Distillability Problem of 4×4 Monomial Matrices

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Abstract: It is a fundamental problem in quantum information whether a particular quantum state of a composite system is entangled. It has enormous potential in quantum error correction, quantum cryptography, and quantum teleportation applications. This problem can be transferred in the form of a mathematical conjecture called the distillation conjecture. In the first section of this paper, relevant physical and mathematical information is presented, including basic linear algebra knowledge, the statement, and concrete applications of multiple mathematical knowledge like conjugate, eigenvalue, and singular value. Then, we introduce the distillation conjecture in a mathematical version for a more precise mathematical analysis. In an effort to make more significant headway in proving the conjecture, we selected some theories and findings relating to the Kronecker product, Kronecker sum, eigenvalue, and singular value, then evaluated and grouped them. In addition, we provided multiple proofs of the conjecture under varying conditions and made numerous attempts and hypotheses regarding how to establish the conjecture.

1. Introduction

1.1. Definition of leaner algebra
Linear algebra is the mathematical discipline concerned with linear equations and functions expressed as matrices and vectors. Simply said, linear algebra enables readers to comprehend and conduct mathematical operations on geometric notions, such as planes in higher-dimensional space.

1.2. Definition of matrix
A matrix is a container for a collection of vectors. A matrix operation entails performing a stretch and rotation transformation on a collection of vectors. Because the vectors are described by their x and y values, these two transformations alter the vectors' x and y values. The matrix was designed to explain linear plane transformations.

1.3. The history of quantum physics
Quantum was initially explained using classical mechanics, atomic interpretation, and electromagnetism. (The quantum is the smallest fundamental unit that is non-divisible). The thermal radiation and Planck

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energy quantum hypothesis were investigated in the 19th century. In the beginning, Kirchhoff conducted experiments with black bodies, which demonstrated that the radiation was simply connected to the temperature but not its density or characteristics. Then, Wien discovered the radiation energy formula. Einstein recognized Planck's significance in physic disciplines at the close of the 19th century and re-established new hypotheses [1-10]. He believed that the quantum was unbroken and inseparable. In certain circumstances, radiation particles can appear in the form of "light quantum." M physicists in the 19th century believed that the atom was the smallest undivided unit. The original theory was proposed by Perrin, who believed that the atom was surrounded by charged particles, with electrons on the exterior [1-10]. Thomson imagined charged particles encircled by an electron ring. Later, Marsden conducted experiments disproving Thomson's idea [1-10]. Finally, Bohr demonstrated that only electrons emitted or absorbed energy during the transition and demonstrated that atoms are stable. The middle twentieth century saw the birth of quantum mechanics and matrix mechanics. At that time, electrons circled the nucleus at distinct harmonic frequencies, exposing Bohr's theory of inherent inconsistencies. Mechanic equations were set down as equations. The matrix form of quantum mechanics is thus established. Inspired by Einstein, Bonn established that physics exclusively dealt with observable quantities and, in cooperation with Jordan, published quantum mechanics with mathematical matrices as evidence.

2. Main Body

2.1. Basic definition

The conjugate, transpose, and conjugate transpose of the matrix A will be denoted by $A^*, A^t,$ and $A^\dagger$, respectively.

2.1.1. Trace

A square matrix can have its sum of elements on the diagonal determined with the use of the trace function.

The definition of the trace of a square matrix with dimensions n by n is as follows:

$$\mathrm{Tr}(A) = \sum_{i=1}^{n} a_{ii} = a_{11} + a_{22} + \cdots \ldots\ldots + a_{nn} \tag{1}$$

Where a_{ii} represents the entry that is located on the i-th row and i-th column of A, the values entering A can be either real numbers or, more generally speaking, complex numbers. In other words, the trace is the sum of all the entries on the main diagonal. When working with matrices that are not square, the trace cannot be defined.

Example: Let $A = \begin{bmatrix} 3 & 4 & 5 \\ 2 & 9 & 6 \\ 1 & 8 & 7 \end{bmatrix}$. The trace of A is Tr (A)=$\sum_{i=1}^{3} a_{ii} = a_{11} + a_{22} + a_{33} = 3 + 9 + 7 = 19$.

