Gradient-flow equations for general Quantum Field Theories

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Colin William Poole

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Abstract

Given our understanding of renormalizable quantum field theories, in which coupling constants vary under Renormalization Group (RG) flow, it is natural to ask whether there exist any underlying principles governing such changes. The a-theorem [1] is conjectured to be one such principle: the existence of a function of the couplings in a theory, satisfying a gradient-flow equation with positive-definite metric on the space of couplings, places constraints on RG flow. Furthermore, at RG fixed points, this "a-function" reduces to the coefficient of the Euler density in the trace anomaly of the corresponding conformal field theory, suggesting a way of counting the degrees of freedom in a quantum theory. In [8], the strongest version of the a-theorem was shown to hold perturbatively, the associated a-function was constructed for general scalar-fermion theories, and the resulting constraints on RG functions were derived. Progress has also been made on formulating an analogous function in six dimensions, and equivalent statements are expected to hold for any even number of spacetime dimensions [36].

In this thesis, our principal aim is to investigate the consequences of such gradient-flow equations, and their associated a-function, in various spacetime dimensions. We extend the results of [9] to general gauge theories, and deduce the implications of a conjectured all-orders expression for the a-function, valid for supersymmetric gauge theories. We then turn to six dimensions and find, as in four dimensions, that a modification in the formulation of the a-function is required due to the presence of a global symmetry. We also reveal some puzzling implications regarding the presence of one-particle-reducible contributions to RG functions, and comment on a proposed solution. Finally, we turn to three dimensions, where there is no trace anomaly, and hence no natural candidate quantity to which the a-function may be related. Nevertheless, we show that one may still construct a function, satisfying the same gradient-flow equation, for both non-supersymmetric and supersymmetric theories; we then show how this function gives new relations between the underlying Feynman integrals.

Declaration

I hereby declare that all work described in this thesis is the result of my own research unless reference to others is given. None of this material has previously been submitted to this or any other university. All work was carried out in the Theoretical Physics Division of the Department of Mathematical Sciences, University of Liverpool, during the period of October 2013 until August 2017.

Publication List

This thesis contains material that has appeared in the following publications by the author:

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- J. A. Gracey, I. Jack and C. Poole, *The a-function in six dimensions*,
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- J. A. Gracey, I. Jack, C. Poole and Y. Schroeder, *a-function for* $\mathcal{N} = 2$ *supersymmetric gauge theories in three dimensions*,
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Acknowledgements

Right, time for the fun bit.

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Finally, I thank my parents, for teaching me my most important lesson, though they may not have realised it at the time. At the age of 9, I had a rather

unpleasant teacher, whose sheer inadequacy was revealed during a discussion at Parents' evening. Having been asked if there was additional help and/or support available, the teacher replied with the following delightful representation of their ability as an educator:

Well what d'you want him to be, a rocket scientist?

My parents told me about this, and pointed out that all rocket scientists did, in fact, attend primary school. Whether I've fallen short of, met, or exceeded this expectation by finishing a PhD in theoretical physics is for others to decide, but I have never forgotten that, irrespective of circumstance, we are in no way predestined to fall short of our aspirations. I'm sure there are plenty of such sorry excuses for teachers around, and I can only hope that they don't succeed in discouraging their students with their own grossly limited preconceptions.

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For Linz.

Chapter 1

Introduction

To date, our most successful attempt at describing the universe posits that all physical phenomena are consequences of four fundamental interactions: gravity, electromagnetism, the weak interaction and the strong interaction. The theoretical framework for gravity is General Relativity (GR), in which gravitational attraction is a consequence of spacetime curvature caused by a distribution of matter and energy, with the precise relation being described by the Einstein Field Equations. The theoretical framework for the other three interactions is Quantum Field Theory (QFT), in which one constructs a Lagrangian density consisting of derivatives and products of fields, such that the various terms are invariant under certain transformations, and then substitutes into a Path integral over the classical fields; the fields then describe quantized particles, and each transformation is associated with an interaction. While GR may also be formulated in terms of a Lagrangian density, the corresponding Path integral leads to a non-renormalizable QFT, and so gravitational effects must be treated as a low-energy effective description.

Historically, the most confusing and objectionable feature introduced into the QFT framework was renormalization, as the procedure seemed little more than an ad hoc method of subtracting infinity from a divergent quantity to yield a finite answer. Such concerns were laid to rest with the introduction of the Renormalization Group (RG), whereby it was realised that one may relate a QFT at a particular energy scale Λ to a self-similar description at another energy scale $\bar{\Lambda}$, in such a way that the parameters of the theory at Λ are functions of the parameters at $\bar{\Lambda}$, and depend on the scaling parameter relating the two descriptions. From this perspective, renormalization is simply the modification required to ensure that a theory is indeed self-similar under such rescaling.

The most striking physical consequence of the renormalization procedure is the scale-dependent behaviour of the parameters in the theory, known (ironically, in retrospect) as coupling constants. Each interaction in the Lagrangian density has an associated coupling constant, and these couplings govern the rate at which particles interact. Since these couplings depend on the renormalization scale, the rate of particle interaction changes as one performs the same collisions at different energy scales, leading to different experimental measurements of the couplings. QFT predicts the value of the coupling at a new energy scale by solving the Renormalization Group Equation (RGE), in which a particular function β encodes the rate of change of the coupling as one varies the scaling parameter. Crucially, this β-function does not itself explicitly depend on the scaling parameter, as this would spoil the self-similarity of the theory.

Given the relations between couplings at different energy scales, it is possible to interpret QFTs as points on a manifold, with couplings $\{g_I\}$ acting as coordinates; the β-functions of the couplings β^I are then said to induce a *flow* on the coupling space, known as RG flow. An RG fixed point is defined as a point on the manifold where $\beta^I = 0$; at such points, RG flow ceases and one obtains a QFT with the additional property of scale-invariance.[1] The coupling-space manifold can therefore be viewed as a collection of scale-invariant QFTs, connected via RG flow and acting as endpoints for the flow of general QFTs. One may then ask, do there exist restrictions on how RG flow may occur, and what are the consequences of such restrictions?

Constraints on RG flow were investigated by Zamolodchikov [2], who was able to show that the RG flow of two-dimensional QFTs is an irreversible process. To demonstrate this, Zamolodchikov constructed a function C of the couplings, which decreases monotonically under RG flow; at RG fixed points, C is stationary, and equals the central charge of a corresponding Conformal Field Theory (CFT). Since contributions to the central charge from any field in a unitary CFT are manifestly positive, C can be said to count the massless degrees of freedom in the theory, and the monotonic behaviour of C under RG flow establishes the empirically-intuitive result that lower-energy descriptions of physical processes have fewer degrees of freedom than do higher-energy descriptions. Furthermore, since RG fixed points define scale-invariant QFTs (SFTs) and C always equals the central charge of a CFT at a fixed point, the existence of C provides evidence that scale invariance in fact implies conformal invariance, at least for two-dimensional QFTs.

[1] Intriguingly, all Lorentz-invariant, unitary, scale-invariant QFTs appear to also be invariant under special conformal transformations, and hence one may speculate as to whether scale invariance automatically implies conformal invariance. This question shall be addressed in chapter 2, where we draw attention to the relevance of the a-theorem in excluding RG flows that end with scale- but not conformally-invariant QFTs.

Zamolodchikov's results, which came to be known as the c-theorem, can be summarised as follows[2]:

Theorem (c-theorem)**.** *For any two-dimensional Quantum Field Theory, there exists a function C of the couplings such that*

- *at RG fixed points, C is equal to the central charge c of the corresponding Conformal Field Theory;*
- *C is monotonically decreasing under RG flow.*

As suggested, this theorem can indeed be used to demonstrate the coarse-graining of degrees of freedom in lower-energy physical descriptions [3], as well as the equivalence of scale- and conformal-invariance [4], for two-dimensional QFTs. One would therefore hope that attempting to reformulate Zamolodchikov's argument for four-dimensional QFTs would lead to proofs for analogous statements. Unfortunately, a crucial step in the argument relies on the positivity of the two-point function for the trace of the Energy-Momentum tensor and its relation to the "c-function" C. To see how this affects attempts to derive a four-dimensional version of the c-theorem, it is useful to follow a re-derivation of the c-theorem using QFT in curved spacetime (QFTCS).

Despite the difficulties inherent in attempting to combine gravity with the Standard Model, or even to simply describe gravity using QFT methods, formulating QFT in a curved spacetime background has proven to be an extremely useful first step. QFTCS has many highly non-trivial consequences, from the prediction of black hole radiation [5] to the inability to consider field quanta as definite, observer-independent particles [6]. The non-trivial Riemann tensor associated with a general spacetime also results in new curvature anomalies: a symmetry unbroken by quantization may instead be broken by a general curved background. The most relevant such anomaly for our purposes is the trace anomaly, where the expectation value of the trace of the energy-momentum tensor for a CFT acquires terms proportional to various curvature scalars[3]. It turns out that the coefficient of the trace anomaly for a two-dimensional CFT is proportional to the central charge, suggesting that there is a connection between QFTCS and the c-theorem.

[2]Zamolodchikov also showed that near RG fixed points, C is not only monotonically decreasing, but in fact obeys a gradient-flow equation with positive-definite metric; in [11], the gradient-flow behaviour was proven true non-perturbatively along the whole RG flow, and so the c-theorem is sometimes implicitly extended to include this.

[3]There is extensive literature on the computation of the trace anomaly for various theories, which may be found in [64], [65], and references therein.

When working with QFTCS, the RG can be extended by promoting the RG scale and the couplings of the theory to spacetime-dependent fields in their own right; the resulting Local Renormalization Group (LRG) then specifies a relation between local Weyl rescalings, running couplings, and curvature anomalies, reducing to the standard RG when one instead considers global Weyl rescaling and non-spacetime-dependent couplings. The idea of using spacetime-dependent couplings and RG scale was first introduced by Drummond and Shore [10], after which the consistency of the LRG (and reinterpretation in terms of Weyl rescalings) was established by Osborn [7, 8], requiring local counterterms proportional to derivatives of the couplings. It was also shown that since Weyl rescalings form an Abelian group, the commutator of two Weyl rescalings must vanish, and so one can derive relations between the various terms in the LRG, known as Weyl consistency conditions. The counterterms related to derivatives of the couplings then lead to highly non-trivial consistency conditions, one of which is a gradient-flow equation relating the derivatives of some function of the couplings to the β-functions of the theory.[4]

The symmetric part of the tensor that appears in this gradient-flow equation acts as a metric on coupling space. In the case of a two-dimensional QFT, one can then use Ward identities for the two-point function of the trace of the energy-momentum tensor to express the metric in terms of manifestly positive-definite quantities; furthermore, one can derive the existence of a function of the couplings, the total derivative of which is given by contracting the gradient-flow equation with the β-functions. This function is then equivalent to Zamolodchikov's c-function, up to a term related to an arbitrary local contribution to the action that vanishes at fixed points. The metric is the same as the Zamolodchikov metric, completing the re-derivation of the c-theorem and establishing a further interpretation of C as a monotonic interpolation between the trace anomalies of CFTs.

The shift in perspective from the central charge to the coefficient in the trace anomaly is what facilitates the search for higher-dimensional analogues of the c-theorem. In four dimensions, the trace anomaly contains four curvature terms, corresponding to the Euler density (a-anomaly), the square of the Ricci scalar (b-anomaly), the square of the Weyl tensor (c-anomaly), and the d'Alembertian of the Ricci scalar. The final anomaly can be removed simply by adding a local counterterm to the Lagrangian density [68], and hence can be dropped from our

[4]Strictly speaking, the β-function appearing in the gradient-flow equation is a modified β-function, containing an additional contribution from global symmetries in the Lagrangian density; we shall see the importance of such a modification in perturbative calculations at three loops and beyond.

considerations. Naïvely, there are then three potential candidates for a four-dimensional analogue of the c-theorem, however the Weyl consistency condition for the b-anomaly leads to a function that vanishes at fixed points, and it can be directly shown that the c-anomaly does not decrease monotonically under RG flow [67]. By this reasoning, Cardy conjectured [1] that the remaining a-anomaly may be the correct quantity for establishing a four-dimensional analogue of the c-theorem; this is in fact a straightforward generalisation, since the Euler density in two dimensions is simply a multiple of the Ricci scalar. This conjecture, known as the a-theorem, has three progressively stronger formulations:

Conjecture (a-theorem)**.** *For any four-dimensional Quantum Field Theory, there exists a function A of the couplings such that*

- *at RG fixed points, A is equal to the coefficient a of the Euler density in the trace anomaly of the corresponding Conformal Field Theory.*

Furthermore,

- *(weak) given two RG fixed points at energy scales $\mu_{UV} > \mu_{IR}$, the function A satisfies $a_{UV} - a_{IR} > 0$;*
- *(stronger) A is monotonically decreasing under RG flow;*
- *(strong) A obeys a gradient-flow equation with positive-definite metric.*

The Weyl consistency condition for the a-anomaly is a four-dimensional generalization of the previous two-dimensional gradient-flow equation, and hence to prove the a-theorem one only need establish the positive-definiteness of the metric. Unfortunately, the Ward identities involving the metric include terms related to three-point functions for which there is no guarantee of positivity - this is the crucial difference compared to the two-dimensional version, where we had only manifestly-positive two-point functions. It is at least possible to salvage a perturbative proof of the strong a-theorem, valid near RG fixed points: since the terms related to the three-point function are proportional to the β-functions, they must be sufficiently small near an RG fixed point that the metric is dominated by the positive two-point terms.

Proof of the strong a-theorem aside, the gradient-flow equation and associated "a-function" are sufficiently interesting in their own right due to the constraints they place on RG quantities for general QFTs. In this thesis, our main objective is to calculate the a-function and investigate the associated constraints, in the manner of Jack and Osborn [8,9]. The calculation is facilitated by a further property of the gradient-flow equation: the a-function and metric may be expanded perturbatively, and their values inferred from RG quantities in flat spacetime, rather than the curved spacetime counterparts. By expressing RG quantities in a schematic way, we may derive the a-function for a general QFT in a given number of spacetime dimensions, then relate it to any particular QFT by inserting the required fields and couplings into the general field multiplets and tensor couplings.

The thesis is structured as follows. After a detailed exposition of the background material in chapter 2, we start chapter 3 by extending the known four-dimensional results to include gauge interactions, and compare the constraints on the general three-loop gauge β-function to the explicit calculation by Gracey, Jones and Pickering [23]. In chapter 4, we test the equivalent conjecture in higher even-dimensional spacetime by moving on to six dimensions and calculating the a-function for a general ϕ^3 theory up to terms involving the three-loop β-function; this calculation allows us to verify that the β-function in the gradient-flow equation must be modified in a way analogous to the four-dimensional case. In chapter 5, despite the lack of trace anomaly and associated Weyl consistency condition in three dimensions [42], we posit that a gradient-flow equation of the same form will produce consistency conditions for general three-dimensional QFTs. Throughout all chapters, the question of scheme-dependence is addressed, and we ensure that every consistency condition imposed by the gradient-flow equation is a scheme-independent result. We also find that the calculations in each chapter lead to unexpected consequences, such as the manifest symmetry of the metric beyond leading order (chapter 3), the apparent existence of one-particle-reducible (1PR) contributions to the β-function for general theories (chapter 4), and new relations between Feynman integrals beyond what one may derive using traditional reduction techniques such as integration by parts (chapter 5). The thesis will conclude with a summary of the results in each chapter, a outline of the solution to the 1PR issue, and a tentative connection to other proposed odd-dimensional analogues of the a-theorem. For completeness, we include an appendix listing the equations derived from the gradient-flow equation in each chapter.

Chapter 2

Background

Essentially, every result contained in this thesis is predicated on the existence, for a general QFT, of a function $A(g)$ of the couplings, satisfying a gradient-flow equation of the form

$$\partial_I A = T_{IJ}\beta^J \quad \Longleftrightarrow \quad dA = dg^I T_{IJ}\beta^J, \tag{2.1}$$

where the upper-case Latin indices I, J run over all marginal couplings in the theory. The gradient-flow equation lies at the intersection of several QFT topics, the most prominent being Renormalization, Conformal Field Theory, and Quantum Field Theory in Curved Spacetime. Due to the copious quantity of literature that has accrued in each of these topics over the last 40-50 years, it would be impractical to give a truly comprehensive introduction. Instead, we opt to provide a streamlined path through each topic in turn, attempting to highlight the necessary features, and culminating in a self-contained derivation of the gradient-flow equation in two- and four-dimensional spacetime.[1] We begin with the Path integral formulation of QFT in four-dimensional spacetime, defining the generating functional and Green function first for a free scalar theory, then extending the formulation to introduce fermions and gauge fields. We then show how interactions may be considered using perturbation theory and derive the associated momentum-space Feynman rules, before moving on to dimensional regularization and renormalization. After introducing the RG equation and associated RG quantities, we discuss the concept of RG flow and fixed points, making connections to scale invariance, conformal invariance and CFTs. From here, we again extend the Path integral formulation to curved spacetime and derive the local

[1]The extension to higher even-dimensional spacetime is completely analogous, though the complexity of deriving the Weyl consistency conditions rapidly increases due to the number of curvature terms present in the action.

RG equation; finally, after an aside on two-dimensional spacetime, we may derive the gradient-flow equations that form the basis of our calculations in subsequent chapters.

2.1 Quantum Field Theory

In this section, we primarily follow the treatment of QFT found in Bailin and Love [59], with supplementary material on Dimensional Regularization found in Collins [61].

2.1.1 Prelude: Quantum Mechanics

The Path integral formulation of QFT is defined by analogy with Feynman's Path integral formulation of standard Quantum Mechanics (QM), where one begins with time-dependent operators $\hat{Q}(t)$ and $\hat{P}(t)$ in the Heisenberg picture, corresponding to a generalised coordinate and conjugate momentum. These operators define eigenstates $|q,t\rangle$ and corresponding eigenvalues q according to $\hat{Q}(t)\,|q,t\rangle = q\,|q,t\rangle$. For a system with Hamiltonian operator $\mathcal{H}(\hat{P},\hat{Q})$, the probability amplitude of an initial state $|q,t\rangle$ transitioning to a final state $|q',t'\rangle$ is given (up to a normalization factor) by the functional integral

$$\langle q',t'|q,t\rangle \sim \int \mathcal{D}q \int \mathcal{D}p\, e^{\frac{i}{\hbar}\int_t^{t'} dt\,(p\dot{q}-H(p,q))},$$

known as the Path integral, where $H(p,q)$ is the classical Hamiltonian corresponding to the operator $\mathcal{H}$. The Path integral is taken over all functions $p(t)$, and all functions $q(t)$ satisfying $q(t)=q$, $q(t')=q'$. For a system with Hamiltonian of the form $\mathcal{H}(\hat{P},\hat{Q}) = \frac{1}{2m}\hat{P}^2 + V(\hat{Q})$, one may formally carry out the p-integral to obtain

$$\langle q',t'|q,t\rangle \sim \int \mathcal{D}q\, e^{\frac{i}{\hbar}\int_t^{t'} dt\,L(q,\dot{q})},$$

recasting QM in terms of a (classical) Lagrangian $L(q,\dot{q}) = \frac{1}{2}m\dot{q}^2 - V(q)$ rather than a Hamiltonian. By including a source term[2] $-J(t)\hat{Q}(t)$ in the Hamiltonian operator, the Path integral is modified to

$$\langle q',t'|q,t\rangle_J \sim \int \mathcal{D}q \int \mathcal{D}p\, e^{\frac{i}{\hbar}\int_t^{t'} dt\,(p\dot{q}-H(p,q)+Jq)},$$

[2] So called because in classical mechanics, the presence of such a term generates a current contribution to the equations of motion.

where $H(p,q)$ is now the Hamiltonian without source. Finally, one defines the generating functional $W[J]$ according to

$$W[J] \sim \lim_{\substack{t\to-\infty \\ t'\to\infty}} \langle q', t'|q, t\rangle_J \sim \int \mathcal{D}q \int \mathcal{D}p \, e^{\frac{i}{\hbar}\int_{-\infty}^{\infty} dt\,(p\dot{q}-H(p,q)+Jq)},$$

with the functional integrals again taken over all functions $p(t)$, and all functions $q(t)$ satisfying $\lim_{t\to-\infty} q(t) = q$, $\lim_{t'\to\infty} q(t') = q'$. Attempting to carry out the p integration when the Hamiltonian (with source) takes the restricted form above no longer works, since the integral cannot be transformed into a Gaussian; nevertheless, we may analytically continue the time variable to imaginary values in order to define a convergent functional in Euclidean space, before carrying out the integration and reverting back to real time. This analytic continuation is known as a Wick rotation, and is used to unambiguously define the generating functional as

$$W[J] \sim \int \mathcal{D}q \, e^{\frac{i}{\hbar}\int_{-\infty}^{\infty} dt\,(L(q,\dot{q})+Jq)}$$

for a Hamiltonian of the restricted form with a source term. From here, we may obtain an expression for the ground-state expectation value of a product of operators in the presence of sources[3]. By the completeness relation,

$$\langle q', t'|q, t\rangle = \prod_{j=1}^{n} \int dq_j \, \langle q', t'|q_n, t_n\rangle \, \langle q_n, t_n|q_{n-1}, t_{n-1}\rangle \ldots \langle q_1, t_1|q, t\rangle \,,$$

where we have employed a time-ordering $t' > t_n > t_{n-1} > \cdots > t_1 > t$. Inserting two operators $\hat{Q}(t_a), \hat{Q}(t_b)$ with $t_b > t_a$, we find

$$\langle q', t'|\hat{Q}(t_b)\hat{Q}(t_a)|q, t\rangle = \prod_{j=1}^{n} \int dq_j \, q_b q_a \, \langle q', t'|q_n, t_n\rangle \, \langle q_n, t_n|q_{n-1}, t_{n-1}\rangle \ldots \langle q_1, t_1|q, t\rangle \,.$$

Since the eigenvalues q_a, q_b are just commuting numbers, the expression for $t_a > t_b$ is the same. Taking the limit $n \to \infty$, we define the Path integral representation for the Time-ordered product of two operators as

$$\langle q', t'|T\{\hat{Q}(t_b)\hat{Q}(t_a)\}|q, t\rangle \sim \int \mathcal{D}q \int \mathcal{D}p \, q_a q_b \, e^{\frac{i}{\hbar}\int_t^{t'} dt\,(p\dot{q}-H(p,q))},$$

[3] Justified by a touch of foresight - the vacuum expectation value of Time-ordered operators in a QFT is of paramount importance.

where the Time-ordering operation is defined by

$$T\{\hat{Q}(t_b)\hat{Q}(t_a)\} = \begin{cases} \hat{Q}(t_b)\hat{Q}(t_a) & t_b > t_a, \\ \hat{Q}(t_a)\hat{Q}(t_b) & t_a > t_b. \end{cases}$$

The Time-ordered expectation value of any number of operators $\hat{Q}(t_k)$ is then given by the obvious generalisation

$$\langle q', t'|T\{\prod_{k=1}^{n} \hat{Q}(t_k)\}|q, t\rangle \sim \int \mathcal{D}q \int \mathcal{D}p \left(\prod_{k=1}^{n} q_k\right) e^{\frac{i}{\hbar}\int_t^{t'} dt\,(p\dot{q}-H(p,q))}$$

This expression can then be connected to the generating functional $W[J]$ via functional differentiation. Given a functional of the form

$$F[J(t)] = e^{\int g(t)J(t)}$$

the functional derivative is

$$\frac{\delta F[J(t)]}{\delta J[t']} = g(t')e^{\int g(t)J(t)} = g(t')F[J(t)].$$

Applying this to the generating functional, we find that

$$\frac{\delta^n W[J(t)]}{\delta J[t_n]\cdots\delta J[t_1]} \sim \int \mathcal{D}q \int \mathcal{D}p \left(\prod_{k=1}^{n} q_k\right) e^{\frac{i}{\hbar}\int_t^{t'} dt\,(p\dot{q}-H(p,q)+Jq)},$$

and so setting the source term to zero gives an expression for Time-ordered expectation values in terms of the generating functional,

$$\langle 0|T\{\prod_{k=1}^{n} \hat{Q}(t_k)\}|0\rangle = \left.\frac{\delta^n W[J(t)]}{\delta J[t_n]\cdots\delta J[t_1]}\right|_{J(t)=0}$$

where $|0\rangle$ denotes the asymptotic ground state. For a Hamiltonian of the special form above, the p integral can again be evaluated, yielding analogous formulae with the replacement $(p\dot{q} - H(p,q)) \to L(q,\dot{q})$.

2.1.2 Extending to Quantum Field Theory

The intent of QFT is to describe special-relativistic particles and their interactions using the language of field theory, in which a function f, known as a field, assigns a certain quantity to every spacetime event. Different types of field are typically indicated by various letters and indices, such as scalar fields ϕ, vector fields A_μ, tensor fields $F_{\mu\nu}$, and spinor fields ψ_a. Classically, such relativistic field theories are most conveniently described by a manifestly Lorentz-invariant Lagrangian density $\mathscr{L} \equiv \mathscr{L}(f, \partial_\mu f)$, defined such that $L = \int d^3\mathbf{x}\, \mathscr{L}$, motivating the Lagrangian description provided by the Path integral. One may of course define conjugate momenta as the derivative of $\mathscr{L}$ with respect to generalised field velocity, $\pi_f = \frac{\partial \mathscr{L}}{\partial(\partial_0 f)}$, then define a Hamiltonian density $\mathscr{H}$ for the classical theory as the Legendre transform of the Lagrangian density,

$$\mathscr{H} \equiv \pi_f(\partial_0 f) - \mathscr{L}.$$

Not only will the Hamiltonian density facilitate the definition of Path integrals in QFT, it allows one to address fundamental questions of consistency, such as whether the energy spectrum of the theory is bounded below.

Having established the Path integral formulation of QM, QFT is obtained by generalising the QM expressions to the case of fields. The generalised coordinate operators $\hat{Q}(t)$ in QM are replaced by operator-valued distributions $\hat{f}(\mathbf{x}, t) \equiv \hat{f}(x^\mu)$ obeying certain commutation relations. After quantization (that is, specifying commutation relations for the fields and substituting into the Path integral), the excitations of these quantized fields will then correspond to various types of particle. The field operators in QFT are again taken to be in the Heisenberg picture, and so we again define eigenstates according to $\hat{f}(\mathbf{x}, t)\,|f(\mathbf{x}), t\rangle = f(\mathbf{x})\,|f(\mathbf{x}), t\rangle$. Unfortunately, defining the Path integral contribution for each type of field is sufficiently subtle that we must consider them separately; nevertheless, the goal is to define a generating functional for each field, such that the total generating functional takes the form

$$W[J_f] \sim \left(\prod_f \int \mathcal{D}f\right) e^{\frac{i}{\hbar}\int d^4x\,\left(\mathscr{L}(f,\partial_\mu f)+\sum_f J_f f\right)}. \tag{2.2}$$

Once obtained, we may specialise to the case of a free field theory and obtain the vacuum expectation value of a Time-ordered product of fields by functional differentiation of the generating functional, where the generating functional is understood as being defined via Wick rotation from the corresponding Euclidean theory. In a free theory, the only Time-ordered product with a non-zero vacuum expectation value is the product of two fields. This "two-point function" may then be identified as a Green function: a fundamental solution to the differential equation

$$\mathscr{D}_x \mathscr{G}(x, x') + \delta(x - x') = 0, \tag{2.3}$$

where the linear differential operator $\mathscr{D}$ taken at spacetime point x^μ is the differential operator appearing in the corresponding classical equation of motion. By extension, the vacuum expectation value of a Time-ordered product of n fields is referred to as an n-point Green function. Later, we shall refine the construction of the generating functional to isolate the so-called connected and one-particle-irreducible (1PI) Green functions, which play a crucial rôle in the renormalization of the theory.

Scalar fields

Scalar fields offer the most straightforward generalisation of the QM Path integral, since the field operators obey simple canonical commutation relations and the eigenvalues are commuting numbers (c-numbers). By direct analogy, we *define* the generating functional in the presence of a source, up to a normalisation constant, as

$$W[J] \sim \lim_{\substack{t\to-\infty \\ t'\to\infty}} \langle \phi', t' | \phi, t \rangle_J \sim \int \mathcal{D}\phi \int \mathcal{D}\pi e^{\frac{i}{\hbar}\int d^4x (\pi \partial_0 \phi - \mathscr{H} + J\phi)},$$

where the functional integrals are taken over all functions $\pi(\mathbf{x}, t)$ and all functions $\phi(\mathbf{x}, t)$ obeying $\lim_{t\to-\infty} \phi(\mathbf{x}, t) = \phi(\mathbf{x})$, $\lim_{t'\to\infty} \phi(\mathbf{x}', t') = \phi(\mathbf{x}')$. The generating functional is normalised so that $W[0] = 1$, and we define the asymptotic ground state as $|0\rangle \equiv \lim_{t\to-\infty} |\phi, t\rangle$[4]. Since the classical Lagrangian density for a scalar field is

$$\mathscr{L}_{scalar} = \frac{\hbar^2}{2} \partial_\mu \phi \partial^\mu \phi + \bar{F}(\phi) \equiv \frac{\hbar^2}{2} (\partial_0 \phi)^2 + F(\phi, \nabla\phi),$$

the corresponding Hamiltonian density takes the form

$$\mathscr{H}_{scalar} = \frac{1}{2\hbar^2} \pi^2 - F(\phi, \nabla\phi),$$

and so we may perform the π integral to obtain the desired form for the QFT generating functional,

$$W[J] \sim \int \mathcal{D}\phi \, e^{\frac{i}{\hbar}\int d^4x (\mathscr{L}_{scalar} + J\phi)}, \tag{2.4}$$

again normalised so that $W[0] = 1$. As with the QM Path integral, ambiguities in the oscillatory functional integral in Minkowski spacetime are resolved by analytic continuation to the exponentially dampened functional integral in Euclidean spacetime.

[4] The state $|0\rangle$ is also called the vacuum state, and is the unique Lorentz-invariant state of a QFT such that it is annihilated by all field annihilation operators.

Continuing the analogy with standard QM, the Time-ordered vacuum expectation value of n scalar fields (n-point Green functions) are given by

$$\left(\frac{i}{\hbar}\right)^n \langle 0|T\{\hat{\phi}(x_1)\ldots\hat{\phi}(x_n)\}|0\rangle = \left.\frac{\delta^n W[J]}{\delta J(x_1)\cdots\delta J(x_n)}\right|_{J(x)=0}; \tag{2.5}$$

these n-point functions can be evaluated exactly for a free scalar theory with Lagrangian density

$$\mathscr{L}_{Scalar} = \frac{\hbar^2}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2.$$

For the free theory, we may use integration by parts to rewrite (2.4) in the form

$$W[J] \sim \int \mathcal{D}\phi\, e^{i\int d^4x'\int d^4x\left(-\frac{1}{2}\phi(x')A(x',x)\phi(x)\right)+\frac{i}{\hbar}\int d^4x J(x)\phi(x)},$$

where we have introduced the bilinear differential operator

$$A(x',x) = \frac{1}{\hbar}\left(\hbar^2\eta^{\mu\nu}\partial_\mu^{x'}\partial_\nu^x + m^2\right)\delta(x'-x).$$

One may now "complete the square" (in analogy with the discrete case) and introduce a field redefinition $\phi \to \phi - A^{-1}J$ to obtain

$$\begin{aligned} W[J] &\sim \det(iA)\, e^{\frac{i}{\hbar^2}\frac{1}{2}\int d^4x'\int d^4x J(x')A^{-1}(x',x)J(x)} \\ &= e^{\frac{i}{\hbar^2}\frac{1}{2}\int d^4x'\int d^4x J(x')A^{-1}(x',x)J(x)}, \end{aligned}$$

where the equality in the last line follows from absorbing the determinant into the overall normalization. By using an integral representation of the δ-function, we may also rewrite A as

$$\begin{aligned} A(x',x) &= \frac{1}{\hbar}\left(\hbar^2\eta^{\mu\nu}\partial_\mu^{x'}\partial_\nu^x + m^2\right)\int \frac{d^4p}{(2\pi\hbar)^4}\, e^{\frac{i}{\hbar}p_\mu(x'-x)^\mu} \\ &= \int \frac{d^4p}{(2\pi\hbar)^4}\, e^{\frac{i}{\hbar}p_\mu(x'-x)^\mu}\frac{-p^2+m^2}{\hbar}, \end{aligned}$$

and so the inverse operator A^{-1} is given by

$$A^{-1}(x',x) = \int \frac{d^4p}{(2\pi\hbar)^4}\, e^{\frac{i}{\hbar}p_\mu(x'-x)^\mu}\frac{\hbar}{-p^2+m^2}.$$

The final form of the generating functional for a free scalar theory is then

$$W[J] = e^{-\frac{i}{\hbar}\frac{1}{2}\int d^4x'\int d^4x J(x')\Delta_F(x'-x)J(x)}, \tag{2.6}$$

where the Feynman propagator $\Delta_F = \frac{-1}{\hbar}A^{-1}$ is given by[5]

$$\Delta_F(x'-x) = \int \frac{d^4p}{(2\pi\hbar)^4}\, e^{\frac{i}{\hbar}p_\mu(x'-x)^\mu}\tilde{\Delta}_F(p), \quad \tilde{\Delta}_F(p) = \frac{1}{p^2-m^2}. \tag{2.7}$$

Spinor fields

Spinor fields are multiple-component complex-valued fields that describe fermions, first introduced by Dirac in his attempts at generalising the Schrödinger equation to include special relativistic effects. Unlike scalar fields, spinor fields obey *anti*commutation relations, and consequently our reformulation in terms of a Path integral must be modified to account for anticommuting (Grassmann) complex-valued variables θ_i. To define the Path integral for spinors, we do so first for real Grassmann variables, generalise to Grassmann fields, then extend to the complex case.

Any collection of n Grassmann variables satisfies

$$\{\theta_i, \theta_j\} = 0 \qquad \forall\, i,j = 1,\ldots,n.$$

Integration over Grassmann variables is defined (without summation over i) by Berezin integration,

$$\int d\theta_i = 0, \quad \int d\theta_i\theta_i = 1,$$

one consequence of which is that integration and differentiation are effectively the same operation, and so one must integrate with respect to every variable in order to yield a non-zero quantity. Since the Path integral is fundamentally based on Gaussian integrals, we need to evaluate

$$I_n \equiv \int d\theta_1 \ldots d\theta_n\, e^{-\frac{1}{2}\Theta^T A\Theta},$$

where Θ is a column vector with entries $(\theta_1,\ldots,\theta_n)$, and A is (necessarily) an antisymmetric $n\times n$ matrix with n even[6]. By Taylor-expanding the exponential,

[5]The Feynman propagator in Minkowski spacetime is conventionally defined as including a small imaginary contribution $+i\epsilon$ in the denominator in order to circumvent the poles at $p^2 = m^2$; this term is implicit throughout.

[6]Consider the expression $\Theta^T A\Theta$, where A is an arbitrary matrix. Decomposing A into its respective symmetric and antisymmetric parts A_s and A_a, we have $\Theta^T A\Theta = \Theta^T A_s\Theta + \Theta^T A_a\Theta$; expanding $\Theta^T A_s\Theta$ gives a sum of terms of the form θ_i^2 and $\theta_i\theta_j + \theta_j\theta_i$, which vanish by the anticommutation relations. If n is odd, det$A$ vanishes and $I_n = 0$.

we see that all contributions vanish except

$$I_n = \int d\theta_1 \dots d\theta_n \frac{1}{\left(\frac{n}{2}\right)!} \left(-\frac{1}{2}\Theta^T A \Theta\right)^{\frac{n}{2}} = (\det A)^{\frac{1}{2}} .$$

Since the Berezin integral vanishes when n is odd, this identity holds for any n, and can therefore be used to define a Path integral over spinor fields $\rho(x)$ as a Grassmann-valued Gaussian integral, by analogy with the procedure for scalar fields:

$$\int \mathcal{D}\rho \, e^{-\frac{1}{2}\int dx' \int dx \, (\rho(x')A(x',x)\rho(x))} = (\det A)^{\frac{1}{2}} .$$

The extension to complex Grassmann variables is straightforward: defining new variables

$$\theta_i \equiv \tilde{\theta}_j + i\tilde{\theta}_k, \quad \theta_i^* \equiv \tilde{\theta}_j - i\tilde{\theta}_k, \quad \tilde{\theta}_{j,k} \in \mathbb{R},$$

we introduce integration over independent complex Grassmann variables, with real and imaginary parts $\Re(\theta_i)$ and $\Im(\theta_i)$ respectively, as

$$\int d\theta_i^* d\theta_i = 2 \int d\Re(\theta_i) d\Im(\theta_i),$$

and the Gaussian integral with complex vectors Θ becomes

$$\int d\theta_1^* d\theta_1 \dots \int d\theta_n^* d\theta_n \, e^{-\Theta^T A \Theta} = \det A,$$

for a skew-Hermitian matrix A, with the corresponding complex functional integral

$$\int \mathcal{D}\rho^* \mathcal{D}\rho \, e^{-\int dx' \int dx (\rho^*(x')A(x',x)\rho(x))} = \det A.$$

Therefore, we define the generating functional over spinors ψ, $\bar{\psi}$ in Minkowski spacetime, with Grassmann-valued source terms $\bar{\sigma}$, σ respectively, by direct analogy with the scalar case,

$$W[\sigma, \bar{\sigma}] \sim \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \, e^{\frac{i}{\hbar}\int d^4x \left(\mathscr{L}_{fermion} + \bar{\psi}\sigma + \bar{\sigma}\psi\right)}, \tag{2.8}$$

again normalised such that $W[0, 0] = 1$. Due to the presence of Grassmann fields, the definition of the Time-ordered vacuum expectation value is slightly different. Similar to the scalar case, the Time-ordering operation for spinors is defined as

$$T\{\hat{\psi}(x_a)\hat{\psi}(x_b)\} = \begin{cases} \hat{\psi}(x_a)\hat{\psi}(x_b) & t_a > t_b, \\ -\hat{\psi}(x_b)\hat{\psi}(x_a) & t_b > t_a, \end{cases} \tag{2.9}$$

where the extra minus sign accounts for anticommutation. The Time-ordered vacuum expectation value is then

$$\langle 0|T\{\hat{\psi}(x_1)\cdots\hat{\psi}(x_n)\hat{\bar{\psi}}(y_1)\cdots\hat{\bar{\psi}}(y_n)\}|0\rangle = \frac{\delta^{2n}W[\sigma,\bar{\sigma}]}{\delta\bar{\sigma}(x_1)\cdots\delta\bar{\sigma}(x_n)\delta\sigma(y_1)\cdots\delta\sigma(y_n)}.$$

As with ordinary Grassmann differentiation, the functional derivatives also anticommute.

In the case of a free theory with

$$\mathscr{L}_{fermion} = \bar{\psi}_a\left(i\hbar\gamma^{\mu}_{ab}\partial_\mu - m\delta_{ab}\right)\psi_b \equiv \bar{\psi}_a A_{ab}\psi_b,$$

we may proceed by introducing a linear term into the complex Grassmann integral and completing the square to obtain

$$W[\sigma,\bar{\sigma}] = e^{-\frac{i}{\hbar}\int d^4x'\int d^4x\left(\bar{\sigma}(x')A^{-1}(x',x)\sigma(x)\right)},$$

where in the fermion case we have

$$A(x',x) = -\left(i\gamma^\mu\partial^x_\mu + m\right)\delta(x'-x).$$

Making use of the integral representation of δ, we obtain the generating functional

$$W[\sigma,\bar{\sigma}] = e^{-\frac{i}{\hbar}\int d^4x'\int d^4x(\bar{\sigma}(x')S_F(x'-x)\sigma(x))}, \tag{2.10}$$

where the Feynman propagator for fermions is

$$S_F(x'-x) = \int\frac{d^4p}{(2\pi\hbar)^4}\,e^{\frac{i}{\hbar}p_\mu(x'-x)^\mu}\tilde{S}_F(p), \quad \tilde{S}_F(p) = \frac{\not{p}+m}{p^2-m^2}. \tag{2.11}$$

Gauge fields

Classical electromagnetism, summarised by Maxwell's equations, exhibits Lorentz invariance rather than Galilean invariance, and is hence automatically consistent with special relativity. By defining the electromagnetic four-potential $A_\mu = (\varphi, \mathbf{A})$ and electromagnetic field strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, one may describe Maxwell's (source-free) equations of electromagnetism in a manifestly Lorentz-invariant way using the Lagrangian density

$$\mathscr{L}_{Maxwell} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}.$$

Since only the derivative of the vector field A_μ appears, the theory is invariant under redefinitions $A_\mu \to A_\mu + \partial_\mu \Lambda$, for any arbitrary function $\Lambda \equiv \Lambda(x)$; the vector field is then referred to as a gauge field, and the Lagrangian density is said to be invariant under the gauge transformation $A_\mu \to A_\mu + \partial_\mu \Lambda$. Since electrons act as sources of electromagnetic fields, we can attempt to augment the Maxwell Lagrangian density with fermions:

$$\mathscr{L}_{source} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}.$$

For constant Λ, the spinor field terms are also invariant under a gauge transformation $\psi \to e^{-ig\Lambda}\psi$, and since $e^{-ig\Lambda}$ represents an element of the symmetry group $U(1)$, $\mathscr{L}_{source}$ is said to be invariant under global $U(1)$ gauge transformations. Unlike $\mathscr{L}_{Maxwell}$, $\mathscr{L}_{source}$ is *not* invariant for functions $\Lambda(x)$, but can be made so by introducing the gauge-covariant derivative $D_\mu = \partial_\mu + igA_\mu$, yielding the QED Lagrangian density

$$\begin{aligned}\mathscr{L}_{QED} &= \bar{\psi}(i\gamma^\mu D_\mu - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \\ &= \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - g\bar{\psi}\gamma^\mu\psi A_\mu.\end{aligned}$$

$\mathscr{L}_{QED}$ is said to be invariant under local $U(1)$ gauge transformations, and we see that the Lagrangian density necessarily includes a contribution to the potential, indicating an interaction between the spinor ψ and the gauge field A. It is the success of QED that established the philosophical principle of attempting to describe fundamental interactions using a Lagrangian density that is invariant under local gauge transformations, as the concept of local gauge invariance (plus renormalizability) fixes the possible interaction terms. Extending gauge transformations to the more general case $\psi \to e^{-igT^a\Lambda^a}\psi$, $A^a_\mu \to A^a_\mu + \partial_\mu \Lambda^a + gf^{abc}\Lambda^b A^c_\mu$, where the generators T^a satisfy the commutation relations $\left[T^a, T^b\right] = if^{abc}T^c$ and f^{abc} is a totally antisymmetric tensor, the generators T^a then form a representation of the Lie algebra corresponding to a non-Abelian symmetry group, and we may construct a Lagrangian density that is invariant under local non-Abelian gauge transformations,

$$\mathscr{L}_{NA} = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi - \frac{1}{4}G^a_{\mu\nu}G^{\mu\nu}_a,$$

with gauge-covariant derivative $D_\mu = \partial_\mu + igT^a A^a_\mu$ and field strength tensor $G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - gf^{abc}A^b_\mu A^c_\nu$. The purely-gauge part of $\mathscr{L}_{NA}$ is known as the

Yang-Mills Lagrangian density,

$$\mathscr{L}_{YM} = -\frac{1}{4} G^a_{\mu\nu} G^{\mu\nu}_a,$$

and is sufficient to define the Path integral over gauge fields. However, attempting to simply define the generating functional as

$$W[J^\mu_a] \overset{?}{\sim} \int \mathcal{D}A \, e^{\frac{i}{\hbar}\int d^4x \left(\mathscr{L}_{YM} + J^\mu_a A^a_\mu\right)} \tag{2.12}$$

leads to divergent results, so the Path integral must be further modified.

The selection of a particular function Λ when computing physical results is known as gauge-fixing, and it is this procedure which causes the most significant differences in formulating a Path integral for gauge fields. The functional integral is intended to sum over all *inequivalent* fields, a criterion that is trivially satisfied for scalars and spinors, but not gauge fields due to the existence of gauge transformations. For QED, or indeed any Abelian gauge theory with symmetry group $U(1)^n$, one can simply introduce a gauge-fixing term $-\frac{1}{2\xi}(\partial_\mu A^\mu_a)^2$ to the Lagrangian density; different values of the parameter ξ then correspond to different constraints on the four-potential A^μ_a, for example the limit $\xi \to 0$ (the Landau gauge) is classically equivalent to the Lorenz gauge $\partial_\mu A^\mu_a = 0$ [7]. The procedure for non-Abelian gauge theories is more complicated: since the Jacobian of the transformation to gauge-fixed fields is non-trivial for non-Abelian symmetry groups, there is no unique correspondence between the gauge-fixed fields A^a_μ and physical states, hence using $\mathscr{L}_{YM}$ plus the gauge-fixing term will still over-count the fields. This is solved by the Fadeev-Popov method[8], in which the Jacobian is converted to an exponential and expressed as a new Path integral over anticommuting scalar fields, known as ghosts. Ghosts do not appear in physical states (as they would violate the Spin-Statistics theorem), but appear as part of a perturbative expansion, where the ghosts systematically remove contributions associated with the over-counting of gauge field configurations.

Applying the Fadeev-Popov method, we define a gauge-fixing procedure according to

$$F_a(A^\mu_b) \equiv \partial_\mu A^\mu_a - f_a(x) = 0,$$

[7] Technically, this term does not entirely fix the gauge. Classically, two solutions to Maxwell's equations in the Lorenz gauge are related by a shift B_μ satisfying $\partial^2 B_\mu = 0$. However, Maxwell's equations automatically imply the continuity equation $\partial_\mu j^\mu = 0$, which forces $\partial_\mu B^\mu = 0$, completely fixing the gauge. QED, the quantum version of electromagnetism, is similar: the gauge is completely fixed as a consequence of the Ward-Takahashi identity.

[8] The Fadeev-Popov method in fact only completes the gauge-fixing of non-Abelian theories locally. Globally, one encounters the Gribov ambiguity, which we shall neglect.

and an associated functional integral over elements of the gauge group $\mathbf{U}$,

$$\Delta[A_a^\mu] = \int \mathcal{D}\mathbf{U} \prod_a \delta\left[F_a(A_b^{\mu\mathbf{U}})\right],$$

such that it is invariant under gauge transformations. We can also define an inverse functional Δ^{-1} such that $\Delta^{-1}[A_\mu^a]\Delta[A_\mu^a] = 1$, then introduce this factor into the Path integral (2.12), giving

$$\int \mathcal{D}A\, \Delta^{-1}[A_\mu^a] \int \mathcal{D}\mathbf{U} \prod_a \delta\left[F_a(A_b^{\mu\mathbf{U}})\right] e^{\frac{i}{\hbar}\int d^4x(\mathscr{L}_{YM})}.$$

By gauge invariance, we obtain

$$\int \mathcal{D}A\, e^{\frac{i}{\hbar}\int d^4x(\mathscr{L}_{YM})} = \int \mathcal{D}\mathbf{U} \int \mathcal{D}A\, \Delta^{-1}[A_\mu^a] \prod_a \delta\left[F_a(A_b^\mu)\right] e^{\frac{i}{\hbar}\int d^4x(\mathscr{L}_{YM})},$$

and should be able to absorb the $\mathbf{U}$ integral into the overall normalization. Since the functional δ-function restricts $\mathbf{U}$ to an infinitesimal region around the identity operator, we may parametrise $\mathcal{D}\mathbf{U} = \prod_c \mathcal{D}\Lambda_c$ and perform a change of variables to obtain

$$\int \mathcal{D}\mathbf{U} = \int \prod_c \mathcal{D}\Lambda_c = \int \prod_c \mathcal{D}F_c \det\left(\frac{\delta\Lambda_b(x)}{\delta F_c(x')}\right),$$

hence the functional integral over gauge transformations becomes

$$\begin{aligned}\Delta[A_a^\mu] &= \int \prod_c \mathcal{D}F_c \det\left(\frac{\delta\Lambda_b(x)}{\delta F_c(x')}\right) \prod_a \delta\left[F_a\right] \\ &= \det\left(\frac{\delta\Lambda_b(x)}{\delta F_a(x')}\right)\Bigg|_{F_a=0},\end{aligned}$$

and the inverse functional appearing in the Path integral is therefore

$$\Delta^{-1}[A_\mu^a] = \det\left(\frac{\delta F_a(x')}{\delta\Lambda_b(x)}\right)\Bigg|_{F_a=0}.$$

The Path integral can be multiplied by another constant term,

$$\int \left(\prod_c \mathcal{D}f_c\right) e^{-\frac{i}{2\xi\hbar}\int d^4x f_a^2(x)},$$

which imposes the gauge-fixing term via the functional δ:

$$\int \mathcal{D}A\, e^{\frac{i}{\hbar}\int d^4x(\mathscr{L}_{YM})} \sim \int \mathcal{D}A \det\left(\frac{\delta F_a(x')}{\delta\Lambda_b(x)}\right) e^{\frac{i}{\hbar}\int d^4x\left(\mathscr{L}_{YM}-\frac{1}{2\xi}(\partial_\mu A_a^\mu)^2\right)}.$$

Finally, the determinant itself can be rewritten as a functional over Grassmann variables (similar to the case of spinors),

$$\det\left(\frac{\delta F_a(x')}{\delta\Lambda_b(x)}\right) \sim \int \mathcal{D}\eta^*\mathcal{D}\eta\, e^{-\frac{i}{\hbar}\left(\int d^4x \int d^4x' \eta_a^*(x')\frac{\delta F_a(x')}{\delta\Lambda_b(x)}\eta_a(x)\right)},$$

so by evaluating the functional derivative of the gauge-fixing condition and integrating by parts, we obtain the correct Path integral over inequivalent gauge field configurations,

$$\int \mathcal{D}A\, e^{\frac{i}{\hbar}\int d^4x(\mathscr{L}_{YM})} \sim \int \mathcal{D}A \int \mathcal{D}\eta^*\mathcal{D}\eta\, e^{\frac{i}{\hbar}\int d^4x\left(\mathscr{L}_{YM}+\mathscr{L}_{FP}-\frac{1}{2\xi}(\partial_\mu A_a^\mu)^2\right)}, \tag{2.13}$$

where the Fadeev-Popov ghost term is given by

$$\mathscr{L}_{FP} = \partial_\mu\eta_a^*\left(\partial^\mu\eta_a + g f_{abc}\eta_b A_c^\mu\right).$$

For general non-Abelian gauge theories, we therefore define the generating functional as

$$W[J_\mu^a] \sim \int \mathcal{D}A \int \mathcal{D}\eta^*\mathcal{D}\eta\, e^{\frac{i}{\hbar}\int d^4x\left(\mathscr{L}_{YM}+\mathscr{L}_{FP}-\frac{1}{2\xi}(\partial_\mu A_a^\mu)^2+J_\mu^a A_a^\mu\right)}, \tag{2.14}$$

again normalized so that $W[0] = 1$. We immediately see that for Abelian gauge theories with $f_{abc} = 0$, $\mathscr{L}_{FP}$ can be factored out and the ghost integral absorbed into the normalization constant, hence there are no ghost contributions and the gauge-fixing term alone is sufficient to define the Path integral.

It is worth noting that while the generating functional constructed using the Fadeev-Popov method is manifestly Lorentz-invariant, the gauge-fixing term hides the manifest gauge-independence, making it difficult to prove that the renormalizability of the theory to all orders is a gauge-independent result. This difficulty was circumvented by the discovery of BRST symmetry, a type of supersymmetry under which the action (but not the Lagrangian density) of a non-Abelian gauge theory is invariant. Requiring that a non-Abelian gauge theory be invariant under BRST transformations is in fact sufficient to determine the correct Lagrangian density, and an extension of the BRST idea enables one to prove renormalizability of non-Abelian gauge theories in a gauge-independent way. Details of this procedure, along with associated references, may be found in Weinberg [60].

Having finally defined the Path integral over general gauge fields, and defining the Time-ordered product of gauge fields to be the same as for scalar fields, the time-ordered vacuum expectation value is given by

$$\left(\frac{i}{\hbar}\right)^n \langle 0|T\{A^{a_1}_{\mu_1}(x_1)\cdots A^{a_n}_{\mu_n}(x_n)\}|0\rangle = \left.\frac{\delta^n W[J^a_\mu]}{\delta J^{a_1}_{\mu_1}\cdots\delta J^{a_n}_{\mu_n}}\right|_{J^a_\mu=0}. \tag{2.15}$$

Specialising once again to the free theory, we find that the ghost sector of (2.14) decouples and simply corresponds to a free massless complex-valued scalar field; that is, the Feynman propagator for the ghost field is

$$\Delta^{Ghost}_F(x'-x) = \int \frac{d^4p}{(2\pi\hbar)^4} e^{\frac{i}{\hbar}p_\mu(x'-x)^\mu}\tilde{\Delta}^{Ghost}_F(p), \quad \tilde{\Delta}^{Ghost}_F(p) = \frac{1}{p^2} \tag{2.16}$$

For the gauge sector, we proceed as before in rewriting the generating functional in the form

$$W[J^a_\mu] \sim \int \mathcal{D}A\, e^{\frac{i}{\hbar}\int d^4x' \int d^4x \left(\frac{1}{2}A^a_\mu(x')C^{\mu\nu}_{ab}(x',x)A^b_\nu(x)\right)+\frac{i}{\hbar}\int d^4x J^a_\mu A^\mu_a},$$

where

$$C^{\mu\nu}_{ab}(x',x) = \delta_{ab}\left[\eta^{\mu\nu}\eta^{\rho\sigma} + \left(\frac{1}{\xi}-1\right)\eta^{\mu\rho}\eta^{\nu\sigma}\right]\partial^{x'}_\rho\partial^x_\sigma\ \delta(x'-x),$$

and so after completing the square once again we are left with the exact expression for the free generating functional,

$$W[J^a_\mu] = e^{-\frac{1}{2}\frac{i}{\hbar}\int d^4x' \int d^4x J^a_\mu(x')\left(C^{-1}\right)^{\mu\nu}_{ab}J^b_\nu(x)}.$$

To find C^{-1}, we use the integral representation of δ, giving

$$\tilde{C}^{\mu\nu}_{ab} = \delta_{ab}\tilde{C}^{\mu\nu}, \quad \tilde{C}^{\mu\nu} = \left[\left(1-\frac{1}{\xi}\right)p^\mu p^\nu - \eta^{\mu\nu}p^2\right].$$

Defining the transverse and longitudinal projection operators $\tilde{C}_T$, $\tilde{C}_L$ as

$$\tilde{C}^{\mu\nu}_T = \eta^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}, \quad \tilde{C}^{\mu\nu}_L = \frac{p^\mu p^\nu}{p^2},$$

we see that

$$\tilde{C}^{\mu\nu} = \left[-p^2\tilde{C}^{\mu\nu}_T - \frac{1}{\xi}p^2\tilde{C}^{\mu\nu}_L\right],$$

and can invert this by finding a tensor of the form

$$\tilde{C}^{-1}_{\mu\nu} = \left[\frac{\alpha}{p^2}\tilde{C}^T_{\mu\nu} + \frac{\beta}{p^2}\tilde{C}^L_{\mu\nu}\right]$$

such that $\tilde{C}^{\mu\nu}\tilde{C}^{-1}_{\mu\nu} = d$, the number of spacetime dimensions. Consequently, we find $\alpha = -1$, $\beta = -\xi$, and so

$$\tilde{D}_{\mu\nu} \equiv \tilde{C}^{-1}_{\mu\nu} = \frac{1}{p^2}\left[-\tilde{C}^T_{\mu\nu} - \xi\tilde{C}^L_{\mu\nu}\right] = \frac{1}{p^2}\left[(1-\xi)\frac{p_\mu p_\nu}{p^2} - \eta_{\mu\nu}\right].$$

Therefore, the exact expression for the gauge sector of the free-theory generating functional is given by

$$W[J^a_\mu] = e^{-\frac{1}{2}\frac{i}{\hbar}\int d^4x' \int d^4x J^a_\mu(x') D^{\mu\nu}_{ab}(x',x) J^b_\nu(x)}, \tag{2.17}$$

where the Feynman propagator for the gauge field, $D^{\mu\nu}_{ab}$, is given by

$$D^{\mu\nu}_{Fab}(x',x) = \int \frac{d^4p}{(2\pi\hbar)^4}\, e^{-ip_\mu(x'-x)^\mu}\, \tilde{D}^{\mu\nu}_{Fab}(p), \tag{2.18}$$

$$\tilde{D}^{\mu\nu}_{Fab}(p) = \frac{\delta_{ab}}{p^2}\left[(1-\xi)\frac{p^\mu p^\nu}{p^2} - \eta^{\mu\nu}\right]. \tag{2.19}$$

2.1.3 Perturbation Theory and Feynman Rules

So far, we have defined a suitable Path integral for scalars (2.4), spinors (2.8), and gauge fields (2.14), permitting a description of quantum-mechanical phenomena that is consistent with Special Relativity. For each field type, there exists an exact expression for the generating functional of the corresponding free theory, given by (2.6), (2.10), and (2.17) respectively. This is of course insufficient, as the intent of QFT is to describe particles and their interactions. In order to study theories with interactions, we may use Perturbation Theory: the Path integral is split into free and interacting parts, then the interaction term is Taylor-expanded in powers of the couplings, and the full theory is treated as a small perturbation away from the free theory. As long as the couplings in the interaction terms are relatively small, this perturbative description of the theory will provide a good approximation, and the calculation of physical processes should be in excellent agreement with experimental results.

Applying perturbation theory to QFT is not a straightforward matter. The n-point Green functions $\mathscr{G}^n(x_1,\ldots,x_n)$ of a theory are given by functional derivatives of the generating functional $W[J]$, and describe the propagation of fields between separated spacetime events; the Green functions therefore become singular if one attempts to functionally differentiate more than once with respect to a source at any spacetime event x_i. When using perturbation theory, we shall see that the formal expressions attempt to do exactly this, and so we immediately encounter singularities. After introducing the Feynman diagram representation of the Taylor expansion, we shall see that these divergences occur whenever a diagram contains a loop, and that the divergences may be regularised in order to continue formal manipulations. The easiest way of isolating these divergences is to introduce the notion of one-particle-irreducible diagrams and reformulate the Feynman rules in momentum space; the divergences may then be regularized using Dimensional Regularization.

We begin with the so-called ϕ^4 theory:

$$\mathscr{L}_{scalar} = \frac{\hbar^2}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4$$

Recall the definition of the Path integral for a scalar field,

$$W[J] \sim \int \mathcal{D}\phi \, e^{\frac{i}{\hbar}\int d^4x(\mathscr{L}_{scalar}+J\phi)}.$$

Since $\frac{\delta W[J]}{\delta J(x)} = \frac{i}{\hbar}\phi(x)W[J]$, we can identify the field operator with the functional derivative

$$\phi(x) \equiv -i\hbar\frac{\delta}{\delta J(x)}.$$

Now, if we split the Lagrangian density into free and interacting parts,

$$\mathscr{L}_{scalar} \equiv \mathscr{L}_{free} + \mathscr{L}_{int}, \quad \mathscr{L}_{int} = -\frac{\lambda}{4!}\phi^4,$$

the Path integral may be split accordingly,

$$\begin{aligned} W[J] &\sim \int \mathcal{D}\phi \, e^{\frac{i}{\hbar}\int d^4x(\mathscr{L}_{scalar}+J\phi)} \\ &\sim \int \mathcal{D}\phi \, e^{\frac{i}{\hbar}\int d^4x\left(\mathscr{L}_{free}+J\phi\right)} \, e^{\frac{i}{\hbar}\int d^4x(\mathscr{L}_{int})}. \end{aligned}$$

After replacing ϕ by the functional derivative, the interaction term no longer depends on ϕ, and so may be factored out of the Path integral to give

$$W[J] \sim e^{\frac{i}{\hbar}\int d^4x\mathscr{L}_{int}\left(-i\hbar\frac{\delta}{\delta J(x)}\right)} \, W_0[J], \tag{2.20}$$

where $W_0[J]$ is the generating functional for the free theory. The exponential operator may now be Taylor-expanded as

$$\begin{aligned} e^{\frac{i}{\hbar}\int d^4x\mathscr{L}_{int}\left(-i\hbar\frac{\delta}{\delta J(x)}\right)} &= \mathbb{1} + \frac{i}{\hbar}\int d^4x\mathscr{L}_{int}\left(-i\hbar\frac{\delta}{\delta J(x)}\right) \\ &+ \left(\frac{i}{\hbar}\right)^2 \int d^4x d^4y \, \mathscr{L}_{int}\left(-i\hbar\frac{\delta}{\delta J(x)}\right)\mathscr{L}_{int}\left(-i\hbar\frac{\delta}{\delta J(y)}\right) \\ &+ \dots \end{aligned}$$

and so the perturbative expansion of the generating functional for the interacting theory is

$$W[J] \sim \left\{ \mathbb{1} + \frac{i}{\hbar} \int d^4x \mathscr{L}_{int} \left(-i\hbar \frac{\delta}{\delta J(x)} \right) + \ldots \right\} W_0[J],$$

where the ϕ^4 interaction term becomes

$$\mathscr{L}_{int} = -\frac{\lambda}{4!} \left(-i\hbar \frac{\delta}{\delta J(x)} \right)^4 = -\frac{\lambda \hbar^4}{4!} \frac{\delta^4}{(\delta J(x))^4}.$$

Since perturbation theory is constructed for all QFTs in this manner, the only functional derivatives that are ever required in perturbative QFT are of the following two forms,

$$\frac{\delta}{\delta J(x)} \int dy\, f(y) J(y) = f(x), \qquad \frac{\delta}{\delta J(x)} e^{\int dy\, f(y) J(y)} = f(x)\, e^{\int dy\, f(y) J(y)},$$

where the required derivatives depend on the type of interaction. Applying these expressions to the generating functional for ϕ^4 theory, we find

$$W[J] \sim \left\{ 1 - \frac{i}{\hbar} \frac{\lambda}{4!} \int d^4x \Big[3 \left(i\hbar \Delta_F(0) \right)^2 \tag{2.21}$$

$$- 6 i\hbar \Delta_F(0) \int d^4y_1 d^4y_2\, i\Delta_F(x - y_1) i\Delta_F(x - y_2) J(y_1) J(y_2) \tag{2.22}$$

$$+ \int d^4y_1 d^4y_2 d^4y_3 d^4y_4 i\Delta_F(x - y_1) i\Delta_F(x - y_2) i\Delta_F(x - y_3) \tag{2.23}$$

$$i\Delta_F(x - y_4) J(y_1) J(y_2) J(y_3) J(y_4) \Big] + \mathcal{O}(\lambda^2) \right\} W_0[J], \tag{2.24}$$

where the $\Delta_F(0)$ terms arise from taking multiple functional derivatives at the same spacetime point x. We see that the leading-order effects due to interactions of quantum fields are now present, represented by terms proportional to the coupling λ. By setting $J = 0$, we find

$$\mathscr{G}^{(0)} \equiv W[0] \sim 1 - \frac{i}{\hbar} \frac{\lambda}{8} \int d^4x \, \left(i\hbar \Delta_F(0) \right)^2 + \mathcal{O}(\lambda^2),$$

The terms proportional to λ are known as vacuum bubbles[9], and represent fluctuations of the vacuum state due to the presence of interacting quantum fields. Their contribution to $W[0]$ is clearly non-zero (and is in fact infinite due to the divergent quantity $\Delta_F(0)$), but recall that there is a normalization constant N

[9]The reason being obvious once we introduce Feynman diagram notation.

implicit in the definition of $W[J]$. Writing this explicitly, we have

$$\mathscr{G}^{(0)} \equiv W[0] = N\left(1 - \frac{i}{\hbar}\frac{\lambda}{8}\int d^4x\ (i\hbar\Delta_F(0))^2 + \mathcal{O}(\lambda^2)\right), \tag{2.25}$$

and so if $W[0]$ is normalized to be 1, the normalization constant is

$$N = \left(1 - \frac{i}{\hbar}\frac{\lambda}{8}\int d^4x\ (i\hbar\Delta_F(0))^2 + \mathcal{O}(\lambda^2)\right)^{-1} \tag{2.26}$$

$$= 1 + \frac{i}{\hbar}\frac{\lambda}{8}\int d^4x\ (i\hbar\Delta_F(0))^2 + \mathcal{O}(\lambda^2). \tag{2.27}$$

If we continue to calculate the n-point Green functions for the interacting theory, for example

$$\begin{aligned}\mathscr{G}^{(2)}(x_1,x_2) = N\Big[& i\hbar\Delta_F(x_1-x_2) - \frac{i}{\hbar}\frac{\lambda}{8}\int d^4x\ (i\hbar\Delta_F(0))^2\ i\hbar\Delta_F(x_1-x_2) \\ & - \frac{i}{\hbar}\frac{\lambda}{2}\int d^4x\, i\hbar\Delta_F(x_1-x)\, i\hbar\Delta_F(0)\, i\hbar\Delta_F(x-x_2) + \mathcal{O}(\lambda^2)\Big] \\ = {}& i\hbar\Delta_F(x_1-x_2) \\ & - \frac{i}{\hbar}\frac{\lambda}{2}\int d^4x\, i\hbar\Delta_F(x_1-x)\, i\hbar\Delta_F(0)\, i\hbar\Delta_F(x-x_2) + \mathcal{O}(\lambda^2)\end{aligned}$$

we see that the effect of the normalization constant N is to remove vacuum bubble contributions to the Green functions. This effect continues to all orders of perturbation theory: by expressing the Green functions in the form

$$\mathscr{G}^{(n)}(x_1,\ldots,x_n) = \frac{\langle 0|T\left\{\phi(x_1)\cdots\phi(x_n)e^{\frac{i}{\hbar}\int d^4x\mathscr{L}_{int}\left(-i\hbar\frac{\delta}{\delta J(x)}\right)}\right\}|0\rangle}{\langle 0|T\left\{e^{\frac{i}{\hbar}\int d^4x\mathscr{L}_{int}(\phi)}\right\}|0\rangle},$$

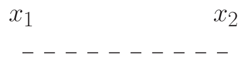
one can apply Wick's theorem [62] to the numerator, showing that it may be rewritten as

$$\begin{aligned}&\langle 0|T\left\{\phi(x_1)\cdots\phi(x_n)e^{\frac{i}{\hbar}\int d^4x\mathscr{L}_{int}\left(-i\hbar\frac{\delta}{\delta J(x)}\right)}\right\}|0\rangle \\ &\qquad = \langle 0|T\left\{e^{\frac{i}{\hbar}\int d^4x\mathscr{L}_{int}(\phi)}\right\}|0\rangle\,\tilde{\mathscr{G}}^{(n)}(x_1,\ldots,x_n),\end{aligned}$$

where $\tilde{\mathscr{G}}^{(n)}(x_1,\ldots,x_n)$ are the Green functions with all vacuum bubble contributions removed. The extra factor now cancels the denominator, and so we have

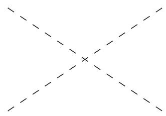
$$\mathscr{G}^{(n)}(x_1,\ldots,x_n) = \tilde{\mathscr{G}}^{(n)}(x_1,\ldots,x_n).$$

It is possible to develop a diagrammatical representation of the terms appearing in the Green functions. These diagrams are known as Feynman diagrams, and the implied correspondence between contributions to the Green functions and Feynman diagrams are known as Feynman rules. The most basic Feynman rule is that for the propagator $\Delta_F(x_1 - x_2)$,



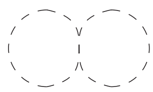
$$\longleftrightarrow \quad i\hbar\Delta_F(x_2 - x_1)$$

and can be thought of as representing the propagation of a scalar field from x_1 to x_2; indeed, this is exactly the meaning of the 2-point Green function for the free scalar theory. We may also specify the Feynman rule for a ϕ^4 interaction:



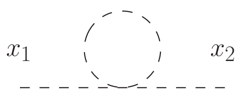
$$\longleftrightarrow \quad -\frac{i}{\hbar}\frac{\lambda}{4!}\int d^4x$$

For ϕ^4 theory, these are the only required rules, and suffice to construct the n-point Green functions to any order of perturbation theory, where the combinatoric factors that appear in $\mathscr{G}$ and N are given by the Wick expansion of the Time-ordered product. The vacuum bubbles appearing in the normalization factor N are therefore Feynman diagrams that form completely closed loops, such as



$$\longleftrightarrow \quad -\frac{i}{\hbar}\frac{\lambda}{4!}\int d^4x i\hbar\Delta_F(x-x)\, i\hbar\Delta_F(x-x),$$

and the 2-point Green function for an interacting theory includes a correction with a loop,



$$\longleftrightarrow \quad -\frac{i}{\hbar}\frac{\lambda}{4!}\int d^4x\, i\hbar\Delta_F(x_1 - x)\, i\hbar\Delta_F(x-x)\, i\hbar\Delta_F(x-x_2).$$

Hence, the divergent quantity $\Delta_F(0) = \Delta_F(x-x)$ occurs once for each loop in the corresponding Feynman diagram.

We now wish to isolate these divergences by restricting our attention to One-Particle-Irreducible (1PI) Green functions. The full Green function $\mathscr{G}^{(n)}$ can be reconstructed from the so-called connected Green functions $G^{(n)}$ [62]. By defining a new generating functional $X[J]$ such that

$$W[J] \equiv e^{\frac{i}{\hbar}X[J]}$$

and expressing $X[J]$ as a functional expansion

$$\frac{i}{\hbar}X[J] \equiv \sum_{n=1}^{\infty} \left(\frac{i}{\hbar}\right)^n \frac{1}{n!} \int d^4x_1 \ldots d^4x_n \, G^{(n)}(x_1, \ldots, x_n) J(x_1) \cdots J(x_n),$$

we see that the functions $G^{(n)}$ are given by

$$\left(\frac{i}{\hbar}\right)^n G^{(n)}(x_1, \ldots, x_n) = \frac{i}{\hbar} \frac{\delta^n X[J]}{\delta J(x_1) \cdots \delta J(x_n)}\bigg|_{J=0}. \tag{2.28}$$

$G^{(n)}$ are known as n-point connected Green functions, and are comprised of all terms in the full n-point Green functions whose Feynman diagram representation connects all n points via propagators and interaction vertices; consequently, it is easy to construct the connected Green functions to any order of perturbation theory by simply using the Feynman rules for the theory. The connected Green functions $G^{(n)}$ may in turn be reconstructed from the 1PI Green functions $\Gamma^{(n)}$, defined as the functional expansion of the effective action.

To define $\Gamma^{(n)}$, we start with the so-called classical field,

$$\phi_c(x) \equiv \frac{\delta X[J]}{\delta J(x)}. \tag{2.29}$$

If one considers the free scalar theory, the generating functional $X[J]$ is simply

$$X[J] = -\frac{1}{2} \int d^4x' d^4x \; J(x') \Delta_F(x' - x) J(x).$$

Functionally differentiating gives

$$\phi_c(x) = -\int d^4y \; \Delta_F(x - y) J(y),$$

and so by definition of a Green function (2.3) we have

$$\left(\hbar^2 \partial^2 + m^2\right) \phi_c(x) = \int d^4y \; \delta(x - y) J(y) = J(x), \tag{2.30}$$

that is, the quantity $\phi_c(x)$ satisfies the classical equation of motion for a free scalar field with source. In a general interacting theory, the one-point Green function is given by

$$\frac{i}{\hbar}\mathscr{G}^{(1)}(x) \;=\; \frac{i}{\hbar}\langle 0|\phi(x)|0\rangle \;=\; \frac{\delta W[J]}{\delta J(x)}\bigg|_{J=0},$$

so using the chain rule for functional differentiation we find

$$\frac{\delta W[J]}{\delta J(x)} = \left(\frac{i}{\hbar}\frac{\delta X[J]}{\delta J(x)}\right) W[J] = \frac{i}{\hbar}\phi_c(x)W[J],$$

and hence the classical field is the normalized vacuum expectation value of the corresponding quantum field,

$$\phi_c(x) = \frac{\langle 0|\phi(x)|0\rangle_J}{W[0]} = \frac{\langle 0|\phi(x)|0\rangle_J}{\langle 0|0\rangle_J}.$$

We now define the effective action Γ as

$$\Gamma[\phi_c] \equiv X[J] - \int d^4x\ J(x)\phi_c(x), \tag{2.31}$$

with the intent of expressing Γ solely in terms of the classical field ϕ_c[10]. Taking again the case of a free scalar field, where the source is related to the classical field by the classical equation of motion, one can easily manipulate the effective action to give

$$\begin{aligned}\Gamma[\phi_c] &= -\frac{1}{2}\int d^4x' d^4x\ J(x')\Delta_F(x'-x)J(x) - \int d^4x\ J(x)\phi_c(x)\\ &= -\frac{1}{2}\int d^4x\ J(x)\phi_c(x)\\ &= -\frac{1}{2}\int d^4x\ \phi_c(x)\hbar^2\partial^2\phi_c(x) - \frac{1}{2}\int d^4x\ m^2\phi_c^2(x)\\ &= \int d^4x\ \left[\frac{\hbar^2}{2}\partial_\mu\phi_c(x)\partial^\mu\phi_c(x) - \frac{1}{2}m^2\phi_c^2(x)\right],\end{aligned}$$

reproducing the action for a classical free scalar field theory. For an interacting theory, the relation between ϕ_c and J must be evaluated perturbatively. From the perturbative expansion of the generating functional $W[J]$, and using the approximations $\ln(1+x) = x + \mathcal{O}(x^2)$ and $N = 1 + \mathcal{O}(\lambda)$, we find

$$\begin{aligned}X[J] &\equiv -i\hbar\ln W[J]\\ &= \ln N - \frac{1}{2}\int d^4x' d^4x\ J(x')\Delta_F(x'-x)J(x)\\ &\quad - \frac{\lambda}{4!}\int d^4x\Big[3\left(i\hbar\Delta_F(0)\right)^2\\ &\quad - 6i\hbar\Delta_F(0)\int d^4y_1 d^4y_2\, i\Delta_F(x-y_1)i\Delta_F(x-y_2)J(y_1)J(y_2)\end{aligned}$$

[10]It is easy to see that this should be possible, since $\frac{\delta\Gamma[\phi_c]}{\delta J(x)} = 0$; that is, the variation of the functional depends only on the classical field.

$$+\int d^4y_1 d^4y_2 d^4y_3 d^4y_4 i\Delta_F(x-y_1) i\Delta_F(x-y_2)$$
$$i\Delta_F(x-y_3) i\Delta_F(x-y_4) J(y_1)J(y_2)J(y_3)J(y_4)\Big] + \mathcal{O}(\lambda^2),$$

so the classical field in the interacting theory becomes

$$\begin{aligned}
\phi_c(x) &\equiv \frac{\delta X[J]}{\delta J(x)} \\
&= -\int d^4x \ \Delta_F(x-y)J(y) \\
&\quad + \frac{\lambda}{2} i\hbar\Delta_F(0) \int d^4y_1 d^4y_2 \ i\hbar\Delta_F(x-y_1)\, i\hbar\Delta_F(y_1-y_2) J(y_1)J(y_2) \\
&\quad - \frac{\lambda}{6}\int d^4y_1 d^4y_2 d^4y_3 d^4y_4 i\hbar\Delta_F(x-y_1)\, i\hbar\Delta_F(y_1-y_2) \\
&\qquad i\hbar\Delta_F(y_1-y_3)\, i\hbar\Delta_F(y_1-y_4) J(y_2)J(y_3)J(y_4) \quad + \mathcal{O}(\lambda^2),
\end{aligned}$$

hence we may apply the Klein-Gordon operator to obtain

$$\begin{aligned}
\left(\hbar^2\partial_x^2 + m^2\right)\phi_c(x) = J(x) &- \frac{\lambda}{2} i\hbar\Delta_F(0)\int d^4y \ i\hbar\Delta_F(x-y)J(y) \\
&+ \frac{\lambda}{6}\int \prod_{k=1}^{3} d^4y_k \, i\hbar\Delta_F(x-y_k)J(y_k) + \mathcal{O}(\lambda^2).
\end{aligned}$$

Since $J(x) = (\hbar^2\partial_x^2 + m^2)\,\phi_c(x) + \mathcal{O}(\lambda)$, we can substitute and integrate by parts to find

$$J(x) = \left(\hbar^2\partial_x^2 + m^2\right)\phi_c(x) - \frac{\lambda}{2} i\hbar\Delta_F(0)\phi_c(x) + \frac{\lambda}{6}\phi_c^3(x) + \mathcal{O}(\lambda^2)$$

Now, from the definitions of the effective action and the classical field, we find

$$\begin{aligned}
\Gamma[\phi_c] = -i\hbar\ln N &+ \frac{1}{2}\int d^4y_1 d^4y_2 \ J(y_1)\Delta_F(y_1-y_2)J(y_2) - \frac{\lambda}{8}\int d^4x \ \left(i\hbar\Delta_F(0)\right)^2 \\
&- \frac{1}{2} i\hbar\Delta_F(0)\int d^4x d^4y_1 d^4(y_2)\, i\hbar\Delta_F(y_1-x)\, i\hbar\Delta_F(x-y_2)J(y_1)J(y_2) \\
&+ \frac{\lambda}{8}\int d^4x d^4y_1 d^4(y_2) d^4y_3 d^4(y_4)\, i\hbar\Delta_F(y_1-x)\, i\hbar\Delta_F(y_2-x) \\
&\qquad i\hbar\Delta_F(y_3-x)\, i\hbar\Delta_F(y_4-x)\; J(y_1)J(y_2)J(y_3)J(y_4) + \mathcal{O}(\lambda^2),
\end{aligned}$$

which can now be rewritten in terms of ϕ_c as

$$\begin{aligned}
\Gamma[\phi_c] = -i\hbar\ln N &- \frac{1}{2}\int d^4x\, \phi_c(x)\left(\hbar^2\partial^2 + m^2\right)\phi_c(x) \\
&- \frac{\lambda}{8}\int d^4x \ \left(i\hbar\Delta_F(0)\right)^2 + \frac{\lambda}{4} i\hbar\Delta_F(0)\int d^4x\, \phi_c^2(x) - \frac{\lambda}{24}\int \mathrm{d}^4x\, \phi_c^4(x)
\end{aligned}$$

$$+ \mathcal{O}(\lambda^2). \tag{2.32}$$

Finally, we obtain the desired 1PI Green functions $\Gamma^{(n)}$ from the functional expansion

$$\Gamma[\phi_c] \equiv \sum_{n=1}^{\infty} \frac{i^n}{n!} \int d^4x_1 \cdots \int d^4x_n \, \Gamma^{(n)}(x_1, \ldots, x_n) \phi_c(x_1) \cdots \phi_c(x_n),$$

that is,

$$\Gamma^{(n)}(x_1, \ldots, x_n) = \left. \frac{i^n \delta^n \Gamma[\phi_c]}{\delta\phi_c(x_1) \cdots \delta\phi_c(x_n)} \right|_{\phi_c=0}. \tag{2.33}$$

The first non-trivial 1PI Green functions are therefore

$$\begin{aligned} \Gamma^{(2)}(x_1, x_2) &= \left(\hbar^2 \partial^2_{x_2} + m^2 - \frac{\lambda}{2} \, i\hbar\Delta_F(0) \right) \delta(x_1 - x_2), \\ \Gamma^{(4)}(x_1, \ldots, x_4) &= -\lambda \int d^4x \, \delta(x_1 - x)\delta(x_2 - x)\delta(x_3 - x)\delta(x_4 - x). \end{aligned} \tag{2.34}$$

In terms of Feynman diagrams (given by $i\Gamma^{(n)}$), it can be shown that the n-point 1PI Green functions consist of all n-point connected Green functions that:

- have all external propagators removed;
- can be constructed using Feynman rules such that each diagram cannot be separated into two diagrams by cutting a single line.

