

# Centre-of-mass for the finite speed of light

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**Abstract.** In 1632, Galilei was aware of relativity of velocity and that this implies relativity of spaces-of-locations. During centuries the relativity of spaces-of-locations was ignored. Professor Harald Keres considered the space-of-locations as a congruence of world-lines, and there is no universal absolute three-dimensional space-of-locations. In applications, velocities relative to centre-of-mass are important. But the concept of centre-of-mass is impossible within relativity theory postulating that each pair of reference systems is related by the Lorentz isometry group transformation. We show that centre-of-mass of many-body interacting (bound) system for the case of finite light-speed is a well defined concept within the group-free approach using algebra epimorphisms as splits. We consider the Keres space-of-locations as the Grassmann factor-algebra of differential forms where a material body with a positive mass is interpreted as idempotent algebra epimorphism of the Grassmann algebra of spacetime onto the Grassmann factor-algebra of corresponding space-of-locations of that material body. A material body as a reference system is a group-free split, and this allows us to express all motions, velocities, accelerations and rotations, as *relative* with respect to the choice of variable reference system.

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Dedicated *in memoriam* of Professor Harald Keres (1912-2010)  
who insisted on relativity of all kinds of motion  
and initiated investigations of Einstein's gravity in Estonia

## 1. Harald Keres (1912-2010)

I am familiar with Professor Harald Keres publications since 40 years. I was immediately strongly convinced by the Keres crystal clear concept of *relativity* of spaces of locations, relativity of diachronic, relativity of 'simul-localidad' (relativity to be in the same location). I like his group-free definition of arbitrary (not necessarily constant) *relative* velocity (for infinite light-speed) of the Keres  $x$ -space relative to  $y$ -space [8, §2.6 p 68 formula (34)]. Moreover, during all these years I was permanently impressed by the Keres insistence that *every* motion must be relative motion, that relativity theory must not end with relativity of arbitrary velocity, but must include also relativity of acceleration and relativity of rotation (general relativity) [9]. The Keres group-free approach strongly motivated my research on group-free relativity for finite light-speed, i.e the Lorentz-isometry-group-free relativity theory [15, Oziewicz Z arXiv:math-ph/0608062], and [16, 17, 18, 19, 20, 21].

The Keres lecture in 1970 at Kiev's Symposium on *Philosophical Questions of Relativistic Physics and Cosmology* [9] is of fundamental importance. In this lecture Keres states that the concept of inertial reference system is a convention, because every reference system can be considered as being inertial. Keres stresses that every motion, including acceleration and rotation must be considered to be relative, to be dependent on the free choice of the variable reference system, identified with a material body of a positive mass.



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There does not exist a universal (unique) absolute three-dimensional space (of locations), common for all phenomena, and therefore there does not exist universal absolute acceleration and rotation. Every motion is relative.

Harald Keres [9] p 142

The above opinion is contrasted with the long standing in history of physics opinion by Langevin in 1911 [11, 12] that acceleration and rotation must have an absolute meaning, reference-system-free, independent of the choice of the three-dimensional space-of-locations.

There is no absolute space (of locations), and we only conceive of relative motion; and yet in most cases mechanical facts are enunciated as if there is an absolute space (of locations) to which they can be referred.

Henri Poincaré (1854-1912), Science and Hypothesis  
Chapter 6: Classical Mechanics 1902

Galileo Galilei (1564-1642) was the first who observed in 1632 relativity of velocity [7]. This implies immediately relativity of location: are we changing the location or do we stay in the same place? – this is relative. There does not exist a concept of location in Nature. Location is a mathematical convention and not a physical concept. In reality what counts are events (space-time of events) and the location of an event in some space-of-locations is an irrelevant convention [19, 20].

Einstein [4, 5] defined acceleration as the Christoffel derivative applied to one-body system, acceleration  $=\nabla_X X$  is a geodesic vector field (see also [25] §14, [2, 3]). No reference body is involved, so it is acceleration with respect to what? According to Keres one must specify another material body, another the Keres  $x$ -space, with respect to which relative velocity of one-body  $y$ -space are to be defined (velocity as a change of location, but location of what reference body?), and then relative acceleration is expected to be given by a derivative of relative velocity for two-body system. All this is beautifully explained in the Keres publications ([8] §2.8, [9, 10]).

Unfortunately, instead of generalizing *relativity* of velocity (supposed to be done by special relativity in terms of an isometry group), by including the *relativity* of acceleration, the Christoffel derivative applied to one-body system was like throwing the baby out with the bathwater: it made acceleration as defined by Einstein, Weyl, and later on by Ehlers, not relative but an absolute concept, neither the second reference material body nor the centre-of-mass were involved.

Keres considered the Galilean relativity of three-dimensional spaces-of-locations. Galilean relativity is due to the relative velocity among two material bodies with positive masses. A material body is identified with a congruence of time-like curves as ‘the Lagrangian description’ of the motion of a fluid material particle in terms of a congruence of curves, in fact, due to Euler and not to Lagrange. Such congruence is called by Keres to be  $x$ -space, and for a pair of such congruences, for  $x$ -space and  $y$ -space, Keres defined arbitrary relative velocity between them (for infinite light-speed). In the Keres notation  $x$  denotes the conserved zero-grade scalar fields [8, 10].

We present the Keres group-free concept of arbitrary relative velocity as Eulerian in terms of derivations of commutative *algebra*  $\mathcal{F}$  of ‘scalar fields’. An algebra derivation corresponds to a geometric vector field, but we avoid geometric interpretation considering an algebra as a primordial concept within manifold-free approach. The idea to represent material bodies in terms of a Lie algebra of derivations goes back to Leonhard Euler in 1757 (for velocity of a fluid) [6], and in the case of the finite light-speed to Hermann Minkowski in 1908.

Relativity of velocity needs at least two-body system, and each such system possesses a mass-centre, therefore the genuine relativity of velocity should include the concept of velocity *relative* to mass-centre rather than between bodies. Section 7 is devoted to the problem of a mass-centre for the finite light-speed.