2.1.2. Transpose

When working with matrices, the transpose operation is analogous to the inverse one. In a more straightforward explanation, the rows and columns of the initial matrix have been inverted. Tr(A)=Tr(A^T)

Example: Let $A = \begin{bmatrix} 3 & 4 & 5 \\ 2 & 9 & 6 \\ 1 & 8 & 7 \end{bmatrix}$. The transposition of A is $A^T = \begin{bmatrix} 3 & 2 & 1 \\ 4 & 9 & 8 \\ 5 & 6 & 7 \end{bmatrix}$.

2.1.3. Conjugate

Conjugate matrices are a notion that describes two matrices whose real numbers are the same, but their imagery numbers are negative. This concept is based on the fact that conjugate matrices finish complex values and divide them into real numbers and imagery numbers. Additionally, a few expressions can be found in the conjugate matrix. Example: Let A = $\begin{bmatrix} 3+i & 4 & 5 \\ 2 & 9+2i & 6 \\ 1 & 8 & 7+3i \end{bmatrix}$. The conjugate of A is $A^* = \begin{bmatrix} 3-i & 4 & 5 \\ 2 & 9-2i & 6 \\ 1 & 8 & 7-3i \end{bmatrix}$.

2.1.4. Eigenvalue and eigenvector

Assume A is an n by n matrix. For some scalar, an eigenvector of A is a nonzero vector v in Rn such that Av = v. A scalar eigenvalue is one for which the equation Av = v has a nontrivial solution. If Av = v for v is not equal to 0, we can say that is the eigenvalue for v and that v is an eigenvector for. The set of all possible answers to the equation can be represented by the empty space in the matrix $A - \lambda I$. Therefore, we can consider this to be a subspace of $\mathbb{R}^n$, which is referred to as the eigenspace of A corresponding to λ. The eigenspace is made up of the zero vector in addition to all of the eigenvectors that are associated with λ.

Example: Let $= \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$. An eigenvalue of A is 2, and we need to find a basis for the eigenspace that it corresponds to. From $A - 2I = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{bmatrix}$ and row, the augmented matrix is reduced for $(A - 2I)x = 0$: $\begin{bmatrix} 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \end{bmatrix} - \begin{bmatrix} 2 & -1 & 6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Because the equation $(A - 2I)x = 0$ contains free variables, it is now abundantly evident that the value 2 does, in fact, constitute one of the eigenvalues of A. The general solution is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$, and x_2 and x_3 are free. The eigenspace is a subspace of R3 that has two dimensions in size. An example of a basis is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$.

Application: The fundamental ideas of eigenvalues and eigenvectors are helpful in many areas of pure and practical mathematics and the contexts in which they are found. Eigenvalues play an important role in the engineering design process and the study of differential equations and continuous dynamical systems. Additionally, eigenvalues naturally occur in a variety of scientific disciplines, including physics and chemistry.

2.1.5. Singular value

Let A be an $m \times n$ matrix for the sake of this definition. If this is the case, $A^T A$ is symmetric and can be diagonalized in an orthogonal fashion. Let $\{v_1, \ldots, v_n\}$ be an orthonormal basis for $\mathbb{R}^n$ that consists of eigenvectors of $A^T A$, and let $\lambda_1, \ldots, \lambda_n$ be the associated eigenvalues of $A^T A$. In other words, let $\{v_1, \ldots, v_n\}$ be an orthonormal basis for $\mathbb{R}^n$. Then, for $1 \le i \le n$,

$$\|Av_i\|^2 = (Av_i)^T Av_i = v_i{}^T A^T A v_i = v_i{}^T (\lambda_i v_i) = \lambda_I \quad (2)$$

Therefore, there is not a single negative eigenvalue in the $A^T A$ matrix. It is possible for us to assume that the eigenvalues are organized in such a way that $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n \ge 0$ by simply renumbering the terms if this turns out to be necessary. The singular values of A are the square roots of the eigenvalues of $A^T A$, and they are indicated by $\sigma_1, \ldots, \sigma_n$. These values are arranged in descending order. That is, $\sigma_i = \sqrt{\lambda_i}$ for $1 \le i \le n$. The singular values of A are the lengths of the vectors $Av_1, \ldots, Av_n$, according to Equation 2.

Application: Calculating the pseudoinverse, approximating matrices, and calculating a matrix's rank, range, and null space are all examples of mathematical applications of the singular value decomposition or SVD.