Furthermore, we see that one of the leading-order effects associated with interactions is a shift in the effective mass of the scalar field,

$$m^2 \to m^2 - \frac{\lambda}{2} \, i\hbar\Delta_F(0) + \mathcal{O}(\lambda^2).$$

This is a quantum correction to the mass of a classical self-interacting scalar field theory, as indicated by the presence of $\hbar$, and is formally infinite due to the divergent term $\Delta_F(0)$ associated with a loop. By extending to higher orders of perturbation theory, we also find that the scalar coupling λ receives divergent quantum corrections from loop diagrams. Having established that quantum corrections are associated with increasing powers of $\hbar$[11], we may now adopt the standard convention of setting $\hbar = 1$ in order to simplify expressions.

By constructing the 1PI Green functions, we have successfully isolated the divergences that arise in the perturbative expansion of an interacting scalar QFT. The case of spinors and gauge theories proceeds analogously, by replacing the

[11] Equivalently, higher loop orders in the 1PI Green functions.

corresponding interaction term with functional derivatives and Taylor-expanding the exponential operator that acts on the generating functional for the free theory. One can then define Feynman rules for the associated propagators and interactions, and the n-point, l-loop, 1PI Green functions for the interacting theory can be systematically constructed by connecting propagators and vertices to make all possible diagrams with n external lines and l loops, such that the diagram cannot be split into two disconnected pieces by cutting a single line.

The final step in our construction is to reformulate the Feynman rules in momentum space: not only are the rules more simple (as can be seen by comparing the position- and momentum-space Feynman propagators), but they allow one to immediately apply dimensional regularization to the corresponding integrals. Using the standard shorthand $a \cdot b = a_\mu b^\mu$, we define

$$\tilde{\Gamma}^{(n)}(p_1, \ldots, p_n)\ (2\pi)^4\ \delta(p_1 + \ldots + p_n) = \int d^4x_1 \cdots d^4x_n\ e^{i(p_1 \cdot x_1 + \ldots + p_n \cdot x_n)} \Gamma^{(n)}(x_1, \ldots, x_n),$$

hence by using Fourier transforms of $\Delta_F(x' - x)$ and $\delta(x' - x)$, we find

$$\begin{aligned} i\tilde{\Gamma}^{(2)}(p, -p) &= \left(\frac{i}{p^2 - m^2}\right)^{-1} + \frac{1}{2}(-i\lambda) \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2} + \mathcal{O}(\lambda^2), \\ i\tilde{\Gamma}^{(4)}(p_1, \ldots, p_4) &= -i\lambda + \mathcal{O}(\lambda^2). \end{aligned} \tag{2.35}$$

The associated momentum-space Feynman rules for a scalar theory are then

- $\frac{i}{p^2 - m^2}$ for each scalar propagator with momentum p,
- $-i\lambda$ for each quartic vertex, plus conservation of momentum entering the vertex,
- $\int \frac{d^4k}{(2\pi)^4}$ for every closed scalar loop.

By comparison with the original Lagrangian density, the Feynman rules can be "read off" by simply multiplying the corresponding quantities by i. For the scalar case, the Feynman rule for a scalar propagator is $i\tilde{\Delta}_F(p)$, and the rule for a scalar vertex is $-i\lambda$; this extends to spinors and gauge fields, so that we have the following momentum-space rules:

- $i\tilde{S}_F(p) = \frac{i(\not{p} + m)}{p^2 - m^2}$ for each fermion propagator with momentum p,
- $i\tilde{D}_F^{\mu\nu}(p) = \frac{i}{p^2}\left[(1 - \xi)\frac{p^\mu p^\nu}{p^2} - \eta^{\mu\nu}\right]$ for each gauge propagator with momentum p,
- $i\tilde{g}$ for each (symmetrised) interaction term with factor $\tilde{g}$,

- an additional factor -1 for every closed fermion loop,
- an additional factor -1 for every closed ghost loop.

$\tilde{g}$ may contain additional matrices, for example the QED interaction $-e\bar{\psi}\not{A}\psi$ leads to the Feynman rule $-ie\gamma_\mu$. The factor -1 for fermion and ghost loops arises from their anticommuting nature.

2.1.4 Regularization and Renormalization

Having defined the 1PI Green functions and momentum-space Feynman rules, we may now regulate the divergences that arise in any perturbative QFT, then remove the divergences via renormalization. In principle one should be able to use any regularization method, renormalize, then recover the same result after removing the regulator. However, certain regulators violate the symmetries present in the original theory, and so we must compensate for these violations before removing the regulator. By far the most commonly used regularization method is Dimensional Regularization, since it preserves both gauge invariance and Poincaré invariance, and hence is convenient for regulating non-Abelian gauge theories; furthermore, Dimensional Regularization acts very much like standard integration, via a series of identities proven in [61].

Regularization

Dimensional Regularization is an analytic continuation of the usual notion of integration, defined by extending the dimension of a Euclidean (Wick-rotated) momentum integral to a complex-valued parameter d. This is motivated by noticing that a divergent integral in a certain number of spacetime dimensions would be convergent in a lower number of spacetime dimensions, for example the four-dimensional Euclidean integral

$$\int_E \frac{d^4p}{(2\pi)^4} \frac{1}{p^2 - m^2} \sim \mathcal{O}(|p|^2)$$

diverges quadratically for large Euclidean momentum p, but the corresponding two-dimensional integral

$$\int_E \frac{d^2p}{(2\pi)^2} \frac{1}{p^2 - m^2} \sim \mathcal{O}(\ln |p|)$$

only diverges logarithmically, and the integral converges for any dimension $d < 2$. Dimensional Regularization extends d to a complex number, defines the integral in a convergent region, then extends this definition to a meromorphic function of d with poles when d is a positive integer greater than some number.

Given a Euclidean vector p_E and function $f(p_E)$, we define d-dimensional integration as the functional

$$I_d[f(p_E)] \equiv \int d^d p_E \, f(p_E) \tag{2.36}$$

such that the functional I obeys linearity, translation and rotation invariance, and is homogeneous of degree $-d$. By expanding $f(p)$ in terms of a set of basis functions

$$f_{s,q}(p) \equiv e^{-s^2(p+q)^2},$$

we may apply translation invariance and scaling to obtain

$$\int d^d p \, f_{s,q}(p) = s^{-d} \int d^d p \, e^{-p^2},$$

expressing every basis function in terms of one integral. If we require that this integral can be converted to a spherical integral analogous to the integer-dimensional Gaussian, we find

$$\begin{aligned}
\int d^{d_1}p d^{d_2}q \, e^{-(p^2+q^2)} &= \int d^{d_1+d_2} r \left| \frac{\partial(r, \theta_1, \ldots, \theta_{d_1+d_2-1})}{\partial(p_1, \ldots, p_{d_1}, q_1, \ldots, q_{d_2})} \right| e^{-r^2} \\
&= S_{(d_1+d_2-1)} \int_0^\infty d|r| \, |r|^{d_1+d_2-1} e^{-|r|^2} \\
&= \frac{2\pi^{\frac{d_1+d_2}{2}}}{\Gamma(\frac{d_1+d_2}{2})} \frac{\Gamma(\frac{d_1+d_2}{2})}{2} \\
&= \pi^{\frac{d_1+d_2}{2}},
\end{aligned}$$

hence we may set the overall normalization of I according to

$$\int d^d p \, e^{-p^2} = \pi^{\frac{d}{2}}. \tag{2.37}$$

The functions we are required to integrate are tensor functions involving the loop momentum p and a finite number of external momenta $q_1, \ldots, q_J$. The tensor functions can be decomposed into products of vectors with scalar function coefficients, then each component of the tensor function can be separately evaluated; consequently, we need to define the integral of a scalar function $f(p, q_1, \ldots, q_J) \equiv f(p^2, p \cdot q_k, q_k^2)$. By isolating a finite, J-dimensional subspace spanned by q_k, we may split p into parallel and orthogonal components $p_P \in \text{Span}\{q_k\}$, $p_O \notin$

Span$\{q_k\}$, then define

$$\int d^d p\, f(p) = \int dp^1 \cdots dp^J \int d^{d-J} p_O\, f(p)$$

such that the J-dimensional integral is a standard, finite-dimensional integral. The orthogonal integral may be performed first by changing to spherical coordinates,

$$\int d^{d-J} p_O\, f(p) = S_{d-J-1} \int_0^\infty d|p_O|\, |p_O|^{d-J-1} f(p)$$

and so we may define the d-dimensional integral of a scalar function in terms of ordinary, finite-dimensional integration:

$$\int d^d p\, f(p) \equiv I_{d,J}[f(p)] = \frac{2\pi^{\frac{d-J}{2}}}{\Gamma\left(\frac{d-J}{2}\right)} \int dp^1 \cdots dp^J \int_0^\infty d|p_O|\, |p_O|^{d-J-1} f(p). \quad (2.38)$$

The orthogonal integral diverges at 0 when $d \leq J$, and diverges at ∞ when $d \geq J$. The divergence at 0 is more critical, since it is guaranteed to occur for $d \leq 0$, independent of the function $f(p)$.

To extend the definition to $d \leq 0$, consider the integral

$$\int d^d p\, f(p^2) \equiv I_{d,0}[f(p^2)] = \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)} \int_0^\infty d|p|\, |p|^{d-1} f(p^2),$$

where $f(p^2)$ decays sufficiently rapidly as $|p| \to \infty$. This may be split into two regions at some arbitrary value C,

$$I_{d,0}[f(p^2)] = \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)} \left\{ \int_C^\infty d|p|\, |p|^{d-1} f(p^2) + \int_0^C d|p|\, |p|^{d-1} f(p^2) \right\},$$

in which the integral over $[C, \infty)$ is finite. By adding and subtracting the function at 0, we find

$$\begin{aligned}
I_{d,0}[f(p^2)] &= \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)} \left\{ \int_C^\infty d|p|\, |p|^{d-1} f(p^2) + \int_0^C d|p|\, |p|^{d-1} \left[f(p^2) - f(0) + f(0) \right] \right\} \\
&= \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)} \left\{ \int_C^\infty d|p|\, |p|^{d-1} f(p^2) + \int_0^C d|p|\, |p|^{d-1} \left[f(p^2) - f(0) \right] + f(0) \frac{C^d}{d} \right\} \\
&= \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)} \left\{ \int_C^\infty d|p|\, |p|^{d-1} f(p^2) + \int_0^C d|p|\, |p|^{d-1} \left[f(p^2) - f(0) \right] \right\} \\
&\quad + \frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2}+1)} f(0) C^d.
\end{aligned}$$

Taking the limit $C \to \infty$, we find that for $-2 < Re(d) < 0$

$$\int d^d p\, f(p^2) = \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)} \left\{ \int_0^\infty d|p|\, |p|^{d-1} \left[f(p^2) - f(0) \right] \right\},$$

and for $d = 0$

$$\int d^0 p\, f(p^2) = f(0).$$

This procedure can be extended to any region $Re(d) \in (-2l-2, -2l)$ by continuing to isolate and subtract the $p = 0$ divergences that arise when $d = -2l, l \in \mathbb{N}$. This gives

$$\int d^d p\, f(p^2) = \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)} \left\{ \int_0^\infty d|p|\, |p|^{d-1} \left[f(p^2) - \sum_{k=0}^{l} \frac{(p^2)^k}{k!} f^{(k)}(0) \right] \right\} \tag{2.39}$$

for $Re(d) \in (-2l-2, -2l)$, and

$$\int d^{-2l} p\, f(p^2) = (-\pi)^{-l} f^{(l)}(0) \tag{2.40}$$

for $d = -2l$. These definitions form the crux of the method of Dimensional Regularization: the d-dimensional integral of any scalar function is defined by the analytic continuation from a region $Re(d) \in (-2l-2, -2l)$ in which the integral converges, and the integral of any tensor function is defined by decomposing each component of the tensor into a basis of terms with scalar function coefficients, then integrating the associated scalar functions. While this derivation assumes $f(0)$ is analytic, one may treat functions with power-law singularities in the same manner, systematically subtracting the singular behaviour to yield a finite integral. A noteworthy example is the case

$$f(p^2) = \frac{(p^2)^\alpha}{(p^2 + m^2)},$$

which diverges as $p \to 0$ if $m = 0$. Repeatedly differentiating f eventually gives

$$f^{(\alpha)}(p^2) = \frac{\alpha!}{p^2 + m^2} + \mathcal{O}(p^2),$$

so by definition we have

$$\int d^d p\, f(p^2) = \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)} \left\{ \int_0^\infty d|p|\, |p|^{d-1} \left[f(p^2) - \frac{(p^2)^\alpha}{m^2} \right] \right\}, \tag{2.41}$$

for some region of convergence $Re(d) \in (-2\alpha - 2, -2\alpha)$. Recall that the d-dimensional integral is defined to obey linearity, so when $m \neq 0$ we may rewrite

(2.41) as

$$\int d^dp\, f(p^2) = \int d^dp\, f(p^2) - \frac{1}{m^2}\int d^dp\,(p^2)^\alpha.$$

We therefore obtain the necessary identity

$$\int d^dp\,(p^2)^\alpha = 0. \tag{2.42}$$

We may now use Dimensional Regularization to regulate the divergent integrals that appear in Feynman diagrams. Consider the first quantum correction to $i\tilde{\Gamma}^{(2)}$ for an interacting scalar theory,

$$\int \frac{d^4k}{(2\pi)^4}\, i\tilde{\Delta}_f(0) = \int \frac{d^4k}{(2\pi)^4}\,\frac{i}{k^2-m^2}. \tag{2.43}$$

There is an implicit factor $i\epsilon$ in the denominator, which shifts the poles to $k_0 = \pm\left(\sqrt{\vec{k}^2+m^2} - i\epsilon\right)$ and allows one to Wick rotate the k_0 integral anticlockwise so that it lies on the imaginary axis. Having done this, we define the Euclidean momentum p according to $k_0 = ip_0, \vec{k} = \vec{p}$, giving

$$\int \frac{d^4k}{(2\pi)^4}\,\frac{i}{k^2-m^2} = \int \frac{d^4p}{(2\pi)^4}\,\frac{1}{p^2+m^2}.$$

Extending to d dimensions, we see that the Euclidean integral is a special case of (2.41), which may be evaluated by a change of variables $|p|^2 = m^2q$:

$$\begin{aligned}\int d^dp\,\frac{(p^2)^\alpha}{(p^2+m^2)} &= \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}\left\{\int_0^\infty d|p|\,|p|^{d-1}\left[\frac{(p^2)^\alpha}{(p^2+m^2)} - \frac{(p^2)^\alpha}{m^2}\right]\right\}\\ &= \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}\int_0^\infty d|p|\,\left[\frac{-1}{m^2}\frac{|p|^{2\left(\frac{d}{2}+\alpha\right)}}{|p|^2+m^2}\right]\\ &= \frac{\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}(-1)(m^2)^{\frac{d}{2}+\alpha-1}\int_0^\infty dq\,\frac{q^{\frac{d}{2}+\alpha}}{1+q}.\end{aligned}$$

The q-integral takes the form of the Euler beta function,

$$B(x,y) \equiv \int_0^z dq\,\frac{q^{x-1}}{(1+q)^{x+y}},$$

specifically $B(\frac{d}{2}+\alpha+1, -\frac{d}{2}-\alpha)$. Since the Euler beta function satisfies $B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$, we have

$$\int d^dp\,\frac{(p^2)^\alpha}{(p^2+m^2)} = \frac{\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}(-1)(m^2)^{\frac{d}{2}+\alpha-1}B\left(\frac{d}{2}+\alpha+1, -\frac{d}{2}-\alpha\right)$$

$$= \frac{\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}(-1)(m^2)^{\frac{d}{2}+\alpha-1}\Gamma\left(\frac{d}{2}+\alpha+1\right)\Gamma\left(-\frac{d}{2}-\alpha\right)$$
$$= \frac{\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}(m^2)^{\frac{d}{2}+\alpha-1}\Gamma\left(\frac{d}{2}+\alpha\right)\Gamma\left(1-\frac{d}{2}-\alpha\right).$$

The dimensionally regularized version of the quantum correction is therefore given by the $\alpha = 0$ case of the above integral[12]:

$$\int \frac{d^dk}{(2\pi)^d}\frac{i}{k^2-m^2} = \frac{m^{d-2}}{(4\pi)^{\frac{d}{2}}}\Gamma\left(1-\frac{d}{2}\right). \tag{2.44}$$

We wish to analyse the behaviour of this integral near $d = 4$, corresponding to the four-dimensional Minkowski spacetime in which our QFT is formulated. Since the mass parameter m is not a dimensionless quantity, we may not yet expand the integral to extract its singular behaviour: first, we must introduce an arbitrary mass scale μ such that

$$m^{d-2} = m^2(m^2)^{\frac{d}{2}-2} = m^2(\mu^2)^{\frac{d}{2}-2}\left(\frac{m^2}{\mu^2}\right)^{\frac{d}{2}-2}.$$

The d-dimensional integral is now

$$\int \frac{d^dk}{(2\pi)^d}\frac{i}{k^2-m^2} = \frac{m^2}{(4\pi)^2}(\mu^2)^{\frac{d}{2}-2}\left(\frac{m^2}{4\pi\mu^2}\right)^{\frac{d}{2}-2}\Gamma\left(1-\frac{d}{2}\right),$$

the last two terms of which may be expanded to give

$$\int \frac{d^dk}{(2\pi)^d}\frac{i}{k^2-m^2} = -\frac{m^2}{(4\pi)^2}(\mu^2)^{\frac{d}{2}-2}\left[\frac{1}{2-\frac{d}{2}}+\gamma+1-\ln\left(\frac{m^2}{4\pi\mu^2}\right)+\mathcal{O}\left(\frac{d}{2}-2\right)\right], \tag{2.45}$$

indicating the presence of a simple pole at $d = 4$. As a foreshadowing of the renormalization procedure, we also see that the classical mass parameter m^2 appears multiplying the singular expression.

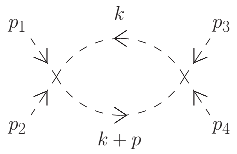
This particular integral is relatively straightforward to evaluate, as it contains only one propagator; moreover, it leads directly to more general expressions by utilizing various properties of d-dimensional integration. In general, a Feynman diagram will contain integration over multiple propagators, and the resulting integral cannot be evaluated in the same manner. There are various methods one may use to evaluate such diagrams, of which a conceptually simple method is the

[12]Note also that when $m = 0$, this result is consistent with (2.42).

technique of Feynman parametrization. By using the identity

$$\frac{1}{AB} = \int_0^1 du \, \frac{1}{[uA + (1-u)B]^2},$$

one may convert a product of propagators into a single integral, the result of which may be obtained from the one-propagator integral (2.44)[13]. As an example, consider now the first quantum correction to $i\tilde{\Gamma}^{(4)}$,



$$\longleftrightarrow \quad \lambda^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - m^2} \frac{1}{(k+p)^2 - m^2},$$

where local conservation of momentum sets $p \equiv p_1 + p_2 = -(p_3 + p_4)$. Applying dimensional regularization and Feynman parametrization to the integral, we find

$$\begin{aligned}
&\int \frac{d^dk}{(2\pi)^d} \frac{1}{k^2 - m^2} \frac{1}{(k+p)^2 - m^2} \\
&= \int \frac{d^dk}{(2\pi)^d} \int_0^1 du \, \frac{1}{[u(k^2 - m^2) + (1-u)\{(k+p)^2 - m^2\}]^2} \\
&= \int \frac{d^dk}{(2\pi)^d} \int_0^1 du \, \frac{1}{[k^2 + 2k \cdot p(1-u) + p^2(1-u) - m^2]^2} \\
&= \int \frac{d^dk}{(2\pi)^d} \int_0^1 du \, \frac{1}{[(k + (1-u)p)^2 - \tilde{m}^2]^2},
\end{aligned}$$

where $\tilde{m} = m^2 - u(1-u)p^2$. From (2.44), we may derive that

$$\int \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2 - m^2)^2} = -i \frac{\partial}{\partial m^2} \int \frac{d^dk}{(2\pi)^d} \frac{i}{k^2 - m^2} = i \frac{(m^2)^{\frac{d}{2}-2}}{(4\pi)^{\frac{d}{2}}} \Gamma\left(2 - \frac{d}{2}\right),$$

and so by first performing the momentum integral we find

$$\int \frac{d^dk}{(2\pi)^d} \frac{1}{k^2 - m^2} \frac{1}{(k+p)^2 - m^2} = \int_0^1 du \int \frac{d^dk}{(2\pi)^d} \frac{1}{[(k + (1-u)p)^2 - \tilde{m}^2]^2} \tag{2.46}$$

$$= \int_0^1 du \int \frac{d^dk}{(2\pi)^d} \frac{1}{[k^2 - \tilde{m}^2]^2} \tag{2.47}$$

$$= \int_0^1 du \, i \, \frac{\Gamma\left(2 - \frac{d}{2}\right)}{(4\pi)^{\frac{d}{2}}} (\tilde{m}^2)^{\frac{d}{2}-2} \tag{2.48}$$

[13] Relating multiple-propagator integrals to the one-propagator integral automatically implies that the resulting dimensionally regularized integrals are well-defined and obey the usual properties.

$$= i\,\frac{\Gamma\left(2-\frac{d}{2}\right)}{(4\pi)^{\frac{d}{2}}}\int_0^1 du\,\left[m^2 - u(1-u)p^2\right]^{\frac{d}{2}-2}. \tag{2.49}$$

By again introducing an arbitrary mass scale, we may expand the u integral around $d = 4$ to obtain

$$\begin{aligned}
&\int \frac{d^dk}{(2\pi)^d}\,\frac{1}{k^2-m^2}\,\frac{1}{(k+p)^2-m^2} \\
&\quad= i\,\frac{\Gamma\left(2-\frac{d}{2}\right)}{(4\pi)^2}(\mu^2)^{\frac{d}{2}-2}\int_0^1 du\,\left[\frac{m^2-u(1-u)p^2}{4\pi\mu^2}\right]^{\frac{d}{2}-2} \\
&\quad= i\,\frac{\Gamma\left(2-\frac{d}{2}\right)}{(4\pi)^2}(\mu^2)^{\frac{d}{2}-2}\left\{1+\left(\frac{d}{2}-2\right)\int_0^1 du\,\ln\left(\frac{m^2-u(1-u)p^2}{4\pi\mu^2}\right)\right. \\
&\qquad\qquad\qquad\left.+\,\mathcal{O}\left(\frac{d}{2}-2\right)^2\right\}.
\end{aligned}$$

The logarithmic integral contributes to the finite part of the Feynman diagram, and its precise value depends on the relation between m^2 and p^2. Since we are concerned only with the singular part of the diagram, we simply expand $\Gamma\left(2-\frac{d}{2}\right)$, giving

$$\lambda^2\int \frac{d^dk}{(2\pi)^d}\,\frac{1}{k^2-m^2}\,\frac{1}{(k+p)^2-m^2} = \frac{i\lambda^2}{(4\pi)^2}(\mu^2)^{\frac{d}{2}-2}\left\{\frac{1}{2-\frac{d}{2}}+\text{finite}\right\}. \tag{2.50}$$

This is the dimensionally-regularized value of the first quantum correction to the classical interaction between four scalar fields. We again see that a parameter from the classical Lagrangian density, in this case the coupling constant λ, appears multiplying the singular expression.

Conceptually, this procedure should allow one to regularize any integral that appears in the Feynman diagram expansion of a perturbative QFT. Practically, the ability to evaluate such integrals depends on the ability to perform the integral over the Feynman parameter; this is difficult at higher loop orders, where one must introduce multiple Feynman parameters via generalized versions of the identity above, and even more difficult for more general QFTs where there are (for example) multiple scalar fields with different masses.

There is one final issue present in the definition of Dimensional Regularization, which is how to correctly handle the γ-matrices present when performing integrals over fermion propagators. While one may define spinor fields in any integer number of spacetime dimensions, their definition is predicated on the corresponding representation of the Dirac algebra, generated by matrices γ obeying the anticommutation and Hermiticity relations

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{1}, \qquad (\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0 = \begin{cases} \gamma^0, & \mu = 0 \\ -\gamma^i, & \mu = i \neq 0 \end{cases} \tag{2.51}$$

In order to correctly apply Dimensional Regularization to integrals containing γ-matrices, we must extend the anticommutation relation to d dimensions, and provide a definition of the trace operation. Since d-dimensional vectors and matrices formally contain an infinite number of components, the γ-matrices can first be constructed inductively for finite components, then extended to infinite components.

Assume there exists a $2^{\frac{d}{2}} \equiv 2^\omega$-dimensional representation γ^μ_ω, satisfying (2.51) for all $0 \leq \mu \leq 2\omega - 1$. If we define matrices $\gamma^\mu_{\omega+1}$ by the matrix direct sum

$$\gamma^\mu_{\omega+1} = \gamma^\mu_\omega \oplus \gamma^\mu_\omega \equiv \begin{pmatrix} \gamma^\mu_\omega & 0 \\ 0 & \gamma^\mu_\omega \end{pmatrix},$$

then we see that for $0 \leq \mu, \nu \leq 2\omega - 1$,

$$\left\{\gamma^\mu_{\omega+1}, \gamma^\nu_{\omega+1}\right\} = \{\gamma^\mu_\omega, \gamma^\nu_\omega\} \oplus \{\gamma^\mu_\omega, \gamma^\nu_\omega\} = 2\eta^{\mu\nu}\left(\mathbb{1}_\omega \oplus \mathbb{1}_\omega\right) = 2\eta^{\mu\nu}\mathbb{1}_{\omega+1}$$

and

$$\begin{aligned} \left(\gamma^\mu_{\omega+1}\right)^\dagger &= \gamma^0_{\omega+1}\gamma^\mu_{\omega+1}\gamma^0_{\omega+1} \\ &= \gamma^0_\omega\gamma^\mu_\omega\gamma^0_\omega \oplus \gamma^0_\omega\gamma^\mu_\omega\gamma^0_\omega \\ &= \begin{cases} \gamma^0_\omega \oplus \gamma^0_\omega = \gamma^0_{\omega+1}, & \mu = 0 \\ -\left(\gamma^i_\omega \oplus \gamma^i_\omega\right) = -\gamma^i_{\omega+1}, & 1 \leq \mu = i \leq \omega - 1 \end{cases} \end{aligned}$$

Generally, $\gamma^\mu_{\omega+1}$ has two more entries than γ^μ_ω, so we must define these new entries in such a way that $\gamma^\mu_{\omega+1}$ satisfies the anticommutation and Hermiticity relations for $0 \leq \mu \leq 2\omega + 1$. By noting that we may define an additional matrix

$$\hat{\gamma}_\omega = i^{\omega-1}\gamma^0_\omega \cdots \gamma^{2\omega-1}_\omega$$

satisfying

$$(\hat{\gamma}_\omega)^\dagger = \hat{\gamma}_\omega, \qquad (\hat{\gamma}_\omega)^2 = \mathbb{1}_\omega, \qquad \{\hat{\gamma}_\omega, \gamma^\mu_\omega\} = 0,$$

we find that if the new entries in $\gamma^\mu_{\omega+1}$ are given by

$$\gamma^{2\omega}_{\omega+1} = \begin{pmatrix} 0 & \hat{\gamma}_\omega \\ -\hat{\gamma}_\omega & 0 \end{pmatrix}, \qquad \gamma^{2\omega+1}_{\omega+1} = \begin{pmatrix} 0 & i\hat{\gamma}_\omega \\ i\hat{\gamma}_\omega & 0 \end{pmatrix},$$

then $\gamma^\mu_{\omega+1}$ is a representation of the Dirac algebra, satisfying

$$\left\{\gamma^\mu_{\omega+1}, \gamma^\nu_{\omega+1}\right\} = 2\eta^{\mu\nu}\mathbb{1}_{\omega+1}, \qquad \left(\gamma^\mu_{\omega+1}\right)^\dagger = \gamma^0_{\omega+1}\gamma^\mu_{\omega+1}\gamma^0_{\omega+1}$$

for all $0 \leq \mu, \nu \leq 2\omega + 1$. To complete the construction, we simply need to define a base case for $\omega = 1$; that is, a basis of two-dimensional matrices satisfying the Dirac algebra, such as the Pauli matrices. The infinite-dimensional γ-matrices are then given by the limit $\omega \to \infty$:

$$\gamma^\mu = \bigoplus_{k=0}^{\infty} \gamma^\mu_\omega = \begin{pmatrix} \gamma^\mu_\omega & 0 & \\ 0 & \gamma^\mu_\omega & \\ & & \ddots \end{pmatrix}. \tag{2.52}$$

By construction, these matrices satisfy the Dirac algebra, and hence the d-dimensional γ-matrices may simply be manipulated in the same manner as their finite-dimensional counterparts. The final step is to define the trace over γ^μ, which is done by imposing linearity and cyclicity[14]:

$$\mathrm{tr}\left(a\gamma^\mu + b\gamma^\nu\right) = a\mathrm{tr}\left(\gamma^\mu\right) + b\mathrm{tr}\left(\gamma^\nu\right), \qquad \mathrm{tr}\left(\gamma^\mu\gamma^\nu\right) = \mathrm{tr}\left(\gamma^\nu\gamma^\mu\right).$$

By imposing these conditions, we find

$$\begin{aligned} \mathrm{tr}\left(\gamma^\mu\gamma^\nu\right) &= \mathrm{tr}\left(2\eta^{\mu\nu}\mathbb{1}_d - \gamma^\nu\gamma^\mu\right) \\ &= 2\eta^{\mu\nu}\mathrm{tr}\left(\mathbb{1}_d\right) - \mathrm{tr}\left(\gamma^\nu\gamma^\mu\right) \\ &= 2\eta^{\mu\nu}\mathrm{tr}\left(\mathbb{1}_d\right) - \mathrm{tr}\left(\gamma^\mu\gamma^\nu\right), \end{aligned}$$

and so

$$\mathrm{tr}\left(\gamma^\mu\gamma^\nu\right) = \eta^{\mu\nu}\mathrm{tr}\left(\mathbb{1}_d\right).$$

Since we require that the d-dimensional trace coincide with the corresponding result in integer dimensions, we define

$$\mathrm{tr}\left(\mathbb{1}_d\right) \equiv 2^{\frac{d}{2}}, \tag{2.53}$$

completing the treatment of γ-matrices in Dimensional Regularization[15].

[14] Imposing both linearity and cyclicity for all products of γ-matrices in d dimensions is in fact too strict a requirement, as it implies $\mathrm{tr}(\gamma_5\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma\ldots) = 0$, where the trace contains a product of γ_5 with $4, 6, 8, \ldots$ other γ-matrices. In order for such a product to be non-zero, and expressible in terms of a totally antisymmetric tensor $\epsilon_{\mu\nu\rho\sigma}$, the cyclicity condition must be relaxed.

[15] As indicated above, this treatment does not lead to a Lorentz-invariant, d-dimensional definition of γ^5, nor by extension the ϵ-tensor. The conventional resolution is to define γ^5 as usual, then restrict its anticommutation relation to the desired finite-dimensional subspace.

Renormalization

We have seen how to extract the divergent behaviour of the integrals arising in a perturbative QFT, using the method of Dimensional Regularization. We have also seen that each regularized integral is accompanied by a parameter from the classical Lagrangian density; specifically, the first divergence in $i\tilde{\Gamma}^{(2)}$ includes a factor m^2, and the first divergence in $i\tilde{\Gamma}^{(4)}$ includes a factor λ^2. Consequently, if the *quantum* versions of these parameters were to differ from the classical parameters, it may be possible to cancel the divergences. The idea behind Renormalization is that it is indeed possible to define new "renormalized" parameters, such that when the 1PI Green functions are expressed in terms of the renormalized parameters, the divergences disappear.

The parameters that appear in the classical Lagrangian density are referred to as bare parameters[16], representing the associated quantity in a theory without quantum fluctuations. Since these fluctuations cannot be "switched off", one never actually measures the bare parameters: one in fact only measures quantities at some particular energy scale μ_R. Our aim is therefore to redefine the bare fields and couplings in the Lagrangian density in terms of new fields and couplings multiplied by renormalization factors Z, such that physical results are given in terms of these new parameters at the scale μ_R. When working in perturbation theory, defining $Z = 1 + \delta Z$ allows one to rewrite the new Lagrangian density such that it takes the same form as the bare Lagrangian density, plus new potential contributions called counterterms; by deriving new Feynman rules for these counterterms, $\delta Z = \sum\limits_{n=1}^{\infty} \delta Z^{(n)}$ can be computed order-by-order, by requiring that the sum of all diagrams and counterterms at n-loops be finite.

As a quick example of this procedure, we can again consider ϕ^4 theory. We now refer to the classical Lagrangian density as the bare Lagrangian density,

$$\mathscr{L}_B = \frac{1}{2}\partial_\mu \phi_B \partial^\mu \phi_B - \frac{1}{2} m_B^2 \phi_B^2 - \frac{\lambda_B}{4!}\phi_B^4,$$

and define renormalized quantities according to[17]

$$\phi_B = Z_\phi^{\frac{1}{2}} \phi, \qquad m_B = Z_m^{\frac{1}{2}} Z_\phi^{-\frac{1}{2}} m, \qquad \lambda_B = Z_\lambda Z_\phi^{-2} \lambda. \tag{2.54}$$

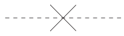
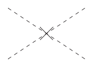
[16]The name "bare parameter" originates from the probing of an electron at different energy scales. Due to quantum fluctuations in the electromagnetic field, an electron will appear to be surrounded by a cloud of particles that are rapidly created and annihilated, which partially screen the electron and affect measurements of the electric charge. If interactions do not occur, then no particles are created and the electron is left unscreened, or bare.

[17]The form of these quantities is chosen purely to simplify the counterterm Lagrangian density, and hence simplify the Feynman rules for counterterms.

Taking $Z = 1 + \delta Z$ and expanding $\mathscr{L}_B$, we find

$$\begin{aligned}
\mathscr{L}_B &= \frac{1}{2}\partial_\mu\left(Z_\phi^{\frac{1}{2}}\phi\right)\partial^\mu\left(Z_\phi^{\frac{1}{2}}\phi\right) - \frac{1}{2}\left(Z_m^{\frac{1}{2}}Z_\phi^{-\frac{1}{2}}m\right)^2\left(Z_\phi^{\frac{1}{2}}\phi\right)^2 - \frac{1}{4!}Z_\lambda Z_\phi^{-2}\lambda\left(Z_\phi^{\frac{1}{2}}\phi\right)^4 \\
&= \left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4\right) + \left(\frac{1}{2}\delta Z_\phi\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\delta Z_m m^2\phi^2 - \frac{\lambda}{4!}\delta Z_\lambda\phi^4\right) \\
&= \mathscr{L} + \delta\mathscr{L}, \qquad (2.55)
\end{aligned}$$

where we have defined the renormalized Lagrangian density $\mathscr{L}$ to take the same form as $\mathscr{L}_B$. By calculating again the Green functions of this theory, we may derive Feynman rules for the terms in the counterterm Lagrangian density $\delta\mathscr{L}$:

-  $\longleftrightarrow i\left(\delta Z_\phi p^2 - \delta Z_m m^2\right)$, a new vertex connecting two propagators;
-  $\longleftrightarrow -i\delta Z_\lambda\lambda$, a new vertex connecting four propagators.

Using these new rules, we wish to calculate the renormalized n-point 1PI Green functions $i\tilde{\Gamma}^{(n)}$ as before, by summing over all possible 1PI Feynman diagrams including diagrams containing counterterms, with the additional constraint that $i\tilde{\Gamma}^{(n)}$ is finite for all n. At one-loop level, and using (2.45), we therefore find that

$$\begin{aligned}
i\tilde{\Gamma}^{(2)} &= -i\left(p^2 - m^2\right) - \frac{i\lambda}{2}\left(\int\frac{d^dk}{(2\pi)^d}\frac{i}{k^2 - m^2}\right) + i\left(\delta Z_\phi p^2 - \delta Z_m m^2\right) + \mathcal{O}(\lambda^2) \\
&= -ip^2\left(1 - \delta Z_\phi\right) + im^2\left(1 - \delta Z_m + \frac{\lambda\left(\mu^2\right)^{\frac{d}{2}-2}}{2(4\pi)^2}\frac{1}{2 - \frac{d}{2}} + \text{finite}\right) + \mathcal{O}(\lambda^2).
\end{aligned}$$

From this expression, we see that if we take

$$\delta Z_\phi^{(1)} = \text{finite}, \qquad \delta Z_m^{(1)} = \frac{\lambda\left(\mu^2\right)^{\frac{d}{2}-2}}{32\pi^2}\frac{1}{2 - \frac{d}{2}} + \text{finite}, \qquad (2.56)$$

where the finite parts of δZ are as-yet-unspecified, then

$$\lim_{d\to 4} i\tilde{\Gamma}^{(2)} = \text{finite} + \mathcal{O}(\lambda^2),$$

and hence is finite to first order in perturbation theory. Likewise, using (2.50), we find

$$\begin{aligned}
i\tilde{\Gamma}^{(4)} &= -i\lambda + \frac{3}{2}(-i\lambda)^2\int\frac{d^dk}{(2\pi)^d}\frac{i}{k^2 - m^2}\frac{i}{(k+p)^2 - m^2} + (-i\delta Z_\lambda) + \mathcal{O}(\lambda^3) \\
&= -i\lambda\left(1 + \delta Z_\lambda - \frac{3}{2}\frac{\lambda(\mu^2)^{\frac{d}{2}-2}}{(4\pi)^2}\frac{1}{2 - \frac{d}{2}} + \text{finite}\right) + \mathcal{O}(\lambda^3),
\end{aligned}$$

and so if we take

$$\delta Z_\lambda^{(1)} = \frac{3\lambda(\mu^2)^{\frac{d}{2}-2}}{32\pi^2}\frac{1}{2-\frac{d}{2}} + \text{finite}, \tag{2.57}$$

then

$$\lim_{d\to 4}\tilde{\Gamma}^{(4)} = \text{finite} + \mathcal{O}(\lambda^3).$$

It is worth noting that we defined the renormalization factors Z to be simple (albeit divergent) numerical coefficients, yet $\delta Z_m^{(1)}$, $\delta Z_\lambda^{(1)}$ appear to have non-zero mass dimension when analytically continued to d dimensions. We have neglected to ensure that the coupling λ in the d-dimensional theory is in fact a dimensionless coupling, as is required to perform a perturbative expansion. Since the action S of a theory must be dimensionless, and the action is defined as

$$S[\mathscr{L}] = \int d^d x\, \mathscr{L},$$

with the measure having mass dimension $\left[d^d x\right] = -d$, each term in $\mathscr{L}$ must have mass dimension $[\mathscr{L}] = d$. Since the mass dimension of a mass term is simply 1 by definition, we can read off the mass dimensions of each term in $\mathscr{L}$:

$$d = \begin{cases} [m^2\phi^2] = 2\,[m] + 2\,[\phi] = 2 + 2\,[\phi] & \implies [\phi] = \frac{d-2}{2} \\ [\partial_\mu\phi\partial^\mu\phi] = 2\,[\partial] + 2\,[\phi] = 2\,[\partial] + d - 2 & \implies [\partial] = 1 \\ [\lambda\phi^4] = [\lambda] + 4\,[\phi] = [\lambda] + 2d - 4 & \implies [\lambda] = 4 - d \end{cases}$$

We may therefore define a coupling $\hat{\lambda} = \lambda\mu^{d-4}$ that is dimensionless in d dimensions, in terms of which $\delta Z_m^{(1)}$, $\delta Z_\lambda^{(1)}$ are then themselves explicitly dimensionless:

$$\delta Z_m^{(1)} = \frac{\hat{\lambda}}{32\pi^2}\frac{1}{2-\frac{d}{2}} + \text{finite}, \quad \delta Z_\lambda^{(1)} = \frac{3\hat{\lambda}}{32\pi^2}\frac{1}{2-\frac{d}{2}} + \text{finite}. \tag{2.58}$$

In general, a QFT is considered perturbatively renormalizable if its 1PI Green functions can be rendered finite by a suitable choice of renormalization factors Z. Each factor $Z = 1 + \delta Z$ leads to a counterterm in the Lagrangian density proportional to δZ, which may then be computed order-by-order. The renormalizability of a theory is then classified as follows:

- Finite - no counterterms are required.
- Super-renormalizable - a finite number of Feynman diagrams require counterterms.
- Strictly Renormalizable - infinitely many Feynman diagrams require coun-

terterms, but all counterterms are determined by a finite set of renormalization factors.

- Non-renormalizable - infinitely many Feynman diagrams require counterterms, and infinitely many renormalization factors are required to generate all counterterms.

One can determine if a particular n-point 1PI Green function $\tilde{\Gamma}^{(n)}$ is divergent by power-counting[18]. Each basic d-dimensional integration measure raises the large-momentum behaviour by a power d, and each scalar propagator lowers the large-momentum behaviour by a power 2, hence we may denote the superficial degree of divergence, D, as

$$D = dL - 2I,$$

where L is the number of independent loop momenta and I the number of internal lines (propagators). Each Feynman diagram is made of L loops, V vertices, I internal lines, and n external lines, with local momentum conservation at each vertex and overall momentum conservation for the external lines. Each vertex therefore constrains one of the internal momenta by momentum conservation, while overall momentum conservation reduces the number of constraints by one, leaving

$$L = I - V + 1.$$

Each $\tilde{\Gamma}^{(n)}$ has n external lines; since the vertices of a ϕ^k theory have k lines and each propagator connects two of these lines, the number of external lines obeys

$$n = kV - 2I.$$

Combining these three relations, we may express the superficial degree of divergence of graphs in $\tilde{\Gamma}^{(n)}$ as

$$D = d - \frac{d-2}{2}n + \frac{k(d-2) - 2d}{2}V.$$

The sign of the term in front of V then determines how D grows as one includes more vertices. For a ϕ^k theory to be renormalizable, we therefore require

$$k \leq \frac{2d}{d-2}.$$

When this inequality is saturated, D becomes independent of the number of

[18]Power-counting is an extension of our analysis of the divergent scalar integrals at large momenta, combining dimensional analysis with basic graph theory.

vertices in a particular Feynman diagram, and one finds that

$$D \geq 0 \qquad \forall a = \frac{E}{2} \leq k, \;\; a \in \mathbb{N}.$$

In the four-dimensional ϕ^4 case where $d = k = 4$, the superficially divergent 1PI Green functions are therefore $\tilde{\Gamma}^{(0)}$, $\tilde{\Gamma}^{(2)}$ and $\tilde{\Gamma}^{(4)}$, which we have seen are rendered finite (at one loop) by the normalization condition on the generating functional, plus the renormalization of the mass m and quartic coupling λ. When $n \geq 6$, $\tilde{\Gamma}^{(n)}$ may still contain subgraphs that diverge, but these graphs must take the form of terms that occur in $\tilde{\Gamma}^{(2)}$ and $\tilde{\Gamma}^{(4)}$, and so the renormalization of such terms is predetermined. The important point here is that there are only *finitely* many superficially divergent 1PI Green functions, and there are sufficiently many parameters in the Lagrangian density such that these functions (and by extension all n-point Green functions) may be rendered finite by renormalization.

This power-counting method extends readily to general scalar-fermion theories, simply by analysing interaction terms that contain s scalars and f fermions. A fermion propagator has high-momentum behaviour $\sim |p|^{-1}$, so the superficial degree of divergence of a graph with I_s internal scalar lines and I_f internal fermion lines is now

$$D = dL - 2I_s - I_f,$$

while momentum conservation implies

$$L = I_s + I_f - V + 1$$

as before. A 1PI Green function $\tilde{\Gamma}^{(E)}$ has $E = E_s + E_f$ external lines, obeying

$$E_s + E_f + 2\left(I_s + I_f\right) = \left(s + f\right) V,$$

so in order to correctly reduce to a theory of only scalars or fermions, there exists a manifestly positive $V(s, f)$ such that

$$E_s + 2I_s = sV(s, f), \qquad E_f + 2I_f = fV(s, f).$$

We may therefore rewrite D as

$$D = d - \frac{d-2}{2}E_s - \frac{d-1}{2}E_f + \left[s\left(\frac{d-2}{2}\right) + f\left(\frac{d-1}{2}\right) - d \right] V(s, f),$$

and so a renormalizable scalar-fermion interaction must satisfy

$$s\left(\frac{d-2}{2}\right)+f\left(\frac{d-1}{2}\right)\leq d. \tag{2.59}$$

In four dimensions, this inequality is saturated by the cases $s = 4, f = 0$ (ϕ^4 theory), and $s = 1, f = 2$ (the Yukawa interaction). Note however that there are valid solutions to the inequality with $f = 1$; such solutions are not compatible with conservation of angular momentum, and so one cannot merely apply power-counting to deduce all renormalizable interaction terms that obey the desired symmetries of the Lagrangian density. This becomes crucial when considering non-Abelian gauge theories, where one must ensure that all interactions and counterterms also obey gauge invariance.

As a final comment on the calculation of δZ in perturbation theory, note that the finite parts have so far been treated schematically. Since the purpose of renormalization is merely to render the Green functions finite, *any* choice of finite part in δZ is valid. A prescription for choosing the finite parts of renormalization factors is known as a renormalization scheme; there exist various commonly-chosen schemes, each designed to satisfy a particular theoretical preference. For higher-order perturbative calculations, the most commonly chosen scheme is Minimal Subtraction (MS), and is defined such that the finite part of any δZ is simply zero, hence one only subtracts the divergent part of $\tilde{\Gamma}^{(n)}$. There is also modified Minimal Subtraction ($\overline{\text{MS}}$), motivated by noticing that the appearance of a (relatively large) term $\gamma - \ln(4\pi)$ in all one-loop calculations is simply a numerical artefact of the Taylor expansion of $(4\pi)^{2-\frac{d}{2}}\,\Gamma\left(2-\frac{d}{2}\right)$. Such artefacts may be systematically removed by rescaling $\mu \to \mu\left(\frac{e^\gamma}{4\pi}\right)^{-\frac{1}{2}}$, and serves to simplify perturbative calculations even further. The notion of a change in renormalization scheme and its relation to the couplings in a QFT will be illustrated below, and is of paramount importance throughout the results presented in this thesis.

2.1.5 The Renormalization Group

We have successfully constructed a perturbative expansion for interacting QFTs, regularized by analytic continuation of the number of spacetime dimensions d, and renormalized by adding counterterm interactions that cancel the divergences when $d \to 4$. The physical content of the theory is summarised by the renormalized 1PI Green functions $\tilde{\Gamma}^{(n)}\left(\phi, \psi, A, g, \xi; \mu\right)$, evaluated at a particular energy scale $\mu = \mu_R$[19]. This fixing raises one final issue in our construction of a QFT: what happens to $\tilde{\Gamma}^{(n)}$ as one varies the energy scale? By first relating the bare Green functions $\tilde{\Gamma}_B$ to the renormalized Green functions $\tilde{\Gamma}$ at some arbitrary scale μ, and noting that $\tilde{\Gamma}_B$ is independent of this scale, we may derive a differential equation for $\tilde{\Gamma}$ that is itself invariant under changes in μ. This equation is called the Renormalization Group equation (RGE), and implicitly defines various functions that describe how certain quantities change as one varies μ.

To derive the RGE, recall the definition of the scalar generating functional (2.4), and the relation between the bare and renormalized Lagrangian density,

$$\mathscr{L}_B + J_B \phi_B = \mathscr{L} + \delta\mathscr{L} + J\phi.$$

To ensure that the bare and renormalized generating functionals W_B and W maintain the same form, we require that their respective source terms are related by

$$J_B = Z^{-\frac{1}{2}} J.$$

It then follows that, since

$$\phi_B = Z^{\frac{1}{2}} \phi$$

and

$$\frac{\delta}{\delta J(x)} \int \mathcal{D}\phi\, f(\phi) e^{i\int d^4x\, J(x)\phi(x)} = i\phi(x) \int \mathcal{D}\phi\, f(\phi) e^{i\int d^4x\, J(x)\phi(x)},$$

the bare and renormalized Green functions $\mathscr{G}_B$ and $\mathscr{G}$, derived from W_B and W respectively, are related by

$$\mathscr{G}_B^{(n)}(x_1, \ldots, x_n) = Z^{\frac{n}{2}} \mathscr{G}^{(n)}(x_1, \ldots, x_n).$$

In direct analogy, we find

$$X_B[J_B] = -i \ln\left(W_B[J_B]\right) = -i \ln\left(W[J]\right) = X[J],$$

and so

$$(\phi_c)_B = \frac{\delta X_B[J_B]}{\delta J_B} = Z^{\frac{1}{2}} \frac{\delta X[J]}{\delta J} = Z^{\frac{1}{2}} \phi_c,$$

therefore the effective action satisfies

$$\Gamma_B\left[(\phi_c)_B\right] = \Gamma[\phi_c].$$

[19]Since Dimensional Regularization introduces an arbitrary mass scale μ, and $\tilde{\Gamma}^{(n)}$ includes terms of the form $g \ln\left(\frac{m^2}{\mu^2}\right)$, the perturbation series for small g will still contain large coefficients unless we set $\mu \equiv \mu_R \sim m$.

Applying the functional expansion, we see that this imposes the desired connection between bare and renormalized 1PI Green functions,

$$\tilde{\Gamma}^{(n)} = Z^{\frac{n}{2}} \tilde{\Gamma}^{(n)}_B.$$

This expression generalises easily to fermions and gauge fields in exactly the same manner, hence a theory with all three field types will have 1PI Green functions satisfying

$$\tilde{\Gamma}^{(n)}\left(\hat{g}, \xi, m, \mu\right) = Z_\phi^{\frac{n_\phi}{2}} Z_\psi^{\frac{n_\psi}{2}} Z_A^{\frac{n_A}{2}} \tilde{\Gamma}^{(n)}_B\left(g_B, \xi_B, m_b\right), \tag{2.60}$$

where the $n = n_\phi + n_\psi + n_A$ external legs are comprised of n_ϕ scalars, n_ψ fermions, and n_A gauge fields, and there is an implicit sum over all dimensionless couplings $\hat{g}$ in the theory. We may now act on both sides of this equation with the differential operator $\mu \frac{d}{d\mu}$, with the understanding that all bare quantities are independent of μ and the renormalization factors depend on μ implicitly: the left-hand-side gives

$$\mu \frac{d}{d\mu} \tilde{\Gamma}^{(n)}\left(\hat{g}, \xi, m, \mu\right) = \left(\mu \frac{\partial}{\partial \mu} + \mu \frac{\partial \hat{g}}{\partial \mu} \frac{\partial}{\partial \hat{g}} + \mu \frac{\partial \xi}{\partial \mu} \frac{\partial}{\partial \xi} + \mu \frac{\partial m}{\partial \mu} \frac{\partial}{\partial m}\right) \tilde{\Gamma}^{(n)},$$

while the right-hand-side gives

$$\begin{aligned} &\mu \frac{d}{d\mu}\left(Z_\phi^{\frac{n_\phi}{2}} Z_\psi^{\frac{n_\psi}{2}} Z_A^{\frac{n_A}{2}} \tilde{\Gamma}^{(n)}_B\right) \\ &= \left(\mu \frac{\partial Z_\phi}{\partial \mu} \frac{\partial}{\partial Z_\phi} + \mu \frac{\partial Z_\psi}{\partial \mu} \frac{\partial}{\partial Z_\psi} + \mu \frac{\partial Z_A}{\partial \mu} \frac{\partial}{\partial Z_A}\right)\left(Z_\phi^{\frac{n_\phi}{2}} Z_\psi^{\frac{n_\psi}{2}} Z_A^{\frac{n_A}{2}} \tilde{\Gamma}^{(n)}_B\right) \\ &= \left(n_\phi \frac{1}{2} \mu \frac{\partial Z_\phi}{\partial \mu} Z_\phi^{-1} + n_\psi \frac{1}{2} \mu \frac{\partial Z_\psi}{\partial \mu} Z_\psi^{-1} + n_A \frac{1}{2} \mu \frac{\partial Z_A}{\partial \mu} Z_A^{-1}\right) \tilde{\Gamma}^{(n)}. \end{aligned}$$

Rearranging, we obtain the Renormalization Group Equation,

$$\left(\mu \frac{\partial}{\partial \mu} + \beta(\hat{g}) \frac{\partial}{\partial \hat{g}} + \beta(\xi) \frac{\partial}{\partial \xi} - \gamma_m m \frac{\partial}{\partial m} - n_\phi \gamma_\phi - n_\psi \gamma_\psi - n_A \gamma_A\right) \tilde{\Gamma}^{(n)} = 0, \tag{2.61}$$

where we have defined the β-functions, mass anomalous dimension, and field anomalous dimensions as

$$\begin{aligned} \beta(\hat{g}) &= \mu \frac{\partial \hat{g}}{\partial \mu}, & \beta(\xi) &= \mu \frac{\partial \xi}{\partial \mu}, & \gamma_m &= -\frac{\mu}{m} \frac{\partial m}{\partial \mu}, \\ \gamma_\phi &= \frac{1}{2} \mu \frac{\partial Z_\phi}{\partial \mu} Z_\phi^{-1}, & \gamma_\psi &= \frac{1}{2} \mu \frac{\partial Z_\psi}{\partial \mu} Z_\psi^{-1}, & \gamma_A &= \frac{1}{2} \mu \frac{\partial Z_A}{\partial \mu} Z_A^{-1}. \end{aligned} \tag{2.62}$$

These functions are known as RG functions, quantities, or coefficients, and describe the effects of a change in the RG scale on the couplings, gauge-fixing parameter, and canonical scaling dimensions as a result of interactions. By considering the behaviour of $\tilde{\Gamma}^{(n)}$ as one scales the external momenta, one may relate

$\tilde{\Gamma}^{(n)}(p_1,\ldots,p_n)$ at some scale p to $\tilde{\Gamma}^{(n)}(sp_1,\ldots,sp_n)$ at the new scale sp, such that any change in the couplings and scaling dimensions is given by the RG quantities.

Since we have already calculated the one-loop renormalization factors for a ϕ^4 theory, we can easily demonstrate how to calculate the RG quantities. Recall that, at one loop, the renormalization constants in the $\overline{\text{MS}}$ scheme are

$$Z_\phi = 1 + \mathcal{O}\left(\delta Z_\phi^{(2)}\right), \qquad Z_m = 1 + \frac{\hat{\lambda}}{16\pi^2}\frac{1}{4-d}, \qquad Z_\lambda = 1 + \frac{3\hat{\lambda}}{16\pi^2}\frac{1}{4-d}.$$

We may use the μ-independence of the bare parameters to generate relations between the Zs and the RG quantities, then keep only terms up to a particular order of $\hat{\lambda}$. Beginning with the bare coupling, we have

$$\begin{aligned}
0 &= \mu\frac{d\lambda_B}{d\mu} = \mu\frac{d}{d\mu}\left(Z_\lambda Z_\phi^{-2}\mu^{4-d}\hat{\lambda}\right) \\
&= (4-d)Z_\lambda Z_\phi^{-2}\mu^{4-d}\hat{\lambda} + Z_\lambda Z_\phi^{-2}\mu^{4-d}\beta_{\hat{\lambda}} + \left(\beta_{\hat{\lambda}}\frac{\partial Z_\lambda}{\partial\hat{\lambda}}\right)Z_\phi^{-2}\mu^{4-d}\hat{\lambda} \\
&\quad - 2Z_\lambda Z_\phi^{-3}\left(\beta_{\hat{\lambda}}\frac{\partial Z_\phi}{\partial\hat{\lambda}}\right)\mu^{4-d}\hat{\lambda} \\
&= \mu^{4-d}Z_\lambda Z_\phi^{-2}\left[(4-d)\hat{\lambda} + \beta_{\hat{\lambda}} + Z_\lambda^{-1}\left(\beta_{\hat{\lambda}}\frac{\partial Z_\lambda}{\partial\hat{\lambda}}\right)\hat{\lambda} - 2Z_\phi^{-1}\left(\beta_{\hat{\lambda}}\frac{\partial Z_\phi}{\partial\hat{\lambda}}\right)\hat{\lambda}\right].
\end{aligned}$$

The terms involving Zs are at least of order $\hat{\lambda}^2$, so by postulating

$$\beta_{\hat{\lambda}} = (d-4)\hat{\lambda} + b_1\hat{\lambda}^2 + \mathcal{O}(\hat{\lambda}^3),$$

substituting in the one-loop expressions for Z_λ, Z_ϕ, and keeping terms only up to $\hat{\lambda}^2$, we find

$$b_1 = (4-d)\delta Z_\lambda^{(1)}\hat{\lambda} = \frac{3}{16\pi^2}\hat{\lambda}^2,$$

therefore the one-loop β-function for λ in $d=4$ dimensions is

$$\beta_\lambda = 3\left(\frac{\lambda}{4\pi}\right)^2 + \mathcal{O}(\lambda^3). \tag{2.63}$$

Next we use the bare mass, giving

$$\begin{aligned}
0 &= \mu\frac{dm_B}{d\mu} = \mu\frac{d}{d\mu}\left(Z_m^{\frac{1}{2}}Z_\phi^{-\frac{1}{2}}m\right) \\
&= \frac{1}{2}Z_m^{-\frac{1}{2}}\left(\beta_{\hat{\lambda}}\frac{\partial Z_m}{\partial\hat{\lambda}}\right)Z_\phi^{-\frac{1}{2}}m - \frac{1}{2}Z_m^{\frac{1}{2}}Z_\phi^{-\frac{3}{2}}\left(\beta_{\hat{\lambda}}\frac{\partial Z_\phi}{\partial\hat{\lambda}}\right)m + Z_m^{\frac{1}{2}}Z_\phi^{-\frac{1}{2}}\mu\frac{\partial m}{\partial\mu} \\
&= m^{-1}Z_m^{-\frac{1}{2}}Z_\phi^{\frac{1}{2}}\left[\frac{1}{2}Z_m^{-1}\beta_{\hat{\lambda}}\frac{\partial Z_m}{\partial\hat{\lambda}} - \frac{1}{2}Z_\phi^{-1}\beta_{\hat{\lambda}}\frac{\partial Z_\phi}{\partial\hat{\lambda}} - \gamma_m\right].
\end{aligned}$$

The terms involving Zs are at least of order $\hat{\lambda}$, so we find that the one-loop mass anomalous dimension in $d = 4$ dimensions is given by[20]

$$\begin{aligned}\gamma_m &= \left[-\frac{1}{2}(4-d)\delta Z_m^{(1)} + \mathcal{O}(\hat{\lambda}^2) \right] \Big|_{d=4} \\ &= -\frac{1}{8\pi}\left(\frac{\lambda}{4\pi}\right) + \mathcal{O}(\lambda^2). \end{aligned} \tag{2.64}$$

Finally, the field anomalous dimension is easily calculable from its definition. Since Z_ϕ has no one-loop coefficient in ϕ^4 theory, the field anomalous dimension at one loop is simply

$$\gamma_\phi = \frac{1}{2}\left(\beta_{\hat{\lambda}} \frac{\partial Z_\phi}{\partial \hat{\lambda}}\right) Z_\phi^{-1} = \mathcal{O}(\lambda^2). \tag{2.65}$$

Having determined the β-function for the scalar coupling λ, we may approximate the change in λ from the current scale $\hat{\lambda}(\mu)$ to a new energy scale $\hat{\lambda}(s\mu)$ by solving the partial differential equation

$$s\frac{\partial \bar{\lambda}(s)}{\partial s} = \beta_{\hat{\lambda}}\left(\bar{\lambda}(s)\right), \qquad \begin{cases} \bar{\lambda}(1) = \hat{\lambda}(\mu) = \hat{\lambda}, \\ \bar{\lambda}(s) = \hat{\lambda}(s\mu). \end{cases}$$

This can be solved at one loop by simple separation of variables, giving

$$\bar{\lambda}(s)\big|_{1-loop} = \frac{\hat{\lambda}}{1 - \hat{\lambda} b_1 \ln s}, \qquad b_1 = \frac{3}{16\pi^2}.$$

Due to the positive sign of b_1, we see that the coupling decreases as one reduces the energy scale, with the limit

$$\lim_{s\to 0} \bar{\lambda}(s) = 0.$$

We say that the coupling *decreases under RG flow* from the UV (high energy) to the IR (low energy). The coupling flows to an RG fixed point at $\lambda = 0$, where $\beta_\lambda = 0$ and the coupling no longer changes. In the simple ϕ^4 case this is the only RG fixed point, but in theories with various interactions it is possible to find non-trivial fixed points where $g^I \neq 0$. In general, a QFT with multiple couplings $g^I \in \{g_1, g_2, \ldots\}$ satisfying $\beta(g^I) = 0$ is referred to as a scale-invariant QFT

[20] If we also keep track of the $\mathcal{O}(\hat{\lambda}^2)$ terms, we find that there is a simple pole in γ_m, the coefficient of which is a combination of simple- and double-pole coefficients in δZ; since γ is finite, this combination of terms must vanish, imposing relations between the simple pole at a particular loop order and the double pole at the next loop order.

(SFT); since RG flow ends at fixed points[21], we may interpret general QFTs as points on a manifold in *coupling space*, parametrised by RG flows between SFTs.

[21]There is a subtlety here, based on the effects of global symmetries in a QFT, which shall be clarified in the next section.

2.2 Constraints on Renormalization-Group flow

In this section, we follow the basic treatment of Quantum Field Theory in Curved Spacetime found in Parker and Toms [65], supplemented by Shore's pedagogical review of the c-theorem, the a-theorem, and the Local Renormalization Group [67]; we also use Osborn's original paper on Weyl consistency conditions [7].

We have finally arrived at the central topic of this thesis: do there exist constraints on RG flows, and what are the consequences of such constraints? There is no a priori reason to think that the only possible behaviour of an RG flow is to approach a fixed point; it is conceivable that an RG flow could instead approach a closed trajectory in coupling space (a limit cycle), or indeed display no asymptotic behaviour at all (ergodic, or chaotic, flow). Physically, such behaviour would be highly unusual, as it would suggest that the high-energy limit of a physical system could be described completely in terms of low-energy properties, akin to describing the small-scale electromagnetic interactions between water molecules purely in terms of the flow velocity, density and pressure of the large-scale water continuum.

Constraints on the possible RG flows of general two-dimensional QFTs were first established by Zamolodchikov [2]. By interpreting the β-functions of a theory as components of a vector field β^I on the space of couplings $\{g^I\}$, Zamolodchikov reasoned that since β^I generates RG flow in coupling space, and RG flow relates the correlation functions of a theory at two different energy scales, there should be an inherent irreversibility to RG flow. Heuristically, this is because the correlation functions are defined at a particular energy scale μ, and it is not meaningful to then attempt to measure correlations at any energy scale $\mu' > \mu$: some information about the UV theory must therefore be lost under RG flow to the IR.

To establish the irreversibility of RG flow, Zamolodchikov defined various functions, based on two-point correlation functions of the energy-momentum tensor. Given a two-dimensional Euclidean QFT and introducing complex coordinates $z = x + iy,\ \bar{z} = x - iy$, if one defines a function C as the linear combination

$$C = 2F - H - \frac{3}{8}G,$$

where

$$F = (2\pi)^2 z^4 \langle T_{zz}(x) T_{\bar{z}\bar{z}}(0) \rangle \,,$$
$$G = (2\pi)^2 x^4 \langle T_{z\bar{z}}(x) T_{z\bar{z}}(0) \rangle \,,$$

$$H = (2\pi)^2 z^2 x^2 \langle T_{zz}(x) T_{z\bar{z}}(0) \rangle ,$$

then C satisfies

$$|z| \frac{\partial C}{\partial |z|} = -\frac{3}{2} G,$$

that is, C decreases as one increases the distance scale $|z|$. Since the trace of the energy-momentum tensor for a general QFT (see (2.66) below) contains an operator anomaly

$$\langle T^{\mu}{}_{\mu} \rangle = 4 \langle T_{z\bar{z}} \rangle = \beta^I \mathcal{O}_I,$$

we may define a new tensor structure (up to some arbitrary constant of proportionality)

$$G_{IJ} \sim (2\pi)^2 z^2 \bar{z}^2 \langle \mathcal{O}_I(x) \mathcal{O}_J(0) \rangle ,$$

then by fixing $|z| = \mu^{-1}$, we may use the RG equation to reformulate the constraint on C as

$$\beta^I \partial_I C \equiv \beta^I \frac{\partial C}{\partial g^I} = \beta^I G_{IJ} \beta^J \geq 0.$$

Since G_{IJ} is manifestly positive-definite, C is stationary only at RG fixed points, where $\beta^I = 0$ and hence $\langle T^{\mu}{}_{\mu} \rangle = 0$. Consequently, at a fixed point g^* we have $C(g^*) = 2F$, and so by using the Operator Product Expansion (OPE) for the energy-momentum tensor of a CFT

$$\langle T_{zz}(x) T_{\bar{z}\bar{z}}(0) \rangle = \frac{1}{(2\pi)^2} \frac{c}{2z^4} + \mathcal{O}(|z|^{-2}),$$

we find that the function C is equal to the central charge of the corresponding CFT,

$$C(g^*) = c.$$

Finally, near a fixed point, expanding C and β^I gives

$$\begin{aligned}
G_{IJ} &= 12\delta_{IJ} + \mathcal{O}(g^2), \\
\beta^I &= \epsilon(g^I) g^I - \frac{1}{2} C_{IJK} g^J g^K + \mathcal{O}(g^3), \\
C(g) &= C(g^*) + 6\epsilon(g^I) g_I g^I - 2C_{IJK} g^I g^J g^K + \mathcal{O}(g^4).
\end{aligned}$$

C, β^I and G_{IJ} are then related by the gradient-flow equation,

$$\frac{\partial C}{\partial g^I} = G_{IJ} \beta^J.$$

As outlined in the introduction, the existence of the function C satisfying these properties is known as the c-theorem[22], and provides a strict constraint on the possible RG flows of a two-dimensional QFT: the couplings of the theory must run in such a way that a particular function C of the couplings decreases monotonically under RG flow. The consequences of the c-theorem have been investigated by numerous others, of which two key results are:

- Polchinski [4] demonstrated that, for a general two-dimensional QFT, one may always redefine $T_{\mu\nu}$ so that it has a canonical scaling dimension. The c-theorem then implies $T^{\mu}{}_{\mu} = 0$ as an operator identity, hence scale invariance implies conformal invariance in two dimensions.

- Friedan and Konechny [11] demonstrated that the gradient-flow equation in fact holds non-perturbatively, away from RG fixed points.

Unfortunately, Zamolodchikov's argument does not generalize beyond two dimensions. It was shown by Cardy that, for any theory in dimension $d > 2$, the standard assumptions of renormalizability, reflection-positivity, translation and rotation invariance are insufficient to construct a function C whose derivative is determined entirely by the two-point function $\langle T^{\mu}{}_{\mu} T^{\nu}{}_{\nu} \rangle$, and so is not guaranteed to be monotonic. Instead, Cardy considered the idea of interpreting C in terms of the trace anomaly of a QFT in curved spacetime (QFTCS), which for a QFT in two dimensions is

$$\langle T^{\mu}{}_{\mu} \rangle = -\frac{c}{24\pi} R,$$

where R is the Ricci scalar of the spacetime and c is again the central charge of the corresponding CFT. In general, the d-dimensional trace anomaly takes the form

$$\langle T^{\mu}{}_{\mu} \rangle = \beta^{I} \langle \mathcal{O}_{I} \rangle + \tilde{c}\, (C)^{\frac{d}{2}} - \tilde{a}\, E_{d} + \sum_{i} b_{i} f_{i}\, (\phi, g_{\mu\nu})\,, \tag{2.66}$$

where we have the usual operator anomaly, plus new gravitational anomalies proportional to powers of the Weyl tensor,

$$C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - \frac{2}{(d-2)} \left(g_{\mu[\rho} R_{\sigma]\nu} - g_{\nu[\rho} R_{\sigma]\mu} \right) + \frac{2}{(d-1)(d-2)} R g_{\mu[\rho} g_{\sigma]\nu},$$

the Euler density,

$$E_{d} = \begin{cases} \frac{1}{2^{n}} \epsilon^{\mu_1\nu_1\ldots\mu_n\nu_n} \epsilon^{\rho_1\sigma_1\ldots\rho_n\sigma_n} R_{\mu_1\nu_1\rho_1\sigma_1} \cdots R_{\mu_n\nu_n\rho_n\sigma_n} & d = 2n, n \in \mathbb{Z}_{+} \\ 0 & d = 2n+1, n \in \mathbb{Z}_{+} \end{cases}$$

[22]Specifically, Zamolodchikov's construction proves the monotonicity of C and the relation $C(g^{*}) = c$ non-perturbatively, while the gradient-flow equation is only valid near RG fixed points.

and a number of other scalars f_i of dimension d, such as $R\phi^2$. The gravitational anomalies are independent of the interactions in the theory, and hence could be related to a c-theorem for general QFTs. When $d = 2$, the Weyl tensor vanishes identically, the Euler density is exactly the Ricci scalar, and the only other possible contribution ($f_i \sim \xi R$) vanishes identically[23], hence at an RG fixed point we must have

$$\tilde{a} \sim c.$$

Cardy therefore conjectured [1] that a suitable generalization of the c-theorem should involve a function A (hereafter referred to as the a-function), satisfying the same properties as the two-dimensional c-function, such that at RG fixed points $A(g^*) = \tilde{a}$, the coefficient of the Euler density in the trace anomaly. The existence of this function is known as the a-theorem, and would be a valid generalization of the c-theorem to any even-dimensional spacetime.

There have been several attempts to prove the a-theorem, or at least one of the progressively-stronger formulations outlined in the introduction:

- Jack and Osborn showed [8] that the desired a-function exists, decreases monotonically under RG flow, is equal to the coefficient $\tilde{a}$ of the Euler density at RG fixed points, and obeys a gradient-flow equation near RG fixed points, with a metric that is positive-definite at leading order in perturbation theory.

- Anselmi, Freedman, Grisaru and Johansen provided a non-perturbative proof of the weak a-theorem for $\mathcal{N} = 1$ supersymmetric theories [12]. The argument was dramatically simplified by Intriligator and Wecht [13], with refinements by Kutasov [14], using the idea of a-maximization.

- Komargodski and Schwimmer provided a non-perturbative proof of the weak a-theorem for general four-dimensional QFTs [16], by coupling the QFT to a dilaton and analysing the four-point function of $T^{\mu}{}_{\mu}$.

Progress has also been made on various corollaries and implications of the a-theorem, for example Luty, Polchinski and Rattazzi proved that the weak a-theorem is sufficient to rule out limit cycles and ergodic RG flow, as well as guaranteeing that scale-invariance implies conformal invariance in four dimensions [17]; Jack and Osborn also derived a consistency condition that, if true, leads to an all-orders expression for the a-function in four-dimensional $\mathcal{N} = 1$

[23]The Lagrangian density of any d-dimensional QFTCS with scalar fields contains a contribution $\xi R\phi^2$; a minimally-coupled theory is one with $\xi = 0$, whereas a conformally-coupled theory is one with $\xi = \frac{(d-2)}{4(d-1)}$. In two dimensions, the two are therefore equivalent.

supersymmetric theories without gauge interactions [9]. Further work has been done on explicitly constructing an a-function in six dimensions [37, 38], in order to verify that the metric is positive-definite at leading order, and attempts have been made to find an equivalent function for odd-dimensional QFTs [43–45].[24]

The constraints on RG flows provided by Jack and Osborn's construction are sufficiently powerful to derive highly non-trivial consistency conditions amongst the various β-functions of a general four-dimensional QFT, and it is these constraints which form our primary interest. The constraints are derived by considering the effects of an infinitesimal Weyl transformation on the QFTCS, then using the commutativity of Weyl transformations to derive relations between RG quantities; consequently, these constraints are known as Weyl consistency conditions. Key to this method is the idea of extending couplings to spacetime-dependent functions $g^I \to g^I(x)$, leading to an extension of the usual RG formulation known as the Local Renormalization Group (LRG). This chapter, therefore, shall conclude with a summary on the extension of QFT to an arbitrary curved spacetime, followed by a re-derivation of the Weyl consistency conditions.

2.2.1 Extending to curved spacetime

It is relatively straightforward to define a classical field theory in a general curved spacetime: one need only identify those features in the action $S\left(\phi, \partial_\mu \phi\right)$ that correspond to a flat background and replace them with the associated covariant quantities, such that the resulting curved spacetime action $S\left(\phi, \nabla_\mu \phi, g_{\mu\nu}\right)$ is invariant under general coordinate transformations. Assuming that the usual variational principle applies to the new curved spacetime action, we then obtain generally covariant equations of motion from the variation of S with respect to the fields ϕ. A new feature is that the total variation δS also contains variations with respect to the metric,

$$\delta S = \int d^d x \left(\frac{\delta S}{\delta \phi_i} \delta \phi_i + \frac{\delta S}{\delta g_{\mu\nu}} \delta g_{\mu\nu} \right),$$

and so there should also be a corresponding equation of motion from varying the metric. To see what this represents for a general theory, consider the Einstein-Hilbert action with matter,

$$S \equiv S_{EH} + S_M = \int d^d x \sqrt{|g|} \left[\frac{1}{2\kappa} \left(2\Lambda - R\right) + \mathscr{L}_M \right]. \tag{2.67}$$

[24]See chapters 4 and 5 respectively.

Under variations of the metric, we have

$$\frac{\delta\sqrt{|g|}}{\delta g^{\mu\nu}} = -\frac{1}{2}\sqrt{|g|}g_{\mu\nu}, \qquad \frac{\delta R}{\delta g^{\mu\nu}} = R_{\mu\nu} + \nabla_\lambda A^\lambda_{\mu\nu},$$

hence (dropping the total derivative)

$$\delta S = \int d^d x\,\sqrt{|g|}\frac{1}{2\kappa}\left[R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} - \frac{2\kappa}{\sqrt{|g|}}\frac{\delta\left(\sqrt{|g|}\mathscr{L}_M\right)}{\delta g^{\mu\nu}}\right]\delta g^{\mu\nu}.$$

Varying the action with respect to the metric therefore yields the Einstein field equations,

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu},$$

where we define the energy-momentum tensor as

$$T_{\mu\nu} \equiv \frac{2}{\sqrt{|g|}}\frac{\delta S_M}{\delta g^{\mu\nu}}, \qquad T^{\mu\nu} \equiv -\frac{2}{\sqrt{|g|}}\frac{\delta S_M}{\delta g_{\mu\nu}}. \tag{2.68}$$

This definition of $T_{\mu\nu}$ is manifestly covariant and symmetric, hence we shall adopt it as our definition for any classical field theory in curved spacetime with action S_M. Variation of S_M with respect to the metric in a general curved spacetime will therefore not necessarily vanish, but requiring diffeomorphism invariance of the action ensures that the variation with respect to an infinitesimal coordinate transformation $\delta_0 g_{\mu\nu}$ *must* vanish. Given an infinitesimal coordinate transformation of the form

$$x'^\mu = x^\mu - \epsilon^\mu(x),$$

and expanding the tensor transformation law

$$g'_{\mu\nu}(x') = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}g_{\rho\sigma}(x),$$

we find that

$$\delta_0 g_{\mu\nu} = g'_{\mu\nu}(x) - g_{\mu\nu}(x) = \epsilon^\rho\partial_\rho g_{\mu\nu} + g_{\mu\rho}\partial_\nu\epsilon^\rho + g_{\nu\rho}\partial_\mu\epsilon^\rho.$$

After some algebraic manipulation, this is equivalent to

$$\delta_0 g_{\mu\nu} \equiv \mathcal{L}_\epsilon g_{\mu\nu} = \nabla_\mu\epsilon_\nu + \nabla_\nu\epsilon_\mu,$$

so the variation of S_M with respect to the metric, assuming ϵ^μ and $\nabla_\nu \epsilon^\mu$ vanish sufficiently rapidly, becomes

$$\begin{aligned} 0 = \delta S &= \int d^d x \, \frac{\delta S}{\delta g_{\mu\nu}} \delta_0 g_{\mu\nu} \\ &= -\int d^d x \sqrt{|g|} T^{\mu\nu} \nabla_\mu \epsilon_\nu \\ &= \int d^d x \sqrt{|g|} \left(\nabla_\mu T^{\mu\nu} \right) \epsilon_\nu , \end{aligned}$$

and so the equation of motion for variation with respect to the metric is simply the covariant analogue of local conservation of energy,

$$\nabla_\mu T^{\mu\nu} = 0. \tag{2.69}$$

Furthermore, a classical theory in curved spacetime is conformally invariant if the theory is invariant under local Weyl transformations[25], given by

$$g_{\mu\nu} \to \Omega^2(x) g_{\mu\nu} \equiv e^{2\sigma(x)} g_{\mu\nu}$$

with some associated transformation of the fields. An infinitesimal Weyl transformation corresponds to

$$\delta g_{\mu\nu} = 2\sigma(x) g_{\mu\nu},$$

hence

$$0 = \delta S = -\frac{1}{2} \int d^d x \sqrt{|g|} T^{\mu\nu} \delta g_{\mu\nu} = -\int d^d x \sqrt{|g|} \sigma(x) T^\mu{}_\mu .$$

A conformally invariant classical field theory in curved spacetime therefore has a traceless energy-momentum tensor,

$$T^\mu{}_\mu = 0. \tag{2.70}$$

When considering a quantum theory in curved spacetime, we follow the same philosophy as for a classical theory; that is, to take the Path integral formulation of a QFT, extend the classical action as above, define a suitable Path integral measure for curved spacetime, and then define operators and Green functions

[25]There is a slight subtlety in what is meant when a theory is said to be conformally invariant. A conformal transformation is a coordinate transformation that preserves angles, hence a theory invariant under general coordinate transformations (diffeomorphisms) is automatically invariant under conformal transformations. However, conformal transformations do not preserve distances, so the action of a theory will differ by a local scale transformation. Therefore, the action of the theory will take the same form if and only if the theory is *also* invariant under local Weyl transformations; such a theory is then called a conformal theory.

using covariant functional derivatives,

$$\frac{\delta}{\delta f(x)} \to \frac{1}{\sqrt{|g|}} \frac{\delta}{\delta f(x)}$$

This procedure is facilitated by using an equivalent formulation of the Path integral known as the Schwinger action principle, which is more easily adaptable to a general spacetime. Succinctly, for a theory in a region of curved spacetime Ω_{12} bounded by two constant-time hypersurfaces $\partial\Omega_1$, $\partial\Omega_2$, if the classical action

$$S_{12} = \int_{\Omega_{12}} d^d x \sqrt{|g|} \mathscr{L}$$

for some Hermitian Lagrangian density $\mathscr{L}$ has a variation given by

$$\delta S_{12} = \int_{\Omega_{12}} d^d x \sqrt{|g|} \left(\delta\mathscr{L} + \nabla_\mu A^\mu \right),$$

where

$$A = \int_{\partial\Omega} d\sigma_x \, n_\mu A^\mu$$

is the generator of unitary transformations on $\partial\Omega$, then the Schwinger action principle states that the variation of the transition amplitude between a state $|1\rangle$ on $\partial\Omega_1$ and a state $|2\rangle$ on $\partial\Omega_2$ is given by

$$\delta \langle 2|1\rangle = i \langle 2| \delta S_{12} |1\rangle . \tag{2.71}$$

By including source terms, one may recover the usual definitions of Green functions, classical fields, the effective action, *etc*, valid now for a general curved spacetime. Applying this to the generating functional of a QFT, where we take $|1\rangle = |in\rangle$ and $|2\rangle = |out\rangle$ to be the asymptotic vacuum states of some particular observer[26], we find that a variation in the vacuum-to-vacuum amplitude is given by

$$\delta W = \delta \langle out|in\rangle = i \langle out| \delta S |in\rangle ,$$

and hence the variation in the generator of connected Green functions, $X = -i \ln W$, is[27]

$$\delta X = -iW^{-1}\delta W = \frac{\langle out| \delta S |in\rangle}{W} \equiv \langle \delta S \rangle . \tag{2.72}$$

[26]Recall that in a general curved spacetime, there is no uniquely-defined, observer-independent state that one may identify as the vacuum: the particle number operator is not invariant under Bogolyubov transformations relating different observers.

