r^2)^3}\int_0^r \frac{r'^4 dr'}{(a^2-r'^2)^3} \tag{D-7}$$

which using tabulated forms[37] yields

$$\int_0^\infty r^2 dr\, \mathscr{I}_I^2(r) = \frac{\pi}{512a^5} \tag{D-8}$$

Eqs. (D-6) and (D-8) in Eq. (D-5) yields

$$W_d^{(\phi)} = \frac{Q^2\beta_\phi^{\ 2}}{6\pi a} \tag{D-9}$$

where the dimensionless quantity

$$\beta_\phi \equiv \frac{a\omega_\phi}{c} \quad . \tag{D-10}$$

Substituting Eq. (B-15) into Eq. (D-2) yields a tabulated form[38] for the radial integral for the energy $W_d^{(\psi)}$ associated with the ψ rotation. Using this form and $\int_0^\pi \sin^3\theta d\theta = 4/3$ yields

$$W_d^{(\psi)} = \frac{Q^2 \beta_\psi^{\;2}}{6\pi a} \qquad \text{(D-11)}$$

where the dimensionless quantity

$$\beta_\psi \equiv \frac{a\omega_\psi}{c} \quad . \qquad \text{(D-12)}$$

APPENDIX E

Action and Angular Momenta in the Vortonic Electromagnetic Fields

This analysis starts with the definition of the action

$$S = \int\int \mathscr{L} \, dVdt \quad , \tag{E-1}$$

where

$$\mathscr{L} = \frac{E^2 - B^2}{8\pi} - \rho\phi + \frac{\vec{j}\cdot\vec{A}}{c} \tag{E-2}$$

is the standard Lagrangian density for Maxwell's equations.[39] While it is known that for the general case, which includes both electric and magnetic charges, there are problems with the Lagrangian formulation,[40] we do this analysis for $\Theta = \pi/2$, the electric charge configuration for which the above Lagrangian applies. Once we have the desired results, we assert that because Maxwell's equations are invariant with respect to the angle Θ,[15] we are no longer constrained to $\Theta = \pi/2$, but may take Θ as a free parameter.

We now must separate out the appropriate piece of the action associated with the rotations; $\int dt$ in Eq. (E-1) will then be taken over one period. Thus,

$$\int dt \rightarrow \frac{2\pi}{\omega_\phi} , \frac{2\pi}{\omega_\psi} \tag{E-3}$$

for ordinary rotations around the z-axis and for toroidal rotations (also associated with the z-axis) respectively.

Using the results of Appendix D, we know from Eq. (31) that at the minimum of the energy (which condition is specified for the vorton configuration) the first term in Eq. (E-2) will lead to a null integral. We also discard the second term because it is not associated with the rotations. The action which we will investigate, then, is

$$S^{(\phi)} \equiv \frac{2\pi}{\omega_\phi c} \iiint \vec{j}^{(\phi)} \cdot \vec{A}^{(\phi)} \, dV \tag{E-4}$$

for the (ordinary) ϕ-rotation and

$$S^{(\psi)} \equiv \frac{2\pi}{\omega_\psi c} \iiint \vec{j}^{(\psi)} \cdot \vec{A}^{(\psi)} \, dV \tag{E-5}$$

for the (toroidal) ψ-rotation. The superscripts are used to label the quantities appropriately. We note that there are no cross terms in the actions; i.e., $\vec{j}^{(\phi)} \cdot \vec{A}^{(\psi)} = \vec{j}^{(\psi)} \cdot \vec{A}^{(\phi)} = 0$.

Since the angles ϕ and ψ range from 0 to 2π, the associated angular momenta will be given by

$$L^{(\phi)} = \frac{1}{2\pi} S^{(\phi)} = \frac{1}{\omega_\phi c} \iiint \vec{j}^{(\phi)} \cdot \vec{A}^{(\phi)} \, dV \tag{E-6}$$

and

$$L^{(\psi)} = \frac{1}{2\pi} S^{(\psi)} = \frac{1}{\omega_\psi c} \iiint \vec{j}^{(\psi)} \cdot \vec{A}^{(\psi)} \, dV \quad . \tag{E-7}$$

The current densities to be used in Eqs. (E-6) and (E-7) are given by

$$\vec{j}^{(\phi)} = q\omega_\phi h_\phi 1_\phi \tag{E-8}$$

and

$$\vec{j}^{(\psi)} = q\omega_\psi h_\psi 1_\psi \tag{E-9}$$

respectively.

To evaluate $L^{(\phi)}$ in Eq. (E-6) we first calculate $\vec{A}^{(\phi)}$. Using Smythe[41] in a manner similar to that in Appendix B, we obtain:

$$\Delta A_\phi^{(\phi)} = \frac{\mu \Delta I \sin\theta'}{2} \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \left(\frac{r}{r'}\right)^n P_n^1(\cos\theta')\, P_n^1(\cos\theta) \tag{E-10}$$

for $r < r'$ and

$$\Delta A_\phi^{(\phi)} = \frac{\mu \Delta I \sin\theta'}{2} \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \left(\frac{r'}{r}\right)^{n+1} P_n^1(\cos\theta')\, P_n^1(\cos\theta) \tag{E-11}$$

for $r > r'$. Proceeding as in Appendix B, Eqs. (E-10) and (E-11) yield

$$A_\phi^{(\phi)} = \frac{8\omega_\phi \sin\theta\, a^3 Q}{3\pi c} \left\{ 2r\mathscr{I}_I - r\mathscr{I}_{II} \right\} \tag{E-12}$$

where $\mathscr{I}_I$ and $\mathscr{I}_{II}$ are defined by Eqs. (B-6) and (B-7) respectively. Using the evaluations of $\mathscr{I}_I$ and $\mathscr{I}_{II}$ given in Appendix B, Eq. (E-12) yields

$$A_\phi^{(\phi)} = \frac{2\omega_\phi a^3 Q \sin\theta}{\pi r c} \left\{ \frac{-1}{a^2+r^2} + \frac{1}{ra} \tan^{-1} \frac{r}{a} \right\} . \tag{E-13}$$