2.1.6. Kronecker product

The Kronecker product has a number of important features, two of which are

$(A \otimes B)^* = A^* \otimes B^*$, and $(A \otimes B)(C \otimes D) = AC \otimes BD$. The second equality suggests that in the case where x i is an eigenvector of $A \in \mathbb{C}^{m \times m}$ with an eigenvalue of *i*, and y_j is an eigenvector of $B \in \mathbb{C}^{n \times n}$ with an eigenvalue of μ_j, then $(A \otimes B)(x_i \otimes y_j) = (Ax_i \otimes By_j) = (\lambda_i x_i \otimes \mu_j y_j) = \lambda_i \mu_j (x_i \otimes y_j)$. Hence, $\lambda_i \mu_j$ is an eigenvalue of $A \otimes B$ with an eigenvector that looks like this: $x_i \otimes y_j$. In fact, the mn eigenvalues of $A \otimes B$ are exactly $\lambda_i \mu_j$, with $i = 1,2,\cdots,m$ and $j = 1,2,\cdots,n$, respectively.

2.1.7. Vector operator

The vector operator concatenates the columns of a matrix into a single lengthy vector. Here's how it works: if $A = [a_1, a_2, \dots, a_m]$, then $\mathrm{vec}(A) = [{a_1}^T {a_2}^T \dots {a_m}^T]^T$; The Kronecker product and the vector operator have a fruitful interaction: for every A, X, and B, their product AXB is defined,

$$\mathrm{vec}(AXB) = (B^T \otimes A)\mathrm{vec}(X) \tag{3}$$

With the help of this relation, we are able to express a linear system $AXB = C$ using the conventional form "Ax=b".

2.1.8. Kronecker sum

The formula for calculating the Kronecker sum of $A \in \mathbb{C}^{m \times m}$ and $B \in \mathbb{C}^{n \times n}$ is as follows: $A \oplus B = A \otimes I_n + I_m \otimes B$. The eigenvalues of $A \oplus B$ can be written as $\lambda_{ij} = \lambda_i(A) + \lambda_j(B)$ where i=1:m and j=1:n. The eigenvalues of A are denoted by $\lambda_i(A)$, while the eigenvalues of B are denoted by $\lambda_j(B)$.The Kronecker sum is calculated by applying the vector operator to the matrix AX+XB, which results in the following expression:

$$\mathrm{vec}(AX + XB) = (I_m \otimes A + B^T \otimes I_n)\mathrm{vec}(X) = (B^T \oplus A)\mathrm{vec}(X) \tag{4}$$

Infinite difference discretization is done for partial differential equations, such as the case when Poisson's equation is discretized on a square by the typical five-point operator, and the Kronecker sum structure also manifests itself.

2.1.9. The eigenvalues and eigenvectors of A and A^T

Our example demonstrated that Eigenvalues A and A^T were identical, but Eigenvectors A and A^T were distinct (though in some ways linked), and that Eigenvalues B and B^T were identical but they were completely unrelated. Why is this the case?

It should not be too difficult for us to find the answer to the eigenvalue question because it is deduced from the characteristics of both the determinant and the transpose. Remember the following two pieces of information: $(A + B)^T = A^T + B^T$; $\det(A) = \det(A^T)$. By computing the characteristic polynomial of a matrix, we may determine its eigenvalues; more specifically, we can determine that $\det(A) = \det(A^T)$. What polynomial best describes the characteristics of A^T? Consider:

$$\det(A^T - \lambda I) = \det(A^T - \lambda I^T) = \det((A - \lambda I)^T = \det(A - \lambda I) \tag{5}$$

Therefore, it is plain to see that the characteristic polynomial for A^T is the same as the one for A. As a result, their eigenvalues are identical to one another.

Where do we stand with their individual eigenvectors? Is there any connection between the two? The answer is a straightforward "No".

These are A and the trace's eigenvalues and eigenvectors.

It is important to take note that the eigenvalues of A are -12 and 12, and that the trace is 6, whereas the eigenvalues of B are -1, 2, and 3, and that the trace of B is 4, respectively. Is there a connection that we can make?

It would appear that the trace is the same as the sum of the eigenvalues! Why does this happen to be the case?

The response to this question is a little beyond the scope of this article; we can justify a portion of this fact, and another portion of it, we'll just state as being true without providing any justification.

