

Une approche probabiliste des théories conformes des champs de Toda

A probabilistic approach to Toda conformal field theories

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Titre : Une approche probabiliste des théories conformes des champs de Toda

Mots clés : Théorie conforme des champs, Théories de Toda, Champ libre gaussien, Chaos multiplicatif gaussien, W-algèbres.

Résumé : Les théories de Toda forment une famille de théories quantiques des champs en dimension deux introduites dans la littérature physique en tant que modèles possédant, au-delà de la symétrie conforme, une symétrie étendue qui est codée par des W-algèbres.

L'objet de cette thèse est de proposer un cadre mathématique dédié à l'étude de ces théories. En nous appuyant sur des objets probabilistes fondamentaux – le champ libre gaussien et le chaos multiplicatif gaussien – nous pourrions ainsi implémenter de manière rigoureuse des méthodes issues de la physique et par là même calculer certaines quantités cruciales dans la compréhension de ces théories : les fonctions de corrélation.

Pour ce faire nous exploiterons dans un premier temps les symétries de ces modèles

en montrant que les fonctions de corrélation définies de manière probabiliste sont soumises à des contraintes appelées identités de Ward. L'existence de telles contraintes nous permettra de donner une expression explicite pour une famille de fonctions de corrélation fondamentales – les constantes de structure – et de décrire une procédure récursive – le bootstrap conforme – permettant de calculer d'autres fonctions de corrélation à partir de celles-ci.

Au cours de cette démarche nous mettrons en évidence certains liens inattendus entre des concepts issus de la théorie des probabilités et d'autres provenant de la théorie conforme des champs, notamment en décrivant une décomposition de chemin brownien qui nous permettra le calcul de coefficients de réflexion des théories de Toda en termes probabilistes.

Title: A probabilistic approach to Toda conformal field theories

Keywords: Conformal field theory, Toda theories, Gaussian free field, Gaussian multiplicative chaos, W-algebras.

Abstract: This thesis is dedicated to the study of a family of two-dimensional quantum field theories called Toda conformal field theories. Initially introduced in the physics literature, we propose here a mathematical definition of such models based on two key probabilistic objects: Gaussian free fields and Gaussian multiplicative chaos.

Within this probabilistic framework we recover properties expected in the physics and related to the enhanced symmetries enjoyed by Toda conformal field theories, which in turn leads to the computation of a family of correlation functions —one of the main quantities of interest.

The computation of these correlation functions is based on the exploitation of the sym-

metries —encoded by W-algebras— of these models in that they provide constraints on the correlation functions. These constraints are the Ward identities, whose existence will be the starting point to the derivation of a family of structure constants, that represent fundamental correlation functions. A recursive procedure known as conformal bootstrap then allows to compute other correlation functions based on the expression of these structure constants.

Along the implementation of this program we will shed light on some unexpected interplays between probability theory and conformal field theory. One instance of such connections is the probabilistic description of Toda reflection coefficients, based on a generalized Brownian path decomposition.

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Introduction

1 - Introduction - Français

Les théories conformes des champs de Toda forment une famille de théories quantiques des champs en dimension deux, famille indexée par les algèbres de Lie semi-simples et complexes. La théorie de Liouville, dont l'étude a contribué à des progrès majeurs dans la compréhension de la géométrie aléatoire en dimension deux – et ce aussi bien dans le cadre de la physique que dans une perspective mathématique –, en est un des exemples fondamentaux. Toutefois au-delà de la symétrie conforme portée par la théorie de Liouville, les théories de Toda possèdent un niveau additionnel de symétrie : la W -symétrie ou symétrie de spin supérieur.

Ce degré de symétrie supplémentaire rend l'étude des théories de Toda plus complexe que celle de la théorie de Liouville, et malgré de nombreux progrès effectués dans cette direction en physique, le calcul de certains objets fondamentaux de la théorie – les fonctions de corrélation d'opérateurs vertex – reste une question ouverte.

Cette thèse est dédiée à l'étude mathématique des théories conformes des champs de Toda. Pour ce faire nous décrivons un cadre probabiliste au sein duquel leurs fonctions de corrélation sont définies en tant qu'objets mathématiques rigoureux. Cet environnement probabiliste s'appuie notamment sur deux objets fondamentaux : les champs libres gaussiens et le chaos multiplicatif gaussien.