In Eq. (E-6) using $\vec{j}^{(\phi)}$ given by Eq. (E-8) and the $A_\phi^{(\phi)}$ given by Eq. (E-13) one, after some calculations, obtains

$$L^{(\phi)} = \frac{Q^2 \beta_\phi}{3\pi c} . \tag{E-14}$$

Of course, one can also obtain this same result directly by noting that the self-energy content of a current distribution is given by [42]

$$\frac{1}{2c} \iiint \vec{j} \cdot \vec{A}\, dB \quad ; \tag{E-15}$$

equating this energy to the evaluation of the magnetic energy given by Eq. (D-9) yields the value of the action integral, and hence the value of the angular momentum.

Another approach to obtain the (ordinary) angular momentum associated with the vorton is to evaluate it directly in the manner of Thompson's calculation.[43] That is, we simply find the amount of angular momentum (circulating around the z-axis) which is residing in the electromagnetic field. Since the electromagnetic fields associated with vortonic charge distribution have already been obtained, we can evaluate Poynting's vector

$$\vec{g} = \frac{1}{4\pi c}\,\vec{E} \times \vec{H} \quad ; \tag{E-16}$$

an integral of the appropriate component of $\vec{g}$ over 3-space will yield the z-projection of the ordinary angular momentum. That is, using $\vec{L} = \vec{r} \times \vec{p}$, we simply write

$$L^{(\phi)} = \frac{1}{4\pi c} \iiint D_r H_\theta r \, dV \tag{E-17}$$

where D_r is obtained from Eq. (A-8) and H_θ is given by Eq. (B-5). (As in Appendix D, we shall assume the Gaussian free-space relationships $\vec{E} = \vec{D}$ and $\vec{B} = \vec{H}$). It is a straightforward matter to multiply out the product $D_r H_\theta$ and perform the integrals term by term; doing this, one again obtains Eq. (E-14). The conformity of these two different approaches gives one additional confidence that the action formulation of Eqs. (E-4) and E-5) is correct.

We can easily obtain $L^{(\psi)}$ by using Eq. (E-7). Again employing Eq. (E-15), this time with Eq. (D-11), yields

$$L^{(\psi)} = \frac{Q^2 \beta_\psi}{3\pi c} \quad . \tag{E-18}$$

Since it is not clear how to define $L^{(\psi)}$ in terms of $\vec{g}$, this calculation is not done.

While it is tedious, for the sake of completeness, the integral in Eq. (E-7) was also evaluated directly, again obtaining Eq. (E-18). To do this $\vec{A}^{(\psi)}$ was determined in the same fashion as was $\vec{A}^{(\phi)}$ above. For the interested reader we record the components of $\vec{A}^{(\psi)}$ here:

$$A_\rho^{(\psi)} = \frac{-4a^2 Q\omega_\psi P_2^1(\cos\theta)}{3\pi c r^2} \left[\frac{a^2}{2(a^2 + r^2)} + 1 - \frac{3a}{2r} \tan^{-1} \frac{r}{a} \right] \tag{E-19}$$

$$A_\phi^{(\psi)} = 0 \quad , \tag{E-20}$$

and

$$A_z^{(\psi)} = \frac{4Q\omega_\psi}{3\pi c} \left\{ \frac{-a^2}{(a^2 + r^2)} + P_2(\cos\theta) \left[\quad \right] \right\} \quad , \tag{E-21}$$

where

$$\left[\quad \right] \equiv \left[- \frac{a^4}{r^2(a^2 + r^2)} - \frac{2a^2}{r^2} + \frac{3a^3}{r^3} \tan^{-1} \frac{r}{a} \right] \quad . \tag{E-22}$$

For this calculation, the direction cosines

$$\gamma_{\rho\psi} = 1_\rho \cdot 1_\psi = \frac{\partial\rho/\partial\psi}{h_\psi} = \frac{-\sinh\sigma \sin\psi}{\cosh\sigma - \cos\psi} \tag{E-23}$$

and

and

$$\gamma_{z\psi} = 1_z \cdot 1_\psi = \frac{\partial z/\partial\psi}{h_\psi} = \frac{\cosh\sigma\ \cos\psi - 1}{\cosh\sigma - \cos\psi} \qquad \text{(E-24)}$$

are also useful. In performing the integrations in spherical coordinates these quantities will combine with

$$h_\psi = \frac{a}{\cosh\sigma - \cos\psi} \qquad \text{(E-25)}$$

to yield

$$h_\psi \gamma_{\rho\psi} = - \frac{r^2}{a} \cos\theta\ \sin\theta \qquad \text{(E-26)}$$

and

$$h_\psi \gamma_{z\psi} = \frac{1}{2a} \left[2(r\ \sin\theta)^2 - (a^2 + r^2) \right] \quad . \qquad \text{(E-27)}$$

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