[27]Note here that we follow the notation of [65], in which the curved spacetime vacuum expectation value is defined with an explicit factor W in the denominator.

By considering variations with respect to the metric, we find

$$\delta X = \int d^d x \, \frac{\delta X}{\delta g^{\mu\nu}} \delta g^{\mu\nu} = \frac{1}{2} \int d^d x \sqrt{|g|} \, \frac{2}{\sqrt{|g|}} \frac{\delta X}{\delta g^{\mu\nu}} \delta g^{\mu\nu},$$

and since

$$\langle \delta S \rangle = W^{-1} \int \mathcal{D}\phi \, \delta S e^{iS} = \frac{1}{2} \int d^d x \sqrt{|g|} \left(\frac{\int \mathcal{D}\phi \, T_{\mu\nu} e^{iS}}{W} \right) \delta g^{\mu\nu},$$

we may define the expectation value of the energy-momentum tensor for a QFTCS as

$$\langle T_{\mu\nu} \rangle = \frac{2}{\sqrt{|g|}} \frac{\delta X}{\delta g^{\mu\nu}}, \qquad \langle T^{\mu\nu} \rangle = -\frac{2}{\sqrt{|g|}} \frac{\delta X}{\delta g_{\mu\nu}} \tag{2.73}$$

which again by diffeomorphism invariance implies the equation of motion

$$\nabla_\mu \langle T^{\mu\nu} \rangle = 0.$$

This definition is again manifestly covariant and symmetric, and one may easily extend to an n-point correlator by repeated functional differentiation:

$$\langle T_{\mu\nu}(x_1) \cdots T_{\alpha\beta}(x_n) \rangle = \frac{2}{\sqrt{|g_{x_1}|}} \cdots \frac{2}{\sqrt{|g_{x_n}|}} \frac{\delta^n X}{\delta g^{\mu\nu}(x_1) \cdots g^{\alpha\beta}(x_n)}. \tag{2.74}$$

By analogy with the classical case, one might expect that $\langle T_{\mu\nu} \rangle$ is traceless for a conformally invariant theory, but this would only be true if the Path integral measure were conformally invariant. The effects of a Weyl transformation on the measure may be established by using the one-loop effective action of a QFTCS.

The one-loop effective action and the trace anomaly

As usual, we begin our discussion with the classical theory. The action of a free scalar field may be extended to curved spacetime by the minimal substitution procedure outlined above, giving

$$S = \int d^d x \sqrt{|g|} \left(\frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} m^2 \phi^2 \right). \tag{2.75}$$

However, if we were to apply a Weyl transformation of the form

$$g_{\mu\nu} \to g'_{\mu\nu} = \Omega^2(x) g_{\mu\nu}, \qquad \phi \to \phi' = \Omega^k(x) \phi, \tag{2.76}$$

we would find that

$$\begin{aligned}S' &= \int d^dx\sqrt{|g'|}\left(\frac{1}{2}g'^{\mu\nu}\nabla'_\mu\phi'\nabla'_\nu\phi' - \frac{1}{2}m^2\phi'^2\right)\\ &= \int d^dx\sqrt{|g|}\Omega^{d-2+2k}\,\frac{1}{2}g^{\mu\nu}\left(\partial_\mu\phi\partial_\nu\phi + k\partial_\mu\ln\Omega\,\partial_\nu(\phi^2) + k^2\phi^2\partial_\mu\ln\Omega\,\partial_\nu\ln\Omega\right)\\ &\quad + \int d^dx\sqrt{|g|}\Omega^{d+2k}\frac{1}{2}m^2\phi^2.\end{aligned}$$

The action cannot be invariant, since the kinetic and mass terms scale differently, and the additional terms in the second line cannot be expressed as a total derivative. This may be remedied by the introduction of a new term in the action, consistent with diffeomorphism invariance:

$$S = \int d^dx\sqrt{|g|}\left(\frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}m^2\phi^2 - \frac{1}{2}\,\xi R\phi^2\right). \tag{2.77}$$

The effect of a Weyl transformation on the Ricci scalar may be deduced from seeing how the Christoffel symbols transform, then constructing the Riemann tensor and contracting indices as usual. The final result is [65]

$$\begin{aligned}R' =& \Omega^{-2}\left(R + 2(d-1)g^{\mu\nu}\left[\partial_\mu\partial_\nu\ln\Omega - \Gamma^\lambda_{\ \mu\nu}\partial_\lambda\ln\Omega\right]\right.\\ &\left. + (d-1)(d-2)g^{\mu\nu}\partial_\mu\ln\Omega\partial_\nu\ln\Omega\right),\end{aligned}$$

which may then be substituted into the transformed action, giving

$$\begin{aligned}S' &= \int d^dx\sqrt{|g'|}\left(\frac{1}{2}g'^{\mu\nu}\nabla'_\mu\phi'\nabla'_\nu\phi' - \frac{1}{2}m^2\phi'^2 - \frac{1}{2}\,\xi R'\phi'^2\right)\\ &= \int d^dx\sqrt{|g|}\,\Omega^{d-2+2k}\left(\frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}\,\xi R\phi^2\right)\\ &\quad - \int d^dx\sqrt{|g|}\Omega^{d+2k}\frac{1}{2}m^2\phi^2\\ &\quad + \int d^dx\sqrt{|g|}\,\Omega^{d-2+2k}\left(\frac{1}{2}kg^{\mu\nu}\partial_\mu\ln\Omega\,\partial_\nu(\phi^2)\right.\\ &\qquad - \xi(d-1)\phi^2g^{\mu\nu}\left[\partial_\mu\partial_\nu\ln\Omega - \Gamma^\lambda_{\ \mu\nu}\partial_\lambda\ln\Omega\right]\\ &\qquad \left. + \frac{1}{2}\left[k^2 - \xi(d-1)(d-2)\right]\phi^2g^{\mu\nu}\partial_\mu\ln\Omega\,\partial_\nu\ln\Omega\right).\end{aligned}$$

We see immediately that we require $m = 0$ and $k = \frac{2-d}{2}$, if the theory is to be conformally invariant. Finally, if we choose $\xi = \frac{d-2}{4(d-1)}$, the $(\partial\ln\Omega)^2$ term drops out, leaving

$$S' = \int d^dx\sqrt{|g|}\left(\frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}\,\xi R\phi^2\right)$$

$$
\begin{aligned}
&+\int d^dx\sqrt{|g|}\left(\frac{2-d}{4}g^{\mu\nu}\partial_\mu\ln\Omega\,\partial_\nu(\phi^2)\right.\\
&\qquad\left.-\frac{d-2}{4}\phi^2 g^{\mu\nu}\left[\partial_\mu\partial_\nu\ln\Omega-\Gamma^\lambda_{\ \mu\nu}\partial_\lambda\ln\Omega\right]\right)\\
&=S-\frac{d-2}{4}\int d^dx\,\partial_\mu\left(\sqrt{|g|}g^{\mu\nu}\phi^2\partial_\nu\ln\Omega\right).
\end{aligned}
$$

Therefore, the action for a classical conformally-invariant scalar field in a general curved spacetime is

$$
S=\int d^dx\sqrt{|g|}\,\frac{1}{2}\left(g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi-\frac{d-2}{4(d-1)}R\phi^2\right), \tag{2.78}
$$

given a Weyl transformation of the form

$$
g_{\mu\nu}\to g'_{\mu\nu}=\Omega^2(x)g_{\mu\nu},\qquad \phi\to\phi'=\Omega^{\frac{2-d}{2}}(x)\phi. \tag{2.79}
$$

Consider now the action for a classical, conformally-invariant, multi-component scalar field $\phi^i(x)$ in curved spacetime with source $J_i(x)$,

$$
\begin{aligned}
S_J&=S+\int d^dx\sqrt{|g|}J_i(x)\phi^i(x)\\
&=\int d^dx\sqrt{|g|}\left(\frac{1}{2}\nabla_\mu\phi^i(x)\nabla^\mu\phi^j(x)\delta_{ij}-\frac{1}{2}\xi R\phi^i\phi^j\delta_{ij}+J_i(x)\phi^i(x)\right)\\
&=-\frac{1}{2}\int d^dxd^dx'\sqrt{|g|}\sqrt{|g'|}\phi^i(x)D_{ij}(x,x')\phi^j(x')+\int d^dx\sqrt{|g|}J_i(x)\phi^i(x),
\end{aligned} \tag{2.80}
$$

where

$$
D_{ij}(x,x')\equiv\delta(x,x')D_{ij}=\delta(x,x')\left(\Box_x+\xi R(x)\right)\delta_{ij},
$$

and

$$
\xi=\frac{(d-2)}{4(d-1)},\qquad \Box_x=\nabla_\mu(x)\nabla^\mu(x),\qquad \delta(x,x')=\frac{1}{\sqrt{|g|}}\delta(x-x').
$$

The action is quadratic in ϕ, hence the Path integral can be written in an exact form, as in the flat spacetime case. Introducing a length factor l so that l^2D_{ij} is dimensionless, we find

$$
W[J]=\int\mathcal{D}\phi\,e^{iS_J}=\left(\det l^2D_{ij}(x,x')\right)^{-\frac{1}{2}}e^{-\frac{i}{2}\int d^dxd^dx'\sqrt{|g|}\sqrt{|g'|}J_i(x)(D^{-1})^{ij}(x,x')J_j(x')},
$$

and so the connected generating functional is

$$X[J] = \frac{1}{2}\int d^dx d^dx' \sqrt{|g|}\sqrt{|g'|} J_i(x)(D^{-1})^{ij}(x,x')J_j(x') + \frac{i}{2}\ln\det\left(l^2 D_{ij}(x,x')\right).$$

Since quantities defined by covariant functional derivatives of the Path integral in curved spacetime take the same form as in flat spacetime, we may immediately write the effective action as

$$\Gamma[\phi_c] = X[J] - \int d^dx \sqrt{|g|}\, J_i(x)\phi_c^i(x).$$

The classical field $\phi_c^i(x)$ is

$$\phi_c^i(x) = \frac{1}{\sqrt{|g|}}\frac{\delta X}{\delta J_i(x)} = -\int d^dx' \sqrt{|g'|}\left(D^{-1}\right)^{ij}(x,x')J_j(x') = -\left(D^{-1}\right)^{ij} J_j(x),$$

so pre-multiplying by D_{ij} gives the source

$$J_i(x) = -D_{ij}\phi_c^j(x),$$

which may be substituted into the effective action to obtain

$$\Gamma[\phi_c^i(x)] = S + \frac{i}{2}\ln\det\left(l^2 D_{ij}(x,x')\right).$$

The first term is the usual classical action for the conformally-invariant scalar field in curved spacetime, whereas the second term is a quantum correction (as could be seen immediately by restoring factors of $\hbar$). The second term is known as the one-loop effective action,

$$\Gamma^{(1)} \equiv \frac{i}{2}\ln\det\left(l^2 D_{ij}(x,x')\right), \tag{2.81}$$

and should contain all leading-order quantum effects for the conformally-invariant scalar field in curved spacetime. In particular, we may isolate the contribution to $\langle T^\mu{}_\mu\rangle$ from the one-loop effective action simply by choosing $X = \Gamma^{(1)}$. Under an infinitesimal Weyl transformation, we therefore have

$$\delta\Gamma^{(1)} = \frac{i}{2}\delta\left[\ln\det\left(l^2 D_{ij}(x,x')\right)\right] = -\int d^dx\sqrt{|g|}\sigma(x)\left\langle T^\mu{}_\mu\right\rangle. \tag{2.82}$$

In solving the Path integral, we have implicitly assumed that the measure $\mathcal{D}\phi$ is such that, as in flat spacetime, the integral reduces to a product of Gaussian integrals when D_{ij} is a diagonalizable operator. This can be justified by considering a set of eigenfunctions $f_N(x)$ of the operator D, satisfying

$$Df_N(x) = (\Box_x + \xi R(x))\, f_N(x) = \lambda_N f_N(x).$$

If the eigenfunctions are orthonormal and complete, satisfying the relations

$$\int d^d x \sqrt{|g|} f_N^*(x) f_{N'}(x) = l^2 \delta_{NN'}, \quad \sum_N f_N^*(x) f_N(x') = l^2 \delta(x, x'), \tag{2.83}$$

then the set of eigenfunctions forms a basis, and we may express the scalar field as

$$\phi(x) = \sum_N \phi_N f_N(x)$$

for some dimensionless coefficients ϕ_N. The action of the classical theory then becomes

$$\begin{aligned}
S &= -\frac{1}{2} \int d^d x d^d x' \sqrt{|g|}\sqrt{|g'|} \phi(x) \delta(x, x') \left(\Box_x + \xi R(x)\right) \phi(x') \\
&= -\frac{1}{2} \int d^d x d^d x' \sqrt{|g|}\sqrt{|g'|} \left(\sum_N \phi_N^* f_N^*(x) \right) \left(\Box_x + \xi R(x)\right) \left(\sum_M \phi_M f_M(x) \right) \\
&= -\frac{1}{2} \sum_N \sum_M \lambda_M \left(\phi_N^* \phi_M\right) \int d^d x \sqrt{|g|}\, f_N^*(x) f_M(x) \\
&= -\frac{1}{2} \sum_N l^2 \lambda_N \phi_N^2.
\end{aligned}$$

If we define the measure as

$$\mathcal{D}\phi \equiv \prod_N \frac{d\phi_N}{\sqrt{-2\pi i}}, \tag{2.84}$$

then the Path integral becomes

$$\begin{aligned}
W &= \int \mathcal{D}\phi\, e^{iS} = \int \prod_N \frac{d\phi_N}{\sqrt{-2\pi i}}\, e^{-\frac{i}{2}\sum\limits_N l^2 \lambda_N \phi_N^2} \\
&= \prod_N \int \frac{d\phi_N}{\sqrt{-2\pi i}}\, e^{-\frac{i}{2} l^2 \lambda_N \phi_N^2} = \prod_N \left(l^2 \lambda_N\right)^{-\frac{1}{2}} \\
&= \left(\det l^2 D\right)^{-\frac{1}{2}}
\end{aligned}$$

as expected; the field theory is then recovered in the formal limit $N \to \infty$. Considering now the effects of a local Weyl transformation, the massless scalar action is invariant if the scalar field transforms as

$$\phi(x) \to e^{\frac{d-2}{2}\sigma(x)} \phi(x)$$

hence the conformally-transformed field may be expanded in the form

$$\tilde{\phi}(x) = \sum_N \phi_N \tilde{f}_N(x), \qquad \tilde{f}_N(x) = e^{\frac{d-2}{2}\sigma(x)} f_N(x). \tag{2.85}$$

The functions $\tilde{f}$ do not satisfy the orthonormality and completeness relations, and hence do not form a basis. However, since

$$\begin{aligned}\sum_N \tilde{f}_N^*(x)\tilde{f}_N(x') &= e^{\frac{d-2}{2}\sigma(x)} e^{\frac{d-2}{2}\sigma(x')} \sum_N f_N^*(x) f_N(x') \\ &= l^2 e^{\sigma(x)} e^{\sigma(x')} \tilde{\delta}(x,x')\end{aligned}$$

and

$$\begin{aligned}\int d^dx\sqrt{|\tilde{g}|}\,\tilde{f}_N^*(x)\tilde{f}_{N'}(x) &= \int d^dx\sqrt{|g|}\,e^{d\sigma(x)} e^{\frac{d-2}{2}\sigma(x)} f_N^*(x) e^{\frac{d-2}{2}\sigma(x)} f_{N'}(x) \\ &= \int d^dx\sqrt{|g|}\,e^{2\sigma(x)} f_N^*(x) f_{N'}(x),\end{aligned}$$

it is easy to see that we may instead define a new set of basis functions for the conformally-transformed spacetime, $g_N(x)$, such that

$$\tilde{f}_N(x) = e^{\sigma(x)} g_N(x). \tag{2.86}$$

Consequently, the conformally-transformed field may be expressed as

$$\tilde{\phi}(x) = \sum_N \phi_N \tilde{f}_N(x) = \sum_N \tilde{\phi}_N g_N(x),$$

and so the expansion coefficients ϕ, $\tilde{\phi}$ are related by

$$\sum_N \phi_N g_N(x) = \sum_N e^{-\sigma(x)} \tilde{\phi}_N g_N(x).$$

Using the orthonormality relation, we may apply the integral operator $\int d^dx\sqrt{|\tilde{g}|}\,g_N^*(x)$ to relate ϕ_N and $\tilde{\phi}_N$:

$$\begin{aligned}\int d^dx\sqrt{|\tilde{g}|}\,g_N^*(x)\left(\sum_{N'} \phi_{N'} g_{N'}(x)\right) &= \sum_{N'} \phi_{N'} \int d^dx\sqrt{|\tilde{g}|}\,g_N^*(x) g_{N'}(x) \\ &= l^2 \sum_{N'} \phi_{N'} \delta_{NN'} \\ &= l^2 \phi_N;\end{aligned}$$

$$\int d^dx\sqrt{|\tilde{g}|}\, g_N^*(x)\left(\sum_{N'} e^{-\sigma(x)}\tilde{\phi}_{N'}g_{N'}(x)\right) = \sum_{N'}\int d^dx\sqrt{|\tilde{g}|}\, e^{-\sigma(x)}g_N^*(x)g_{N'}(x);$$

therefore

$$\phi_N = \sum_{N'} C_{NN'}\tilde{\phi}_{N'}, \quad C_{NN'} \equiv l^{-2}\int d^dx\sqrt{|\tilde{g}|}\, e^{-\sigma(x)}g_N^*(x)g_{N'}(x). \tag{2.87}$$

Under a local Weyl transformation, the measure for the Path integral therefore transforms as

$$\prod_N d\phi_N = (\det C_{NN'})\prod_{N'} d\tilde{\phi}_{N'},$$

where conformal invariance requires $\det C_{NN'} = 1$. Converting this determinant to an exponential in the generating functional, we find

$$\tilde{W} = \int \mathcal{D}\tilde{\phi}\, e^{i(S - i\ln\det C_{NN'})},$$

and so the associated variation in the one loop effective action is

$$\delta\Gamma^{(1)} = -i\ln\det C_{NN'}.$$

Reverting back to the original basis f_N and expanding the infinitesimal Weyl variation to first order, we find

$$\begin{aligned} C_{NN'} &= l^{-2}\int d^dx\sqrt{|g|}\,\left(1-\sigma(x)\right) f_N^*(x)f_{N'}(x) \\ &= \delta_{NN'} - l^{-2}\int d^dx\sqrt{|g|}\,\sigma(x)f_N^*(x)f_{N'}(x) \\ \Longrightarrow\ \delta\Gamma^{(1)} &= -i\ln\det C_{NN'} = \int d^dx\sqrt{|g|}\sigma(x)\left(il^{-2}\sum_N f_N^*(x)f_N(x)\right), \end{aligned}$$

hence the trace of the energy momentum tensor for a conformally-invariant scalar field in curved spacetime is

$$\langle T^\mu{}_\mu\rangle = -il^{-2}\sum_N f_N^*(x)f_N(x). \tag{2.88}$$

The sum on the right-hand-side is a divergent quantity, and therefore requires regularization. The most convenient method for our purposes is Heat Kernel regularization, of which a comprehensive treatment is given by DeWitt in [66]. If we define the Heat kernel $K(\tau; x, x')$ as

$$K(\tau; x, x') \equiv l^{-2} \sum_N e^{-i\tau\lambda_N} f_N^*(x) f_N(x'),$$

then the sum corresponds to $K(0; x, x)$. The Heat kernel satisfies the partial differential equation

$$\begin{aligned} i\frac{\partial}{\partial\tau} K^i_{\ j}(\tau; x, x') &= D^i_{\ k} K^k_{\ j}(\tau; x, x') \\ \lim_{\tau\to 0} K^i_{\ j}(\tau; x, x') &= \delta^i_{\ j}\delta(x, x'), \end{aligned}$$

and in a general curved spacetime has the solution

$$K(\tau; x, x') = \frac{i}{(4\pi i\tau)^{\frac{d}{2}}} e^{\frac{s(x,x')}{2i\tau}} \Delta^{\frac{1}{2}}(x, x') \sum_{k=0}^{\infty} (i\tau)^k E_k(x, x'),$$

where

$$s(x, x') = \frac{1}{2} g_{\mu\nu}(x - x')^\mu (x - x')^\nu$$

is the geodetic interval, and $\Delta^{\frac{1}{2}}$ is the operator square-root of the Van Vleck-Morette determinant,

$$\Delta(x, x') = (-1)^d \frac{\det\left(\nabla_\mu \nabla_{\nu'} s(x, x')\right)}{\sqrt{|g_x|}\sqrt{|g_{x'}|}}.$$

For small τ, and in the coincidence limit $x' \to x$, the Heat kernel has the asymptotic expansion

$$K(\tau; x, x) \sim \frac{i}{(4\pi i\tau)^{\frac{d}{2}}} \sum_{k=0}^{\infty} (i\tau)^k E_k(x), \qquad E_k(x) \equiv \lim_{x'\to x} E_k(x, x'),$$

hence the regularized trace of the energy-momentum tensor is

$$\langle T^\mu_{\ \mu}\rangle = \lim_{\tau\to 0} -i\, \mathrm{tr} K(\tau; x, x) = \mathcal{O}\left(\tau^{-\frac{d}{2}}\right) + \frac{1}{(4\pi)^{\frac{d}{2}}} \mathrm{tr}\, E_{\frac{d}{2}}(x), \tag{2.89}$$

where $\mathcal{O}\left(\tau^{-\frac{d}{2}}\right)$ are singular contributions that should be removed after renormalization. The functions $E_k(x)$ are generated as coincidence limits of a recurrence relation for $E_k(x, x')$, derived by substituting the Heat kernel into the heat equation. Given a differential operator of the form

$$D_{ij} = \delta_{ij}\Box_x + Q_{ij}(x), \qquad i, j = 1, \ldots, k$$

where $Q(x)$ contains no spacetime derivatives, the recurrence relation is

$$-\Delta^{\frac{1}{2}}\left[(k+1)E_{k+1}+\nabla^{\mu}s\right]=\Box\left(\Delta^{\frac{1}{2}}E_k\right)+\Delta^{\frac{1}{2}}QE_k,$$

hence the first three functions $E_0(x)$, $E_1(x)$ and $E_2(x)$ are given by

$$\begin{aligned}
E_0(x) &= \lim_{x'\to x} E_0(x,x') = 1,\\
E_1(x) &= \lim_{x'\to x} -\left(\Box\Delta^{\frac{1}{2}}+Q\right),\\
E_2(x) &= \lim_{x'\to x} \frac{1}{6}\left(\Box^2\Delta^{\frac{1}{2}}+2(\Box\Delta^{\frac{1}{2}})^2+6(\Box\Delta^{\frac{1}{2}})Q+\Box Q+3Q^2\right).
\end{aligned}$$

Utilising relations for covariant derivatives of s and $\Delta^{\frac{1}{2}}$,

$$s=\frac{1}{2}\nabla_{\mu}s\nabla^{\mu}s, \qquad d\Delta^{\frac{1}{2}}=2\nabla_{\mu}\Delta^{\frac{1}{2}}\nabla^{\mu}s+\Delta^{\frac{1}{2}}\Box s,$$

and after extensive use of the Riemann tensor defined as a commutation relations for covariant derivatives,

$$\left[\nabla_{\nu},\nabla_{\rho}\right]A_{\mu}=R^{\lambda}{}_{\mu\nu\rho}A_{\lambda},$$

we find

$$\begin{aligned}
\lim_{x'\to x}\Box\,\Delta^{\frac{1}{2}} &= -\frac{1}{6}R,\\
\lim_{x'\to x}\Box^2\Delta^{\frac{1}{2}} &= \frac{1}{30}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}-\frac{1}{30}R_{\mu\nu}R^{\mu\nu}+\frac{1}{36}R^2-\frac{1}{5}\Box R,
\end{aligned}$$

hence

$$\begin{aligned}
E_0(x) &= 1,\\
E_1(x) &= \frac{1}{6}R\,\mathbb{1}_k-Q,\\
E_2(x) &= \left(\frac{1}{180}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}-\frac{1}{180}R_{\mu\nu}R^{\mu\nu}+\frac{1}{72}R^2-\frac{1}{30}\Box R\right)\mathbb{1}_k\\
&\quad+\frac{1}{6}\Box Q-\frac{1}{6}RQ+\frac{1}{2}Q^2.
\end{aligned}$$

We can therefore read off the trace anomaly for a two-dimensional scalar CFT, with $Q=\xi R\mathbb{1}_k|_{d=2}=0$, as

$$\langle T^{\mu}{}_{\mu}\rangle\Big|_{d=2}=\frac{1}{4\pi}\operatorname{tr}E_1(x)=\frac{k}{24\pi}R, \tag{2.90}$$

and for a four-dimensional scalar CFT, with $Q = \xi R \mathbb{1}_k|_{d=4} = \frac{1}{6} R \mathbb{1}_k$, as

$$\begin{aligned} \langle T^\mu{}_\mu \rangle \big|_{d=4} &= \frac{1}{16\pi^2} \operatorname{tr} E_2(x) \\ &= \frac{k}{16\pi^2} \left(\frac{1}{180} R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - \frac{1}{180} R_{\mu\nu} R^{\mu\nu} - \frac{1}{180} \Box R \right) \end{aligned} \tag{2.91}$$

Since $\langle T^\mu{}_\mu \rangle$ for a classically conformally invariant theory is a function of curvature scalars that are non-zero in a general curved spacetime, we see that the associated QFTCS cannot be conformally invariant: this is the trace anomaly, and is in fact only non-zero in even-dimensional spacetime. Fermion contributions to the trace anomaly may be calculated in a similar manner, by extending spinors and γ-matrices to curved spacetime and constructing the effective action for a free Dirac spinor; gauge contributions may be calculated by extending a Yang-Mills theory to curved spacetime, then constructing an effective action that is invariant under gauge transformations. In both cases, the intent is to derive the analogous operator D_{ij} that appears in (2.81), then calculate the associated Heat kernel coefficient $E_2(x)$; full details are given in [65].

2.2.2 Weyl consistency conditions

We have seen that extending a CFT to curved spacetime introduces a gravitational anomaly in the trace of the energy-momentum tensor, as a consequence of the measure transforming non-trivially under a local Weyl transformation. Generally, the gravitational part of the trace anomaly of a QFT in d dimensions is a linear combination of curvature invariants with dimension d, for example in two dimensions the anomaly is simply proportional to R, while in four dimensions the anomaly may contain terms proportional to R^2, $R_{\mu\nu}R^{\mu\nu}$, $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ and $\Box R$. Such terms can be rewritten using a basis of curvature scalars that includes the Euler density and powers of the Weyl tensor; we shall see that the coefficients of terms in such a basis have more convenient behaviour under RG flow.

Recall that the gravitational contribution to the trace anomaly is derived by considering the effects of a Weyl rescaling on the vacuum generating functional of a classically conformal theory, and consequently takes the form of a functional derivative. It is in fact possible to express the operational contribution $\beta^I \langle \mathcal{O}_I \rangle$ in the same way, by allowing the couplings to be spacetime-dependent functions, $g^I \equiv g^I(x)$: the couplings act as source terms $\int d^dx \sqrt{|g|}\, g^I(x) \mathcal{O}_I$ for the composite operators $\mathcal{O}_I$, and so the composite operator is given by the functional derivative

$$\mathcal{O}_I \equiv \frac{1}{\sqrt{|g|}} \frac{\delta S}{\delta g^I}. \tag{2.92}$$

Considering now a variation of the vacuum generating functional X with respect to the "local" couplings, we find

$$\delta X = \int d^d x \, \frac{\delta X}{\delta g^I} \, \delta g^I = \int d^d x \sqrt{|g|} \left(\frac{1}{\sqrt{|g|}} \frac{\delta X}{\delta g^I} \right) \delta g^I,$$

so by making use of (2.72), the expectation value of the composite operator may be defined as

$$\langle \mathcal{O}_I \rangle = \frac{1}{\sqrt{|g|}} \frac{\delta X}{\delta g^I}, \tag{2.93}$$

and pre-multiplying the functional derivative by β^I yields the operational contribution. We would (naïvely) expect that acting on the vacuum generating functional with these functional derivatives would give precisely the trace anomaly in the form

$$\langle T^{\mu}{}_{\mu} \rangle = \beta^I \langle \mathcal{O}_I \rangle + \mathcal{A}, \tag{2.94}$$

but this neglects the effect local couplings may have on the finiteness of n-point correlation functions, and hence the renormalization of the theory. In order to maintain renormalizability, the action of the original theory must be augmented with additional terms corresponding to derivatives of the couplings; this is in keeping with the general philosophy of renormalization, where one includes *all* operators of the correct dimension.

There is a deep relation between a consistent theory in curved spacetime with local couplings (from which one can extract the trace anomaly) and the Renormalization Group. The classical action of such a theory may be parametrised as

$$S = S_{con} + \int d^d x \sqrt{|g|} \left(g^I \mathcal{O}_I + \mathscr{B}^{\alpha} \mathscr{R}_{\alpha} + \partial_\mu \sigma \mathscr{Z}^{\mu} \right), \tag{2.95}$$

where S_{con} is classically conformally-invariant, and $\mathscr{B}^{\alpha}\mathscr{R}_{\alpha}$, $\partial_\mu \sigma \mathscr{Z}^{\mu}$ are understood as containing all curvature scalars and coupling-derivatives of the correct dimension, multiplied by some appropriate tensor structure. Defining new functional derivative operators

$$\begin{aligned} \Delta^g_\sigma &\equiv \int d^d x \sqrt{|g|} \left(\sigma(x) g^{\mu\nu} \right) \frac{2}{\sqrt{|g|}} \frac{\delta}{\delta g^{\mu\nu}}, \\ \Delta^\beta_\sigma &\equiv \int d^d x \sqrt{|g|} \left(\sigma(x) \beta^I \right) \frac{1}{\sqrt{|g|}} \frac{\delta}{\delta g^I}, \end{aligned} \tag{2.96}$$

there should exist a quantity $\mathcal{A}_\partial$ satisfying

$$\left[\Delta^g_\sigma - \Delta^\beta_\sigma\right] X = \int d^dx \sqrt{|g|}\, \mathcal{A}_\partial, \tag{2.97}$$

where X is understood to be derived from a suitably renormalized (2.95). $\mathcal{A}_\partial$ contains the gravitational contribution $\mathcal{A}$, as well as extra contributions proportional to derivatives of the local couplings $g^I(x)$ and Weyl rescaling function $\sigma(x)$; in the limit where the couplings are constant, (2.97) should reduce to an integral version of (2.94). If, however, one instead considers a global Weyl rescaling $\sigma(x) \equiv \sigma$, the $\partial_\mu \sigma$ term vanishes and we may rewrite (2.97) as

$$\int d^dx \sqrt{|g|} \left(g^{\mu\nu} \frac{2}{\sqrt{|g|}} \frac{\delta}{\delta g^{\mu\nu}} - \beta^I \frac{1}{\sqrt{|g|}} \frac{\delta}{\delta g^I} - \mathscr{B}^\alpha \frac{1}{\sqrt{|g|}} \frac{\delta}{\delta \mathscr{B}^\alpha} \right) X = 0,$$

Assuming all operators in (2.95) are marginal, so that the bare couplings may be expressed in the form

$$g^I_B = \mu^{-\epsilon} Z^I{}_J g^J, \quad \mathscr{B}^\alpha_B = \mu^{-\epsilon} Z^\alpha{}_\beta \mathscr{B}^\beta, \tag{2.98}$$

we find that the associated vacuum generating functional obeys

$$\left[\mu \frac{\partial}{\partial \mu} + \int d^dx \sqrt{|g|}\, g^{\mu\nu} \frac{2}{\sqrt{|g|}} \frac{\delta}{\delta g^{\mu\nu}} \right] X = 0; \tag{2.99}$$

that is, a global Weyl rescaling is effectively an inverse RG scaling[28]. We therefore find that, for a global Weyl rescaling, (2.97) is in fact a generalization of the standard RG equation for the vacuum generating functional, extended to local couplings:

$$\mu \frac{d}{d\mu} X = \left[\mu \frac{\partial}{\partial \mu} + \int d^dx \left(\beta^I \frac{\delta}{\delta g^I} + \mathscr{B}^\alpha \frac{\delta}{\delta \mathscr{B}^\alpha} \right) \right] X = 0. \tag{2.100}$$

Consequently, (2.97) is referred to as the Local Renormalization Group (LRG) equation: it contains all information on the RG flow of the theory in a curved spacetime with local couplings, and this information may be obtained by appropriate functional differentiation.

The LRG, and its definition in terms of Weyl transformations, underpins our attempts to impose constraints on RG flow. A crucial property of Weyl

[28]Heuristically, this makes sense. Recall that RG flow is usually depicted as a "zooming out" process, in which one describes the higher-energy theory in terms of lower-energy degrees of freedom. Conversely, a Weyl rescaling is a "zooming in" process, in which one describes the larger-length theory in terms of shorter lengths.

transformations is that they form an Abelian group, and so the commutator of two Weyl rescaling operations must vanish. Since the LRG describes the behaviour of a theory under a Weyl transformation, imposing that the commutator of two such transformations must be zero leads to highly non-trivial relations between the various terms present in $\mathcal{A}_\partial$. These relations are known as Weyl consistency conditions, and the key equation (2.1) stated at the very beginning of this chapter is precisely one such condition. In the case of two spacetime dimensions, the Weyl consistency conditions may be combined with relations between the tensor structures in $\mathcal{A}_\partial$ and the two-point functions; the end result is a re-derivation of Zamolodchikov's c-theorem, outlined at the beginning of this section.

To see how this works, consider the action (2.95) with $d = 2$. We must include all curvature scalars and coupling-derivatives of the correct dimension, hence in this case we have (up to total derivatives)

$$\mathscr{B}^\alpha \mathscr{R}_\alpha = cR + \tfrac{1}{2}\chi_{IJ}\partial_\mu g^I \partial^\mu g^J, \tag{2.101}$$

where $c \equiv c(g)$ may be a function of the couplings. (2.97) then takes the form

$$\left[\Delta^g_\sigma - \Delta^\beta_\sigma\right] X = \int d^dx\sqrt{|g|}\left[\sigma(x)\left(cR + \tfrac{1}{2}\chi_{IJ}\partial_\mu g^I\partial^\mu g^J\right) + \partial_\mu\sigma\left(\omega_I\partial^\mu g^I\right)\right], \tag{2.102}$$

having defined a new tensor structure ω_I according to

$$\mathscr{Z}^\mu \equiv \omega_I \partial^\mu g^I.$$

If we apply another Weyl rescaling, using the functional derivatives (2.96) with a new infinitesimal transformation $\sigma'(x)$, we find

$$\begin{aligned}\Delta^g_{\sigma'}\left[\Delta^g_\sigma - \Delta^\beta_\sigma\right] X = \int d^dx\sqrt{|g|}\Big\{&\sigma'\sigma(2-d)\left[cR + \tfrac{1}{2}\chi_{IJ}\partial_\mu g^I\partial^\mu g^J\right] \\ &+ \sigma'\nabla_\mu\sigma\left[(2-d)\omega_I\partial^\mu g^I - 2(d-1)\partial^\mu c\right] \\ &- \nabla_\mu\sigma'\nabla^\mu\sigma\left[2(d-1)c\right]\Big\};\end{aligned}$$

$$\begin{aligned}\Delta^\beta_{\sigma'}\left[\Delta^g_\sigma - \Delta^\beta_\sigma\right] X = \int d^dx\sqrt{|g|}\Big\{&\sigma'\sigma\left[\beta^I\partial_K\chi_{IJ}\partial_\mu g^J\partial^\mu g^K + \beta^I\chi_{IJ}\nabla^2 g^J\right] \\ &+ \sigma'\nabla_\mu\sigma\left[\beta^I\left(\partial_I\omega_J - \chi_{IJ} - \partial_J\omega_I\right)\partial^\mu g^J + \nabla^\mu(\beta^I\omega_I)\right] \\ &- \nabla_\mu\sigma'\nabla^\mu\sigma\left[\beta^I\omega_I\right]\Big\};\end{aligned}$$

Setting $d = 2$, and using that the variation of the Ricci scalar under an infinitesimal Weyl transformation is $\delta R = -2\sigma R - 2\nabla^2\sigma$, the commutator of two Weyl rescaling operations with parameters $\sigma(x)$, $\sigma'(x)$ is given by

$$\left[\Delta^g_\sigma - \Delta^\beta_\sigma, \Delta^g_{\sigma'} - \Delta^\beta_{\sigma'}\right] X = \int d^d x \sqrt{|g|} \left(\sigma' \nabla_\mu \sigma - \sigma \nabla_\mu \sigma'\right) \mathscr{X}^\mu, \tag{2.103}$$

where

$$\mathscr{X}_\mu \equiv \partial_\mu c - \chi_{IJ} \beta^J \partial_\mu g^I + \beta^J \partial_J \omega_I \partial^\mu g^I - \beta^J \partial_I \omega_J \partial^\mu g^I + \nabla^\mu (\beta^I \omega_I). \tag{2.104}$$

Imposing the vanishing of (2.103) in a curved spacetime for arbitrary Weyl rescaling σ then imposes that $\mathscr{X}_\mu = 0$. Factoring out the $\partial_\mu g^I$ term, $\mathscr{X}_\mu$ will vanish for arbitrary spacetime-dependent coupling $g^I(x)$ if

$$\partial_I c = \chi_{IJ} \beta^J - \left(\beta^J \partial_J \omega_I + \partial_I \beta^J \omega_J\right). \tag{2.105}$$

Finally, defining a new quantity $\tilde{c} \equiv c + \omega_I \beta^I$, this condition may be rewritten as

$$\partial_I \tilde{c} = \chi_{IJ} \beta^J + \left(\partial_I \omega_J - \partial_J \omega_I\right) \beta^J, \tag{2.106}$$

which then satisfies

$$\beta^I \partial_I \tilde{c} = \beta^I \chi_{IJ} \beta^J. \tag{2.107}$$

We see immediately that (2.106) is in the form (2.1), with $A = \tilde{c}$, $T_{IJ} = \chi_{IJ} + 2\partial_{[I}\omega_{J]}$, and that $\tilde{c}$ is stationary when $\beta^I = 0$; if χ_{IJ} is positive-definite, $\tilde{c}$ satisfies the required properties of the c-function for two-dimensional theories. This may indeed be shown by calculating anomalous Ward identity for the two-point function of the composite operator $\mathcal{O}_I$, making use of the commutation relations

$$\left[\frac{\delta}{\delta g^{\mu\nu}}, \Delta^g_\sigma - \Delta^\beta_\sigma\right] = 0, \qquad \left[\frac{\delta}{\delta g^I}, \Delta^g_\sigma - \Delta^\beta_\sigma\right] = \partial_I \beta^J \frac{\delta}{\delta g^J}. \tag{2.108}$$

Using the LRG (2.97) and the commutators (2.108), and returning to flat spacetime with non-position-dependant couplings, we find that [7]

$$\begin{aligned} &\left[\Delta^g_\sigma - \Delta^\beta_\sigma\right] \langle \mathcal{O}_I(x) \mathcal{O}_J(y) \rangle + \partial_I \beta^K \langle \mathcal{O}_K(x) \mathcal{O}_J(y) \rangle \\ &+ \partial_J \beta^K \langle \mathcal{O}_I(x) \mathcal{O}_K(y) \rangle + \partial_I \partial_J \beta^K \langle \mathcal{O}_K(x) \rangle \delta(x,y) = \chi_{IJ} \nabla^2 \delta(x,y) \end{aligned} \tag{2.109}$$

and so χ_{IJ} is proportional to the manifestly-positive two-point function $\langle \mathcal{O}_I(x) \mathcal{O}_J(y) \rangle$; $\tilde{c}$ is therefore a suitable function of the couplings in the theory, completing the derivation of the c-theorem. There is in fact an arbitrariness present in this derivation, in that if one adds a local functional to the action,

$$\delta S = \int d^d x \sqrt{|g|} \left(\tfrac{1}{2} b R - \tfrac{1}{2} a_{IJ} \partial_\mu g^I \partial^\mu g^J \right), \tag{2.110}$$

then the derivation goes through as before, with the corresponding shifts

$$\tilde{c} \to \tilde{c} + \beta^I a_{IJ} \beta^J, \qquad \omega_I \to \omega_I - \partial_I b + a_{IJ} \beta^J, \tag{2.111}$$

under which (2.107) is invariant. The effects of these shifts vanish when $\beta^I = 0$, and hence serve to parametrize the scheme-dependence of $\tilde{c}$ between RG fixed points.

The importance of the approach using Weyl consistency conditions is that the method may be applied to higher-dimensional theories, and hence facilitates the search for a proof of the a-theorem. When $d = 4$, (2.95) contains more curvature terms, coupling-derivatives, and mixed terms such as $R X_{IJ} \partial_\mu g^I \partial^\mu g^J$, but the derivation of the Weyl consistency conditions is otherwise the same as in $d = 2$. In (2.97), $\mathcal{A}_\partial$ is now understood as containing derivatives up to $\nabla^2 \sigma$, and so the right-hand-side of (2.103) has several contributions proportional to $\sigma' \partial_\mu \sigma - \sigma \partial_\mu \sigma'$, $\partial_\mu \sigma' \partial_\nu \sigma - \partial_\nu \sigma' \partial_\mu \sigma$, and $\partial_\mu \sigma' \nabla^2 \sigma - \nabla^2 \sigma' \partial_\mu \sigma$; requiring each of these terms to vanish separately imposes a large number of relations, from which one may derive a four-dimensional analogue of (2.106).

Again, we begin with the action (2.95) with $d = 4$. A suitable basis of dimension-four operators is

$$\begin{aligned} \mathscr{B}^\alpha \mathscr{R}_\alpha = {} & c\, C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} - a E_4 + b R^2 + \tfrac{1}{2} A_{IJ} \nabla^2 g^I \nabla^2 g^J \\ & + \tfrac{1}{2} B_{IJK} \partial_\mu g^I \partial_\mu g^J \nabla^2 g^K + \tfrac{1}{2} C_{IJKL} \partial_\mu g^I \partial^\mu g^J \partial_\nu g^K \partial^\nu g^L \\ & + \tfrac{1}{3} E_I \partial_\mu R \partial^\mu g^I + \tfrac{1}{6} F_{IJ} R \partial_\mu g^I \partial^\mu g^J + \tfrac{1}{2} G_{IJ} G^{\mu\nu} \partial_\mu g^I \partial_\nu g^J, \end{aligned} \tag{2.112}$$

and so (2.97) takes the form

$$\begin{aligned} \left[\Delta^g_\sigma - \Delta^\beta_\sigma \right] X = \int d^d x \sqrt{|g|} \Big\{ & \sigma(x) \mathscr{B}^\alpha \mathscr{R}_\alpha \\ & + \partial_\mu \sigma \left(G^{\mu\nu} W_I \partial_\nu g^I + R H_I \partial^\mu g^I + \mathscr{Z}^\mu \right) \\ & + \nabla^2 \sigma \left(R D + \mathscr{Y} \right) \Big\}, \end{aligned} \tag{2.113}$$

where we define

$$\begin{aligned} \mathscr{Z}^\mu &= S_{IJ} \partial_\mu g^I \nabla^2 g^J + T_{IJK} \partial^\mu g^I \partial_\nu g^J \partial^\nu g^K, \\ \mathscr{Y} &= U_I \nabla^2 g^I + V_{IJ} \partial_\mu g^I \partial^\mu g^J. \end{aligned} \tag{2.114}$$

Making use of the variations under Weyl rescaling

$$\begin{aligned}\delta C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} &= -4\sigma C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}, \qquad \delta G^{\mu\nu} = -4\sigma G^{\mu\nu} - 2\left(\nabla^\mu\nabla^\nu - g^{\mu\nu}\nabla^2\right)\sigma,\\ \delta\nabla^2 &= -2\sigma\nabla^2 + 2\partial_\mu\sigma\nabla^\mu, \qquad \delta E_4 = -4\sigma E_4 + 8G^{\mu\nu}\nabla_\mu\nabla_\nu\sigma,\\ \delta R &= -2\sigma R - 6\nabla^2\sigma,\end{aligned}$$

the derivation of Weyl consistency conditions then proceeds as before; full details of the calculation can be found in [7–9]. Most importantly, we find the following two conditions:

$$\begin{aligned}8\partial_I a &= G_{IJ}\beta^J - \left(\beta^J\partial_J\omega_I + \partial_I\beta^J\omega_J\right),\\ G_{IJ} + 2A_{IJ} + 2\partial_I\beta^K A_{KJ} + \beta^K B_{IJK} &= \partial_I\beta^K S_{KJ} + \partial_J\beta^K S_{IK} + \beta^K\partial_K S_{IJ}.\end{aligned} \tag{2.115}$$

Hence, we see that the quantity $\tilde{a} = 8a + \omega_I\beta^I$ satisfies four-dimensional versions of (2.106),

$$\partial_I\tilde{a} = G_{IJ}\beta^J + \left(\partial_I\omega_J - \partial_J\omega_I\right)\beta^J, \tag{2.116}$$

and (2.107),

$$\beta^I\partial_I\tilde{a} = \beta^I G_{IJ}\beta^J, \tag{2.117}$$

which are again in the desired form (2.1). Unfortunately, we cannot immediately prove the a-theorem in the same manner as the c-theorem: the second consistency condition relates G_{IJ} to B_{IJK}, and since the latter is related to the three-point function $\langle\mathcal{O}_I\mathcal{O}_J\mathcal{O}_K\rangle$, G_{IJ} cannot be manifestly positive-definite.

Before we begin our investigations into the constraints placed on RG flow by (2.1), there is one final aspect of the LRG that must be taken into account. For a general theory with field multiplets ϕ^i, $i = 1, \ldots, n$, there is a global $\mathcal{O}(n)$ symmetry[29] corresponding to the permutation of these fields. In order for the operator term $g^I\mathcal{O}_I$ to be invariant under such permutations, the coupling g^I must acquire some compensating transformation; that is, for some field variation $\delta\phi = -\epsilon\phi$, there is an associated coupling variation $\delta g^I = -(\epsilon g)^I$, where ϵ is an element of the Lie algebra $\mathfrak{o}(n)$. There is an anomalous current $\langle J_\mu\rangle$ induced by such a symmetry, which may be derived from the classical action by promoting the symmetry to a local symmetry $\epsilon \to \epsilon(x)$, and introducing a local auxiliary term $A^\mu(x)$ that acts as a source. A^μ may then be treated as another local

[29] While this argument holds when the symmetry group is a general Lie group, the only case we consider in this thesis is that of permuting field multiplets in six-dimensional ϕ^3 theory: see chapter 4.

coupling, so that the current is defined via the usual variation

$$\langle J_\mu(x)\rangle = -\frac{1}{\sqrt{|g|}}\frac{\delta X}{\delta A^\mu(x)}.$$

There is then an associated contribution to (2.97), generated by a new operator

$$\Delta_\sigma^A = \int d^dx\sqrt{|g|}\left(\sigma(x)\beta_\mu^A\frac{\delta}{\delta A_\mu} - \partial_\mu\sigma\mathcal{S}\frac{\delta}{\delta A_\mu}\right), \tag{2.118}$$

where $\beta_\mu^A \equiv \rho_I D_\mu g^I$, and $D_\mu g^I \equiv \partial_\mu g^I + A_\mu g^I$ is the gauge-covariant derivative. The trace anomaly then acquires new terms of the form

$$\langle T^\mu{}_\mu\rangle \to \langle T^\mu{}_\mu\rangle + \tfrac{1}{4}F^{\mu\nu}\cdot\kappa\cdot F_{\mu\nu} + \tfrac{1}{2}F^{\mu\nu}\cdot\zeta_{IJ}\partial_\mu g^I\partial_\nu g^J + \nabla_\mu\left(F^{\mu\nu}\cdot\eta_I\partial_\nu g^I\right), \tag{2.119}$$

where $F_{\mu\nu}$ is the field-strength tensor associated with $A_\mu(x)$. The presence of this extra operation modifies the Weyl consistency conditions, which should now take the form

$$\left[\Delta_\sigma^g - \Delta_\sigma^\beta - \Delta_\sigma^A, \Delta_{\sigma'}^g - \Delta_{\sigma'}^\beta - \Delta_{\sigma'}^A\right]X = 0. \tag{2.120}$$

By defining shifted functions

$$B^I = \beta^I - (\mathcal{S}g)^I, \quad P^I = \rho^I + \partial_I\mathcal{S}, \quad B_\mu^A = \beta_\mu^A + D_\mu\mathcal{S}, \tag{2.121}$$

the $\mathcal{S}$ contributions may in fact be absorbed into other terms, so that the LRG takes its usual form, with the β-functions β^I replaced by their gauge-covariant analogues B^I. The Weyl consistency conditions are then modified, such that

$$8\partial_I a = G_{IJ}B^J - \left(\beta^J\partial_J\omega_I + \partial_I\beta^J\omega_J\right) - (P_I g)^J\omega_J, \qquad B^I P_I = 0, \tag{2.122}$$

so by again introducing $\tilde{a} = 8a + B^I\omega_I$, we have an equation in the form (2.1),

$$B^I\partial_I\tilde{a} = B^I G_{IJ}B^J. \tag{2.123}$$

The presence of global symmetries is intimately connected to the existence of limit cycles in RG flows, and the question of whether scale-invariance implies conformal-invariance. Since the β-functions in the LRG are replaced by corresponding B-functions, the trace anomaly becomes

$$\langle T^\mu{}_\mu\rangle = B^I\langle\mathcal{O}_I\rangle + B_\mu^A\langle J^\mu\rangle + (\text{curvature}) + (\partial g)\,, \tag{2.124}$$

and so in the limit of flat spacetime with non-position-dependant couplings, the vanishing of $\langle T^\mu{}_\mu\rangle$ is governed by the B-functions, rather than the β-functions as expected. In [15], it was shown that for a QFT with a limit cycle (so that $\beta^I \neq 0$), the theory necessarily has $B^I = 0$ along the limit cycle, and satisfies $B^I = \beta^I$ at RG fixed points; combined with (2.123), this demonstrates that the question of scale- implying conformal-invariance can be generalised to include theories with limit cycles, by identifying couplings related by a global symmetry transformation, such that the RG flows are instead generated by B^I.

Chapter 3

Four Dimensions

The four-dimensional a-theorem, as originally conjectured by Cardy [1], is the most phenomenologically relevant case that we consider in this thesis, since any derived constraints on RG flow can be applied directly to the Standard Model of particle physics. The Standard Model is a four-dimensional, perturbatively renormalizable quantum field theory, based on the spontaneously broken symmetry group $SU(3)_c \times SU(2)_L \times U(1)_Y$, and contains a scalar, fermions, and gauge bosons; the Standard Model Lagrangian therefore contains each possible marginal coupling in four dimensions, associated with gauge interactions, Yukawa interactions and scalar self-interactions. Recent work on the a-theorem by Komargodski and Schwimmer has led to a proof [16] of the weak formulation, subject to certain assumptions on the four-point function $\langle T^\mu{}_\mu T^\nu{}_\nu T^\rho{}_\rho T^\sigma{}_\sigma \rangle$ highlighted in [67]. In [17], it was shown that the weak formulation is in fact sufficient to rule out the existence of theories that are scale-invariant but not conformally-invariant, extending the result of [4] to four dimensions.

We are interested in the strong a-theorem, hence the starting point of our investigations is the existence [8] of a function $A(g)$ of the couplings, which at RG fixed points is proportional to the Euler density coefficient a in the trace anomaly, and which obeys the gradient-flow equation (2.1). The definitions, calculation methods and results in this chapter are a slight generalization of those found in our published version [19], with more emphasis placed on completely arbitrary renormalization schemes (most noticeably in section 3.2). Our method is essentially that of Wallace and Zia [18], suitably generalized and adapted to the case of multiple fields and couplings. We shall reproduce the perturbative proof of the strong a-theorem, by showing the metric $G_{IJ} = T_{(IJ)}$ is positive-definite at leading order, as a byproduct of our consistency condition calculations.

3.1 General gauge theories

We begin by considering a general renormalizable gauge theory, with a simple gauge group $G \subset [U(n_\psi) \cap O(n_\phi)]$, and containing n_ϕ real scalars and n_ψ two-component Weyl fermions ψ_i. The basic set of couplings for the theory is then $\{g, Y_a, \bar{Y}_a, \lambda_{abcd}\}$, where $\bar{Y}_a = Y_a^*$ and $Y_a^T = Y_a$; the couplings correspond to the gauge interactions, the Yukawa interaction $\frac{1}{2}\psi_i^T C(Y_a)_{ij}\psi_j\phi_a + h.c.$, and the quartic scalar interaction $\frac{1}{4!}\lambda_{abcd}\phi_a\phi_b\phi_c\phi_d$. The hermitian gauge generators for the scalar and fermion fields are denoted t_A^ϕ, t_A^ψ respectively, where $A = 1, \ldots, n_v$ and $n_v = \dim G$ is the dimension of the representation; the generators satisfy $[t_A, t_B] = if_{ABC}t_C$, and gauge invariance requires the identities

$$Y_a t_A^\psi + t_A^{\psi T} Y_a = (t_A^\phi)_{ab} Y_b, \quad (t_A^\phi)_{ae}\lambda_{ebcd}\phi_a\phi_b\phi_c\phi_d = 0.$$

The form of our results may be simplified by assembling the Yukawa couplings and gauge generators into matrices:

$$y_a = \begin{pmatrix} Y_a & 0 \\ 0 & \bar{Y}_a \end{pmatrix}, \qquad \hat{y}_a = \begin{pmatrix} \bar{Y}_a & 0 \\ 0 & Y_a \end{pmatrix} = \sigma_1 y_a \sigma_1 \,,$$

$$T_A = \begin{pmatrix} t_A^\psi & 0 \\ 0 & -t_A^{\psi *} \end{pmatrix}, \qquad \hat{T}_A = \sigma_1 T_A \sigma_1 = -T_A^T \,.$$

Here, σ_1 is the first Pauli matrix. To realise this form of the Yukawa coupling and gauge generators, the Weyl fermions are consequently assembled into Majorana spinors $\Psi = \begin{pmatrix} \psi_i \\ -C^{-1}\bar{\psi}^{iT} \end{pmatrix}$. Finally, to remove factors of $\frac{1}{16\pi^2}$ that appear in the β-functions at each loop order, we perform a trivial rescaling of the couplings according to

$$\lambda_{abcd} \to 16\pi^2 \lambda_{abcd} \,, \quad Y_a \to 4\pi Y_a, \quad g \to 4\pi g \,.$$

3.1.1 Leading and Next-to-leading order

To evaluate the A-function perturbatively at lowest order, we require only the one-loop gauge β-function, since (as we shall see) all terms generated by the one-loop Yukawa and scalar β-functions are of higher loop order.[1] The one-loop gauge β-function is simply

$$\beta_g^{(1)} = e_1^{(1)} g^3, \tag{3.1}$$

[1] The precise ordering of contributions is gauge-Yukawa-scalar, for example the three-loop gauge, two-loop Yukawa and one-loop scalar β-functions all give contributions to the A-function at the same loop order. This is referred to as the "3-2-1" phenomenon, as detailed in [20].

where $e_1^{(1)} = -\frac{1}{3}(11C_G - 2R^\psi - \frac{1}{2}R^\phi)$, and the various invariants are defined by

$$\mathrm{tr}[t_A^\psi t_B^\psi] = R^\psi \delta_{AB}, \quad \mathrm{tr}[t_A^\phi t_B^\phi] = R^\phi \delta_{AB}, \quad f_{ACD} f_{BCD} = C_G \delta_{AB}. \tag{3.2}$$

The only terms of the correct loop order that may contribute to A are of the form $g^2 \mathrm{tr}[t_a t_a]$, hence we may define

$$A^{(2)} = a_1^{(2)} n_v g^2. \tag{3.3}$$

Expanding (2.1), we wish to solve

$$dA^{(2)} = dg T_{gg}^{(1)} \beta_g^{(1)}. \tag{3.4}$$

Substituting in (3.1) and (3.3), we find

$$2a_1^{(2)} n_v g \, dg = e_1^{(1)} T_{gg}^{(1)} g^3 \, dg,$$

hence to ensure the coefficients match, $T_{gg}^{(1)}$ must take the form

$$T_{gg}^{(1)} = \sigma_1^{(1)} \frac{n_v}{g^2}. \tag{3.5}$$

It is easy to see that for the metric $G_{IJ} = T_{(IJ)}$ to be positive-definite, $\sigma_1^{(1)} > 0$ is a necessary condition; we shall soon find that it is in fact a sufficient condition. Having found $T_{gg}^{(1)}$, the A-function coefficient is therefore given by

$$a_1^{(2)} = \tfrac{1}{2} e_1^{(1)} \sigma_1^{(1)}, \tag{3.6}$$

and hence the leading-order A-function for a general four-dimensional gauge theory is

$$A^{(2)} = \tfrac{1}{2} e_1^{(1)} \sigma_1^{(1)} n_v g^2. \tag{3.7}$$

While one may substitute in the exact values of the β-function coefficients as calculated in a particular renormalization scheme, one can just as easily leave the coefficients arbitrary. By doing so, it becomes possible to investigate the scheme-dependence of the coefficients in the A-function, as well as any consistency conditions on β-function coefficients that arise as a consequence of (2.1).

We now turn to the next-to-leading order A-function. Expanding (2.1) to this order, and recalling that y and $\hat{y}$ are not independent, we now wish to solve

$$d_y A^{(3)} \equiv d(y_a)_{ij} \frac{\partial}{\partial (y_a)_{ij}} A^{(3)} = dy T_{yy}^{(2)} \beta_y^{(1)},$$

$$d_g A^{(3)} \equiv \; dg \frac{\partial}{\partial g} A^{(3)} = dg T^{(1)}_{gg} \beta^{(2)}_g + dg T^{(2)}_{gg} \beta^{(1)}_g, \tag{3.8}$$

where we define

$$\frac{\partial}{\partial (y_a)_{ij}} (y_b)_{kl} \; \equiv \; \tfrac{1}{2} \delta_{ab} \left(\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} \right), \tag{3.9}$$

and the Yukawa metric term takes the form

$$T^{(2)}_{yy} = \mu \, \delta_{yy} \tag{3.10}$$

such that

$$dy T^{(2)}_{yy} \beta^{(1)}_y = \mu \, d(y_a)_{ij} (\beta^{(1)}_{y\,a})_{ij}. \tag{3.11}$$

At this order, there exist potential mixed Yukawa-gauge terms, and so this is the first order at which one may find non-trivial constraints on the β-functions; we shall see that this is indeed the case.

In addition to the one-loop gauge β-function, we now require the one-loop Yukawa β-function[2],

$$\begin{aligned} \beta^{(1)}_{y\,a} &= \sum_{i=1}^{5} c^{(1)}_i (C^{(1)}_i)_a \\ &= c^{(1)}_1 y_b \hat{y}_a y_b + c^{(1)}_2 \left(y_a \hat{y}_b y_b + y_b \hat{y}_b y_a \right) + c^{(1)}_3 \mathrm{tr}[y_a \hat{y}_b] y_b \\ &\quad + c^{(1)}_4 g^2 (y_a C^{\psi} + \hat{C}^{\psi} y_a) + c^{(1)}_5 g^2 C^{\phi}_{ab} y_b, \end{aligned} \tag{3.12}$$

and the two-loop gauge β-function,

$$\begin{aligned} \beta^{(2)}_g = \; & e^{(2)}_1 g^5 + e^{(2)}_2 \frac{g^5}{n_v} \mathrm{tr}[(C^{\psi})^2] + e^{(2)}_3 \frac{g^5}{n_v} \mathrm{tr}[(C^{\phi})^2] \\ & + e^{(2)}_4 \frac{g^3}{n_v} \mathrm{tr}[C^{\psi} \hat{y}_a y_a] + e^{(2)}_5 \frac{g^3}{n_v} \mathrm{tr}[\hat{y}_a C^{\phi}_{ab} y_b]. \end{aligned} \tag{3.13}$$

Given the terms that arise in each β-function, we may express $A^{(3)}$ in the form

$$\begin{aligned} A^{(3)} &= \sum_{i=1}^{8} a^{(3)}_i A^{(3)}_i + \left(\beta^I g_{IJ} \beta^J \right)^{(3)} \\ &= \sum_{i=1}^{8} a^{(3)}_i A^{(3)}_i + \alpha \frac{n_v}{g^2} \beta^{(1)}_g \beta^{(1)}_g, \end{aligned} \tag{3.14}$$

[2] Here and elsewhere, we adopt the convention of dropping explicit fermion indices, with the understanding that products of Yukawa couplings read left-to-right, for example $y_b \hat{y}_a y_b \equiv (y_b)_{ik} (\hat{y}_a)_{kl} (y_b)_{lj}$. Fully contracted fermion indices are implied by a trace, i.e. $\mathrm{tr}[y_a \hat{y}_a] \equiv (y_a)_{ij} (\hat{y}_a)_{ji}$.

where

$$\begin{aligned}
&A_1^{(3)} = \mathrm{tr}[y_a \hat{y}_b y_a \hat{y}_b], && A_2^{(3)} = \mathrm{tr}[y_a \hat{y}_a y_b \hat{y}_b], && A_3^{(3)} = \mathrm{tr}[y_a \hat{y}_b] tr[y_a \hat{y}_b], \\
&A_4^{(3)} = g^2 \mathrm{tr}[y_a \hat{y}_a \hat{C}^\psi], && A_5^{(3)} = g^2 \mathrm{tr}[\hat{y}_a C^\phi_{ab} y_b], && A_6^{(3)} = g^4 \mathrm{tr}[(C^\psi)^2], \\
&A_7^{(3)} = g^4 \mathrm{tr}[(C^\phi)^2], && A_8^{(3)} = n_v g^4. && \qquad (3.15)
\end{aligned}$$

The additional term with coefficient α represents the arbitrariness $A \to A + \beta^I g_{IJ} \beta^J$ present in solutions to (2.1), where g_{IJ} is itself an arbitrary tensor structure. Substituting (3.1), (3.5), (3.10), (3.12), (3.13) and (3.14) into (3.8) gives a system of ten equations:

$$\begin{aligned}
&4a_1^{(3)} = \mu c_1^{(1)}, && 4a_2^{(3)} = 2\mu c_2^{(1)}, && 4a_3^{(3)} = \mu c_3^{(1)}, \\
&2a_4^{(3)} g^2 = 2\mu c_4^{(1)} g^2, && 2a_5^{(3)} g^2 = \mu c_5^{(1)} g^2, && 2a_4^{(3)} g\, dg = \sigma_1^{(1)} e_4^{(2)} g\, dg, \\
&2a_5^{(3)} g\, dg = \sigma_1^{(1)} e_5^{(2)} g\, dg, && 4a_6^{(3)} g^3\, dg = \sigma_1^{(1)} e_2^{(2)} g^3\, dg, && 4a_7^{(3)} g^3\, dg = \sigma_1^{(1)} e_3^{(2)} g^3\, dg,
\end{aligned}$$

$$4(a_8^{(3)} + \alpha (e_1^{(1)})^2) g^3\, dg \;=\; \sigma_1^{(1)} e_1^{(2)} n_v g^3\, dg + e_1^{(1)} T_{gg}^{(2)} g^3\, dg. \qquad (3.16)$$

From these equations, we see that $T_{gg}^{(2)}$ takes the form

$$T_{gg}^{(2)} = \sigma_1^{(2)} n_v, \qquad (3.17)$$

and so the coefficients of the next-to-leading order A-function are:

$$\begin{aligned}
&a_1^{(3)} = \tfrac{1}{4}\mu c_1^{(1)}, && a_2^{(3)} = \tfrac{1}{2}\mu c_2^{(1)}, \\
&a_3^{(3)} = \tfrac{1}{4}\mu c_3^{(1)}, && a_4^{(3)} = \mu c_4^{(1)} = \tfrac{1}{2}\sigma_1^{(1)} e_4^{(2)}, \\
&a_5^{(3)} = \tfrac{1}{2}\mu c_5^{(1)} = \tfrac{1}{2}\sigma_1^{(1)} e_5^{(2)}, && a_6^{(3)} = \tfrac{1}{4}\sigma_1^{(1)} e_2^{(2)}, \\
&a_7^{(3)} = \tfrac{1}{4}\sigma_1^{(1)} e_3^{(2)}, && a_8^{(3)} = \tfrac{1}{4}(\sigma_1^{(1)} e_1^{(2)} + \sigma_1^{(2)} e_1^{(1)}) - \alpha (e_1^{(1)})^2. \qquad (3.18)
\end{aligned}$$

We now see how consistency conditions may arise: both $a_4^{(3)}$ and $a_5^{(3)}$ are given by two equations, each equation expressed in terms of a particular β-function coefficient. Consequently, in deducing the function A satisfying (2.1), we have found that there exist two extra equalities:

$$2\mu c_4^{(1)} = \sigma_1^{(1)} e_4^{(2)}, \quad \mu c_5^{(1)} = \sigma_1^{(1)} e_5^{(2)}. \qquad (3.19)$$

These equations hold regardless of the explicit values of the A-function or metric coefficients. We may now go further and eliminate the metric coefficients, leaving behind a consistency condition on the β-function coefficients themselves,

$$e_4^{(2)}c_5^{(1)} = 2e_5^{(2)}c_4^{(1)}. \tag{3.20}$$

Since $\beta_y^{(1)}$, $\beta_g^{(1)}$ and $\beta_g^{(2)}$ are all scheme-independent, this consistency condition holds independent of renormalization scheme. While this scheme-independence is trivial, we shall later demonstrate that more complicated consistency conditions are also scheme-independent, by constructing the effects of a coupling redefinition (corresponding to a change in renormalization scheme) and showing that the consistency conditions are invariant under such changes. Finally, we may easily see that this consistency condition is indeed satisfied by the β-function coefficients [21],

$$c_1^{(1)} = 2, \;\; c_2^{(1)} = \tfrac{1}{2}, \;\; c_3^{(1)} = \tfrac{1}{2}, \;\; c_4^{(1)} = -3, \;\; c_5^{(1)} = 0, \tag{3.21}$$

$$e_1^{(2)} = -\tfrac{1}{3}C_G(34C_G - 10R^\psi - R^\phi), e_2^{(2)} = -1, e_3^{(2)} = -4, e_4^{(2)} = -\tfrac{1}{2}, e_5^{(2)} = 0. \tag{3.22}$$

Substituting these values into (3.19) then fixes the ratio of the coefficients in $T_{gg}^{(1)}$ and $T_{yy}^{(2)}$,

$$\mu = \frac{e_4^{(2)}}{2c_4^{(1)}}\sigma_1^{(1)} = \frac{1}{12}\sigma_1^{(1)} \tag{3.23}$$

and so the leading-order positivity of G_{yy} is determined by that of G_{gg}.

It is worth noting that had we neglected the tensor structures with coefficients $c_5^{(1)}$, $e_5^{(2)}$, justified solely by the vanishing of the coefficients in $\overline{\text{MS}}$, we would not have deduced the existence of this consistency condition. Consequently, at higher loop orders we shall retain all diagrams whose contributions do not *manifestly* vanish, for example by being one-particle-reducible, as the vanishing of such diagrams may be a scheme-dependent result.

3.1.2 Next-to-next-to-leading order

So far, the leading order A-function required the one-loop gauge β-function, and the next-to-leading order A-function required the two-loop gauge and one-loop Yukawa β-functions. Following the "3-2-1" phenomenon [20], at next-to-next-to-leading order, we shall require the three-loop gauge, two-loop Yukawa and one-loop scalar β-functions. This is also the first order at which there exist potential off-diagonal terms in the expansion of (2.1), namely $T_{gy}^{(3)}$ and $T_{yg}^{(3)}$.[3] For simplicity, we shall therefore neglect $\beta_g^{(3)}$ initially, deducing first the Yukawa- and scalar-dependant terms in $A^{(4)}$, before augmenting the system of equations with

[3]Below this order, the lack of such off-diagonal terms justifies the use of the term "metric" to refer to T_{IJ} directly, rather than the symmetric part $G_{IJ} = T_{(IJ)}$.

scalar/Yukawa contributions from $d_g A^{(4)}$. Expanding (2.1) to the required order, we therefore wish to solve

$$d_\lambda A^{(4)} = d\lambda T^{(3)}_{\lambda\lambda}\beta^{(1)}_\lambda, \tag{3.24}$$

$$d_y A^{(4)} = dy T^{(2)}_{yy}\beta^{(2)}_y + dy T^{(3)}_{yy}\beta^{(1)}_y + dy T^{(3)}_{yg}\beta^{(1)}_g, \tag{3.25}$$

emphasising again the first occurrence of off-diagonal contributions.

The general two-loop Yukawa β-function was calculated in [22], and takes the form

$$\beta^{(2)}_{y\,a} = \sum_{i=1}^{30} c^{(2)}_i (C^{(2)}_i)_a, \tag{3.26}$$

where the tensor structures $C^{(2)}_i$ (with $(C^{(2)}_i)_a$ implied) are given by

$$\begin{aligned}
C^{(2)}_1 &= y_b\hat{y}_c y_a\hat{y}_b y_c, & C^{(2)}_2 &= y_b\hat{y}_a y_c \mathrm{tr}[\hat{y}_b y_c],\\
C^{(2)}_3 &= y_b\hat{y}_c y_a\hat{y}_c y_b, & C^{(2)}_4 &= \lambda_{abcd} y_b\hat{y}_c y_d,\\
C^{(2)}_5 &= (C^\phi_{ab} y_c\hat{y}_b y_c)g^2, & C^{(2)}_6 &= (y_b\hat{y}_a y_c C^\phi_{bc})g^2,\\
C^{(2)}_7 &= (C^\phi_{ab}\mathrm{tr}[y_b\hat{y}_c]y_c)g^2, & C^{(2)}_8 &= y_a\hat{y}_b y_c\hat{y}_c y_b + y_b\hat{y}_c y_c\hat{y}_b y_a,\\
C^{(2)}_9 &= y_a\hat{y}_b y_c\hat{y}_b y_c + y_c\hat{y}_b y_c\hat{y}_b y_a, & C^{(2)}_{10} &= y_a\hat{y}_b\mathrm{tr}[y_b\hat{y}_c]y_c + y_c\mathrm{tr}[\hat{y}_c y_b]\hat{y}_b y_a,\\
C^{(2)}_{11} &= y_b\hat{y}_a y_c\hat{y}_c y_b + y_b\hat{y}_c y_c\hat{y}_a y_b, & C^{(2)}_{12} &= (\hat{C}^\psi y_a\hat{y}_b y_b + y_b\hat{y}_b y_a C^\psi)g^2,\\
C^{(2)}_{13} &= (y_a C^\psi\hat{y}_b y_b + y_b\hat{y}_b\hat{C}^\psi y_a)g^2, & C^{(2)}_{14} &= (y_a\hat{y}_b\hat{C}^\psi y_b + y_b C^\psi\hat{y}_b y_a)g^2,\\
C^{(2)}_{15} &= (\hat{T}_c y_a\hat{y}_b\hat{T}_c y_b + y_b T_c\hat{y}_b y_a T_c)g^2, & C^{(2)}_{16} &= (y_a\hat{y}_b C^\phi_{bc} y_c + y_c C^\phi_{cb}\hat{y}_b y_a)g^2,\\
C^{(2)}_{17} &= C^\phi_{ab}(y_b\hat{y}_c y_c + y_c\hat{y}_c y_b)g^2, & C^{(2)}_{18} &= (y_b\hat{y}_a y_b C^\psi + \hat{C}^\psi y_b\hat{y}_a y_b)g^2,\\
C^{(2)}_{19} &= (y_b\hat{y}_a\hat{C}^\psi y_b + y_b C^\psi\hat{y}_a y_b)g^2, & C^{(2)}_{20} &= (y_a C^\psi + \hat{C}^\psi y_a)g^4,\\
C^{(2)}_{21} &= \hat{C}^\psi y_a C^\psi g^4, & C^{(2)}_{22} &= (y_a(C^\psi)^2 + (\hat{C}^\psi)^2 y_a)g^4,\\
C^{(2)}_{23} &= C^\phi_{ab}(y_b C^\psi + \hat{C}^\psi y_b), & C^{(2)}_{24} &= \mathrm{tr}[y_a\hat{y}_b y_c\hat{y}_c]y_b,\\
C^{(2)}_{25} &= \mathrm{tr}[y_a\hat{y}_c y_b\hat{y}_c]y_b, & C^{(2)}_{26} &= \mathrm{tr}[\hat{C}^\psi y_a\hat{y}_b]y_b g^2,\\
C^{(2)}_{27} &= C^\phi_{ab}C^\phi_{bc} y_c g^4, & C^{(2)}_{28} &= C^\phi_{ab} y_b g^4,\\
C^{(2)}_{29} &= \lambda_{acde}\lambda_{cdeb} y_b, & C^{(2)}_{30} &= y_b\hat{y}_a y_c\hat{y}_b y_c + y_c\hat{y}_b y_c\hat{y}_a y_b.
\end{aligned} \tag{3.27}$$

Note that the coefficients $c^{(2)}_{20}$, $c^{(2)}_{28}$ may in principle have three contributions, proportional to each of the group-theoretic constants C_G, R^ψ, R^ϕ. Similarly, the one-loop scalar β-function is given by

$$\beta^{(1)}_{\lambda\,abcd} = \sum_{i=1}^{5} d^{(1)}_i (D^{(1)}_i)_{abcd}, \tag{3.28}$$

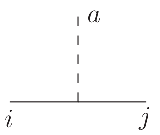
where (again with $(D_i^{(1)})_{abcd}$ implied)

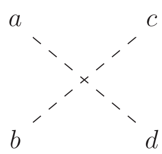
$$
\begin{aligned}
D_1^{(1)} &= \lambda_{abef}\lambda_{efcd} + \lambda_{acef}\lambda_{efbd} + \lambda_{adef}\lambda_{efbc}, \\
D_2^{(1)} &= \lambda_{ebcd}\mathrm{tr}[y_e\hat{y}_a] + \lambda_{aecd}\mathrm{tr}[y_e\hat{y}_b] + \lambda_{abed}\mathrm{tr}[y_e\hat{y}_c] + \lambda_{abce}\mathrm{tr}[y_e\hat{y}_d], \\
D_3^{(1)} &= \mathrm{tr}[y_a\hat{y}_b y_c\hat{y}_d], \\
D_4^{(1)} &= (\lambda_{ebcd}C^\phi_{ea} + \lambda_{aecd}C^\phi_{eb} + \lambda_{abed}C^\phi_{ec} + \lambda_{abce}C^\phi_{ed})g^2, \\
D_5^{(1)} &= ((t^\phi_A t^\phi_B)_{ab}(t^\phi_A t^\phi_B)_{cd} + (t^\phi_A t^\phi_B)_{ac}(t^\phi_A t^\phi_B)_{bd} + (t^\phi_A t^\phi_B)_{ad}(t^\phi_A t^\phi_B)_{bc})g^4, \qquad (3.29)
\end{aligned}
$$

and the coefficients may easily be calculated in $\overline{\mathrm{MS}}$:

$$
d_1^{(1)} = 1, \quad d_2^{(1)} = \tfrac{1}{2}, \quad d_3^{(1)} = -12, \quad d_4^{(1)} = -3, \quad d_5^{(1)} = 12. \qquad (3.30)
$$

As mentioned previously, the metric at this order becomes more complex; consequently, the contributions to the A-function itself also become more complex, to the point where writing the explicit tensor structures is cumbersome and uninformative. To alleviate this, we shall employ a diagrammatic notation, based on traditional Feynman diagrams. Each tensor coupling is represented by its associated Feynman diagram vertex representation, with the indices labelling each leg:

$(y_a)_{ij} \to$ 

$\lambda_{abcd} \to$ 

The gauge generators t^ϕ_A, T_A and quadratic Casimir operators C^ϕ, C^ψ are represented as

$(t^\phi_A)_{ab} \to$ 

, $(T_A)_{ij} \to$ 

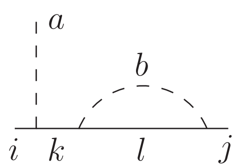
,

$C^\phi_{ab} \to$ 

, $C^\psi_{ij} \to$ 

,

and contracted lines represent contracted indices,

$(y_a)_{ik}(\hat{y}_b)_{kl}(y_b)_{lj} \equiv y_a\hat{y}_b y_b \to$ 

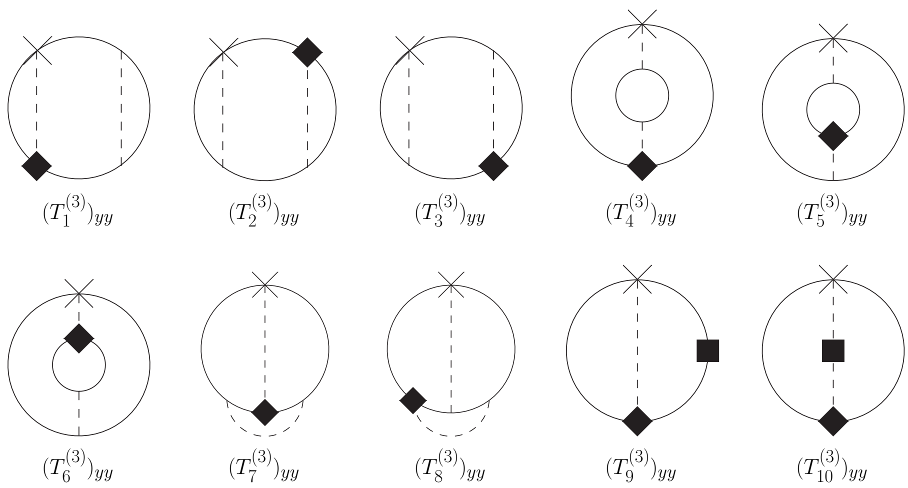


Table 3.1: Contributions to $T_{yy}^{(3)}$

Using this notation, we may now express the "metric" term $T_{yy}^{(3)}$ as

$$T_{yy}^{(3)} = \sum_{i=1}^{10} t_i^{(3)} (T_i^{(3)})_{yy}; \tag{3.31}$$

the individual tensor structures $(T_i^{(3)})_{yy}$ are then given in Table 3.1, contracted in the form $dyT_{yy}^{(3)}\beta_y^{(1)}$, where a cross represents dy and a diamond $\beta_y^{(1)}$. As an example, according to the notation discussed above, the diagram labelled $(T_1^{(3)})_{yy}$ corresponds to $d(y_a)_{ij}(\beta_{y\,a}^{(1)})_{jk}(y_b)_{kl}(\hat{y}_b)_{li}$, and so the tensor structure $(T_1^{(3)})_{yy}$ itself (contracting $d(y_a)_{ij}$ and $(\beta_{y\,b}^{(1)})_{kl}$) would be

$$(T_1^{(3)})_{yy} = \delta_{ab}\delta_{jk}(y_c)_{lm}(\hat{y}_c)_{mi}.$$

Similarly, we may express the potential off-diagonal terms in the form

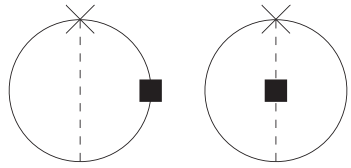
$$T_{yg}^{(3)} = \sum_{i=1}^{2} \tau_i^{(3)} (T_i^{(3)})_{yg}, \tag{3.32}$$

where the tensor structures $(T_1^{(3)})_{yg}$, $(T_2^{(3)})_{yg}$ are inferred from Table 3.2.