Au cours de ce chapitre introductif nous nous attachons à expliquer de manière plus approfondie quelles sont ces théories de Toda et de quelle manière il est possible de les définir de manière rigoureuse. Ce faisant, nous décrivons comment le cadre probabiliste proposé permet de donner un sens mathématique à certaines considérations issues de la physique. Une attention particulière sera accordée au calcul d'une famille de fonctions de corrélation pour la théorie de Toda associée à l'algèbre de Lie $\mathfrak{g} = \mathfrak{sl}_3$.

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1.1 . Les théories conformes de champs de Toda

Que sont les théories conformes des champs de Toda ? Avant de nous plonger dans l'étude de cette famille de modèles il convient de répondre à cette question élémentaire mais non moins fondamentale. C'est l'objet de cette section introductive au cours de laquelle nous présenterons ces théories et les enjeux qui leur sont attachés. Dans cette perspective nous nous appuierons en premier lieu sur l'étude de la théorie de Liouville, qui est l'exemple de théorie de Toda le mieux compris à l'heure actuelle. Dans un second temps nous décrirons le cadre plus général de la théorie conforme des champs en dimension deux et expliquerons comment les théories de Toda s'inscrivent dans cet environnement.

1.1.1 . La théorie conforme des champs de Liouville

Dans un article fondateur [111], le physicien russe Alexander Polyakov pose en 1981 les fondements d'un champ de recherche qui va dès lors devenir particulièrement fertile, l'étude de la gravité quantique en dimension deux, en décrivant une manière naturelle de définir une notion de surface aléatoire. Dans ce travail majeur est introduite la *théorie conforme des champs de Liouville*, qui est alors envisagée comme un modèle proposant une manière canonique de tirer au hasard une géométrie sur une surface de Riemann dont la topologie serait fixée [123]. Au-delà d'être un objet d'études fondamental en soi, la théorie de Liouville est désormais considérée comme étant une composante essentielle dans la compréhension de certaines théories des cordes mais également de la géométrie aléatoire en dimension deux, et en partie pour cette raison a été étudiée très minutieusement que ce soit au sein de la communauté physique ou mathématique. Ces liens sont par exemple mis en avant dans [79] où les relations entre la théorie de Liouville et la théorie des cordes sont décrites plus en détail, tandis que des détails supplémentaires sur la notion de gravité quantique en dimension deux peuvent être trouvés dans [44].

En outre la théorie de Liouville est intrinséquement reliée à une famille de modèles possédant le même niveau de symétrie : les *théories conformes des champs*. Un exemple de tels liens est la relation Knizhnik-Polyakov-Zamolodchikov [80], fondamentale dans l'étude de modèles de physique statistique à criticalité dans un environnement aléatoire. Mais au-delà de ses liens avec cette famille de modèles, la théorie de Liouville trouve également de nombreuses applications en dehors de ce cadre, que ce soit dans l'étude de modèles de matrices aléatoires [142, 137, 24] ou dans celle de fractales aléatoires [80, 50, 117]. Notre propos ici n'est pas de passer en revue toutes les applications possibles et envisageables de la théorie de Liouville ; le lecteur curieux pourra en avoir un aperçu en consultant par exemple [103] où sont également discutées certaines extensions de la théorie de Liouville. Nous redirigeons également vers l'article [131] où les différentes perspectives communément adoptées en physique pour traiter de la théorie de Liouville sont mises en évidence.

1.1.1.1 La théorie de Liouville par l'intégrale de chemin

Il existe plusieurs approches différentes menant à la définition de la théorie de Liouville, dont un postulat en physique stipule qu'elles sont équivalentes. Dans ce manuscrit nous utiliserons la définition de la théorie de Liouville par *intégrale de chemin*, une méthode

communément employée en physique pour définir une théorie quantique des champs en partant d'une théorie déterministe. Elle peut être pensée comme une manière de quantifier le principe de moindre action.

Illustrons cette méthode dans le cas de la théorie de Liouville sur la sphère de dimension deux $\mathbb{S}^2$. La théorie classique considère des solutions du problème d'uniformisation, c'est-à-dire la question de trouver une métrique riemannienne sur la sphère qui lui donnerait une courbure constante. Ce problème peut se formuler de manière quantitative en utilisant les règles de transformation sous changement conforme de la métrique de l'opérateur de Laplace-Beltrami Δ_g et de la courbure scalaire de Ricci R_g : si g est une métrique riemannienne sur $\mathbb{S}^2$ et que $g' = e^{2\varphi}g$ est une autre métrique conformément équivalente à g , alors les quantités Δ_g, R_g et $\Delta_{g'}, R_{g'}$ sont reliées par

$$\Delta_{g'} = e^{-2\varphi} \Delta_g \quad \text{et} \quad R_{g'} = e^{-2\varphi} (-2\Delta_g \varphi + R_g). \quad (1.1.1)$$

