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Geometric measure of quantum discord for entanglement of Dirac fields in noninertial frames

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ABSTRACT

We investigate the geometric measure of quantum discord of all possible bipartite divisions of a tripartite system of Dirac fields in noninertial frames. As a comparison, we calculate the geometric measure of entanglement. We discuss the properties of geometric measure of quantum discord and geometric measure of entanglement for three qubit–qubit subsystems with acceleration parameter and the parameter describing the degree of entanglement the system in detail. We have found a conservative relationship involving two of three geometric discords in some condition and another conservative relationship involving three geometric discords for initially maximally entangled states. By the way, we also report some conservative relationships of concurrence, mutual information and geometric measure of entanglement for two bipartite subsystems.

1. Introduction

The theory of relativity and quantum theory form the cornerstones of modern physics. In addition, the combination of quantum theory and information theory further yields quantum information theory. Furthermore, the integration of quantum information and relativity theory creates the theory of relativistic quantum information [1–3], which combines general relativity, quantum field theory and quantum information theory. Evidently, the study on the theory of quantum information in a relativistic framework is not only helpful for understanding some fundamental questions in quantum information theory, but also is practical, because many contemporary experiments on quantum-information processing use photons or other particles that have relativistic velocities. In recent years, the theory of relativistic quantum information has become a focus of research in quantum information science for both conceptual and experimental reasons.

Recently, much effort has been made in the study of entanglement shared between inertial and noninertial observers by discussing how the Unruh effect and Hawking effect will influence the degree of entanglement. Following some seminal work performed in this regard [4,5], many authors focus on the study of entanglement between quantum field modes as observed by relatively accelerating observers. For example, Q. Pan and J. Jing studied the degradation of non-maximal entanglement of scalar and Dirac fields in non-inertial frames [6]. Mi-Ra Hwang et al. studied tripartite entanglement in a noninertial frame using π-tangle [7]. J. Wang and J. Jing investigated multipartite entanglement of fermionic systems in noninertial frames also using π-tangle [8]. Due to the resemblance between the Unruh effect [9] and Hawking radiation [10], some authors also studied the degradation of entanglement occurred in black-hole physics. Q. Pan and J. Jing have investigated the effect of the Hawking temperature on the entanglement and teleportation for the scalar field in a most general, static and asymptotically flat black hole with spherical symmetry [11]. They also studied entanglement redistribution in the Schwarzschild spacetime [12].

Though most studies in noninertial systems have focused on the entanglement, however, it has been found that the entanglement is not the only characteristic of a quantum system, and it has no advantage for some quantum information tasks. In some cases [13–15], although there is no entanglement, certain quantum information processing tasks can still be done efficiently by using quantum correlation [16–18], which is believed to be more workable than the entanglement. Quantum correlation may be quantified by quantum discord rather than by entanglement. The quantum discord, initially introduced by Ollivier and Zurek [16] and by Henderson and Vedral [17], is a measure of quantum correlations that extends beyond entanglement. M. Ali et al. found the Quantum discord for two-qubit X states [19]. J. Wang et al. studied

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classical correlation and quantum discord sharing of Dirac fields in noninertial frames [20]. C.C. Rulli et al. studied the global quantum discord in multipartite systems [21]. A. Datta investigated the quantum discord between relatively accelerated observers [22].

The above studies show the calculation of quantum discord involves a difficult optimization procedure: it is difficult to obtain analytical results except for a few families of two-qubit states [19,23,24]. Therefore, Dakić et al. proposed a geometric measure of quantum discord [25], which also was called a geometric discord [26–28].

Before we start discussing our subject, we notice that recently there was a debate on the geometric measure of quantum discord [29]. M. Piani argued that the geometric measure of quantum discord is not a good measure for the quantum correlations. A detailed discussion about this issue in detail is beyond the scope of this paper. Here, we will still use the geometric discord to study bipartite correlations presented in three two-qubit subsystems of the tripartite system in a noninertial frame.

In this paper we are going to consider the following situation. Alice and Rob share an entangled state initially when they are not moving relatively. Subsequently, Rob moves with a uniform acceleration with respect to Alice. This system is a bipartite from an inertial perspective, but from a noninertial perspective an extra set of complementary modes in Rindler region II becomes relevant. Therefore, we calculate the geometric measure of quantum discord in all possible bipartite divisions of the tripartite system: the mode A described by Alice, the mode I in Rindler region I (described by Rob), and the complementary mode II in Rindler region II. For comparison, we also derived the analytic expressions of geometric measure of entanglement [30,31] as a function of Rob's acceleration for the same system. Our results revealed that geometric measure of quantum discord gives the similar global properties as geometric measure of entanglement does for the system under consideration, but in some respect, the description of the system using geometric discord are more detailed than using other measures. More important, we report some conservative relationship of geometric discord, concurrence and mutual information in noninertial frame.

This paper is organized as follows. In the next section, we give a short review of geometric measure of quantum discord and geometric measure of entanglement. In Section 3 we derive the analytic expressions of geometric discord. Section 4 devotes to calculate the geometric measure of entanglement and compares two kinds of geometric measures. A detailed discussion and summary are given in Section 5.

2. Brief review of geometric measure of quantum discord and geometric measure of entanglement

For convenience of later use, we give a brief review of geometric measure of quantum discord and geometric measure of entanglement, respectively.