To begin, we are aware that $\mathrm{tr(AB) = tr(BA)}$. Second, we make the unsupported assertion that given a square matrix A, we are able to locate a square matrix P such that $\mathrm{P^{-1}AP}$ is an upper triangular matrix with the eigenvalues of A on the diagonal. This assertion is made despite the absence of any supporting evidence. Therefore, the total of the eigenvalues is denoted by $\mathrm{tr(P^{-1}AP)}$; in addition, we are aware that $\mathrm{tr(P^{-1}AP) = tr(P^{-1}PA) = tr(A)}$. Therefore, the total of the eigenvalues constitutes the trace of A.

2.1.10. The eigenvalues and eigenvectors of A and the determinant

Once more, the eigenvalues of variable A are 6 and 12, and the determinant of variable A is 72. The values -1, 2, and 3 are the eigenvalues of B, and the value -6 is the determinant of B. It would appear that the determinant is the product of the eigenvalues.

This is absolutely correct, and the justification for this can be found in our argument presented earlier. It is common knowledge that the product of a triangular matrix's diagonal elements constitutes the determinant of such a matrix. Therefore, if we are given a matrix A, we may find P such that $\mathrm{P^{-1}AP}$ is an upper triangular shape with the eigenvalues of A along the diagonal. Therefore, the product of the eigenvalues is denoted by the notation $\mathrm{det(P^{-1}AP)}$. We know that $\mathrm{det(P^{-1}AP) = det(P^{-1}PA) = det(A)}$. The product of the eigenvalues is hence the determinant of the variable A.

2.2. Distillability problem

The hypothesis for the distillability problem is as follows: $\mathrm{A, B, I} \in \mathbb{C}^{\mathrm{d\times d}}, \mathrm{d} \geq 4$, with the matrix being:

$$\mathrm{X = A \otimes I + I \otimes B} \quad (6)$$

Where:

$$\mathrm{TrA = TrB = 0, TrA^{\dagger}A + TrB^{\dagger}B = \frac{1}{d}}. \quad (7)$$

Define the set χ_{d}, the elements of which are determined by both equations. The singular values of $\mathrm{X} \in \chi_{\mathrm{d}}$ are $\sigma_1, \ldots, \sigma_{\mathrm{d}^2}$ in descending order. Therefore,

$$\sup_{\mathrm{X}\in\chi}(\sigma_1^2 + \sigma_2^2) \leq \frac{1}{2} \quad (8)$$

Here are some examples to further understand Equations (6) and (7):

Let $\mathrm{A}=\begin{pmatrix} 0 & a_1 & 0 & 0 \\ a_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_3 \\ 0 & 0 & a_4 & 0 \end{pmatrix}$ and $\mathrm{B}=\begin{pmatrix} 0 & b_1 & 0 & 0 \\ b_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & b_3 \\ 0 & 0 & b_4 & 0 \end{pmatrix}$. Then,

$$\mathrm{X = A \otimes I + I \otimes B}$$

$$= \begin{pmatrix} 0 & a_1 & 0 & 0 \\ a_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_3 \\ 0 & 0 & a_4 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & b_1 & 0 & 0 \\ b_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & b_3 \\ 0 & 0 & b_4 & 0 \end{pmatrix}$$

$$
=\begin{pmatrix}
0 & b_1 & 0 & 0 & a_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
b_2 & 0 & 0 & 0 & 0 & a_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & b_3 & 0 & 0 & a_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & b_4 & 0 & 0 & 0 & 0 & a_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
a_2 & 0 & 0 & 0 & 0 & b_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & a_2 & 0 & 0 & b_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & a_2 & 0 & 0 & 0 & 0 & b_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & a_2 & 0 & 0 & b_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b_1 & 0 & 0 & a_3 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b_2 & 0 & 0 & 0 & 0 & a_3 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b_3 & 0 & 0 & a_3 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b_4 & 0 & 0 & 0 & 0 & a_3\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_4 & 0 & 0 & 0 & 0 & b_1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_4 & 0 & 0 & b_2 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_4 & 0 & 0 & 0 & 0 & b_3\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_4 & 0 & 0 & b_4 & 0
\end{pmatrix}
$$