Précisons que ces relations ne sont valables que dans le cadre de la dimension deux. Une conséquence de l'équation (1.1.1) est la propriété que pour la métrique conforme $g' = e^{2\Phi}g$ avoir courbure scalaire constante égale à -2Λ revient à satisfaire la relation

$$-2\Delta_g \Phi + R_g + 2\Lambda e^{2\Phi} = 0. \quad (1.1.2)$$

Ce qui n'est rien d'autre que l'équation de Liouville.

Le problème d'uniformisation admet une reformulation faible comme suit : si Φ est une solution de l'équation de Liouville il s'agit donc d'un point critique de l'action de Liouville

$$S_L^0(\phi, g) := \frac{1}{4\pi} \int_{\mathbb{S}^2} \left(\langle \partial_g \phi(x), \partial_g \phi(x) \rangle_g + R_g \phi(x) + \Lambda e^{2\phi(x)} \right) v_g(dx). \quad (1.1.3)$$

Remarquons que dans le cas de la sphère, si nous supposons que Λ est positif (autrement dit que la courbure est négative) le théorème de Gauss-Bonnet fournit une obstruction topologique à l'existence d'une telle solution. Afin de pallier cette limitation il est nécessaire de supposer le champ irrégulier, et une manière standard de le faire est de lui imposer des singularités coniques, c'est-à-dire de demander que le champ se comporte comme $\Phi(z) \sim -\alpha_k \ln |z - z_k|$ avec $\alpha_k < 2$ autour de certains points $(z_k)_{1 \leq k \leq N}$ de $\mathbb{S}^2$. Sous l'hypothèse que $\sum_{k=1}^N \alpha_k > 4$ (ce qui demande en particulier d'avoir au moins $N = 3$ singularités) le problème d'uniformisation admet une solution à courbure constante négative que nous notons Φ^* comme il est montré par exemple dans [134].

En suivant cette approche le champ classique est purement déterministe. Ce n'est plus le cas dans la théorie quantique des champs qui lui est associée, dans laquelle cette fonction sur la sphère est désormais aléatoire et fluctue autour du champ classique. Ce degré d'aléa est mesuré par l'intermédiaire de la *constante de couplage* $\gamma > 0$ (qui joue en quelque sorte le rôle de la constante de Planck $\hbar$) : dans la limite semi-classique où γ tend vers 0 ce champ aléatoire devrait redevenir déterministe et égal au champ classique Φ^* – ce qui a été rigoureusement prouvé dans [86]. En utilisant cette constante de couplage nous pouvons introduire une version quantifiée de l'action de Liouville (1.1.3) :

$$S_L(\phi, g) := \frac{1}{4\pi} \int_{\mathbb{S}^2} \left(\langle \partial_g \phi(x), \partial_g \phi(x) \rangle_g + Q R_g \phi(x) + \Lambda e^{\gamma \phi(x)} \right) v_g(dx) \quad (1.1.4)$$

où $Q = \frac{\gamma}{2} + \frac{2}{\gamma}$ est appelée la *charge de fond*. Le champ aléatoire de la théorie de Liouville est alors tiré selon une mesure sur un espace fonctionnel $\mathcal{F} \subset L^2(\mathbb{S}^2 \rightarrow \mathbb{R})$ dont la définition prend formellement la forme

$$\langle F(\Phi) \rangle_{\gamma, \mu} := \frac{1}{\mathcal{Z}} \int_{\mathcal{F}} F(\phi) e^{-S_L(\phi, g)} D\phi \quad (1.1.5)$$

pour toute *observable* du champ F , c'est-à-dire toute fonctionnelle $F : \mathcal{F} \rightarrow \mathbb{R}$. Dans cette expression la mesure apparaissant à droite correspondrait à la "mesure uniforme" sur $\mathcal{F}$, qui ne fait pas vraiment sens d'un point de vue mathématique. Le fonction de partition $\mathcal{Z}$ correspond à la masse totale $\mathcal{Z} = \int_{\mathcal{F}} e^{-S_L(\phi, g)} D\phi$, rendant ainsi la mesure définie par l'intermédiaire de l'équation (1.1.5) une mesure de probabilité. Ceci correspond à la définition de la théorie de Liouville par l'intégrale de chemin. Cette approche est évidemment purement heuristique et n'est pas vraiment rigoureuse mathématiquement parlant. Toutefois il est possible de donner un sens à cette intégrale de chemin en interprétant celle-ci dans un cadre probabiliste comme expliqué dans [40].