Quantum discord is a quantum-versus-classical paradigm for correlations [32–34] and is not in the entanglement-versus-separability framework [35,36]. The quantum discord of a bipartite state ρ on a system $H^a \otimes H^b$ with marginals ρ^a and ρ^b can be expressed as

$$Q(\rho) = \min_{\Pi^a}\{I(\rho) - I(\Pi^a(\rho))\}. \tag{1}$$

Here the minimum is over von Neumann measurements (one-dimensional orthogonal projectors summing up to the identity) $\Pi^a = \{\Pi^a_k\}$ on subsystem a, and

$$\Pi^a(\rho) = \sum_k (\Pi^a_k \otimes I^b)\rho(\Pi^a_k \otimes I^b) \tag{2}$$

is the resulting state after the measurement. $I(\rho) = S(\rho^a) + S(\rho^b) - S(\rho)$ is the quantum mutual information, $S(\rho) = -\operatorname{tr}\rho\ln\rho$ is the von Neumann entropy, and I^b is the identity operator on H^b. Then, Dakić et al. proposed the following geometric measure of quantum discord [25]:

$$D(\rho) = \min_{\chi}\|\rho - \chi\|_2^2, \tag{3}$$

where the minimum is over the set of zero-discord states [i.e., $Q(\chi) = 0$] and $\|A\|_2 := \sqrt{\operatorname{tr}(A^\dagger A)}$ is the Frobenius or Hilbert–Schmidt norm. The density operator of any two-qubit state can be expressed as

$$\rho = \frac{1}{4}\left(\mathbf{I}^A \otimes \mathbf{I}^B + \sum_{i=1}^{3}(x_i\sigma_i \otimes \mathbf{I}^B + \mathbf{I}^A \otimes y_i\sigma_i) + \sum_{i,j=1}^{3} t_{ij}\sigma_i \otimes \sigma_j\right), \tag{4}$$

where $\{\sigma_i,\ i = 1, 2, 3\}$ denote the Pauli spin matrices. Then, the geometric measure of quantum discord of any two-qubit state is evaluated as

$$D(\rho) = \frac{1}{4}(\|\mathbf{x}\|^2 + \|\mathbf{T}\|^2 - k_{max}), \tag{5}$$

where $\mathbf{x} := (x_1, x_2, x_3)^t$ is a column vector, $\|\mathbf{x}\|^2 := \sum_i x_i^2$, $x_i = \operatorname{tr}(\rho(\sigma_i \otimes \mathbf{I}^b))$, $T := (t_{ij})$ is a matrix and $t_{ij} = \operatorname{tr}(\rho(\sigma_i \otimes \sigma_j))$, k_{max} is the largest eigenvalue of matrix $\mathbf{x}\mathbf{x}^t + \mathbf{T}\mathbf{T}^t$.

Since Dakić et al. proposed the geometric measure of quantum discord, many authors extended Dakić's results to the general bipartite states. Luo and Fu evaluated the geometric measure of quantum discord for an arbitrary state and obtained an explicit formula

$$D(\rho) = \operatorname{tr}(\mathbf{C}\mathbf{C}^t) - \max_A\ \operatorname{tr}(\mathbf{A}\mathbf{C}\mathbf{C}^t\mathbf{A}^t), \tag{6}$$

where $\mathbf{C} = (c_{ij})$ is an $m^2 \times n^2$ matrix, given by the expansion $\rho = \sum c_{ij} X_i \otimes Y_j$ in terms of orthonormal operators $X_i \in L(H^a), Y_j \in L(H^b)$ and $A = (a_{ki})$ is an $m \times m^2$ matrix given by $a_{ki} = \operatorname{tr}|k\rangle\langle k|X_i = \langle k|X_i|k\rangle$ for any orthonormal basis $|k\rangle$ of H^a. They also gave a tight lower bound for geometric discord of arbitrary bipartite states [37]. Recently, a different tight lower bound for geometric discord of arbitrary bipartite states was given by S. Rana et al. [26], and Ali Saif M. Hassan et al. [38] independently. Alternatively, D. Girolami et al. found an explicit expression of geometric discord for two-qubit system and extended it to $(2\bigotimes d)$-dimensional systems [27]. T. Tufarelli et al. also gave another formula of geometric discord for qubit–qudit system, which is available to $(2\bigotimes d)$-dimensional systems including $d = \infty$ [28].

On the other hand, geometric measure of entanglement was first proposed by T.C. Wei et al. [30,31]. For pure states it is defined as follows:

$$E_g(|\psi\rangle) = 1 - \Lambda^2_{\max} = 1 - \max_{\phi}|\langle\psi|\phi\rangle|^2 \tag{7}$$

where $|\phi\rangle$ is an arbitrary separable pure state and the maximization is done over the set of $|\phi\rangle$. For mixed states ρ, the geometric measure of entanglement was originally defined via the convex roof construction, in the same way as was done for the entanglement of formation:

$$E_g(\rho) = \min\sum_i p_i E_g(|\psi_i\rangle) \tag{8}$$

with minimization over all pure state decompositions of ρ. Calculation of geometric measure needs to find the entanglement eigenvalue Λ_{max}. For bipartite pure states, this is a linear problem. However, generally speaking, for the case of three or more parts, the eigenproblem becomes nonlinear. Fortunately, using a theorem stating that any reduced $(n-1)$-qubit state uniquely determines the geometric measure of the original n-qubit pure state [39], L. Tamaryan et al. obtained an analytic expressions for geometric measure of three-qubit pure states [40]. For arbitrary two-qubit mixed states, the following analytical expression has been obtained [30],

$$E_g(\rho) = \frac{1}{2}\left[1 - \sqrt{1 - C(\rho)^2}\right], \tag{9}$$

here the concurrence $C(\rho)$ is given by

$$C(\rho) = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}, \tag{10}$$

where λ_i are the square roots of the eigenvalues of $\rho \cdot \tilde{\rho}$ in decreasing order, and $\tilde{\rho}$ is defined as $\tilde{\rho} = (\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y)$. For most quantum states, except for a few special cases, such as generalized Werner states, isotropic states [30], no exact expression for geometric measure of entanglement is known. Therefore, A. Streltsov et al. developed a numerical algorithm to calculate the geometric measure of entanglement for any multipartite mixed state [41].