$$
A^{\dagger}A=\begin{pmatrix} 0 & \overline{a_2} & 0 & 0\\ \overline{a_1} & 0 & 0 & 0\\ 0 & 0 & 0 & \overline{a_4}\\ 0 & 0 & \overline{a_3} & 0 \end{pmatrix}\begin{pmatrix} 0 & a_1 & 0 & 0\\ a_2 & 0 & 0 & 0\\ 0 & 0 & 0 & a_3\\ 0 & 0 & a_4 & 0 \end{pmatrix}=\begin{pmatrix} \overline{a_2}a_2 & 0 & 0 & 0\\ 0 & \overline{a_1}a_1 & 0 & 0\\ 0 & 0 & \overline{a_4}a_4 & 0\\ 0 & 0 & 0 & \overline{a_3}a_3 \end{pmatrix}
$$

$$
B^{\dagger}B=\begin{pmatrix} 0 & \overline{b_2} & 0 & 0\\ \overline{b_1} & 0 & 0 & 0\\ 0 & 0 & 0 & \overline{b_4}\\ 0 & 0 & \overline{b_3} & 0 \end{pmatrix}\begin{pmatrix} 0 & b_1 & 0 & 0\\ b_2 & 0 & 0 & 0\\ 0 & 0 & 0 & b_3\\ 0 & 0 & b_4 & 0 \end{pmatrix}=\begin{pmatrix} \overline{b_2}b_2 & 0 & 0 & 0\\ 0 & \overline{b_1}b_1 & 0 & 0\\ 0 & 0 & \overline{b_4}b_4 & 0\\ 0 & 0 & 0 & \overline{b_3}b_3 \end{pmatrix}
$$

Therefore, $TrA = TrB = 0, TrA^{\dagger}A + TrB^{\dagger}B = \sum_{i=1}^{4}(\overline{a_i}a_i + \overline{b_i}b_i) = \frac{1}{4}$.

To examine the conjecture, we must first grasp the following theorems:

2.2.1. The theorem of Gershgorin's Circle:

R_i is the absolute value sum of the non-diagonal items in the i-th row: $R_i = \sum_{j\neq i}|a_{ij}|$

Let $D(a_{ii}, R_i) \subseteq \mathbb{C}$ be a closed disc with radius R_i and centered at a_{ii}. A Gershgorin disc is one such disc.

Every eigenvalue of A is represented by at least one of the Gershgoin discs $D(a_{ii}, R_i)$.

Assume that is one of A's eigenvalues. Choose an eigenvector with the equation $x = (x_j)$ where one of the components, x_i, has the value one, and the remaining components have absolute values that are either less than or equal to 1: $x_i = 1$ and $|x_j| \leq 1$ for $j \neq i$. When any eigenvector is divided by the component of that eigenvector that has the highest modulus, there is always an x that can be discovered. Since Ax=λx, and notably $\sum_j a_{ij}x_j = \lambda x_i = \lambda$.

As a result of splitting the total and remembering that $x_i = 1$, we get $\sum_{j\neq i} a_{ij}x_j + a_{ii} = \lambda$.

As a result, using the triangle inequality

$$|\lambda - a_{ii}| = |\sum_{j\neq i} a_{ij}x_j| \leq \sum_{j\neq i}|a_{ij}||x_j| \leq \sum_{j\neq i}|a_{ij}| = R_i. \quad (9)$$

2.2.2. Theorem of Brauer

Let K be a field in which, for each integer r greater than 0, there exists an integer $\psi(r)$ such that for n greater than or equal to $\psi(r)$, every equation is true.

$$a_1 x_1^r + \cdots + a_n x_n^r = 0, \text{and } a_i \in K, i = 1, \ldots, n \quad (10)$$

In K, there is a nontrivial solution. Then, given a homogeneous polynomial $f_1, \ldots, f_k$ of degrees $r_1, \ldots, r_k$ with coefficients in K, there exists a number $\omega(r_1, \ldots, r_k, I)$ such that for $n \geq \omega(r_1, \ldots, r_k, I)$, there exists an I-dimensional affine subspace M of K^n (regarded as a vector space over K) fulfilling

$$f_1(x_1, \ldots, x_n) = f_k(x_1, \ldots, x_n) = 0, \forall (x_1, \ldots, x_n) \in M \quad (11)$$

2.2.3. Examples of proofs

Several proven examples of 4×4 monomial matrices are discussed.