Finally, at next-to-next-to-leading order, we may parametrize the A-function as

$$A^{(4)} = \sum_{i=1}^{27} a_i^{(4)} A_i^{(4)} + (\beta^I g_{IJ} \beta^J)^{(4)} + \mathcal{O}(g^6), \tag{3.33}$$

where the pure-gauge terms $\mathcal{O}(g^6)$ have not been considered fully, and there may now be multiple arbitrary contributions subsumed in the $\beta^I g_{IJ} \beta^J$ term. The



$(T_1^{(3)})_{yg}$ $(T_2^{(3)})_{yg}$

Table 3.2: Contributions to $T_{yg}^{(3)}$

tensor structures $A_i^{(4)}$ are listed in Table 3.3,[4] and the explicit expressions may be reconstructed from the diagrammatic representation. The arbitrary terms with which we shall be concerned are

$$\beta_y^{(1)} g_{yy}^{(2)} \beta_y^{(1)} = a(\beta_{y\,a}^{(1)})_{ij}(\beta_{y\,a}^{(1)})_{ij},$$
$$\beta_g^{(2)} g_{gg}^{(1)} \beta_g^{(1)} + \beta_g^{(1)} g_{gg}^{(1)} \beta_g^{(2)} = 2e_1^{(1)} \alpha n_v g \beta_g^{(2)}(y), \tag{3.34}$$

considering only the Yukawa-dependent parts of $\beta_g^{(2)}$.

We many now solve (3.24), (3.25). Like the lowest-order Yukawa metric term, the lowest-order scalar metric term $T_{\lambda\lambda}^{(3)}$ is simply

$$T_{\lambda\lambda}^{(3)} = \lambda\,\delta_{\lambda\lambda}, \tag{3.35}$$

so substituting (3.28), (3.33) and (3.35) into (3.24) gives five simple linear equations,

$$3a_1^{(4)} = 3\lambda d_1^{(1)}, \qquad 2a_2^{(4)} = 4\lambda d_2^{(1)}, \qquad a_3^{(4)} = \lambda d_3^{(1)},$$
$$2a_{12}^{(4)} g^2 = 4\lambda d_4^{(1)} g^2, \qquad a_{21}^{(4)} g^4 = 3\lambda d_5^{(1)} g^4. \tag{3.36}$$

These equations clearly provide solutions to the coefficients $a_{1-3}^{(4)}$, $a_{12}^{(4)}$, $a_{21}^{(4)}$; since the tensor structures $A_2^{(4)}$, $A_3^{(4)}$ also contain Yukawa couplings, there may exist a consistency condition relating the respective scalar and Yukawa β-functions, obtained by also solving (3.25). Substituting (3.1), (3.10), (3.12), (3.26), (3.31), (3.32) and (3.33) into (3.25) gives a complex system of 37 equations (A.1), with solution

$$a_1^{(4)} = -\frac{\mu}{48}\, c_4^{(2)}, \qquad a_2^{(4)} = -\frac{\mu}{48}\, c_4^{(2)},$$
$$a_3^{(4)} = \frac{\mu}{4}\, c_4^{(2)}, \qquad a_4^{(4)} = \frac{\mu}{6}\, c_1^{(2)},$$
$$a_5^{(4)} = -\frac{\mu}{3}\, c_8^{(2)}, \qquad a_6^{(4)} = 0,$$
$$a_7^{(4)} = \mu\,(c_{10}^{(2)} - c_8^{(2)}), \qquad a_8^{(4)} = \frac{\mu}{12}(4c_{10}^{(2)} - 4c_8^{(2)} - c_{24}^{(2)}),$$

[4]Note that $A_4^{(4)}$ is a non-planar diagram.

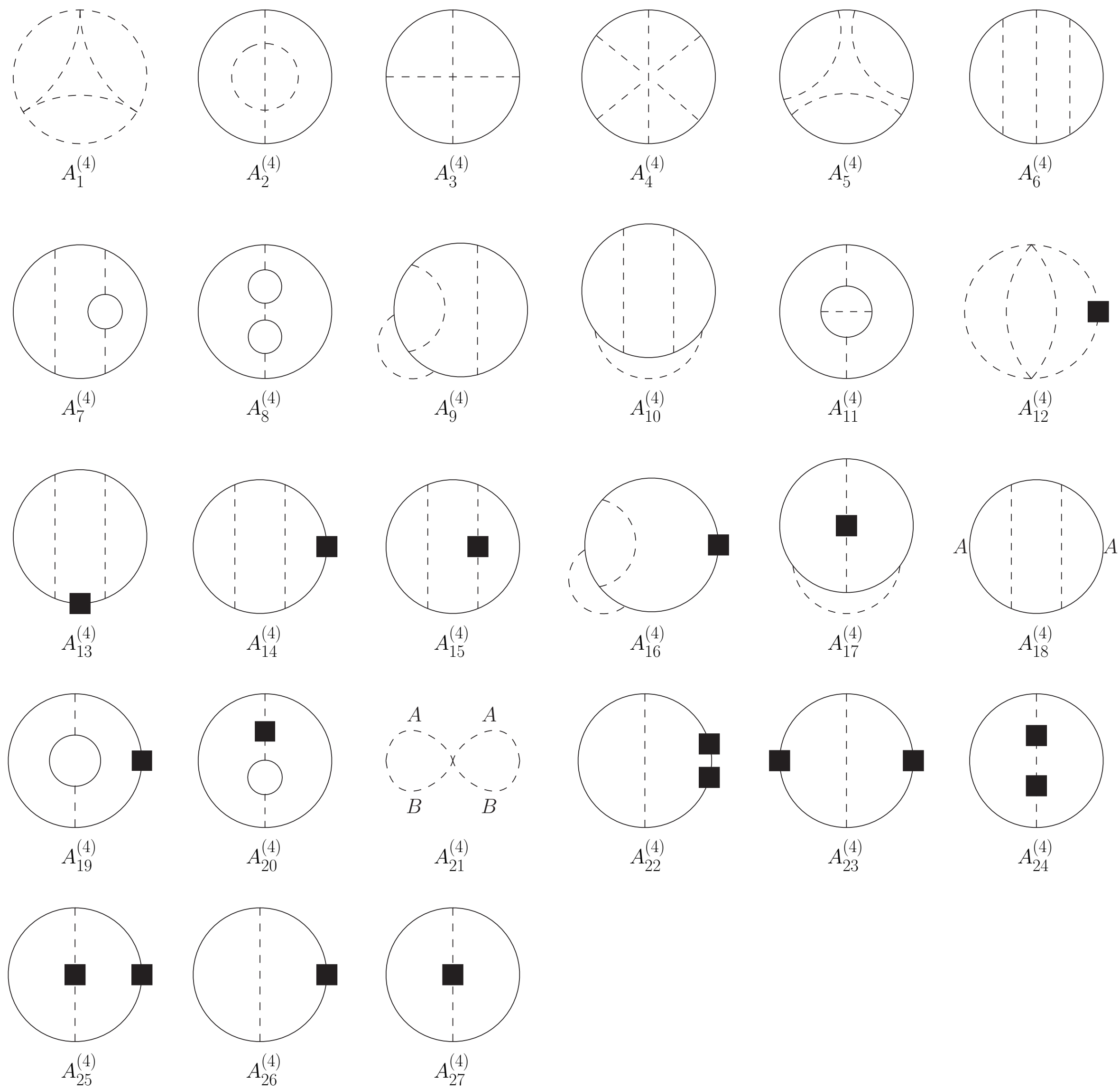


Table 3.3: Contributions to $A^{(4)}$, neglecting pure-gauge terms

$$
\begin{aligned}
a_9^{(4)} &= \mu\,(c_9^{(2)} - 4c_8^{(2)}), & a_{10}^{(4)} &= \frac{\mu}{2}\,(c_3^{(2)} - 4c_{11}^{(2)}), \\
a_{11}^{(4)} &= \frac{\mu}{2}\,(4c_{10}^{(2)} - 4c_8^{(2)} - c_{11}^{(2)}), & a_{12}^{(4)} &= \frac{\mu}{8}\,c_4^{(2)}, \\
a_{13}^{(4)} &= \frac{\mu}{2}\,(12c_8^{(2)} - c_{12}^{(2)} + c_{13}^{(2)} + c_{14}^{(2)}), & a_{14}^{(4)} &= \mu\,(6c_8^{(2)} + c_{14}^{(2)}), \\
a_{15}^{(4)} &= \mu\,c_{16}^{(2)}, & a_{16}^{(4)} &= \mu\,(24c_8^{(2)} - 6c_9^{(2)} + 6c_{11}^{(2)} + c_{19}^{(2)}), \\
a_{17}^{(4)} &= \frac{\mu}{2}\,c_6^{(2)}, & a_{18}^{(4)} &= \frac{\mu}{2}\,c_{15}^{(2)}, \\
a_{19}^{(4)} &= \mu\,(12c_8^{(2)} - 6c_{10}^{(2)} - c_{12}^{(2)} + c_{14}^{(2)}), & a_{20}^{(4)} &= \frac{\mu}{4}\,(c_7^{(2)} + c_{16}^{(2)} - c_{17}^{(2)}), \\
a_{21}^{(4)} &= -\frac{3\mu}{4}\,c_4^{(2)}, & a_{22}^{(4)} &= \mu\,(6c_{12}^{(2)} - 36c_8^{(2)} - 6c_{14}^{(2)} + c_{22}^{(2)}), \\
a_{23}^{(4)} &= \frac{\mu}{2}\,(12c_{12}^{(2)} - 72c_8^{(2)} - 12c_{14}^{(2)} + c_{21}^{(2)}), & a_{24}^{(4)} &= \frac{\mu}{2}\,c_{27}^{(2)}, \\
a_{25}^{(4)} &= \mu\,(6c_{17}^{(2)} - 6c_{16}^{(2)} + c_{23}^{(2)}), & a_{26}^{(4)} &= \mu\,c_{20}^{(2)} + \frac{\tau_1^{(3)}}{2}e_1^{(1)},
\end{aligned}
$$

$$a_{27}^{(4)} = \frac{\mu}{2}\, c_{28}^{(2)} + \frac{\tau_2^{(3)}}{2} e_1^{(1)}, \tag{3.37}$$

$$\lambda = -\frac{\mu}{48}\, c_4^{(2)} \tag{3.38}$$

$$\begin{aligned}
t_1^{(3)} &= 2a, & t_2^{(3)} + t_3^{(3)} &= -4\mu c_8^{(2)} + 4a,\\
t_4^{(3)} &= 2\mu\, (c_{10}^{(2)} - c_8^{(2)}) + a, & t_5^{(3)} + t_6^{(3)} &= \mu\, (2c_{10}^{(2)} - 2c_8^{(2)} - c_{24}^{(2)}) + 2a,\\
t_7^{(3)} &= -2\mu c_{11}^{(2)} + 4a, & t_8^{(3)} &= 4\mu(c_9^{(2)} - 4c_8^{(2)}) + 8a,\\
t_9^{(3)} &= 4\mu(6c_8^{(2)} - c_{12}^{(2)} + c_{14}^{(2)}) - 12a, & t_{10}^{(3)} &= 2\mu(c_{16}^{(2)} - c_{17}^{(2)}).
\end{aligned} \tag{3.39}$$

One can again eliminate all A-function and metric coefficients to leave behind a system of six consistency conditions,

$$\begin{aligned}
4c_2^{(2)} + c_3^{(2)} + 4c_9^{(2)} - 16c_{10}^{(2)} - c_{30}^{(2)} &= 0\\
c_3^{(2)} + 16c_8^{(2)} - 4c_9^{(2)} - 4c_{11}^{(2)} - c_{30}^{(2)} &= 0\\
c_{11}^{(2)} - 2c_{24}^{(2)} + c_{25}^{(2)} &= 0\\
12c_8^{(2)} - 2c_{12}^{(2)} + 2c_{14}^{(2)} - 6c_{24}^{(2)} - c_{26}^{(2)} &= 0\\
c_5^{(2)} - c_6^{(2)} + 4c_{16}^{(2)} - 4c_{17}^{(2)} &= 0\\
c_{18}^{(2)} - c_{19}^{(2)} + 6c_{25}^{(2)} + 2c_{26}^{(2)} &= 0
\end{aligned} \tag{3.40}$$

where the first three conditions do not involve gauge terms, and are equivalent to those derived in [9] for a scalar/fermion theory. Combining with the solution of (3.24) gives one extra condition relating the scalar and Yukawa β-function coefficients,

$$d_2^{(1)} c_4^{(2)} = d_3^{(1)} c_{29}^{(2)}. \tag{3.41}$$

It is worth noting that, as indicated in the previous section, we have retained all potential non-vanishing terms in our system of equations. Nevertheless, for presentation purposes, we have opted to substitute in the explicit values of the one-loop Yukawa coefficients, listed in (3.21); we stress that the number of derived consistency conditions has not been reduced.

To verify these consistency conditions, we first check that they are satisfied by the $\overline{\text{MS}}$ values of the coefficients, then deduce the effects of a change of renormalization scheme and ensure any induced changes in the coefficients cancel when substituted in to the conditions; this demonstrates that the conditions are correct, and hold in an arbitrary renormalization scheme. The $\overline{\text{MS}}$ values of the two-loop Yukawa β-function coefficients are [22]

$$c_1^{(2)} = 2, \quad c_2^{(2)} = -1, \quad c_3^{(2)} = -2, \quad c_4^{(2)} = -2, \quad c_5^{(2)} = -12, \quad c_6^{(2)} = 6,$$
$$c_7^{(2)} = 0, \quad c_8^{(2)} = -\tfrac{1}{8}, \quad c_9^{(2)} = 0, \quad c_{10}^{(2)} = -\tfrac{3}{8}, \quad c_{11}^{(2)} = -1, \quad c_{12}^{(2)} = 0,$$
$$c_{13}^{(2)} = -\tfrac{7}{4}, \quad c_{14}^{(2)} = -\tfrac{1}{4}, \quad c_{15}^{(2)} = 6, \quad c_{16}^{(2)} = \tfrac{9}{2}, \quad c_{17}^{(2)} = 0, \quad c_{18}^{(2)} = 3,$$
$$c_{19}^{(2)} = 5,$$
$$c_{20}^{(2)} = -\tfrac{1}{12}(194C_G - 20R^\psi - 11R^\phi),$$
$$c_{21}^{(2)} = 0, \quad c_{22}^{(2)} = -\tfrac{3}{2}, \quad c_{23}^{(2)} = 6, \quad c_{24}^{(2)} = -\tfrac{3}{4}, \quad c_{25}^{(2)} = -\tfrac{1}{2}, \quad c_{26}^{(2)} = \tfrac{5}{2},$$
$$c_{27}^{(2)} = -\tfrac{21}{2},$$
$$c_{28}^{(2)} = \tfrac{1}{12}(147C_G - 12R^\psi - 3R^\phi),$$
$$c_{29}^{(2)} = \tfrac{1}{12}, \quad c_{30}^{(2)} = 0, \tag{3.42}$$

and combined with the $\beta_\lambda^{(1)}$ coefficients (3.30), we see that all seven consistency conditions are indeed satisfied.

To derive the effects of a coupling redefinition, we begin with the vector β-function $\beta^I \equiv \beta^I(g)$, where $g \equiv g^I$ contains all couplings in the theory. $\beta^I(g)$ therefore represents the β-functions of a theory evaluated in a particular scheme, for example $\overline{\text{MS}}$. By definition,

$$\beta^I(g) = \mu \frac{\partial}{\partial \mu} g^I,$$

hence if we were to consider a new β-function $\beta^{'I}$ as a function of new couplings $g^{'I}$, related to the original couplings by some finite shift of the form $g^{'I} = g^I + \delta g^I$, we may write

$$\begin{aligned}\beta^{'I}(g^{'}) = \mu \frac{\partial}{\partial \mu} g^{'I} = \mu \frac{\partial}{\partial \mu} (g^I + \delta g^I) = \mu \frac{\partial g^J}{\partial \mu} \frac{\partial}{\partial g^J} (g^I + \delta g^I) \\ = \beta^J \frac{\partial}{\partial g^J} (g^I + \delta g^I) = \beta^I + \beta^J \frac{\partial}{\partial g^J} \delta g^I.\end{aligned}$$

We may also perform a Taylor expansion of $\beta^{'I}$ as a function of the new couplings,

$$\beta^{'I}(g^{'}) = \beta^{'I}(g + \delta g) = \left[\mathbb{1} + \delta g^J \frac{\partial}{\partial g^J} + \frac{1}{2} \left(\delta g^J \frac{\partial}{\partial g^J} \right)^2 + \ldots \right] \beta^{'I}(g),$$

where $\beta^{'I}(g) = \beta^I(g) + \delta\beta^I(g)$ therefore defines the effects of a coupling redefinition on the β-function, $\delta\beta^I(g)$. Comparing the two expressions, we obtain the effect of a coupling redefinition on the β-functions:

$$\delta\beta^I(g) = \beta^J \frac{\partial}{\partial g^J}\delta g^I - \delta g^J \frac{\partial}{\partial g^J}\beta^I \\ - \frac{1}{2}\left(\delta g^J \frac{\partial}{\partial g^J}\right)^2 \beta^I - \delta g^J \frac{\partial}{\partial g^J}(\delta\beta^I) - \dots \tag{3.43}$$

Finally, following [61], it is possible to identify a finite shift in the couplings with a change in the finite part of the renormalization constants, hence (3.43) gives the result of a change in renormalization scheme. It is to be understood that (3.43) is a somewhat schematic result, that in principle contains higher order corrections; the precise effects of a scheme change at a particular order are given by extracting from (3.43) all contributions of the correct loop order. We may also immediately see that the lowest possible order of change expressed in (3.43) is $(\delta\beta^I)^{(2)}$, hence all one-loop β-function results are scheme-independent.

From (3.43), given a change of couplings of the form $y^{'} = y + (\delta y)^{(1)}$, $g^{'} = g + (\delta g)^{(1)}$, we may extract the effects of a coupling redefinition on the two-loop gauge β-function,

$$\delta\beta_g^{(2)} = \beta_g^{(1)}\frac{\partial}{\partial g}(\delta g)^{(1)} - (\delta g)^{(1)}\frac{\partial}{\partial g}\beta_g^{(1)}, \tag{3.44}$$

and the two-loop Yukawa β-function,

$$\delta\beta_y^{(2)} = \left(\beta_y^{(1)}\frac{\partial}{\partial y} + \beta_g^{(1)}\frac{\partial}{\partial g}\right)(\delta y)^{(1)} - \left((\delta y)^{(1)}\frac{\partial}{\partial y} + (\delta g)^{(1)}\frac{\partial}{\partial g}\right)\beta_y^{(1)}. \tag{3.45}$$

The most general possible coupling redefinition may then be parametrised as

$$(\delta g)^{(1)} = \delta_1^{(1)} g^3, \qquad (\delta y)^{(1)} = \sum_{i=1}^{5} \epsilon_i^{(1)} C_i^{(1)}, \tag{3.46}$$

with $C_i^{(1)}$ defined in (3.12). Consequently, there is no change in $\beta_g^{(2)}$ (establishing its scheme-independence), and the corresponding changes in $\beta_y^{(2)}$ are given by

$$\begin{aligned}
\delta c_2^{(2)} &= 2(c_3^{(1)}\epsilon_1^{(1)} - c_1^{(1)}\epsilon_3^{(1)}), & \delta c_6^{(2)} &= 2(c_5^{(1)}\epsilon_1^{(1)} - c_1^{(1)}\epsilon_5^{(1)}),\\
\delta c_7^{(2)} &= 2(c_5^{(1)}\epsilon_3^{(1)} - c_3^{(1)}\epsilon_5^{(1)}), & \delta c_9^{(2)} &= 2(c_1^{(1)}\epsilon_2^{(1)} - c_2^{(1)}\epsilon_1^{(1)}),\\
\delta c_{10}^{(2)} &= 2(c_3^{(1)}\epsilon_2^{(1)} - c_2^{(1)}\epsilon_3^{(1)}), & \delta c_{11}^{(2)} &= 2(c_2^{(1)}\epsilon_1^{(1)} - c_1^{(1)}\epsilon_2^{(1)}),\\
\delta c_{13}^{(2)} &= 2(c_4^{(1)}\epsilon_2^{(1)} - c_2^{(1)}\epsilon_4^{(1)}), & \delta c_{14}^{(2)} &= 2(c_4^{(1)}\epsilon_2^{(1)} - c_2^{(1)}\epsilon_4^{(1)}),\\
\delta c_{16}^{(2)} &= 2(c_5^{(1)}\epsilon_2^{(1)} - c_2^{(1)}\epsilon_5^{(1)}), & \delta c_{19}^{(2)} &= 2(c_4^{(1)}\epsilon_1^{(1)} - c_1^{(1)}\epsilon_4^{(1)}),\\
\delta c_{20}^{(2)} &= 2(e_1^{(1)}\epsilon_4^{(1)} - c_4^{(1)}\delta_1^{(1)}), & \delta c_{24}^{(2)} &= 4(c_2^{(1)}\epsilon_3^{(1)} - c_3^{(1)}\epsilon_2^{(1)}),
\end{aligned}$$

$$\delta c_{25}^{(2)} = 2(c_1^{(1)}\epsilon_3^{(1)} - c_3^{(1)}\epsilon_1^{(1)}) \qquad \delta c_{26}^{(2)} = 4(c_4^{(1)}\epsilon_3^{(1)} - c_3^{(1)}\epsilon_4^{(1)}),$$
$$\delta c_{28}^{(2)} = 2(e_1^{(1)}\epsilon_5^{(1)} - c_5^{(1)}\delta_1^{(1)}), \tag{3.47}$$

with $\delta c_i^{(2)} = 0$ implied for any coefficient not in this list. These changes may now be substituted into (3.40), (3.41), and we see that the consistency conditions are indeed scheme-independent. Crucially, since $c_4^{(2)}$ is scheme-independent, we may substitute its value into (3.38), and combining with (3.19) we find

$$\lambda = \tfrac{1}{24}\mu = \tfrac{1}{288}\sigma_1^{(1)}, \tag{3.48}$$

verifying that the positivity of $\sigma_1^{(1)}$ is indeed a sufficient condition for the leading-order positive-definiteness of G_{IJ}. The leading-order metric coefficients were originally calculated separately in [8], and in our conventions are given by

$$\sigma_1^{(1)} = 2, \qquad \mu = \tfrac{1}{6}, \qquad \lambda = \tfrac{1}{144}, \tag{3.49}$$

in accordance with our consistency conditions.

Having investigated the construction of $A^{(4)}$ using the scalar and Yukawa β-functions, we now turn to contributions from the gauge β-function. Expanding (2.1) to this order gives, in addition to (3.24), (3.25),

$$d_g\, A^{(4)} = dg\, T_{gg}^{(1)}\beta_g^{(3)} + dg\, T_{gg}^{(2)}\beta_g^{(2)} + dg\, T_{gg}^{(3)}\beta_g + dg\, T_{gy}^{(3)}\beta_y^{(1)}. \tag{3.50}$$

We now wish to answer three questions:

- Are we free to impose that T_{IJ} be symmetric at this order; that is, are we free to choose $T_{yg}^{(3)} = T_{gy}^{(3)}$?
- Does the A-function (3.33) satisfy (3.50), and what consistency conditions are therefore imposed on the coefficients of $\beta_g^{(3)}$?
- Does knowledge of the one-loop scalar and two-loop Yukawa β-function coefficients provide enough information to *determine* the scalar- and Yukawa-dependent parts of $\beta_g^{(3)}$, in conjunction with the A-function (3.33)?

The third question is of particular note, as it implies that the constraints on renormalization group flow provided by a function satisfying (2.1) are sufficient to determine higher-order β-functions without having to perform ever-more-complex loop integrals. Depending on the consistency conditions and the underlying Feynman integrals of the associated β-function contributions, it may even be possible to predict the simple poles of highly non-trivial integrals at higher loop orders without any integration at all.

Since we are concerned with the terms in $A^{(4)}$ that are not pure-gauge, we may express the three-loop gauge β-function in the form

$$\beta_g^{(3)} = g \sum_{i=1}^{16} e_i^{(3)} A_{i+11}^{(4)} + \mathcal{O}(g^7), \tag{3.51}$$

the contributions to $T_{gg}^{(3)}$ as

$$T_{gg}^{(3)} = \sigma_1^{(3)} \text{tr}\left[C^\psi \hat{y}_a y_a\right] + \sigma_2^{(3)} \text{tr}\left[\hat{y}_a C^\phi_{ab} y_b\right] + \mathcal{O}(g^2), \tag{3.52}$$

and the potential off-diagonal metric terms $T_{gy}^{(3)}$ as

$$T_{gy}^{(3)} = \sum_{i=1}^{2} \tilde{\tau}_i^{(3)} (T_i^{(3)})_{gy}, \tag{3.53}$$

where the tensor structures $(T_i^{(3)})_{gy}$ are obtained from $(T_i^{(3)})_{yg}$, by replacing the derivatives in Table 3.2 with β-functions; imposing symmetry at this order would therefore simply require $\tau_i^{(3)} = \tilde{\tau}_i^{(3)}$. Substituting the A-function (3.33), metric terms (3.5), (3.17), (3.52), (3.53), and β-functions (3.1), (3.12), (3.13), (3.51) into (3.50) then gives a new set of 16 equations (see (A.2)), leading to 12 additional consistency conditions:

$$\begin{aligned}
c_4^{(2)} + 4e_{10}^{(3)} &= 0,\\
12e_1^{(3)} + e_{10}^{(3)} &= 0,\\
c_{15}^{(2)} - 12e_7^{(3)} &= 0,\\
c_{27}^{(2)} - 6e_{13}^{(3)} &= 0,\\
c_5^{(2)} - 4c_{17}^{(2)} + 24e_4^{(3)} - 12e_6^{(3)} &= 0,\\
c_{21}^{(2)} - 2c_{22}^{(2)} + 6e_{11}^{(3)} - 6e_{12}^{(3)} &= 0,\\
c_5^{(2)} + c_6^{(2)} - 4c_7^{(2)} - 24e_6^{(3)} + 96e_9^{(3)} &= 0,\\
6c_5^{(2)} - 3c_6^{(2)} + 4c_{23}^{(2)} - 36e_6^{(3)} - 12e_{14}^{(3)} &= 0,\\
c_{12}^{(2)} - c_{13}^{(2)} + c_{14}^{(2)} + 12e_2^{(3)} - 12e_3^{(3)} &= 0,\\
6c_9^{(2)} - 6c_{11}^{(2)} + 4c_{14}^{(2)} - c_{19}^{(2)} - 24e_3^{(3)} + 6e_5^{(3)} &= 0,\\
18c_{10}^{(2)} - 3c_{12}^{(2)} + 3c_{14}^{(2)} - c_{22}^{(2)} + 18e_8^{(3)} - 3e_{11}^{(3)} &= 0,\\
18c_8^{(2)} + 3c_{14}^{(2)} + c_{22}^{(2)} - 18c_{24}^{(2)} - 3c_{26}^{(2)} - 18e_3^{(3)} - 3e_{11}^{(3)} &= 0.
\end{aligned} \tag{3.54}$$

To simplify these conditions, we have used the relations between leading-order metric coefficients (3.48), and inserted the (scheme-independent) one-loop β-function coefficients. We see immediately that these consistency conditions do indeed relate the coefficients of $\beta_g^{(3)}$ to the lower-order $\beta_y^{(2)}$, allowing one to deduce some higher-order β-function coefficients using the existence of an A-function satisfying (2.1).

We may again verify that these conditions hold in $\overline{\text{MS}}$ by using the $\beta_y^{(2)}$ coefficients in (3.42), and the $\beta_g^{(3)}$ coefficients (first calculated in [23]) listed below,

$$\begin{aligned}
&e_1^{(3)} = -\tfrac{1}{24}, && e_2^{(3)} = \tfrac{1}{16}, && e_3^{(3)} = \tfrac{3}{16}, && e_4^{(3)} = \tfrac{1}{4},\\
&e_5^{(3)} = \tfrac{3}{4}, && e_6^{(3)} = -\tfrac{1}{2}, && e_7^{(3)} = \tfrac{1}{2}, && e_8^{(3)} = \tfrac{7}{16},\\
&e_9^{(3)} = -\tfrac{1}{16}, && e_{10}^{(3)} = \tfrac{1}{2}, && e_{11}^{(3)} = -\tfrac{5}{8}, && e_{12}^{(3)} = -\tfrac{1}{8},\\
&e_{13}^{(3)} = -\tfrac{7}{4}, && e_{14}^{(3)} = -4, && e_{15}^{(3)} = -3C_G, && e_{16}^{(3)} = \tfrac{9}{8}C_G,
\end{aligned} \tag{3.55}$$

and can therefore check that they hold in arbitrary schemes by deducing the effects of a coupling redefinition, then showing that the consistency conditions are invariant under such changes. Discarding purely-gauge terms, the effects of a coupling redefinition of the form (3.46), plus

$$(\delta g)^{(2)} = \mathcal{O}(g^5) + \delta_4^{(2)} \frac{g^3}{n_v} \text{tr}\left[C^\psi \hat{y}_a y_a\right] + \delta_5^{(2)} \frac{g^3}{n_v} \text{tr}\left[\hat{y}_a C^\phi_{ab} y_b\right], \tag{3.56}$$

on (3.51) is

$$\begin{aligned}
\delta e_2^{(3)} &= 2\left(c_2^{(1)}\delta_4^{(2)} - e_4^{(2)}\epsilon_2^{(1)}\right), & \delta e_3^{(3)} &= 2\left(c_2^{(1)}\delta_4^{(2)} - e_4^{(2)}\epsilon_2^{(1)}\right),\\
\delta e_4^{(3)} &= 4\left(c_2^{(1)}\delta_5^{(2)} - e_5^{(2)}\epsilon_2^{(1)}\right), & \delta e_5^{(3)} &= 2\left(c_1^{(1)}\delta_4^{(2)} - e_4^{(2)}\epsilon_1^{(1)}\right),\\
\delta e_6^{(3)} &= 2\left(c_1^{(1)}\delta_5^{(2)} - e_5^{(2)}\epsilon_1^{(1)}\right), & \delta e_8^{(3)} &= 2\left(c_3^{(1)}\delta_4^{(2)} - e_4^{(2)}\epsilon_3^{(1)}\right),\\
\delta e_9^{(3)} &= 2\left(c_3^{(1)}\delta_5^{(2)} - e_5^{(2)}\epsilon_3^{(1)}\right), & \delta e_{11}^{(3)} &= 2\left(c_4^{(1)}\delta_4^{(2)} - e_4^{(2)}\epsilon_4^{(1)}\right),\\
\delta e_{12}^{(3)} &= 2\left(c_4^{(1)}\delta_4^{(2)} - e_4^{(2)}\epsilon_4^{(1)}\right), & \delta e_{13}^{(3)} &= 2\left(c_5^{(1)}\delta_5^{(2)} - e_5^{(2)}\epsilon_5^{(1)}\right),
\end{aligned}$$
$$\delta e_{14}^{(3)} = 2\left(c_5^{(1)}\delta_4^{(2)} + 2c_4^{(1)}\delta_5^{(2)} - e_4^{(2)}\epsilon_5^{(1)} - 2e_5^{(2)}\epsilon_4^{(1)}\right), \tag{3.57}$$

and $\delta e_i^{(3)} = 0$ otherwise; upon substituting into (3.54), we find that the conditions are indeed invariant. Furthermore, upon attempting to fix the off-diagonal metric as symmetric, no additional consistency conditions are generated: there is sufficient freedom in both $A^{(4)}$ and $T_{IJ}^{(3)}$ to impose symmetry.

3.2 $\mathcal{N} = 1$ supersymmetric gauge theory

We now turn to the case of a general $\mathcal{N} = 1$ supersymmetric gauge theory. In a supersymmetric theory, the Lagrangian density exhibits additional symmetry under exchange of bosonic and fermionic fields; a theory possessing k such sets of transformations is then referred to as an $\mathcal{N} = k$ supersymmetric theory. Using the superspace formalism, the component fields of such theories may be assembled into supermultiplets, and their possible interactions then described by the superpotential. Attempting to construct an interacting supersymmetric QFT places strong constraints on the masses and couplings in the theory, for example the quartic scalar coupling in an $\mathcal{N} = 1$ gauge theory must satisfy

$$\lambda_{ij}{}^{kl} = Y^{ijm}\bar{Y}_{mkl} - g^2\left[(R_A)^i{}_k(R_A)^j{}_l + (R_A)^i{}_l(R_A)^j{}_k\right] \tag{3.58}$$

effectively reducing the number of couplings. Furthermore, the renormalization of supersymmetric theories is constrained to the extent that there exist non-renormalization theorems [63], in which some terms undergo no renormalization at all. The form of certain RG functions is then dramatically simplified, and we shall make use of two key results:

- The Yukawa β-function is determined entirely by the chiral superfield anomalous dimension γ: $\beta_Y^{ijk} = Y^{ljk}\gamma_l{}^i + Y^{ilk}\gamma_l{}^j + Y^{ijl}\gamma_l{}^k$.
- There exists, in a particular renormalization scheme, an exact expression for the gauge β-function, known as the $NSVZ$ β-function.

If we define the general gauge β-function according to

$$\beta_g = f(g)\tilde{\beta}_g, \quad \tilde{\beta}_g \equiv \left(Q - \frac{2}{n_v}\mathrm{tr}\left[\gamma C(R)\right]\right), \tag{3.59}$$

with $Q = T_R - 3C_G$, then the $NSVZ$ β-function is given by

$$(\beta_g)_{NSVZ} = f(g)|_{NSVZ}\tilde{\beta}_g, \quad f(g)|_{NSVZ} = \frac{g^3}{1 - 2g^2C_G}. \tag{3.60}$$

This result was first derived in [24], for the special case with no chiral superfields, and then extended to a general theory in [25]. One final, crucial difference in the supersymmetric case is the regularization method. Supersymmetry imposes the equality of bosonic and fermionic degrees of freedom, and requires the introduction of auxiliary fields to ensure that this remains the case off-shell (that is, without imposing the equations of motion). Since the number of fermionic degrees of freedom, and hence the form of the auxiliary fields, depends on the dimensionality of the gamma matrices, supersymmetry transformations require a fixed number of spacetime dimensions, and so one cannot use Dimensional Regularization without manifestly breaking supersymmetry. Consequently, the preferred regularization method for supersymmetric theories is Dimensional Reduction [26], in which one extends momentum integrals to d dimensions as usual, but keeps all other tensors in the intended number of dimensions, preserving supersymmetry[5]. After performing any required tensor manipulations, the final scalar integrals may also be evaluated d-dimensionally to yield the required n-point contributions; the theory may then be renormalized as usual.

In principle, the results for $\mathcal{N}=1$ are contained in the general solution (3.33) with coefficients (3.37), given the appropriate choice of fields and couplings[6]; however, there are some interesting theoretical developments in the supersymmetric case that demand attention. Firstly, there is a proposed *all-orders* expression for A [12–14], which when perturbatively expanded should reproduce the result obtained by specialising (3.33) to the supersymmetric case. Secondly, in [9] a sufficient condition on the chiral superfield anomalous dimension γ was derived, which guarantees the validity of the all-orders expression, and which can be used to derive constraints on the anomalous dimension. We shall consider each of these points in turn.

3.2.1 An all-orders expression for A

The couplings in a general $\mathcal{N}=1$ theory are $g^I = \left\{g, Y^{ijk}, \bar{Y}_{ijk}\right\}$, where $\bar{Y}_{ijk} = (Y^{ijk})^*$, hence (2.1) can be written as

$$\begin{aligned} d_Y A &= dY\, T_{Y\bar{Y}}\, \beta_{\bar{Y}} + dY\, \tilde{T}_{Yg}\, \tilde{\beta}_g, \\ d_{\overline{Y}} A &= d\bar{Y}\, T_{\bar{Y}Y}\, \beta_Y + d\bar{Y}\, \tilde{T}_{\bar{Y}g}\, \tilde{\beta}_g, \\ d_g A &= dg\, T_{gY}\, \beta_Y + dg\, T_{g\bar{Y}}\, \beta_{\bar{Y}} + dg\, \tilde{T}_{gg} \tilde{\beta}_g, \end{aligned} \tag{3.61}$$

where we have used the definition of β_g in (3.59), and absorbed the $f(g)$ pre-factor into the tensor T. These equations can, of course, be solved perturbatively as in the non-supersymmetric case. Alternatively, by introducing the gaugino field λ_A,

[5] The extent to which Dimensional Reduction actually preserves supersymmetry is debated, as one may still encounter a supercurrent anomaly at sufficiently-high loop order [63].

[6] When performing this reduction, one must ensure that the choice of regularization method used for the non-supersymmetric theory is compatible with supersymmetry.

choosing the non-supersymmetric field multiplets to be

$$\phi_a \to \begin{pmatrix} \varphi_i \\ \bar{\varphi}^i \end{pmatrix}, \quad \bar{\varphi}^i = (\varphi_i)^*, \quad \psi_i \to \begin{pmatrix} \psi_i \\ \lambda_A \end{pmatrix}, \quad i = 1 \dots n_C, \tag{3.62}$$

and expanding the Yukawa couplings $y_a\phi_a = y^i\varphi_i + \bar{y}_i\bar{\varphi}^i$ according to

$$y^i \to \begin{pmatrix} Y^{ijk} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2}g(R_B)_j{}^i \\ 0 & 0 & \sqrt{2}g(R_A^T)^i{}_k & 0 \end{pmatrix},$$

$$\bar{y}_i \to \begin{pmatrix} 0 & \sqrt{2}g(R_B^T)^j{}_i & 0 & 0 \\ \sqrt{2}g(R_A)_i{}^k & 0 & 0 & 0 \\ 0 & 0 & \bar{Y}_{ijk} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{3.63}$$

where the non-supersymmetric gauge generators are given by

$$t_A^\varphi \to \begin{pmatrix} R_A & 0 \\ 0 & -R_A^T \end{pmatrix}, \quad t_A^\psi \to \begin{pmatrix} R_A & 0 \\ 0 & R_A^{\text{ad}} \end{pmatrix}, \quad (R_A^{\text{ad}})_{BC} = -if_{ABC}, \tag{3.64}$$

then we may simply expand our non-supersymmetric A-function (3.33) and simplify to obtain the supersymmetric result (again neglecting purely-gauge contributions). Expressing the supersymmetric A-function in the form

$$A^{(4)}_{SUSY} = \sum_{i=1}^{8} s_i^{(4)} S_i^{(4)} + \alpha\beta_Y^{ijk}\beta^{\bar{Y}}_{ijk}, \tag{3.65}$$

where the tensor structures $S_i^{(4)}$ are given in Table 3.4, and expanding out the non-supersymmetric tensor structures in Table 3.3 using (3.62 – 3.64), we find

$$\begin{aligned}
A_1^{(4)} &\to 2S_1^{(4)} - 18S_3^{(4)} + 12S_4^{(4)} + 12S_5^{(4)} - 24S_6^{(4)} - 6(C_G - 2T_R)S_7^{(4)} + 8S_8^{(4)} + \mathcal{O}(g^6), \\
A_2^{(4)} &\to 6S_2^{(4)} + 12S_3^{(4)} - 24S_4^{(4)} - 36S_5^{(4)} + 6(C_G - 2T_R)S_7^{(4)} + \mathcal{O}(g^6), \\
A_3^{(4)} &\to -5S_3^{(4)} + 16S_5^{(4)} - 16S_6^{(4)} + 4C_GS_7^{(4)} + 2S_8^{(4)} + \mathcal{O}(g^6), \\
A_4^{(4)} &\to 18S_5^{(4)} - 12S_6^{(4)} + 6C_GS_7^{(4)} + 2S_8^{(4)} + \mathcal{O}(g^6), \\
A_5^{(4)} &\to 2S_1^{(4)} - 12S_3^{(4)} + 24S_5^{(4)} + \mathcal{O}(g^6), \\
A_6^{(4)} &\to 2S_2^{(4)} - 8S_4^{(4)} + 8S_6^{(4)} - 16T_RS_7^{(4)} + \mathcal{O}(g^6), \\
A_7^{(4)} &\to 2S_2 - 4S_3^{(4)} - 12S_4^{(4)} + 24S_5^{(4)} + 16S_6^{(4)} - 8T_RS_7^{(4)} + \mathcal{O}(g^6), \\
A_8^{(4)} &\to 2S_1^{(4)} - 24S_3^{(4)} + 96S_5^{(4)} + \mathcal{O}(g^6),
\end{aligned}$$

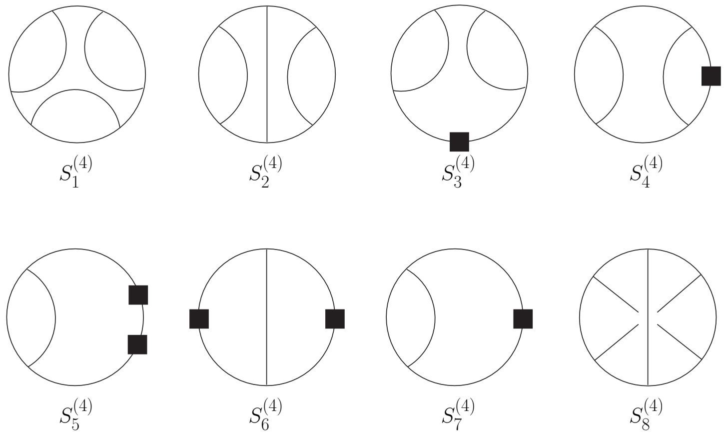


Table 3.4: Contributions to $A^{(4)}_{SUSY}$

$$
\begin{aligned}
A_9^{(4)} &\to 2S_3^{(4)} + 4S_4^{(4)} - 4S_5^{(4)} - 8S_6^{(4)} + 4T_R S_7^{(4)} + \mathcal{O}(g^6),\\
A_{10}^{(4)} &\to -2S_3^{(4)} + 8S_5^{(4)} + \mathcal{O}(g^6),\\
A_{11}^{(4)} &\to 8S_3^{(4)} - 32S_5^{(4)} + \mathcal{O}(g^6),\\
A_{12}^{(4)} &\to -6S_4^{(4)} - 12S_5^{(4)} + \mathcal{O}(g^6),\\
A_{13}^{(4)} &\to -2S_3^{(4)} + 8S_5^{(4)} + \mathcal{O}(g^6),\\
A_{14}^{(4)} &\to -2S_4^{(4)} + 4S_6^{(4)} - 4C_G S_7^{(4)} + \mathcal{O}(g^6),\\
A_{15}^{(4)} &\to -2S_4^{(4)} + 4S_5^{(4)} + 4S_6^{(4)} + \mathcal{O}(g^6),\\
A_{16}^{(4)} &\to -2S_5^{(4)} - 4S_6^{(4)} + 2C_G S_7^{(4)} + \mathcal{O}(g^6),\\
A_{17}^{(4)} &\to -8S_5^{(4)} + \mathcal{O}(g^6),\\
A_{18}^{(4)} &\to -\tfrac{1}{2}S_3^{(4)} - 2C_G S_7^{(4)} + \mathcal{O}(g^6),\\
A_{19}^{(4)} &\to -2S_4^{(4)} + 4S_5^{(4)} + 8S_6^{(4)} - 4C_G S_7^{(4)} + \mathcal{O}(g^6),\\
A_{20}^{(4)} &\to -2S_3^{(4)} + 16S_5^{(4)} + \mathcal{O}(g^6),\\
A_{21}^{(4)} &\to 3S_5^{(4)} - 2S_6^{(4)} + \tfrac{1}{2}C_G S_7^{(4)} + \mathcal{O}(g^6),\\
A_{22}^{(4)} &\to 2S_5^{(4)} + \mathcal{O}(g^6),\\
A_{23}^{(4)} &\to 2S_6^{(4)} + \mathcal{O}(g^6),\\
A_{24}^{(4)} &\to 2S_5^{(4)} + \mathcal{O}(g^6),\\
A_{25}^{(4)} &\to 2S_6^{(4)} + \mathcal{O}(g^6),\\
A_{26}^{(4)} &\to -2S_7^{(4)} + \mathcal{O}(g^6),\\
A_{27}^{(4)} &\to -2S_7^{(4)} + \mathcal{O}(g^6),\\
A_{28}^{(4)} &\to \tfrac{3}{2}S_1^{(4)} + 3S_2^{(4)} + 12S_3^{(4)} + 24S_4^{(4)} + 24S_5^{(4)} + 48S_6^{(4)} + \mathcal{O}(g^6).
\end{aligned}
\tag{3.66}
$$

Using (3.37), the $A^{(4)}_{SUSY}$ coefficients $s_i^{(4)}$ are therefore

$$
s_1^{(4)} = -\frac{1}{24}\mu c_4^{(2)} - \frac{4}{3}\mu c_8^{(2)} + \frac{2}{3}\mu c_{10}^{(2)} - \frac{1}{6}\mu c_{24}^{(2)} + \frac{3}{2}a,
$$

$$
\begin{aligned}
s_2^{(4)} &= -\frac{1}{8}\mu c_4^{(2)} - 2\mu c_8^{(2)} + 2\mu c_{10}^{(2)} + 3a, \\
s_3^{(4)} &= -\mu c_3^{(2)} - \frac{9}{8}\mu c_4^{(2)} - \frac{1}{2}\mu c_7^{(2)} - 20\mu c_8^{(2)} + 2\mu c_9^{(2)} + 4\mu c_{10}^{(2)} + \mu c_{12}^{(2)} - \mu c_{13}^{(2)} \\
&\quad - \mu c_{14}^{(2)} - \frac{1}{4}\mu c_{15}^{(2)} - \frac{1}{2}\mu c_{16}^{(2)} + \frac{1}{2}\mu c_{17}^{(2)} + 2\mu c_{24}^{(2)} + 12a, \\
s_4^{(4)} &= -\frac{1}{2}\mu c_4^{(2)} - 40\mu c_8^{(2)} + 4\mu c_9^{(2)} + 2\mu c_{12}^{(2)} - 4\mu c_{14}^{(2)} - 2\mu c_{16}^{(2)} + 24a, \\
s_5^{(4)} &= 3\mu c_1^{(2)} + 4\mu c_3^{(2)} + \frac{3}{4}\mu c_4^{(2)} - 4\mu c_6^{(2)} + 4\mu c_7^{(2)} - 8\mu c_8^{(2)} + 8\mu c_9^{(2)} \\
&\quad - 32\mu c_{10}^{(2)} - 12\mu c_{11}^{(2)} + 4\mu c_{12}^{(2)} + 4\mu c_{13}^{(2)} - 4\mu c_{14}^{(2)} + 8\mu c_{16}^{(2)} \\
&\quad - 4\mu c_{17}^{(2)} - 2\mu c_{19}^{(2)} + 2\mu c_{22}^{(2)} - 8\mu c_{24}^{(2)} + \mu c_{27}^{(2)} + 24a, \\
s_6^{(4)} &= -2\mu c_1^{(2)} - 2\mu c_4^{(2)} - 32\mu c_8^{(2)} + 16\mu c_9^{(2)} - 32\mu c_{10}^{(2)} - 24\mu c_{11}^{(2)} + 4\mu c_{12}^{(2)} \\
&\quad - 8\mu c_{16}^{(2)} + 12\mu c_{17}^{(2)} - 4\mu c_{19}^{(2)} + \mu c_{21}^{(2)} + 2\mu c_{23}^{(2)} + 48a, \\
s_7^{(4)} &= (-8\mu c_8^{(2)} + 4\mu c_9^{(2)} - 8\mu c_{10}^{(2)})T_R + (\mu c_1^{(2)} + \frac{5}{8}\mu c_4^{(2)} - 24\mu c_8^{(2)} - 12\mu c_9^{(2)} \\
&\quad + 24\mu c_{10}^{(2)} + 12\mu c_{11}^{(2)} + 4\mu c_{12}^{(2)} - 8\mu c_{14}^{(2)} - \mu c_{15}^{(2)} + 2\mu c_{19}^{(2)})C_G \\
&\quad - 2\mu c_{20}^{(2)}|_{SUSY} - \mu c_{28}^{(2)}|_{SUSY}, \\
s_8^{(4)} &= \frac{1}{3}\mu c_1^{(2)} + \frac{1}{3}\mu c_4^{(2)}, \qquad (3.67)
\end{aligned}
$$

where $c_i^{(2)}|_{SUSY}$ indicates that the Casimir invariants in $c_{20}^{(2)}$, $c_{28}^{(2)}$ are replaced by their supersymmetric counterparts, according to

$$
R^\phi \to 2T_R, \qquad R^\psi \to T_R + C_G. \qquad (3.68)
$$

We may now compare with the exact form for the a-function of an $\mathcal{N} = 1$ supersymmetric theory, first proposed in [12]. For a theory with n_c chiral superfields, the a-function is conjectured to be

$$
A = \frac{1}{12}(n_c + 9n_v) - \frac{1}{2}tr(\gamma^2) + \frac{1}{3}tr(\gamma^3) + \Lambda \circ \beta_{\overline{Y}} + n_v \lambda \tilde{\beta}_g + \beta_Y \circ H \circ \beta_{\overline{Y}}, \quad (3.69)
$$

where Λ and λ are a tensor structure and scalar respectively. This proposal was shown to be consistent with the two-loop anomalous dimension for a general gauge theory [27][7], and with the three-loop anomalous dimension for the Wess-Zumino model [9]. In order to match the coefficients of our reduction (3.67) with a perturbative expansion of (3.69), we must first ensure that the regularization method and renormalization scheme are suitable for supersymmetric theories. In [28], it was shown that one may in fact use Dimensional Reduction to regularize non-supersymmetric theories, settling earlier conjectures about its validity [29];

[7] Our result is essentially a verification of this calculation, beginning instead from a completely general four-dimensional a-function, valid in arbitrary renormalization schemes.

the $\overline{\text{MS}}$ renormalization of this theory is then related by a coupling redefinition to the corresponding dimensionally-regularized theory [30], again renormalized in $\overline{\text{MS}}$. The $\beta_y^{(2)}$ coefficients in Dimensional Reduction are given [31] by a coupling redefinition of the form (3.46), with non-zero variations

$$\delta_1^{(1)} = \frac{1}{6} C_G, \quad \epsilon_4^{(2)} = -\frac{1}{2}, \quad \epsilon_5^{(2)} = 1. \tag{3.70}$$

The Dimensional Reduction coefficients that differ from the Dimensional Regularization results in (3.37) are therefore

$$\begin{aligned} &c_6^{(2)} = 2 \quad c_7^{(2)} = -1, \quad c_{13}^{(2)} = -\tfrac{5}{4}, \quad c_{14}^{(2)} = \tfrac{1}{4}, \quad c_{16}^{(2)} = \tfrac{7}{2}, \\ &c_{19}^{(2)} = 7, \quad c_{20}^{(2)} = -\tfrac{1}{12}\left(138 C_G - 12 R^{\psi} - 9 R^{\phi}\right), \\ &c_{26}^{(2)} = \tfrac{7}{2}, \quad c_{28}^{(2)} = \tfrac{1}{12}(59 C_G + 4 R^{\psi} + R^{\phi}), \end{aligned} \tag{3.71}$$

hence the $A_{SUSY}^{(4)}$ coefficients become

$$\begin{aligned} &s_1^{(4)} = \tfrac{1}{8}\mu + \tfrac{3}{2}a, \quad s_2^{(4)} = -\tfrac{1}{4}\mu + 3a, \quad s_3^{(4)} = 2\mu + 12a, \quad s_4^{(4)} = -2\mu + 24a, \\ &s_5^{(4)} = 10\mu + 24a, \quad s_6^{(4)} = -4\mu + 48a, \quad s_7^{(4)} = -\tfrac{3}{2}\mu Q, \quad s_8^{(4)} = 0. \end{aligned} \tag{3.72}$$

Expanding out the all-orders expression (3.69), using the anomalous dimension coefficients

$$\begin{aligned} \gamma^{(1)} &= \tfrac{1}{2}\bar{Y}_{ikl}Y^{klj} - 2g^2 C(R)_i{}^j, \\ \gamma^{(2)} &= -\tfrac{1}{2}\bar{Y}_{ikl}Y^{kmn}\bar{Y}_{mnp}Y^{lpj} + 2g^2\bar{Y}_{ikl}C(R)_m{}^k Y^{mlj} - g^2\bar{Y}_{ikl}Y^{klm}C(R)_m{}^j \\ &\quad + 4g^4 C(R)_i{}^k C(R)_k{}^j + 2Qg^4 C(R)_i{}^j, \end{aligned} \tag{3.73}$$

calculated with Dimensional Reduction and $\overline{\text{MS}}$ [32], and the $NSVZ$ formula for β_g[8], we see that (3.69) and (3.72) are indeed consistent, up to the purely-gauge terms that we have neglected. Two consistency checks in this result are:

- $s_8^{(4)} = 0$ identically, consistent with the absence of non-planar terms in (3.69) at this order;
- $s_7^{(4)}$ is proportional to Q, consistent with the expansion of the $\lambda\tilde{\beta}_g$ term in (3.69).

[8] Recall that the $NSVZ$ formula is predicated on a particular renormalization scheme, but since the gauge β-function is scheme-independent at two loops we are free to use it at this order [33].

3.2.2 The Λ-equation

We have seen that the conjectured all-orders expression for A, (3.69), is consistent with the reduction of (3.33) to the supersymmetric case, modulo purely-gauge terms that have been ignored throughout, where RG quantities are calculated using Dimensional Reduction with minimal subtraction, and we make use of the $NSVZ$ form of the gauge β-function. While this afford some comfort in the validity of (3.69), there is in fact a more promising approach, which serves to establish (3.69) to all orders.

In [9], it was shown that, subject to the constraint $\Lambda \circ \beta_{\bar{Y}} = \beta_Y \circ \bar{\Lambda}$, applying the chain rule to (3.69) gives[9]

$$\begin{aligned} d_Y A &= \mathrm{tr}\left[(d_Y\gamma)\left(\frac{1}{2}\bar{Y}\cdot\Lambda - 2\lambda C(R) - \gamma + \gamma^2\right)\right] + (d_Y\Lambda)\circ\beta_{\bar{Y}} + n_v(d_Y\lambda)\tilde{\beta}_g,\\ d_{\bar{Y}} A &= \mathrm{tr}\left[(d_{\bar{Y}}\gamma)\left(\frac{1}{2}\bar{\Lambda}\cdot Y - 2\lambda C(R) - \gamma + \gamma^2\right)\right] + \beta_Y\circ(d_{\bar{Y}}\bar{\Lambda}) + n_v(d_{\bar{Y}}\lambda)\tilde{\beta}_g,\\ d_g A &= \mathrm{tr}\left[(d_g\gamma)\left(\frac{1}{2}\bar{Y}\cdot\Lambda - 2\lambda C(R) - \gamma + \gamma^2\right)\right] + (d_g\Lambda)\circ\beta_{\bar{Y}} + n_v(d_g\lambda)\tilde{\beta}_g. \end{aligned} \tag{3.74}$$

Substituting these equations into (3.61), we see that if Λ, λ are required to satisfy

$$\frac{1}{2}\overline{Y}\cdot\Lambda - 2\lambda C(R)g^2 = \gamma - \gamma^2 + \Theta\cdot\beta_{\overline{Y}} + \Phi\tilde{\beta}_g, \tag{3.75}$$

then (3.69) will satisfy (3.61) with T_{IJ} satisfying

$$\begin{aligned} dY\, T_{Y\bar{Y}}\,\beta_{\bar{Y}} &= \mathrm{tr}\left[(d_Y\gamma)\Theta\cdot\beta_{\bar{Y}}\right] + (d_Y\Lambda)\circ\beta_{\bar{Y}},\\ d\bar{Y}\, T_{\bar{Y}Y}\,\beta_Y &= \mathrm{tr}\left[(d_{\bar{Y}}\gamma)\beta_Y\cdot\bar{\Theta}\right] + \beta_Y\circ(d_{\bar{Y}}\Lambda),\\ dY\,\tilde{T}_{Yg}\,\tilde{\beta}_g &= \mathrm{tr}\left[(d_Y\gamma)\theta\tilde{\beta}_g\right] + n_v(d_Y\lambda)\tilde{\beta}_g,\\ d\bar{Y}\,\tilde{T}_{\bar{Y}g}\,\tilde{\beta}_g &= \mathrm{tr}\left[(d_{\bar{Y}}\gamma)\theta\tilde{\beta}_g\right] + n_v(d_{\bar{Y}}\lambda)\tilde{\beta}_g,\\ dg\,T_{gY}\,\tilde{\beta}_Y &= 0,\\ dg\,T_{g\bar{Y}}\,\tilde{\beta}_{\bar{Y}} &= \mathrm{tr}\left[(d_g\gamma)\,\Theta\cdot\beta_{\bar{Y}}\right] + (d_g\Lambda)\circ\beta_{\bar{Y}},\\ dg\,\tilde{T}_{gg}\,\tilde{\beta}_g &= \mathrm{tr}\left[(d_g\gamma)\theta\tilde{\beta}_g\right] + n_v(d_g\lambda)\tilde{\beta}_g. \end{aligned} \tag{3.76}$$

and therefore will provide a proof of the strong a-theorem for a general supersymmetric gauge theory, to all orders of perturbation theory. Note also that there is

[9]Note that there is a normalization factor $\Lambda \to \frac{1}{6}\Lambda$, for ease of comparison with [9, 19].

some freedom in T_{IJ}: by using the identity

$$\mathrm{tr}\left[(d_g\gamma)\,\frac{1}{2}\bar{Y}\cdot\Lambda\right]-\mathrm{tr}\left[(d_g\gamma)\,\frac{1}{2}\bar{\Lambda}\cdot Y\right]=\beta_Y\circ(d_g\bar{\Lambda})-(d_g\Lambda)\circ\beta_{\bar{Y}} \tag{3.77}$$

T may be rewritten such that

$$\begin{aligned} dg\,T_{gY}\,\beta_Y &= \frac{1}{2}\mathrm{tr}\left[(d_g\gamma)\,\bar{\Theta}\cdot\beta_Y\right]+\frac{1}{2}\,\beta_Y\circ(d_g\bar{\Lambda}) \\ dg\,T_{g\bar{Y}}\,\beta_{\bar{Y}} &= \frac{1}{2}\mathrm{tr}\left[(d_g\gamma)\,\Theta\cdot\beta_{\bar{Y}}\right]+\frac{1}{2}\,(d_g\Lambda)\circ\beta_{\bar{Y}}, \end{aligned} \tag{3.78}$$

and one may again investigate imposing symmetry at non-trivial loop orders. (3.75) shall henceforth be referred to as the "Λ-equation".

In much the same way that the gradient-flow equation (2.1) places restrictions on the form of the β-functions of a theory, the Λ-equation may be used to derive constraints on the form of the anomalous dimension. These constraints have already been derived in [9], for a general non-gauge $\mathcal{N}=1$ theory up to three loops, and it was shown that the constraints were satisfied by the three-loop anomalous dimension, calculated in Dimensional Reduction with minimal subtraction [34]. Here, we wish to extend these results to the gauge case, and to arbitrary renormalization schemes.

At leading order, we may perturbatively expand (3.75) as

$$\frac{1}{2}\overline{Y}\cdot\Lambda^{(1)}-2\lambda^{(1)}C(R)g^2=\gamma^{(1)}. \tag{3.79}$$

Given that the lowest possible order contribution to $\Lambda\circ\beta_{\overline{Y}}$ in (3.69) is simply

$$(\Lambda\circ\beta_{\overline{Y}})^{(3)}=\Lambda_1^{(1)}Y^{ijk}(\beta_{\overline{Y}}^{(1)})_{ijk}, \tag{3.80}$$

we find that

$$\overline{Y}\cdot\Lambda^{(1)}=\Lambda_1^{(1)}\overline{Y}_{ikl}Y^{klj}. \tag{3.81}$$

Parametrising the one-loop anomalous dimension as

$$\gamma^{(1)}=\gamma_1^{(1)}\overline{Y}_{ikl}Y^{klj}+\gamma_2^{(1)}C(R)_i{}^jg^2, \tag{3.82}$$

the solution to (3.75) at leading order is

$$\Lambda_1^{(1)}=2\gamma_1^{(1)},\quad \lambda^{(1)}=-\tfrac{1}{2}\gamma_2^{(1)}, \tag{3.83}$$

with no consistency conditions on $\gamma^{(1)}$.

At next-to-leading order, we may expand (3.75) as

$$\frac{1}{2}\overline{Y}\cdot\Lambda^{(2)} - 2\lambda^{(2)}C(R)g^2 = \gamma^{(2)} - (\gamma^{(1)})^2 + \Theta^{(1)}\cdot\beta_{\overline{Y}}^{(1)} + \Phi^{(1)}\tilde{\beta}_g^{(1)}. \tag{3.84}$$

The next-to-leading contributions to $\Lambda\circ\beta_{\overline{Y}}$ may be parametrized as

$$(\Lambda\circ\beta_{\overline{Y}})^{(4)} = \Lambda_1^{(2)}Y^{ikl}\overline{Y}_{imn}Y^{mnj}(\beta_{\overline{Y}}^{(1)})_{jkl} + \Lambda_2^{(2)}g^2Y^{ijk}C(R)_k{}^l(\beta_{\overline{Y}}^{(1)})_{ijl}, \tag{3.85}$$

hence

$$\begin{aligned}\overline{Y}\cdot\Lambda^{(2)} = \quad & \Lambda_1^{(2)}\left(\tfrac{2}{3}\overline{Y}_{ikl}Y^{lmn}\overline{Y}_{mnp}Y^{pkj} + \tfrac{1}{3}\overline{Y}_{ikl}Y^{klm}\overline{Y}_{mpq}Y^{pqj}\right)\\ & + \Lambda_2^{(2)}\left(\tfrac{2}{3}\overline{Y}_{ikl}C(R)^l{}_mY^{kmj} + \tfrac{1}{3}\overline{Y}_{ikl}Y^{klm}C(R)_m^{\ j}\right).\end{aligned} \tag{3.86}$$

Recalling that λ is a scalar quantity, the most general possible contribution to the λ-term at this order is simply

$$\left(\lambda C(R)g^2\right)^{(2)} = \lambda^{(2)}C(R)g^4. \tag{3.87}$$

The two-loop anomalous dimension is parametrised as

$$\gamma^{(2)} = \sum_{i=1}^{5}\gamma_i^{(2)}\Gamma_i^{(2)}, \tag{3.88}$$

where

$$\begin{aligned}&\Gamma_1^{(2)} = \overline{Y}_{ikl}Y^{lmn}\overline{Y}_{mnp}Y^{pkj}, && \Gamma_2^{(2)} = g^2\,\overline{Y}_{ikl}C(R)^l{}_mY^{kmj},\\ &\Gamma_3^{(2)} = g^2\,\overline{Y}_{ikl}Y^{klm}C(R)_m^{\ j}, && \Gamma_4^{(2)} = g^4\,C(R)_i{}^kC(R)_k{}^j,\\ &\Gamma_5^{(2)} = g^4\,C(R)_i{}^j,\end{aligned} \tag{3.89}$$

and the first contribution from $\Theta\cdot\beta_{\overline{Y}}$ is

$$\Theta^{(1)}\cdot\beta_{\overline{Y}}^{(1)} = \theta_1^{(1)}(\beta_{\overline{Y}}^{(1)})_{ikl}Y^{klj}. \tag{3.90}$$

At this order, we also have the first contribution from $\Phi\tilde{\beta}_g$. Recall $\tilde{\beta}_g$ is defined relative to the gauge β-function as

$$\beta_g = f(g)\,\tilde{\beta}_g, \quad f(g) = g^3 + \mathcal{O}(g^5),$$

where

$$\tilde{\beta}_g = Q - tr[\gamma\,C(R)], \quad Q = T_R - 3C_G,$$

and the $NSVZ$ β-function corresponds to

$$f(g) = \frac{g^3}{1 - 2g^2 C_G}.$$

We may therefore simply parametrise $\tilde{\beta}_g$ and $\Phi^{(1)}$ as

$$\tilde{\beta}_g^{(1)} = \tilde{d}_1^{(1)}, \qquad \Phi^{(1)} = \phi_1^{(1)} g^4 C(R)_i{}^j, \tag{3.91}$$

where $\tilde{d}_1^{(1)}|_{NSVZ} = Q$. Consequently, the first contribution from $\Phi\tilde{\beta}_g$ is

$$\Phi^{(1)}\tilde{\beta}_g^{(1)} = \phi_1^{(1)}\tilde{d}_1^{(1)} g^4\, C(R)_i{}^j. \tag{3.92}$$

With these parametrizations, we obtain from (3.84) a system of six equations,

$$\begin{aligned} \tfrac{1}{3}\Lambda_1^{(2)} &= \gamma_1^{(2)} + 2\gamma_1^{(1)}\theta_1^{(1)}, & \tfrac{1}{6}\Lambda_1^{(2)} &= \gamma_1^{(1)}\theta_1^{(1)} - \left(\gamma_1^{(1)}\right)^2, \\ \tfrac{1}{3}\Lambda_2^{(2)} &= \gamma_2^2 + 2\gamma_2^{(1)}\theta_1^{(1)}, & \tfrac{1}{6}\Lambda_2^{(2)} &= \gamma_3^{(2)} - 2\gamma_1^{(1)}\gamma_2^{(1)} + \gamma_2^{(1)}\theta_1^{(1)}, \\ 0 &= \gamma_4^{(2)} - \left(\gamma_2^{(1)}\right)^2, & -2\lambda^{(2)} &= \gamma_5^{(2)} + \phi_1^{(1)}\tilde{d}_1^{(1)}, \end{aligned} \tag{3.93}$$

from which we obtain three consistency conditions on the anomalous dimension,

$$\begin{aligned} \gamma_1^{(2)} + 2\left(\gamma_1^{(1)}\right)^2 &= 0, \\ \gamma_2^{(2)} - 2\gamma_3^{(2)} + 4\gamma_1^{(1)}\gamma_2^{(1)} &= 0, \\ \gamma_4^{(2)} - \left(\gamma_2^{(1)}\right)^2 &= 0. \end{aligned} \tag{3.94}$$

To check whether these conditions are scheme-independent, we must deduce the effects of a coupling redefinition on the anomalous dimension. Once again, this may be obtained by expanding out (3.43), where $\beta_Y^{ijk} = Y^{ljk}\gamma_l{}^i + Y^{ilk}\gamma_l{}^j + Y^{ijl}\gamma_l{}^k$ by the non-renormalization theorem; the change in the β-function coefficient $\delta\beta$ is then equal to the change $\delta\gamma$ in the corresponding anomalous dimension coefficient. Given a redefinition of the form

$$(\delta Y)^{(1)} = \epsilon_1^{(1)}\bar{Y}_{ikl}Y^{klj} + \epsilon_2^{(1)} g^2 C(R)_i{}^j, \qquad (\delta g)^{(1)} = \delta_1^{(1)} g^3, \tag{3.95}$$

the change in the two-loop anomalous dimension is

$$\begin{aligned} \delta\gamma_2^{(2)} &= 4(\gamma_2^{(1)}\epsilon_1^{(1)} - \gamma_1^{(1)}\epsilon_2^{(1)}), & \delta\gamma_3^{(2)} &= 2(\gamma_2^{(1)}\epsilon_1^{(1)} - \gamma_1^{(1)}\epsilon_2^{(1)}), \\ \delta\gamma_5^{(2)} &= 2(d_1^{(1)}\epsilon_2^{(1)} - \gamma_2^{(1)}\delta_1^{(1)}). \end{aligned} \tag{3.96}$$

Since $\delta\gamma_2^{(2)} = 2\left(\delta\gamma_3^{(2)}\right)$, we see that the consistency conditions are indeed scheme-independent.