3. Geometric discord for Dirac fields in noninertial frames

We assume that Alice and Rob share initially the entangled state in an inertial frame,

$$|\psi\rangle_{AR} = \alpha|0_A 0_R\rangle + \sqrt{1-\alpha^2}|1_A 1_R\rangle. \tag{11}$$

After sharing his own qubit, Rob moves with respect to Alice with a uniform acceleration a. Using the single-mode approximation, Rob's vacuum and one-particle states $|0_R\rangle$ and $|1_R\rangle$ in Minkowski space are transformed into [5]

$$\begin{aligned} |0_R\rangle &\to \cos r|0_I 0_{II}\rangle + \sin r|1_I 1_{II}\rangle, \\ |1_R\rangle &\to |1_I 0_{II}\rangle, \end{aligned} \tag{12}$$

where r is the acceleration parameter, which is in the range $0 \le r \le \pi/4$ for $0 \le a \le \infty$, $|n_I\rangle$ and $|n_{II}\rangle (n = 0, 1)$ are the mode decomposition in the two causally disconnected regions in Rindler space. Using Eq. (12), we obtain

$$\begin{aligned} |\psi\rangle_{A,I,II} = {} & \alpha|0_A\rangle \otimes (\cos r|0_I\, 0_{II}\rangle + \sin r|1_I 1_{II}\rangle) \\ & + \sqrt{1-\alpha^2}|1_A 1_I\, 0_{II}\rangle, \end{aligned} \tag{13}$$

and corresponding density operator

$$\begin{aligned} \rho_{A,I,II} = {} & \alpha^2 \cos^2 r|0_A 0_I 0_{II}\rangle\langle 0_A 0_I 0_{II}| \\ & + \alpha^2 \sin^2 r|0_A 1_I 1_{II}\rangle\langle 0_A 1_I 1_{II}| \\ & + \alpha\sqrt{1-\alpha^2}[(\cos r|0_A 0_I 0_{II}\rangle + \sin r|0_A 1_I 1_{II}\rangle)\langle 1_A 1_I 0_{II}| \\ & + |1_A 1_I 0_{II}\rangle(\cos r\langle 0_A 0_I 0_{II}| + \sin r\langle 0_A 1_I 1_{II}|)] \\ & + \alpha^2 \cos r \sin r(|0_A 1_I 1_{II}\rangle\langle 0_A 0_I 0_{II}| \\ & + |0_A 0_I 0_{II}\rangle\langle 0_A 1_I 1_{II}|) \\ & + (1-\alpha^2)|1_A 1_I 0_{II}\rangle\langle 1_A 1_I 0_{II}|. \end{aligned} \tag{14}$$

Taking the trace over the mode in region II, we obtain a mixed density operator between Alice and mode I,

$$\begin{aligned} \rho_{A,I} = {} & \alpha^2(\cos^2 r|0_A 0_I\rangle\langle 0_A 0_I| + \sin^2 r|0_A 1_I\rangle\langle 0_A 1_I|) \\ & + \alpha\sqrt{1-\alpha^2}\cos r(|1_A 1_I\rangle\langle 0_A 0_I| + |0_A 0_I\rangle\langle 1_A 1_I|) \\ & + (1-\alpha^2)|1_A 1_I\rangle\langle 1_A 1_I|. \end{aligned} \tag{15}$$

Recall that Pauli spin matrices can be expressed by Dirac notation,

$$\sigma_x^a = |0_a\rangle\langle 1_a| + |1_a\rangle\langle 0_a|, \tag{16a}$$

$$\sigma_y^a = i(|1_a\rangle\langle 0_a| - |0_a\rangle\langle 1_a|), \tag{16b}$$

$$\sigma_z^a = |0_a\rangle\langle 0_a| - |1_a\rangle\langle 1_a|, \tag{16c}$$

where σ_x^a $(x = 1, 2, 3)$ is Pauli spin matrix expressed by basis vectors $|0_a\rangle$ and $|1_a\rangle$ of qubit a $(a = A, I, II)$. To calculate $D(\rho_{A,I})$, we first calculate vector $\mathbf{x}$ and matrix T and obtain

$$\mathbf{x} = (0, 0, 2\alpha^2 - 1)^t \tag{17}$$

and

$$T = \begin{pmatrix} 2\alpha\sqrt{1-\alpha^2}\sin r & 0 & 0 \\ 0 & 2\alpha\sqrt{1-\alpha^2}\sin r & 0 \\ 0 & 0 & 2\alpha^2\cos^2 r - 1 \end{pmatrix}. \tag{18}$$

It is easy to find $\mathbf{XX}^t + TT^t$ has two eigenvalues $\{4\alpha^2(1-\alpha^2)\cos^2 r, (2\alpha^2-1)^2 + (2\alpha^2\cos^2 r - 1)^2\}$. Substituting these and $\mathbf{x}$ as well as T into Eq. (5) and doing some simplification, we obtain the geometric measure of quantum discord between mode A and mode I.