## 2. Universal property of the differential and other notations

In what follows  $\mathcal{F}$  denotes an (unital, associative, commutative) algebra of zero-grade scalar fields. We need two Grassmann algebras, the Grassmann algebra of differential multiforms  $\mathcal{F}^\wedge$ , and the Grassmann algebra  $(\text{der } \mathcal{F})^\wedge$  of ‘multivector fields’,

$$\mathcal{F}^{\wedge 0} \equiv (\text{der } \mathcal{F})^{\wedge 0} \equiv \mathcal{F}, \quad (\text{der } \mathcal{F})^{\wedge 1} \equiv \text{der } \mathcal{F} \equiv \text{der}(\mathcal{F}, \mathcal{F}) \quad (2.1)$$

For  $X$  and  $Y \in (\text{der } \mathcal{F})^\wedge$  we use the left Grassmann multiplication

$$\wedge_X Y \equiv X \wedge Y, \quad (\wedge_X) \circ (\wedge_Y) = \wedge_{X \wedge Y}, \quad (\wedge_X)^2 = 0 \quad \text{for grade } X = \text{odd} \quad (2.2)$$

**2.1 Definition** (evaluation). The left *evaluation*, denoted by  $\text{ev}$ , is defined as an involutive pull-back of the left Grassmann wedge product

$$\text{ev}_X \equiv (\wedge_X)^*, \quad (\wedge_X) \circ (\wedge_Y) = \wedge_{X \wedge Y} \iff (\text{ev}_Y) \circ (\text{ev}_X) = \text{ev}_{X \wedge Y} \quad (2.3a)$$

$$\text{grade } X = \text{odd} \implies (\text{ev}_X)^2 = 0 \quad (2.3b)$$

**2.2 Exercise.** Let for  $X \in \text{der } \mathcal{F}$  and  $S \in \mathcal{F}^{\wedge 1}$ ,  $(\text{ev}_X)S = 1$ . Then

$$(\text{ev}_X) \circ \wedge_S \circ (\text{ev}_X) = \text{ev}_X \quad \text{and} \quad \wedge_S \circ (\text{ev}_X) \circ \wedge_S = \wedge_S \quad (2.4)$$

**2.3 Clarification** (universal property of the differential). The differential  $d \in \text{der}(\mathcal{F}, \mathcal{F}^{\wedge 1})$  (extended to graded differential  $d \in \text{der}(\mathcal{F}^\wedge, \mathcal{F}^\wedge)$  with  $d^2 = d \circ d = 0$ ), possesses a universal property: each derivation of an algebra  $\mathcal{F}$ ,  $X \in \text{der}(\mathcal{F}, \mathcal{F})$ , is a composition of a differential with evaluation. Thanks to the Grassmann condition (2.3b) and because the evaluation by a derivation-vector  $\text{ev}_X$  diminish grade by one, we can write for grade  $X = 1$ ,

$$(\text{ev}_X)|\mathcal{F} \equiv 0, \quad X = (\text{ev}_X) \circ d + d \circ (\text{ev}_X) = (d + \text{ev}_X)^2 \in \text{der}(\mathcal{F}, \mathcal{F}) \quad (2.5)$$

**2.4 Definition** (Christoffel 1869). Elwin Bruno Christoffel (1829-1900) introduced a zero-grade derivation of all tensor fields, for  $X \in \text{der } \mathcal{F}$  and  $f \in \mathcal{F}$ , by two conditions [1]

$$\nabla_X \circ d \neq d \circ \nabla_X, \quad \nabla_{fx} = f \nabla_X \quad \text{and} \quad \text{ev} \circ \nabla_X = \nabla_X \circ \text{ev} \quad (2.6)$$

**2.5 Definition** (Ślebodziński 1931). Władysław Ślebodziński (1884-1972) introduced in his PhD Thesis [23] a zero-grade derivation of all tensor fields by two conditions, for  $f \in \mathcal{F}$  and  $X \in \text{der } \mathcal{F}$ ,

$$\mathcal{L}_X \circ d = d \circ \mathcal{L}_X, \quad \mathcal{L}_{fx} \neq f \mathcal{L}_X \quad \text{and} \quad \text{ev} \circ \mathcal{L}_X = \mathcal{L}_X \circ \text{ev} \quad (2.7)$$

In 1932, Van Dantzig called the Ślebodziński derivation (2.7) the Lie derivation and introduced a notation  $\mathcal{L}$ , however, Sophus Lie passed away in 1899 and has nothing to do with this concept of derivation of a tensor field. We keep the name the Ślebodziński derivation (or the Ślebodziński-Lie derivation). From definition (2.7) one can prove the following

$$\mathcal{L}_X = (d + \text{ev}_X)^2 \in \text{der}(\mathcal{F}^\wedge), \quad \text{and} \quad \forall Y \in \text{der } \mathcal{F}, \quad \mathcal{S}_X Y = [X, Y] \quad (2.8)$$

The Ślebodziński conditions (2.7) determine the unique derivation  $\mathcal{L}$ . The Christoffel conditions (2.6) left a lot of freedom, and an extra condition put on by Christoffel,  $\nabla g = 0$ , for  $g$  being the metric tensor, do not determine the Christoffel derivation  $\nabla$  uniquely as was observed by Cartan in 1922.

**2.6 Theorem** (Cartan 1922). *Élie Cartan (1869-1951) observed in 1922 that a derivation  $X \in \text{der } \mathcal{F}$ , extends to a graded derivation of the Grassmann algebra of differential multiforms,  $\text{ev}_X \in \text{der}(\mathcal{F}^\wedge)$ . For any differential multiforms  $\alpha$  and  $\beta$  we have the Cartan graded derivation with the following Cartan theorem,*

$$(\text{ev}_X)(\alpha \wedge \beta) = (\text{ev}_X \alpha) \wedge \beta + (-)^\alpha \alpha \wedge (\text{ev}_X \beta), \quad \text{ev}_X \in \text{der}(\mathcal{F}^\wedge) \quad (2.9)$$

Cartan denoted the above derivation by  $i_X$ , and called *inner product* or inner derivation, however the name *evaluation* is more appropriate.

**2.7 Notation** (algebra map versus derivation of an algebra). In the sequel we use the following notation for grade  $X = 0$ ,

$$\begin{aligned} X \in \text{alg}(\mathcal{F}^\wedge) &\iff X(\alpha \wedge \beta) = (X\alpha) \wedge (X\beta) \\ X^p \in \text{der}(\mathcal{F}^\wedge) &\iff X^p(\alpha \wedge \beta) = (X^p\alpha) \wedge \beta + \alpha \wedge (X^p\beta) \end{aligned} \quad (2.10)$$

**2.8 Definition** (split). The Grassmann algebra map  $X$  as a  $(1, 1)$ -tensor field in terms a vector field  $X$  and a differential one-form  $S_X$  such that  $(\text{ev}_X)S_X \equiv 1$ , given as follows, is said to be a  $X$ -split

$$\begin{aligned} X \equiv (\text{ev}_X) \circ (\wedge_{S_X}) &\in \text{alg}(\mathcal{F}^\wedge) \quad \text{and} \quad X^p \equiv (\wedge_{S_X}) \circ (\text{ev}_X) \in \text{der}(\mathcal{F}^\wedge) \\ X^* \equiv (\text{ev}_{S_X}) \circ (\wedge_X) &\in \text{alg}((\text{der } \mathcal{F})^\wedge) \quad \text{and} \quad X^{p*} \equiv (\wedge_X) \circ (\text{ev}_{S_X}) \in \text{der}((\text{der } \mathcal{F})^\wedge) \end{aligned} \quad (2.11)$$

The Grassmann algebra derivation  $X^p$  is an involute permutation of an algebra map  $X$ .