Finally, at next-to-next-to-leading order, we may perturbatively expand (3.75) as

$$\frac{1}{2}\overline{Y}\cdot\Lambda^{(3)} - 2\lambda^{(3)}C(R)g^2 = \gamma^{(3)} - \gamma^{(2)}\gamma^{(1)} - \gamma^{(1)}\gamma^{(2)} + \Theta^{(2)}\cdot\beta^{(1)}_{\overline{Y}} + \Theta^{(1)}\cdot\beta^{(2)}_{\overline{Y}} + \Phi^{(2)}\tilde{\beta}^{(1)}_g + \Phi^{(1)}\tilde{\beta}^{(2)}_g. \tag{3.97}$$

To facilitate the perturbative expansion of the right-hand-side of (3.75), we first define the tensor structures appearing in the anomalous dimension. The three-loop anomalous dimension takes the form

$$\gamma^{(3)} = \sum_{i=1}^{19}\gamma_i^{(3)}\Gamma_i^{(3)}, \tag{3.98}$$

where the tensor structures $\Gamma_i^{(3)}$ are given by

$$\begin{aligned}
\Gamma_1^{(3)} &= \overline{Y}_{ikl}Y^{lmn}\overline{Y}_{mnp}Y^{pqr}\overline{Y}_{qrs}Y^{skj}, & \Gamma_2^{(3)} &= \overline{Y}_{ikl}Y^{kmn}\overline{Y}_{mnp}Y^{lqr}\overline{Y}_{qrs}Y^{psj},\\
\Gamma_3^{(3)} &= \overline{Y}_{ikl}Y^{kmn}\overline{Y}_{mpq}Y^{pqr}\overline{Y}_{rns}Y^{slj}, & \Gamma_4^{(3)} &= \overline{Y}_{ikl}Y^{kmn}\overline{Y}_{mpq}Y^{lqr}\overline{Y}_{nrs}Y^{psj},\\
\Gamma_5^{(3)} &= g^2\overline{Y}_{ikl}Y^{kmn}\overline{Y}_{mnp}C(R)^p{}_qY^{qlj}, & \Gamma_6^{(3)} &= g^2\overline{Y}_{ikl}Y^{kmn}\overline{Y}_{mnp}C(R)^l{}_qY^{pqj},\\
\Gamma_7^{(3)} &= g^2\overline{Y}_{ikl}Y^{kmn}C(R)_m{}^p\overline{Y}_{npq}Y^{qlj}, & \Gamma_8^{(3)} &= g^2\overline{Y}_{ikl}Y^{kmn}\overline{Y}_{mnp}Y^{lpq}C(R)_q{}^j,\\
\Gamma_9^{(3)} &= g^4\overline{Y}_{ikl}C(R)^k{}_mC(R)^m{}_nY^{nlj}, & \Gamma_{10}^{(3)} &= g^4\overline{Y}_{ikl}C(R)^k{}_mC(R)^l{}_nY^{mnj},\\
\Gamma_{11}^{(3)} &= g^4\overline{Y}_{ikl}C(R)^k{}_mY^{mln}C(R)_n{}^j, & \Gamma_{12}^{(3)} &= g^4\overline{Y}_{ikl}Y^{klm}C(R)_m{}^nC(R)_n{}^j,\\
\Gamma_{13}^{(3)} &= g^4tr[\overline{Y}YC(R)]C(R)_i{}^j, & \Gamma_{14}^{(3)} &= g^4\overline{Y}_{ikl}C(R)^k{}_mY^{mlj},\\
\Gamma_{15}^{(3)} &= g^4\overline{Y}_{ikl}Y^{klm}C(R)_m{}^j, & \Gamma_{16}^{(3)} &= g^6C(R)_i{}^kC(R)_k{}^lC(R)_l{}^j,\\
\Gamma_{17}^{(3)} &= g^6tr[C(R)C(R)]C(R)_i{}^j, & \Gamma_{18}^{(3)} &= g^6C(R)_i{}^kC(R)_k{}^j,\\
\Gamma_{19}^{(3)} &= g^6C(R)_i{}^j.
\end{aligned} \tag{3.99}$$

Additionally, the following six one-particle-reducible tensor structures also appear:

$$\begin{aligned}
\Gamma_\alpha^{(3)} &= \overline{Y}_{ikl}Y^{kmn}\overline{Y}_{mnp}Y^{lpq}\overline{Y}_{qrs}Y^{rsj}, & \Gamma_\beta^{(3)} &= \overline{Y}_{ikl}Y^{klm}\overline{Y}_{mnp}Y^{pqr}\overline{Y}_{qrs}Y^{snj},\\
\Gamma_\gamma^{(3)} &= g^2\overline{Y}_{ikl}C(R)^k{}_mY^{lmn}\overline{Y}_{npq}Y^{pqj}, & \Gamma_\delta^{(3)} &= g^2\overline{Y}_{ikl}Y^{klm}\overline{Y}_{mnp}C(R)^p{}_qY^{nqj},\\
\Gamma_\epsilon^{(3)} &= \overline{Y}_{ikl}Y^{klm}\overline{Y}_{mnp}Y^{npq}\overline{Y}_{qrs}Y^{rsj}, & \Gamma_\zeta^{(3)} &= g^2\overline{Y}_{ikl}Y^{klm}\overline{Y}_{mpq}Y^{pqn}C(R)_n{}^j.
\end{aligned} \tag{3.100}$$

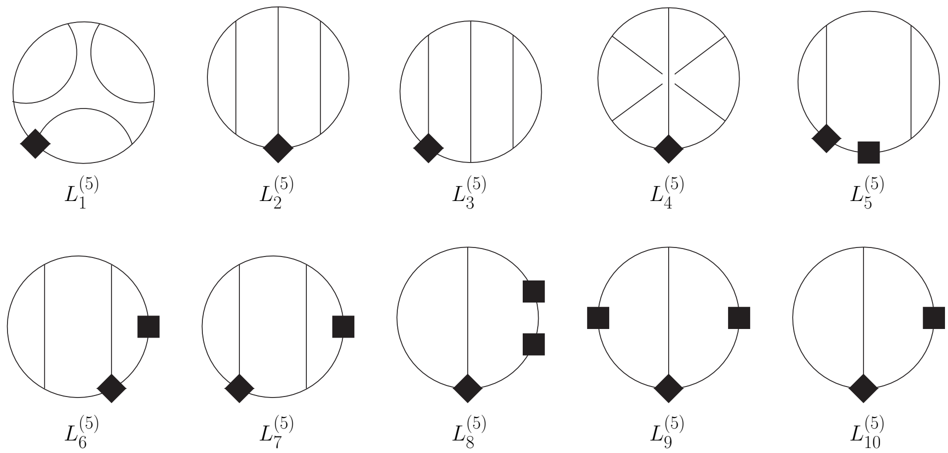


Table 3.5: Next-to-next-to-leading order contributions to $\Lambda \circ \beta_{\overline{Y}}$

At this order, the contributions to $\Lambda \circ \beta_{\overline{Y}}$ may be expressed as

$$(\Lambda \circ \beta_{\overline{Y}})^{(5)} = \sum_{i=1}^{10} \Lambda_i^{(3)} L_i^{(5)}, \tag{3.101}$$

where the tensor structures $L_i^{(5)}$ are depicted diagrammatically in Table 3.5, with vertices representing Y, $\overline{Y}$ in an alternating manner, a diamond representing $\beta_{\overline{Y}}^{(1)}$, and a box representing $g^2C(R)$; for example, the diagram $L_5^{(5)}$ represents the tensor structure

$$L_5^{(5)} \to g^2(\beta_{\overline{Y}}^{(1)})_{ijk} Y^{jkl} \overline{Y}_{lmn} Y^{mnp} C(R)_p{}^i.$$

Consequently, $\overline{Y} \cdot \Lambda^{(3)}$ is given by

$$\begin{aligned}
\overline{Y} \cdot \Lambda^{(3)} = \quad & \Lambda_1^{(3)} \left(\tfrac{2}{3}\Gamma_1^{(3)} + \tfrac{1}{3}\Gamma_\epsilon^{(3)}\right) + \Lambda_2^{(3)} \left(\tfrac{1}{3}\Gamma_2^{(3)} + \tfrac{2}{3}\Gamma_\alpha^{(3)}\right) \\
& + \Lambda_3^{(3)} \left(\tfrac{2}{3}\Gamma_3^{(3)} + \tfrac{1}{3}\Gamma_\beta^{(3)}\right) + \Lambda_4^{(3)}\Gamma_4^{(3)} \\
& + \Lambda_5^{(3)} \left(\tfrac{2}{3}\Gamma_5^{(3)} + \tfrac{1}{3}\Gamma_\zeta^{(3)}\right) + \Lambda_6^{(3)} \left(\tfrac{1}{3}\Gamma_6^{(3)} + \tfrac{1}{3}\Gamma_8^{(3)} + \tfrac{1}{3}\Gamma_\gamma^{(3)}\right) \\
& + \Lambda_7^{(3)} \left(\tfrac{2}{3}\Gamma_7^{(3)} + \tfrac{1}{3}\Gamma_\delta^{(3)}\right) + \Lambda_8^{(3)} \left(\tfrac{2}{3}\Gamma_9^{(3)} + \tfrac{1}{3}\Gamma_{12}^{(3)}\right) \\
& + \Lambda_9^{(3)} \left(\tfrac{2}{3}\Gamma_{10}^{(3)} + \tfrac{1}{3}\Gamma_{11}^{(3)}\right) + \Lambda_{10}^{(3)} \left(\tfrac{2}{3}\Gamma_{14}^{(3)} + \tfrac{1}{3}\Gamma_{15}^{(3)}\right).
\end{aligned} \tag{3.102}$$

The most general contribution from the λ-term is given by

$$\left(\lambda C(R) g^2\right)^{(3)} = \lambda_1^{(3)}\Gamma_{13}^{(3)} + \lambda_2^{(3)}\Gamma_{17}^{(3)} + \lambda_3^{(3)} C(R) g^6, \tag{3.103}$$

and the contributions from $\Theta \cdot \beta_{\overline{Y}}$ are

$$(\Theta \cdot \beta_{\overline{Y}})^{(3)} = \quad \Theta^{(1)} \cdot \beta_{\overline{Y}}^{(2)} + \Theta^{(2)} \cdot \beta_{\overline{Y}}^{(1)}$$

$$\begin{aligned}
= \quad & \theta_1^{(1)}\Big(2\gamma_1^{(2)}\Gamma_3^{(3)} + 2\gamma_3^{(2)}\Gamma_5^{(3)} + 2\gamma_2^{(2)}\Gamma_7^{(3)} + 2\gamma_4^{(2)}\Gamma_9^{(3)} + \gamma_4^{(2)}\Gamma_{12}^{(3)} \\
& \quad + 2\gamma_5^{(2)}\Gamma_{14}^{(3)} + \gamma_5^{(2)}\Gamma_{15}^{(3)} + \gamma_1^{(2)}\Gamma_\alpha^{(3)} + \gamma_2^{(2)}\Gamma_\gamma^{(3)} + \gamma_3^{(2)}\Gamma_\zeta^{(3)}\Big) \\
& + \theta_1^{(2)}\Big(2\gamma_1^{(1)}\Gamma_\alpha^{(3)} + 2\gamma_2^{(1)}\Gamma_\gamma^{(3)} + \gamma_1^{(1)}\Gamma_\epsilon^{(3)} + \gamma_2^{(1)}\Gamma_\zeta^{(3)}\Big) \\
& + \theta_2^{(2)}\Big(2\gamma_1^{(1)}\Gamma_\beta^{(3)} + 2\gamma_2^{(1)}\Gamma_\delta^{(3)} + \gamma_1^{(1)}\Gamma_\epsilon^{(3)} + \gamma_2^{(1)}\Gamma_\zeta^{(3)}\Big) \\
& + \theta_3^{(2)}\Big(\gamma_1^{(1)}\Gamma_1^{(3)} + \gamma_1^{(1)}\Gamma_2^{(3)} + \gamma_2^{(1)}\Gamma_5^{(3)} + \gamma_2^{(1)}\Gamma_6^{(3)} + \gamma_2^{(1)}\Gamma_8^{(3)} + \gamma_1^{(1)}\Gamma_\beta^{(3)}\Big) \\
& + \theta_4^{(2)}\Big(\gamma_1^{(1)}\Gamma_1^{(3)} + 2\gamma_1^{(1)}\Gamma_3^{(3)} + \gamma_2^{(1)}\Gamma_5^{(3)} + 2\gamma_2^{(1)}\Gamma_7^{(3)}\Big) \\
& + \theta_5^{(2)}\Big(2\gamma_1^{(1)}\Gamma_8^{(3)} + 2\gamma_2^{(1)}\Gamma_{11}^{(3)} + \gamma_2^{(1)}\Gamma_{12}^{(3)} + \gamma_1^{(1)}\Gamma_\zeta^{(3)}\Big) \\
& + \theta_6^{(2)}\Big(\gamma_1^{(1)}\Gamma_5^{(3)} + \gamma_1^{(1)}\Gamma_6^{(3)} + \gamma_2^{(1)}\Gamma_9^{(3)} + \gamma_2^{(1)}\Gamma_{10}^{(3)} + \gamma_2^{(1)}\Gamma_{11}^{(3)} + \gamma_2^{(1)}\Gamma_\delta^{(3)}\Big).
\end{aligned} \tag{3.104}$$

The last contributions are from $\Phi\tilde{\beta}_g$, and take the form

$$\begin{aligned}
\left(\Phi\tilde{\beta}_g\right)^{(3)} = \quad & \Phi^{(1)}\tilde{\beta}_g^{(2)} + \Phi^{(2)}\tilde{\beta}_g^{(1)} \\
= \quad & \phi_1^{(1)}\Big(\tilde{d}_1^{(2)}\Gamma_{19}^{(3)} + \tilde{d}_2^{(2)}\Gamma_{18}^{(3)} + \tilde{d}_3^{(2)}\Gamma_{17}^{(3)}\Big) \\
& + \phi_1^{(2)}\tilde{d}_1^{(1)}\Gamma_{14}^{(3)} + \phi_2^{(2)}\tilde{d}_1^{(1)}\Gamma_{15}^{(3)} + \phi_3^{(2)}\tilde{d}_1^{(1)}\Gamma_{18}^{(3)} + \phi_4^{(2)}\tilde{d}_1^{(1)}\Gamma_{19}^{(3)}.
\end{aligned} \tag{3.105}$$

Given these parametrizations, we obtain from (3.97) a system of 25 equations, listed in (A.3), which impose the following consistency conditions:

$$\begin{aligned}
\gamma_{16}^{(3)} + 2(\gamma_1^{(1)})^2 &= 0, \\
2\gamma_1^{(3)} - 4\gamma_2^{(3)} - \gamma_3^{(3)} + 8(\gamma_1^{(1)})^3 &= 0, \\
2\gamma_2^{(1)}\gamma_1^{(3)} - 2\gamma_1^{(1)}\gamma_6^{(3)} - \gamma_1^{(1)}\gamma_7^{(3)} - 4(\gamma_1^{(1)})^2\gamma_2^{(2)} &= 0, \\
\gamma_2^{(1)}\gamma_1^{(3)} - \gamma_1^{(1)}\gamma_5^{(3)} + \gamma_1^{(1)}\gamma_6^{(3)} - \gamma_1^{(1)}\gamma_8^{(3)} - 2(\gamma_1^{(1)})^2\gamma_2^{(2)} - 12(\gamma_1^{(1)})^3\gamma_2^{(1)} &= 0, \\
\gamma_2^{(1)}\gamma_6^{(3)} - \gamma_2^{(1)}\gamma_8^{(3)} - \gamma_1^{(1)}\gamma_9^{(3)} + 2\gamma_1^{(1)}\gamma_{12}^{(3)} - 2\gamma_1^{(1)}\gamma_2^{(1)}\gamma_2^{(2)} - 16(\gamma_1^{(1)})^2(\gamma_2^{(1)})^2 &= 0.
\end{aligned} \tag{3.106}$$

The scheme-independence of the consistency conditions may again be verified by expanding (3.43) to third order, albeit with some subtleties. The two-loop redefinition of the gauge coupling may simply be parametrised as

$$(\delta g)^{(2)} = \delta_1^{(2)} g^5 + \delta_2^{(2)} g^5 \mathrm{tr}\,[C(R)C(R)] + \delta_3^{(2)}\mathrm{tr}\left[\bar{Y}YC(R)\right], \tag{3.107}$$

and part of the two-loop redefinition for the Yukawa coupling as

$$(\delta Y)^{(2)} = \sum_{i=1}^{5} \epsilon_i^{(2)} \left(Y * \Gamma_i^{(2)} \right) + \ldots, \tag{3.108}$$

where $(Y * a)^{ijk} \equiv Y^{ljk} a_l{}^i + Y^{ilk} a_l{}^j + Y^{ijl} a_l{}^k$, and $\Gamma_i^{(2)}$ are defined in (3.89). However, including only these terms leads to nineteen $\beta_Y^{(3)}$ contributions of the form $Y^{ilm} a_l{}^j b_m{}^k$, which are not determined solely by the anomalous dimension, violating the non-renormalization theorem. This may be corrected by including extra terms in $(\delta Y)^{(2)}$, such that

$$(\delta Y)^{(2)} = \sum_{i=1}^{5} \epsilon_i^{(2)} \left(Y * \Gamma_i^{(2)} \right) + \sum_{i=1}^{5} \rho_i^{(2)} \Gamma_{Ri}^{(2)}, \tag{3.109}$$

where

$$\begin{aligned}
\Gamma_{R1}^{(2)} &= Y^{ijl} \bar{Y}_{lmn} Y^{mnp} \bar{Y}_{pqr} Y^{qrj}, & \Gamma_{R2}^{(2)} &= Y^{ilp} \bar{Y}_{lmn} Y^{mnj} \bar{Y}_{pqr} Y^{qrk}, \\
\Gamma_{R3}^{(2)} &= g^2\, Y^{ilp} \bar{Y}_{lmn} Y^{mnj} C(R)_p{}^k, & \Gamma_{R4}^{(2)} &= g^2\, Y^{ilm} C(R)_l{}^j \bar{Y}_{mpq} Y^{pqk}, \\
\Gamma_{R5}^{(2)} &= g^4\, Y^{ilm} C(R)_l{}^j C(R)_m{}^k, & &
\end{aligned} \tag{3.110}$$

and each $\Gamma_{Ri}^{(2)}$ is understood as being symmetrised over the three indices. The coefficients $\rho_{2-5}^{(2)}$ may then be chosen to cancel off the unwanted contributions to $\beta_Y^{(3)}$, preserving the non-renormalization theorem under a change of renormalization scheme. We also expect that the anomalous dimension will consist only of 1PI contributions, allowing us to fix $\rho_1^{(2)}$. Given these criteria, the coefficients $\rho_i^{(2)}$ must take the form

$$\begin{aligned}
\rho_1^{(2)} &= \tfrac{3}{2} (\epsilon_1^{(1)})^2, & \rho_2^{(2)} &= (\epsilon_1^{(1)})^2, & \rho_3^{(2)} &= \epsilon_1^{(1)} \epsilon_2^{(1)}, \\
\rho_4^{(2)} &= \epsilon_1^{(1)} \epsilon_2^{(1)}, & \rho_5^{(2)} &= (\epsilon_2^{(1)})^2, & &
\end{aligned} \tag{3.111}$$

and the anomalous dimension coefficients transform as

$$\begin{aligned}
\delta\gamma_1^{(3)} &= 4\gamma_1^{(1)} \epsilon_1^{(2)} - 4\gamma_1^{(2)} \epsilon_1^{(1)} - 8\gamma_1^{(1)} (\epsilon_1^{(1)})^2, \\
\delta\gamma_2^{(3)} &= 2\gamma_1^{(1)} \epsilon_1^{(2)} - 2\gamma_1^{(2)} \epsilon_1^{(1)} - 4\gamma_1^{(1)} (\epsilon_1^{(1)})^2, \\
\delta\gamma_3^{(3)} &= 0, \\
\delta\gamma_4^{(3)} &= 0, \\
\delta\gamma_5^{(3)} &= 4\gamma_3^{(2)} \epsilon_1^{(1)} - 4\gamma_1^{(1)} \epsilon_3^{(2)} + 4\gamma_2^{(1)} \epsilon_1^{(2)} - 4\gamma_1^{(2)} \epsilon_2^{(1)} + 2\gamma_1^{(1)} \epsilon_2^{(2)} - 2\gamma_2^{(2)} \epsilon_1^{(1)} \\
&\quad + 4\gamma_1^{(1)} \epsilon_1^{(1)} \epsilon_2^{(1)} - 8\gamma_2^{(1)} (\epsilon_1^{(1)})^2, \\
\delta\gamma_6^{(3)} &= 2\gamma_2^{(1)} \epsilon_1^{(2)} - 2\gamma_1^{(2)} \epsilon_2^{(1)} + 2\gamma_1^{(1)} \epsilon_2^{(2)} - 2\gamma_2^{(2)} \epsilon_1^{(1)} - 8\gamma_2^{(1)} (\epsilon_1^{(1)})^2,
\end{aligned}$$

$$
\begin{aligned}
\delta\gamma_7^{(3)} &= 4\gamma_2^{(2)}\epsilon_1^{(1)} - 4\gamma_1^{(1)}\epsilon_2^{(2)} + 4\gamma_2^{(1)}\epsilon_1^{(2)} - 4\gamma_1^{(2)}\epsilon_2^{(1)},\\
\delta\gamma_8^{(3)} &= 2\gamma_2^{(1)}\epsilon_1^{(2)} - 2\gamma_1^{(2)}\epsilon_2^{(1)} + 4\gamma_1^{(1)}\epsilon_3^{(2)} - 4\gamma_3^{(2)}\epsilon_1^{(1)} - 4\gamma_1^{(1)}\epsilon_1^{(1)}\epsilon_2^{(1)}\\
&\quad - 8\gamma_2^{(1)}(\epsilon_1^{(1)})^2,\\
\delta\gamma_9^{(3)} &= 4\gamma_4^{(2)}\epsilon_1^{(1)} - 4\gamma_1^{(1)}\epsilon_4^{(2)} + 2\gamma_2^{(1)}\epsilon_2^{(2)} - 2\gamma_2^{(2)}\epsilon_2^{(1)} - 8\gamma_2^{(1)}\epsilon_1^{(1)}\epsilon_2^{(1)}\\
&\quad + 6\gamma_1^{(1)}(\epsilon_2^{(1)})^2,\\
\delta\gamma_{10}^{(3)} &= 2\gamma_2^{(1)}\epsilon_2^{(2)} - 2\gamma_2^{(2)}\epsilon_2^{(1)} - 8\gamma_2^{(1)}\epsilon_1^{(1)}\epsilon_2^{(1)} + 4\gamma_1^{(1)}(\epsilon_2^{(1)})^2,\\
\delta\gamma_{11}^{(3)} &= 2\gamma_2^{(1)}\epsilon_2^{(2)} - 2\gamma_2^{(2)}\epsilon_2^{(1)} + 4\gamma_2^{(1)}\epsilon_3^{(2)} - 4\gamma_3^{(2)}\epsilon_2^{(1)} - 20\gamma_2^{(1)}\epsilon_1^{(1)}\epsilon_2^{(1)}\\
&\quad - 10\gamma_1^{(1)}(\epsilon_2^{(1)})^2,\\
\delta\gamma_{12}^{(3)} &= 2\gamma_4^{(2)}\epsilon_1^{(1)} - 2\gamma_1^{(1)}\epsilon_4^{(2)} + 2\gamma_2^{(1)}\epsilon_3^{(2)} - 2\gamma_3^{(2)}\epsilon_2^{(1)} - 6\gamma_2^{(1)}\epsilon_1^{(1)}\epsilon_2^{(1)}\\
&\quad + 3\gamma_1^{(1)}(\epsilon_2^{(1)})^2,\\
\delta\gamma_{13}^{(3)} &= 2d_3^{(2)}\epsilon_2^{(1)} - 2\gamma_2^{(1)}\delta_3^{(2)},\\
\delta\gamma_{14}^{(3)} &= 4\gamma_5^{(2)}\epsilon_1^{(1)} - 4\gamma_1^{(1)}\epsilon_5^{(2)} + 2d_1^{(1)}\epsilon_2^{(2)} - 2\gamma_2^{(2)}\delta_1^{(1)} + 8\gamma_1^{(1)}\delta_1^{(1)}\epsilon_2^{(1)}\\
&\quad - 8\gamma_2^{(1)}\delta_1^{(1)}\epsilon_1^{(1)},\\
\delta\gamma_{15}^{(3)} &= 2\gamma_5^{(2)}\epsilon_1^{(1)} - 2\gamma_1^{(1)}\epsilon_5^{(2)} + 2d_1^{(1)}\epsilon_3^{(2)} - 2\gamma_3^{(2)}\delta_1^{(1)} + 4\gamma_1^{(1)}\delta_1^{(1)}\epsilon_2^{(1)}\\
&\quad - 4\gamma_2^{(1)}\delta_1^{(1)}\epsilon_1^{(1)} - 2d_1^{(1)}\epsilon_1^{(1)}\epsilon_2^{(1)},\\
\delta\gamma_{16}^{(3)} &= 0,\\
\delta\gamma_{17}^{(3)} &= 2d_2^{(2)}\epsilon_2^{(1)} - 2\gamma_2^{(1)}\delta_2^{(2)},\\
\delta\gamma_{18}^{(3)} &= 4d_1^{(1)}\epsilon_4^{(2)} - 4\gamma_4^{(2)}\delta_1^{(1)} - 2d_1^{(1)}(\epsilon_2^{(1)})^2,\\
\delta\gamma_{19}^{(3)} &= 2d_1^{(2)}\epsilon_2^{(1)} - 2\gamma_2^{(1)}\delta_1^{(2)} + 4d_2^{(1)}\epsilon_5^{(2)} - 4\gamma_5^{(2)}\delta_1^{(1)} - 8d_1^{(1)}\delta_1^{(1)}\epsilon_2^{(1)}\\
&\quad + 7\gamma_2^{(1)}(\delta_1^{(1)})^2,
\end{aligned}
\tag{3.112}
$$

from which we can see that the consistency conditions (3.106) are scheme-independent. Strangely, although the 1PR contributions proportional to $\Gamma_\epsilon^{(3)}$ and $\Gamma_\zeta^{(3)}$ vanish, we are in fact left with two antisymmetric 1PR contributions to the anomalous dimension,

$$(\delta\gamma^{(3)}) \ni \gamma_{R1}^{(3)}\left(\Gamma_\alpha^{(3)} - \Gamma_\beta^{(3)}\right) + \gamma_{R2}^{(3)}\left(\Gamma_\gamma^{(3)} - \Gamma_\delta^{(3)}\right) \tag{3.113}$$

where

$$\gamma_{R1}^{(3)} = \gamma_1^{(2)}\epsilon_1^{(1)} - \gamma_1^{(1)}\epsilon_1^{(2)} + 2\gamma_1^{(1)}(\epsilon_1^{(1)})^2, \quad \gamma_{R2}^{(3)} = \gamma_2^{(2)}\epsilon_1^{(1)} - \gamma_1^{(1)}\epsilon_2^{(2)} + 2\gamma_2^{(1)}(\epsilon_1^{(1)})^2. \tag{3.114}$$

While the presence of such 1PR contributions does not affect our consistency conditions, it is nonetheless unexpected. We shall defer further discussion of 1PR contributions arising from a coupling redefinition until the end of the next chapter.

3.3 Summary

In this chapter, our intent was to extend the work of [9] to the case of a general gauge theory; that is, we wished to calculate the A-function, deduce the associated consistency conditions, and show that these conditions are independent of the chosen renormalization scheme. We have calculated the A-function (3.33, 3.37) up to four loops, as well as the associated tensor T_{IJ} (3.1, 3.2, 3.39), using gauge, Yukawa and scalar β-functions with arbitrary coefficients, up to three, two and one loop respectively. At this order, the first off-diagonal tensor components $T^{(3)}_{yg}$, $T^{(3)}_{gy}$ appear, and we have shown that one is free to impose that these terms be equal, fixing the tensor $T^{(3)}_{IJ}$ to be equal to the coupling-space metric $G^{(3)}_{IJ}$ for a general gauge theory in an arbitrary renormalization scheme. This is in contrast to [9], where attempting to fix $T^{(4)}_{IJ} = G^{(4)}_{IJ}$ for a general scalar-fermion theory at the first non-trivial order appeared to be valid only in particular schemes, and is in fact not valid for $\overline{\text{MS}}$; we shall re-visit the question of imposing symmetry in the next chapter.

As well as calculating the A-function and metric, we have deduced all associated consistency conditions. The majority of these conditions are relations between the coefficients of various β-functions at different loop orders, which we have shown are all satisfied in $\overline{\text{MS}}$. It is therefore possible to determine some higher-order β-function coefficients purely using lower-order calculations, and we demonstrated this by matching our predictions for $\beta^{(3)}_g$ to the explicit $\overline{\text{MS}}$ calculations of [23]. Furthermore, we have shown that these conditions are invariant under general changes in the β-functions as a consequence of varying the couplings. Such variations correspond to a change in renormalization scheme, hence the consistency conditions are scheme-independent. Interestingly, we also found scheme-independent consistency conditions between the leading-order metric coefficients, such that they must appear in a fixed ratio. Consequently, the leading-order positive-definiteness of the metric $T_{IJ} = G_{IJ}$ is in fact determined entirely by the positivity of any one of these coefficients. The three leading-order coefficients were of course calculated in [8], but it is pleasing to discover that in fact only one of these calculations was necessary.

The final part of our work in this chapter involved extending the Λ-equation to general gauge theories, providing evidence that a conjectured all-orders expression for A in an $\mathcal{N} = 1$ supersymmetric gauge theory is correct. This can in fact be accomplished by simply including an additional gauge-dependent tensor structure, and we have again demonstrated that the existence of a function Λ satisfying this equation leads to consistency conditions between coefficients of the chiral superfield anomalous dimension γ at different loop orders, up to three loops. We have verified that these consistency conditions are satisfied by γ, as computed using Dimensional Reduction with $\overline{\text{MS}}$, and we have again shown that the conditions are invariant under changes in γ induced by a coupling redefinition, hence hold in an arbitrary renormalization scheme. One unexpected feature of this calculation is the prediction of 1PR contributions to the anomalous dimension in a non-minimal renormalization scheme; we stress that such terms do not affect the consistency of the conjectured all-order a-function for supersymmetric theories, but are nevertheless troublesome. As mentioned, we shall comment further on such apparent contributions in the next chapter, where a similar phenomenon arises.

Since our work has been a purely perturbative analysis, we have made no comment on attempts to prove the all-orders positive-definiteness of the metric for general gauge theories, nor the all-orders validity of the Λ-equation for supersymmetric gauge theories. Progress on proving positive-definiteness of the metric is still lacking, and more recent techniques used to prove the weak a-theorem [16] appear insufficient in demonstrating the existence of a monotonically-decreasing A-function that is valid away from RG fixed points. However, a full superspace LRG analysis of a general $\mathcal{N} = 1$ supersymmetric theory was conducted in [35], and one of the Weyl consistency conditions appears to be of the same form as the Λ-equation, up to some as-yet-undetermined function; it would therefore be of interest to elucidate the precise connection between the two.

Chapter 4

Six Dimensions

In chapter 3, we investigated the phenomenologically-relevant case of four dimensions, following on from the work initiated in [8] and extended in [9]. This work was based on extending a theory to curved spacetime with position-dependent couplings, then using Weyl consistency conditions in order to derive constraints on renormalization group quantities; the approach was earlier used in [7] to re-derive the two-dimensional c-theorem and provide a potential answer to Cardy's conjecture [1]. As it happens, a similar approach can be applied to theories in six dimensions [36], demonstrating the existence of a function A satisfying (2.1). The precise definitions of A and T_{IJ} differ from those in two and four dimensions, but the structure is nevertheless the same: there exists a function that, for sufficiently weak coupling, behaves monotonically under renormalization group flow, and at RG fixed points is proportional to the coefficient of the Euler density in the six-dimensional trace anomaly. In [36, 37], an attempt was made to calculate A and T_{IJ} to leading order, and it was found that the metric was negative-definite, in contrast to the two- and four-dimensional cases. This was later remedied in [38], by instead constructing a one-parameter family of functions satisfying (2.1) with a metric that is positive-definite at leading order.

In [39], we endeavoured to derive the consequences of an a-function satisfying (2.1), by calculating the a-function and all required β-functions for a general six-dimensional scalar theory; an extensive list of further references regarding the β-function calculations may be found there. Unlike the four-dimensional calculations of the previous chapter, the results of this chapter are essentially identical to the published counterpart. Irrespective of questions regarding the leading-order metric, we shall proceed in the six-dimensional case as in the four-dimensional case, constructing the A-function order by order. We shall continue up to five-loop contributions $A^{(5)}$, requiring knowledge of the β-function up to three loops. At this order, we will be able to test the effects of a global symmetry on the construction of A, namely whether the β-functions in (2.1) must be modified as in four dimensions.

4.1 ϕ^3 theory

We begin with the lagrangian for a general ϕ^3 theory in six dimensions,

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu \phi^i \partial^\mu \phi^i - \tfrac{1}{3!} g^{ijk} \phi^i \phi^j \phi^k, \tag{4.1}$$

containing a scalar field multiplet ϕ^i and tensor coupling g^{ijk} with β-function $\beta(g^{ijk}) \equiv \beta_{ijk}$[1]. We shall denote various 3-index tensor structures using lower-case labelling, for example $g^{ijk}_{(1a)} = g^{ilm} g^{jmn} g^{knl}$, and various 2-index tensor structures using upper-case labelling, for example $g^{ij}_{(1A)} = g^{ikl} g^{jkl}$. From the 2-index structures we also define associated 3-index structures according to $g^{ijk}_{(1A)} = g^{il}_{(1A)} g^{ljk}$. In this way, we may easily identify anomalous dimension contributions to β_{ijk} by their upper-case label, and non-anomalous dimension contributions by their lower-case label. While this may seem an unnecessary distinction, it will be of use when considering potential contributions to a v-term[2] as in four dimensions, resulting from the presence of a global symmetry, and first introduced at three loops.

The various L-loop β-functions take the form

$$\beta^{(L)}_{ijk} = c_{(La)} g^{ijk}_{(La)} + \cdots + c_{(LA)} g^{ijk}_{(LA)} + \ldots \tag{4.2}$$

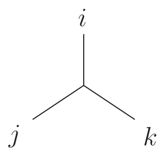
Any tensor structures that are not totally symmetric under exchange of their indices are implicitly accompanied by their symmetrised partners, for example

$$\beta^{(L)}_{ijk} \ \ni \ g^{ijk}_{(1A)} = g^{il}_{(1A)} g^{ljk} \ \xrightarrow{implies} \ g^{ijk}_{(1A)} = g^{il}_{(1A)} g^{ljk} + g^{jl}_{(1A)} g^{ilk} + g^{kl}_{(1A)} g^{ijl}. \tag{4.3}$$

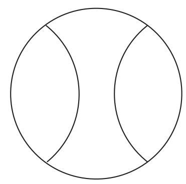
[1] Throughout this chapter, we shall be somewhat imprecise about the position of indices: all β-functions are to be understood as contravariant β-functions β^I.

[2] The name v-term is a reference to the notation of [9], in which elements of the Lie algebra corresponding to the global symmetry $\mathcal{S}$ are denoted by v, such that $B^I = \beta^I - (vg)^I$.

As in the previous chapter, we shall make use of a diagrammatic notation, in which the scalar coupling is depicted thus:

$g_{ijk} \rightarrow$ 

A corresponding term in the leading order A-function can hence be represented as

$g^{ijk}g^{jkl}g^{lmn}g^{mni} \rightarrow$  ,

where contracted lines represent contracted indices.

4.1.1 The A-function at leading- and next-to-leading order

Since there is only one tensor coupling, and hence only one β-function, the construction of A is much simpler than in four dimensions. The one-loop β-function $\beta^{(1)}_{ijk}$ is given by

$$\beta^{(1)}_{ijk} = c_{(1a)} g^{ijk}_{(1a)} + c_{(1A)} g^{ijk}_{(1A)}, \tag{4.4}$$

where the tensor structures are given by

$$g^{ijk}_{(1a)} = g^{ilm} g^{jmn} g^{knl}, \quad g^{ij}_{(1A)} = g^{ikl} g^{jkl}. \tag{4.5}$$

Expanding (2.1) at lowest order, we wish to solve

$$dA^{(3)} = dg T^{(2)}_{gg} \beta^{(1)}_{g}, \tag{4.6}$$

after defining

$$\begin{aligned} \frac{\partial}{\partial g^{ijk}} g^{i'j'k'} = \tfrac{1}{6} [\, & \delta_{ii'}(\delta_{jj'}\delta_{kk'} + \delta_{jk'}\delta_{kj'}) \\ & + \delta_{ij'}(\delta_{ji'}\delta_{kk'} + \delta_{jk'}\delta_{ki'}) \\ & + \delta_{ik'}(\delta_{ji'}\delta_{kj'} + \delta_{jj'}\delta_{ki'})\,] \,. \end{aligned} \tag{4.7}$$

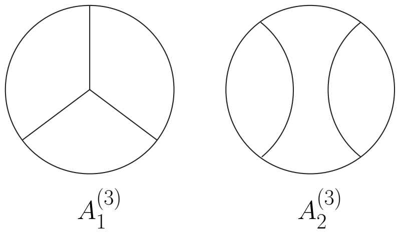


Table 4.1: Leading order terms in 6D A-function

and parametrising the leading order $A^{(3)}$ (shown diagrammatically in Table 4.1) as

$$\begin{aligned} A^{(3)} &= a_1^{(3)} A_1^{(3)} + a_2^{(3)} A_2^{(3)} \\ &= a_1^{(3)} (g^{klm} g^{knp} g^{lpq} g^{mqn}) + a_2^{(3)} (g^{klm} g^{lmn} g^{npq} g^{pqk}). \end{aligned} \tag{4.8}$$

With this normalization, the lowest order metric is simply $T_{IJ}^{(2)} = \lambda\, \delta_{IJ}$, and the coefficients $a_i^{(3)}$ are given by

$$a_1^{(3)} = \tfrac{\lambda}{4} c_{(1a)}, \quad a_1^{(3)} = \tfrac{3\lambda}{4} c_{(1A)}. \tag{4.9}$$

As expected, the construction of A at leading order is trivial, giving no consistency conditions on the β-function coefficients. From here, we shall use the actual values of the one-loop β-function coefficients,

$$c_{(1a)} = -1, \quad c_{(1A)} = \tfrac{1}{12} \tag{4.10}$$

valid in any renormalization scheme, in order to reduce the number of explicit coefficients that appear in various expressions and consistency conditions; note we have suppressed a factor of $\frac{1}{64\pi^3}$ at each loop order. By doing so, the leading order A-function can be expressed as

$$A^{(3)} = -\tfrac{\lambda}{4} \left(A_1^{(3)} - \tfrac{1}{4} A_2^{(3)} \right). \tag{4.11}$$

At next-to-leading order, the construction becomes slightly more complex. As in four dimensions, there is an arbitrariness in the definition of a function A satisfying (2.1), namely $A \to A + \beta^I g_{IJ} \beta^J$ for any g_{IJ}. At this order, the most general arbitrary term is $g_{IJ}^{(2)} = \alpha^{(4)} \delta_{IJ}$.

Expanding (2.1) at next-to-leading order, we wish to solve

$$dA^{(4)} = dg T_{gg}^{(2)} \beta_g^{(2)} + dg T_{gg}^{(3)} \beta_g^{(1)}, \tag{4.12}$$

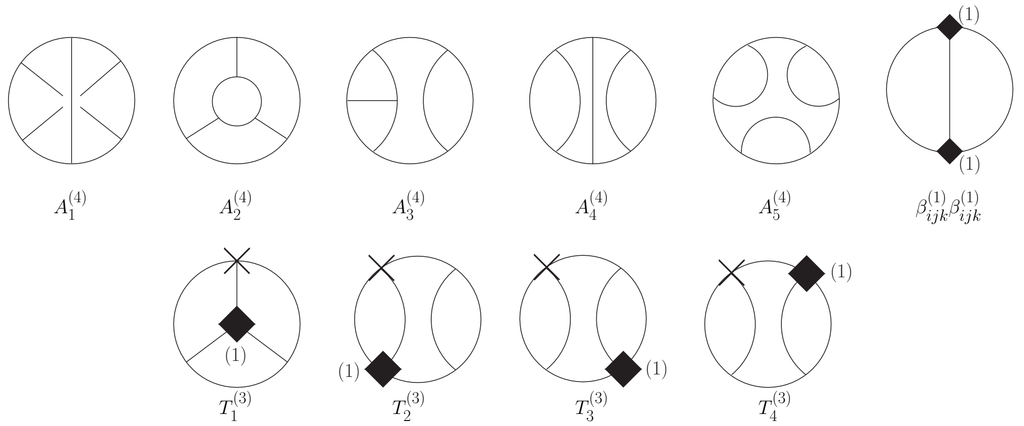


Table 4.2: Contributions to $A^{(4)}$ and $T_{IJ}^{(3)}$

where

$$A^{(4)} = \sum_{i=1}^{5} a_i^{(4)} A_i^{(4)} + \alpha^{(4)} \beta_{ijk}\beta_{ijk} \tag{4.13}$$

and

$$T_{gg}^{(3)} = \sum_{i=1}^{4} t_i^{(3)} (T_i^{(3)})_{gg}. \tag{4.14}$$

The relevant tensor structures are depicted in Table 4.2. As in four dimensions, it is easiest to depict the tensor structures $(T_i^{(4)})_{gg}$ as contractions between dg and β_{ijk}; that is, each metric diagram in Table 4.2 is of the form $(dg)^{abc}(T_i^{(4)})_{abc}{}^{def}\beta_{def}^{(1)}$. The two-loop β-function is given by

$$\beta_{ijk}^{(2)} = c_{(2b)} g_{(2b)}^{ijk} + c_{(2c)} g_{(2c)}^{ijk} + c_{(2d)} g_{(2d)}^{ijk} + c_{(2B)} g_{(2B)}^{ijk} + c_{(2C)} g_{(2C)}^{ijk}, \tag{4.15}$$

where

$$g_{(2b)}^{ijk} = g^{jpq} g^{kpr} g_{(1a)}^{iqr}, \quad g_{(2c)}^{ijk} = g^{ipr} g^{jpq} g_{1A}^{qs} g^{ksr}, \quad g_{(2d)}^{ijk} = g^{imn} g^{jpq} g^{krs} g^{nqs} g^{mpr}, \tag{4.16}$$

and

$$g_{(2B)}^{ij} = g^{ipq} g_{(1a)}^{jpq}, \quad g_{(2C)}^{ij} = g^{imn} g^{jmq} g_{(1A)}^{nq}. \tag{4.17}$$

Substituting $A^{(4)}$, $T_{IJ}^{(3)}$ and $\beta_{ijk}^{(2)}$ into (4.12), we obtain the A-function coefficients

$$a_1^{(4)} = \tfrac{\lambda}{6} c_{(2d)}, \quad a_2^{(4)} = \tfrac{\lambda}{2} c_{(2b)} + \alpha^{(4)}, \quad a_3^{(4)} = \tfrac{3\lambda}{2} c_{(2c)} - \tfrac{1}{2}\alpha^{(4)},$$
$$a_4^{(4)} = -\tfrac{\lambda}{4} c_{(2c)} + \tfrac{1}{24}\alpha^{(4)}, \quad a_5^{(4)} = -\tfrac{\lambda}{24}(2c_{(2c)} - c_{(2B)}) + \tfrac{1}{48}\alpha^{(4)}, \tag{4.18}$$

and metric coefficients

$$t_1^{(3)} = -6\alpha^{(4)}, \quad t_2^{(3)} = -3\lambda c_{(2c)} + \tfrac{1}{2}\alpha^{(4)}, \quad t_3^{(3)} + t_4^{(3)} = -3\lambda(c_{(2c)} - c_{(2B)}) + \alpha^{(4)}, \tag{4.19}$$

subject to the consistency condition

$$6c_{(2C)} + c_{(2c)} + c_{(2B)} = 0. \tag{4.20}$$

It is easy to see that this consistency condition is satisfied for the $\overline{\text{MS}}$ values of the $\beta^{(2)}_{ijk}$ coefficients,

$$c_{(2b)} = -\tfrac{1}{4}, \quad c_{(2c)} = \tfrac{7}{72}, \quad c_{(2d)} = -\tfrac{1}{2}, \quad c_{(2B)} = \tfrac{1}{18}, \quad c_{(2C)} = -\tfrac{11}{432}, \tag{4.21}$$

and we shall see later that (4.20) is in fact scheme-independent. Interestingly, the scheme-independence of this condition will be crucial for answering the question of whether T_{IJ} is symmetric beyond the manifest symmetry displayed at leading order and next-to-leading order.

Substituting in the $\overline{\text{MS}}$ values of $c_{(2i)}$, the next-to-leading order A-function can therefore be expressed as

$$A^{(4)} = \lambda\left(-\tfrac{1}{12}A_1^{(4)} - \tfrac{1}{8}A_2^{(4)} + \tfrac{7}{48}A_3^{(4)} - \tfrac{7}{288}A_4^{(4)} - \tfrac{5}{864}A_5^{(4)}\right) + \alpha^{(4)}\beta_{ijk}\beta_{ijk}. \tag{4.22}$$

4.1.2 The A-function at next-to-next-to-leading order

Despite the conceptual simplicity of solving the required equation, the construction of A beyond next-to-leading order is highly non-trivial. As mentioned at the beginning of this chapter, we must investigate whether the β appearing in (2.1) must be modified as in four dimensions; the work of [38] strongly suggests that such a modification should indeed take place, as their construction is analogous to the two- and four-dimensional cases.

It was shown in [8] that for a four-dimensional theory with general couplings g^I and a global symmetry in the kinetic term of the lagrangian density $\mathscr{L}$, the β^I in (2.1) must be replaced by a generalization,

$$\beta^I \to B^I = \beta^I - (vg)^I\,, \tag{4.23}$$

where v is an element of the Lie algebra of the global symmetry group. The analogous case in six dimensions, where the kinetic term in (4.1) is invariant under $O(N)$ transformations of the scalar fields ϕ^i, is a shift in the β-function defined by

$$\beta^{ijk} \to B^{ijk} = \beta^{ijk} - v^{l(i}g^{jk)l} \tag{4.24}$$

for some antisymmetric tensor v^{ij}. The only tensor structures present in the theory that can give such contributions to B^{ijk} are antisymmetric combinations of structures that appear in the anomalous dimension; such antisymmetric contributions are first possible only at three loops.

Expanding (2.1) and taking the modification (4.24) into account, we therefore wish to solve

$$dA^{(5)} = dgT^{(2)}_{gg}B^{(3)}_g + dgT^{(3)}_{gg}\beta^{(2)}_g + dgT^{(4)}_{gg}\beta^{(1)}_g. \tag{4.25}$$

In what follows, we shall again use the explicit values for scheme-independent β-function coefficients in order to simplify the consistency conditions derived from (4.25). The scheme-independent coefficients are

$$c_{(1a)} = -1, \;\; c_{(1A)} = \tfrac{1}{12}, \;\; c_{(2b)} = -\tfrac{1}{4}, \;\; c_{(2d)} = -\tfrac{1}{2}, \;\; c_{(2C)} = -\tfrac{11}{432}, \tag{4.26}$$

and we shall postpone the derivation of their scheme-independence until the next section.

The three-loop β-function is given by

$$\beta^{(3)}_{ijk} = c_{(3e)}g^{ijk}_{(3e)} + \ldots + c_{(3u)}g^{ijk}_{(3u)} + c_{(3D)}g^{ijk}_{(3D)} + \ldots + c_{(3L)}g^{ijk}_{(3L)}. \tag{4.27}$$

Defining two new useful tensor structures

$$g^{ijkl}_{22} = g^{ijm}g^{klm}, \quad g^{ij}_{(2D)} = g^{im}_{(1A)}g^{mj}_{(1A)}, \tag{4.28}$$

we may express the non-anomalous dimension terms as

$$\begin{aligned}
&g^{ijk}_{(3e)} = g^{ilm}_{(2b)}g^{jlkm}_{22}, && g^{ijk}_{(3f)} = g^{lmi}_{(2b)}g^{jlkm}_{22}, && g^{ijk}_{(3g)} = g^{ipq}g^{jpr}_{(1a)}g^{kqr}_{(1a)},\\
&g^{ijk}_{(3h)} = g^{ilm}_{(1a)}g^{nq}_{(1A)}g^{jln}g^{kmq}, && g^{ijk}_{(3i)} = g^{pq}_{(1A)}g^{ipr}_{(1a)}g^{jqkr}_{22}, && g^{ijk}_{(3j)} = g^{ilm}g^{jln}g^{kmq}g^{nq}_{(2B)},\\
&g^{ijk}_{(3k)} = g^{mil}_{(2c)}g^{jlkm}_{22}, && g^{ijk}_{(3l)} = g^{ilm}_{(2c)}g^{jlkm}_{22}, && g^{ijk}_{(3m)} = g^{iln}g^{jmq}g^{klm}g^{nq}_{(2C)},\\
&g^{ijk}_{(3n)} = g^{iln}g^{jmq}g^{klm}g^{nq}_{(2D)}, && g^{ijk}_{(3o)} = g^{pq}_{(1A)}g^{rs}_{(1A)}g^{jpkr}_{22}g^{iqs}, && g^{ijk}_{(3p)} = g^{ist}g^{qt}_{(1A)}g^{jrps}_{22}g^{kpqr}_{22},\\
&g^{ijk}_{(3q)} = g^{irs}_{(1a)}g^{jqps}_{22}g^{kpqr}_{22}, && g^{ijk}_{(3r)} = g^{ipq}g^{jrs}g^{knps}_{22}g^{rnq}_{(1a)}, && g^{ijk}_{(3s)} = g^{ilm}_{(2d)}g^{jlkm}_{22},\\
&g^{ijk}_{(3t)} = g^{ipq}g^{jrps}_{22}g^{knrl}_{22}g^{nqls}_{22}, && g^{ijk}_{(3u)} = g^{kpq}g^{jrsn}_{22}g^{ispl}_{22}g^{rlqn}_{22}, &&
\end{aligned} \tag{4.29}$$

and the anomalous dimension terms as

$$\begin{aligned}
&g^{ij}_{(3D)} = g^{ipq} g^{qjp}_{(2b)}, && g^{ij}_{(3E)} = g^{ipq} g^{jpq}_{(2b)}, && g^{ij}_{(3F)} = g^{ipq} g^{jpq}_{(2c)}, \\
&g^{ij}_{(3G)} = g^{imn} g^{mp}_{(1A)} g^{pnj}_{(1a)}, && g^{ij}_{(3G')} = g^{inp}_{(1a)} g^{mp}_{(1A)} g^{jmn}, && g^{ij}_{(3H)} = g_{22}^{ipjq} g^{pq}_{(2B)}, \\
&g^{ij}_{(3I)} = g^{ipq} g^{jpq}_{(2d)}, && g^{ij}_{(3J)} = g^{imn} g^{mp}_{(1A)} g^{nq}_{(1A)} g^{jpq}, && g^{ij}_{(3K)} = g_{22}^{ipjq} g^{pq}_{(2C)}, \\
&g^{ij}_{(3L)} = g_{22}^{injq} g^{nq}_{(2D)}.
\end{aligned} \tag{4.30}$$

Note that $g^{ij}_{(3G)} = g^{ji}_{(3G')}$ are the only non-symmetric tensor structures that contribute to the anomalous dimension, hence we postulate that the antisymmetric tensor v defined by (4.24) takes the form

$$v^{ij} = c^{v}_{(3G)} \left(g_{(3G)} - g_{(3G')} \right)^{ij} . \tag{4.31}$$

Finally, it is possible that in general non-minimal renormalization schemes one may obtain contributions to $\beta^{(3)}_{ijk}$ from one-particle-reducible (1PR) diagrams that are not present in $\overline{\text{MS}}$. We shall therefore introduce four new anomalous dimension contributions,

$$\begin{aligned}
&g^{ij}_{(3M)} = g^{il}_{(2B)} g^{lj}_{(1A)}, && g^{ij}_{(3M')} = g^{il}_{(1A)} g^{lj}_{(2B)}, \\
&g^{ij}_{(3N)} = g^{il}_{(2C)} g^{lj}_{(1A)}, && g^{ij}_{(3N')} = g^{il}_{(1A)} g^{lj}_{(2C)},
\end{aligned} \tag{4.32}$$

with associated β-function contributions

$$\beta^{(3)}_{ijk} \to \beta^{(3)}_{ijk} + c_{(3M)} g^{ijk}_{(3M)} + c_{(3M')} g^{ijk}_{(3M')} + c_{(3N)} g^{ijk}_{(3N)} + c_{(3N')} g^{ijk}_{(3N')}. \tag{4.33}$$

The reason for including these four terms in particular will become clear when we discuss scheme dependence. Since the new tensor structures satisfy $g^{ij}_{(3M)} = g^{ji}_{(3M')}$ and $g^{ij}_{(3N)} = g^{ji}_{(3N')}$, there may also be new contributions to v in the non-minimal scheme:

$$v^{ij} \to v^{ij} + c^{v}_{(3M)} \left(g_{(3M)} - g_{(3M')} \right)^{ij} + c^{v}_{(3N)} \left(g_{(3N)} - g_{(3N')} \right)^{ij} . \tag{4.34}$$

At next-to-next-to leading order, the arbitrariness implicit in the definition of A must itself be expanded beyond leading order. By doing so, we find the most general possible arbitrariness to be

$$\left(\beta^I g_{IJ} \beta^J \right)^{(5)} = 2 \beta^{(1)}_g g^{(2)}_{gg} \beta^{(2)}_g + \beta^{(1)}_g g^{(3)}_{gg} \beta^{(1)}_g , \tag{4.35}$$

where

$$g^{(2)}_{gg} = \alpha^{(4)}, \quad g^{(3)}_{gg} = \alpha^{(5)}_1 T^{(3)}_1 + \alpha^{(5)}_2 T^{(3)}_2 + \alpha^{(5)}_3 T^{(3)}_3 . \tag{4.36}$$

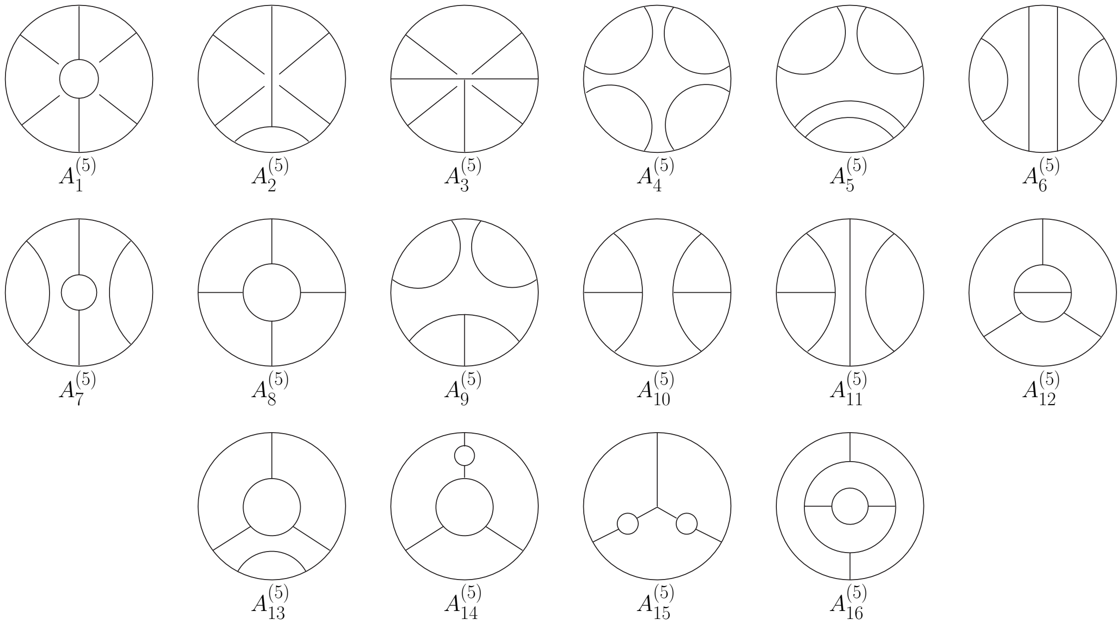


Table 4.3: Contributions to $A^{(5)}$

Diagrammatically, the three[3] new arbitrary terms correspond to replacing the derivatives dg^{ijk}, displayed on the metric terms in Table 4.2, with $\beta^{(1)}_{ijk}$. The next-to-next-to leading order contribution to A may therefore be parametrised as

$$\begin{aligned} A^{(5)} &= \sum_{i=1}^{16} a_i^{(5)} A_i^{(5)} + 2\alpha^{(4)} \beta^{(1)}_{ijk} \beta^{(2)}_{ijk} \\ &\quad + \sum_{j=1}^{3} \alpha_i^{(5)} \beta^{(1)}_{abc} (T_i^{(3)})_{abc}{}^{def} \beta^{(1)}_{def}, \end{aligned} \tag{4.37}$$

where

$$\begin{aligned} A_1^{(5)} &= g^{ijk} g^{ijk}_{(3I)}, & A_9^{(5)} &= g^{ijk} g^{ijk}_{(3n)}, \\ A_2^{(5)} &= g^{ijk} g^{ijk}_{(3s)}, & A_{10}^{(5)} &= g^{ijk} g^{ijk}_{(3j)}, \\ A_3^{(5)} &= g^{ijk} g^{ijk}_{(3u)}, & A_{11}^{(5)} &= g^{ijk} g^{ijk}_{(3m)}, \\ A_4^{(5)} &= g^{ij}_{(1A)} g^{jk}_{(1A)} g^{kl}_{(1A)} g^{li}_{(1A)}, & A_{12}^{(5)} &= g^{ijk} g^{ijk}_{(3e)}, \\ A_5^{(5)} &= g^{ijk} g^{ijk}_{(3L)}, & A_{13}^{(5)} &= g^{ijk} g^{ijk}_{(3l)}, \\ A_6^{(5)} &= g^{ijk} g^{ijk}_{(3K)}, & A_{14}^{(5)} &= g^{ijk} g^{ijk}_{(3E)}, \\ A_7^{(5)} &= g^{ijk} g^{ijk}_{(3J)}, & A_{15}^{(5)} &= g^{ijk} g^{ijk}_{(3o)}, \\ A_8^{(5)} &= g^{ijk} g^{ijk}_{(3t)}, & A_{16}^{(5)} &= g^{ijk} g^{ijk}_{(3F)}, \end{aligned} \tag{4.38}$$

[3]Three arbitrary terms is sufficient, because $\beta^{(1)}_{abc}(T_3^{(3)})_{abc}{}^{def}\beta^{(1)}_{def} = \beta^{(1)}_{abc}(T_4^{(3)})_{abc}{}^{def}\beta^{(1)}_{def}$; obviously, this may not hold at higher orders.

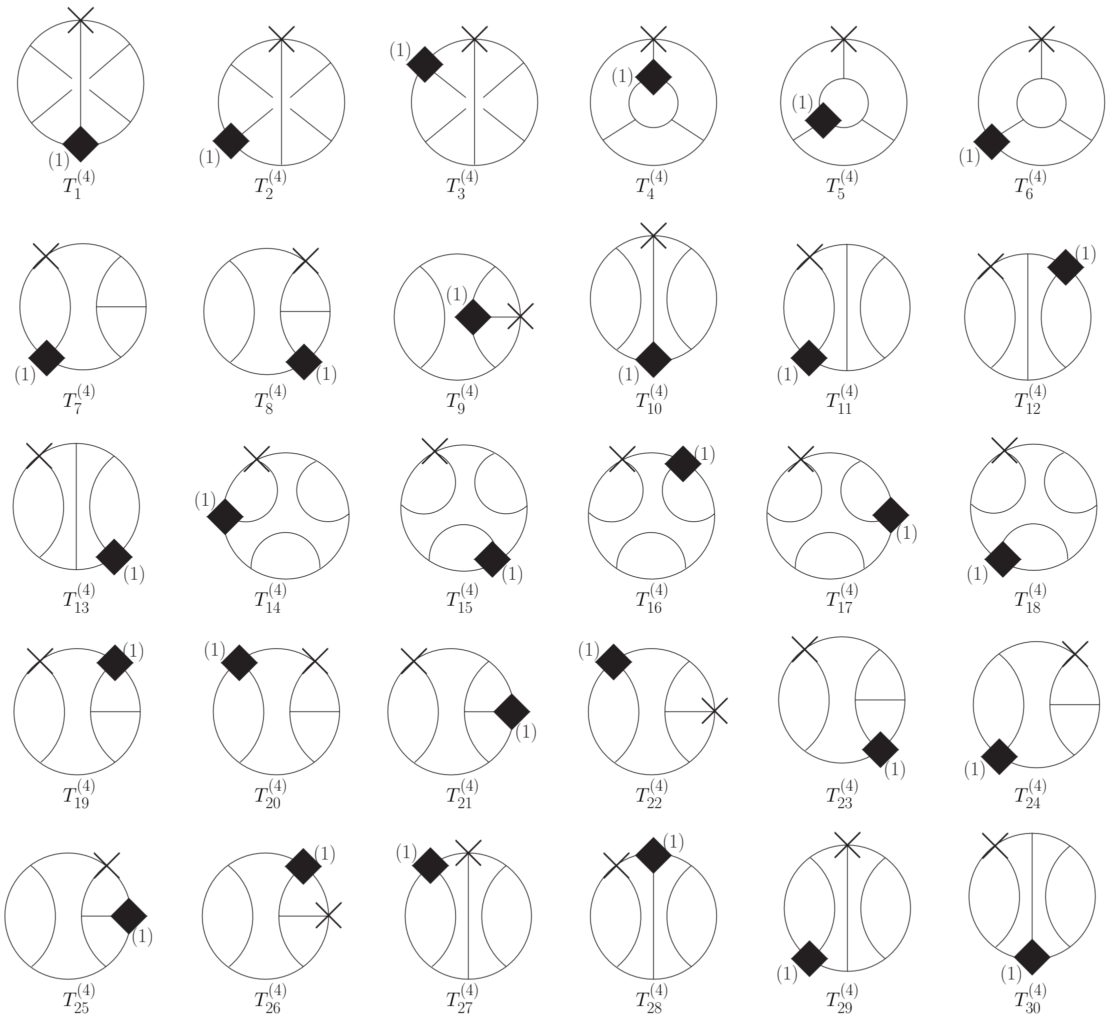


Table 4.4: Next-to-next-to-leading-order metric terms $T_{IJ}^{(4)}$

as shown in Table 4.3. The metric $T_{gg}^{(4)}$ may likewise be expressed as

$$T_{gg}^{(4)} = \sum_{i=1}^{30} t_i^{(4)} (T_i^{(4)})_{gg}, \tag{4.39}$$

shown in Table 4.4, and again depicted most efficiently as a contraction between dg and β_g.

Substituting the three-loop β-function (4.27), v term (4.31), 1PR contributions (4.32) and (4.34), A-function contributions (4.37) and metric terms (4.39), plus the relevant lower-order contributions, into the next-to-next-to-leading order equation (4.25), we obtain a very large system of linear equations, detailed in (B.1). After eliminating all A-function and metric coefficients, we obtain several consistency conditions:

$$c_{(3q)} - c_{(3s)} - 12c_{(3I)} + 6c_{(2B)} = 0,$$
$$c_{(3r)} - 2c_{(3s)} + 12c_{(3p)} - 12c_{(2c)} = 0,$$

$$
\begin{aligned}
c_{(3e)} - c_{(3g)} - 24c_{(3h)} - 144c_{(3o)} - 72Z + 3(c_{(2B)} + 2c_{(2c)}) &= 0, \\
c_{(3e)} - c_{(3g)} - 6c_{(3i)} + 6c_{(3k)} + 72Z - 3c_{(2B)} - 144c_{(2c)}^2 &= 0, \\
2c_{(3h)} + 6c_{(3m)} - 12c_{(3n)} + 18c_{(3o)} + c_{(3D)} + 12c_{(3F)} + 72c_{(3J)} & \\
+36c_{(3K)} - 72c_{(3L)} - \tfrac{11}{144}[1 + 24(c_{(2B)} - c_{(2c)})] &= 0, \\
c_{(3e)} - \tfrac{1}{2}c_{(3f)} + 6c_{(3k)} - 12c_{(3l)} &= 0, \\
c_{(3j)} + 6c_{(3m)} + 6c_{(3H)} + 36c_{(3K)} - 12c_{(2B)}c_{(2c)} &= 0, \\
c_{(3h)} - c_{(3i)} + c_{(3l)} - c_{(3D)} - 12c_{(3F)} + 12Z - 12c_{(2c)}(c_{(2B)} + 2c_{(2c)}) &= 0, \qquad (4.40)
\end{aligned}
$$

and

$$
\begin{aligned}
12c^v_{(3G)} + 24c^v_{(3M)} + 144c^v_{(3N)} = 6(c_{(3G)} - c_{(3G')}) + 12(c_{(3M)} - c_{(3M')}) \\
+ 72(c_{(3N)} - c_{(3N')}) - c_{(3j)} - 6c_{(3m)} - 12c_{(2c)}^2,
\end{aligned}
\qquad (4.41)
$$

where

$$
Z = c_{(3G)} + c_{(3G')} - c_{(3o)} + \tfrac{1}{6}c_{(3E)} - 2c_{(3F)} + 12c_{(3J)}. \qquad (4.42)
$$

The conditions (4.40) are satisfied by the $\overline{\text{MS}}$ values of the β-function coefficients, where the two-loop results were given earlier in (4.21) and the three-loop results are given by

$$
\begin{aligned}
&c_{(3e)} = -\tfrac{3}{8}, && c_{(3f)} = \tfrac{1}{4}, && c_{(3g)} = \tfrac{5}{16}, && c_{(3h)} = -\tfrac{47}{864}, \\
&c_{(3i)} = -\tfrac{47}{432}, && c_{(3j)} = \tfrac{23}{288}, && c_{(3k)} = \tfrac{5}{27}, && c_{(3l)} = \tfrac{11}{216}, \\
&c_{(3m)} = -\tfrac{19}{324}, && c_{(3n)} = \tfrac{11}{1728}, && c_{(3o)} = \tfrac{11}{1728}, && c_{(3p)} = \tfrac{11}{144}, \\
&c_{(3q)} = -\tfrac{1}{16}, && c_{(3r)} = -\tfrac{23}{24} + \zeta(3), && c_{(3s)} = -\tfrac{29}{48} + \tfrac{1}{2}\zeta(3), && c_{(3t)} = -1, \\
&c_{(3u)} = \tfrac{1}{3} - \zeta(3),
\end{aligned}
\qquad (4.43)
$$

and

$$
\begin{aligned}
&c_{(3D)} = \tfrac{7}{864}, && c_{(3E)} = \tfrac{71}{1728}, && c_{(3F)} = -\tfrac{103}{10368}, && c_{(3G)} = c_{(3G')} = -\tfrac{1}{108}, \\
&c_{(3H)} = -\tfrac{121}{5184}, && c_{(3I)} = \tfrac{7}{96} - \tfrac{1}{24}\zeta(3), && c_{(3J)} = \tfrac{23}{62208}, && c_{(3K)} = \tfrac{103}{7776}, \\
&c_{(3L)} = -\tfrac{13}{31104}, && c_{(3M)} = c_{(3M')} = 0, && c_{(3N)} = c_{(3N')} = 0.
\end{aligned}
\qquad (4.44)
$$

Here, $\zeta(z)$ is the Riemann ζ-function.

The isolated condition (4.41) is in some sense the most important consistency condition derived from (4.25): we see that the only occurrence of the 1PR terms in the consistency conditions is in (4.41), alongside the v term contributions.

Furthermore, we will show in the next section that while the rest of the conditions are scheme-independent, the RHS of (4.41) is only scheme-independent if we include precisely these 1PR terms. In addition, by substituting in the $\overline{\text{MS}}$ values of the β-function coefficients, we see that (4.41) is only satisfied if

$$c^v_{(3G)} + 2c^v_{(3M)} + 12c^v_{(3N)} = \tfrac{137}{10368}. \tag{4.45}$$

Since the 1PR coefficients vanish in $\overline{\text{MS}}$, it seems natural to assume that the associated 1PR contributions to v also vanish, and hence we predict

$$\left(c^v_{(3G)}\right)_{\overline{\text{MS}}} = \tfrac{137}{10368}. \tag{4.46}$$

This prediction may, in principle, be verified by direct calculation, using the methods in [15, 40].

Given that the consistency conditions are satisfied in $\overline{\text{MS}}$, we may list the explicit values for the coefficients $a^{(5)}_i$,

$$\begin{aligned} a^{(5)}_1 &= \left(\tfrac{9}{64} - \tfrac{1}{16}\zeta(3)\right)\lambda, & a^{(5)}_9 &= \tfrac{47}{1152}\lambda,\\ a^{(5)}_2 &= \left(-\tfrac{29}{48} + \tfrac{1}{2}\zeta(3)\right)\lambda, & a^{(5)}_{10} &= -\tfrac{23}{576}\lambda,\\ a^{(5)}_3 &= \left(\tfrac{1}{8} - \tfrac{3}{8}\zeta(3)\right)\lambda, & a^{(5)}_{11} &= -\tfrac{7}{128}\lambda,\\ a^{(5)}_4 &= -\tfrac{145}{82944}\lambda, & a^{(5)}_{12} &= \tfrac{107}{96}\lambda,\\ a^{(5)}_5 &= -\tfrac{5}{41472}\lambda, & a^{(5)}_{13} &= -\tfrac{5}{32}\lambda,\\ a^{(5)}_6 &= \tfrac{29}{2304}\lambda, & a^{(5)}_{14} &= -\tfrac{35}{128}\lambda,\\ a^{(5)}_7 &= 0, & a^{(5)}_{15} &= \tfrac{101}{3456}\lambda,\\ a^{(5)}_8 &= -\tfrac{1}{8}\lambda, & a^{(5)}_{16} &= -\tfrac{5}{2304}\lambda, \end{aligned} \tag{4.47}$$

up to the arbitrariness defined in (4.35). Before giving the $\overline{\text{MS}}$ values for the metric coefficients, we shall discuss the issues regarding symmetry. As can be seen from the diagrams in Table 4.4, at this order the metric $T^{(4)}_{IJ}$ is *not* manifestly symmetric; symmetry of $T^{(4)}_{IJ}$ requires each pair of terms $\{T^{(4)}_{17}, T^{(4)}_{18}\}$, $\ldots\{T^{(4)}_{29}, T^{(4)}_{30}\}$ to have equal coefficients:

$$\begin{aligned} &t^{(4)}_{17} = t^{(4)}_{18}, \quad t^{(4)}_{19} = t^{(4)}_{20}, \quad t^{(4)}_{21} = t^{(4)}_{22}, \quad t^{(4)}_{23} = t^{(4)}_{24},\\ &t^{(4)}_{25} = t^{(4)}_{26}, \quad t^{(4)}_{27} = t^{(4)}_{28}, \quad t^{(4)}_{29} = t^{(4)}_{30}. \end{aligned} \tag{4.48}$$

It turns out that due to the arbitrariness present in the solution of the equations derived from (4.25), it is in fact possible to impose that these coefficients be equal, subject to an additional consistency condition,

$$\begin{aligned}c_{(3m)} - 2c_{(3n)} + \tfrac{1}{6}c_{(3E)} - c_{(3H)} - 12c_{(3L)} - Z \\ +2(c_{(3\rho)} + c_{(3\sigma)}) + 12(c_{(3\tau)} + c_{(3\chi)}) = \tfrac{11}{36}(c_{(2B)} - 2c_{(2c)}).\end{aligned} \tag{4.49}$$

This condition is indeed satisfied by the $\overline{\text{MS}}$ values of the β-function coefficients, and hence one is free to impose that T_{IJ} be symmetric up to this order. With symmetry imposed, the $\overline{\text{MS}}$ values of the metric coefficients are

$$\begin{aligned}
t_1^{(4)} &= \tfrac{13}{8}\lambda - \tfrac{3}{2}\lambda\zeta(3) - 3\alpha_1^{(4)} - t_3^{(4)}, \\
t_2^{(4)} &= \tfrac{1}{4}\lambda - 2\alpha_1^{(4)}, \\
t_4^{(4)} &= -\tfrac{161}{48}\lambda + \tfrac{11}{2}\alpha_1^{(4)} + 24\alpha_2^{(5)}, \\
t_5^{(4)} &= -\tfrac{89}{24}\lambda + 11\alpha_1^{(4)} + 48\alpha_2^{(5)}, \\
t_6^{(4)} &= -\tfrac{31}{24}\lambda + 4\alpha_1^{(4)} + 24\alpha_2^{(5)}, \\
t_7^{(4)} &= -\tfrac{13}{24}\lambda + \tfrac{1}{3}\alpha_1^{(4)} - 2\alpha_2^{(5)}, \\
t_8^{(4)} &= \tfrac{49}{144}\lambda - \tfrac{7}{12}\alpha_1^{(4)} - 4\alpha_2^{(5)}, \\
t_9^{(4)} &= -\tfrac{11}{96}\lambda - 2\alpha_2^{(5)}, \\
t_{10}^{(4)} &= \tfrac{1}{3}\alpha_2^{(5)}, \\
t_{11}^{(4)} &= \tfrac{391}{1728}\lambda - \tfrac{11}{72}\alpha_1^{(4)} + \tfrac{1}{3}\alpha_2^{(5)}, \\
t_{12}^{(4)} &= \tfrac{11}{432}\lambda + \tfrac{4}{3}\alpha_2^{(5)} - t_{13}^{(4)}, \\
t_{14}^{(4)} &= \tfrac{1}{192}\lambda + \tfrac{1}{3}\alpha_2^{(5)}, \\
t_{15}^{(4)} &= -\tfrac{299}{1728}\lambda - t_{16}^{(4)} - 2t_{17}^{(4)} + \tfrac{11}{36}\alpha_1^{(4)} + \tfrac{5}{3}\alpha_2^{(5)}, \\
t_{17}^{(4)} &= t_{18}^{(4)}, \\
t_{19}^{(4)} &= t_{20}^{(4)} = -\tfrac{59}{72}\lambda + \tfrac{2}{3}t_4^{(3)} - 12(t_{16}^{(4)} + t_{17}^{(4)}) + \tfrac{11}{12}\alpha_1^{(4)} + 6\alpha_2^{(5)}, \\
t_{21}^{(4)} &= t_{22}^{(4)} = \tfrac{115}{288}\lambda - \tfrac{7}{6}\alpha_1^{(4)} - 8\alpha_2^{(5)}, \\
t_{23}^{(4)} &= t_{24}^{(4)} = \tfrac{73}{48}\lambda - \tfrac{2}{3}t_4^{(3)} + 12(t_{16}^{(4)} + t_{17}^{(4)}) - \tfrac{25}{12}\alpha_1^{(4)} - 12\alpha_2^{(5)}, \\
t_{25}^{(4)} &= t_{26}^{(4)} = \tfrac{101}{288}\lambda - \tfrac{7}{12}\alpha_1^{(4)} - 6\alpha_2^{(5)}, \\
t_{27}^{(4)} &= t_{28}^{(4)} = \tfrac{373}{1728}\lambda - \tfrac{11}{36}t_4^{(3)} + 2(t_{16}^{(4)} + t_{17}^{(4)}) - \tfrac{11}{72}\alpha_1^{(4)} - \alpha_2^{(5)}, \\
t_{29}^{(4)} &= t_{30}^{(4)} = -\tfrac{11}{48}\lambda + \tfrac{11}{36}t_4^{(3)} - 2(t_{16}^{(4)} + t_{17}^{(4)}) + \tfrac{11}{72}\alpha_1^{(4)} + 2\alpha_2^{(5)}.
\end{aligned} \tag{4.50}$$

In (4.19), we saw that the next-to-leading order metric $T_{IJ}^{(3)}$ had two free parameters $t_1^{(3)}$ and $t_4^{(3)}$, of which $t_1^{(3)}$ was identified with the freedom $\alpha^{(4)}$ in A, leaving one new free parameter. At next-to-next-to-leading order we find similar behaviour, in that the freedom in A corresponds to various linear combinations of $t_1^{(3)}$ and $t_{10}^{(4)}$,

$$\alpha_1^{(5)} = \tfrac{7}{12}t_1^{(3)} - 36t_{10}^{(4)}, \quad \alpha_2^{(5)} = 3t_{10}^{(4)}, \quad \alpha_3^{(5)} = -\tfrac{11}{144}t_1^{(3)} + 6t_{10}^{(4)}, \tag{4.51}$$

and there is additional arbitrariness introduced by the free parameters $t_3^{(4)}$, $t_{13}^{(4)}$, $t_{16}^{(4)}$, $t_{17}^{(4)}$. If symmetry is not imposed on the metric, then the coefficients (4.50) would also display the full arbitrariness in A as given in (4.35), however after imposing symmetry we are left with two free parameters $\alpha^{(4)}$ and $\alpha_2^{(5)}$, corresponding to choosing

$$\alpha_1^{(5)} = -12\alpha_2^{(5)} - \tfrac{7}{2}\alpha^{(4)}, \quad \alpha_3^{(5)} = 2\alpha_2^{(5)} + \tfrac{11}{24}\alpha^{(4)}. \tag{4.52}$$

In [9], an attempt to impose symmetry for four-dimensional scalar/fermion theories was made, but there did not appear to be enough freedom in the A-function to allow this for arbitrary schemes. Given our success at the equivalent loop order in six dimensions, and our ability to impose symmetry in four dimensions for the off-diagonal terms $T_{yg}^{(3)}$, $T_{gy}^{(3)}$, we feel that there is some support for the idea of revisiting the four-dimensional case.

4.2 Scheme-dependence and one-particle-reducible contributions

We now turn to the question of scheme-dependence, specifically the effects of a coupling redefinition on the β-function coefficients, and hence whether the consistency conditions are scheme-independent as expected. The coupling redefinition takes the form

$$g^{ijk} \to \bar{g}^{ijk} \equiv \bar{g}^{ijk}(g), \tag{4.53}$$

and its effect on the β-function can be deduced (as in four dimensions, see (3.43)) from the identity

$$\bar{\beta}_{ijk}(\bar{g}) = \mu\frac{d}{d\mu}\bar{g}^{ijk} = \beta \cdot \frac{\partial}{\partial g}\bar{g}^{ijk}(g), \tag{4.54}$$

where we have introduced the shorthand notation $a \cdot b \equiv a^{ijk}b_{ijk}$. Our investigations into scheme-dependence have revealed some rather surprising and counter-intuitive results regarding potential 1PR contributions to the β-function in non-minimal schemes.

At lowest order, expressing the coupling in the new scheme as a shift of the form $\bar{g} = g + \delta g$, (4.54) can be expanded to give

$$\delta\beta^{(2)}_{ijk} = \beta^{(1)} \cdot \frac{\partial}{\partial g} (\delta g)^{(1)}_{ijk} - (\delta g)^{(1)} \cdot \frac{\partial}{\partial g} \beta^{(1)}_{ijk}. \tag{4.55}$$

The most general one-loop redefinition that can be made is

$$(\delta g)^{(1)}_{ijk} = \delta_1 g^{ijk}_{(1a)} + \delta_2 g^{ijk}_{(1A)}, \tag{4.56}$$

where the non-symmetric tensor structures are again to be understood as being accompanied by their symmetrised partners. Substituting this redefinition and the one-loop β-function into (4.55) gives the following changes at two loops:

$$\delta c_{(2B)} = -\tfrac{1}{6}\Delta, \quad \delta c_{(2c)} = \tfrac{1}{6}\Delta, \quad \Delta = \delta_1 + 12\delta_2. \tag{4.57}$$

We can now see that the two-loop consistency condition (4.20) is invariant under the most general possible coupling redefinition, and is hence scheme-independent.

At the next order, we must track not only two-loop redefinitions of g, but also the higher-order effects of one-loop redefinitions. Consequently, expanding (4.54) and using a coupling redefinition $\bar{g} = g^{ijk} + (\delta g)^{(1)}_{ijk} + (\delta g)^{(2)}_{ijk}$, we find

$$\begin{aligned}
\delta\beta^{(3)}_{ijk} \quad = \quad & \beta^{(1)} \cdot \frac{\partial}{\partial g} (\delta g)^{(2)}_{ijk} - (\delta g)^{(2)} \cdot \frac{\partial}{\partial g} \beta^{(1)}_{ijk} \\
& + \beta^{(2)} \cdot \frac{\partial}{\partial g} (\delta g)^{(1)}_{ijk} - (\delta g)^{(1)} \cdot \frac{\partial}{\partial g} \beta^{(2)}_{ijk} - \frac{1}{2} \left(\delta g^{(1)} \cdot \frac{\partial}{\partial g} \right)^2 \beta^{(1)}_{ijk} \\
& - \delta g^{(1)} \cdot \frac{\partial}{\partial g} \left[\beta^{(1)} \cdot \frac{\partial}{\partial g} (\delta g)^{(1)}_{ijk} - (\delta g)^{(1)} \cdot \frac{\partial}{\partial g} \beta^{(1)}_{ijk} \right].
\end{aligned} \tag{4.58}$$

The general one-loop redefinition is given above in (4.56), and the general two-loop redefinition is

$$\begin{aligned}
(\delta g)^{(2)}_{ijk} = \; & \epsilon_1 g^{ijk}_{(2b)} + \epsilon_2 g^{ijk}_{(2c)} + \epsilon_3 g^{ijk}_{(2d)} + \epsilon_4 g^{ijk}_{(2e)} \\
& + \epsilon_5 g^{ijk}_{(2f)} + \epsilon_6 g^{ijk}_{(2B)} + \epsilon_7 g^{ijk}_{(2C)} + \epsilon_8 g^{ijk}_{(2D)}.
\end{aligned} \tag{4.59}$$

Note that the general two-loop redefinition allows for three 1PR structures, defined by

$$g^{ijk}_{(2e)} = g^{ilm} g^{lj}_{(1A)} g^{mk}_{(1A)}, \quad g^{ijk}_{(2f)} = g^{ijl}_{(1a)} g^{lk}_{(1A)}, \quad g^{ij}_{(2D)} = g^{ik}_{(1A)} g^{kj}_{(1A)}. \tag{4.60}$$

These redefinitions will generate the 1PR diagrams $g^{ijk}_{(3M)}, g^{ijk}_{(3M')}, g^{ijk}_{(3N)}, g^{ijk}_{(3N')}$ specified earlier, as well as 11 additional three-loop 1PR diagrams,

$$
\begin{aligned}
&g^{ijk}_{(3\alpha)} = g^{il}_{(1A)} g^{klj}_{(2b)}, && g^{ijk}_{(3\beta)} = g^{jl}_{(1A)} g^{lik}_{(2c)}, && g^{ijk}_{(3\gamma)} = g^{il}_{(1A)} g^{jlk}_{(2c)},\\
&g^{ijk}_{(3\delta)} = g^{il}_{(2B)} g^{ljk}_{(1a)}, && g^{ijk}_{(3\epsilon)} = g^{il}_{(2C)} g^{ljk}_{(1a)}, && g^{ijk}_{(3\zeta)} = g^{il}_{(1A)} g^{jm}_{(1A)} g^{lmk}_{(1a)},\\
&g^{ijk}_{(3\eta)} = g^{il}_{(2D)} g^{ljk}_{(1a)}, && g^{ijk}_{(3\kappa)} = g^{il}_{(1A)} g^{jm}_{(2B)} g^{lmk}, && g^{ijk}_{(3\lambda)} = g^{im}_{(1A)} g^{jl}_{(2C)} g^{lmk},\\
&g^{ijk}_{(3\mu)} = g^{il}_{(1A)} g^{jm}_{(2D)} g^{lmk}, && g^{ijk}_{(3\nu)} = g^{il}_{(1A)} g^{lm}_{(2D)} g^{mjk},
\end{aligned}
\tag{4.61}
$$

with associated β-function coefficients $c_{(3\alpha)}, \ldots, c_{(3\nu)}$.