$$D(\rho_{A,I}) = \begin{cases} 2\alpha^2(1-\alpha^2)\cos^2 r, \\ \quad 4 + \alpha^4[15 + \cos(4r)] \ge 16\alpha^2; \\ \frac{1}{8}\{4 + \alpha^4[7 + \cos(4r)] \\ \quad - 8\alpha^2[1 - (1-\alpha^2)\cos(2r)]\}, \\ \quad \text{otherwise.} \end{cases} \tag{19}$$

To investigate quantum discord in this system in more detail we must further consider other two quantum correlations of bipartite mixed-state subsystems besides the quantum discord between the Minkowski mode A and Rindler mode I. In the following, we calculate the geometric measure of quantum discord between mode A and mode II as well as between mode I and mode II. Tracing over the mode in region I(A), we obtain the density operators $\rho_{A,II}$ and $\rho_{I,II}$

$$\begin{aligned} \rho_{A,II} = {} & \alpha^2\cos^2 r|0_A 0_{II}\rangle\langle 0_A 0_{II}| \\ & + [(1-\alpha^2)|1_A 0_{II}\rangle + \alpha\sqrt{1-\alpha^2}\sin r|0_A 1_{II}\rangle]\langle 1_A 0_{II}| \\ & + \alpha\sin r(\sqrt{1-\alpha^2}|1_A 0_{II}\rangle + \alpha\sin r|0_A 1_{II}\rangle)\langle 0_A 1_{II}|, \end{aligned} \tag{20}$$

$$\begin{aligned} \rho_{I,II} = {} & \alpha^2\cos^2 r|0_I 0_{II}\rangle\langle 0_I 0_{II}| + \alpha^2\sin^2 r|1_I 1_{II}\rangle\langle 1_I 1_{II}| \\ & + \alpha^2\cos r\sin r(|1_I 1_{II}\rangle\langle 0_I 0_{II}| + |0_I 0_{II}\rangle\langle 1_I 1_{II}|) \\ & + (1-\alpha^2)|1_I 0_{II}\rangle\langle 1_I 0_{II}|. \end{aligned} \tag{21}$$

Using the same procedure as calculating the geometric measure of quantum discord between mode A and mode I, we obtain geometric measure of quantum discord between mode A and mode II and one between mode I and mode II,

$$D(\rho_{A,II}) = \begin{cases} 2\alpha^2(1-\alpha^2)\sin^2 r, \\ \quad 4 + \alpha^4[15 + \cos(4r)] \ge 16\alpha^2; \\ \frac{1}{8}\{4 + \alpha^4[7 + \cos(4r)] \\ \quad - 8\alpha^2[1 + (1-\alpha^2)\cos(2r)]\}, \\ \quad \text{otherwise.} \end{cases} \tag{22}$$

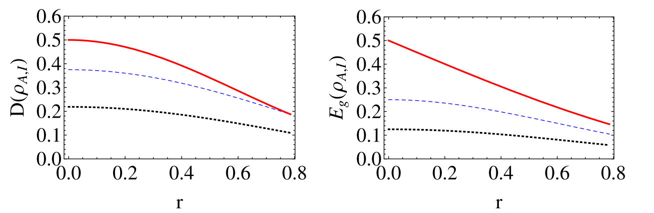

Fig. 1. (Color online.) Plots of geometric measure of quantum discord and entanglement $D(\rho_{A,I})$ and $E_g(\rho_{A,I})$ as functions of r between Alice and Rob in region I for $|\alpha| = 1/2$ (dashed and blue); $|\alpha| = 1/\sqrt{2}$ (thick and red); $|\alpha| = \sqrt{7/8}$ (dotted and black).

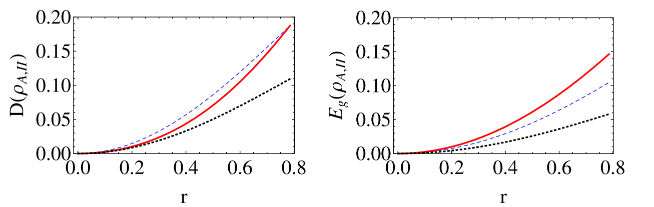

Fig. 2. (Color online.) Plots of geometric measure of quantum discord and entanglement $D(\rho_{A,II})$ and $E_g(\rho_{A,II})$ as functions of r between Alice and anti-Rob in region II for $|\alpha| = 1/2$ (dashed and blue); $|\alpha| = 1/\sqrt{2}$ (thick and red); $|\alpha| = \sqrt{7/8}$ (dotted and black).

$$D(\rho_{I,II}) = \begin{cases} \frac{1}{2}\alpha^4 \sin^2(2r), \\ \quad \alpha^2\{\alpha^2[5+\cos(4r)] - 2(1-\alpha^2)\cos(2r) - 6\} \geq -2; \\ \frac{1}{2}\{1+\alpha^2[3\alpha^2 - 3 - (1-\alpha^2)\cos(2r)]\}, \\ \quad \text{otherwise}. \end{cases} \tag{23}$$