**2.9 Exercise** (split is idempotent). For  $X$ -split and  $X^*$ -split the following holds

$$X \circ X = X, \quad X^p \circ X^p = X^p, \quad X^p \circ X = 0 = X \circ X^p \quad (2.12a)$$

$$X^p + X = (\text{ev}_X + \wedge_{S_X})^2 = \text{id}_{\mathcal{F}^\wedge} \quad (2.12b)$$

$$(X^p + X)^* = X^{*p} + X^* = (\text{ev}_{S_X} + \wedge_X)^2 = \text{id}_{(\text{der } \mathcal{F})^\wedge} \quad (2.12c)$$

### 3. From the Keres space-of-locations to algebra derivation

Keres introduced three-dimensional space-of-locations as a congruence of time-like world-curves in four-dimensional space-time manifold [8, §2.1], [10, §3]. Each congruence corresponds to a material particle-body with a positive mass. Having explicitly a pair of two congruences, *two* spaces-of-locations, called *x*-space and *y*-space, Keres defined conceptually an *arbitrary velocity* of one space *relative* to another space, i.e an arbitrary velocity explicitly as *relative* velocity, without invoking the group theory concept, i.e group-free relative velocity. This definition allows us to define also acceleration and rotation of the Keres *y*-space relative to *x*-space, i.e acceleration and rotation explicitly as *relative* acceleration and *relative* rotation ([8] §2.8).

One can equivalently understand each congruence of world-lines in Eulerian way as integral curves of the corresponding vector field or synonymously as the integral curves of an ordinary differential equation. This allows us to replace congruence by an algebra derivation, and thus a material particle is the same as a derivation of an algebra  $\mathcal{F}$ , where  $\{x^1, x^2, x^3\}$  describe the corresponding conserved scalar ‘integrals of motion’ of a particle. With analogous conserved scalar ‘integrals of motion’ of the second particle in ‘adopted coordinate system’ we have the following two-body free-system as a pair of derivations of an algebra  $\mathcal{F}$ ,

$$t, x^1, x^2, x^3, y^1, y^2, y^3 \in \mathcal{F} \quad (3.1a)$$

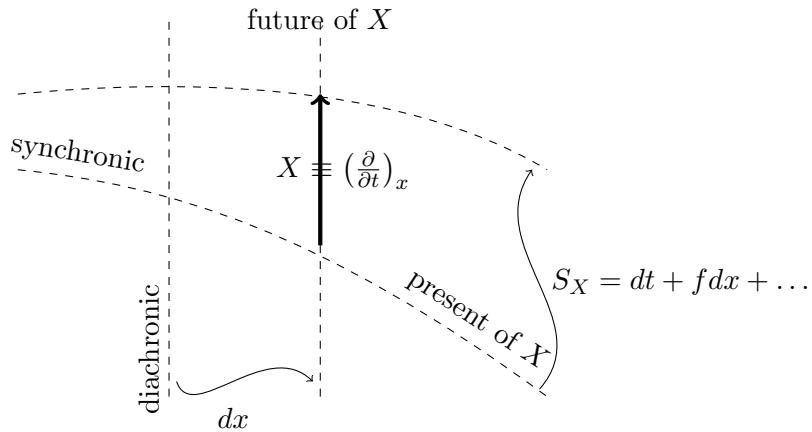
$$X = \left( \frac{\partial}{\partial t} \right)_{x^1, x^2, x^3} \quad \text{and} \quad Y = \left( \frac{\partial}{\partial t} \right)_{y^1, y^2, y^3} \in \text{der } \mathcal{F} \quad (3.1b)$$

$$(\text{ev}_X)(dt) = X t \equiv 1 = (\text{ev}_Y)dt, \quad (\text{ev}_X)(dx^i) \equiv 0, \quad (\text{ev}_Y)(dy^i) \equiv 0 \quad (3.1c)$$

Every material body in (3.1b) is basis-free because derivation of an algebra  $\mathcal{F}$  is an example of a basis-free tensor field.

In the present section for the Keres two-body system (3.1b), we set the Galilean simultaneity

$$S_X = S_Y = dt \implies X^{p*} Y = X \quad (3.2)$$



**Figure 1.** The reference system  $X$  is a material body (with a positive mass) as a permanent diachronic process-derivation  $X \in \text{der } \mathcal{F}$  from past to near future. Process  $X$  is going alongside of the crests of a wave of  $X$ -locations  $\{dx\}$ ,  $(\text{ev}_X)(dx) = X x = 0$ . Synchronic  $X$ -simultaneity of a reference system  $X$  is a differential one-form  $S_X = dt + f dx + \dots$ , such that  $(\text{ev}_X)S_X \equiv 1$ .

**3.1 Definition** (relative-velocity, [8] §2.6 formula (34)). An arbitrary (not necessarily constant) velocity  $v_{XY}$  of a material body  $Y$  (the Keres  $y$ -space) relative to a reference system  $X$  (relative to the Keres observer  $x$ -space) is defined as an  $X^*$ -split of  $Y$ ,

$$v_{XY} \equiv X^* Y, \quad Y = (X^{*p} + X^*)Y = X + v_{XY}, \quad v_{XY} \equiv Y - X, \quad (dt)v_{XY} = 0 \quad (3.3)$$

Above definition is valid only for absolute simultaneity  $dt$  (3.1c), and only in this particular case the relative velocity is reciprocal,  $v_{XY} = -v_{YX}$ . The above concept of relative velocity is a group-free concept.

**3.2 Corollary.** The scalar components,  $v_{XY} x^i$  and  $v_{XY} y^i$ , are exactly the Keres expressions for the velocity of  $y$ -space relative to  $x$ -space, and vice versa, see [8, §2.6] and [10, p 351 before formula (10)],

$$v_{XY} x^i = Y x^i = \left( \frac{\partial x^i}{\partial t} \right)_{y^1, y^2, y^3} \quad \text{and} \quad v_{XY} y^i = -X y^i = - \left( \frac{\partial y^i}{\partial t} \right)_{x^1, x^2, x^3} \quad (3.4)$$

#### 4. Galileo Galilei 1632 and Leonhard Euler 1757

The Keres definition of relative split-velocity [8, §2.6 formula (34)], Definition 3.1, expressed the observation made by Galileo Galilei in 1632 about the relativity of velocity. Note that the same material body  $Y$  possesses infinite many velocities with respect to many different reference systems, possesses many distinct splits

$$Y = X + v_{XY} = Z + v_{ZY} = \dots \quad (4.1)$$

If  $v_{XY}x = v = \text{const}$ , then  $x' = x - vt$  does the job, however the definition (3.3) is not restricted to constant velocities.