Substituting (4.4), (4.15), (4.56) and (4.59) into (4.58), we obtain the effects of a general coupling redefinition on β_g at three loops. The changes in β-function coefficients are

$$
\begin{aligned}
\delta c_{(3e)} &= 0,\\
\delta c_{(3f)} &= 0,\\
\delta c_{(3g)} &= -2\epsilon_1 - 2c_{(2b)}\delta_1 + \delta_1^2,\\
\delta c_{(3h)} &= \tfrac{1}{6}\epsilon_1 - \epsilon_2 - c_{(2c)}\delta_1 - 2c_{(2b)}\delta_2 - \tfrac{1}{6}\delta_1^2,\\
\delta c_{(3i)} &= \tfrac{1}{3}\epsilon_1 - 2\epsilon_2 + 2\epsilon_5 - 2c_{(2c)}\delta_1 - 4c_{(2b)}\delta_2 - \tfrac{1}{3}\delta_1^2 - 2\delta_1\delta_2,\\
\delta c_{(3j)} &= -2\epsilon_2 + 2\epsilon_6 - 2c_{(2c)}\delta_1 + 2c_{(2B)}\delta_1 - \tfrac{1}{3}\delta_1^2 - 4\delta_1\delta_2,\\
\delta c_{(3k)} &= \tfrac{1}{3}\epsilon_1 + 2\epsilon_2 + 2c_{(2c)}\delta_1 - 4c_{(2b)}\delta_2,\\
\delta c_{(3l)} &= \tfrac{1}{6}\epsilon_1 + \epsilon_2 + c_{(2c)}\delta_1 - 2c_{(2b)}\delta_2,\\
\delta c_{(3m)} &= \tfrac{1}{3}\epsilon_2 + 2\epsilon_7 - 4c_{(2c)}\delta_2 + 2c_{(2C)}\delta_1 - 2\left(\tfrac{1}{3}\delta_1\delta_2 + 4\delta_2^2\right),\\
\delta c_{(3n)} &= \tfrac{1}{3}\epsilon_2 + 2\epsilon_8 - 4c_{(2c)}\delta_2 - \tfrac{2}{3}\delta_1\delta_2 - 7\delta_2^2,\\
\delta c_{(3o)} &= \tfrac{1}{3}\epsilon_2 + \epsilon_4 - 4c_{(2c)}\delta_2 - \tfrac{2}{3}\delta_1\delta_2 - 5\delta_2^2,\\
\delta c_{(3p)} &= \tfrac{1}{3}\epsilon_3 - 4c_{(2d)}\delta_2,\\
\delta c_{(3q)} &= -\epsilon_3 - c_{(2d)}\delta_1,\\
\delta c_{(3r)} &= -2\epsilon_3 - 2c_{(2d)}\delta_1,\\
\delta c_{(3s)} &= \epsilon_3 + c_{(2d)}\delta_1,\\
\delta c_{(3t)} &= 0,\\
\delta c_{(3u)} &= 0,
\end{aligned}
\tag{4.62}
$$

$$
\begin{aligned}
\delta c_{(3D)} &= -\tfrac{1}{3}\epsilon_1 - 2\epsilon_6 + 4c_{(2b)}\delta_2 - 2c_{(2B)}\delta_1 + 4\delta_1\delta_2 + \tfrac{1}{3}\delta_1^2,\\
\delta c_{(3E)} &= -\tfrac{1}{6}\epsilon_1 - 2\epsilon_6 + 2c_{(2b)}\delta_2 - 2c_{(2B)}\delta_1 + \tfrac{1}{4}\delta_1^2 + 4\delta_1\delta_2,\\
\delta c_{(3F)} &= -\tfrac{1}{6}\epsilon_2 + \tfrac{1}{6}\epsilon_6 + 2c_{(2c)}\delta_2 - 2c_{(2B)}\delta_2 + 4\delta_2^2 + \tfrac{1}{3}\delta_1\delta_2,
\end{aligned}
$$

$$
\begin{aligned}
\delta c_{(3G)} = \delta c_{(3G')} &= -\tfrac{1}{6}\epsilon_2 + \tfrac{1}{3}\epsilon_6 - \epsilon_7 - \tfrac{1}{6}\epsilon_5 + 2c_{(2c)}\delta_2 \\
&\qquad - 4c_{(2B)}\delta_2 - c_{(2C)}\delta_1 + \tfrac{1}{2}\delta_1\delta_2 + 8\delta_2^2, \\
\delta c_{(3H)} &= -\tfrac{1}{3}\epsilon_6 - 2\epsilon_7 + 4c_{(2B)}\delta_2 - 2c_{(2C)}\delta_1, \\
\delta c_{(3I)} &= -\tfrac{1}{6}\epsilon_3 + 2c_{(2d)}\delta_2, \\
\delta c_{(3J)} &= -\tfrac{1}{6}\epsilon_4 + \tfrac{1}{6}\epsilon_7 - 2c_{(2C)}\delta_2 - \tfrac{1}{6}\delta_2^2, \\
\delta c_{(3K)} &= 0, \\
\delta c_{(3L)} &= \tfrac{1}{3}\epsilon_7 - \tfrac{1}{3}\epsilon_8 - 4c_{(2C)}\delta_2 - \tfrac{1}{6}\delta_2^2, \\
\delta c_{(3M)} &= -\tfrac{1}{12}\epsilon_5 + \tfrac{1}{12}\epsilon_6 - 2\epsilon_8 - c_{(2B)}\delta_2 + 4\delta_2^2 + \tfrac{1}{12}\delta_1\delta_2, \\
\delta c_{(3M')} &= -\tfrac{1}{12}\epsilon_5 - \tfrac{1}{12}\epsilon_6 - 2\epsilon_8 + c_{(2B)}\delta_2 + 2\delta_2^2 + \tfrac{1}{12}\delta_1\delta_2, \\
\delta c_{(3N)} &= -\tfrac{1}{6}\epsilon_4 + \tfrac{1}{12}\epsilon_7 + \tfrac{1}{3}\epsilon_8 - c_{(2C)}\delta_2 - \tfrac{1}{2}\delta_2^2 \\
\delta c_{(3N')} &= -\tfrac{1}{6}\epsilon_4 - \tfrac{1}{12}\epsilon_7 + \tfrac{1}{3}\epsilon_8 + c_{(2C)}\delta_2 - \tfrac{1}{6}\delta_2^2
\end{aligned} \tag{4.63}
$$

and for the other 1PR coefficients

$$
\begin{aligned}
\delta c_{(3\alpha)} &= -2\epsilon_5 + 2\delta_1\delta_2, \quad \delta c_{(3\beta)} = \tfrac{1}{6}\epsilon_5 - \tfrac{1}{6}\delta_1\delta_2, \quad \delta c_{(3\gamma)} = 2\epsilon_4 + \tfrac{1}{3}\epsilon_5 - \tfrac{1}{3}\delta_1\delta_2 - 2\delta_2^2, \\
\delta c_{(3\delta)} &= -2\epsilon_5 + 2\delta_1\delta_2, \quad \delta c_{(3\epsilon)} = \tfrac{1}{3}\epsilon_5 - \tfrac{1}{3}\delta_1\delta_2, \quad \delta c_{(3\zeta)} = -\epsilon_4 + \delta_2^2, \\
\delta c_{(3\eta)} &= \tfrac{1}{6}\epsilon_5 - \tfrac{1}{6}\delta_1\delta_2, \quad \delta c_{(3\kappa)} = -4\epsilon_4 + 4\delta_2^2, \quad \delta c_{(3\lambda)} = \tfrac{2}{3}\epsilon_4 - \tfrac{2}{3}\delta_2^2, \\
\delta c_{(3\mu)} &= \tfrac{1}{3}\epsilon_4 - \tfrac{1}{3}\delta_2^2, \quad \delta c_{(3\nu)} = \tfrac{1}{6}\epsilon_8 - \tfrac{1}{4}\delta_2^2.
\end{aligned} \tag{4.64}
$$

Using these changes, we see that the consistency conditions in (4.40) are scheme-independent; more interestingly, the extra condition (4.49), required for imposing symmetry of $T_{IJ}^{(4)}$, is also scheme-independent if the two-loop consistency conditions is satisfied, and hence we may always impose symmetry of $T_{IJ}^{(4)}$.

The last consistency condition, (4.41), is somewhat surprising. Substituting in the results of a scheme change, we see that the RHS is indeed scheme-independent, and hence the $\overline{\text{MS}}$ result in (4.45),

$$
c^v_{(3G)} + 2c^v_{(3M)} + 12c^v_{(3N)} = \tfrac{137}{10368}, \tag{4.65}
$$

is in fact a scheme-independent result. However, the invariance of the RHS relies on the changes to the 1PR coefficients $c_{(3M)}$, etc. Consequently, we find that in a general renormalization scheme, these particular 1PR coefficients may in fact be non-zero. One may suppose that this is fine for a *completely* general renormalization scheme, but for any practical non-minimal renormalization schemes, such as momentum subtraction (MOM), the required coupling redefinitions δ_i, ϵ_i will be such that all 1PR coefficients vanish regardless. By looking at the results of a coupling redefinition on the 11 extra 1PR terms listed in (4.64), it is obvious that these terms only vanish if

$$\epsilon_4 = \delta_2^2, \quad \epsilon_5 = \delta_1\delta_2, \quad \epsilon_8 = \tfrac{3}{2}\delta_2^2. \tag{4.66}$$

However, these relations also impose

$$\begin{aligned} \delta c_{(3M)} &= -\delta c_{(3M')} = \tfrac{1}{12}\epsilon_6 + \delta_2^2 - c_{(2B)}\delta_2, \\ \delta c_{(3N)} &= -\delta c_{(3N')} = \tfrac{1}{12}\epsilon_7 - \tfrac{1}{6}\delta_2^2 - c_{(2C)}\delta_2, \end{aligned} \tag{4.67}$$

which may not necessarily vanish. One is of course free to impose values of δ_2, ϵ_6, ϵ_7 such that these coefficient changes do vanish, but such a choice of coupling redefinition may not correspond to any natural renormalization prescription.

To test whether these 1PR terms are likely to vanish in a standard non-minimal scheme, we shall use the example of MOM. We know already that there should be no issues at two loops: the MOM β-function coefficients that differ from the $\overline{\text{MS}}$ coefficients are

$$c_{(2B)}^{\text{MOM}} = \tfrac{1}{36} - \tfrac{2}{81}\pi^2 + \tfrac{1}{27}\psi'\left(\tfrac{1}{3}\right), \quad c_{(2c)}^{\text{MOM}} = \tfrac{1}{8} + \tfrac{2}{81}\pi^2 - \tfrac{1}{27}\psi'\left(\tfrac{1}{3}\right), \tag{4.68}$$

where $\psi(z)$ is the Euler ψ-function defined by

$$\psi(z) \equiv \frac{d}{dz}\ln\Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)}, \tag{4.69}$$

and $\Gamma(z)$ is the usual Euler Γ-function. This corresponds to coupling redefinitions (4.57) such that

$$\Delta = \tfrac{1}{6} + \tfrac{4}{27}\pi^2 - \tfrac{2}{9}\psi'\left(\tfrac{1}{3}\right). \tag{4.70}$$

The exact one-loop redefinitions required have been calculated as

$$\delta_1 = \tfrac{3}{2} + \tfrac{4}{27}\pi^2 - \tfrac{2}{9}\psi'\left(\tfrac{1}{3}\right), \quad \delta_2 = -\tfrac{1}{9}. \tag{4.71}$$

Similarly, the two-loop redefinitions required to obtain the three-loop MOM coefficients are

$$\begin{aligned} \epsilon_1 &= \tfrac{51}{32} + \tfrac{11}{54}\pi^2 - \tfrac{11}{36}\psi'\left(\tfrac{1}{3}\right), \\ \epsilon_2 &= -\tfrac{703}{1728} - \tfrac{41}{972}\pi^2 + \tfrac{41}{648}\psi'\left(\tfrac{1}{3}\right), \\ \epsilon_3 &= \tfrac{59}{48} - \tfrac{1}{2}\zeta(3) - \tfrac{7}{27}\pi^2 + \tfrac{1}{144}\ln(3)^2\sqrt{3}\pi \\ &\quad - \tfrac{1}{12}\ln(3)\sqrt{3}\pi - \tfrac{29}{3888}\sqrt{3}\pi^3 + 3s_2\left(\tfrac{\pi}{6}\right) - 6s_2\left(\tfrac{\pi}{2}\right) \end{aligned}$$

$$
\begin{aligned}
&\quad - 5s_3\left(\tfrac{\pi}{6}\right) + 4s_3\left(\tfrac{\pi}{2}\right) + \tfrac{7}{18}\psi'\left(\tfrac{1}{3}\right),\\
\epsilon_4 &= \tfrac{1}{81},\\
\epsilon_5 &= -\tfrac{1}{6} - \tfrac{4}{243}\pi^2 + \tfrac{2}{81}\psi'\left(\tfrac{1}{3}\right),\\
\epsilon_6 &= -\tfrac{215}{864},\\
\epsilon_7 &= \tfrac{791}{10368},\\
\epsilon_8 &= \tfrac{1}{54},
\end{aligned} \tag{4.72}
$$

where $s_n(z)$ is defined by

$$
s_n(z) = \frac{1}{\sqrt{3}}\,\mathcal{I}\left[\mathrm{Li}_n\left(\frac{e^{iz}}{\sqrt{3}}\right)\right], \tag{4.73}
$$

and $\mathrm{Li}_n(z)$ is the polylogarithm function. We can see immediately that the one- and two-loop redefinitions required to obtain the MOM results satisfy (4.66), hence the extra 1PR terms $c_{(3\alpha-3\nu)}$ vanish. However, given these redefinitions, the remaining contributions to $\delta c_{(3M)}$, etc do *not* vanish. Consequently, there exist 1PR contributions to $\beta_g^{(3)}$ in a well-understood and commonly-used non-minimal renormalization scheme; until now, it was expected that all such 1PR contributions should vanish. One may suppose that there could be different values of the two-loop redefinitions that reproduce the MOM β-function coefficients and allow $\delta c_{(3M)}$, etc to vanish, however ϵ_6 may be fixed by directly calculating $c_{(3L)}^{\mathrm{MOM}}$, and ϵ_7 by calculating $c_{(3J)}^{\mathrm{MOM}}$, forcing the non-zero values

$$
\begin{aligned}
c_{(3M)} &= -c_{(3M')} = -\tfrac{23}{10368},\\
c_{(3N)} &= -c_{(3N')} = \tfrac{61}{41472}.
\end{aligned} \tag{4.74}
$$

4.3 Summary

In this chapter, we have done a perturbative analysis of the A-function for a general six-dimensional ϕ^3 theory, prompted by the recent work of [36–38]. We have calculated A up to five loops, and the associated tensor T_{IJ} up to four loops, using the scalar β-function up to three loops; this has allowed us to test various non-trivial aspects of the a-theorem in six dimensions, such as the ability to impose symmetry of T_{IJ}, and the expected modification $\beta \to B$ in the gradient-flow equation due to the global $\mathcal{O}(N)$ symmetry of the theory. We have deduced the associated consistency conditions relating the various β-function coefficients, and verified that they are satisfied in $\overline{\mathrm{MS}}$. One of these conditions allows us to predict the coefficient of the shift $\beta \to B$, which first appears at three loops, and we have shown that it is indeed non-zero. While symmetry of T_{IJ} up to three loops is manifest, we have found that we may impose that $T^{(4)}_{IJ} = G^{(4)}_{IJ}$, as long as an additional consistency condition is satisfied; this too has been shown to hold in $\overline{\text{MS}}$. We have again deduced the effects of a coupling redefinition on β, and demonstrated that the consistency conditions are invariant under such changes, including the extra condition required to impose symmetry of $T^{(4)}_{IJ}$, hence we are free to impose symmetry in an arbitrary renormalization scheme. This is again in contrast to the scheme-dependence found in the four-dimensional scalar-fermion calculations of [9].

When attempting to investigate the effects of a coupling redefinition on the β-function, we encountered an unexpected issue: a completely general coupling redefinition appears to lead to 1PR contributions to $\beta^{(3)}$, arising as antisymmetric terms in the anomalous dimension. Unlike the earlier four-dimensional $\mathcal{N} = 1$ case, the non-zero redefinitions in a six-dimensional ϕ^3 theory are crucial to the scheme-independence of one of the consistency conditions, and we have verified that the β-function coefficients of these terms are non-zero in a well-known and commonly-utilised non-minimal renormalization scheme, MOM. The existence of such terms was eventually resolved in [41], where it was noticed that in the context of RG flow, the anomalous dimension possesses an arbitrariness corresponding to antisymmetric contributions. The authors showed how to define the anomalous dimension for a general theory, such that any antisymmetric contributions are automatically absorbed into this arbitrariness, hence removing the 1PR contributions to the β-function. The authors also explicitly demonstrated that the anomalous dimension remains symmetric in the case of four-dimensional $\mathcal{N} = 1$ supersymmetry, the graphs of which are topologically equivalent to those in a chiral ϕ^3 theory. It remains to be seen what effect this new formalism has on the v term, and hence on the required shift to the B-function. It is quite remarkable that, while it is comparatively straightforward to construct a six-dimensional analogue of the A-function for a ϕ^3 theory, pursuing its consequences has led to a necessary clarification in the definition of an anomalous dimension for general quantum field theories, and how it transforms under a coupling redefinition.

Chapter 5

Three Dimensions

So far, we have considered even-dimensional quantum field theories, in which the trace anomaly contains various curvature invariants. The most critical of these invariants, common to all dimensions, is the Euler density, as the A-function satisfying the a-theorem in even dimensions reduces to the coefficient of the Euler density at fixed points of the renormalization group (RG) flow. Unfortunately, the Euler density for an odd-dimensional spacetime vanishes identically [65], hence it would seem that there is no candidate quantity for an a-theorem in three dimensions. Furthermore, attempts to derive a gradient-flow equation in three dimensions using local renormalization group (LRG) methods have so far been unsuccessful, though it was shown that some of the same restrictions on the form of RG quantities do hold [42]. A different approach was proposed in [43], in which evidence was given that for $\mathcal{N} = 2$ theories, a function F, related to the free energy of the corresponding Euclidean CFT, may satisfy the weak a-theorem. This was formalised in [44], and extended to non-supersymmetric theories in [45], at least for non-interacting theories; the "F-function" was then shown to obey a gradient-flow equation in the vicinity of an RG fixed point. However, as of yet there are very few perturbative calculations of F for interacting theories, and there is no direct analogue of the a-function defined (up to the usual arbitrariness) away from RG fixed points.

Despite this lack of theoretical justification, we were (surprisingly) able to construct such a function, obeying the same gradient-flow equation as in even dimensions. The results in this chapter are based on the work carried out in three papers, [46–48], in which we investigate such constructions in full generality, valid (as usual) for arbitrary renormalization schemes, for a wide range of three-dimensional theories. The possibility of constructing such a function, at least for particular theories, was first investigated using the two-loop β-function calculations in [49], and the four-loop β-function contributions containing more than one type of coupling [46]; the theories considered were abelian Chern-Simons theory, non-abelian $SU(N)$ Chern-Simons theory, and non-abelian $\mathcal{N}=1$ supersymmetric Chern-Simons theory (the third being a special case of the second). Each theory is, of course, perturbatively renormalizable, with n pairs of scalars and fermions (ϕ_i, ψ_i) transforming according to the fundamental representation of a global $SU(n)$ symmetry (in the supersymmetric example the field pairs are contained in complex supermultiplets). Furthermore, the gauge coupling g in Chern-Simons theories is a topological quantity with $\beta(g) \equiv 0$ [50], and hence the gauge β-function plays no rôle in our construction.

5.1 Evidence for the existence of an A-function

5.1.1 Leading order

We begin with the abelian Chern-Simons lagrangian

$$\begin{aligned}\mathcal{L} \;=\; & \tfrac{1}{2}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho + |D_\mu\phi_j|^2 + i\overline{\psi}_j \not{D}\psi_j + \alpha\overline{\psi}_j\psi_j\phi_k^*\phi_k + \beta\overline{\psi}_j\psi_k\phi_k^*\phi_j \\ & + \tfrac{1}{4}\gamma(\overline{\psi}_j\psi_k^*\phi_j\phi_k + \overline{\psi}_j^*\psi_k\phi_j^*\phi_k^*) - h(\phi_j^*\phi_j)^3, \end{aligned} \tag{5.1}$$

where $D_\mu = \partial_\mu - igA_\mu$ and $\not{D} = \gamma^\mu D_\mu$. The n pairs of scalars and fermions (ϕ_i, ψ_i) have charge g, and there are five dimensionless couplings $\{\alpha, \beta, \gamma, h, g\}$. By observing the powers of couplings that appear in the respective β-functions, it is clear that the "3-2-1" phenomenon will once again occur[1], and hence to construct the lowest order A-function we need only concern ourselves with $\{\beta_\alpha, \beta_\beta, \beta_\gamma\}$, referred to collectively as Yukawa β-functions.

The two-loop Yukawa β-functions are [49]:

$$\begin{aligned}\beta_\alpha^{(2)} = \; & \left(\tfrac{8}{3}n+2\right)\alpha^3 + \tfrac{16}{3}\alpha^2\beta + \left(\tfrac{8}{3}n+3\right)\alpha\beta^2 + (n+2)\beta^3 + \tfrac{1}{4}\left(\tfrac{8}{3}n+\tfrac{17}{3}\right)\alpha\gamma^2 \\ & + \tfrac{3}{4}(n+2)\beta\gamma^2 + 3\beta^2g^2 + \tfrac{1}{4}\gamma^2g^2 - \tfrac{2}{3}(20n+31)\alpha g^4 - 8\beta g^4 - 8(n+2)g^6,\end{aligned}$$

$$\begin{aligned}\beta_\beta^{(2)} = \; & \left(\tfrac{8}{3}n+6\right)\alpha^2\beta + \left(3n+\tfrac{16}{3}\right)\alpha\beta^2 + \left(\tfrac{2}{3}n+1\right)\beta^3 + \tfrac{3}{4}(n+2)\alpha\gamma^2 \\ & + \tfrac{1}{4}\left(\tfrac{8}{3}n+\tfrac{17}{3}\right)\beta\gamma^2 - 3n\beta^2g^2 + \tfrac{1}{4}(n+2)\gamma^2g^2 - \tfrac{2}{3}(8n+31)\beta g^4,\end{aligned}$$

[1] Strictly speaking, due to the absence of odd-loop contributions in three dimensions, this would more correctly be a "6-4-2" phenomenon, or even more strictly speaking a "4-2" phenomenon due to the lack of gauge β-function.

$$\beta_\gamma^{(2)} = \left(\tfrac{8}{3}n+6\right)\alpha^2\gamma + \left(6n+\tfrac{34}{3}\right)\alpha\beta\gamma + \left(\tfrac{8}{3}n+6\right)\beta^2\gamma + \tfrac{1}{6}(n+1)\gamma^3 \\ + 4\alpha\gamma g^2 + 2(n+1)\beta\gamma g^2 - \tfrac{2}{3}(2n-5)\gamma g^4. \tag{5.2}$$

Note, a factor of $\frac{1}{8\pi}$ for each loop order is supressed. We are attempting to find a function $A \equiv A(\alpha,\beta,\gamma,h,g)$ satisfying the gradient-flow equation (2.1), hence at lowest order A must satisfy

$$\begin{pmatrix} \partial_\alpha A^{(5)} \\ \partial_\beta A^{(5)} \\ \partial_\gamma A^{(5)} \end{pmatrix} = \begin{pmatrix} T^{(3)}_{\alpha\alpha} & T^{(3)}_{\alpha\beta} & T^{(3)}_{\alpha\gamma} \\ T^{(3)}_{\beta\alpha} & T^{(3)}_{\beta\beta} & T^{(3)}_{\beta\gamma} \\ T^{(3)}_{\gamma\alpha} & T^{(3)}_{\gamma\beta} & T^{(3)}_{\gamma\gamma} \end{pmatrix} \begin{pmatrix} \beta_\alpha^{(2)} \\ \beta_\beta^{(2)} \\ \beta_\gamma^{(2)} \end{pmatrix}. \tag{5.3}$$

This can be solved as in even dimensions by postulating terms that may appear in A and substituting into (5.3). The solution (up to irrelevant purely-gauge terms $\sim g^8$) is

$$A^{(5)} = \tfrac{n}{4}\left(\tfrac{8}{3}n+2\right)\alpha^4 + \tfrac{1}{6}\left(n^2+3n+3\right)\beta^4 + \tfrac{1}{96}(n+1)^2\gamma^4 + \left(\tfrac{8}{3}n+2\right)\alpha^3\beta \\ + \tfrac{1}{3}(3n^2+8n+3)\beta^3\alpha + \tfrac{1}{3}(4n^2+9n+8)\alpha^2\beta^2 \\ + \tfrac{1}{12}(4n+9)(n+1)(\alpha^2+\beta^2)\gamma^2 + \tfrac{1}{12}(9n+17)(n+1)\alpha\beta\gamma^2 \\ + (1-n^2)\beta^3 g^2 + \tfrac{1}{2}(n+1)\alpha\gamma^2 g^2 + \tfrac{1}{4}(n+1)^2\beta\gamma^2 g^2 \\ - \tfrac{n}{3}(20n+31)\alpha^2 g^4 - \tfrac{1}{3}(8n^2+31n+12)\beta^2 g^4 \\ - \tfrac{n}{3}(2n-5)\gamma^2 g^4 - \tfrac{2}{3}(20n+31)\alpha\beta g^4 - 8n(n+2)\alpha g^6, \tag{5.4}$$

with corresponding lowest-order metric

$$T^{(3)}_{IJ} = \begin{pmatrix} n & 1 & 0 \\ 1 & n & 0 \\ 0 & 0 & \tfrac{1}{4}(n+1) \end{pmatrix}. \tag{5.5}$$

A and T_{IJ} are in fact only determined up to an overall scale, and the lack of Euler density prevents this scale being fixed. Nonetheless, we have found a function A of the couplings in the theory, obeying the gradient-flow equation (2.1) and with a positive-definite metric for $n \neq 1$ (the case $n = 1$ is where α and β are equivalent, and can be treated separately with the same conclusion).

Next, we consider the non-abelian $SU(N)$ Chern-Simons lagrangian

$$\mathcal{L} = \tfrac{1}{2}\epsilon^{\mu\nu\rho}A^A_\mu\partial_\nu A^A_\rho + \tfrac{1}{6}g f^{ABC}\epsilon^{\mu\nu\rho}A^A_\mu A^B_\nu A^C_\rho + |\mathcal{D}_\mu\phi_j|^2 + i\overline{\psi}_j \not{\mathcal{D}}\psi_j$$

$$+\alpha\overline{\psi}_j\psi_j\phi_k^*\phi_k+\beta\overline{\psi}_j\psi_k\phi_k^*\phi_j+\tfrac{1}{4}\gamma(\overline{\psi}_j\psi_k^*\phi_j\phi_k+\overline{\psi}_j^*\psi_k\phi_j^*\phi_k^*)-h(\phi_j^*\phi_j)^3, \tag{5.6}$$

with

$$\mathcal{D}_\mu\phi_i=\partial_\mu\phi_i-igT^A_{jk}A^A_\mu\phi_k,\quad \mathcal{D}_\mu\psi_i=\partial_\mu\psi_i-igT^A_{jk}A^A_\mu\psi_k,$$

where T^A_{ij} are generators for the fundamental representation of $SU(N)$ satisfying $\left[T^A,T^B\right]=if^{ABC}T^C$. As in the abelian case, we need only consider the Yukawa β-functions at lowest order:

$$\begin{aligned}\beta_\alpha^{(2)}=&\left(\tfrac{8}{3}n+2\right)\alpha^3+\tfrac{16}{3}\alpha^2\beta+\left(\tfrac{8}{3}n+3\right)\alpha\beta^2+(n+2)\beta^3+\tfrac{1}{4}\left(\tfrac{8}{3}n+\tfrac{17}{3}\right)\alpha\gamma^2\\&+\tfrac{3}{4}(n+2)\beta\gamma^2-\alpha\beta g^2+\frac{n^2-3}{2n}\beta^2g^2+\frac{n^2-1}{8n}\gamma^2g^2\\&-\frac{40n^3-17n^2-40n+62}{12n^2}\alpha g^4-\frac{5n^3+6n^2-18n+8}{4n^2}\beta g^4\\&+\frac{3n^4-4n^3+5n^2-8n+16}{8n^3}g^6,\end{aligned}$$

$$\begin{aligned}\beta_\beta^{(2)}=&\left(\tfrac{8}{3}n+6\right)\alpha^2\beta+\left(3n+\tfrac{16}{3}\right)\alpha\beta^2+\left(\tfrac{2}{3}n+1\right)\beta^3+\tfrac{3}{4}(n+2)\alpha\gamma^2\\&+\tfrac{1}{4}\left(\tfrac{8}{3}n+\tfrac{17}{3}\right)\beta\gamma^2+n\alpha\beta g^2+\beta^2g^2+\frac{n^2-1}{4n}\gamma^2g^2-\frac{5(n^2-4)}{4n}\alpha g^4\\&-\frac{22n^3-23n^2-64n+62}{12n^2}\beta g^4-\frac{(n^2-4)(n-2)}{2n^2}g^6,\end{aligned}$$

$$\begin{aligned}\beta_\gamma^{(2)}=&\left(\tfrac{8}{3}n+6\right)\alpha^2\gamma+\left(6n+\tfrac{34}{3}\right)\alpha\beta\gamma+\left(\tfrac{8}{3}n+6\right)\beta^2\gamma+\tfrac{1}{6}(n+1)\gamma^3\\&+\frac{(n-1)(n+2)}{n}\alpha\gamma g^2+\frac{(n-1)(2n+1)}{n}\beta\gamma g^2\\&-\frac{(n-1)(2n^2-2n+5)}{6n^2}\gamma g^4.\end{aligned}\tag{5.7}$$

Postulating a form for A and substituting into (5.3) then gives

$$\begin{aligned}A^{(5)}=&\tfrac{n}{4}\left(\tfrac{8}{3}n+2\right)\alpha^4+\tfrac{1}{6}\left(n^2+3n+3\right)\beta^4+\tfrac{1}{96}(n+1)^2\gamma^4\\&+\left(\tfrac{8}{3}n+2\right)\alpha^3\beta+\tfrac{1}{3}(3n^2+8n+3)\beta^3\alpha+\tfrac{1}{3}(4n^2+9n+8)\alpha^2\beta^2\\&+\tfrac{1}{12}(4n+9)(n+1)(\alpha^2+\beta^2)\gamma^2+\tfrac{1}{12}(9n+17)(n+1)\alpha\beta\gamma^2\\&+(n^2-1)\Big[\frac{n+2}{8n}\alpha\gamma^2g^2+\frac{2n+1}{8n}\beta\gamma^2g^2+\tfrac{1}{2}\alpha\beta^2g^2+\frac{1}{2n}\beta^3g^2\\&-\frac{20n-1}{12n}\alpha^2g^4-\frac{11n^2-4n-12}{12n^2}\beta^2g^4-\frac{2n^2-2n+5}{48n^2}\gamma^2g^4\\&-\frac{15n^2+40n-62}{12n^2}\alpha\beta g^4+\frac{3n^2-8n+16}{8n^2}\alpha g^6\\&-\frac{4n^3-11n^2-8n+16}{8n^3}\beta g^6\Big].\end{aligned}\tag{5.8}$$

with the same metric (5.5) as in the abelian case. A quick consistency check between the two cases is that the non-gauge terms are necessarily identical, since both cases should reduce to the same non-gauge theory when $g \to 0$.

The final case we shall consider at leading order is non-abelian $\mathcal{N} = 1$ supersymmetry, with action

$$\begin{aligned} S = \int d^3x d^2\theta \Big[&-\tfrac{1}{4}(D^\alpha \Gamma^{A\beta})(D_\beta \Gamma^A_\alpha) - \tfrac{1}{6} g f^{ABC} (D^\alpha \Gamma^{A\beta}) \Gamma^B_\alpha \Gamma^C_\beta \\ &- \tfrac{1}{24} g^2 f^{ABC} f^{ADE} \Gamma^{B\alpha} \Gamma^{C\beta} \Gamma^D_\alpha \Gamma^E_\beta \\ &- \tfrac{1}{2}(D^\alpha \overline{\Phi}_j + ig \overline{\Phi}_k T^A_{kj} \Gamma^\alpha_A)(D_\alpha \Phi_j - ig \Gamma^B_\alpha T^B_{jl} \Phi_l) \\ &+ \tfrac{1}{4}\eta_0 (\overline{\Phi}_j \Phi_j)^2 + \tfrac{1}{4}\eta_1 (\overline{\Phi}_j T^A_{jk} \Phi_k)^2 \Big], \end{aligned} \tag{5.9}$$

where $\Gamma^{A\alpha}$ is a real gauge superfield and Φ is a complex supermultiplet. This action contains couplings $\{\eta_0, \eta_1, g\}$, of which g is again irrelevant for our purposes; a full notational discussion is given in [49]. In principle, since we are dealing with supersymmetry, one should now consider regularization by dimensional reduction rather than dimensional regularization, however at lowest order the two methods will of course give identical results. The abelian and non-abelian $SU(N)$ cases could therefore be derived from the earlier non-supersymmetric results by an appropriate choice of fields and couplings.

The β-functions for a general non-abelian theory are given by [49]

$$\begin{aligned} \beta^{(2)}_{\eta_1} = &\big[(R_{31} + \tfrac{1}{2}R_{t1} + T_R C_R + 2C_R^2)\eta_1 + \tfrac{1}{4}T_R C_A g^2 - \tfrac{1}{2}R_{f1}(\eta_1 + g^2) \\ &- \tfrac{1}{4}C_R C_A(5\eta_1 - 3g^2) + \tfrac{1}{8}C_A^2(\eta_1 - 3g^2)\big](\eta_1^2 - g^4) \\ &+ [T_R(\eta_1^2 + \eta_1 g^2 + g^4) + C_R(3\eta_1^2 + 4\eta_1 g^2 + 3g^4) \\ &- \tfrac{1}{4}C_A(5\eta_1^2 + 8\eta_1 g^2 + 7g^4)]R_{21}(\eta_1 - g^2) \\ &+ \big[(6R_{21} + 10C_R + 3T_R - \tfrac{3}{2}C_A)\eta_1^2 + (2n + 11)\eta_1\eta_0 \\ &+ 2C_R\eta_1 g^2 - (2R_{21} + \tfrac{1}{2}C_A)g^4\big]\eta_0, \end{aligned}$$

$$\begin{aligned} \beta^{(2)}_{\eta_0} = &\big[(R_{30} + \tfrac{1}{2}R_{t0})\eta_1 - \tfrac{1}{2}R_{f0}(\eta_1 + g^2)\big](\eta_1^2 - g^4) \\ &+ [T_R(\eta_1^2 + \eta_1 g^2 + g^4) + C_R(3\eta_1^2 + 4\eta_1 g^2 + 3g^4) \\ &- \tfrac{1}{4}C_A(5\eta_1^2 + 8\eta_1 g^2 + 7g^4)]R_{20}(\eta_1 - g^2) \\ &+ [7C_R\eta_1\eta_0 + 3(n+2)\eta_0^2 + C_R\eta_0 g^2 \\ &+ 2R_{20}(3\eta_1^2 - g^4) + (2C_R + 2T_R - C_A)C_R(\eta_1^2 - g^4)]\eta_0, \end{aligned} \tag{5.10}$$

where the coefficients R_{Xi} are defined by

$$T_X = R_{X0}T_0 + R_{X1}T_1, \tag{5.11}$$

and

$$\begin{aligned} T_0 &= (\overline{\Phi}\Phi)^2, & T_3 &= (\overline{\Phi}T^AT^BT^C\Phi)^2, \\ T_1 &= (\overline{\Phi}T^A\Phi)^2, & T_t &= (\overline{\Phi}T^A\Phi)(\overline{\Phi}T^BT^C\Phi)\mathrm{tr}(T^A\{T^B, T^C\}), \\ T_2 &= (\overline{\Phi}T^AT^B\Phi)^2, & T_f &= f^{EAC}f^{EDB}(\overline{\Phi}T^AT^B\Phi)(\overline{\Phi}T^CT^D\Phi). \end{aligned} \tag{5.12}$$

Finally, group-theoretic invariants are defined as usual by

$$C_R\,\mathbb{1} = T^AT^A, \quad T_R\delta^{AB} = \mathrm{tr}(T^AT^B), \quad C_A\delta^{AB} = f^{ACD}f^{BCD} \tag{5.13}$$

If one takes the symmetry group to be $SU(N)$ or $Sp(N)$, there exist relations between T_0 and T_1, such that the matter couplings η_0, η_1 may be replaced by a single coupling. These cases are therefore trivial: the A-function is simply the integral of the new β-function with respect to its coupling, and the metric $T_{IJ} \sim \delta_{IJ}$ is positive-definite to all orders of perturbation theory. The simplest supersymmetric theory with a potentially non-trivial A-function is therefore $SO(N)$.

The group-theoretic invariants and coefficients R_{Xi} for $SO(N)$ are given by

$$\begin{aligned} C_R &= \tfrac{1}{2}(n-1)T_R, & C_A &= (n-2)T_R \\ R_{20} &= \tfrac{1}{4}(n-1)T_R^2, & R_{30} &= R_{f0} = -\tfrac{1}{8}T_R^3(n-1)(n-2), \\ R_{21} &= -\tfrac{1}{2}T_R(n-2), & R_{31} &= \tfrac{1}{4}T_R^2(n^2-3n+3), \\ R_{t0} &= R_{t1} = 0, & R_{f1} &= \tfrac{1}{4}T_R^2(n-2)(n-3), \end{aligned} \tag{5.14}$$

where the precise value of T_R depends on a choice of scale parameter for the representation matrices and structure constants [51]. Hence, the β-functions are

$$\begin{aligned} \beta^{(2)}_{\eta_0} &= (n-1)\big[\tfrac{1}{8}T_R^3(5\eta_1^2 + 6\eta_1 g^2 + 5g^4)(\eta_1 - g^2) + T_R^2(3\eta_1^2 - 2g^4)\eta_0 \\ &\quad + \tfrac{1}{2}T_R(7\eta_1 + g^2)\eta_0^2\big] + 3(n+2)\eta_0^3, \\ \\ \beta^{(2)}_{\eta_1} &= T_R^2\big[\tfrac{5}{4}\eta_1^3 + \tfrac{1}{2}(n-2)\eta_1^2 g^2 + \tfrac{1}{4}(3-4n)\eta_1 g^4 + \tfrac{1}{2}(n-2)g^6\big] \\ &\quad + \big[\tfrac{1}{2}(n+14)\eta_1^2 + (n-1)\eta_1 g^2 + \tfrac{1}{2}(n-2)g^4\big]T_R\eta_0 + (2n+11)\eta_1\eta_0^2. \end{aligned} \tag{5.15}$$

By the same method, we therefore find an A-function

$$\begin{aligned}A^{(5)} = &\frac{3(n+1)(n+2)}{n-1}\eta_0^4 + \tfrac{2}{3}(n+1)T_R g^2\eta_0^3 - \tfrac{1}{2}(7n+10)T_R^2 g^4\eta_0^2 \\ &- \tfrac{3}{2}(n+3)T_R^3 g^6\eta_0 + 6(n+2)T_R\eta_0^3\eta_1 + \tfrac{13}{2}(n+2)T_R^2\eta_0^2\eta_1^2 \\ &+ (n-1)T_R^2 g^2\eta_0^2\eta_1 - \tfrac{1}{2}(5n-2)T_R^3 g^4\eta_0\eta_1 + \tfrac{3}{2}(n-1)T_R^3 g^2\eta_0\eta_1^2 \\ &+ \tfrac{5}{2}(n+2)T_R^3\eta_0\eta_1^3 + \tfrac{5}{16}(n+2)T_R^4\eta_1^4 + \tfrac{1}{12}(7n-13)T_R^4 g^2\eta_1^3 \\ &- \tfrac{1}{8}(13n-10)T_R^4 g^4\eta_1^2 + \tfrac{1}{4}(n-7)T_R^4 g^6\eta_1. \qquad (5.16)\end{aligned}$$

with associated positive-definite metric

$$T^{(3)} = \begin{pmatrix} \frac{4(n+1)}{(n-1)} & 2T_R \\ 2T_R & 3T_R^2 \end{pmatrix} \qquad (5.17)$$

at lowest order.

5.1.2 Next-to-leading order

We have so far shown that an A-function satisfying (2.1) exists at lowest order, with positive-definite metric, for a range of three-dimensional theories. Due to the simplicity of the construction at lowest order, we have no guarantee that A can be constructed at next-to-leading order, where the hitherto-unconsidered scalar β-function β_h must be taken into account.

To investigate whether it is possible in three dimensions for (2.1) to hold beyond leading order, we return to the abelian case described by lagrangian (5.1). The two-loop scalar β-function is given by [49]

$$\begin{aligned}\beta_h^{(2)} = &12(3n+11)h^2 + 4h[4n\alpha^2 + 8\alpha\beta + (n+3)\beta^2] \\ &+ (n+4)h\gamma^2 - 4(5n+16)hg^4 \\ &- [4n\alpha^4 + 16\alpha^3\beta + 4(n+5)\alpha^2\beta^2 + 4(n+3)\alpha\beta^3 + (n+3)\beta^4] \\ &- [(n+6)\alpha^2 + (3n+11)(\alpha\beta + \tfrac{1}{2}\beta^2)]\gamma^2 - \tfrac{1}{16}(n+3)\gamma^4 - 2(\alpha+\beta)\gamma^2 g^2 \\ &+ 4(n\alpha^2 + 2\alpha\beta + \beta^2)g^4 - \gamma^2 g^4 + 8(n\alpha+\beta)g^6 + 4(2n+7)g^8, \qquad (5.18)\end{aligned}$$

and the A-function must in principle satisfy a differential equation of the form

$$\partial_h A^{(7)} = T_{hh}^{(5)}\beta_h^{(2)} + \sum_{Y=\alpha,\beta,\gamma} T_{hY}^{(5)}\beta_Y^{(2)} + \sum_{Y=\alpha,\beta,\gamma} T_{hY}^{(3)}\beta_Y^{(4)}. \qquad (5.19)$$

However, it turns out that extra terms involving Yukawa β-functions are not necessary. Keeping only the term $T^{(5)}_{hh}\beta^{(2)}_h$, we see that the A-function will acquire the additional contribution $A \to A + A_h$, where

$$\begin{aligned}A_h = T^{(5)}_{hh}\Big[&4(3n+11)h^3 + 2h^2[4n\alpha^2 + 8\alpha\beta + (n+3)\beta^2]\\&+ \tfrac{1}{2}(n+4)h^2\gamma^2 - 2(5n+16)h^2g^4\\&- h[4n\alpha^4 + 16\alpha^3\beta + 4(n+5)\alpha^2\beta^2 + 4(n+3)\alpha\beta^3 + (n+3)\beta^4]\\&- h[(n+6)\alpha^2 + (3n+11)(\alpha\beta + \tfrac{1}{2}\beta^2)]\gamma^2 - \tfrac{1}{16}(n+3)h\gamma^4 - 2h(\alpha+\beta)\gamma^2g^2\\&+ 4h(n\alpha^2 + 2\alpha\beta + \beta^2)g^4 - h\gamma^2g^4 + 8h(n\alpha+\beta)g^6 + 4h(2n+7)g^8\Big].\end{aligned} \tag{5.20}$$

If A_h is correct, then differentiating with respect to the Yukawa couplings and using (2.1) at next-to-leading order should produce the correct coefficients for the mixed Yukawa-scalar terms αh^2, $\alpha^3 h$, etc. We have computed the relevant terms in the four-loop Yukawa β-functions,

$$\begin{aligned}\beta^{(4)}_\alpha = {}& h^2[\tfrac{8}{3}(n+1)(n+2)\alpha + 2(n+2)\beta]\\&- \tfrac{2}{3}(n+2)h\Big\{4(n+1)\alpha^3 + 10(n+2)\alpha^2\beta + (2n+9)\alpha\beta^2 + (n+3)\beta^3\\&+ \tfrac{1}{4}[(2n+11)\alpha + (3n+11)\beta]\gamma^2\Big\} + \ldots,\end{aligned}$$

$$\begin{aligned}\beta^{(4)}_\beta = {}& \tfrac{2}{3}(n+2)(n+4)h^2\beta\\&- \tfrac{2}{3}(n+2)h\Big\{2(n+6)\alpha^2\beta + (3n+10)\alpha\beta^2 + (n+3)\beta^3\\&+ \tfrac{1}{4}[3(n+4)\alpha + (3n+11)\beta]\gamma^2\Big\} + \ldots,\end{aligned}$$

$$\begin{aligned}\beta^{(4)}_\gamma = {}& \tfrac{2}{3}(n+2)(n+4)h^2\gamma\\&- \tfrac{4}{3}(n+2)h\gamma\left[(n+6)\alpha^2 + (3n+11)(\alpha\beta + \tfrac{1}{2}\beta^2)\right] + \ldots,\end{aligned} \tag{5.21}$$

where the ellipses indicate pure-Yukawa terms. We can now see that A_h is indeed correct, provided $T^{(5)}_{hh} = \frac{1}{6}(n+1)(n+2)$. Given this value of $T^{(5)}_{hh}$, we also see that

$$\begin{pmatrix}\partial_\alpha A^{(7)}_h\\ \partial_\beta A^{(7)}_h\\ \partial_\gamma A^{(7)}_h\end{pmatrix} = \begin{pmatrix} n & 1 & 0\\ 1 & n & 0\\ 0 & 0 & \frac{1}{4}(n+1)\end{pmatrix}\begin{pmatrix}\beta^{(4)}_\alpha\\ \beta^{(4)}_\beta\\ \beta^{(4)}_\gamma\end{pmatrix}, \tag{5.22}$$

hence higher-order metric contributions such as $T^{(5)}_{\alpha\alpha}$ do not contribute to A_h. In summary, we have shown that the A-function can be extended consistently beyond leading order, with positive-definite lowest-order metric

$$T^{(3)}_{IJ} = \begin{pmatrix} n & 1 & 0 & 0 \\ 1 & n & 0 & 0 \\ 0 & 0 & \frac{1}{4}(n+1) & 0 \\ 0 & 0 & 0 & \frac{1}{6}(n+1)(n+2) \end{pmatrix}. \tag{5.23}$$

5.2 Leading order construction for general abelian Chern-Simons theory

The preceding examples all provide reasonable evidence that an A-function satisfying (2.1) can be constructed for a general three-dimensional theory. We shall therefore consider a general abelian Chern-Simons theory, with lagrangian

$$\begin{aligned} \mathcal{L} =& \tfrac{1}{2}[\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho + (D_\mu\phi_i)^2 + i\overline{\psi}_a D\psi_a] \\ &+ \tfrac{1}{4}Y_{abij}\psi_a\psi_b\phi_i\phi_j - \tfrac{1}{6!}h_{ijklmn}\phi_i\phi_j\phi_k\phi_l\phi_m\phi_n \end{aligned} \tag{5.24}$$

Here and throughout the remainder of the chapter, fermion contractions are represented by letters near the beginning of the alphabet, $\{a, b, c, \dots\}$, and scalar contractions by letters near the middle, $\{i, j, k, \dots\}$. Recall that in three dimensions, $\overline{\psi} = \psi^{*T}$, therefore there is no obstacle to decomposing ψ into real Majorana fields, and so we are free to choose a real basis for both scalar and fermion fields. As usual, $D_\mu = \partial_\mu - iEA_\mu$ where E is a charge matrix, denoted E^ϕ, E^ψ for scalars and fermions respectively. Finally, gauge invariance implies

$$\begin{aligned} E^\psi_{ac}Y_{cbij} + E^\psi_{bc}Y_{acij} + E^\phi_{im}Y_{abmj} + E^\phi_{jm}Y_{abim} = 0, \\ E^\phi_{ip}h_{pjklmn} + \text{perms} = 0. \end{aligned} \tag{5.25}$$

The L-loop Yukawa β-function takes the form

$$(\beta^{(L)}_Y)_{abij} = \sum_{p=1}^{n_L} c^{(L)}_p (U^{(L)}_p)_{abij}, \tag{5.26}$$

where each $U^{(L)}_p$ denotes one of the n_L possible L-loop tensor structures; note that we introduce here the convention of symmetrizing over inequivalent external lines, and include a weighting factor such that each tensor structure has a "weight" of one. To construct A for this theory, we begin with $\beta^{(2)}_Y$, in which there are $n_2 = 29$ tensor structures. The gauge identities in (5.25) relate some of the possible tensor structures; it is therefore necessary to select a basis of structures that include gauge terms. We have done so for the tensor structures containing two gauge insertions, but not for those with four or six gauge insertions, as the latter have a one-to-one correspondence with potential A-function contributions and are hence somewhat trivial. The two-loop tensor structures with no gauge insertions are

$$\begin{aligned}
(U_1^{(2)})_{abij} &= \tfrac{1}{4}[Y_{acil}Y_{cdjm}Y_{dblm} + Y_{aclm}Y_{cdjm}Y_{dbil} + Y_{acjl}Y_{cdim}Y_{dblm} + Y_{aclm}Y_{cdim}Y_{dbjl}],\\
(U_2^{(2)})_{abij} &= Y_{aclm}Y_{cdij}Y_{dblm},\\
(U_3^{(2)})_{abij} &= Y_{cdik}Y_{abkl}Y_{cdlj},\\
(U_4^{(2)})_{abij} &= \tfrac{1}{2}[Y_{acij}Y_{cdlm}Y_{dblm} + Y_{adlm}Y_{dclm}Y_{cbij}],\\
(U_5^{(2)})_{abij} &= \tfrac{1}{2}[Y_{abik}Y_{cdkl}Y_{dclj} + Y_{cdil}Y_{dclk}Y_{abkj}],
\end{aligned} \tag{5.27}$$

those with two gauge insertions are

$$\begin{aligned}
(U_6^{(2)})_{abij} &= \tfrac{1}{4}[Y_{acil}(E^{\psi 2})_{cd}Y_{dblj} + (a \leftrightarrow b, \quad i \leftrightarrow j)],\\
(U_7^{(2)})_{abij} &= \tfrac{1}{4}[Y_{acil}E^{\psi}_{cd}E^{\phi}_{lm}Y_{dbmj} + (a \leftrightarrow b, \quad i \leftrightarrow j)],\\
(U_8^{(2)})_{abij} &= \tfrac{1}{4}[E^{\psi}_{ac}Y_{cdil}Y_{delj}E^{\psi}_{be} + (a \leftrightarrow b, \quad i \leftrightarrow j)],\\
(U_9^{(2)})_{abij} &= \tfrac{1}{4}[E^{\psi}_{ac}Y_{cdil}E^{\phi}_{ml}Y_{dbmj} + (a \leftrightarrow b, \quad i \leftrightarrow j)],\\
(U_{10}^{(2)})_{abij} &= \tfrac{1}{4}[E^{\psi}_{ac}Y_{cdil}E^{\psi}_{de}Y_{eblj} + (a \leftrightarrow b, \quad i \leftrightarrow j)],\\
(U_{11}^{(2)})_{abij} &= \tfrac{1}{4}[(E^{\phi 2})_{im}Y_{acml}Y_{cblj} + (a \leftrightarrow b, \quad i \leftrightarrow j)],\\
(U_{12}^{(2)})_{abij} &= \tfrac{1}{4}[E^{\psi}_{ac}E^{\phi}_{im}Y_{deml}Y_{eblj} + (a \leftrightarrow b, \quad i \leftrightarrow j)],
\end{aligned} \tag{5.28}$$

and those with either four or six insertions are

$$\begin{aligned}
(U_{13}^{(2)})_{abij} &= \tfrac{1}{2}[(E^{\psi 3})_{ac}Y_{cdij}E^{\psi}_{bd} + (a \leftrightarrow b)],\\
(U_{14}^{(2)})_{abij} &= \tfrac{1}{2}[(E^{\psi 2})_{ac}Y_{cdij}(E^{\psi 2})_{db} + (a \leftrightarrow b)],\\
(U_{15}^{(2)})_{abij} &= \tfrac{1}{4}[(E^{\phi 2})_{ik}E^{\psi}_{ac}Y_{cdkj}E^{\psi}_{bd} + (a \leftrightarrow b, \quad i \leftrightarrow j)],\\
(U_{16}^{(2)})_{abij} &= \tfrac{1}{4}[(E^{\phi 2})_{ik}(E^{\psi 2})_{ac}Y_{cbkj} + (a \leftrightarrow b, \quad i \leftrightarrow j)],\\
(U_{17}^{(2)})_{abij} &= (E^{\phi 2})_{ik}(E^{\phi 2})_{jl}Y_{abkl},\\
(U_{18}^{(2)})_{abij} &= (E^{\phi 2})_{ij}(E^{\phi 2})_{kl}Y_{abkl},\\
(U_{19}^{(2)})_{abij} &= (E^{\psi 2})_{ab}(E^{\psi 2})_{cd}Y_{cdij},\\
(U_{20}^{(2)})_{abij} &= \mathrm{tr}(E^{\phi 2})E^{\psi}_{ac}Y_{cdij}E^{\psi}_{bd},\\
(U_{21}^{(2)})_{abij} &= \mathrm{tr}(E^{\psi 2})E^{\psi}_{ac}Y_{cdij}E^{\psi}_{bd},\\
(U_{22}^{(2)})_{abij} &= \tfrac{1}{2}[(E^{\phi 4})_{ik}Y_{abkj} + (i \leftrightarrow j)],\\
(U_{23}^{(2)})_{abij} &= \tfrac{1}{2}[(E^{\psi 4})_{ac}Y_{cbij} + (a \leftrightarrow b)],
\end{aligned}$$

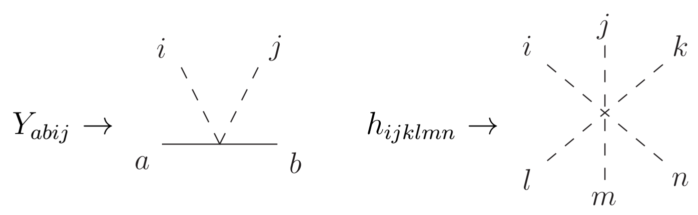
$$
\begin{aligned}
(U^{(2)}_{24})_{abij} &= \tfrac{1}{2}[\mathrm{tr}(E^{\phi 2}) + \mathrm{tr}(E^{\psi 2})][(E^{\phi 2})_{ik}Y_{abkj} + (i \leftrightarrow j)],\\
(U^{(2)}_{25})_{abij} &= \tfrac{1}{2}[\mathrm{tr}(E^{\phi 2}) + \mathrm{tr}(E^{\psi 2})][(E^{\psi 2})_{ac}Y_{cbij} + (a \leftrightarrow b)],\\
(U^{(2)}_{26})_{abij} &= (E^{\phi 2})_{ij}(E^{\psi 4})_{ab},\\
(U^{(2)}_{27})_{abij} &= (E^{\phi 4})_{ij}(E^{\psi 2})_{ab},\\
(U^{(2)}_{28})_{abij} &= \mathrm{tr}(E^{\phi 2})(E^{\phi 2})_{ij}(E^{\psi 2})_{ab},\\
(U^{(2)}_{29})_{abij} &= \mathrm{tr}(E^{\psi 2})(E^{\phi 2})_{ij}(E^{\psi 2})_{ab}.
\end{aligned}
\tag{5.29}
$$

As mentioned above, we have selected a basis of terms with two gauge insertions. There are in fact thirteen such terms, most of which manifestly give no contribution to β_Y. Due to each diagram being logarithmically divergent, we may set external momentum to zero, after which there are three circumstances in which a diagram gives no contribution:

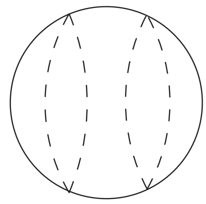
- The diagram is one-particle-reducible.
- The diagram is proportional to a trace over a single γ matrix.
- The diagram has a charge matrix E^ϕ on an external scalar line.

The first is guaranteed in $\overline{\mathrm{MS}}$, the second follows since $tr(\gamma^\mu) = 0$ by Lorentz invariance, and the third follows since such diagrams contain the term $\epsilon_{\mu\nu\rho}p_\nu p_\rho = 0$, arising from the gauge propagator. Consequently, we have removed as many of these terms as possible from our chosen basis. Furthermore, the four non-vanishing contributions in the basis each correspond to a single Feynman diagram, simplifying our results as much as possible.

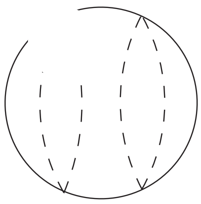
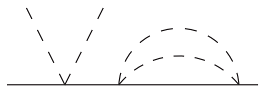
Now that we have a complete list of the tensor structures that appear in $\beta^{(2)}_Y$, we may construct the lowest order contributions to the A-function. As in the previous chapters, we once again introduce our diagrammatic notation. The Yukawa and scalar couplings will be represented by vertices, with the fermion and scalar legs indicated thus:



A corresponding term in the lowest order a-function can hence be represented as

$$Y_{acij}Y_{cbij}Y_{bdlm}Y_{dalm} \rightarrow$$  ,

where contracted lines indicate contracted indices. Differentiating the terms in A with respect to each coupling corresponds to removing each associated vertex, leaving a structure that may appear in the β-functions:

 $\rightarrow$  $\rightarrow \quad Y_{acij}Y_{cdkl}Y_{dbkl} \in \beta_Y^{(2)}$

The lowest-order A-function can be parametrised by

$$A^{(5)} = \sum_{r=1}^{9} a_r^{(5)} A_r^{(5)} + \sum_{r=13}^{29} a_r^{(5)} A_r^{(5)}, \tag{5.30}$$

where diagrams constructed using $U_{10-12}^{(2)}$ have been removed from the corresponding basis of gauge-dependent A-function terms, as they give no contribution to $\beta_Y^{(2)}$. The remaining terms have been split into contributions with up to two gauge insertions $(1-9)$ and four or six gauge insertions $(13-29)$. The first nine terms are given by

$$\begin{aligned}
A_1^{(5)} &= Y_{abij}Y_{bckl}Y_{cdik}Y_{dajl}, \quad A_2^{(5)} = Y_{abij}Y_{bckl}Y_{cdij}Y_{dakl}, \\
A_3^{(5)} &= Y_{abij}Y_{cdjk}Y_{abkl}Y_{cdli}, \quad A_4^{(5)} = Y_{acij}Y_{cbij}Y_{bdlm}Y_{dalm}, \\
A_5^{(5)} &= Y_{abik}Y_{bakj}Y_{cdil}Y_{dclj}, \quad A_6^{(5)} = Y_{abij}(E^{\psi 2})_{bc}Y_{cdjk}Y_{daki}, \\
A_7^{(5)} &= Y_{abij}(E^{\phi 2})_{jk}Y_{bckl}Y_{cali}, \quad A_8^{(5)} = Y_{abij}E^{\psi}_{bc}E^{\phi}_{jk}Y_{cdkl}Y_{dali}, \\
A_9^{(5)} &= Y_{abij}E^{\psi}_{bc}Y_{cdjk}E^{\psi}_{de}Y_{eaki},
\end{aligned} \tag{5.31}$$

and are depicted diagrammatically in Table (5.1), where insertions of a single gauge matrix are indicated by a blob on the corresponding scalar or fermion line. The remaining A-function terms are given by

$$A_i^{(5)} = (U_i^{(2)})_{abij}Y_{abij}, \quad i = 13, \ldots, 29. \tag{5.32}$$

We now deduce the coefficients $a_r^{(5)}$ in terms of $c_r^{(5)}$ by using equation (2.1) in

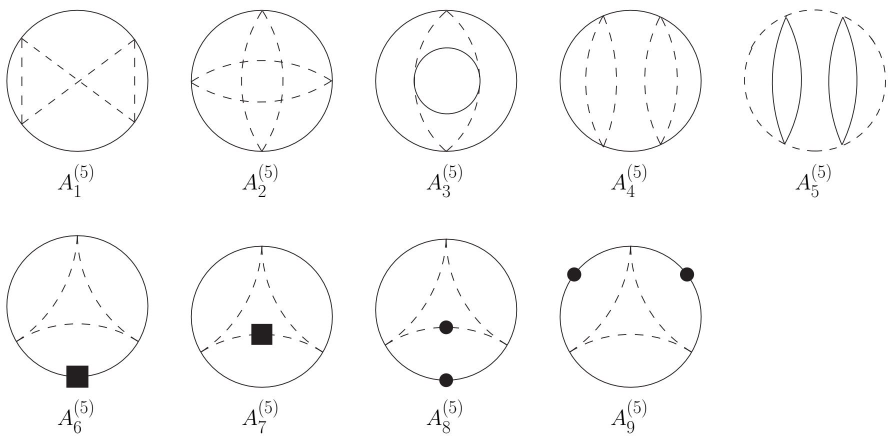


Table 5.1: Contributions to 3D A-function from $\beta_Y^{(2)}$

the form

$$\frac{\partial A^{(5)}}{\partial Y_{abij}} = \mu \, \beta^{(2)}_{abij}, \tag{5.33}$$

where we define

$$\frac{\partial}{\partial Y_{abij}} Y_{a'b'i'j'} = \tfrac{1}{4}(\delta_{aa'}\delta_{bb'} + \delta_{ab'}\delta_{ba'})(\delta_{ii'}\delta_{jj'} + \delta_{ij'}\delta_{ji'}). \tag{5.34}$$

With this normalization, the lowest order metric is simply $T^{(3)}_{IJ} = \mu \, \delta_{IJ}$, and the first five A-function coefficients are[2]

$$a_r^{(5)} = \tfrac{1}{4}\mu c_r^{(2)}, \quad r = 1, \ldots 5. \tag{5.35}$$

From this, we can see immediately that for a general non-gauge theory in three dimensions (in which one only has the first five terms in $\beta_Y^{(2)}$), the lowest order A-function is in fact trivial: there is a simple one-to-one correspondence between the tensor structures $U^{(2)}_{1-5}$ and the A-function contributions $A^{(5)}_{1-5}$. Despite the apparent non-triviality displayed in the non-gauge parts of examples (5.1) and (5.6), the coefficients in $\beta_\alpha^{(2)}, \beta_\beta^{(2)}, \beta_\gamma^{(2)}$ were in fact guaranteed to appear in ratios that allow the existence of an A-function.

The next four coefficients involve the terms with two gauge insertions, and their evaluation is non-trivial. The four A-function terms $A^{(5)}_{6-9}$ form a basis for terms involving two gauge insertions, but upon differentiating lead to eight tensor structures, one more than the seven structures listed in (5.28). We are therefore required to substitute in the expression for this eighth tensor structure in terms of the basis, then rewrite the differentiated A-function term as a linear combination

[2] Recall that A and T_{IJ} are determined only up to the overall scale μ.

of the other seven. Having done so, the system of equations can be solved, giving

$$a_6^{(5)} = \mu c_6^{(2)}, \quad a_7^{(5)} = 0, \quad a_8^{(5)} = \mu c_6^{(2)}, \quad a_9^{(5)} = \mu c_8^{(2)}, \tag{5.36}$$

plus the consistency conditions

$$c_6^{(2)} = c_7^{(2)} = -\tfrac{1}{2} c_9^{(2)}, \quad c_8^{(2)} = c_7^{(2)} + \tfrac{1}{2} c_{10}^{(2)}. \tag{5.37}$$

The existence of consistency conditions shows that unlike the non-gauge terms, the gauge parts of the A-functions were not guaranteed to work. We shall see shortly that the consistency conditions (5.37) are indeed satisfied; that this is the case for a general theory shows that the existence of an A function for the specific theories considered previously is in fact non-trivial.

The final coefficients in the general A-function are simply given by

$$\begin{aligned} a_r^{(5)} &= \tfrac{1}{2}\mu c_r^{(2)}, \quad r = 13, \dots 25, \\ a_r^{(5)} &= \mu c_r^{(2)}, \quad r = 26, \dots 29, \end{aligned} \tag{5.38}$$

due to the one-to-one correspondence between the remaining tensor structures (5.29) and the A-function contributions. The general solution to the A-function parametrised as (5.30) is therefore given by (5.35), (5.36) and (5.38), subject to the consistency conditions (5.37).

Since the existence of A is predicated on the consistency conditions (5.37) being satisfied, we must of course calculate the actual values of the coefficients $c_p^{(2)}$ in $\beta_Y^{(2)}$. The coefficients have each been calculated using dimensional regularisation with $\overline{\text{MS}}$, and are given by

$$\begin{aligned} &c_1^{(2)} = 8, && c_2^{(2)} = 2, && c_3^{(2)} = 2, && c_4^{(2)} = \tfrac{2}{3}, && c_5^{(2)} = \tfrac{2}{3}, && c_6^{(2)} = 8, \\ &c_7^{(2)} = 8, && c_8^{(2)} = 8, && c_9^{(2)} = -16, && c_{10}^{(2)} = 0, && c_{11}^{(2)} = 0, && c_{12}^{(2)} = 0, \\ &c_{13}^{(2)} = 24, && c_{14}^{(2)} = 4, && c_{15}^{(2)} = 24, && c_{16}^{(2)} = -16, && c_{17}^{(2)} = -8, && c_{18}^{(2)} = -8, \\ &c_{19}^{(2)} = -8, && c_{20}^{(2)} = 2, && c_{21}^{(2)} = 2, && c_{22}^{(2)} = -\tfrac{40}{3}, && c_{23}^{(2)} = -\tfrac{4}{3}, && c_{24}^{(2)} = -\tfrac{8}{3}, \\ &c_{25}^{(2)} = -\tfrac{2}{3}, && c_{26}^{(2)} = -32, && c_{27}^{(2)} = -32, && c_{28}^{(2)} = -8, && c_{29}^{(2)} = -8. \end{aligned} \tag{5.39}$$

These coefficients are of course scheme-independent. We can clearly see that the consistency conditions (5.37) are therefore satisfied in any renormalization scheme, and hence we have constructed the A-function at leading order for a general abelian Chern-Simons theory in three dimensions, with positive-definite metric.

5.3 Next-to-leading-order construction for general scalar-fermion theory

In section 5.1, the A-function constructed for each example considered was shown to hold beyond leading order, at least when considering $\beta_h^{(2)}$ and the mixed scalar-Yukawa terms in $\beta_Y^{(4)}$. Having constructed the general leading-order A-function, we shall now endeavour to show that a general construction holds beyond leading order. The next-to-leading order A-function, $A^{(7)}$, can be split into multiple contributions:

$$A^{(7)} = A_h^{(7)} + A_{hY}^{(7)} + A_Y^{(7)} + a(\beta_Y^{(2)})_{abij}(\beta_Y^{(2)})_{abij}. \tag{5.40}$$

$A_h^{(7)}$, $A_{hY}^{(7)}$ and $A_Y^{(7)}$ are the pure scalar, mixed scalar-Yukawa and pure Yukawa contributions respectively, and the last term is the expected arbitrariness in the definition of a function satisfying (2.1), analogous to the case in even dimensions.

5.3.1 Scalar and mixed scalar-Yukawa terms

To test the viability of constructing A for a general abelian theory beyond leading order, we can again look at $\beta_h^{(2)}$ and the scalar-Yukawa parts of $\beta_Y^{(4)}$. This enables us to construct the first two terms in (5.40), which may be parametrised as

$$A_h^{(7)} = a_{h_1}^{(7)} A_{h_1}^{(7)}, \quad A_{hY}^{(7)} = \sum_{i=2}^{14} a_{h_i}^{(7)} A_{h_i}^{(7)}, \tag{5.41}$$

and are depicted in Table 5.2. The explicit expressions for each term in $A^{(7)}$ are rather unwieldy, containing many index contractions, but if desired can easily be reconstructed from the diagrams.

The scalar-Yukawa part of $\beta_Y^{(4)}$ takes the form

$$\beta_Y^{(4)}(h) = \sum_{i=1}^{13} c_{H_i} U_{H_i}^{(4)}. \tag{5.42}$$

Rather than explicitly listing each tensor structure in $\beta_Y^{(4)}$, we have simply labelled the vertices of the terms in $A^{(7)}$, such that when each labelled vertex is removed one obtains the relevant tensor structure $U_{H_i}^{(4)}$[3]. Calculating the coefficients $c_{H_{1-13}}^{(4)}$

[3]Again, the tensor structures are implied to have a weight of one, and be symmetrized over inequivalent external lines.

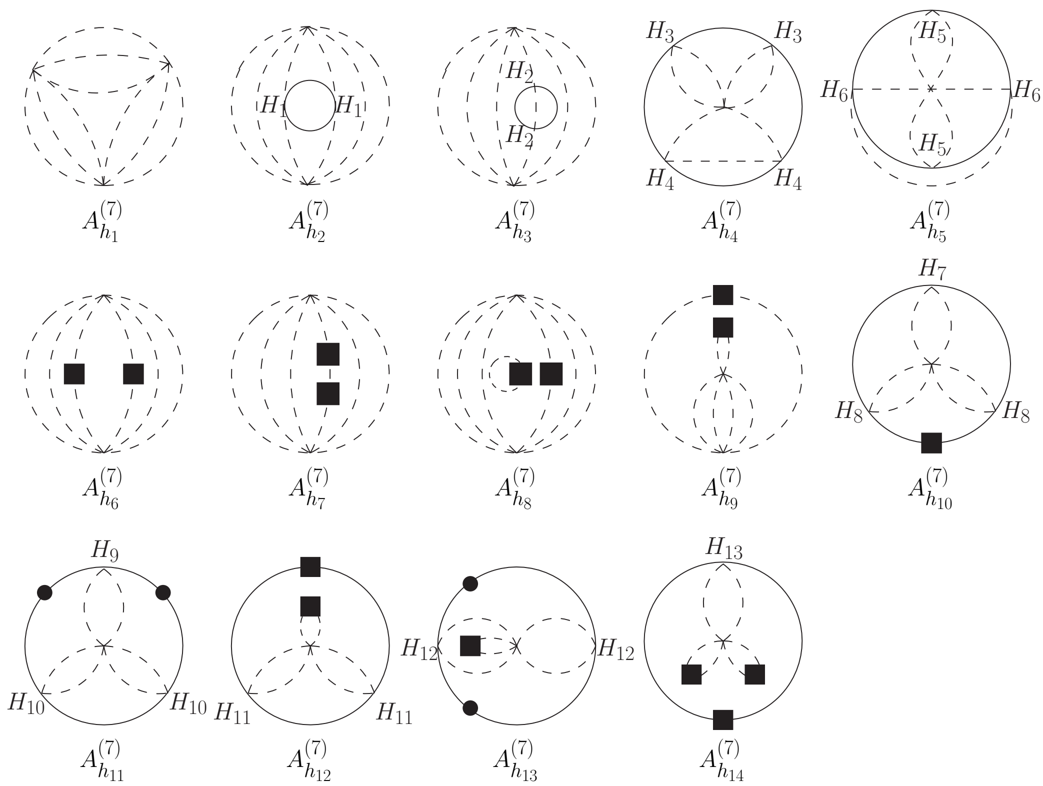


Table 5.2: $A_h^{(7)}$ and $A_{hY}^{(7)}$ parts of 3D A-function

in $\overline{\text{MS}}$, we find that [47]

$$\begin{aligned}
&c_{H_1}^{(4)} = \tfrac{1}{3}, \quad c_{H_2}^{(4)} = \tfrac{2}{45}, \quad c_{H_3}^{(4)} = -8, \quad c_{H_4}^{(4)} = -8, \quad c_{H_5}^{(4)} = -8, \quad c_{H_6}^{(4)} = -8,\\
&c_{H_7}^{(4)} = 4, \quad c_{H_8}^{(4)} = 8, \quad c_{H_9}^{(4)} = -4, \quad c_{H_{10}}^{(4)} = -8, \quad c_{H_{11}}^{(4)} = 0, \quad c_{H_{12}}^{(4)} = 16,\\
&c_{H_{13}}^{(4)} = 16.
\end{aligned} \tag{5.43}$$

The L-loop scalar β-function takes the form

$$(\beta_h^{(L)})_{ijklmn} = \sum_{p=1}^{m_L} d_p^{(L)} (V_p^{(L)})_{ijklmn}, \tag{5.44}$$

where each $V_p^{(L)}$ denotes one of the m_L possible L-loop tensor structures. The $m_2 = 14$ structures in $\beta_h^{(2)}$ are

$$\begin{aligned}
(V_1^{(2)})_{ijklmn} &= \tfrac{1}{6!}(h_{ijkpqr}h_{lmnpqr} + \text{perms}),\\
(V_2^{(2)})_{ijklmn} &= \tfrac{1}{6!}(h_{ijklpq}Y_{abmp}Y_{abnq} + \text{perms}),\\
(V_3^{(2)})_{ijklmn} &= \tfrac{1}{6!}(h_{ijklmp}Y_{abpq}Y_{abnq} + \text{perms}),\\
(V_4^{(2)})_{ijklmn} &= \tfrac{1}{6!}(Y_{abij}Y_{bckl}Y_{cdmp}Y_{dapn} + \text{perms}),\\
(V_5^{(2)})_{ijklmn} &= \tfrac{1}{6!}(Y_{abij}Y_{bcmp}Y_{cdkl}Y_{dapn} + \text{perms}),
\end{aligned}$$

$$
\begin{aligned}
(V_6^{(2)})_{ijklmn} &= \tfrac{1}{6!}(h_{ijklpq}(E^{\phi 2})_{pm}(E^{\phi 2})_{qn} + \text{perms}),\\
(V_7^{(2)})_{ijklmn} &= \tfrac{1}{6!}(h_{ijklmp}(E^{\phi 4})_{pm} + \text{perms}),\\
(V_8^{(2)})_{ijklmn} &= \tfrac{1}{6!}(h_{ijklmp}(E^{\phi 2})_{pm}[\text{tr}(E^{\phi 2}) + \text{tr}(E^{\psi 2})] + \text{perms}),\\
(V_9^{(2)})_{ijklmn} &= \tfrac{1}{6!}(h_{ijklpq}(E^{\phi 2})_{qp}(E^{\phi 2})_{mn} + \text{perms}),\\
(V_{10}^{(2)})_{ijklmn} &= \tfrac{1}{6!}(Y_{abij}E^{\psi 2}_{bc}Y_{cdkl}Y_{damn} + \text{perms}),\\
(V_{11}^{(2)})_{ijklmn} &= \tfrac{1}{6!}(Y_{abij}E^{\psi}_{bc}Y_{cdkl}E^{\psi}_{de}Y_{eamn} + \text{perms}),\\
(V_{12}^{(2)})_{ijklmn} &= \tfrac{1}{6!}(E^{\phi 2}_{ij}Y_{abkl}E^{\psi 2}_{bc}Y_{camn} + \text{perms}),\\
(V_{13}^{(2)})_{ijklmn} &= \tfrac{1}{6!}(E^{\phi 2}_{ij}Y_{abkl}E^{\psi}_{bc}Y_{cdmn}E^{\psi}_{da} + \text{perms}),\\
(V_{14}^{(2)})_{ijklmn} &= \tfrac{1}{6!}(E^{\phi 2}_{ij}E^{\phi 2}_{kl}Y_{abmn}E^{\psi 2}_{ba} + \text{perms}), \qquad (5.45)
\end{aligned}
$$

where, as with the tensor structures in $\beta_Y^{(2)}$ containing four or six gauge insertions, we have simply listed all structures that correspond to non-manifestly-vanishing contributions from Feynman diagrams. We have again calculated the coefficients in $\overline{\text{MS}}$ [47]:

$$
\begin{aligned}
&d_1^{(2)} = \tfrac{20}{3}, && d_2^{(2)} = 30, && d_3^{(2)} = 4, && d_4^{(2)} = -360, && d_5^{(2)} = -360,\\
&d_6^{(2)} = -120, && d_7^{(2)} = -40, && d_8^{(2)} = -8, && d_9^{(2)} = -120, && d_{10}^{(2)} = 360,\\
&d_{11}^{(2)} = -360, && d_{12}^{(2)} = 0, && d_{13}^{(2)} = 720, && d_{14}^{(2)} = 1440. && \qquad (5.46)
\end{aligned}
$$

We may now attempt to construct part of A beyond leading order. The key equation (2.1) takes the form

$$
\begin{aligned}
d_h A^{(7)} &= dh\, T^{(5)}_{hh}\beta^{(2)}_h && (5.47)\\
d_Y A^{(7)} &= dY\, T^{(5)}_{YY}\beta^{(2)}_Y + dY\, T^{(3)}_{YY}\beta^{(4)}_Y. && (5.48)
\end{aligned}
$$

Recall that $T^{(3)}_{IJ} = \mu\,\delta_{IJ}$, and $T^{(5)}_{hh}$ will likewise be found to be proportional to the unit tensor with some coefficient λ. Working with a general theory, it now becomes obvious why no mixed terms in the metric were required in order to solve (5.19): any such mixed metric terms of the correct loop order must necessarily contain "pinched loops"[4], which vanish in dimensionally regularised theories. Solving (5.47) therefore gives the A-function coefficients

$$
\begin{aligned}
a^{(7)}_{h_1} &= \tfrac{1}{3}\lambda d^{(2)}_1,\\
a^{(7)}_{h_i} &= \tfrac{1}{2}\lambda d^{(2)}_i, \quad i = 2\text{–}3, 6\text{–}9,\\
a^{(7)}_{h_i} &= \lambda d^{(2)}_i, \quad i = 4\text{–}5, 10\text{–}14, \qquad (5.49)
\end{aligned}
$$

[4]By pinched loops, we mean diagrams that may be separated into two disconnected regions by cutting at a vertex - such diagrams must contain a factor of the form $\int \mathrm{d}^d p\, (p^2)^\alpha \equiv 0$.

with $T^{(5)}_{hh} = \lambda$. These values, together with the schematic form of $\beta^{(4)}_Y(h)$, can be substituted into (5.48), and the existence of a solution then predicts the following relations between coefficients in $\beta^{(2)}_h$ and $\beta^{(4)}_Y$:

$$\begin{aligned}
&\mu c^{(4)}_{H_1} = \lambda d^{(2)}_2, \quad &&\mu c^{(4)}_{H_2} = \lambda d^{(2)}_3, \quad &&\mu c^{(4)}_{H_3} = 2\lambda d^{(2)}_4, \quad &&\mu c^{(4)}_{H_4} = 2\lambda d^{(2)}_4,\\
&\mu c^{(4)}_{H_5} = 2\lambda d^{(2)}_5, &&\mu c^{(4)}_{H_6} = 2\lambda d^{(2)}_5, &&\mu c^{(4)}_{H_7} = \lambda d^{(2)}_{10}, &&\mu c^{(4)}_{H_8} = 2\lambda d^{(2)}_{10},\\
&\mu c^{(4)}_{H_9} = \lambda d^{(2)}_{11}, &&\mu c^{(4)}_{H_{10}} = 2\lambda d^{(2)}_{11}, &&\mu c^{(4)}_{H_{11}} = 2\lambda d^{(2)}_{12}, &&\mu c^{(4)}_{H_{12}} = 2\lambda d^{(2)}_{13},\\
&\mu c^{(4)}_{H_{13}} = \lambda d^{(2)}_{14}.
\end{aligned} \tag{5.50}$$

Comparing the $\beta^{(4)}_Y(h)$ coefficients in (5.43) and the $\beta^{(2)}_h$ coefficients in (5.46), we see that the predictions are indeed correct, provided we have $\lambda = \frac{\mu}{90}$. This linear relation between metric coefficients demonstrates that A continues to exist beyond leading order, up to an overall scale μ, with positive-definite metric

$$T^{(3)}_{IJ} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{90} \end{pmatrix}. \tag{5.51}$$

The existence of a solution to the scalar-dependent part (5.41) of $A^{(7)}$, up to the overall scale μ and with correctly-predicted relations between β-function coefficients, therefore provides very strong evidence that the construction of $A^{(7)}$ will hold for the general theory beyond leading order, involving the full $\beta^{(4)}_Y$.