For the special case that Alice and Rob shared initially the maximally entangled state, which means $|\alpha| = \frac{1}{\sqrt{2}}$, Eqs. (19), (22), (23) reduce to

$$D(\rho_{A,I}) = \frac{1}{4}\cos^2 r\left(1+\cos^2 r\right); \tag{24a}$$

$$D(\rho_{A,II}) = \frac{1}{4}\sin^2 r\left(1+\sin^2 r\right); \tag{24b}$$

$$D(\rho_{I,II}) = \frac{1}{4}\sin^2 r. \tag{24c}$$

To analyze the properties of $D(\rho_{A,I})$, $D(\rho_{A,II})$ and $D(\rho_{I,II})$ we can calculate derivatives of them with respect to r and α^2. We find that $D(\rho_{A,I})$ is monotone decreasing function of r, but $D(\rho_{A,II})$ and $D(\rho_{I,II})$ are monotone increasing functions of r for all values of $0 \leq |\alpha| \leq 1$, respectively. The behaviors of $D(\rho_{A,I})$, $D(\rho_{A,II})$ and $D(\rho_{I,II})$ with $|\alpha|$ are more complicated than that of them with r. $D(\rho_{A,I})$, $D(\rho_{A,II})$, $D(\rho_{A,II})$ increase with increasing of $|\alpha|$ for some α and r, but otherwise they are decrease with increasing of $|\alpha|$. To further demonstrate these properties, we plot $D(\rho_{x,y})$ (hereafter in this paper {x, y} = {A, I}, {A, II} and {I, II}) as functions of acceleration parameter r for some typical values of $|\alpha|$ on the left sides in Figs. 1–3, and $D(\rho_{x,y})$ as functions of parameter α^2 for some typical values of acceleration parameter r on the left sides in Figs. 4–6, respectively.

4. Comparison with geometric measure of entanglement

For further understanding the quantum discord of the system consisted of Alice and Rob's modes I and II, we shall calculate the geometric measure of entanglement for the same system and compare the results of two kind geometric measures.

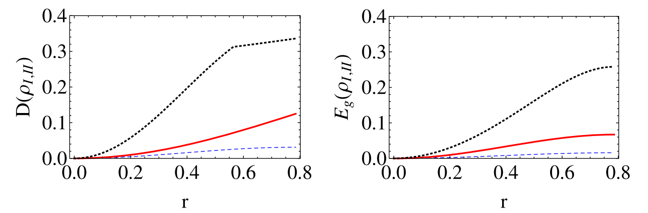

Fig. 3. (Color online.) Plots of geometric measure of quantum discord and entanglement $D(\rho_{I,II})$ and $E_g(\rho_{I,II})$ as functions of r between the modes in regions I and II for $|\alpha| = 1/2$ (dashed and blue); $|\alpha| = 1/\sqrt{2}$ (thick and red); $|\alpha| = \sqrt{7/8}$ (dotted and black).

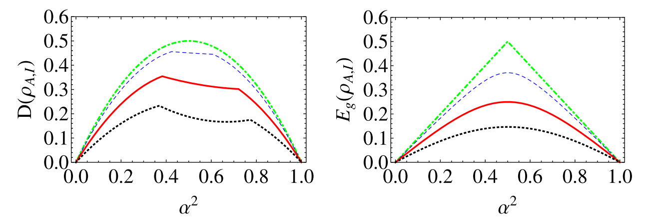

Fig. 4. (Color online.) Plots of geometric measure of quantum discord and entanglement $D(\rho_{A,I})$ and $E_g(\rho_{A,I})$ as functions of α^2 between Alice and Rob in region I for acceleration parameter $r = 0$ (dot-dashed and green); $r = \pi/12$ (dashed and blue); $r = \pi/6$ (thick and red); $r = \pi/4$ (dotted and black).

In the following, we calculate geometric measures of entanglement for $\rho_{A,I}$, $\rho_{A,II}$, $\rho_{I,II}$, respectively. It is easy to find $C(\rho_{A,I}) = 2\alpha\sqrt{1-\alpha^2}\cos r$, $C(\rho_{A,II}) = 2\alpha\sqrt{1-\alpha^2}\sin r$ and $C(\rho_{I,II}) = \alpha^2\sin(2r)$. Substituting these concurrences into Eq. (9), we obtain E_g for $\rho_{A,I}$, $\rho_{A,II}$, $\rho_{I,II}$, respectively,

$$E_g(\rho_{A,I}) = \frac{1}{2}\left(1-\sqrt{1-4\alpha^2(1-\alpha^2)\cos^2 r}\right), \tag{25}$$

$$E_g(\rho_{A,II}) = \frac{1}{2}\left(1-\sqrt{1-4\alpha^2(1-\alpha^2)\sin^2 r}\right), \tag{26}$$

$$E_g(\rho_{I,II}) = \frac{1}{2}\left(1-\sqrt{1-\alpha^4\sin^2(2r)}\right). \tag{27}$$

For the special case $|\alpha| = \frac{1}{\sqrt{2}}$, Eqs. (25), (26), (27) reduce to

$$E_g(\rho_{A,I}) = \frac{1}{2}(1-\sin r); \tag{28a}$$

$$E_(\rho_{A,II}) = \frac{1}{2}(1-\cos r); \tag{28b}$$

$$E_g(\rho_{I,II}) = \frac{1}{2}\left(1-\sqrt{1-\sin^2 r\cos^2 r}\right). \tag{28c}$$

To compare the geometric measure of the quantum discord with geometric measure of entanglement we also plot $E_g(\rho_{x,y})$ as functions of acceleration parameter r for some typical values of α^2 on the right sides in Figs. 1–3, and $E_g(\rho_{x,y})$ as functions of parameter α^2 for some typical values of acceleration parameter r on the right sides in Figs. 4–6, respectively. Figs. 1–3 show that $E_g(\rho_{A,I})$ is decreasing function of r, but $E_g(\rho_{A,II})$ and $E_g(\rho_{I,II})$ are increasing functions of r for $\alpha^2 \leq 1$, respectively. Meanwhile, Figs. 4–6 show that $E_g(\rho_{A,I})$ and $E_g(\rho_{A,II})$ first increase to their maximum values at $\alpha = 1/\sqrt{2}$, then decrease to 0 at $|\alpha| = 1$, respectively, but $E_g(\rho_{I,II})$ always increases with the increase of $|\alpha|$.