**4.1 Comment.** A velocity (relative or not) seen as a vector field must be an algebra derivation  $v \in \text{der } \mathcal{F}$ . An acceleration is a derivation of a velocity vector field, a derivation of a derivation. However before invention by Christoffel in 1869, the concept of a derivation of a vector field was not known, and therefore Newton in 1687, and Euler in 1757, defined acceleration not as a derivation of a vector field, but as a second derivation of an algebra of scalar fields. Newtonian definition of acceleration is still in University Physics textbooks as a composition of two algebra derivations. Let  $Y \in \text{der } \mathcal{F}$  and  $c^* \in \text{alg}(\mathcal{F}^\wedge, \dots)$  be an algebra epimorphism to algebra on integral curve of  $Y$ . Then acceleration is commonly defined as a composition of derivations,

$$\left( \frac{d}{dt} \circ \frac{d}{dt} \right) \circ c^* = c^* \circ (Y \circ Y) \notin \text{der } \mathcal{F} \quad (4.2)$$

A derivation of an algebra of scalar fields is a basis-free vector field, but composition of derivations is not. Thus Newton's and Euler's acceleration is not a tensor field. Einstein replaced the composition of derivations by the Christoffel derivation. The Christoffel derivation of a vector is a vector, and Albert Einstein [4, 5] defined acceleration as the Christoffel derivative of a velocity vector (giving the geodesic vector field).

The Keres relative velocity (3.3)-(4.1) is the same as the Euler definition in 1757 of a fluid, and fluid's *relative* velocity. The Euler derivative as a sum of vector fields, (4.3) below, is known as substantial or material, or barycentric, or hydrodynamic derivative. In the fluid context a material body is a *vector* field on space-time  $\in \text{der } \mathcal{F}$ . The Euler derivative is interpreted in Fluid Mechanics textbooks in the following way: "the total rate of change is a sum of the local rate of change plus the convective rate of change"

$$Y \equiv \frac{\partial}{\partial t} + \sum v^i \frac{\partial}{\partial x^i} = X + v_{XY}, \quad \frac{\partial}{\partial t} \equiv \left( \frac{\partial}{\partial t} \right)_{x^1, x^2, x^3} \quad (4.3)$$

In fact, the Euler derivative should be interpreted as a definition of a velocity  $v$  of a fluid  $Y$  relative to a reference system  $X$  in adopted coordinates  $X = (\partial_t)_x$ . Given vector fields  $X$  and  $v$ , then a fundamental theorem of ordinary differential equations assure that for a sum  $Y = X + v$  there exist local adopted coordinates that  $Y$  takes a form as in (3.1b).

For an observer  $X$  being the reference system,  $X|_x = 0$ , if  $x$  is in the Keres  $x$ -space, cf (3.1b),  $v_{XX} = 0_X$ . Therefore, the velocity of  $Y$  relative to  $X$  is

$$v_{XY}|_x = (Y - X)|_x = Y|_x - X|_x = Y|_x \quad (4.4)$$

Euler defined acceleration of a fluid following Newton as the second derivative of scalar fields, i.e not as a basis-free tensor,

$$a_{\text{Newton}}|_x = Y \circ Y, \quad a_{\text{Euler}} = Y \circ (Y - X) = (X + v) \circ v \notin \text{der } \mathcal{F} \quad (4.5a)$$

$$a_{\text{Euler}}|_{x^i} = \{(X + v) \circ v\}|_{x^i} = (X + v)v^i = \frac{\partial v^i}{\partial t} + \sum v^j \frac{\partial v^i}{\partial x^j} \quad (4.5b)$$

## 5. Simultaneity differential form

The Riemannian metric tensor  $g$  can be considered as an algebra morphism from the Grassmann algebra of multivector fields to the Grassmann algebra of differential multiforms, and his inverse  $g^{-1}$ , if it exists, as an algebra morphism from the Grassmann algebra of differential forms to multivector fields. Metric tensor must not be interpreted as a distance between events. Distance between points of a manifold is an *integral* over curve.

**5.1 Notation.** We found convenient to use the following notation for the Grassmann algebra map with volume dependent scalar field  $\det S$ . The ‘reciprocity’ symmetry (5.1b) is postulated

$$S \equiv \frac{1}{\det g} g \in \text{alg}((\text{der } \mathcal{F})^\wedge, \mathcal{F}^\wedge), \quad (\det S)(\det g)^3 = 1, \quad \text{with} \quad \det g = -c^2 \quad (5.1a)$$

$$S_{X \wedge Y} = S_X \wedge S_Y, \quad \gamma_{XY} \equiv S_{X \otimes Y} \equiv (\text{ev}_Y)S_X = (\text{ev}_X)S_Y = S_{Y \otimes X} \equiv \gamma_{YX} \quad (5.1b)$$

**5.2 Axiom** (absence of privileged reference material body). The relativity postulate of the absence of a privileged reference body is expressed in the following way. Each time-like material body  $X$  (with a positive mass) as a traveller-process-derivation to his near future, possesses its own simultaneity differential Pfaff form,  $g_X$  or  $S_X$ , that is ambient-dependent, i.e metric-tensor  $g$ -dependent, such that for all *time-like* material bodies  $X$  the following axiom holds (i.e  $\det g$  is the reference-body-free),

$$X \xrightarrow{g} g_X, \quad (g_X)X = \det g, \quad (\text{ev}_X)S_X \equiv S_X X = S_{X \otimes X} \equiv 1 \quad (5.2)$$

## 6. Arbitrary relative split-velocity for the finite light-speed

In this section we drop the Galilean condition of absolute simultaneity and we will consider the case when  $dt' \wedge dt \neq 0$ .

$$X = \left( \frac{\partial}{\partial t} \right)_{x^1, x^2, x^3} \quad \text{and} \quad Y = \left( \frac{\partial}{\partial t'} \right)_{y^1, y^2, y^3} \in \text{der } \mathcal{F} \equiv \text{der}(\mathcal{F}, \mathcal{F}) \quad (6.1)$$

**6.1 Example.** In adopted coordinates in (6.1) we have

$$S_X = dt + f_i dx^i, \quad S_Y = dt' + g_i dy^i \quad (6.2)$$

The adopted dual bases of vector fields associative to simultaneity differential forms are as follows

$$\begin{aligned} X_i &\equiv \frac{\partial}{\partial x^i} - \left( S_X \frac{\partial}{\partial x^i} \right) X, \quad S_X X_i \equiv 0, \quad (dx^i) X_i = \delta_j^i \\ Y_i &\equiv \frac{\partial}{\partial y^i} - \left( S_Y \frac{\partial}{\partial y^i} \right) Y, \quad S_Y Y_i \equiv 0, \quad (dy^i) Y_j = \delta_j^i \end{aligned} \quad (6.3)$$

$$\text{id} = X \otimes S_X + X_i \otimes dx^i = Y \otimes S_Y + Y_i \otimes dy^i \quad (6.4)$$

Each material body as in (6.1) is coordinate-free and basis-free because derivation of an algebra  $\mathcal{F}$  is an example of a coordinate-free tensor field.