Un exemple classique d'une telle traduction d'intégrale de chemin dans un langage probabiliste est donné par la mesure de Wiener $W(dw)$, vue ici comme une mesure de probabilité sur l'espace $C_0^0(\mathbb{R}^+, \mathbb{R})$ des fonctions w continues de $\mathbb{R}^+$ dans $\mathbb{R}$ et vérifiant $w(0) = 0$. Cette mesure peut formellement être définie par

$$W(dw) = \frac{1}{\mathcal{Z}} e^{-S(w)} dw, \quad \text{où} \quad S(w) := \frac{1}{2} \int_{\mathbb{R}^+} w'(t)^2 dt.$$

En écrivant $S(w) = \frac{1}{2} \int_{\mathbb{R}^+} w''(t)w(t)dt$ on a donc par analogie avec une mesure gaussienne, pour f, g convenables

$$\int_{C_0^0(\mathbb{R}^+, \mathbb{R})} \left(\int_{\mathbb{R}^+} f''(t)w(t)dt \right) \left(\int_{\mathbb{R}^+} g''(t)w(t)dt \right) W(dw) = \left(\int_{\mathbb{R}^+} f''(t)g(t)dt \right).$$

En prenant $f(u) = u \wedge s$ et $g(u) = u \wedge t$ on retrouve bien la propriété usuelle du mouvement brownien :

$$\int_{C_0^0(\mathbb{R}^+, \mathbb{R})} w(s)w(t)dt W(dw) = s \wedge t.$$

Le formalisme de l'intégrale de chemin est également similaire à celui des mesures de Gibbs utilisé notamment en physique statistique, dans lequel l'action jouerait le rôle de l'Hamiltonien, et où l'intégrale portant sur un espace fonctionnel serait remplacée par une somme sur l'ensemble des configurations possibles.

Cette approche est naturelle à bien des égards ; notons simplement qu'une de ses propriétés est que la mesure ainsi définie tend à favoriser des champs ϕ pour lesquels l'action est petite, et cela est d'autant plus vrai que la constante de couplage γ est petite. La limite semi-classique $\gamma \rightarrow 0$ correspond alors à l'analogue infini-dimensionnel de la méthode de Laplace pour une fonction d'une variable réelle. Sur l'exemple du mouvement brownien cette limite semi-classique consiste à considérer la limite $\gamma \rightarrow 0$ d'expressions de la forme

$$\frac{1}{\mathcal{Z}_\gamma} \int_{C_0^0(\mathbb{R}^+, \mathbb{R})} F(w) e^{-\frac{1}{\gamma^2} S(w)} dw,$$

ces dernières définissant la loi d'un mouvement brownien de variance γ . Le champ classique est donc la fonction constante $w = 0$, qui est bien le minimum de l'action S .

1.1.1.2 Approches mathématiques de la théorie de Liouville

Tout d'abord étudiée dans la littérature physique, la communauté mathématique s'est par la suite emparée de la théorie de Liouville et différentes approches ont été proposées pour en décrire un cadre mathématique, en témoigne la diversité des méthodes employées pour définir une notion de "surface aléatoire canonique", amenant ainsi à de nombreux développements dans la compréhension de la géométrie aléatoire en dimension deux.

De la même manière que le mouvement brownien peut se construire en tant que limite d'échelle de marches aléatoires, une surface aléatoire peut en analogie être construite en considérant une limite d'échelle de surfaces discrètes "naturelles". La sphère brownienne introduite dans [93] est définie dans cette perspective, correspondant à une limite d'échelle de modèles de cartes planaires aléatoires [88, 96]. Des travaux ultérieurs [18, 19] ont permis d'étendre cette construction à d'autres surfaces que la sphère.

Il est possible également de définir une surface aléatoire directement dans le continu en s'appuyant sur les propriétés qu'un tel objet se doit de satisfaire. Une notion de "surface quantique" fut ainsi introduite dans [49], intrinsèquement reliée à d'autres notions fondamentales liées à la géométrie aléatoire en dimension deux : les *champs libres gaussiens*, les *évolutions de Schramm-Loewner* [121] et les *arbres aléatoires continus*. Ces surfaces quantiques, définies initialement en tant que surfaces de Riemann équipées d'une forme volume, furent ultérieurement dotées d'une structure d'espace métrique grâce à la construction de la *métrique de la gravité quantique de Liouville* [42, 47, 70]. Plus de détails sur cette approche de la géométrie aléatoire peuvent être trouvées dans [69] ou [43], ainsi que dans les références s'y trouvant.