1. Let A=$\begin{pmatrix} 0 & a_1 & 0 & 0 \\ a_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_3 \\ 0 & 0 & a_4 & 0 \end{pmatrix}$ and B=$\begin{pmatrix} 0 & b_1 & 0 & 0 \\ b_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & b_3 \\ 0 & 0 & b_4 & 0 \end{pmatrix}$. If we discover that $X^\dagger X$ can be represented as $Y_1 \oplus Y_2$, it's simple to observe that Y_1 and Y_2 are nearly identical by changing a_1 with a_3 and a_2 with a_4, from Y_1 to Y_2. By combining $\sum_{i=1}^{4}(|a_i|^2 + |b_i|^2) = \frac{1}{4}$ and $\det(\lambda I - Y_1)$, we discover that the sum of the two highest eigenvalues of Y_1 is not more than $\frac{1}{2}$, which can be derived by $TrA = TrB = 0, TrA^\dagger A + TrB^\dagger B = \frac{1}{4}$.

2. If we define A as $\begin{pmatrix} 0 & a_1 & 0 & 0 \\ a_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_3 \\ 0 & 0 & a_4 & 0 \end{pmatrix}$ and B as $\text{diag}(b_1, b_2, b_3, b_4)$, we are able to discover that $X^\dagger X$ may be represented as $H_1 \oplus H_2$. After going over several instances of the sum of the two largest eigenvalues, we are able to demonstrate that the sum of the two largest eigenvalues cannot be greater than half of the original value.

3. Let A=$\begin{pmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 \\ 0 & 0 & a_3 & 0 \\ 0 & 0 & 0 & a_4 \end{pmatrix}$ and B=$\begin{pmatrix} 0 & b_1 & 0 & 0 \\ 0 & 0 & b_2 & 0 \\ 0 & 0 & 0 & b_1 e^{i\theta_1} \\ b_2 e^{i\theta_2} & 0 & 0 & 0 \end{pmatrix}$, and then we will assume that $b_1,\ b_2 \geq 0$ and $\theta_1 = 0$. It is then enough to show that $X^\dagger X = \oplus_{i=1}^{4} M_i$. If we start with a matrix of size 4 by 4, denoted as $M_i^{\sim}$, then we can use the eigenvalues of $M_i^{\sim}$ to determine the two greatest eigenvalues of $M_i^{\sim}$, which will assist us in proving the assumption.

The proof that the conjecture is correct in the case where both A and B are monomial matrices have already been carried out.

Let's assume that V=$\begin{pmatrix} a_1 & 0 & 0 & 0 \\ 0 & -a_1 & 0 & 0 \\ 0 & 0 & 0 & a_2 \\ 0 & 0 & a_3 & 0 \end{pmatrix}$, W=$\begin{pmatrix} 0 & b_1 & 0 & 0 \\ b_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & b_3 \\ 0 & 0 & b_4 & 0 \end{pmatrix}$, X=$\begin{pmatrix} 0 & c_1 & 0 & 0 \\ 0 & 0 & c_2 & 0 \\ 0 & 0 & 0 & c_3 \\ c_4 & 0 & 0 & 0 \end{pmatrix}$. If either A or B is to be either of these three matrices, then the conjecture would be correct. We can use A and B to figure out $X^\dagger X$, which is the primary tactic for proving them. $P^\dagger X^\dagger X P$ and $X^\dagger X$ would be the same for any unitary matrix P. $P^\dagger X^\dagger X P$ can be conveniently written as the combination of four 4×4 positive semidefinite matrices by simply locating the appropriate P. In this case, it is sufficient to determine either the sum of the maximum of two eigenvalues in each of the four matrices or the sum of the maximum of two eigenvalues in any of the four matrices.

3. Conclusion

One of the most fundamental and fundamentally significant components of the theory of quantum information processing is the concept of entanglement, which can be considered one of entanglement's most fundamental features. Because of this, determining whether or not the quantum state of a composite system is entangled is of the utmost importance. We have shown that in order to prove the distillation conjecture, it is a good idea to take into consideration the Gershgorin circle theorem, Brauer's theorem, the properties of the Kronecker product, and some crucial conclusions on the Kronecker sum's trace, eigenvalues, and singular values. We have also shown that in order to prove the distillation conjecture, it is a good idea to take into consideration the properties of the Kronecker product. We have also demonstrated that, in order to demonstrate that the distillation conjecture is true, it is necessary to take into account the characteristics of the Kronecker apparatus. In addition, there exist proofs under three different conditions that establish that the conjecture is right when both A and B are monomial matrices. These proofs demonstrate that the conjecture holds true when both A and B are monomial matrices (work that will be done in the future, detailed in greater detail).

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