One can express the derivation  $X$  in a basis adopted for  $Y$  with  $S_Y$  and vice versa,

$$X = (S_Y X) \{Y + (X y^i) Y_i\}, \quad Y = (S_X Y) \{X + (Y x^i) X_i\}. \quad (6.5)$$

The scalar components,  $X y^i$  (and  $Y x^i$ ), are exactly the Keres expressions for the velocity of  $x$ -space relative to  $y$ -space ([10] p 351 before formula (10))

$$X y^i = \left( \frac{\partial y^i}{\partial t} \right)_{x^1, x^2, x^3} \quad \text{and} \quad Y x^i = \left( \frac{\partial x^i}{\partial t'} \right)_{y^1, y^2, y^3} \quad (6.6)$$

**6.2 Definition** (basis-free split-velocity for finite-light-speed). The above expressions (6.5)–(6.6) lead to the following basis-free definition of the velocity of the  $x$ -space *relative* to the observer  $y$ -space in the case of the finite light-speed. This definition explicitly depends on the simultaneity map  $S$  that can be identified with an environment,

$$\begin{aligned} \text{Velocity of } X \text{ relative to } Y \otimes S_Y \text{ is } v_{YX} &\equiv \frac{1}{\gamma} X - Y, \quad S_Y v_{YX} = 0 \\ \text{Velocity of } Y \text{ relative to } X \otimes S_X \text{ is } v_{XY} &\equiv \frac{1}{\gamma} Y - X, \quad S_X v_{XY} = 0 \end{aligned} \quad (6.7)$$

**6.3 Proposition.** *Speed of relative velocities is the same,  $(v_{YX})^2 = (v_{XY})^2$ .*

*Proof.* We have

$$\begin{aligned} (v_{YX})^2 / \det g &\equiv S_v v = (\gamma^{-1} S_X - S_Y) v = \gamma^{-1} S_X v = \gamma^{-1} S_X (\gamma^{-1} X - Y) \\ &= - (1 - 1/\gamma^2) = (v_{XY})^2 / \det g \end{aligned} \quad (6.8)$$

□

If simultaneity forms are not parallel,  $S_X \wedge S_Y \neq 0$ , then the relative velocities cannot be reciprocal,

$$\gamma = (1 + v^2 / \det g)^{-\frac{1}{2}}, \quad v_{YX} \wedge v_{XY} = - (1 - 1/\gamma^2) X \wedge Y \quad (6.9a)$$

$$(S_X \wedge S_Y)(X \wedge Y) = (S_Y Y)(S_X X) - (S_Y X)(S_X Y) = -(\gamma^2 - 1) \quad (6.9b)$$

## 7. Centre-of-mass

*Relativity* of velocity is not complete without of the concept of centre-of-mass. A material body possesses many different velocities relative to choices of different reference bodies, and this fact is crystal clear in the Keres publications. But many-body interacting system possesses a centre-of-mass and more important are velocities relative to the centre-of-mass. Therefore a genuine special relativity would be complete only with the centre-of-mass concept. We consider that all phenomena considered in physics, the Doppler shift, stellar aberration discovered by Bradley, absorptions, scatterings, etc., should be analyzed in terms of velocities relative to centres-of-mass only. This means that the only important boost should be the boost from the centre-of-mass. In our opinion the concept of the centre-of-mass with its own simultaneity is of vital importance for special relativity theory.

However it is well known, see e.g [22, 13, 14] and any textbook on special or general relativity [Landau and Lifshitz *The Classical Theory of Fields*], that within the Lorentz isometry group approach, the concept of intrinsic centre-of-mass does not exist for finite light-speed, and whenever centre-of-mass is used in practice of cosmology the Newtonian limit is accepted. We are showing in the present section that the group-free concept of relative split-velocity, in the spirit of the Keres approach, extended to finite light-speed allows the perfect concept of centre-of-mass and split-velocities relative to centre-of-mass.

It is stressed in many publications that the real problem facing any attempt to construct a relativistic many-particle theory (theory for the finite light-speed) is the lack of a clear notion of simultaneity. Here, within the group-free approach, split-velocities (7.1) offer perfect solution of this problem: one must forget Lorentz isometry group, in the spirit of the group-free approach by Keres.

For a given a priori two-body system as the Keres  $x$ -space and  $y$ -space, a mass-centre is in another the Keres  $z$ -space with a total mass  $M = \sum m_i$  – binding energy, with nontrivial mass-dependence and variable velocity dependence in case of the finite light-speed. One-body

problem will stay always as one-body, however two-body problem is in fact always three-body, because it must involve a set of three the Keres spaces:  $x$ -,  $y$ - and mass-centre  $z$ -space.

We consider a mass-centre with a total mass  $M$ , as a time-like derivation (a vector field)  $Z$  with its own simultaneity differential one-form  $S_Z$ , for a free two-body system  $\{X, Y\}$  with corresponding masses  $\{m_x, m_y\}$ . A mass-centre  $Z$  is a bound system and in order to be ionized – decoupled into free-system there must be an external massless gravity field  $G$  (or a massless electromagnetic field), in terms of energy-momenta. To write down the mathematical expression, we need the Minkowski scalar factors in terms of simultaneity differential forms. For a two-body time-like system,  $X$  with a positive mass  $m_x$ , and  $Y$  with a positive mass  $m_y$ , following Minkowski [1908] we set for a group-free relative split-velocity  $v_{XY}$ ,

$$\gamma_{XY} \equiv S_X Y = S_Y X = \gamma_{YX} = 1/\sqrt{1 - (v_{XY})^2/c^2} \quad (7.1)$$

The energy of the massless radiation  $G$  (the gravity or electromagnetic radiation with  $S_G G \equiv 0$ ) relative to time-like material body  $X$  with a positive mass,  $S_X X \equiv 1$ , was defined by Schrödinger in 1956 as the following evaluation

$$E_{XG} \equiv S_G X = S_X G \quad (7.2)$$

With these definitions we define mass-centre  $Z$  with its own simultaneity differential one-form  $S_Z$ ,  $S_Z Z \equiv 1$ , by the following energy-momenta conservation law, rotations and spins are disregarded not only for simplicity, but more for pedagogical reasons.

**7.1 Definition** (mass-centre). Many-body interacting material system, as a set of time-like vectors-derivations  $\{X_i\}$ , possesses a time-like mass-centre  $Z \in \text{der } \mathcal{F}$  with a total mass  $M$  according to the following definition (equivalent to the law of conservation of energy-momenta). On the right there is the free decoupled many-body system, whereas on the left,  $Z$  is a mass-centre describing the bound system, if radiation is not absent  $G \neq 0$ ,

$$G + M Z = \sum m_i X_i, \quad S_Z = \frac{1}{M} \left( \sum m_i S_{X_i} - S_G \right) \quad (7.3)$$

The above definition gives immediately two expressions for the total mass for finite light-speed, the first expression in terms of split-velocities and radiation energy relative to the mass-centre, whereas the second expresses the same total mass in terms of split-velocities between constitutive particles of the given many-body system.