5.3.2 Pure Yukawa terms

From here, due to the complexity of the calculation, we shall drop gauge interactions. Despite the non-gauge case being trivial at leading order, we shall see that at next-to-leading order there are a large number of consistency conditions that must be satisfied in order for A to exist, and we shall check each condition via direct computation. Combining the non-gauge terms in (5.41) with our results for $A^{(7)}_Y$ below will give a complete calculation of the next-to-leading order A-function for a completely general scalar-fermion theory.

The pure Yukawa contributions lead to a highly non-trivial system of linear equations, listed in Appendix (C.1–C.5). The pure Yukawa terms in A can be parametrised as

$$A^{(7)}_Y = \sum_{i=1}^{52} a^{(7)}_i A^{(7)}_i, \tag{5.52}$$

depicted in Tables 5.3 and 5.4, and the pure Yukawa terms in $\beta_Y^{(4)}$ as

$$\beta_Y^{(4)}(Y) = \sum_{p=1}^{105} c_p^{(4)} (U_p^{(4)})_{abij}, \tag{5.53}$$

where the tensor structures $U_{abij}^{(4)}$ are again understood to be symmetrised over their external legs and to have an overall weight of one. As with (5.41), rather than attempting to list all 105 possible β-function terms, we have simply labelled the corresponding vertices in the tables of tensor structures $A_Y^{(7)}$. Any vertex denoted by an X indicates that there is no associated tensor structure in $\beta_Y^{(4)}(Y)$, according to the usual criteria. Finally, the next-to-leading order metric $T_{YY}^{(5)}$ takes the form

$$T_{YY}^{(5)} = \sum_{i=1}^{18} t_i^{(5)} (T_i^{(5)})_{YY}, \tag{5.54}$$

with the corresponding tensor structures depicted in Table 5.5. As in the four- and six-dimensional cases, the tensor structures are most easily depicted as contractions between dY and β_Y, denoted by a cross and diamond respectively. While the leading order metric $T_{IJ}^{(3)}$ was trivially symmetric, we can see from the diagrams in Table 5.5 that $T_{IJ}^{(5)}$ is also symmetric, confirming our intuition that the three-dimensional A-function has exactly the same behaviour as in four and six dimensions.

The A-function terms have been arranged such that substituting into (5.48) produces consistency conditions in the following order:

- Diagrams 1-6 simply relate the A-function coefficients $a_{1-6}^{(7)}$ to the $\beta_Y^{(4)}$ coefficients, and give no consistency conditions.
- Diagrams 7 and 8 relate tensor structures that appear in $\beta_Y^{(4)}$ to tensor structures that do not appear in $\beta_Y^{(4)}$, hence setting the corresponding β-function coefficients to zero.
- Diagrams 9-21 relate tensor structures that appear in $\beta_Y^{(4)}$ but not in any higher-order metric contributions, giving simple consistency conditions.
- Diagrams 22-47 relate tensor structures which appear both in $\beta_Y^{(4)}$ and in metric contributions, giving non-trivial consistency conditions.
- Diagrams 48-52, along with metric terms $(T_{16-18}^{(5)})_{YY}$ form a closed set of equations independent of the rest of the system.

For convenience, we shall set the scale parameter $\mu = 1$, so that $T_{YY}^{(3)} = \mathbb{1}$; recall that the consequence of the scale factor is simply an overall factor of μ

pre-multiplying A and T_{IJ}. Examples of equations from the first four categories are:

- Substituting $A_1^{(7)}$ gives $6a_1^{(7)} = c_1^{(4)}$, so that $a_1^{(7)} = \frac{1}{6}c_1^{(4)}$ in a similar manner to the lowest-order calculation.
- Substituting $A_7^{(7)}$ gives $2a_7^{(7)} = c_5^{(4)}$ and $4a_7^{(7)} = 0$, hence $c_5^{(4)} = 0$.
- Substituting $A_9^{(7)}$ gives $2a_7^{(7)} = c_7^{(4)}$, $2a_7^{(7)} = c_8^{(4)}$ and $2a_7^{(7)} = c_9^{(4)}$, hence $c_7^{(4)} = c_8^{(4)} = c_9^{(4)}$.
- Substituting $A_{25}^{(7)}$ and $A_{28}^{(7)}$ gives a set of nine equations:

$$
\begin{aligned}
&a_{25}^{(7)} = c_{45}^{(4)}, && a_{25}^{(7)} = c_{46}^{(4)}, && a_{25}^{(7)} = c_{47}^{(4)}, \\
&a_{25}^{(7)} = \tfrac{1}{4}c_1^{(2)}t_5^{(5)}, && a_{25}^{(7)} = \tfrac{1}{2}c_1^{(2)}t_4^{(5)} + c_{48}^{(4)}, && a_{25}^{(7)} = \tfrac{1}{4}c_1^{(2)}t_5^{(5)}, \\
&2a_{28}^{(7)} = c_{58}^{(4)}, && 2a_{28}^{(7)} = \tfrac{1}{4}c_1^{(2)}t_5^{(5)}, && 2a_{28}^{(7)} = \tfrac{1}{2}c_1^{(2)}t_4^{(5)} + \tfrac{1}{4}c_1^{(2)}t_5^{(5)},
\end{aligned} \tag{5.55}
$$

  leading to the consistency conditions $t_4^{(5)} = 0$ and $c_{45}^{(4)} = c_{46}^{(4)} = c_{47}^{(4)} = c_{48}^{(4)} = c_{58}^{(4)}$.

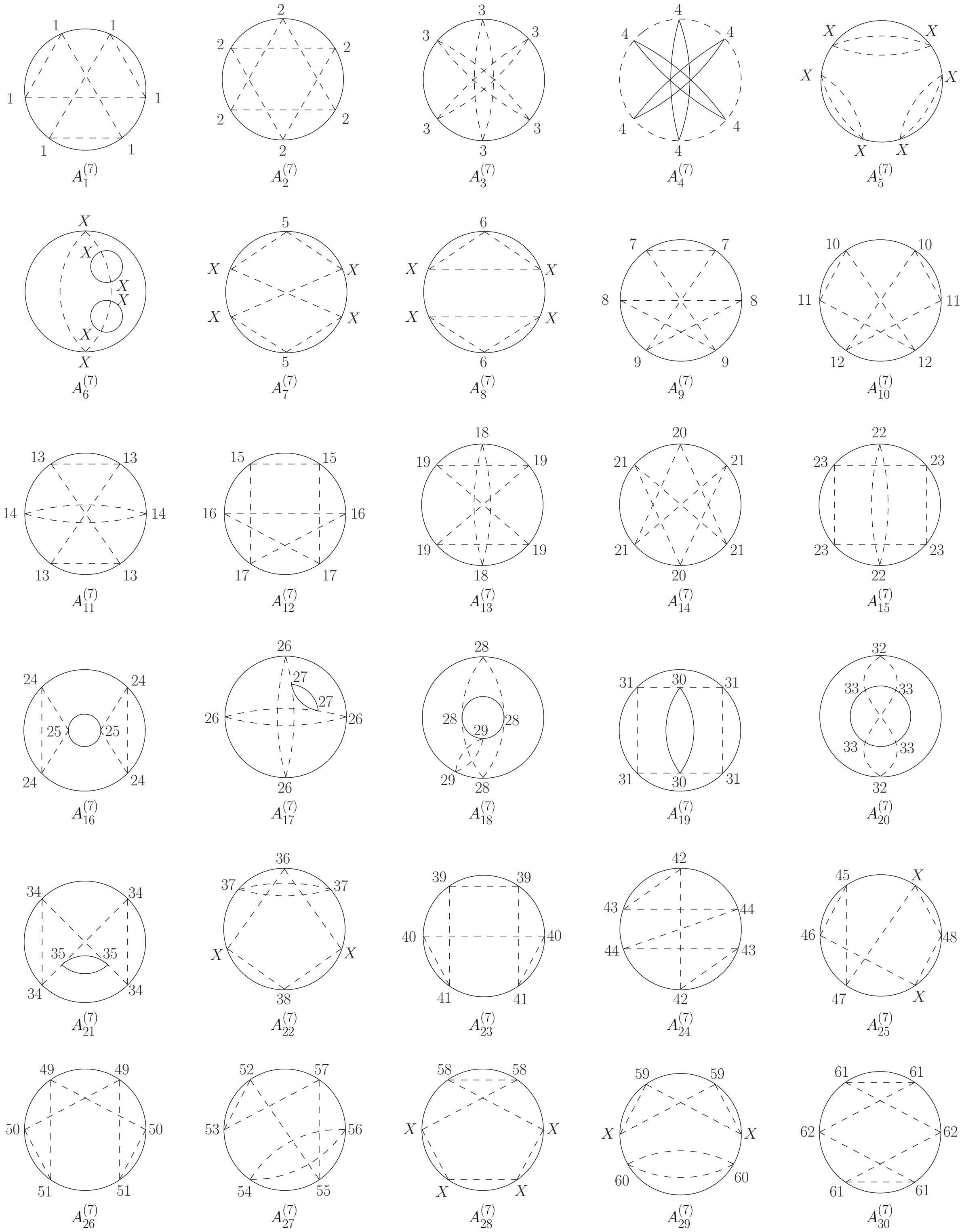


Table 5.3: Contributions to $A_Y^{(7)}$ - terms 1 to 30

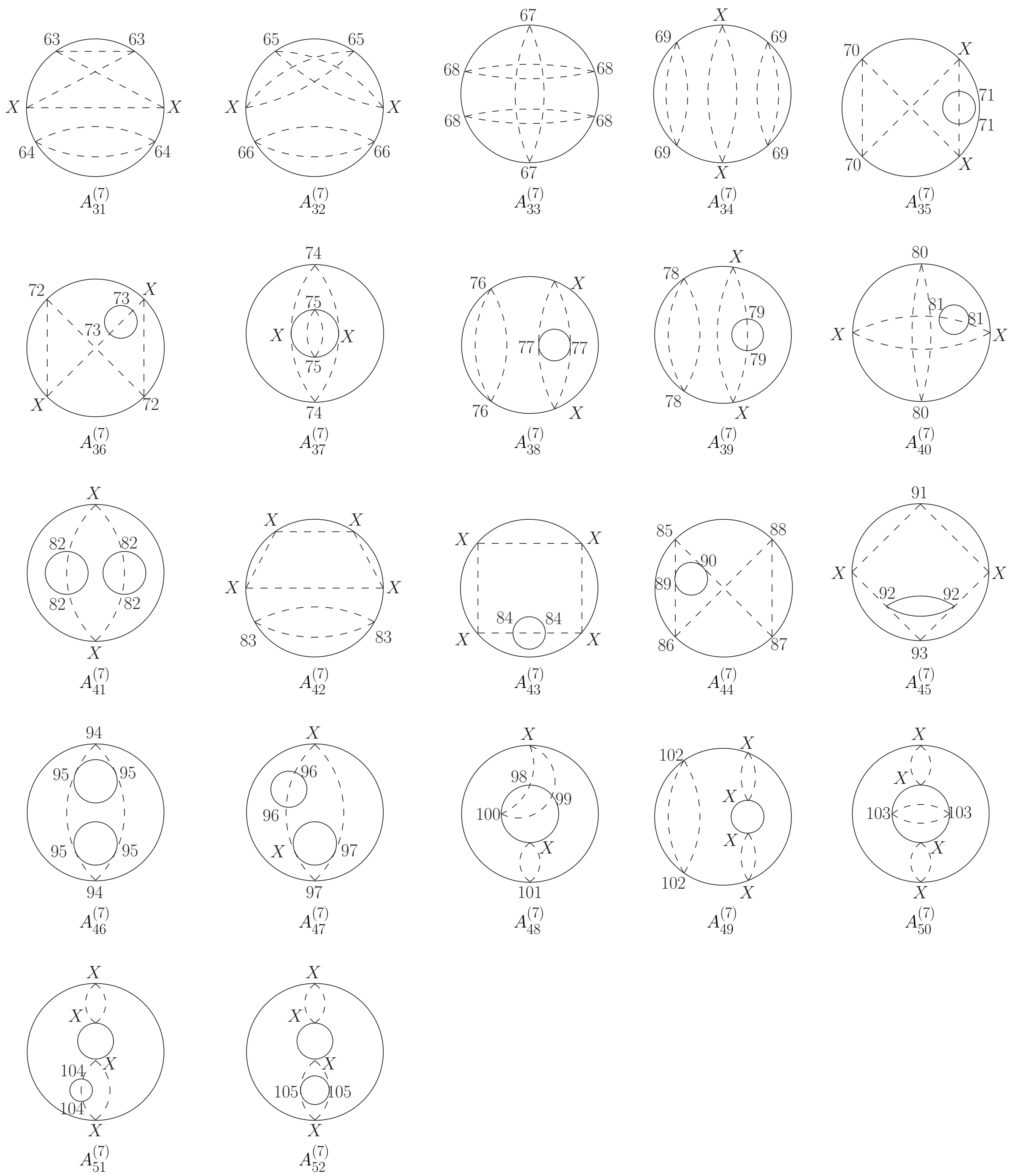


Table 5.4: Contributions to $A_Y^{(7)}$ - terms 31 to 52

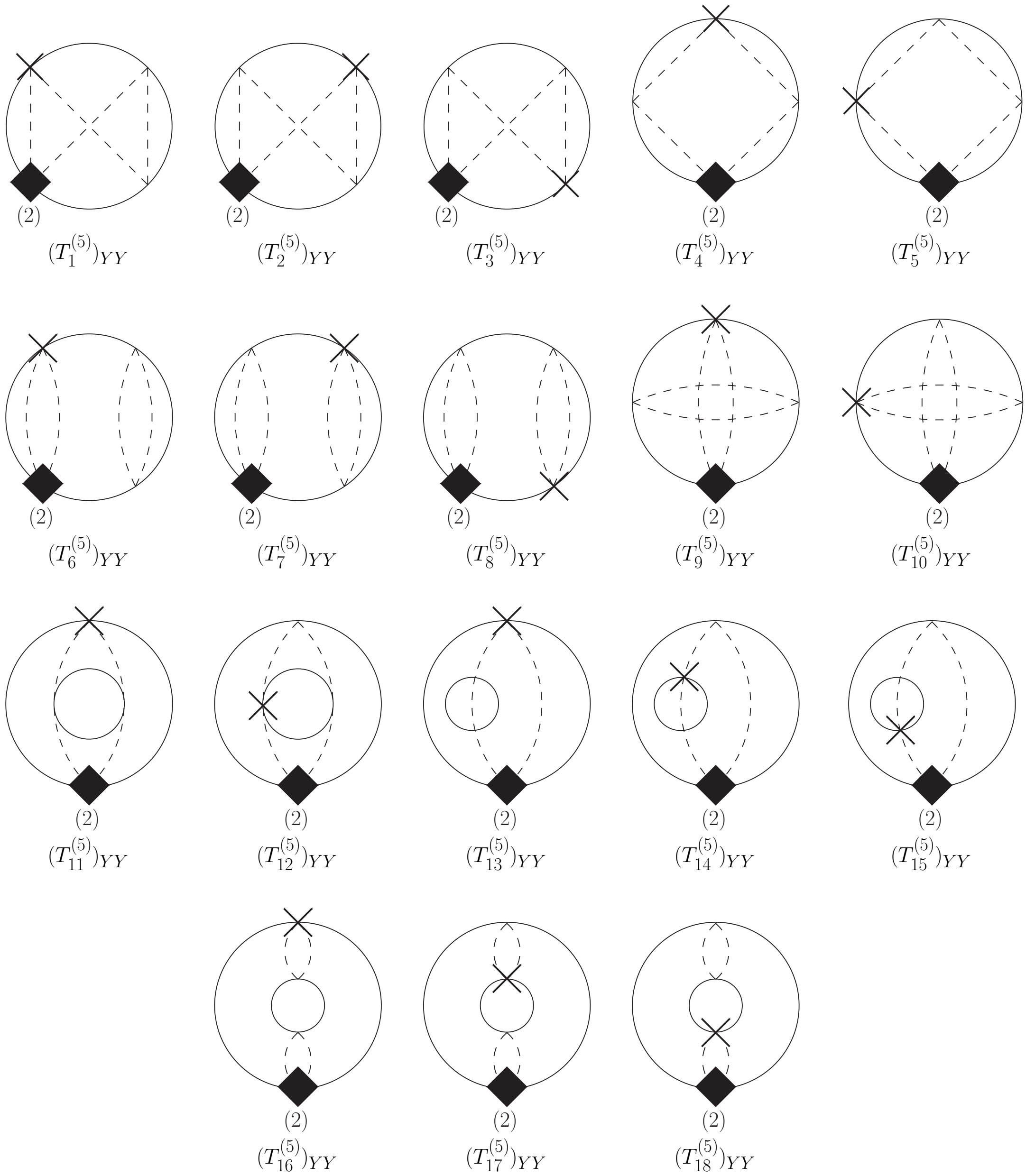


Table 5.5: Contributions to $T^{(5)}_{YY}$

The full list of consistency conditions is given below. In the interest of easily verifying that these consistency conditions hold, we have rewritten certain conditions such that we have expressed as many anomalous dimension type coefficients as possible in terms of non-anomalous dimension coefficients. Our reason for doing so is that the anomalous dimension terms are typically harder to evaluate, as the diagrams are linearly or quadratically divergent, whereas the rest of the terms are logarithmically divergent, and (using integration by parts) can be calculated from the master integrals in [52]. We therefore have a list of consistency conditions relating terms originating from logarithmically divergent diagrams, and twelve predictions for anomalous dimension contributions.

The simple consistency conditions obtained from $A_9^{(7)}$-$A_{21}^{(7)}$ are

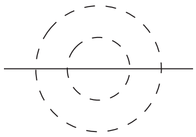
$$\begin{aligned}
c_5^{(4)} &= 0, & c_6^{(4)} &= 0, & c_7^{(4)} &= c_8^{(4)} = c_9^{(4)}, \\
c_{10}^{(4)} &= c_{11}^{(4)} = c_{12}^{(4)}, & c_{13}^{(4)} &= 2c_{14}^{(4)}, & c_{15}^{(4)} &= c_{16}^{(4)} = c_{17}^{(4)}, \\
2c_{18}^{(4)} &= c_{19}^{(4)}, & 2c_{20}^{(4)} &= c_{21}^{(4)}, & 2c_{22}^{(4)} &= c_{23}^{(4)}, \\
c_{24}^{(4)} &= 2c_{25}^{(4)}, & 2c_{26}^{(4)} &= c_{27}^{(4)}, & c_{28}^{(4)} &= c_{29}^{(4)}, \\
2c_{30}^{(4)} &= c_{31}^{(4)}, & 2c_{32}^{(4)} &= c_{33}^{(4)}, & c_{34}^{(4)} &= 2c_{35}^{(4)}, \\
c_{54}^{(4)} &= c_{56}^{(4)}, & c_{89}^{(4)} &= c_{90}^{(4)},
\end{aligned} \tag{5.56}$$

while those resulting from $A_{22}^{(7)}$-$A_{47}^{(7)}$ are

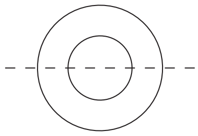
$$\begin{aligned}
c_{40}^{(4)} - c_{39}^{(4)} &= c_{42}^{(4)} - c_{44}^{(4)} = c_{50}^{(4)} - c_{49}^{(4)} \\
&= c_{52}^{(4)} - c_{57}^{(4)} = 3(c_{70}^{(4)} - c_{72}^{(4)}) = c_{87}^{(4)} - c_{86}^{(4)}, \\
c_{40}^{(4)} - c_{41}^{(4)} &= c_{42}^{(4)} - c_{43}^{(4)} = c_{50}^{(4)} - c_{51}^{(4)} = c_{52}^{(4)} - c_{53}^{(4)} \\
&= 6(c_{70}^{(4)} - c_{72}^{(4)}) - c_{61}^{(4)} + c_{62}^{(4)} = c_{87}^{(4)} - c_{88}^{(4)}, \\
c_{55}^{(4)} - 4c_{67}^{(4)} = \tfrac{1}{2}(c_{56}^{(4)} - 2c_{68}^{(4)}), &\quad 3(c_{70}^{(4)} + c_{72}^{(4)}) + c_{85}^{(4)} = c_{88}^{(4)} + 12c_{97}^{(4)}, \\
c_{75}^{(4)} = 12(c_{80}^{(4)} - c_{97}^{(4)}), &\quad c_{85}^{(4)} - c_{89}^{(4)} = 2(2c_{94}^{(4)} - c_{95}^{(4)}),
\end{aligned} \tag{5.57}$$

and

$$\begin{aligned}
c_{52}^{(4)} - c_{55}^{(4)} - 6c_{63}^{(4)} + 12c_{65}^{(4)} + 6c_{70}^{(4)} + c_{85}^{(4)} - c_{87}^{(4)} - 12c_{97}^{(4)} &= 0, \\
3c_{59}^{(4)} + 3c_{70}^{(4)} - 6c_{77}^{(4)} - c_{87}^{(4)} + c_{89}^{(4)} - 6c_{97}^{(4)} &= 0, \\
6c_{63}^{(4)} - 3c_{70}^{(4)} + 3c_{72}^{(4)} - 12c_{77}^{(4)} - c_{85}^{(4)} - c_{88}^{(4)} + 2c_{89}^{(4)} &= 0, \\
3c_{65}^{(4)} + c_{74}^{(4)} - 3c_{77}^{(4)} &= 0, \\
4c_{67}^{(4)} - c_{68}^{(4)} + 2c_{74}^{(4)} + 6c_{80}^{(4)} + 4c_{94}^{(4)} - 2c_{95}^{(4)} - 6c_{97}^{(4)} &= 0, \\
2c_{36}^{(4)} = c_{37}^{(4)} = 2c_{38}^{(4)} = c_{45}^{(4)} = c_{46}^{(4)} = c_{47}^{(4)} = c_{48}^{(4)} = c_{58}^{(4)} = 2c_{91}^{(4)} = c_{92}^{(4)} &= 2c_{93}^{(4)}.
\end{aligned} \tag{5.58}$$



Integral for $c_{69}^{(4)}$



Integral for $c_{82}^{(4)}$

Table 5.6: Feynman integrals for terms undetermined by (5.48)

The condition resulting from $A_{48}^{(7)}$-$A_{52}^{(7)}$ is

$$c_{98}^{(4)} = c_{99}^{(4)} = c_{100}^{(4)} = c_{101}^{(4)} = 6c_{102}^{(4)} = 4c_{103}^{(4)} = 6c_{104}^{(4)} = 4c_{105}^{(4)}, \tag{5.59}$$

and the predictions for twelve out of fourteen anomalous dimension type coefficients are

$$\begin{aligned}
c_{60}^{(4)} &= c_{59}^{(4)} - 2c_{63}^{(4)} + 12c_{69}^{(4)} + 2c_{70}^{(4)} - \tfrac{1}{6}c_{86}^{(4)} - \tfrac{1}{6}c_{87}^{(4)} + \tfrac{1}{3}c_{90}^{(4)} + \tfrac{4}{3}c_{94}^{(4)} - \tfrac{2}{3}c_{95}^{(4)} - 4c_{97}^{(4)}, \\
c_{64}^{(4)} &= -c_{63}^{(4)} + 12c_{69}^{(4)} + c_{70}^{(4)} - \tfrac{1}{6}c_{86}^{(4)} + \tfrac{1}{6}c_{90}^{(4)} + \tfrac{2}{3}c_{94}^{(4)} - \tfrac{1}{3}c_{95}^{(4)} - 2c_{97}^{(4)}, \\
c_{66}^{(4)} &= -\tfrac{1}{2}c_{63}^{(4)} + 6c_{69}^{(4)} + \tfrac{1}{2}c_{70}^{(4)} + \tfrac{1}{6}c_{74}^{(4)} - \tfrac{1}{2}c_{80}^{(4)} + \tfrac{1}{3}c_{94}^{(4)} - \tfrac{1}{6}c_{95}^{(4)} - \tfrac{1}{2}c_{97}^{(4)}, \\
c_{71}^{(4)} &= 12c_{82}^{(4)} - \tfrac{1}{6}c_{86}^{(4)} + \tfrac{1}{6}c_{90}^{(4)} + \tfrac{2}{3}c_{94}^{(4)} - \tfrac{1}{3}c_{95}^{(4)} - 2c_{97}^{(4)}, \\
c_{73}^{(4)} &= 12c_{82}^{(4)} - \tfrac{1}{6}c_{87}^{(4)} + \tfrac{1}{6}c_{90}^{(4)} + \tfrac{2}{3}c_{94}^{(4)} - \tfrac{1}{3}c_{95}^{(4)} - 2c_{97}^{(4)}, \\
c_{76}^{(4)} &= -c_{63}^{(4)} + 6c_{69}^{(4)} + c_{70}^{(4)} + c_{77}^{(4)} + \tfrac{2}{3}c_{94}^{(4)} - \tfrac{1}{3}c_{95}^{(4)} - 2c_{97}^{(4)}, \\
c_{78}^{(4)} &= -\tfrac{1}{6}c_{63}^{(4)} + 2c_{69}^{(4)} + \tfrac{1}{6}c_{70}^{(4)}, \\
c_{79}^{(4)} &= \tfrac{1}{6}c_{63}^{(4)} - \tfrac{1}{6}c_{70}^{(4)} + 2c_{82}^{(4)}, \\
c_{81}^{(4)} &= \tfrac{1}{3}c_{74}^{(4)} + 6c_{82}^{(4)} + \tfrac{2}{3}c_{94}^{(4)} - \tfrac{1}{3}c_{95}^{(4)} - c_{97}^{(4)}, \\
c_{83}^{(4)} &= \tfrac{1}{6}c_{92}^{(4)}, \\
c_{84}^{(4)} &= \tfrac{1}{6}c_{92}^{(4)}, \\
c_{96}^{(4)} &= 6c_{82}^{(4)} + \tfrac{1}{3}c_{94}^{(4)} - \tfrac{1}{6}c_{95}^{(4)} - c_{97}^{(4)},
\end{aligned} \tag{5.60}$$

the undetermined coefficients being $c_{69}^{(4)}$ and $c_{82}^{(4)}$. To obtain all fourteen anomalous dimension coefficients, we therefore need only evaluate the two Feynman integrals in Table 5.6, then deduce their associated β-function coefficients.

We have calculated the non-anomalous dimension terms in $\overline{\text{MS}}$ via integration by parts, using [52][5]:

$c_1^{(4)} = -8,\qquad c_2^{(4)} = 32,\qquad c_3^{(4)} = -4,\qquad c_4^{(4)} = -2,$

[5]With the exception of five integrals involving double propagators that must instead be calculated directly. We shall discuss new relations between these remaining integrals shortly.

$$\begin{aligned}
&c_5^{(4)} = 0, && c_6^{(4)} = 0, && c_7^{(4)} = 4(\pi^2-8), && c_8^{(4)} = 4(\pi^2-8),\\
&c_9^{(4)} = 4(\pi^2-8), && c_{10}^{(4)} = 16, && c_{11}^{(4)} = 16, && c_{12}^{(4)} = 16,\\
&c_{13}^{(4)} = -8, && c_{14}^{(4)} = -4, && c_{15}^{(4)} = 4\pi^2, && c_{16}^{(4)} = 4\pi^2,\\
&c_{17}^{(4)} = 4\pi^2, && c_{18}^{(4)} = 2(\pi^2-8), && c_{19}^{(4)} = 4(\pi^2-8), && c_{20}^{(4)} = 16\left(\tfrac{\pi^2}{3}-2\right),\\
&c_{21}^{(4)} = 32\left(\tfrac{\pi^2}{3}-2\right), && c_{22}^{(4)} = \pi^2, && c_{23}^{(4)} = 2\pi^2, && c_{24}^{(4)} = 16,\\
&c_{25}^{(4)} = 8, && c_{26}^{(4)} = 32, && c_{27}^{(4)} = 64, && c_{28}^{(4)} = 0,\\
&c_{29}^{(4)} = 0, && c_{30}^{(4)} = \pi^2, && c_{31}^{(4)} = 2\pi^2, && c_{32}^{(4)} = 2\pi^2,\\
&c_{33}^{(4)} = 4\pi^2, && c_{34}^{(4)} = 8, && c_{35}^{(4)} = 4, && c_{36}^{(4)} = \pi^2,\\
&c_{37}^{(4)} = 2\pi^2, && c_{38}^{(4)} = \pi^2, && c_{39}^{(4)} = 8, && c_{40}^{(4)} = 16,\\
&c_{41}^{(4)} = 0, && c_{42}^{(4)} = 8, && c_{43}^{(4)} = -8, && c_{44}^{(4)} = 0,\\
&c_{45}^{(4)} = 2\pi^2, && c_{46}^{(4)} = 2\pi^2, && c_{47}^{(4)} = 2\pi^2, && c_{48}^{(4)} = 2\pi^2,\\
&c_{49}^{(4)} = 16, && c_{50}^{(4)} = 24, && c_{51}^{(4)} = 8, && c_{52}^{(4)} = 8,\\
&c_{53}^{(4)} = -8, && c_{54}^{(4)} = 0, && c_{55}^{(4)} = 16, && c_{56}^{(4)} = 0,\\
&c_{57}^{(4)} = 0, && c_{58}^{(4)} = 2\pi^2, && c_{59}^{(4)} = \tfrac{16}{3}, && c_{61}^{(4)} = 0,\\
&c_{62}^{(4)} = 0, && c_{63}^{(4)} = \tfrac{8}{3}, && c_{65}^{(4)} = \tfrac{8}{3}, && c_{67}^{(4)} = 4,\\
&c_{68}^{(4)} = 0, && c_{70}^{(4)} = 8, && c_{72}^{(4)} = \tfrac{16}{3}, && c_{74}^{(4)} = 4,\\
&c_{75}^{(4)} = 0, && c_{77}^{(4)} = 4, && c_{80}^{(4)} = \tfrac{8}{3}, && c_{85}^{(4)} = 0,\\
&c_{86}^{(4)} = 16, && c_{87}^{(4)} = 24, && c_{88}^{(4)} = 8, && c_{89}^{(4)} = 24,\\
&c_{90}^{(4)} = 24, && c_{91}^{(4)} = \pi^2, && c_{92}^{(4)} = 2\pi^2, && c_{93}^{(4)} = \pi^2,\\
&c_{94}^{(4)} = -2, && c_{95}^{(4)} = 8, && c_{97}^{(4)} = \tfrac{8}{3}, && c_{98}^{(4)} = 2\pi^2,\\
&c_{99}^{(4)} = 2\pi^2, && c_{100}^{(4)} = 2\pi^2, && c_{101}^{(4)} = 2\pi^2, && c_{103}^{(4)} = \tfrac{\pi^2}{2},\\
&c_{105}^{(4)} = \tfrac{\pi^2}{2}.
\end{aligned} \tag{5.61}$$

From these results, one can verify that (5.56 – 5.59) are all satisfied. After evaluating the integrals in Table 5.6, subtracting the central two-loop subdivergences and deducing the associated β-function coefficients, we find

$$c_{69}^{(4)} = \tfrac{4}{27}, \quad c_{82}^{(4)} = \tfrac{22}{27}, \tag{5.62}$$

hence the other anomalous dimension coefficients are predicted by (5.60) to be

$$\begin{aligned}
&c_{60}^{(4)} = \tfrac{4}{9}, && c_{64}^{(4)} = -\tfrac{8}{9}, && c_{66}^{(4)} = -\tfrac{4}{9}, && c_{71}^{(4)} = \tfrac{16}{9}\\
&c_{73}^{(4)} = \tfrac{4}{9} && c_{76}^{(4)} = \tfrac{8}{9}, && c_{78}^{(4)} = \tfrac{32}{27}, && c_{79}^{(4)} = \tfrac{20}{27},\\
&c_{81}^{(4)} = -\tfrac{4}{9}, && c_{83}^{(4)} = \tfrac{\pi^2}{3} && c_{84}^{(4)} = \tfrac{\pi^2}{3}, && c_{96}^{(4)} = \tfrac{2}{9},\\
&c_{102}^{(4)} = \tfrac{\pi^2}{3}, && c_{104}^{(4)} = \tfrac{\pi^2}{3}.
\end{aligned} \tag{5.63}$$

We have checked the majority of these predictions by explicit computation, and found them to be correct.

A final check on the consistency conditions is that they should be scheme-independent. This can be verified (as in even dimensions) by deducing the changes in coefficients under a coupling redefinition, corresponding to a change in renormalization scheme, and checking that all consistency conditions are invariant under such changes. The change in $\beta_Y^{(4)}(Y)$ is given by

$$\delta\beta_Y^{(4)} = \beta_Y^{(2)} \cdot \frac{\partial}{\partial Y}\, \delta Y^{(2)} - \delta Y^{(2)} \cdot \frac{\partial}{\partial Y}\, \beta_Y^{(2)}, \tag{5.64}$$

so writing

$$\delta Y^{(2)} = \sum_1^5 \delta_i U_i^{(2)}, \tag{5.65}$$

we see that the induced changes are

$$\begin{array}{llll}
\delta c_{52}^{(4)} = 2(\delta_1 - 4\delta_2) & \delta c_{53}^{(4)} = 2(\delta_1 - 4\delta_2) & \delta c_{54}^{(4)} = 4(4\delta_2 - \delta_1) & \delta c_{55}^{(4)} = 2(4\delta_2 - \delta_1) \\
\delta c_{56}^{(4)} = 4(4\delta_2 - \delta_1) & \delta c_{57}^{(4)} = 2(\delta_1 - 4\delta_2) & \delta c_{59}^{(4)} = \frac{2}{3}\delta_1 - 8\delta_4 & \delta c_{60}^{(4)} = 8\delta_4 - \frac{2}{3}\delta_1 \\
\delta c_{63}^{(4)} = \frac{2}{3}\delta_1 - 8\delta_4 & \delta c_{64}^{(4)} = 8\delta_4 - \frac{2}{3}\delta_1 & \delta c_{65}^{(4)} = \frac{4}{3}\delta_2 - 4\delta_4 & \delta c_{66}^{(4)} = 4\delta_4 - \frac{4}{3}\delta_2 \\
\delta c_{70}^{(4)} = \frac{2}{3}\delta_1 - 8\delta_5 & \delta c_{71}^{(4)} = 8\delta_5 - \frac{2}{3}\delta_1 & \delta c_{72}^{(4)} = \frac{2}{3}\delta_1 - 8\delta_5 & \delta c_{73}^{(4)} = 8\delta_5 - \frac{2}{3}\delta_1 \\
\delta c_{74}^{(4)} = 4(\delta_3 - \delta_2) & \delta c_{75}^{(4)} = 16(\delta_2 - \delta_3) & \delta c_{76}^{(4)} = 4\delta_4 - \frac{4}{3}\delta_3 & \delta c_{77}^{(4)} = \frac{4}{3}\delta_3 - 4\delta_4 \\
\delta c_{78}^{(4)} = \frac{4}{3}(\delta_4 - \delta_5) & \delta c_{79}^{(4)} = \frac{4}{3}(\delta_5 - \delta_4) & \delta c_{80}^{(4)} = \frac{4}{3}\delta_2 - 4\delta_5 & \delta c_{81}^{(4)} = 4\delta_5 - \frac{4}{3}\delta_2 \\
\delta c_{85}^{(4)} = 2(4\delta_3 - \delta_1) & \delta c_{86}^{(4)} = 2(\delta_1 - 4\delta_3) & \delta c_{87}^{(4)} = 2(\delta_1 - 4\delta_3) & \delta c_{88}^{(4)} = 2(\delta_1 - 4\delta_3) \\
\delta c_{89}^{(4)} = 2(4\delta_3 - \delta_1) & \delta c_{90}^{(4)} = 2(4\delta_3 - \delta_1) & \delta c_{96}^{(4)} = 4\delta_5 - \frac{4}{3}\delta_3 & \delta c_{97}^{(4)} = \frac{4}{3}\delta_3 - 4\delta_5,
\end{array} \tag{5.66}$$

while all other coefficients are scheme-independent. One can then verify that (5.56 – 5.60) are invariant under these changes, and hence hold in any renormalization scheme.

By verifying all consistency conditions deduced from (5.48), and checking their scheme-independence, we have verified that it is possible to construct an A-function satisfying (2.1) at next-to-leading order for a general three dimensional scalar-fermion theory, with very strong evidence that the construction exists beyond leading order for a general abelian gauge theory. This function takes the form (5.40), with the scalar terms expressed as (5.41) and the purely Yukawa parts expressed as (5.52). The coefficients for the scalar-dependant part are given by (5.49), with $\overline{\text{MS}}$ values deduced from (5.46) and leading order metric given by

(5.51), while the $\overline{\text{MS}}$ coefficients of the pure Yukawa part are

$$
\begin{aligned}
&a_1^{(7)} = -\tfrac{4}{3}, && a_2^{(7)} = \tfrac{16}{3}, && a_3^{(7)} = -\tfrac{2}{3}, && a_4^{(7)} = -\tfrac{1}{3},\\
&a_5^{(7)} = -\tfrac{7}{162} + \tfrac{1}{24}a, && a_6^{(7)} = \tfrac{11}{162} + \tfrac{1}{24}a, && a_7^{(7)} = 0, && a_8^{(7)} = 0,\\
&a_9^{(7)} = 2\pi^2 - 16, && a_{10}^{(7)} = 8, && a_{11}^{(7)} = -2, && a_{12}^{(7)} = 2\pi^2,\\
&a_{13}^{(7)} = \pi^2 - 8, && a_{14}^{(7)} = \tfrac{8}{3} - 16\pi^2, && a_{15}^{(7)} = \tfrac{\pi^2}{2} && a_{16}^{(7)} = 4,\\
&a_{17}^{(7)} = 8, && a_{18}^{(7)} = 0 && a_{19}^{(7)} = \tfrac{\pi^2}{2}, && a_{20}^{(7)} = \pi^2,\\
&a_{21}^{(7)} = 2, && a_{22}^{(7)} = \pi^2, && a_{23}^{(7)} = -\tfrac{16}{2} + 3a, && a_{24}^{(7)} = -\tfrac{28}{3} + 3a,\\
&a_{25}^{(7)} = 2\pi^2, && a_{26}^{(7)} = -\tfrac{4}{3} + 3a, && a_{27}^{(7)} = -\tfrac{56}{3} + 6a, && a_{28}^{(7)} = \pi^2,\\
&a_{29}^{(7)} = -\tfrac{10}{9} + a, && a_{30}^{(7)} = -\tfrac{28}{3} + 3a, && a_{31}^{(7)} = -\tfrac{16}{9} + a, && a_{32}^{(7)} = -\tfrac{8}{9} + \tfrac{1}{2}a,\\
&a_{33}^{(7)} = -\tfrac{7}{3} + \tfrac{3}{4}a, && a_{34}^{(7)} = -\tfrac{1}{54} + \tfrac{1}{24}a, && a_{35}^{(7)} = \tfrac{8}{9} + a, && a_{36}^{(7)} = \tfrac{2}{9} + a,\\
&a_{37}^{(7)} = -\tfrac{14}{3} + \tfrac{3}{2}a, && a_{38}^{(7)} = -\tfrac{2}{9} + \tfrac{1}{2}a, && a_{39}^{(7)} = \tfrac{10}{27} + \tfrac{1}{6}a, && a_{40}^{(7)} = -\tfrac{2}{9} + \tfrac{1}{2}a,\\
&a_{41}^{(7)} = \tfrac{11}{54} + \tfrac{1}{24}a, && a_{42}^{(7)} = \tfrac{\pi^2}{6}, && a_{43}^{(7)} = \tfrac{\pi^2}{6}, && a_{44}^{(7)} = -\tfrac{8}{3} + 6a,\\
&a_{45}^{(7)} = \pi^2, && a_{46}^{(7)} = -\tfrac{4}{3} + \tfrac{3}{4}a, && a_{47}^{(7)} = \tfrac{1}{9} + \tfrac{1}{2}a, && a_{48}^{(7)} = 2\pi^2,\\
&a_{49}^{(7)} = \tfrac{\pi^2}{12}, && a_{50}^{(7)} = \tfrac{\pi^2}{4}, && a_{51}^{(7)} = \tfrac{\pi^2}{12}, && a_{52}^{(7)} = \tfrac{\pi^2}{4},
\end{aligned}
\tag{5.67}
$$

and the next-to-leading order metric coefficients in (5.54) are

$$
\begin{aligned}
t_1^{(5)} &= -\tfrac{28}{3} + 3a, & t_2^{(5)} &= -\tfrac{40}{3} + 3a, & t_3^{(5)} &= -\tfrac{16}{3} + 3a,\\
t_4^{(5)} &= 0, & t_5^{(5)} &= \pi^2, & t_6^{(5)} &= -\tfrac{1}{9} + \tfrac{1}{4}a,\\
t_7^{(5)} + t_8^{(5)} &= -\tfrac{2}{3} + \tfrac{1}{2}a, & t_9^{(5)} &= -\tfrac{13}{3} + \tfrac{3}{4}a, & t_{10}^{(5)} &= -\tfrac{14}{3} + \tfrac{3}{2}a,\\
t_{11}^{(5)} &= -\tfrac{1}{3} + \tfrac{3}{4}a, & t_{12}^{(5)} &= -\tfrac{20}{3} + \tfrac{1}{2}a, & t_{13}^{(5)} &= \tfrac{11}{9} + \tfrac{3}{4}a,\\
t_{14}^{(5)} + t_{15}^{(5)} &= \tfrac{1}{2}a, & t_{16}^{(5)} &= \tfrac{\pi^2}{4}, & t_{17}^{(5)} &= 0,\\
t_{18}^{(5)} &= \tfrac{\pi^2}{4}.
\end{aligned}
\tag{5.68}
$$

As was found in even dimensions, A and T are determined up to the arbitrariness a present in (5.40); there is also some additional arbitrariness in the definition of $T_{YY}^{(5)}$, since only the sums $t_7^{(5)} + t_8^{(5)}$, $t_{14}^{(5)} + t_{15}^{(5)}$ are determined. Finally, it is worth noting that $t_i^{(5)}$ is scheme-independent for $i = \{4, 5, 13, 14, 15, 16, 17, 18\}$, hence $t_4^{(5)}$ and $t_{17}^{(5)}$ are both zero in an arbitrary scheme.

5.3.3 Relations between Feynman integrals

As mentioned previously, with the exception of five integrals involving double propagators, the non-anomalous dimension contributions to $\beta_Y^{(4)}(Y)$ were all determined from the master integrals in [52], using integration by parts. By this method alone, one can reduce the basis of Feynman diagrams to ten, shown in

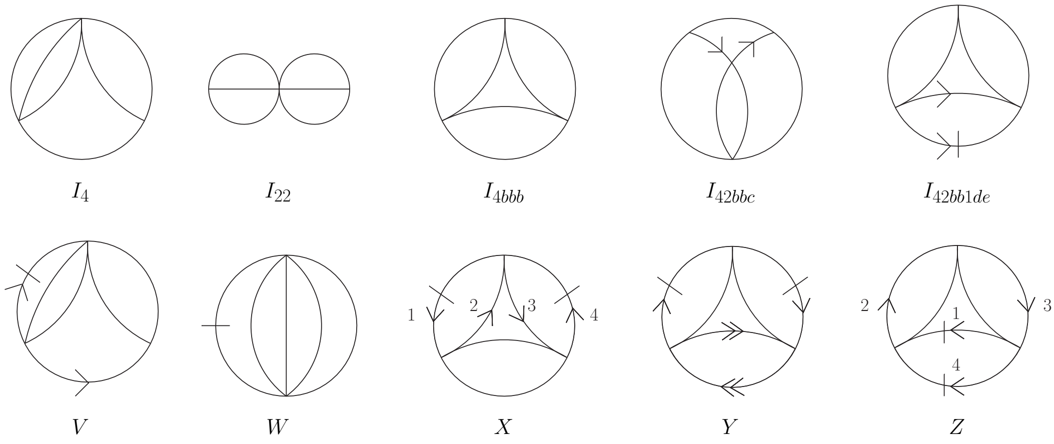


Table 5.7: Feynman integrals that appear in the non-anomalous dimension contributions to $\beta_Y^{(4)}(Y)$

Table 5.7. The first five integrals are named following the conventions of [52], while the latter five require new labels. As far as we can tell, there is no simple process of integration by parts that will further reduce this basis of integrals.

Having constructed an A-function in three dimensions beyond leading order, an intriguing consequence is that such a function can not only be used to predict the coefficients of various terms in β_Y, as seen in (5.60), but can also be used to derive relations between the Feynman diagrams from which these β-function terms originate. The consistency conditions (5.56) are all derived from equations containing no higher-order metric coefficients, hence for these relations to hold there must exist analogous relations between the Feynman diagrams.

To derive the relations between the Feynman diagrams, one need only express the β-function coefficients as multiples of the simple poles in the Feynman diagrams, then use the consistency conditions (5.56). By doing so, we obtain the following conditions:

$$\begin{aligned} I_4 - \tfrac{1}{2}I_{22} &= I_{42bbc} = -2V = -2W, \\ I_{4bbb} &= -2I_{42bb1de} = 4X = -8Y, \\ Z &= 0 \end{aligned} \tag{5.69}$$

We see that all Feynman integrals used to deduce the non-anomalous dimension terms in $\beta_Y^{(4)}(Y)$ have now been reduced to three simple integrals with no momentum running through. These relations appear to be completely new, and appear to showcase a rather remarkable feature: the existence of a function restricting the behaviour of renormalization-group flow can infer new relations between the divergences encountered in Feynman integrals. The conditions for diagrams X and Y are predicated on the vanishing of $t_4^{(5)}$, but as was noted at the end of the last subsection $t_4^{(5)} = 0$ is a scheme-independent result.

Combined with the deduction of the anomalous dimension terms using (5.60), one can summarise the situation as follows: to derive the pure Yukawa part of $\beta_Y^{(4)}$ for a general theory, one may construct the next-to-leading order A-function and evaluate a total of five simple Feynman integrals; the rest follows from simple integration by parts.

5.4 General $\mathcal{N} = 2$ supersymmetric gauge theory

Throughout this chapter, we have attempted to construct, for three-dimensional quantum field theories, an A-function satisfying (2.1). After explicitly demonstrating the existence of such a function for particular theories, we showed that the A-function exists to leading order for a general Abelian theory in three dimensions, and to next-to-leading order for a scalar-Yukawa theory. Clearly, the obvious next step is to include gauge interactions and attempt to construct A for a completely general theory at next-to-leading order, but this would be a very involved calculation. We may instead attempt to demonstrate that A does indeed exist for a general $\mathcal{N} = 2$ supersymmetric gauge theory,[6] providing evidence that it may then be possible to construct A for a general non-supersymmetric gauge theory. In a supersymmetric theory, one may take advantage of the non-renormalization theorem to significantly reduce the number of potential contributions to $\beta_Y^{(4)}$; furthermore, we need only consider Yukawa-dependent terms, as any Yukawa-independent contributions may easily be shown to satisfy (2.1) without imposing any consistency conditions.

The action for a general $\mathcal{N} = 2$ supersymmetric gauge theory takes the form [54]

$$S = S_{SUSY} + S_{GF} + S_{GH}, \tag{5.70}$$

where S_{SUSY} is the supersymmetric action

$$\begin{aligned} S_{SUSY} = &\int d^3x \int d^4\theta \left(k \int_0^1 dt\, \mathrm{Tr}[\overline{D}^\alpha (e^{-tV} D_\alpha e^{tV})] + \Phi^j (e^{V_A R_A})^i{}_j \Phi_i \right) \\ &+ \left(\int d^3x \int d^2\theta\, W(\Phi) + \text{h.c.} \right), \end{aligned} \tag{5.71}$$

[6] The required four-loop β-function, $\beta_Y^{(4)}$, was calculated in [53], with the exception of one term corresponding to a non-planar integral. We shall comment on this integral at the end of this section.

S_{GF} the gauge-fixing term (with gauge-fixing parameter ξ) [55]

$$S_{GF} = -\tfrac{k}{2\xi}\int d^3x d^2\theta \,\mathrm{tr}[f\overline{f}] - \tfrac{k}{2\xi}\int d^3x d^2\overline{\theta}\,\mathrm{tr}[f\overline{f}], \tag{5.72}$$

and S_{GH} the ghost action [69], which we omit as we do need to consider diagrams with ghost propagators. In the action, $V = V_A T_A$ is a vector superfield in the adjoint representation, with T_A being the generators of the fundamental representation satisfying

$$\begin{aligned}[][T_A, T_B] &= i f_{ABC} T_C, \\ Tr\,(T_A T_B) &= \delta_{AB}.\end{aligned} \tag{5.73}$$

Φ is a chiral matter superfield (with the convention $\Phi^i = \Phi_i{}^*$) that may be in a general representation, with gauge matrices R_A satisfying

$$\begin{aligned}[][R_A, R_B] &= i f_{ABC} R_C, \\ Tr\,(R_A R_B) &= T_R \delta_{AB}.\end{aligned} \tag{5.74}$$

$W(\Phi)$ is the superpotential, which for renormalizability must be quartic in three dimensions, and takes the form

$$W(\Phi) = \tfrac{1}{4!} Y^{ijkl}\Phi_i\Phi_j\Phi_k\Phi_l. \tag{5.75}$$

The Yukawa coupling Y^{ijkl} is dimensionless in three dimensions, and we introduce the further convention $\overline{Y}_{ijkl} = (Y^{ijkl})^*$. Finally, to satisfy gauge invariance, the gauge coupling k is quantized such that $2\pi k$ is an integer.

We shall now attempt to construct A. As mentioned previously, the only β-functions in a general $\mathcal{N} = 2$ theory are those corresponding to the Yukawa coupling and its conjugate; therefore, at leading order, we may expand (2.1) as

$$\begin{aligned} d_Y\, A^{(5)} &= dY T^{(3)}_{Y\overline{Y}} \beta^{(2)}_{\overline{Y}}, \\ d_{\overline{Y}}\, A^{(5)} &= d\overline{Y} T^{(3)}_{\overline{Y}Y} \beta^{(2)}_{Y}.\end{aligned} \tag{5.76}$$

Both β-functions are given (as described above) by the chiral superfield anomalous dimension, which at two loops takes the form

$$\gamma_\Phi = \gamma_1^{(2)} \overline{Y}_{iklm} Y^{klmj} + \gamma_2^{(2)} C(R)_i{}^k C(R)_k{}^j + \gamma_3^{(2)} C(R)_i{}^j. \tag{5.77}$$

Since $\overline{\gamma_\Phi} = \gamma_\Phi$, we see that the coefficients of the corresponding tensor structures appearing in $\beta^{(2)}_Y$, $\beta^{(2)}_{\overline{Y}}$ will be equal. The two-loop β-functions may therefore be expressed as

$$\beta_Y^{(2)} = (\gamma_\Phi^{(2)})_m^{(i} Y^{jkl)m} = \sum_{i=1}^{3} c_i^{(2)} C_i^{(2)},$$
$$\beta_{\overline{Y}}^{(2)} = \overline{Y}_{m(ijk} (\gamma_\Phi^{(2)})_{l)}^{m} = \sum_{i=1}^{3} c_i^{(2)} \overline{C}_i^{(2)}, \tag{5.78}$$

with tensor structures

$$\begin{aligned} C_1^{(2)} = &\ Y^{ijkm}\overline{Y}_{mpqr}Y^{pqrl} + Y^{ijml}\overline{Y}_{mpqr}Y^{pqrk} \\ &+ Y^{imkl}\overline{Y}_{mpqr}Y^{pqrj} + Y^{mjkl}\overline{Y}_{mpqr}Y^{pqri}, \\ C_2^{(2)} = &\ Y^{ijkm}C(R)_m{}^n C(R)_n{}^l + Y^{ijml}C(R)_m{}^n C(R)_n{}^k \\ &+ Y^{imkl}C(R)_m{}^n C(R)_n{}^j + Y^{mjkl}C(R)_m{}^n C(R)_n{}^i, \\ C_3^{(2)} = &\ Y^{ijkm}C(R)_m{}^l + Y^{ijml}C(R)_m{}^k \\ &+ Y^{imkl}C(R)_m{}^j + Y^{mjkl}C(R)_m{}^i. \end{aligned} \tag{5.79}$$

Similarly, we may parametrise the leading order A-function as

$$\begin{aligned} A^{(5)} = &\ a_1^{(5)} Y^{ijkl}\overline{Y}_{ijkm}Y^{mpqr}\overline{Y}_{pqrl} + a_2^{(5)} Y^{ijkl}C(R)_l{}^m C(R)_m{}^n \overline{Y}_{ijkn} \\ &+ a_3^{(5)} Y^{ijkl}C(R)_l{}^m \overline{Y}_{ijkm}, \end{aligned} \tag{5.80}$$

and the lowest order metric $T_{IJ}^{(3)}$ such that

$$dY T_{Y\overline{Y}}^{(3)} \beta_{\overline{Y}}^{(2)} = \mu\ (dY)^{ijkl} (\beta_{\overline{Y}}^{(2)})_{ijkl},$$
$$d\overline{Y} T_{\overline{Y}Y}^{(3)} \beta_Y^{(2)} = \overline{\mu}\ (d\overline{Y})_{ijkl} (\beta_Y^{(2)})^{ijkl}. \tag{5.81}$$

Given that the leading order metric in the non-supersymmetric theory is simply proportional to the unit matrix, we would expect to find that $\mu = \overline{\mu}$. Substituting (5.78), (5.80) and (5.81) into (5.76) then gives the solution

$$\begin{aligned} a_1^{(5)} &= 2c_1^{(2)}\mu & a_2^{(5)} &= 4c_2^{(2)}\mu & a_3^{(5)} &= 4c_3^{(2)}\mu \\ &= 2c_1^{(2)}\overline{\mu}, & &= 4c_2^{(2)}\overline{\mu}, & &= 4c_3^{(2)}\overline{\mu}, \end{aligned} \tag{5.82}$$

so we see immediately that indeed $\mu = \overline{\mu}$, and therefore the leading order metric is proportional to the unit matrix as expected. Due to the simplicity of the construction, the terms in $A^{(5)}$ and $\beta_Y^{(2)}$ are in one-to-one correspondence, hence there are no consistency conditions.

We now turn to the next order. The Yukawa-dependant part of the four-loop chiral superfield anomalous dimension can be expressed as

$$\gamma_\Phi^{(4)} = \sum_{i=1}^{12} \gamma_i^{(4)} \Gamma_i^{(4)} + \dots, \tag{5.83}$$

where the tensor structures

$$\begin{aligned}
(\Gamma_1^{(4)})_i^{\ j} &= \overline{Y}_{iklm}(Y\overline{Y})^m{}_n Y^{klnj}, & (\Gamma_2^{(4)})_i^{\ j} &= \overline{Y}_{iklm} Y^{lmpq} \overline{Y}_{pqrs} Y^{rskj},\\
(\Gamma_3^{(4)})_i^{\ j} &= \overline{Y}_{iklm} Y^{klmp} (C(R)C(R))_p{}^j, & (\Gamma_4^{(4)})_i^{\ j} &= \overline{Y}_{iklm}(C(R)C(R))^m{}_p Y^{klpj},\\
(\Gamma_5^{(4)})_i^{\ j} &= \overline{Y}_{iklm} C(R)^m{}_n Y^{klnp} C(R)_p{}^j, & (\Gamma_6^{(4)})_i^{\ j} &= \overline{Y}_{ikmn} C(R)^m{}_p C(R)^n{}_q Y^{kpqj},\\
(\Gamma_7^{(4)})_i^{\ j} &= \overline{Y}_{iklm}(R_A R_B)^m{}_n Y^{klnp}(R_B R_A)_p{}^j, & (\Gamma_8^{(4)})_i^{\ j} &= \overline{Y}_{ikmn}(R_A R_B)^m{}_p (R_A R_B)^n{}_q Y^{kpqj},\\
(\Gamma_9^{(4)})_i^{\ j} &= (\overline{Y}Y)_i{}^k C(R)_k{}^j, & (\Gamma_{10}^{(4)})_i^{\ j} &= \overline{Y}_{iklm} C(R)^m{}_n Y^{klnj},\\
(\Gamma_{11}^{(4)})_i^{\ j} &= \tfrac{1}{2} tr[Y\overline{Y} R_A R_B](\{R_A, R_B\})_i{}^j, & (\Gamma_{12}^{(4)})_i^{\ j} &= \overline{Y}_{ikmn}(R_A)^m{}_p (R_A)^n{}_q C(R)^q{}_r Y^{kprj},
\end{aligned} \tag{5.84}$$

form a basis of terms with four gauge matrices, and we have defined

$$\begin{aligned}
(Y\overline{Y})^i{}_j &= Y^{iklm}\overline{Y}_{klmj}, & (\overline{Y}Y)_i{}^j &= \overline{Y}_{iklm} Y^{klmj},\\
(C(R)C(R))^i{}_j &= C(R)^i{}_k C(R)^k{}_j, & (R_A R_B)^i{}_j &= (R_A)^i{}_k (R_B)^k{}_j.
\end{aligned} \tag{5.85}$$

Strictly speaking, $\Gamma_{12}^{(4)}$ is superfluous as it can be expressed as

$$\Gamma_{12}^{(4)} = \tfrac{1}{24}\Gamma_3^{(4)} - \tfrac{1}{8}\Gamma_4^{(4)} - \tfrac{1}{4}\Gamma_6^{(4)} - \tfrac{1}{2}\Gamma_8^{(4)} - \tfrac{1}{12} C_G \Gamma_9^{(4)}, \tag{5.86}$$

but it appears naturally in several diagrammatic calculations and gives a vanishing contribution to $\gamma^{(4)}$, hence its presence does not affect our calculation of A.

Expanding (2.1), we wish to solve

$$\begin{aligned}
d_Y A^{(7)} &= dY T^{(3)}_{Y\overline{Y}} \beta^{(4)}_{\overline{Y}} + dY T^{(5)}_{Y\overline{Y}} \beta^{(2)}_{\overline{Y}} + dY T^{(5)}_{YY} \beta^{(2)}_{Y},\\
d_{\overline{Y}} A^{(7)} &= d\overline{Y} T^{(3)}_{\overline{Y}Y} \beta^{(4)}_{Y} + d\overline{Y} T^{(5)}_{\overline{Y}Y} \beta^{(2)}_{Y} + d\overline{Y} T^{(5)}_{\overline{Y}\overline{Y}} \beta^{(2)}_{\overline{Y}},
\end{aligned} \tag{5.87}$$

where again the β-functions are given by

$$\begin{aligned}
\beta_Y^{(4)} &= (\gamma_\Phi^{(4)})_m^{(i} Y^{jkl)m} = \sum_{i=1}^{12} c_i^{(4)} C_i^{(4)},\\
\beta_{\overline{Y}}^{(4)} &= \overline{Y}_{m(ijk} (\gamma_\Phi^{(4)})^{\ m}_{l)} = \sum_{i=1}^{12} c_i^{(4)} \overline{C}_i^{(4)}.
\end{aligned} \tag{5.88}$$

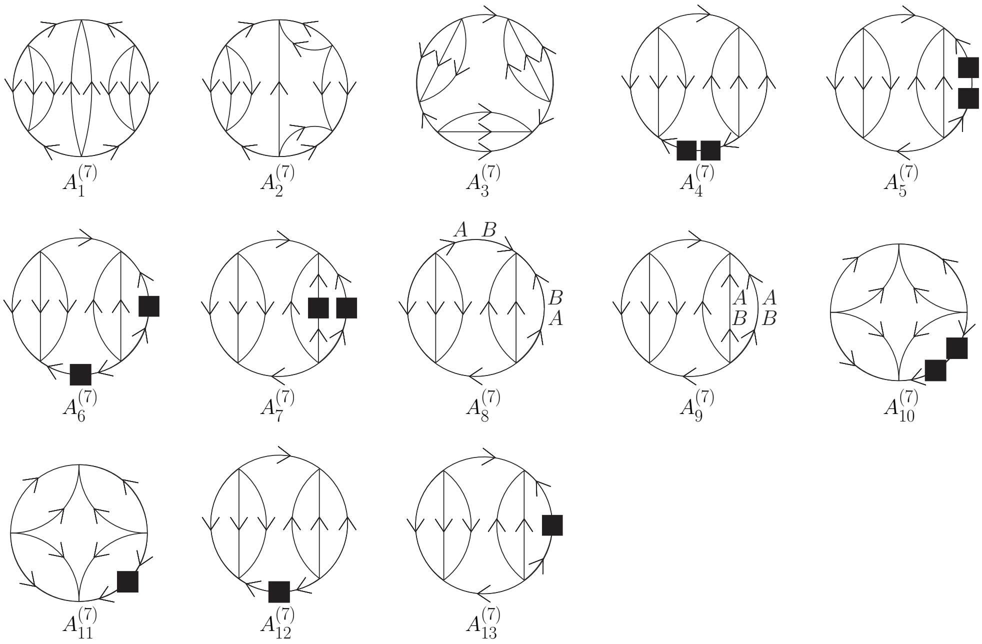


Table 5.8: Contributions to $A^{(7)}$ for $\mathcal{N} = 2$

The A-function at this order takes the form

$$A^{(7)} = \sum_{i=1}^{14} a_i^{(7)} A_i^{(7)} + a(\beta_Y^{(2)})^{ijkl}(\beta_{\overline{Y}}^{(2)})_{ijkl} + \ldots, \tag{5.89}$$

where $A^{(7)}_{1-13}$ are depicted in Table 5.8, $A^{(7)}_{14}$ is given by

$$A_{14}^{(7)} = \tfrac{1}{4}\, tr[Y\overline{Y}\{R_A, R_B\}]\, tr[Y\overline{Y}\{R_A, R_B\}],$$

and we neglect terms that originate from Yukawa-independent contributions to γ_Φ. As previously alluded, for any Yukawa-independent contribution $x^{(4)}X_i{}^j \in \gamma_\Phi^{(4)}$, one may simply add to $A^{(7)}$ a term $a^{(7)}Y^{ijkl}X_{(i}{}^m\overline{Y}_{jkl)m}$, and substituting into (5.87) gives $a^{(7)} = 4\mu x^{(4)}$ with no further consistency conditions. Diagrammatically, Yukawa couplings are represented by the four-point vertices; the convention for chirality is that arrows always point from a Y to a $\overline{Y}$. C_R insertions are represented by boxes, and insertions of gauge matrices R_A, R_B are represented by labels A, B respectively. As an example, diagram $A_9^{(7)}$ corresponds to the tensor structure

$$A_9^{(7)} = Y^{ijkl}\overline{Y}_{jklm}Y^{mnpr}(R_BR_A)_p{}^q(R_BR_A)_r{}^s\overline{Y}_{nqsi}.$$

The leading order metric was shown to be proportional to the unit matrix, $T_{IJ}^{(3)} =$

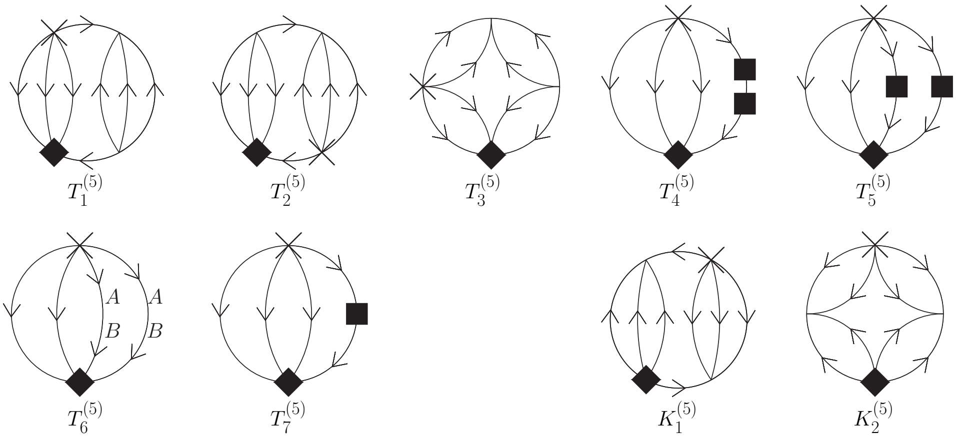


Table 5.9: Contributions to $T^{(5)}$, $K^{(5)}$ for $\mathcal{N} = 2$

$\mu\delta_{IJ}$. The next-to-leading order metric may be expressed as

$$T^{(5)}_{Y\overline{Y}} = \sum_{i=1}^{7} t_i^{(5)} T_i^{(5)}, \qquad T^{(5)}_{YY} = \sum_{i=1}^{2} k_i^{(5)} K_i^{(5)},$$
$$T^{(5)}_{\overline{Y}Y} = \sum_{i=1}^{7} \overline{t}_i^{(5)} \overline{T}_i^{(5)}, \qquad T^{(5)}_{\overline{Y}\overline{Y}} = \sum_{i=1}^{2} \overline{k}_i^{(5)} \overline{K}_i^{(5)}, \tag{5.90}$$

with the tensor structures $T_i^{(5)}$, $K_i^{(5)}$ depicted in Table 5.9, again contracted in the form $dYT^{(5)}_{Y\overline{Y}}\beta^{(2)}_{\overline{Y}}$, etc. The corresponding tensor structures $\overline{T}_i^{(5)}$, $\overline{K}_i^{(5)}$, contracted in the form $d\overline{Y}T^{(5)}_{\overline{Y}Y}\beta^{(2)}_{Y}$, etc may be obtained by reversing the arrows of each term in Table 5.9.

Substituting (5.78), (5.81), (5.88), (5.89) and (5.90) into (5.87) gives a large system of equations (C.6), from which one can deduce the consistency conditions

$$c_5^{(4)} = c_6^{(4)}, \quad c_7^{(4)} = c_8^{(4)}, \tag{5.91}$$

and the metric constraints

$$\begin{aligned} &t_i^{(5)} = \overline{t}_i^{(5)} \quad \forall i \neq 2, \\ &(t_2^{(5)} - \overline{t}_2^{(5)}) + (k_1^{(5)} - \overline{k}_1^{(5)}) = 0, \\ &k_2^{(5)} = \overline{k}_2^{(5)} = 0. \end{aligned} \tag{5.92}$$

The coefficients $t_2^{(5)}$, $\overline{t}_2^{(5)}$, $k_1^{(5)}$, $\overline{k}_1^{(5)}$, while constrained to satisfy the above equality, are otherwise arbitrary. Consequently, we are free to choose that the metric be symmetric by imposing $t_2^{(5)} = \overline{t}_2^{(5)}$, which then forces $k_1^{(5)} = \overline{k}_1^{(5)}$. Having imposed symmetry, we obtain the following solution for the A-function coefficients,

$$\begin{aligned}
&a_1^{(7)} = 2\mu c_1^{(4)}, && a_2^{(7)} = 4\mu c_2^{(4)}, && a_3^{(7)} = \tfrac{2}{9}\mu c_3^{(4)},\\
&a_4^{(7)} = \tfrac{2}{3}\mu(3c_3^{(4)} + c_4^{(4)}), && a_5^{(7)} = 4\mu c_4^{(4)}, && a_6^{(7)} = 4\mu c_5^{(4)},\\
&a_7^{(7)} = 4\mu c_5^{(4)}, && a_8^{(7)} = 4\mu c_7^{(4)}, && a_9^{(7)} = 4\mu c_7^{(4)},\\
&a_{10}^{(7)} = 4\tfrac{c_2^{(2)}}{c_1^{(2)}}\mu c_2^{(4)}, && a_{11}^{(7)} = 4\tfrac{c_3^{(2)}}{c_1^{(2)}}\mu c_2^{(4)}, && a_{12}^{(7)} = \tfrac{2}{3}\mu(3c_9^{(4)} + c_{10}^{(4)}),\\
&a_{13}^{(7)} = 4\mu c_{10}^{(4)}, && a_{14}^{(7)} = 2\mu c_{11}^{(4)}, &&
\end{aligned} \tag{5.93}$$

and metric coefficients

$$\begin{aligned}
&t_1^{(5)} = \tfrac{2}{3c_1^{(2)}}\mu c_1^{(4)} + 4b_1 a, \qquad t_2^{(5)} + k_1^{(5)} = 8c_1^{(2)} a,\\
&t_3^{(5)} = \tfrac{2}{c_1^{(2)}}\mu c_2^{(4)}, \quad t_4^{(5)} = \tfrac{2}{3c_1^{(2)}}\mu(2c_4^{(4)} - \tfrac{c_2^{(2)}}{c_1^{(2)}}c_1^{(4)}) + 4c_2^{(2)} a, \quad t_5^{(5)} = \tfrac{2}{c_1^{(2)}}\mu c_5^{(4)},\\
&t_6^{(5)} = \tfrac{2}{c_1^{(2)}}\mu c_7^{(4)}, \quad t_7^{(5)} = \tfrac{2}{3c_1^{(2)}}\mu(2c_{10}^{(4)} - \tfrac{c_3^{(2)}}{c_1^{(2)}}c_1^{(4)}) + 4c_3^{(2)} a, \quad k_2^{(5)} = 0,
\end{aligned} \tag{5.94}$$

where $\bar{t}_i^{(5)} = t_i^{(5)}$, $\bar{k}_i^{(5)} = k_i^{(5)}$. We see that, as in four and six dimensions, there is a correspondence between the freedom in the definition of A and an arbitrariness in the metric.

As always, due to the method of construction, we expect the consistency conditions (5.91) on the β-function coefficients to be scheme-independent. Given a coupling redefinition $(\delta Y)^{(2)}$, the induced change in $\beta_Y^{(4)}$ is

$$\begin{aligned}
\delta\beta_Y^{(4)} = &\left(\beta_Y^{(2)} \cdot \frac{\partial}{\partial Y} + \beta_{\overline{Y}}^{(2)} \cdot \frac{\partial}{\partial \overline{Y}}\right)(\delta Y)^{(2)}\\
&- \left((\delta Y)^{(2)} \cdot \frac{\partial}{\partial Y} + (\delta\overline{Y})^{(2)} \cdot \frac{\partial}{\partial \overline{Y}}\right)\beta_{\overline{Y}}^{(2)}
\end{aligned} \tag{5.95}$$

hence given a two-loop redefinition of the form

$$(\delta Y)^{(2)} = \sum_{i=1}^{3} \delta_i Y^{m(jkl}(C_i^{(2)})_m{}^{i)}, \tag{5.96}$$

we obtain the following changes in the coefficients of $\beta_Y^{(4)}$,

$$\begin{aligned}
&\delta c_3^{(4)} = 2(b_2\delta_1 - b_1\delta_2), && \delta c_4^{(4)} = 6(b_2\delta_1 - b_1\delta_2),\\
&\delta c_9^{(4)} = 2(b_3\delta_1 - b_1\delta_3), && \delta c_{10}^{(4)} = 6(b_3\delta_1 - b_1\delta_3),
\end{aligned} \tag{5.97}$$

and so we see immediately that (5.91) are indeed scheme-independent.

To verify that the consistency conditions hold, we therefore need only evaluate the $\overline{\text{MS}}$ values of the β-function coefficients.[7] Table 5.10 shows all diagrams that may contribute to the Yukawa-dependent part of $\gamma^{(4)}$, and their individual contributions are listed in Table 5.11. The results are presented such that to obtain the contributions from a particular diagram, one sums the weighted contribution from each master integral (columns I_4, I_{22}, I_{4bbb}, W), then multiples by the symmetry factor and overall group factor; for example, diagram (a) gives a contribution

$$(a) \to -\tfrac{1}{12}\, I_4\, \Gamma_1^{(4)}$$

to $\gamma^{(4)}$, while diagram (l) gives

$$(l) \to (-2I_4 + \tfrac{4}{3} I_{4bbb}) \left(\tfrac{1}{6}\Gamma_3^{(4)} - \tfrac{1}{2}\Gamma_7^{(4)}\right).$$

[7]For a detailed exposition, see chapter four of [48].

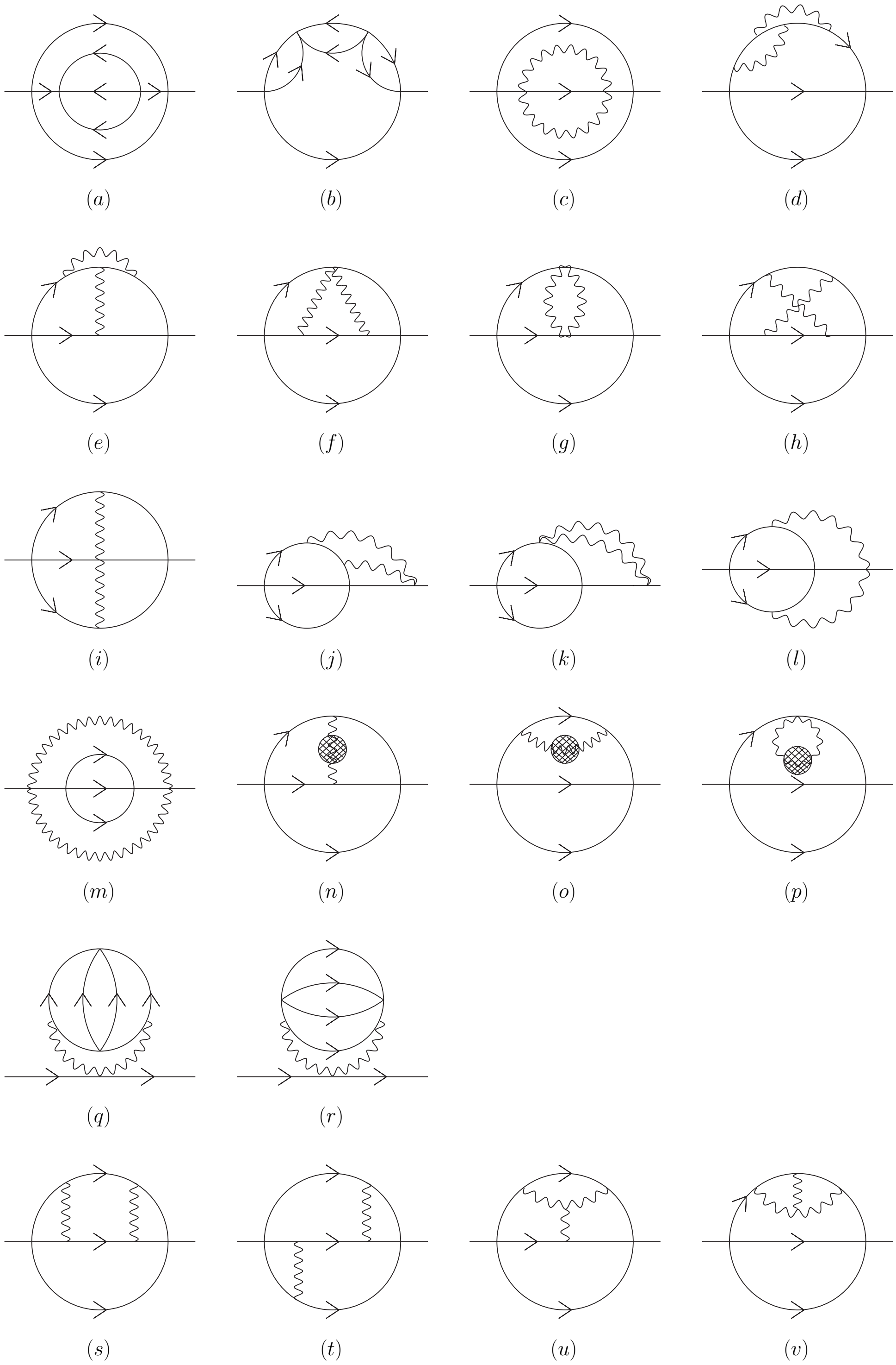


Table 5.10: Four-loop diagrams contributing to the Yukawa-dependent part of the $\mathcal{N} = 2$ superfield anomalous dimension

	symm	I_4	I_{22}	I_{4bbb}	W	overall group factor
(a)	$-\frac{1}{12}$	1	0	0	0	$\Gamma^{(4)}_1$
(b)	$-\frac{1}{8}$	0	0	1	0	$\Gamma^{(4)}_2$
(c)	$-\frac{1}{4}$	-2	0	0	0	$\Gamma^{(4)}_4 - \frac{1}{4}C_G\Gamma^{(4)}_{10}$
(d)	$-\frac{1}{2}$	0	0	1	0	$\Gamma^{(4)}_4 - \frac{1}{2}C_G\Gamma^{(4)}_{10}$
(e)	-1	0	0	$\frac{2}{3}$	0	$-\frac{1}{2}\Gamma^{(4)}_4 - \frac{1}{2}\Gamma^{(4)}_{12} - \frac{1}{12}C_G\Gamma^{(4)}_9 + \frac{1}{4}C_G\Gamma^{(4)}_{10}$
(f)	1	0	0	$-\frac{2}{3}$	0	$\Gamma^{(4)}_8 + \frac{1}{12}C_G\Gamma^{(4)}_9 - \frac{1}{4}C_G\Gamma^{(4)}_{10}$
(g)	$-\frac{1}{4}$	0	0	-2	0	$\Gamma^{(4)}_8 + \frac{1}{12}C_G\Gamma^{(4)}_9 - \frac{1}{4}C_G\Gamma^{(4)}_{10}$
(i)	$\frac{1}{2}$	0	0	$-\frac{2}{3}$	0	$\frac{1}{2}\Gamma^{(4)}_4 + \frac{1}{2}\Gamma^{(4)}_7 - \Gamma^{(4)}_8 + \Gamma^{(4)}_{12}$
(j)	1	-2	0	1	0	$\Gamma^{(4)}_7 - \frac{1}{12}C_G\Gamma^{(4)}_9$
(k)	$-\frac{1}{2}$	-2	0	0	0	$\Gamma^{(4)}_7 - \frac{1}{12}C_G\Gamma^{(4)}_9$
(l)	1	-2	0	$\frac{4}{3}$	0	$\frac{1}{6}\Gamma^{(4)}_3 - \frac{1}{2}\Gamma^{(4)}_7$
(m)	$-\frac{1}{12}$	-2	0	0	0	$\Gamma^{(4)}_3 - \frac{1}{4}C_G\Gamma^{(4)}_9$
(n)	$-\frac{1}{2}$	1	0	$-\frac{1}{2}$	0	$\left(\tilde{T} + \frac{1}{2}C_G\right)\left(\frac{1}{6}\Gamma^{(4)}_9 - \frac{1}{2}\Gamma^{(4)}_{10}\right)$
(o)	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	0	$\left(\tilde{T} + \frac{1}{2}C_G\right)\Gamma^{(4)}_{10}$
(p)	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	$\left(\tilde{T} + \frac{1}{2}C_G\right)\Gamma^{(4)}_{10}$
(q)	$-\frac{1}{12}$	0	1	0	-2	$\Gamma^{(4)}_{11}$
(r)	$\frac{1}{6}$	1	0	0	-1	$\Gamma^{(4)}_{11}$

Table 5.11: Results for diagrams listed in Table 5.10 in terms of master integrals (see Table 5.7) and invariants involving Yukawa couplings of Eq. (5.84)

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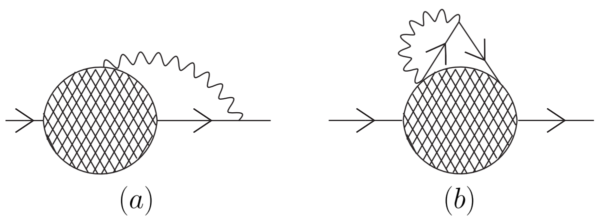

Table 5.12: Classes of diagrams that do not contribute to $\gamma^{(4)}$

Table 5.12 shows diagrams that do not contribute to γ [56]: any diagram of the form 5.12(a) is finite by power counting, whereas the diagrams of the form 5.12(b) contain fewer than two Ds and two $\bar{D}$s and hence give a vanishing contribution. Summing all contributions, we therefore find the $\overline{\rm MS}$ results for the Yukawa-dependent part of $\gamma^{(4)}$, and hence the corresponding coefficients of $\beta_Y^{(4)}$:

$$\begin{aligned} &c_1^{(4)} = \tfrac{2}{3}, \quad c_2^{(4)} = \tfrac{\pi^2}{4}, \quad c_3^{(4)} = \tfrac{4}{3}(1 - \tfrac{\pi^2}{3}), \quad c_4^{(4)} = \tfrac{2\pi^2}{3} - 4, \\ &c_5^{(4)} = 0, \quad c_6^{(4)} = 0, \quad c_7^{(4)} = -\tfrac{\pi^2}{3}, \quad c_8^{(4)} = -\tfrac{\pi^2}{3}, \\ &c_9^{(4)} = \tfrac{2}{3}(T_R - C_G) - \tfrac{\pi^2}{12}T_R + \tfrac{\pi^2}{8}C_G, \\ &c_{10}^{(4)} = -2(T_R - C_G) + \tfrac{\pi^2}{4}T_R - \tfrac{3\pi^2}{8}C_G, \\ &c_{11}^{(4)} = -\tfrac{4}{3}. \end{aligned} \tag{5.98}$$

It is worth noting here that the calculation of contributions from diagram (h) is very involved, requiring the evaluation of a new master integral at three loops (shown below in Figure 5.1). Nevertheless, it is possible to do, and was rather surprisingly found to be finite.