As for the special case $\alpha^2 = 1/2$, we plot $D(\rho_{x,y})$ and $E_g(\rho_{x,y})$ as functions of an acceleration parameter r for $|\alpha| = \frac{1}{\sqrt{2}}$ on the left side and right side, respectively, in Fig. 7, which shows that $D(\rho_{x,y})$ and $E|_g(\rho_{x,y})$ have similar behavior with r, and $D(\rho_{x,y}) \geq E_g(\rho_{x,y})$ always hold for the same $\rho_{x,y}$.

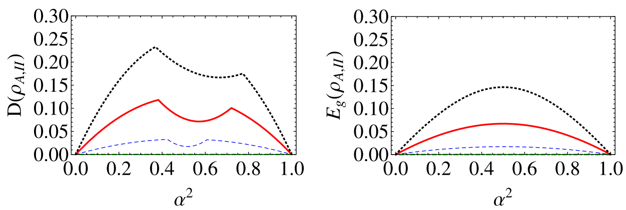

Fig. 5. (Color online.) Plots of geometric measure of quantum discord and entanglement $D(\rho_{A,II})$ and $E_g(\rho_{A,II})$ as functions of α^2 between Alice and anti-Rob in region II for acceleration parameter $r=0$ (dot-dashed and green); $r=\pi/12$ (dashed and blue); $r=\pi/6$ (thick and red); $r=\pi/4$ (dotted and black).

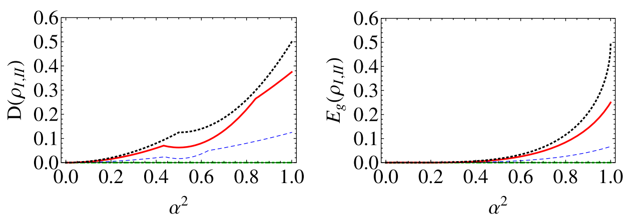

Fig. 6. (Color online.) Plots of geometric measure of quantum discord and entanglement $D(\rho_{I,II})$ and $E_g(\rho_{I,II})$ as functions of α^2 between the modes in regions I and II for acceleration parameter $r=0$ (dot-dashed and green); $r=\pi/12$ (dashed and blue); $r=\pi/6$ (thick and red); $r=\pi/4$ (dotted and black).

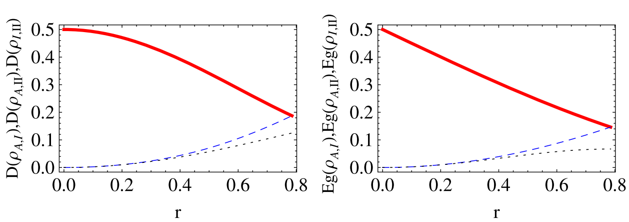

Fig. 7. (Color online.) Plots of geometric measure of quantum discord and entanglement $D(\rho_{A,I}), D(\rho_{A,II}), D(\rho_{I,II})$ and $Eg(\rho_{A,I}), Eg(\rho_{A,II}), Eg(\rho_{I,II})$ as functions of r. $D(\rho_{A,I})$ and $Eg(\rho_{A,I})$: thick and red; $D(\rho_{A,II})$ and $Eg(\rho_{A,II})$: dashed and blue; $D(\rho_{I,II})$ and $Eg(\rho_{I,II})$: dotted, black.

5. Discussion and summary

We now discuss $D(\rho_{x,y})$ and $E_g(\rho_{x,y})$ obtained in Section 3 and Section 4 in detail. First, Figs. 1–3 show $D(\rho_{A,I})$ is the decreasing function of an acceleration parameter r, while $D(\rho_{A,II})$ and $D(\rho_{I,II})$ are increasing functions of r for all α. $D(\rho_{A,I})$ decreases from $2\alpha^2(1-\alpha^2)$ when $r=0$ to $\alpha^2(1-\alpha^2)$ for $\alpha \le \sqrt{7(4-\sqrt{2})}/7 \approx 0.675$ or $\alpha \ge \sqrt{7(4+\sqrt{2})}/7 \approx 0.829$, alternatively to $(2-4\alpha^2+3\alpha^4)/4$ for other cases, respectively, when $r=\pi/4$. We noticed that for the case $\alpha \le \sqrt{7(4-\sqrt{2})}/7$ or $\alpha \ge \sqrt{7(4+\sqrt{2})}/7$ and acceleration go to infinity, both $D(\rho_{A,I})$ and $D(\rho_{A,II})$ equal the half of $D(\rho_{A,I})$ at $r=0$. $D(\rho_{A,II})$ increases from 0 when $r=0$ to the same value of $D(\rho_{A,I})$ at $r=\pi/4$. $D(\rho_{I,II})$ increases from 0 when $r=0$ to $\alpha^4/2$ when $|\alpha| \le 1/\sqrt{2}$, or $(1-3\alpha^2+3\alpha^4)/2$ for $1/\sqrt{2} < |\alpha| \le 1$ at $r=\pi/4$.