We denote by  $\gamma_i \equiv \gamma_{iZ}$  the Minkowski factor relative to mass-centre  $Z$ .

$$\text{Because: } S_{\sum m_i X_i} \sum m_i X_i = \sum m_i m_j \gamma_{ij} \quad \text{and} \quad S_Z \sum m_i X_i = \sum m_i \gamma_i \quad \text{then:} \quad (7.4a)$$

$$M = \sum m_i \gamma_i - E_{ZG}, \quad M^2 = (M S_Z)(M Z) = \sum m_i m_j \gamma_{ij} - 2 \sum m_i E_{iG} \quad (7.4b)$$

**7.2 Lemma** (gravity ionization energy). *The gravity (or alias electromagnetic) ionization energy  $E_{ZG}$  in terms of Minkowski factors is*

$$E_{ZG} = \sqrt{\sum m_i m_j (\gamma_i \gamma_j - \gamma_{ij})} \quad (7.5a)$$

*Proof.* We have

$$M E_{ZG} = S_G \sum m_i X_i = \sum m_i S_i G = \sum m_i S_i (\sum m_j X_j - M Z) \quad (7.6a)$$

$$= \sum m_i m_j \gamma_{ij} - M \sum m_i \gamma_i \quad (7.6b)$$

and

$$\sum m_i m_j \gamma_{ij} = M \left( \sum m_i \gamma_i + E_{ZG} \right) = \left( \sum m_j \gamma_j - E_{ZG} \right) \left( \sum m_i \gamma_i + E_{ZG} \right) \quad \square \quad (7.7)$$

In the Newtonian limit of the infinite light-speed we have

$$\begin{aligned} \sum m_i \gamma_i &\xrightarrow{c \rightarrow \infty} \sum m_i, \quad \sum m_i m_i \gamma_{ij} \xrightarrow{c \rightarrow \infty} \left( \sum m_i \right)^2 \\ E_{ZG} &\xrightarrow{c \rightarrow \infty} 0, \quad M \xrightarrow{c \rightarrow \infty} \sum m_i \end{aligned} \quad (7.8)$$

Lemma 7.2 with expression (7.5a) contains also mutual relative velocities between bodies of the system,  $\gamma_{ij}$ , and it would be better to express it in terms of the velocities relative to mass-centre  $\gamma_i$ . This will not reduce the number of independent variables, because  $\gamma_{ij}$  will be expressed in terms of  $\gamma_i$  and cosines of angles  $v_{Zi} \cdot v_{Zj}$ . The following identity we derived in [15] that holds for non-reciprocal split-velocities and is analogous (but not identical) to the Sommerfeld identity [24] for scalar speed of composed isometric-velocities (reciprocal) within the Lorentz isometry group,

$$\gamma_{ij} = \gamma_i \gamma_j (1 - (v_{Zi} \cdot v_{Zj})/c^2) \implies (E_{ZG})^2 = \sum m_i m_j \sqrt{(\gamma_i^2 - 1)(\gamma_j^2 - 1)} \cos(ij) \quad (7.9)$$

In the Sommerfeld identity in the last factor in (7.9) there is plus sign, because isometric velocity is reciprocal,  $v_{Zi} = -v_{iZ}$ , but this is not possible within the group-free approach, where by definition,  $S_Z v_{ZX} \equiv 0$  and  $S_X v_{XZ} \equiv 0$ , and simultaneity is relative,  $S_Z \wedge S_X \neq 0$  [15, Oziewicz Z arXiv:math-ph/0608062], and [16, 17, 18, 19, 20, 21].

For two-body system  $\{m_x, m_y\}$  with mass-centre  $\{m_x, m_y, M\}$  we have the following system of scalar equations for total mass  $M$ ,

$$G + MZ = m_x X + m_y Y, \quad ME_{ZG} = m_x E_{XG} + m_y E_{YG} \quad (7.10a)$$

$$E_{ZG} + M = m_x \gamma_{ZX} + m_y \gamma_{ZY}, \quad E_{XG} + M \gamma_{ZX} = m_x m + m_y \gamma_{XY} \quad (7.10b)$$

$$E_{YG} + M \gamma_{ZY} = m_x \gamma_{XY} + m_y \quad (7.10c)$$

The total mass of this bound system for finite light-speed is given in terms of three Minkowski factors (three Minkowski factors for two-body system), because we are obliged to consider in fact three-body system  $\{m_x, m_y, M\}$  in space-time.

We will end this section with a comment on maximal binding energy within the above group-free theory. The binding energy is

$$E_b := \text{Binding energy} = \sum m_i - M = E_{ZG} - \sum m_i (\gamma_i - 1) \quad (7.11)$$

Assuming  $\cos(ij) = 1$  in (7.9), we get the the maximal binding energy,

$$E_{ZG}^{\max} = \sum m_i \sqrt{\gamma_i^2 - 1} \implies \text{Maximal } E_b = \sum m_i \left( \sqrt{\gamma_i^2 - 1} - (\gamma_i - 1) \right) < m_i \quad (7.12a)$$

where  $0 \leq \sqrt{\gamma^2 - 1} - (\gamma - 1) < 1$ .

**8. Universe as a bunch of Grassmann factor-algebras of relative spaces-of-locations**  
The Cartan theorem (2.9) implies that the kernel of an algebra derivation,  $\ker(\text{ev}_X)$ , is a Grassmann *factor-algebra*, or quotient-algebra, that we baptize as the Grassmann  $X$ -factor-algebra of relative  $X$ -space, shortly ‘ $X$ -space’ is a  $\mathcal{F}$ -algebra,

$$\alpha \in \ker(\text{ev}_X) \quad \text{and} \quad \beta \in \ker(\text{ev}_X) \implies \alpha \wedge \beta \in \ker(\text{ev}_X) \quad (8.1)$$

An algebra map (morphism from one algebra to another algebra) possesses a kernel, and this kernel is two-sided ideal. Therefore each factor-algebra can be presented as an image of some algebra map (instead of as a kernel of a derivation as we started above), so moreover we have

$$\text{Grassmann } X\text{-factor-algebra-of-locations} = \frac{\text{Grassmann algebra}}{X\text{-simultaneity ideal}} \quad (8.2)$$

For each material body  $X$  of a positive mass, i.e of a positive internal energy,

$$\dim_{\mathcal{F}}(\text{Grassmann algebra}) = 2^4 = 16 \quad \text{and} \quad \dim_{\mathcal{F}}(X\text{-factor-algebra}) = 2^3 = 8 \quad (8.3)$$

**8.1 Example.** One can list some members of the Grassmann  $X$ -factor-algebra of  $X$ -space-of-locations for the Keres congruence in terms of a derivation  $X$  given in (6.1).

$$\begin{aligned} \text{If } dx \text{ is in } \ker(\text{ev}_X), \text{ then } 0 = (\text{ev}_X)dx = \{(\text{ev}_X) \circ d\}x = Xx \\ \text{and } x \text{ is conserved scalar 'integral' of } X. \end{aligned} \quad (8.4)$$

Let  $f \in \mathcal{F}$  denotes any scalar of a zero-grade, then

$$\{\mathcal{F}, f, t, t', x^1, x^2, x^3, y^1, f dx^1, \dots, f dx^1 \wedge dx^2 \wedge dx^3\} \in \ker(\text{ev}_X) \equiv X\text{-factor-algebra} \quad (8.5a)$$

$$X\text{-simultaneity ideal} \ni \{fdt, fdt \wedge dx^1, \dots\} \notin \ker(\text{ev}_X) \quad (8.5b)$$

In particular, the Grassmann  $X$ -factor-algebra contains all  $X$ -electric and  $X$ -magnetic relative fields.