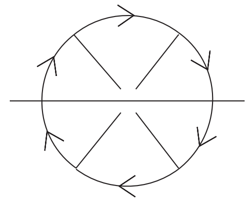

Figure 5.1: Ω - the new non-planar Feynman integral for $\mathcal{N} = 2$ gauge theory

Finally, as in the non-supersymmetric case, one may attempt to derive relations between the underlying Feynman integrals. The basis of Feynman integrals for the β-function of the general $\mathcal{N} = 2$ gauge theory is in fact the same as that of the non-supersymmetric scalar-fermion theory (see Table 5.7), with the addition of one new non-planar diagram Ω, shown in Figure 5.1. This integral has an implied spinor trace over the six outer propagators, where the arrows indicate a factor $\not{k}$ in the numerator, and we use three-dimensional gamma matrices with $\mathrm{tr}(\mathbb{1}) = 2$. Expressing the β-function coefficients in terms of the underlying integrals, one has

$$c_5^{(4)} = 0, \quad c_6^{(4)} = 0, \quad c_7^{(4)} = \tfrac{1}{6}I_{4bbb}, \quad c_8^{(4)} = \tfrac{1}{6}I_{4bbb} + k\omega,$$

where ω is the singular part of the new integral Ω, and k is some non-zero constant.[8] As a consequence of the consistency conditions (5.91), we see immediately

[8] Specifically, k is the product of the symmetry factor for (h), the "weight" factor from the master integral Ω, and the group-theoretic factor proportional to $\Gamma_8^{(4)}$.

that we must have

$$\omega = 0, \tag{5.99}$$

and hence our A-function predicts that the new integral Ω must be finite. It is somewhat remarkable that a simple first-order differential equation with (as of yet) no theoretical justification is sufficiently powerful to determine that a highly non-trivial three-loop integral must in fact be finite.

5.5 Summary

In this chapter, we have investigated the possibility of constructing an A-function for general three-dimensional QFTs, satisfying the same gradient-flow equation as in even dimensions. Generically, such a function is determined only up to an overall multiplication constant, since the Euler density conventionally used to fix this scale in even dimensions vanishes identically in odd dimensions. We first calculated A explicitly for a range of theories, showing in each case that the leading-order metric is positive-definite, up to the overall constant. Next we considered general theories, finding that while the leading-order A-function for a general scalar-fermion theory is somewhat trivial to construct, the Abelian gauge theory leads to a number of simple consistency conditions, which are trivially scheme-independent and satisfied in $\overline{\text{MS}}$. We then demonstrated that the three-dimensional A-function may be constructed beyond leading order, in direct analogy with the even-dimensional case, by deducing the mixed scalar-Yukawa contributions to $A^{(7)}$, and deriving the associated consistency conditions. These conditions were again automatically scheme-independent, and (as in four dimensions) fixed a ratio between the leading-order metric coefficients such that if either coefficient is positive, then the leading-order metric is positive-definite.

Having shown that the A-function continues to exist beyond leading order, we completed the calculation of the next-to-leading order A-function for a general scalar-fermion theory by deducing all purely-Yukawa contributions to $A^{(7)}$. This required determining all possible terms in the A-function, four-loop Yukawa β-function, and next-to-leading-order metric, leading to a very large set of consistency conditions. To verify these conditions, we first calculated all non-anomalous dimension contributions to $\beta_Y^{(4)}$ in $\overline{\text{MS}}$ using integration by parts, and found that each condition was indeed satisfied. While we did not calculate all anomalous dimension contributions, we did calculate two out of the fourteen possible terms, then deduced the remaining twelve using the consistency conditions; we were later able to independently verify most of these predictions. We then ensured that each condition was scheme-independent, again by deducing the effects of a coupling redefinition. Since there were so many simple consistency conditions, it was possible to deduce the relations between the poles of the underlying Feynman integrals; by doing so, we found that the existence of an A-function for a three-dimensional scalar-fermion theory was sufficient to reduce the basis of master integrals, up to four loops, to just five very simple diagrams. Given that there are 105 pure Yukawa terms and a further 6 scalar terms in $\beta_Y^{(4)}$, this is a truly dramatic simplification.

Finally, in order to provide more evidence for the existence of an A-function for general three-dimensional gauge theories beyond leading order, we considered a general $\mathcal{N} = 2$ supersymmetric gauge theory. This was again trivial at leading order, but at next-to-leading order resulted in a number of consistency conditions. The conditions relating β-function coefficients were again satisfied in $\overline{\text{MS}}$ and shown to be scheme-independent via calculating the effects of a coupling redefinition, whereas the conditions on the tensor T_{IJ} automatically imposed symmetry for all but two terms. It is easy to see that these last terms may indeed be chosen so that $T_{IJ} = G_{IJ}$, without imposing any extra consistency conditions. We then again deduced the relations between the required Feynman integrals, of which all but one had been calculated in [53], and found that the required basis of integrals was the same as for the non-supersymmetric scalar-fermion theory, plus the extra integral not previously calculated. Our relations required that this final integral be finite: a rather powerful statement, given that the integral in question is a highly non-trivial, three loop, non-planar integral. The result is even more surprising given that it is ultimately predicated on the as-yet-unjustified existence of an A-function in three dimensions, satisfying the same gradient-flow equation as in even dimensions.

Chapter 6

Conclusions

Throughout this thesis, our aim has been to construct a function satisfying (2.1), then deduce consequences of the existence of such a function, for a range of general QFTs in various numbers of spacetime dimensions. In each case, the existence of this function led to consistency conditions, relating coefficients of the β-functions of each theory across multiple loop orders. By expressing all β-functions as sums of general tensor structures with arbitrary coefficients, and using the equivalence of a coupling redefinition and a change in renormalization scheme, we have been able to show that all such consistency conditions are invariant under the changes induced by a coupling redefinition, and hence hold in arbitrary renormalization schemes. We have also addressed the question of whether the tensor T_{IJ} that appears in (2.1) may in fact be chosen to be symmetric, reducing to the metric G_{IJ}; we have explicitly shown this to be possible in each theory considered. For the six-dimensional ϕ^3 theory, this possibility was contingent on a new consistency condition being satisfied, and we have shown that this is the case.

We have also considered less obvious consequences of the existence of a function satisfying (2.1). In the case of four-dimensional $\mathcal{N} = 1$ supersymmetry, we have extended a proposed all-orders expression for the a-function to a general gauge theory. This proposal is reliant on a new equation being satisfied, the Λ-equation, and we have deduced the implied consistency conditions for the coefficients of the chiral superfield anomalous dimension. In six dimensions, we have shown that the existence of an a-function is sufficient to determine the first non-trivial contribution to B, the shifted β-function that determines whether a theory is conformally invariant. Finally, in three dimensions, we have shown that the consistency conditions are of a simple enough form that one can derive relations between the underlying Feynman integrals themselves, rather than just the coefficients of the β-functions. The power of these conditions was demonstrated by considering a general three-dimensional $\mathcal{N} = 2$ supersymmetric gauge theory, and showing that the final non-planar Feynman integral, left undetermined by [53], must in fact be finite. Since our three-dimensional work was carried out last, we have not gone back and re-assessed whether the consistency conditions of our four- and six-dimensional theories also lead to relations between Feynman integrals; such an undertaking would therefore be a straightforward line of further inquiry.

Throughout our investigations, we have encountered several obvious questions that we feel merit further study. As mentioned above, in each example we have shown that one may impose symmetry of the tensor T_{IJ} up to the order considered, and that this imposition is renormalization scheme-independent. However, it was claimed in [9] that, in the case of a four-dimensional scalar-fermion theory, imposing symmetry at the first non-trivial order is only possible for particular schemes. The first question is therefore, is it always possible to choose $T_{IJ} = G_{IJ}$, and if not, why not? While symmetry of T_{IJ} in our calculations was manifest at lower orders of perturbation theory, and imposed with no consequence in all four- and three-dimensional theories considered, the six-dimensional ϕ^3 theory essentially relied on fixing some of the arbitrariness present in the definition of the a-function away from RG fixed points. It is possible that there is simply insufficient arbitrariness in the a-function for the four-dimensional scalar-fermion theory, but nevertheless we feel that this particular case should be revisited, using the completely general perspective employed in this thesis; at the very least, if the conclusions of [9] are upheld, then the four-dimensional scalar-fermion case would form a concrete counterexample to the suggestion of always being able to impose symmetry of T_{IJ}.

The next question raised was the potential for 1PR contributions to the β-function of six-dimensional ϕ^3 theory in general non-minimal renormalization schemes, arising specifically from antisymmetric contributions to the anomalous dimension. By construction, these 1PR contributions are of the same form as the tensor structures present in the "v-term" (that is, the shift to the B function), and hence would be expected to arise for any theory with a global symmetry, in any number of spacetime dimensions. The work of [41] demonstrated how to remove these antisymmetric contributions, but made no mention of whether this has an effect on the potential 1PR contributions to the v-term. One can therefore ask, after re-defining the anomalous dimension to remove antisymmetric contributions, what effect (if any) does this have on 1PR contributions to the v-term? Our prediction for the coefficient of the 1PI contribution to the v-term was predicated on the vanishing of 1PR contributions in $\overline{\text{MS}}$, hence we cannot make any statement regarding their presence in non-minimal schemes.

Further questions that one may ask are of a much more general nature. We have provided perturbative calculations supporting an extension of the Λ-equation to supersymmetric gauge theories, but this obviously does not address whether the Λ-equation is actually true. As mentioned in the summary of chapter 3, the authors of [35] used Weyl consistency conditions to derive an equation that appears very similar to the Λ-equation. Is their equation precisely the Λ-equation? If so, then the conjectured all-orders expression for the a-function in $\mathcal{N} = 1$ supersymmetric gauge theories will have been established.

A glaring issue is, of course, the question of what we are calculating in three dimensions. There is no Euler anomaly in three-dimensional theories, and so the function we have constructed cannot reduce to the usual coefficient a at RG fixed points. The most promising notion is that we are in fact constructing the F-function proposed by [44, 45]. The F-function is intended to satisfy a three-dimensional analogue of the a-theorem known as the F-theorem, and for free theories has indeed been shown to satisfy the equivalent weak formulation $F_{UV} > F_{IR}$. As noted by the authors of [44, 45], it is difficult to calculate the F-function for interacting theories, or even to evaluate it numerically, and so if our construction is indeed reproducing the F-function, then we have a method of calculating F perturbatively for a completely general, interacting, three-dimensional QFT.

Broadening scope beyond immediate questions, the existence of an a-function may be of use in constructing asymptotically safe theories; that is, theories that possess a non-trivial interacting RG fixed point, rendering them finite at high energy. Classes of such theories have been constructed by [57], and rely on cancellations between β-functions at different loop orders in exactly the same perturbative ordering used to construct an a-function. Given that our consistency conditions are valid for general four-dimensional theories, and are independent of renormalization scheme, one may ask: is it possible to utilize such relations between β-function coefficients to demonstrate that a more general theory is asymptotically safe? If so, it may be possible to entertain the notion of asymptotic safety for theories in spacetime dimensions other than four, using similar consistency conditions.

Finally, we return to the use of the basic consistency conditions that we derive for each case. A major weakness of our approach is that there appears to be no predictive quality to the generation of consistency conditions; although we know that the conditions will relate *some* β-function coefficients, there is no indication of *which* coefficients will appear in any one condition. Furthermore, with present understanding, there does not seem to be any way to determine the number of conditions generated at any loop order. As an intermediate step, instead of counting consistency conditions, one could hope to count the number of combinations of β-function coefficients that are scheme-invariant, of which the consistency conditions are simply linear combinations. Unfortunately, even attempting to count such invariants is counter-intuitive: for example, one might suppose that the number of invariants is simply the number of independent β-function coefficients minus the number of parameters present in the variations induced by a coupling redefinition. However, in chapter 4, we found that ϕ^3 theory has five independent two-loop β-function coefficients and two redefinition parameters, but *four* independent combinations (including scheme-independent coefficients). It has been noticed that, while there are two independent redefinition parameters, the induced changes in the β-function are given in terms of only one linear combination of the parameters. It would be tempting to speculate that the correct counting method is therefore the number of independent coefficients minus the number of *independent linear combinations* of redefinition parameters, but one would need to conduct a much deeper analysis of the invariants to provide evidence that this would be the case.

A potential way forward in counting the number of consistency conditions may lie in treating the problem more mathematically. Some predictions for scheme-independent combinations of β-function coefficients were noted in the PhD thesis of [58], as a consequence of treating Feynman integrals and their subdivergences from the perspective of a Hopf algebra. In particular, Panzer demonstrated that given a Feynman graph G_i with subgraphs g_j and associated β-function contribution $b_iG_i \in \beta$, there will exist scheme-independent combinations of coefficients b_i according to the following criteria:

- If a graph G has a subgraph g such that $G \cong g$, then its associated β-function coefficient is scheme-independent;
- If two graphs G_1, G_2 have corresponding subgraphs g_1, g_2 such that the quotient graphs satisfy $G_1/g_1 \cong g_2$ and $G_2/g_2 \cong g_1$, then $xb_1 + yb_2$ is scheme-independent for some $x, y > 0$;
- If two graphs G_1, G_2 have the same subgraph g, such that the quotient graphs satisfy $G_1/g \cong G_2/g$, then $xb_1 - yb_2$ is scheme-independent for some $x, y > 0$.

One will immediately notice that our consistency conditions are, in many cases, substantially more complex than these pairwise invariants, and so provide ample opportunity to generalise these criteria. If we were to sufficiently generalize the criteria to encompass our consistency conditions, then we would have a new systematic method of deducing scheme-independent combinations of β-function coefficients for general theories; furthermore, combined with the potential relations between the Feynman integrals, it may be possible to develop a novel approach to integral reduction that extends beyond methods reliant on integration by parts.

A: Four-dimensional equations

Here we list the systems of linear equations generated when solving (2.1) for a general four-dimensional gauge theory.

Equations for $A^{(4)}$

$$
\begin{aligned}
&2a_2^{(4)} = \mu c_{29}^{(2)} \\
&4a_3^{(4)} = \mu c_4^{(2)} \\
&6a_4^{(4)} = \mu c_1^{(2)} \\
&6a_5^{(4)} + 12(c_2^{(1)})^2 a = c_2^{(1)}(t_1^{(3)} + t_2^{(3)} + t_3^{(3)}) \\
&2a_6^{(4)} + 4(c_2^{(1)})^2 a = c_2^{(1)} t_1^{(3)} \\
&4a_6^{(4)} + 8(c_2^{(1)})^2 a = 2\mu c_8^{(2)} + c_2^{(1)}(t_2^{(3)} + t_3^{(3)}) \\
&2a_7^{(4)} + 8c_2^{(1)} c_3^{(1)} a = 2\mu c_{10}^{(2)} + c_3^{(1)}(t_2^{(3)} + t_3^{(3)}) \\
&2a_7^{(4)} + 8c_2^{(1)} c_3^{(1)} a = c_3^{(1)} t_1^{(3)} + 2c_2^{(1)} t_4^{(3)} \\
&2a_7^{(4)} + 8c_2^{(1)} c_3^{(1)} a = \mu c_{24}^{(2)} + 2c_2^{(1)}(t_5^{(3)} + t_6^{(3)}) \\
&6a_8^{(4)} + 6(c_3^{(1)})^2 a = c_3^{(1)}(t_4^{(3)} + t_5^{(3)} + t_6^{(3)}) \\
&2a_9^{(4)} + 8c_1^{(1)} c_2^{(1)} a = c_1^{(1)} t_1^{(3)} + c_2^{(1)} t_8^{(3)} \\
&2a_9^{(4)} + 8c_1^{(1)} c_2^{(1)} a = 2\mu c_{11}^{(2)} + c_2^{(1)}(2t_7^{(3)} + t_8^{(3)}) \\
&2a_9^{(4)} + 8c_1^{(1)} c_2^{(1)} a = 2\mu c_9^{(2)} + c_1^{(1)}(t_2^{(3)} + t_3^{(3)}) \\
&4a_{10}^{(4)} + 4(c_1^{(1)})^2 a = 2\mu c_{30}^{(2)} + c_1^{(1)} t_8^{(3)} \\
&2a_{10}^{(4)} + 2(c_1^{(1)})^2 a = \mu c_3^{(2)} + c_1^{(1)} t_7^{(3)} \\
&2a_{11}^{(4)} + 4c_1^{(1)} c_3^{(1)} a = \mu c_{25}^{(2)} + c_1^{(1)}(t_5^{(3)} + t_6^{(3)}) \\
&2a_{11}^{(4)} + 4c_1^{(1)} c_3^{(1)} a = c_1^{(1)} t_4^{(3)} + c_3^{(1)} t_7^{(3)} \\
&2a_{11}^{(4)} + 4c_1^{(1)} c_3^{(1)} a = \mu c_2^{(2)} + c_3^{(1)} t_8^{(3)} \\
&4a_{13}^{(4)} + 16c_2^{(1)} c_4^{(1)} a = 2\mu c_{13}^{(2)} + c_4^{(1)}(t_1^{(3)} + t_2^{(3)} + t_3^{(3)}) + c_2^{(1)} t_9^{(3)} \\
&2a_{14}^{(4)} + 8c_2^{(1)} c_4^{(1)} a = 2\mu c_{14}^{(2)} + c_4^{(1)}(t_2^{(3)} + t_3^{(3)})
\end{aligned}
$$

$$
\begin{aligned}
&2a^{(4)}_{14} + 8c^{(1)}_2 c^{(1)}_4 a = 2\mu c^{(2)}_{12} + c^{(1)}_4 t^{(3)}_1 + c^{(1)}_2 t^{(3)}_9\\
&2a^{(4)}_{15} + 8c^{(1)}_2 c^{(1)}_5 a = 2\mu c^{(2)}_{16} + c^{(1)}_5 (t^{(3)}_2 + t^{(3)}_3)\\
&2a^{(4)}_{15} + 8c^{(1)}_2 c^{(1)}_5 a = 2\mu c^{(2)}_{17} + c^{(1)}_5 t^{(3)}_1 + 2c^{(1)}_2 t^{(3)}_{10}\\
&2a^{(4)}_{16} + 8c^{(1)}_1 c^{(1)}_4 a = 2\mu c^{(2)}_{18} + c^{(1)}_4 t^{(3)}_8 + c^{(1)}_1 t^{(3)}_9\\
&2a^{(4)}_{16} + 8c^{(1)}_1 c^{(1)}_4 a = 2\mu c^{(2)}_{19} + c^{(1)}_4 (2t^{(3)}_7 + t^{(3)}_8)\\
&2a^{(4)}_{17} + 4c^{(1)}_1 c^{(1)}_5 a = \mu c^{(2)}_5 + c^{(1)}_5 t^{(3)}_7 + c^{(1)}_1 t^{(3)}_{10}\\
&2a^{(4)}_{17} + 4c^{(1)}_1 c^{(1)}_5 a = \mu c^{(2)}_6 + c^{(1)}_5 t^{(3)}_8\\
&4a^{(4)}_{18} = 2\mu c^{(2)}_{15}\\
&2a^{(4)}_{19} + 8c^{(1)}_3 c^{(1)}_4 a = 2c^{(1)}_4 t^{(3)}_4 + c^{(1)}_3 t^{(3)}_9\\
&2a^{(4)}_{19} + 8c^{(1)}_3 c^{(1)}_4 a = \mu c^{(2)}_{26} + 2c^{(1)}_4 (t^{(3)}_5 + t^{(3)}_6)\\
&4a^{(4)}_{20} + 8c^{(1)}_3 c^{(1)}_5 a = \mu c^{(2)}_7 + c^{(1)}_5 (t^{(3)}_4 + t^{(3)}_5 + t^{(3)}_6) + c^{(1)}_3 t^{(3)}_{10}\\
&2a^{(4)}_{22} + 4(c^{(1)}_4)^2 a = 2\mu c^{(2)}_{22} + c^{(1)}_4 t^{(3)}_9\\
&2a^{(4)}_{23} + 4(c^{(1)}_4)^2 a = \mu c^{(2)}_{21} + c^{(1)}_4 t^{(3)}_9\\
&2a^{(4)}_{24} + 2(c^{(1)}_5)^2 a = \mu c^{(2)}_{27} + c^{(1)}_5 t^{(3)}_{10}\\
&2a^{(4)}_{25} + 8c^{(1)}_4 c^{(1)}_5 a = 2\mu c^{(2)}_{23} + c^{(1)}_5 t^{(3)}_9 + 2c^{(1)}_4 t^{(3)}_{10}\\
&2a^{(4)}_{26} = 2\mu c^{(2)}_{20} + e^{(1)}_1 \tau^{(3)}_1\\
&2a^{(4)}_{27} = \mu c^{(2)}_{28} + e^{(1)}_1 \tau^{(3)}_2
\end{aligned}
\tag{A.1}
$$

Additional equations for constraints on $\beta^{(3)}_g$

$$
\begin{aligned}
&2a^{(4)}_{12} = \sigma^{(1)}_1 e^{(3)}_1\\
&2a^{(4)}_{13} + 8c^{(1)}_2 c^{(1)}_4 a = \sigma^{(1)}_1 e^{(3)}_2 + \tilde{\tau}^{(3)}_1 c^{(1)}_2\\
&2a^{(4)}_{14} + 8c^{(1)}_2 c^{(1)}_4 a = \sigma^{(1)}_1 e^{(3)}_3 + \tilde{\tau}^{(3)}_1 c^{(1)}_2\\
&2a^{(4)}_{15} + 8c^{(1)}_2 c^{(1)}_5 a = \sigma^{(1)}_1 e^{(3)}_4 + 2\tilde{\tau}^{(3)}_2 c^{(1)}_2\\
&2a^{(4)}_{16} + 8c^{(1)}_1 c^{(1)}_4 a = \sigma^{(1)}_1 e^{(3)}_5 + \tilde{\tau}^{(3)}_1 c^{(1)}_1\\
&2a^{(4)}_{17} + 4c^{(1)}_1 c^{(1)}_5 a = \sigma^{(1)}_1 e^{(3)}_6 + \tilde{\tau}^{(3)}_2 c^{(1)}_1\\
&2a^{(4)}_{18} = \sigma^{(1)}_1 e^{(3)}_7\\
&2a^{(4)}_{19} + 8c^{(1)}_3 c^{(1)}_4 a = \sigma^{(1)}_1 e^{(3)}_8 + \tilde{\tau}^{(3)}_1 c^{(1)}_3\\
&2a^{(4)}_{20} + 4c^{(1)}_3 c^{(1)}_5 a = \sigma^{(1)}_1 e^{(3)}_9 + \tilde{\tau}^{(3)}_2 c^{(1)}_3\\
&4a^{(4)}_{21} = \sigma^{(1)}_1 e^{(3)}_{10}\\
&4a^{(4)}_{22} + 8(c^{(1)}_4)^2 a = \sigma^{(1)}_1 e^{(3)}_{11} + \tilde{\tau}^{(3)}_1 c^{(1)}_4\\
&4a^{(4)}_{23} + 8(c^{(1)}_4)^2 a = \sigma^{(1)}_1 e^{(3)}_{12} + \tilde{\tau}^{(3)}_1 c^{(1)}_4
\end{aligned}
$$

$$
\begin{aligned}
&4a_{24}^{(4)} + 4(c_5^{(1)})^2 a = \sigma_1^{(1)} e_{13}^{(3)} + \tilde{\tau}_2^{(3)} c_5^{(1)} \\
&4a_{25}^{(4)} + 16c_4^{(1)} c_5^{(1)} a = \sigma_1^{(1)} e_{14}^{(3)} + \tilde{\tau}_1^{(3)} c_5^{(1)} + 2\tilde{\tau}_2^{(3)} c_4^{(1)} \\
&4a_{26}^{(4)} + 8e_1^{(1)} e_4^{(2)} \tilde{a} = \sigma_1^{(1)} e_{15}^{(3)} + \sigma_1^{(2)} e_4^{(2)} + \sigma_1^{(3)} e_1^{(1)} \\
&4a_{27}^{(4)} + 8e_1^{(1)} e_5^{(2)} \tilde{a} = \sigma_1^{(1)} e_{16}^{(3)} + \sigma_1^{(2)} e_5^{(2)} + \sigma_2^{(3)} e_1^{(1)}
\end{aligned}
\qquad (A.2)
$$

Equations for $\Lambda^{(3)}$

$$
\begin{aligned}
&\tfrac{1}{3}\Lambda_1^{(3)} = \gamma_1^{(3)} + \theta_3^{(2)}\gamma_1^{(1)} + \theta_4^{(2)}\gamma_1^{(1)} \\
&\tfrac{1}{6}\Lambda_2^{(3)} = \gamma_2^{(3)} + \theta_3^{(2)}\gamma_1^{(1)} \\
&\tfrac{1}{3}\Lambda_3^{(3)} = \gamma_3^{(3)} + \theta_1^{(1)}\gamma_1^{(2)} + \theta_4^{(2)}\gamma_1^{(1)} \\
&\tfrac{1}{2}\Lambda_4^{(3)} = \gamma_4^{(3)} \\
&\tfrac{1}{3}\Lambda_5^{(3)} = \gamma_5^{(3)} + 2\theta_1^{(1)}\gamma_3^{(2)} + \theta_3^{(2)}\gamma_2^{(1)} + \theta_4^{(2)}\gamma_2^{(1)} + \theta_6^{(2)}\gamma_1^{(1)} \\
&\tfrac{1}{6}\Lambda_6^{(3)} = \gamma_6^{(3)} + \theta_3^{(2)}\gamma_2^{(1)} + \theta_6^{(2)}\gamma_1^{(1)} \\
&\tfrac{1}{3}\Lambda_7^{(3)} = \gamma_7^{(3)} + \theta_1^{(1)}\gamma_2^{(2)} + 2\theta_4^{(2)}\gamma_2^{(1)} \\
&\tfrac{1}{6}\Lambda_6^{(3)} = \gamma_8^{(3)} + \theta_3^{(2)}\gamma_2^{(1)} + \theta_5^{(2)}\gamma_1^{(1)} - 2\gamma_1^{(2)}\gamma_2^{(1)} \\
&\tfrac{1}{3}\Lambda_8^{(3)} = \gamma_9^{(3)} + \theta_1^{(1)}\gamma_4^{(2)} + \theta_6^{(2)}\gamma_2^{(1)} \\
&\tfrac{1}{3}\Lambda_9^{(3)} = \gamma_{10}^{(3)} + \theta_6^{(2)}\gamma_2^{(1)} \\
&\tfrac{1}{6}\Lambda_9^{(3)} = \gamma_{11}^{(3)} + 2\theta_5^{(2)}\gamma_2^{(1)} + \theta_6^{(2)}\gamma_2^{(1)} - 2\gamma_2^{(2)}\gamma_2^{(1)} \\
&\tfrac{1}{6}\Lambda_8^{(3)} = \gamma_{12}^{(3)} + \theta_1^{(1)}\gamma_4^{(2)} + \theta_5^{(2)}\gamma_2^{(1)} - 2\gamma_3^{(2)}\gamma_2^{(1)} - 2\gamma_4^{(2)}\gamma_1^{(1)} \\
&0 = \gamma_{13}^{(3)} + 2\lambda_1^{(3)} + \phi_1^{(1)} d_3^{(2)} \\
&\tfrac{1}{3}\Lambda_{10}^{(3)} = \gamma_{14}^{(3)} + 2\theta_1^{(1)}\gamma_5^{(2)} + \phi_1^{(2)} d_1^{(1)} \\
&\tfrac{1}{6}\Lambda_{10}^{(3)} = \gamma_{15}^{(3)} + \theta_1^{(1)}\gamma_5^{(2)} + \phi_2^{(2)} d_1^{(1)} - 2\gamma_5^{(2)}\gamma_1^{(1)} \\
&0 = \gamma_{16}^{(3)} - 2\gamma_4^{(2)}\gamma_2^{(1)} \\
&0 = \gamma_{17}^{(3)} + 2\lambda_2^{(3)} + \phi_1^{(1)} d_2^{(2)} \\
&0 = \gamma_{18}^{(3)} + \phi_3^{(2)} d_1^{(1)} - 2\gamma_5^{(2)}\gamma_2^{(1)} \\
&0 = \gamma_{19}^{(3)} + 2\lambda_3^{(3)} + \phi_4^{(2)} \tilde{d}_1^{(1)} + \phi_1^{(1)} \tilde{d}_1^{(2)} \\
&\tfrac{1}{6}\Lambda_1^{(3)} = \theta_1^{(2)}\gamma_1^{(1)} + \theta_2^{(2)}\gamma_1^{(1)} \\
&\tfrac{1}{6}\Lambda_5^{(3)} = \theta_1^{(1)}\gamma_3^{(2)} + \theta_1^{(2)}\gamma_2^{(1)} + \theta_2^{(2)}\gamma_2^{(1)} + \theta_5^{(2)}\gamma_1^{(1)} - 2\gamma_3^{(2)}\gamma_1^{(1)} \\
&\tfrac{1}{3}\Lambda_2^{(3)} = \theta_1^{(1)}\gamma_1^{(2)} + 2\theta_1^{(2)}\gamma_1^{(1)} - \gamma_1^{(2)}\gamma_1^{(1)} \\
&\tfrac{1}{6}\Lambda_6^{(3)} = 2\theta_2^{(2)}\gamma_1^{(1)} + \theta_3^{(2)}\gamma_1^{(1)} - \gamma_1^{(2)}\gamma_1^{(1)} \\
&\tfrac{1}{6}\Lambda_3^{(3)} = \theta_1^{(1)}\gamma_2^{(2)} + 2\theta_1^{(2)}\gamma_2^{(1)} - \gamma_2^{(2)}\gamma_1^{(1)} \\
&\tfrac{1}{6}\Lambda_7^{(3)} = 2\theta_2^{(2)}\gamma_2^{(1)} + \theta_6^{(2)}\gamma_1^{(1)} - \gamma_2^{(2)}\gamma_1^{(1)}
\end{aligned}
\qquad (A.3)
$$

B: Six-dimensional equations

Here we list the systems of linear equations generated when solving (2.1) for a six-dimensional ϕ^3 theory.

Equations for $A^{(5)}$

$$2a_1^{(5)} + 12\alpha^{(4)}c_{(1A)}c_{(2d)} = c_{(1A)}t_1^{(4)} + c_{(1A)}t_3^{(4)} + c_{(2d)}t_2^{(3)}$$
$$2a_1^{(5)} + 12\alpha^{(4)}c_{(1A)}c_{(2d)} = c_{(2d)}t_3^{(3)} + c_{(2d)}t_4^{(3)} + 3\lambda c_{(3I)}$$
$$4a_1^{(5)} + 24\alpha^{(4)}c_{(1A)}c_{(2d)} = 2c_{(1A)}t_1^{(4)} + 3c_{(1A)}t_2^{(4)} + 2c_{(1A)}t_3^{(4)} + 3\lambda c_{(3p)}$$
$$3a_2^{(5)} + 6\alpha^{(4)}c_{(1a)}c_{(2d)} = c_{(1a)}t_1^{(4)} + c_{(1a)}t_3^{(4)} + 3\lambda c_{(3q)}$$
$$3a_2^{(5)} + 6\alpha^{(4)}c_{(1a)}c_{(2d)} = c_{(2d)}t_1^{(3)} + 3\lambda c_{(3s)}$$
$$2a_2^{(5)} + 4\alpha^{(4)}c_{(1a)}c_{(2d)} = c_{(1a)}t_2^{(4)} + \lambda c_{(3r)}$$
$$8a_3^{(5)} = 3\lambda c_{(3u)}$$
$$8a_4^{(5)} + 8c_{(1A)}c_{(1A)}\alpha_2^{(5)} + 8c_{(1A)}c_{(1A)}\alpha_3^{(5)}$$
$$= c_{(1A)}t_{14}^{(4)} + c_{(1A)}t_{18}^{(4)} + c_{(1A)}t_{15}^{(4)} + c_{(1A)}t_{17}^{(4)} + c_{(1A)}t_{16}^{(4)}$$
$$2a_5^{(5)} + 12\alpha^{(4)}c_{(1A)}c_{(2C)} + 12c_{(1A)}c_{(1A)}\alpha_2^{(5)} + 8c_{(1A)}c_{(1A)}\alpha_3^{(5)}$$
$$= c_{(1A)}t_{28}^{(4)} + 2c_{(1A)}t_{18}^{(4)} + 2c_{(1A)}t_{15}^{(4)} + c_{(2C)}t_2^{(3)} + c_{(2C)}t_3^{(3)} + 3\lambda c_{(3N')} - 3\lambda c_{(3N)}^v$$
$$2a_5^{(5)} + 12\alpha^{(4)}c_{(1A)}c_{(2C)} + 12c_{(1A)}c_{(1A)}\alpha_2^{(5)} + 8c_{(1A)}c_{(1A)}\alpha_3^{(5)}$$
$$= c_{(1A)}t_{11}^{(4)} + c_{(1A)}t_{30}^{(4)} + 2c_{(1A)}t_{17}^{(4)} + 2c_{(1A)}t_{16}^{(4)} + c_{(2C)}t_4^{(3)} + 3\lambda c_{(3N)} + 3\lambda c_{(3N)}^v$$
$$2a_5^{(5)} + 12\alpha^{(4)}c_{(1A)}c_{(2C)} + 12c_{(1A)}c_{(1A)}\alpha_2^{(5)} + 8c_{(1A)}c_{(1A)}\alpha_3^{(5)}$$
$$= c_{(1A)}t_{27}^{(4)} + c_{(1A)}t_{29}^{(4)} + 2c_{(1A)}t_{10}^{(4)} + 2c_{(1A)}t_{14}^{(4)}$$
$$2a_5^{(5)} + 12\alpha^{(4)}c_{(1A)}c_{(2C)} + 12c_{(1A)}c_{(1A)}\alpha_2^{(5)} + 8c_{(1A)}c_{(1A)}\alpha_3^{(5)}$$
$$= c_{(1A)}t_{28}^{(4)} + c_{(1A)}t_{30}^{(4)} + c_{(1A)}t_{12}^{(4)} + c_{(1A)}t_{13}^{(4)} + 3\lambda c_{(3L)}$$
$$4a_6^{(5)} + 48\alpha^{(4)}c_{(1A)}c_{(2C)} + 16c_{(1A)}c_{(1A)}\alpha_3^{(5)}$$
$$= 2c_{(1A)}t_{12}^{(4)} + 2c_{(1A)}t_{13}^{(4)} + 2c_{(2C)}t_3^{(3)} + 2c_{(2C)}t_4^{(3)} + 3\lambda c_{(3K)}$$
$$4a_6^{(5)} + 48\alpha^{(4)}c_{(1A)}c_{(2C)} + 16c_{(1A)}c_{(1A)}\alpha_3^{(5)}$$

$= 2c_{(1A)}t^{(4)}_{27} + 2c_{(1A)}t^{(4)}_{29} + 2c_{(1A)}t^{(4)}_{11} + 2c_{(2C)}t^{(3)}_{2}$

$6a^{(5)}_{7} + 12c_{(1A)}c_{(1A)}\alpha^{(5)}_{2} = c_{(1A)}t^{(4)}_{28} + c_{(1A)}t^{(4)}_{30} + 3\lambda c_{(3J)}$

$2a^{(5)}_{7} + 4c_{(1A)}c_{(1A)}\alpha^{(5)}_{2} = c_{(1A)}t^{(4)}_{10}$

$8a^{(5)}_{8} = 3\lambda c_{(3t)}$

$2a^{(5)}_{9} + 12\alpha^{(4)}c_{(1A)}c_{(2B)} + 2c_{(1A)}c_{(1A)}\alpha^{(5)}_{1} + 4c_{(1A)}c_{(1a)}\alpha^{(5)}_{2} + 4c_{(1a)}c_{(1A)}\alpha^{(5)}_{3}$

$= c_{(1A)}t^{(4)}_{19} + c_{(1a)}t^{(4)}_{18} + c_{(1a)}t^{(4)}_{15} + c_{(2B)}t^{(3)}_{2} + c_{(2B)}t^{(3)}_{3} + 3\lambda c_{(3M')} - 3\lambda c^{v}_{(3M)}$

$2a^{(5)}_{9} + 12\alpha^{(4)}c_{(1A)}c_{(2B)} + 2c_{(1A)}c_{(1A)}\alpha^{(5)}_{1} + 4c_{(1A)}c_{(1a)}\alpha^{(5)}_{2} + 4c_{(1a)}c_{(1A)}\alpha^{(5)}_{3}$

$= c_{(1A)}t^{(4)}_{7} + c_{(1A)}t^{(4)}_{23} + c_{(1a)}t^{(4)}_{17} + c_{(1a)}t^{(4)}_{16} + c_{(2B)}t^{(3)}_{4} + 3\lambda c_{(3M)} + 3\lambda c^{v}_{(3M)}$

$2a^{(5)}_{9} + 12\alpha^{(4)}c_{(1A)}c_{(2B)} + 2c_{(1A)}c_{(1A)}\alpha^{(5)}_{1} + 4c_{(1A)}c_{(1a)}\alpha^{(5)}_{2} + 4c_{(1a)}c_{(1A)}\alpha^{(5)}_{3}$

$= c_{(1A)}t^{(4)}_{20} + c_{(1A)}t^{(4)}_{24} + c_{(1A)}t^{(4)}_{8} + c_{(1a)}t^{(4)}_{14}$

$2a^{(5)}_{9} + 12\alpha^{(4)}c_{(1A)}c_{(2B)} + 2c_{(1A)}c_{(1A)}\alpha^{(5)}_{1} + 4c_{(1A)}c_{(1a)}\alpha^{(5)}_{2} + 4c_{(1a)}c_{(1A)}\alpha^{(5)}_{3}$

$= c_{(1A)}t^{(4)}_{22} + c_{(1A)}t^{(4)}_{26} + 3\lambda c_{(3n)}$

$4a^{(5)}_{10} + 24\alpha^{(4)}c_{(1a)}c_{(2B)} + 4c_{(1a)}c_{(1a)}\alpha^{(5)}_{3}$

$= c_{(1a)}t^{(4)}_{7} + c_{(1a)}t^{(4)}_{20} + c_{(1a)}t^{(4)}_{24} + c_{(2B)}t^{(3)}_{1}$

$4a^{(5)}_{10} + 24\alpha^{(4)}c_{(1a)}c_{(2B)} + 4c_{(1a)}c_{(1a)}\alpha^{(5)}_{3}$

$= c_{(1a)}t^{(4)}_{22} + 2c_{(2B)}t^{(3)}_{1} + 3\lambda c_{(3j)}$

$2a^{(5)}_{11} + 12\alpha^{(4)}c_{(1a)}c_{(2C)} + 24\alpha^{(4)}c_{(1A)}c_{(2B)} + 8c_{(1a)}c_{(1A)}\alpha^{(5)}_{3}$

$= 2c_{(1A)}t^{(4)}_{7} + c_{(1a)}t^{(4)}_{27} + c_{(1a)}t^{(4)}_{29} + 2c_{(2B)}t^{(3)}_{2}$

$2a^{(5)}_{11} + 12\alpha^{(4)}c_{(1a)}c_{(2C)} + 24\alpha^{(4)}c_{(1A)}c_{(2B)} + 8c_{(1a)}c_{(1A)}\alpha^{(5)}_{3}$

$= 2c_{(1A)}t^{(4)}_{20} + 2c_{(1A)}t^{(4)}_{24} + c_{(1a)}t^{(4)}_{11} + c_{(2C)}t^{(3)}_{1}$

$2a^{(5)}_{11} + 12\alpha^{(4)}c_{(1a)}c_{(2C)} + 24\alpha^{(4)}c_{(1A)}c_{(2B)} + 8c_{(1a)}c_{(1A)}\alpha^{(5)}_{3}$

$= c_{(1a)}t^{(4)}_{12} + c_{(1a)}t^{(4)}_{13} + 2c_{(2B)}t^{(3)}_{3} + 2c_{(2B)}t^{(3)}_{4} + 3\lambda c_{(3H)}$

$2a^{(5)}_{11} + 12\alpha^{(4)}c_{(1a)}c_{(2C)} + 24\alpha^{(4)}c_{(1A)}c_{(2B)} + 8c_{(1a)}c_{(1A)}\alpha^{(5)}_{3}$

$= 2c_{(1A)}t^{(4)}_{22} + 2c_{(2C)}t^{(3)}_{1} + 3\lambda c_{(3m)}$

$2a^{(5)}_{12} + 12\alpha^{(4)}c_{(1a)}c_{(2b)} + 2c_{(1a)}c_{(1a)}\alpha^{(5)}_{1} = c_{(1a)}t^{(4)}_{4} + c_{(2b)}t^{(3)}_{1} + 3\lambda c_{(3e)}$

$2a^{(5)}_{12} + 12\alpha^{(4)}c_{(1a)}c_{(2b)} + 2c_{(1a)}c_{(1a)}\alpha^{(5)}_{1} = c_{(1a)}t^{(4)}_{6} + 3\lambda c_{(3g)}$

$4a^{(5)}_{12} + 24\alpha^{(4)}c_{(1a)}c_{(2b)} + 4c_{(1a)}c_{(1a)}\alpha^{(5)}_{1} = c_{(1a)}t^{(4)}_{5} + 2c_{(2b)}t^{(3)}_{1} + 6\lambda c_{(3f)}$

$a^{(5)}_{13} + 6\alpha^{(4)}c_{(1a)}c_{(2c)} + 12\alpha^{(4)}c_{(1A)}c_{(2b)} + 4c_{(1a)}c_{(1A)}\alpha^{(5)}_{1}$

$= c_{(1A)}t^{(4)}_{6} + c_{(1a)}t^{(4)}_{9} + 3\lambda c_{(3h)}$

$a^{(5)}_{13} + 6\alpha^{(4)}c_{(1a)}c_{(2c)} + 12\alpha^{(4)}c_{(1A)}c_{(2b)} + 4c_{(1a)}c_{(1A)}\alpha^{(5)}_{1}$

$= c_{(1A)}t^{(4)}_{5} + c_{(2c)}t^{(3)}_{1} + 3\lambda c_{(3l)}$

$2a^{(5)}_{13} + 12\alpha^{(4)}c_{(1a)}c_{(2c)} + 24\alpha^{(4)}c_{(1A)}c_{(2b)} + 8c_{(1a)}c_{(1A)}\alpha^{(5)}_{1}$

$= 2c_{(1A)}t^{(4)}_{4} + c_{(1A)}t^{(4)}_{5} + 2c_{(2c)}t^{(3)}_{1} + 6\lambda c_{(3k)}$

$$
\begin{aligned}
&2a^{(5)}_{13} + 12\alpha^{(4)}c_{(1a)}c_{(2c)} + 24\alpha^{(4)}c_{(1A)}c_{(2b)} + 8c_{(1a)}c_{(1A)}\alpha^{(5)}_{1} \\
&= c_{(1A)}t^{(4)}_{6} + c_{(1a)}t^{(4)}_{25} + 2c_{(2b)}t^{(3)}_{2} \\
&2a^{(5)}_{13} + 12\alpha^{(4)}c_{(1a)}c_{(2c)} + 24\alpha^{(4)}c_{(1A)}c_{(2b)} + 8c_{(1a)}c_{(1A)}\alpha^{(5)}_{1} \\
&= c_{(1a)}t^{(4)}_{21} + 2c_{(2b)}t^{(3)}_{3} + 2c_{(2b)}t^{(3)}_{4} + 3\lambda c_{(3D)} \\
&2a^{(5)}_{14} + 12\alpha^{(4)}c_{(1A)}c_{(2b)} + 4c_{(1a)}c_{(1A)}\alpha^{(5)}_{1} + 2c_{(1a)}c_{(1a)}\alpha^{(5)}_{2} \\
&= c_{(1A)}t^{(4)}_{4} + c_{(1a)}t^{(4)}_{8} + c_{(2b)}t^{(3)}_{2} \\
&2a^{(5)}_{14} + 12\alpha^{(4)}c_{(1A)}c_{(2b)} + 4c_{(1a)}c_{(1A)}\alpha^{(5)}_{1} + 2c_{(1a)}c_{(1a)}\alpha^{(5)}_{2} \\
&= c_{(1a)}t^{(4)}_{19} + c_{(1a)}t^{(4)}_{23} + c_{(2b)}t^{(3)}_{3} + c_{(2b)}t^{(3)}_{4} + 3\lambda c_{(3E)} \\
&4a^{(5)}_{14} + 24\alpha^{(4)}c_{(1A)}c_{(2b)} + 8c_{(1a)}c_{(1A)}\alpha^{(5)}_{1} + 4c_{(1a)}c_{(1a)}\alpha^{(5)}_{2} \\
&= c_{(1A)}t^{(4)}_{5} + c_{(1A)}t^{(4)}_{6} + c_{(1a)}t^{(4)}_{26} + 6\lambda c_{(3i)} \\
&a^{(5)}_{15} + 12\alpha^{(4)}c_{(1A)}c_{(2c)} + 6c_{(1A)}c_{(1A)}\alpha^{(5)}_{1} + 4c_{(1a)}c_{(1A)}\alpha^{(5)}_{2} \\
&= c_{(1A)}t^{(4)}_{26} + 2c_{(1A)}t^{(4)}_{9} + 3\lambda c_{(3o)} \\
&a^{(5)}_{15} + 12\alpha^{(4)}c_{(1A)}c_{(2c)} + 6c_{(1A)}c_{(1A)}\alpha^{(5)}_{1} + 4c_{(1a)}c_{(1A)}\alpha^{(5)}_{2} \\
&= c_{(1A)}t^{(4)}_{25} + c_{(1a)}t^{(4)}_{10} \\
&2a^{(5)}_{15} + 24\alpha^{(4)}c_{(1A)}c_{(2c)} + 12c_{(1A)}c_{(1A)}\alpha^{(5)}_{1} + 8c_{(1a)}c_{(1A)}\alpha^{(5)}_{2} \\
&= 2c_{(1A)}t^{(4)}_{19} + c_{(1A)}t^{(4)}_{21} + c_{(1a)}t^{(4)}_{30} + 2c_{(2c)}t^{(3)}_{4} + 3\lambda c_{(3G)} + 3\lambda c^{v}_{(3G)} \\
&2a^{(5)}_{15} + 24\alpha^{(4)}c_{(1A)}c_{(2c)} + 12c_{(1A)}c_{(1A)}\alpha^{(5)}_{1} + 8c_{(1a)}c_{(1A)}\alpha^{(5)}_{2} \\
&= c_{(1A)}t^{(4)}_{21} + 2c_{(1A)}t^{(4)}_{23} + c_{(1a)}t^{(4)}_{28} + 2c_{(2c)}t^{(3)}_{3} + 3\lambda c_{(3G')} - 3\lambda c^{v}_{(3G)} \\
&2a^{(5)}_{15} + 24\alpha^{(4)}c_{(1A)}c_{(2c)} + 12c_{(1A)}c_{(1A)}\alpha^{(5)}_{1} + 8c_{(1a)}c_{(1A)}\alpha^{(5)}_{2} \\
&= c_{(1A)}t^{(4)}_{25} + 2c_{(1A)}t^{(4)}_{8} + c_{(1A)}t^{(4)}_{26} + 2c_{(2c)}t^{(3)}_{2} \\
&4a^{(5)}_{16} + 24\alpha^{(4)}c_{(1A)}c_{(2c)} + 8c_{(1A)}c_{(1A)}\alpha^{(5)}_{1} \\
&= c_{(1A)}t^{(4)}_{21} + c_{(2c)}t^{(3)}_{3} + c_{(2c)}t^{(3)}_{4} + 3\lambda c_{(3F)} \\
&4a^{(5)}_{16} + 24\alpha^{(4)}c_{(1A)}c_{(2c)} + 8c_{(1A)}c_{(1A)}\alpha^{(5)}_{1} \\
&= c_{(1A)}t^{(4)}_{25} + c_{(1A)}t^{(4)}_{9} + c_{(2c)}t^{(3)}_{2}
\end{aligned}
\tag{B.1}
$$

C: Three-dimensional equations

Here we list the systems of linear equations generated when solving (2.1) for a general three-dimensional Abelian gauge theory.

Equations for $A^{(7)}$ - Pure Yukawa terms

$$
\begin{aligned}
6a_1^{(7)} &= 4\mu c_1^{(4)} \\
6a_2^{(7)} &= 2\mu c_2^{(4)} \\
6a_3^{(7)} &= \mu c_3^{(4)} \\
6a_4^{(7)} &= \mu c_4^{(4)} \\
6a_5^{(7)} + 12(c_4^{(2)})^2 a &= c_4^{(2)}(t_6^{(5)} + t_7^{(5)} + t_8^{(5)}) \\
6a_6^{(7)} + 12(c_5^{(2)})^2 a &= c_5^{(2)}(t_{13}^{(5)} + t_{14}^{(5)} + t_{15}^{(5)})
\end{aligned}
\tag{C.1}
$$

$$
\begin{aligned}
2a_7^{(7)} &= 2\mu c_5^{(4)} \\
4a_7^{(7)} &= 0 \\
2a_8^{(7)} &= 2\mu c_6^{(4)} \\
4a_8^{(7)} &= 0
\end{aligned}
\tag{C.2}
$$

$$
\begin{aligned}
2a_9^{(7)} &= 4\mu c_7^{(4)} \\
2a_9^{(7)} &= 4\mu c_8^{(4)} \\
2a_9^{(7)} &= 4\mu c_9^{(4)} \\
2a_{10}^{(7)} &= 4\mu c_{10}^{(4)} \\
2a_{10}^{(7)} &= 4\mu c_{11}^{(4)} \\
2a_{10}^{(7)} &= 4\mu c_{12}^{(4)} \\
4a_{11}^{(7)} &= 4\mu c_{13}^{(4)} \\
2a_{11}^{(7)} &= \mu c_{14}^{(4)}
\end{aligned}
$$

$$\begin{aligned}
2a_{12}^{(7)} &= 4\mu c_{15}^{(4)} \\
2a_{12}^{(7)} &= 4\mu c_{16}^{(4)} \\
2a_{12}^{(7)} &= 4\mu c_{17}^{(4)} \\
2a_{13}^{(7)} &= \mu c_{18}^{(4)} \\
4a_{13}^{(7)} &= 4\mu c_{19}^{(4)} \\
2a_{14}^{(7)} &= 2\mu c_{20}^{(4)} \\
4a_{14}^{(7)} &= 4\mu c_{21}^{(4)} \\
2a_{15}^{(7)} &= \mu c_{22}^{(4)} \\
4a_{15}^{(7)} &= 4\mu c_{23}^{(4)} \\
4a_{16}^{(7)} &= 4\mu c_{24}^{(4)} \\
2a_{16}^{(7)} &= \mu c_{25}^{(4)} \\
4a_{17}^{(7)} &= 2\mu c_{26}^{(4)} \\
2a_{17}^{(7)} &= \mu c_{27}^{(4)} \\
4a_{18}^{(7)} &= 2\mu c_{28}^{(4)} \\
2a_{18}^{(7)} &= \mu c_{29}^{(4)} \\
2a_{19}^{(7)} &= \mu c_{30}^{(4)} \\
4a_{19}^{(7)} &= 4\mu c_{31}^{(4)} \\
2a_{20}^{(7)} &= \mu c_{32}^{(4)} \\
4a_{20}^{(7)} &= 4\mu c_{33}^{(4)} \\
4a_{21}^{(7)} &= 4\mu c_{34}^{(4)} \\
2a_{21}^{(7)} &= \mu c_{35}^{(4)}
\end{aligned} \tag{C.3}$$

$$\begin{aligned}
&a_{22}^{(7)} = 2\mu c_{36}^{(4)} \\
&2a_{22}^{(7)} = 2\mu c_{37}^{(4)} \\
&2a_{22}^{(7)} = c_2^{(2)} t_5^{(5)} \\
&a_{22}^{(7)} = 2\mu c_{38}^{(4)} + c_2^{(2)} t_4^{(5)} \\
&2a_{23}^{(7)} + 8(c_1^{(2)})^2 a = 4\mu c_{39}^{(4)} + c_1^{(2)} t_1^{(5)} \\
&2a_{23}^{(7)} + 8(c_1^{(2)})^2 a = 4\mu c_{40}^{(4)} + c_1^{(2)} t_2^{(5)} \\
&2a_{23}^{(7)} + 8(c_1^{(2)})^2 a = 4\mu c_{41}^{(4)} + c_1^{(2)} t_3^{(5)} \\
&2a_{24}^{(7)} + 8(c_1^{(2)})^2 a = 4\mu c_{42}^{(4)} + c_1^{(2)} t_2^{(5)} \\
&2a_{24}^{(7)} + 8(c_1^{(2)})^2 a = 4\mu c_{43}^{(4)} + c_1^{(2)} t_3^{(5)} \\
&2a_{24}^{(7)} + 8(c_1^{(2)})^2 a = 4\mu c_{44}^{(4)} + c_1^{(2)} t_1^{(5)} \\
&a_{25}^{(7)} = 4\mu c_{45}^{(4)}
\end{aligned}$$

$$a_{25}^{(7)} = 4\mu c_{46}^{(4)}$$
$$a_{25}^{(7)} = 4\mu c_{47}^{(4)}$$
$$a_{25}^{(7)} = c_1^{(2)} t_5^{(5)}$$
$$a_{25}^{(7)} = 4\mu c_{48}^{(4)} + 2c_1^{(2)} t_4^{(5)}$$
$$a_{25}^{(7)} = c_1^{(2)} t_5^{(5)}$$
$$2a_{26}^{(7)} + 8(c_1^{(2)})^2 a = 4\mu c_{49}^{(4)} + c_1^{(2)} t_1^{(5)}$$
$$2a_{26}^{(7)} + 8(c_1^{(2)})^2 a = 4\mu c_{50}^{(4)} + c_1^{(2)} t_2^{(5)}$$
$$2a_{26}^{(7)} + 8(c_1^{(2)})^2 a = 4\mu c_{51}^{(4)} + c_1^{(2)} t_3^{(5)}$$
$$a_{27}^{(7)} + 8c_1^{(2)} c_2^{(2)} a = 4\mu c_{52}^{(4)} + c_2^{(2)} t_2^{(5)}$$
$$a_{27}^{(7)} + 8c_1^{(2)} c_2^{(2)} a = 4\mu c_{53}^{(4)} + c_2^{(2)} t_3^{(5)}$$
$$a_{27}^{(7)} + 8c_1^{(2)} c_2^{(2)} a = 2\mu c_{54}^{(4)} + 2c_1^{(2)} t_{10}^{(5)}$$
$$a_{27}^{(7)} + 8c_1^{(2)} c_2^{(2)} a = 4\mu c_{55}^{(4)} + 4c_1^{(2)} t_9^{(5)}$$
$$a_{27}^{(7)} + 8c_1^{(2)} c_2^{(2)} a = 2\mu c_{56}^{(4)} + 2c_1^{(2)} t_{10}^{(5)}$$
$$a_{27}^{(7)} + 8c_1^{(2)} c_2^{(2)} a = 4\mu c_{57}^{(4)} + c_2^{(2)} t_1^{(5)}$$
$$2a_{28}^{(7)} = 4\mu c_{58}^{(4)}$$
$$2a_{28}^{(7)} = c_1^{(2)} t_5^{(5)}$$
$$2a_{28}^{(7)} = 2c_1^{(2)} t_4^{(5)} + c_1^{(2)} t_5^{(5)}$$
$$2a_{29}^{(7)} + 16c_1^{(2)} c_4^{(2)} a = 4\mu c_{59}^{(4)} + c_4^{(2)} (t_1^{(5)} + t_2^{(5)})$$
$$2a_{29}^{(7)} + 16c_1^{(2)} c_4^{(2)} a = c_4^{(2)} t_3^{(5)} + 2c_1^{(2)} t_6^{(5)}$$
$$2a_{29}^{(7)} + 16c_1^{(2)} c_4^{(2)} a = 2\mu c_{60}^{(4)} + c_1^{(2)} (t_7^{(5)} + t_8^{(5)})$$
$$4a_{30}^{(7)} + 16(c_1^{(2)})^2 a = 4\mu c_{61}^{(4)} + c_1^{(2)} (t_2^{(5)} + t_3^{(5)})$$
$$2a_{30}^{(7)} + 8(c_1^{(2)})^2 a = 2\mu c_{62}^{(4)} + c_1^{(2)} t_1^{(5)}$$
$$2a_{31}^{(7)} + 16c_1^{(2)} c_4^{(2)} a = 4\mu c_{63}^{(4)} + c_4^{(2)} (t_2^{(5)} + t_3^{(5)})$$
$$2a_{31}^{(7)} + 16c_1^{(2)} c_4^{(2)} a = c_4^{(2)} t_1^{(5)} + 2c_1^{(2)} t_6^{(5)}$$
$$2a_{31}^{(7)} + 16c_1^{(2)} c_4^{(2)} a = 2\mu c_{64}^{(4)} + 2c_1^{(2)} (t_7^{(5)} + t_8^{(5)})$$
$$2a_{32}^{(7)} + 8c_2^{(2)} c_4^{(2)} a = 2\mu c_{65}^{(4)} + 2c_4^{(2)} t_9^{(5)} + c_4^{(2)} t_{10}^{(5)}$$
$$2a_{32}^{(7)} + 8c_2^{(2)} c_4^{(2)} a = c_2^{(2)} t_6^{(5)} + c_4^{(2)} t_{10}^{(5)}$$
$$2a_{32}^{(7)} + 8c_2^{(2)} c_4^{(2)} a = 2\mu c_{66}^{(4)} + c_2^{(2)} (t_7^{(5)} + t_8^{(5)})$$
$$2a_{33}^{(7)} + 2(c_2^{(2)})^2 a = \mu c_{67}^{(4)} + c_2^{(2)} t_9^{(5)}$$
$$4a_{33}^{(7)} + 4(c_2^{(2)})^2 a = 2\mu c_{68}^{(4)} + c_2^{(2)} t_{10}^{(5)}$$
$$2a_{34}^{(7)} + 4(c_4^{(2)})^2 a = c_4^{(2)} t_6^{(5)}$$
$$4a_{34}^{(7)} + 8(c_4^{(2)})^2 a = 2\mu c_{69}^{(4)} + c_4^{(2)} (t_7^{(5)} + t_8^{(5)})$$
$$2a_{35}^{(7)} + 16c_1^{(2)} c_5^{(2)} a = 4\mu c_{70}^{(4)} + c_5^{(2)} (t_2^{(5)} + t_3^{(5)})$$

$$2a^{(7)}_{35} + 16c^{(2)}_1 c^{(2)}_5 a = c^{(2)}_5 t^{(5)}_1 + 2c^{(2)}_1 t^{(5)}_{13}$$
$$2a^{(7)}_{35} + 16c^{(2)}_1 c^{(2)}_5 a = 2\mu c^{(4)}_{71} + 2c^{(2)}_1 (t^{(5)}_{14} + t^{(5)}_{15})$$
$$2a^{(7)}_{36} + 16c^{(2)}_1 c^{(2)}_5 a = 4\mu c^{(4)}_{72} + c^{(2)}_5 (t^{(5)}_1 + t^{(5)}_3)$$
$$2a^{(7)}_{36} + 16c^{(2)}_1 c^{(2)}_5 a = c^{(2)}_5 t^{(5)}_2 + 2c^{(2)}_1 t^{(5)}_{13}$$
$$2a^{(7)}_{36} + 16c^{(2)}_1 c^{(2)}_5 a = 2\mu c^{(4)}_{73} + 2c^{(2)}_1 (t^{(5)}_{14} + t^{(5)}_{15})$$
$$2a^{(7)}_{37} + 4c^{(2)}_2 c^{(2)}_3 a = \mu c^{(4)}_{74} + c^{(2)}_2 t^{(5)}_{12}$$
$$2a^{(7)}_{37} + 4c^{(2)}_2 c^{(2)}_3 a = c^{(2)}_2 t^{(5)}_{11} + c^{(2)}_3 t^{(5)}_9$$
$$2a^{(7)}_{37} + 4c^{(2)}_2 c^{(2)}_3 a = \mu c^{(4)}_{75} + c^{(2)}_3 t^{(5)}_{10}$$
$$2a^{(7)}_{38} + 8c^{(2)}_3 c^{(2)}_4 a = 2\mu c^{(4)}_{76} + c^{(2)}_3 (t^{(5)}_7 + t^{(5)}_8)$$
$$2a^{(7)}_{38} + 8c^{(2)}_3 c^{(2)}_4 a = c^{(2)}_3 t^{(5)}_6 + 2c^{(2)}_4 t^{(5)}_{11}$$
$$2a^{(7)}_{38} + 8c^{(2)}_3 c^{(2)}_4 a = \mu c^{(4)}_{77} + 2c^{(2)}_4 t^{(5)}_{12}$$
$$2a^{(7)}_{39} + 16c^{(2)}_4 c^{(2)}_5 a = 2\mu c^{(4)}_{78} + 2c^{(2)}_5 (t^{(5)}_7 + t^{(5)}_8)$$
$$2a^{(7)}_{39} + 16c^{(2)}_4 c^{(2)}_5 a = 2c^{(2)}_5 t^{(5)}_6 + 2c^{(2)}_4 t^{(5)}_{13}$$
$$2a^{(7)}_{39} + 16c^{(2)}_4 c^{(2)}_5 a = 2\mu c^{(4)}_{79} + 2c^{(2)}_4 (t^{(5)}_{14} + t^{(5)}_{15})$$
$$2a^{(7)}_{40} + 8c^{(2)}_2 c^{(2)}_5 a = \mu c^{(4)}_{80} + 2c^{(2)}_5 t^{(5)}_{10}$$
$$2a^{(7)}_{40} + 8c^{(2)}_2 c^{(2)}_5 a = 2c^{(2)}_5 t^{(5)}_9 + c^{(2)}_2 t^{(5)}_{13}$$
$$2a^{(7)}_{40} + 8c^{(2)}_2 c^{(2)}_5 a = 2\mu c^{(4)}_{81} + c^{(2)}_2 (t^{(5)}_{14} + t^{(5)}_{15})$$
$$2a^{(7)}_{41} + 4(c^{(2)}_5)^2 a = c^{(2)}_5 t^{(5)}_{13}$$
$$4a^{(7)}_{41} + 8(c^{(2)}_5)^2 a = 2\mu c^{(4)}_{82} + c^{(2)}_5 (t^{(5)}_{14} + t^{(5)}_{15})$$
$$2a^{(7)}_{42} = 2c^{(2)}_4 t^{(5)}_4 + c^{(2)}_4 t^{(5)}_5$$
$$2a^{(7)}_{42} = c^{(2)}_4 t^{(5)}_5$$
$$2a^{(7)}_{42} = 2\mu c^{(4)}_{83}$$
$$2a^{(7)}_{43} = 2c^{(2)}_5 t^{(5)}_4 + c^{(2)}_5 t^{(5)}_5$$
$$2a^{(7)}_{43} = c^{(2)}_5 t^{(5)}_5$$
$$2a^{(7)}_{43} = 2\mu c^{(4)}_{84}$$
$$a^{(7)}_{44} + 8c^{(2)}_1 c^{(2)}_3 a = 4\mu c^{(4)}_{85} + 4c^{(2)}_1 t^{(5)}_{11}$$
$$a^{(7)}_{44} + 8c^{(2)}_1 c^{(2)}_3 a = 4\mu c^{(4)}_{86} + c^{(2)}_3 t^{(5)}_1$$
$$a^{(7)}_{44} + 8c^{(2)}_1 c^{(2)}_3 a = 4\mu c^{(4)}_{87} + c^{(2)}_3 t^{(5)}_2$$
$$a^{(7)}_{44} + 8c^{(2)}_1 c^{(2)}_3 a = 4\mu c^{(4)}_{88} + c^{(2)}_3 t^{(5)}_3$$
$$a^{(7)}_{44} + 8c^{(2)}_1 c^{(2)}_3 a = 2\mu c^{(4)}_{89} + 2c^{(2)}_1 t^{(5)}_{12}$$
$$a^{(7)}_{44} + 8c^{(2)}_1 c^{(2)}_3 a = 2\mu c^{(4)}_{90} + 2c^{(2)}_1 t^{(5)}_{12}$$
$$a^{(7)}_{45} = 2\mu c^{(4)}_{91} + c^{(2)}_3 t^{(5)}_4$$
$$2a^{(7)}_{45} = c^{(2)}_3 t^{(5)}_5$$

$$\begin{aligned}
&2a_{45}^{(7)} = 2\mu c_{92}^{(4)}\\
&a_{45}^{(7)} = 2\mu c_{93}^{(4)}\\
&2a_{46}^{(7)} + 2(c_3^{(2)})^2 a = \mu c_{94}^{(4)} + c_3^{(2)} t_{11}^{(5)}\\
&4a_{46}^{(7)} + 4(c_3^{(2)})^2 a = 2\mu c_{95}^{(4)} + c_3^{(2)} t_{12}^{(5)}\\
&2a_{47}^{(7)} + 8c_3^{(2)} c_5^{(2)} a = c_5^{(2)} t_{12}^{(5)} + c_3^{(2)} t_{13}^{(5)}\\
&2a_{47}^{(7)} + 8c_3^{(2)} c_5^{(2)} a = 2\mu c_{96}^{(4)} + c_3^{(2)} (t_{14}^{(5)} + t_{15}^{(5)})\\
&2a_{47}^{(7)} + 8c_3^{(2)} c_5^{(2)} a = 2\mu c_{97}^{(4)} + 2c_5^{(2)} t_{11}^{(5)} + c_5^{(2)} t_{12}^{(5)}
\end{aligned} \tag{C.4}$$

$$\begin{aligned}
&a_{48}^{(7)} = 4c_1^{(2)} t_{16}^{(5)}\\
&a_{48}^{(7)} = 4c_1^{(2)} t_{18}^{(5)}\\
&a_{48}^{(7)} = 4\mu c_{98}^{(4)}\\
&a_{48}^{(7)} = 4\mu c_{99}^{(4)}\\
&a_{48}^{(7)} = 4\mu c_{100}^{(4)}\\
&a_{48}^{(7)} = \mu c_{101}^{(4)} + 4c_1^{(2)} t_{17}^{(5)}\\
&2a_{49}^{(7)} = 2c_4^{(2)} t_{16}^{(5)}\\
&2a_{49}^{(7)} = 2c_4^{(2)} (t_{17}^{(5)} + t_{18}^{(5)})\\
&2a_{49}^{(7)} = 2\mu c_{102}^{(4)}\\
&2a_{50}^{(7)} = c_2^{(2)} t_{16}^{(5)}\\
&2a_{50}^{(7)} = c_2^{(2)} (t_{17}^{(5)} + t_{18}^{(5)})\\
&2a_{50}^{(7)} = \mu c_{103}^{(4)}\\
&2a_{51}^{(7)} = 2c_5^{(2)} (t_{16}^{(5)} + t_{17}^{(5)})\\
&2a_{51}^{(7)} = 2c_5^{(2)} t_{18}^{(5)}\\
&2a_{51}^{(7)} = 2\mu c_{104}^{(4)}\\
&2a_{52}^{(7)} = c_3^{(2)} (t_{16}^{(5)} + t_{17}^{(5)})\\
&2a_{52}^{(7)} = c_3^{(2)} t_{18}^{(5)}\\
&2a_{52}^{(7)} = \mu c_{105}^{(4)}
\end{aligned} \tag{C.5}$$

Equations for $A^{(7)}$ - $\mathcal{N} = 2$ supersymmetric gauge theory

$$\begin{aligned}
&2a_1^{(7)} + 24(c_1^{(2)})^2 a = 4\mu c_1^{(4)} + 3c_1^{(2)} (t_2^{(5)} + k_1^{(5)})\\
&2a_1^{(7)} + 24(c_1^{(2)})^2 a = 4\mu c_1^{(4)} + 3c_1^{(2)} (\bar{t}_2^{(5)} + \bar{k}_1^{(5)})\\
&a_1^{(7)} + 12(c_1^{(2)})^2 a = 3c_1^{(2)} t_1^{(5)}
\end{aligned}$$

$$a_1^{(7)} + 12(c_1^{(2)})^2 a = 3c_1^{(2)}\bar{t}_1^{(5)}$$
$$a_2^{(7)} = 4\mu c_2^{(4)}$$
$$a_2^{(7)} = 4\mu c_2^{(4)}$$
$$a_2^{(7)} = 2c_1^{(2)}\bar{t}_3^{(5)}$$
$$a_2^{(7)} = 2c_1^{(2)}t_3^{(5)} + 4c_1^{(2)}k_2^{(5)}$$
$$a_2^{(7)} = 4c_1^{(2)}\bar{k}_2^{(5)} + 2c_1^{(2)}\bar{t}_3^{(5)}$$
$$a_2^{(7)} = 2c_1^{(2)}t_3^{(5)}$$
$$3a_3^{(7)} + 12(c_1^{(2)})^2 a = c_1^{(2)}(t_1^{(5)} + t_2^{(5)}) + c_1^{(2)}k_1^{(5)}$$
$$3a_3^{(7)} + 12(c_1^{(2)})^2 a = c_1^{(2)}(\bar{t}_1^{(5)} + \bar{t}_2^{(5)}) + c_1^{(2)}\bar{k}_1^{(5)}$$
$$2a_4^{(7)} + 16c_1^{(2)}c_2^{(2)}a = 4\mu c_3^{(4)} + c_2^{(2)}(t_1^{(5)} + t_2^{(5)} + k_1^{(5)}) + c_1^{(2)}t_4^{(5)}$$
$$2a_4^{(7)} + 16c_1^{(2)}c_2^{(2)}a = 4\mu c_3^{(4)} + c_2^{(2)}(\bar{t}_1^{(5)} + \bar{t}_2^{(5)} + \bar{k}_1^{(5)}) + c_1^{(2)}\bar{t}_4^{(5)}$$
$$a_5^{(7)} + 24c_1^{(2)}c_2^{(2)}a = 4\mu c_4^{(4)} + 3c_2^{(2)}(t_2^{(5)} + k_1^{(5)})$$
$$a_5^{(7)} + 24c_1^{(2)}c_2^{(2)}a = 4\mu c_4^{(4)} + 3c_2^{(2)}(\bar{t}_2^{(5)} + \bar{k}_1^{(5)})$$
$$a_5^{(7)} + 24c_1^{(2)}c_2^{(2)}a = 3c_2^{(2)}\bar{t}_1^{(5)} + 3c_1^{(2)}\bar{t}_4^{(5)}$$
$$a_5^{(7)} + 24c_1^{(2)}c_2^{(2)}a = 3c_2^{(2)}t_1^{(5)} + 3c_1^{(2)}t_4^{(5)}$$
$$a_6^{(7)} = 4\mu c_5^{(4)}$$
$$a_6^{(7)} = 4\mu c_5^{(4)}$$
$$a_6^{(7)} = 2c_1^{(2)}\bar{t}_5^{(5)}$$
$$a_6^{(7)} = 2c_1^{(2)}t_5^{(5)}$$
$$a_7^{(7)} = 4\mu c_6^{(4)}$$
$$a_7^{(7)} = 4\mu c_6^{(4)}$$
$$a_7^{(7)} = 2c_1^{(2)}\bar{t}_5^{(5)}$$
$$a_7^{(7)} = 2c_1^{(2)}t_5^{(5)}$$
$$a_8^{(7)} = 4\mu c_7^{(4)}$$
$$a_8^{(7)} = 4\mu c_7^{(4)}$$
$$a_8^{(7)} = 2c_1^{(2)}\bar{t}_6^{(5)}$$
$$a_8^{(7)} = 2c_1^{(2)}t_6^{(5)}$$
$$a_9^{(7)} = 4\mu c_8^{(4)}$$
$$a_9^{(7)} = 4\mu c_8^{(4)}$$
$$a_9^{(7)} = 2c_1^{(2)}\bar{t}_6^{(5)}$$
$$a_9^{(7)} = 2c_1^{(2)}t_6^{(5)}$$
$$a_{10}^{(7)} = 2c_2^{(2)}t_3^{(5)} + 4c_2^{(2)}k_2^{(5)}$$
$$a_{10}^{(7)} = 4c_2^{(2)}\bar{k}_2^{(5)} + 2c_2^{(2)}\bar{t}_3^{(5)}$$

$$
\begin{aligned}
&a_{10}^{(7)} = 2c_2^{(2)}t_3^{(5)}\\
&a_{10}^{(7)} = 2c_2^{(2)}\bar{t}_3^{(5)}\\
&a_{11}^{(7)} = 2c_3^{(2)}t_3^{(5)} + 4c_3^{(2)}k_2^{(5)}\\
&a_{11}^{(7)} = 4c_3^{(2)}\bar{k}_2^{(5)} + 2c_3^{(2)}\bar{t}_3^{(5)}\\
&a_{11}^{(7)} = 2c_3^{(2)}\bar{t}_3^{(5)}\\
&a_{11}^{(7)} = 2c_3^{(2)}t_3^{(5)}\\
&2a_{12}^{(7)} + 16c_1^{(2)}c_3^{(2)}a = 4\mu c_9^{(4)} + c_3^{(2)}(t_1^{(5)} + t_2^{(5)} + k_1^{(5)}) + c_1^{(2)}t_7^{(5)}\\
&2a_{12}^{(7)} + 16c_1^{(2)}c_3^{(2)}a = 4\mu c_9^{(4)} + c_3^{(2)}(\bar{t}_1^{(5)} + \bar{t}_2^{(5)} + \bar{k}_1^{(5)}) + c_1^{(2)}\bar{t}_7^{(5)}\\
&a_{13}^{(5)} + 24c_1^{(2)}c_3^{(2)}a = 4\mu c_{10}^{(4)} + 3c_3^{(2)}(t_2^{(5)} + k_1^{(5)})\\
&a_{13}^{(5)} + 24c_1^{(2)}c_3^{(2)}a = 4\mu c_{10}^{(4)} + 3c_3^{(2)}(\bar{t}_2^{(5)} + \bar{k}_1^{(5)})\\
&a_{13}^{(5)} + 24c_1^{(2)}c_3^{(2)}a = 3c_3^{(2)}\bar{t}_1^{(5)} + 3c_1^{(2)}\bar{t}_7^{(5)}\\
&a_{13}^{(5)} + 24c_1^{(2)}c_3^{(2)}a = 3c_3^{(2)}t_1^{(5)} + 3c_1^{(2)}t_7^{(5)}
\end{aligned}
\tag{C.6}
$$

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