Figs. 1–3 also show $E_g(\rho_{x,y})$ as functions of r has the same properties of $D(\rho_{x,y})$. It means that $E_g(\rho_{A,I})$ is the decreasing function of an acceleration parameter r, but $E_g(\rho_{A,II})$ and $E_g(\rho_{1,II})$ are increasing functions of r for all α. $E_g(\rho_{A,I})$ decreases from $(1-|1-2\alpha^2|)/2$ when $r=0$ to $(1-\sqrt{1-2\alpha^2+2\alpha^4})/2$ when $r=\pi/4$. $E_g(\rho_{A,II})$ and $E_g(\rho_{I,II})$ increase from 0 at $r=0$ to $(1-\sqrt{1-2\alpha^2+2\alpha^4})/2$ and $(1-\sqrt{1-\alpha^4})/2$, respectively, when $r=\pi/4$. We noticed $D(\rho_{A,I})=D(\rho_{A,II})$ and $E_g(\rho_{A,I})=E_g(\rho_{A,II})$, which are not zero, when $r=\pi/4$.

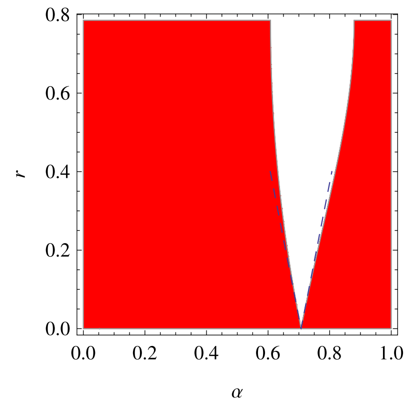

Fig. 8. (Color online.) The plot of the region (red) where conservative relation (31) valids.

Second, we investigate how $D(\rho_{x,y})$ and $E_g(\rho_{x,y})$ vary with parameter α for the same r. In this case, $E_g(\rho_{x,y})$ have simpler relations with $|\alpha|$ than $D(\rho_{x,y})$ have. Figs. 4–6 show $E_g(\rho_{A,I})$ and $E_g(\rho_{A,II})$ increase from 0 when $\alpha=0$ to their maximum values $(1-\sin r)/2$ and $(1-\cos r)/2$ at $|\alpha|=1/\sqrt{2}$, respectively, then decrease to 0 at $|\alpha|=1$ for all r. Meanwhile, $E_g(\rho_{I,II})$ increases from 0 at $r=0$ to $\sin^2 r$ at $r=\pi/4$. The global behavior of $D(\rho_{x,y})$ with $|\alpha|$ is similar to that of $E_g(\rho_{x,y})$ with $|\alpha|$, but it is more complicated than that of $E_g(\rho_{x,y})$ in the details, which can be seen from Figs. 4–6. When α varies from 0 to its maximum value 1, except for the case of $r=0$, $D(\rho_{A,I})$ and $D(\rho_{A,II})$ first increase from 0 to their first maximum, then decrease to their minimums, afterwards increase to their second maximum again, finally reduce to 0 for a fixed r. The positions of maximum and minimum points of $D(\rho_{A,I})$ and $D(\rho_{A,II})$ for α depend on values of α and r. In contrast, $D(\rho_{I,II})$ increases to a maximum, then decreases to a minimum, finally increases to $2\sin^2 r\cos^2 r$ when α increases form 0 to 1.

Third, we find there are some conservative relations for concurrence, mutual information, geometric measure of entanglement and geometric measure of quantum discord. We have obtained concurrences $C(\rho_{A,I}), C(\rho_{A,II})$ and $C(\rho_{I,II})$ in Section 4. It is easy to check that $C(\rho_{A,I}(r))^2+C(\rho_{A,II}(r))^2=C(\rho_{A,I}(0))^2$. In the similar way we have also found $I(\rho_{A,I}(r))+I(\rho_{A,II}(r))=I(\rho_{A,I}(0))$, where $I(\rho_{x,y})=S(\rho_x)+S(\rho_y)-S(\rho_{x,y})$ is the mutual information and $S(\rho)=-\mathrm{Tr}(\rho\log_2\rho)$ is the entropy of the density matrix ρ. Refs. [5,6,12] have calculated the concurrence and the mutual information, but unfortunately, these two conservative relations were not presented. Using the conservative relation of the concurrence $C(\rho)$ and Eq. (9) we can further find the conservative relation of geometric measure of entanglement:

$$\left[E_g(\rho_{A,I}(r))-\frac{1}{2}\right]^2+\left[E_g(\rho_{A,II}(r))-\frac{1}{2}\right]^2 = \left[E_g(\rho_{A,I}(0))-\frac{1}{2}\right]^2+\frac{1}{4}. \tag{29}$$

Geometric measures of quantum discord $D(\rho_{A,I})$ and $D(\rho_{A,II})$ have also a conservative relation under some conditions. From Eq. (19) and Eq. (22) we see when

$$\alpha^4[15+\cos(4r)]+4>16\alpha^2, \tag{30}$$

$$D(\rho_{A,I}(r))+D(\rho_{A,II}(r))=D(\rho_{A,I}(0)). \tag{31}$$

Unlike Eq. (29), which holds for all allowable r and α, Eq. (31) is restricted by Eq. (30). Fortunately, Eq. (31) is valid in the most of the region $0 \le \alpha \le 1$ and $0 \le r \le \pi/4$. We have plotted the valid region of α and r for Eq. (31) in Fig. 8, which shows just in a very narrow strip area around $\alpha = 1/\sqrt{2}$ Eq. (31) is not valid. Therefore, Eq. (31) is not applicable for $\alpha = 1/\sqrt{2}$ except for $r = 0$. It is notable that above conservative relationships do not involve measurements related to $\rho_{I,II}(r)$. This can only be imputed to Unruh effect that makes Rob causally disconnected from mode II. Though, there is no conservative relation between geometric measures of quantum discord $D(\rho_{A,I})$ and $D(\rho_{A,II})$ for $\alpha = 1/\sqrt{2}$, but for this case, there exists a more general conservative relation. From Eqs. (24) we can easily obtain

$$D(\rho_{A,I}(r)) - D(\rho_{A,II}(r)) + 4D(\rho_{I,II}(r)) = D(\rho_{A,I}(0)) = \frac{1}{2}, \quad \text{for } |\alpha| = \frac{1}{\sqrt{2}}. \tag{32}$$

We could understand why $D(\rho_{A,I})$ is the decreasing function of an acceleration parameter r, while $D(\rho_{A,II})$ and $D(\rho_{I,II})$ are increasing functions of r for all α' partially from Eqs. (29), (31), (32).