**8.2 Warning.** We must stress very important peculiarities. Each material body  $X$  is owner of the Grassmann  $X$ -factor-algebra of his own  $X$ -factor-space-of-locations. It is good even simply identify each material body with such Grassmann  $X$ -factor-algebra of his space-of-locations. And when we say in what follows,  $X$ -factor-algebra(-of-locations), this means the *Grassmann  $X$ -factor-algebra* of the Grassmann  $\mathcal{F}$ -algebra (of a space-time). Now, for each material body  $X$  ('descendant' of the Keres congruence as in (6.1)) being the Grassmann  $X$ -factor-algebra, the following two warnings hold.

- Every  $X$ -factor-algebra-of-locations is over the same entire algebra  $\mathcal{F}$  of zero-grade scalars, i.e it is a different  $\mathcal{F}$ -algebra for each body  $X$ , but always over the *same* algebra  $\mathcal{F}$  of scalars.
- Not each two-sided simultaneity-ideal  $I_X$  (two-sided ideal in the Grassmann algebra generated by a simultaneity Pfaffian differential form assigned to a material body  $X$ ) possesses a factor-derivation  $d/I_X$  that we will denote by  $d_X$ , and thus a factor-differential  $(d_X)^2 = 0$ . The necessary and sufficient condition for the existence of  $d_X$  is  $dI_X \subset I_X$ , and then an ideal is said to be closed. This fact is important for the concept of an inertial or non-inertial material bodies. If factor-differential does exist, then and only then the Grassmann  $X$ -factor-algebra-of-locations is invariant (stable) with respect to the factor-differential

$$(d_X)\{\text{Grassmann } X\text{-factor-algebra}\} \subset \{\text{Grassmann } X\text{-factor-algebra}\} \quad (8.6)$$

An ideal  $X$ -simultaneity-ideal in the Grassmann  $\mathcal{F}$ -algebra of the differential multiforms is also known as an exterior differential system in the terminology of Élie Cartan.

So how we can see the Universe? The Universe is 'matter in relative motion', and in absolute motion from past to future. We can consider the Universe as a collection of material bodies 'in relative motions' and each material body as a Grassmann factor-algebra of differential forms, always over the same entire algebra  $\mathcal{F}$  of scalar fields. Universe is a bunch of the Grassmann factor-algebras.

### 8.1. Recapitulation

We formulated the Keres concept of an  $x$ -space-of-locations as a congruence of world-lines in a space-time manifold [10] in terms of a basis-free and a manifold-free Grassmann  $X$ -factor-algebra over an algebra  $\mathcal{F}$ . We stress that each the Keres space-of-locations must include entire algebra  $\mathcal{F}$  of zero-grade scalars. In this realm space-of-locations can be considered as the Grassmann factor-algebra of the Grassmann  $\mathcal{F}$ -algebra – no change of algebra of scalars! The Keres  $x$ -space include *all* coordinates, also  $t \in \{x\text{-space}\}$ , however simultaneity one-form,  $S_X = dt$ , is not in the Keres  $x$ -space,

$$(d_X)f = df - (X f)S_X \in \{x\text{-space}\} \equiv \ker(\text{ev}_X) \quad (8.7)$$

Moreover we warn that the existence of a factor-differential  $d_X$  is not granted a priori.

I learn from Professor the Keres publications [8, 9, 10] two important lessons.

- Three-dimensional space-of-locations is not a certain instant of time as majority of present-day textbooks claim. This wrong claim of what it is ‘the physical space’ dominate in all present-day scientific journals and in scientific conferences on gravity and cosmology. Three-dimensional space-of-locations is not a submanifold of a space-time manifold. Keres defined a space-of-locations as a congruence of world-lines. Therefore the Keres space-of-locations is the *quotient* manifold: entire world-line is just only one single *location*. No definite instant of time has a ‘conscience’ of a location, instead instance tells us about relative simultaneity. An event, as an element of a cross-section of a space-time for a given instant of time, can belong to infinite many different congruences of world-lines, and, as Professor Keres was clearly aware, there is no way to assign to such single event some location in a ‘real physical space-of-locations’. I explained on several illustrative examples in another publication [19], that the space-of-locations must be defined as a **quotient**-manifold and never as a sub-manifold, and of course Professor Keres was clearly aware of this fact,

$$\text{Space-of-locations is a Grassmann factor-algebra} \equiv \frac{\text{Grassmann } \mathcal{F}\text{-algebra}}{\text{Simultaneity ideal}} \quad (8.8)$$

- Keres is explaining that there does *not* exist just one ‘physical space-of-locations’ (as a factor-manifold; in the present paper re-interpreted as a Grassmann factor-algebra). His original enunciation is in terms of different congruences. the Keres clear idea of *many* distinct ‘physical’ spaces-of-locations can be traced and attributed also to observations by Galileo Galilei and by Minkowski, but this idea of relativity of ‘simul-localidad’ was never elucidated so explicitly as in the Keres publications.

## 9. Relativity vs gravity

It is good, at least in the beginning for clear pedagogical reasons, distinguish conceptually the theory of gravitational interaction from the theory of relativity. This is because relativity theory (relativity of locations, relativity of velocities, etc.) is fundamental for *all* kinds of interactions including electromagnetic interactions, weak and nuclear interactions. For historical reasons, relativity theory is divided into two parts, relativity of the concept of location and of velocity, and the second part deals with relativity of acceleration and with relativity of rotation.

To be relative means to be dependent on the irrelevant choice of the reference system. There are still different non-equivalent understandings of the mathematical description of the physical reference system, when the reference system is defined as a material body with a positive mass. Thus within relativity theory, the velocity, acceleration, and rotation of one material body depend on the free choice of another reference body. Often ‘general relativity’ is understood as the theory of gravity only (excluding electromagnetic field), however I think it is more adequate not to mix ‘relativity’ with ‘gravity’, and interpret ‘general relativity’ rather as

relativity of *all* motions, i.e. relativity of location, relativity of velocity, relativity of acceleration and relativity of rotations, without selecting any specific kind of interaction. Thus consideration of the gravitational interaction would be under the name ‘gravitation’, avoiding the double terminology in some journals ‘general relativity and gravitation’.

## 10. Reference system as a monad within manifold-free and group-free approach

The idea to define a material body as a time-like vector field is due to Hermann Minkowski in 1908. Einstein in 1905 adopted alternative and not equivalent definition of a material body as a coordinate system alias a coordinate basis, and this is not used in the present paper; we follow an approach by Minkowski, and the Keres ideas of congruence. There are two advantages to define a material body, and in this way every reference system, as a time-like vector field or as the Keres congruence of world-lines, instead of Einstein’s coordinate system. Theory of reference systems in terms of algebra derivations is known as the *monad* theory.

- (i) Each derivation in (6.1) is an example of a tensor field, thus it is coordinate-free and basis-free, and in this way we can forget irrelevant coordinates and bases.
- (ii) The second advantage to define a material body as a derivation is that this description is manifold-free in the spirit of the non-commutative point-free geometry. The commutative geometry deals with the commutative *algebra* of scalar fields and for historical reasons the name ‘geometry’ dominate over more adequate ‘algebra’. The recent birth (at the end of XX century) of non-commutative ‘geometry’ opens the way to manifold-free algebraic approach. The term ‘geometry’ we are able to exchange for ‘algebra’, and for this ideological exchange non-commutativity or commutativity of the primary algebra of ‘scalars’ is not so important: one can develop manifold-free approach also for commutative differential algebra as it was developed in the recent decades in the realm of manifold-free non-commutative ‘geometry’.

What advantage we see in abandoning, or never using, the space-time *manifold* of events, the world of events, so popular in the present-day textbooks on relativity, on gravity, and on cosmology?

- (i) Each manifold of higher than one dimension must possesses by definition one-dimensional sub-manifolds, and in particular must possess closed curves. Closed curve is natural in the three-dimensional space-of-locations, however it is hard to imagine a real closed curve in a manifold of events.
- (ii) In the present paper a ‘space-time algebra’ is the Grassmann algebra of differential multiforms and the Grassmann algebra of ‘multiderivations’ (alias multivector fields). Such ‘space-time’ *algebra* with the Grassmann functor,  $\mathcal{F} \mapsto \mathcal{F}^\wedge$ , allows us to avoid to think about universe as a four-dimensional manifold of events that has existed since eternity, exists now in this very moment, and implies that the future exists now. In the manifold-free approach universe is a collection of material bodies (and massless radiation) à la Kant and each material body is a process-derivation of a commutative algebra as in (6.1) – reinstating the Keres idea of congruences in terms of algebra derivations.

The manifold-free algebraic approach assumes the commutative algebra  $\mathcal{F}$  as the primary concept, and the ‘coordinates’ in (6.1) are elements of a commutative algebra without of necessity to use the concept of an event, i.e. this is event-free and point-free approach to relativity theory. We avoid the name ‘vector field’ and replace it by adequate ‘derivation of algebra of scalars’ that we use as a synonym of a material body, as it is clear in expressions (6.1),

$$\text{der } \mathcal{F} \quad \ni X \quad \longrightarrow \quad \text{ev}_X \quad \in \text{der}(\mathcal{F}^\wedge) \quad (10.1)$$

Within the Riemannian geometry, with non-singular or with singular metric tensor  $g$ , it is important to consider also the Clifford algebra of differential forms and tensor algebra together

with the Grassmann algebra, however for simplicity we restricted ourselves in the present paper to consider the Grassmann algebra alone.

### 11. Singular metric

During 1957-1964 Professor Harald Keres published four papers in the Transactions of the Institute of Physics and Astronomy of the Estonian Academy of Sciences on the concept of inertial reference system within the Einsteinian gravity. In 1965 he demonstrated that the infinite light-speed limit,  $c \rightarrow \infty$ , of the Einstein gravity contains rotational non-Newtonian fields.

In 1976 Keres reformulated the Newtonian gravity with rotation within singular metric [8, 10]. The way in which Keres introduced singular metric,  $\det g = 0$ , deserves attention. We ask: what vector-derivation should be chosen to be in the kernel  $v \in (\ker g)$ ? Still there is not crystal clear, at least for me, what in good mathematics means the Newtonian limit.

The Galilean-Newtonian limit is considered to be the infinite light-speed limit,  $c \rightarrow \infty$ , however how exactly? Maybe mathematically more precise must be the limit of a concept of simultaneity in terms of the Minkowski factor,  $\gamma \rightarrow 1$ ? But the limit  $\gamma \rightarrow 1$  could have also another interpretation as a limit from bound to a free-system, i.e disconnection of an interaction. Maybe it would be convenient to consider  $\det g$  as a free variable parameter with a limit  $\det g \rightarrow 0$ , but what kernel of a metric tensor,  $g \in \text{alg}((\text{der } \mathcal{F})^\wedge, \mathcal{F}^\wedge)$  defined to be an algebra map, would be the best choice for such Galilean limit?

If a simultaneity algebra map  $S$  is an algebra isomorphism then

$$X \wedge Y \neq 0 \iff S_{X \wedge Y} = S_X \wedge S_Y \neq 0 \quad (11.1)$$

One can expect that the Galilean-Newtonian limit would mean that for a two-body system as in (6.1), simultaneities of these bodies coincide in this limit with simultaneity of mass-centre

$$S_X, S_Y \xrightarrow{\text{Newtonian limit}} S_Z \quad (11.2)$$

The expected kernel of a singular metric could be the Keres relative velocity,  $Y - X \in \ker g$ . An algebra map to be a singular algebra map do not mean to be algebra isomorphism possessing nontrivial kernel as two-sided ideal in the Grassmann algebra. Kernel of an algebra map,  $g \in \text{alg}((\text{der } \mathcal{F})^\wedge, \mathcal{F}^\wedge)$ , is an ideal, and is a factor-space = algebra/sub-algebra.

It is not complicated to define such singular metric, and develop, following the Keres idea, an analogy of Einstein's gravity theory for the singular metric, calculate the Christoffel derivative for the singular metric,  $\nabla g = 0$ . However the choice of the singular metric tensor made by professor Keres for two-body system (6.1) was different from our proposal, the Keres choice was,  $X = Y + v_{XY} \in \ker g$ , [10, §3 p 351 formula (10)].

With the singular metric tensor Keres was able to develop the complete analogy of the Einstein gravity theory for the Newtonian limit in terms of the Christoffel derivative [10].

### 12. The Keres publications are rich in many more fundamental ideas

In the present paper we reminded only the concept of relativity of spaces of locations, the main introductory but of fundamental importance the Keres idea of  $x$ -space as different from  $y$ -space. This allows Professor Keres to define a general (not constant) relative velocity in a group-free way. We pointed that this group-free way can be extended to the case of finite light-speed that is the source of relativity of simultaneities. We also left for another publication the problem of the reduced mass for finite light-speed.

Keres developed in full clarity the concepts of relative acceleration, and relative rotation (of  $x$ -space relative to  $y$ -space) [8, 10], and very deep insight into gravity theory that is outside of the scope of the present paper. I am sure many other fundamental concepts developed by Professor Keres deserve separate papers.

I am convinced that it would be most desirable to edit the (selected) Collected Works of Professor Harald Keres, with translating them from Russian into English to allow these important publications be available to wider number of students and researches. All known to me publications of Professor Keres are in Russian, they were published during the Estonia Soviet period after 1945.

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