Fourth, we noticed that the equation of dividing lines between the red region and the white region in Fig. 8 is

$$F(\alpha, r) = \alpha^4[15 + \cos(4r)] + 4 - 16\alpha^2 = 0, \tag{33}$$

which is just the critical condition that makes $D(\rho_{A,I})$ and $D(\rho_{A,II})$ take the different forms depending on α and r in Eq. (19) and Eq. (22), respectively. It is easy to find: $F_\alpha(\alpha, r) = F_r(\alpha, r) = 0$, where $F_x = \partial F/\partial x$, at points $(\alpha = 1/\sqrt{2},\ r = 0)$ and $(\alpha = 2/\sqrt{7},\ r = \pi/4)$. Therefore, these two points are singular points of the curve expressed by Eq. (33) [42]. We can further obtain $F_{\alpha\alpha}(\alpha, r) = 64$, $F_{rr}(\alpha, r) = 256/49 \approx 5.22449$, $F_{\alpha r}(\alpha, r) = 0$ and $F_{\alpha\alpha}(\alpha, r)F_{rr}(\alpha, r) - F_{\alpha r}(\alpha, r)^2 = 16\,384/49 \approx 334.367 > 0$ at the point $(\alpha = 2/\sqrt{7},\ r = \pi/4)$, so, this point is an isolated point. On the other hand, $F_{\alpha\alpha}(\alpha, r) = 64$, $F_{rr}(\alpha, r) = -4$, $F_{\alpha r}(\alpha, r) = 0$ and $F_{\alpha\alpha}(\alpha, r)F_{rr}(\alpha, r) - F_{\alpha r}(\alpha, r)^2 = -256 < 0$ at $(\alpha = 1/\sqrt{2},\ r = 0)$, which show that the point $(\alpha = 1/\sqrt{2},\ r = 0)$ is a node. The curve has two distinct tangents with the slopes ± 4 at the node as shown in Fig. 8. Using the same procedure, we can also find that the curve expressed as $\alpha^2\{\alpha^2[5 + \cos(4r)] - 2(1 - \alpha^2)\cos(2r) - 6\} + 2 = 0$, which is the critical condition that makes $D(\rho_{I,II})$ take the different forms depending on α and r, has two singular points: one is an isolated point at $(2/\sqrt{7}, \pi/4)$; another is a node, which also at $(1/\sqrt{2}, 0)$ and has two distinct tangents with the slopes ± 4. Above analysis demonstrates that any bipartite states of the tripartite system, which initially maximally entangled, are singular points for the geometric discord. On the contrary, there are no singular points for geometrical measure of entanglement in the case considered here.

Summarizing, based on a general initially entangled state instead of the maximally entangled state shared by Alice and Rob, we have derived analytical expressions of geometric measures of quantum discord for Dirac field in noninertial frames and discussed the behaviors of them with acceleration and entanglement parameter α. To have an insight into our results, we also calculated the geometric measure of entanglement for the same system and compared the results of two kinds of geometric measure. Moreover, we paid special attention to the case of initially maximally entangled state. We found: (1) Both $D(\rho_{A,I})$ and $E_g(\rho_{A,I})$ are decreasing functions of acceleration, on the contrary, $D(\rho_{A,II})$ and $D(\rho_{I,II})$, $E_g(\rho_{A,II})$ and $E_g(\rho_{I,II})$ are increasing functions of acceleration. These properties are independent of α and similar to other measurements, such as tangle and mutual information [5]. (2) $D(\rho_{A,I})$ and $D(\rho_{A,II})$ simultaneously converged to the same non-zero values, which depends on α, when acceleration tends to infinity, so do $E_g(\rho_{A,I})$ and $E_g(\rho_{A,II})$. (3) Even though $D(\rho_{x,y})$ and $E_g(\rho_{x,y})$ globally have the similar trend with α for a given r, but the relation between $D(\rho_{x,y})$ and α is more complicated than that between $E_g(\rho_{x,y})$ and α. (4) Besides conservative relationships of concurrence, mutual information and geometric measure of entanglement related to $\rho_{A,I}$ and $\rho_{A,II}$, which independent of α and acceleration, there also has a conservative relationship between $D(\rho_{A,I})$ and $D(\rho_{A,II})$, which is valid in the most of the area in the α–r plane except for a strip region containing $\alpha = 1/\sqrt{2}$. Additionally, there is a general conservative relationship, as shown by Eq. (32), among $D(\rho_{A,I})$, $D(\rho_{A,II})$ and $D(\rho_{I,II})$ for initially maximally entangled states. Furthermore, we have found that three bipartite states of the tripartite state generated by initially maximally entangled states are singular points for the geometric discord when $r = 0$. Though we cannot further give a deeper physical reasonable explanation for these conservative relationships and singularity at the moment, but with physical intuition, we have a premonition that these results certainly have some potential and important meanings. Therefore, they need to be further studied.

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Appendix A. Supplementary material

Supplementary material related to this article can be found online at http://dx.doi.org/10.1016/j.physletb.2015.02.001.

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