Parallèlement à ces développements David-Guillarmou-Kupiainen-Rhodes-Vargas initièrent dans une série d'articles [40, 68] une étude de la théorie de Liouville plus proche de la définition originelle de Polyakov, et qui est l'approche se rapportant le plus au cadre que nous développerons dans cette thèse. En construisant les fonctions de corrélation de la théorie de Liouville de manière probabiliste, bon nombre de prédictions faites dans la littérature physiques furent rigoureusement établies, établissant ainsi une connection entre les communautés physiques et mathématiques. Entre autres accomplissements se trouve une preuve rigoureuse de la célèbre formule DOZZ [84], prédite en physique [45, 141, 131] il y a près de trente ans, mais également une implémentation rigoureuse de la procédure du *bootstrap conforme* [66], une des composantes les plus importantes dans l'étude de la théorie conforme des champs en dimensions deux du point de vue de la littérature physique.

Ces différentes approches ont conduit à établir diverses notions de "surface aléatoire canonique". Et comme bien souvent en mathématiques, ces définitions *a priori* distinctes correspondent en fait à des points de vue différents sur un seul et même objet. Les liens entre cartes planaires aléatoires et théorie de Liouville ne sont plus à prouver, en témoigne le fait que la sphère brownienne ne soit en fait qu'un cas particulier de surface quantique. Ces connections sont explicitées par exemple dans [97, 98, 99, 71]. Ces surfaces quantiques sont également étroitement liées aux objets considérés dans [40] puisqu'elles coïncident en effet dans certains cas : les articles [7, 28] montrent que ces notions alternatives de surface aléatoire sont en fait équivalentes sur deux exemples de telles surfaces. Ces liens ont permis d'effectuer des progrès fondamentaux dans la compréhension de la géométrie aléatoire en dimension deux, voir e.g. [126, 128, 100, 5]. Ces différentes notions de surface aléatoire sont passées en revue dans [127].

1.1.2 . La théorie conforme des champs en dimension deux

L'étude de la théorie de Liouville comme proposée par David-Guillarmou-Kupiainen-Rhodes-Vargas s'appuie sur le fait que le modèle ainsi défini entre dans le cadre de la théorie conforme des champs en dimension deux. Nous proposons dans un premier temps de décrire dans les grandes lignes cette notion.

1.1.2.1 Sur la théorie conforme des champs

Suite à l'article fondateur de Polyakov [111], Belavin, Polyakov et Zamolodchikov (BPZ) présentèrent dans un travail pionnier en 1984 [12] une procédure systématique pour calculer les fonctions de corrélation de modèles qui comme la théorie de Liouville possèdent certaines symétries conformes. Ces modèles sont appelés des *théories conformes des champs en dimension deux*. La méthode employée par BPZ pour résoudre ces modèles s'appuie sur l'exploitation des contraintes posées par la symétrie conforme par l'intermédiaire de l'étude de l'algèbre de ses générateurs : l'*algèbre de Virasoro*, un postulat stipulant que la théorie des représentations de l'algèbre de Virasoro peut se traduire en contraintes concrètes sur le modèle. Grâce à cette approche les fonctions de corrélation de la théorie peuvent être déterminées en utilisant une procédure récursive universelle en ce qu'elle ne dépend de la théorie qu'à partir de certaines données – sa charge centrale, ses *constantes de structure* et son *spectre*. Cette méthode est le *bootstrap conforme*.

Ces postulats fournissent une méthode de résolution des théories conformes des champs en dimension deux. Toutefois définir cette notion en elle-même n'est pas forcément chose aisée, et différentes perspectives ont été développées à cet effet selon le point de vue que l'on cherche à adopter. Ainsi la théorie conforme des champs peut être considérée de manière purement algébrique par la notion d'algèbres vertex [22, 59] ou dans une perspective plus géométrique au travers de l'étude de fibrés en ligne sur des espaces de modules de courbes complexes [61]. Dans cette même perspective mentionnons que le programme de Langlands géométrique possède également des liens avec la théorie conforme des champs [58]. Une approche axiomatique de cette notion est cependant privilégiée en physique. Citons par exemple l'axiomatique de Segal [122] liée au recollement de surfaces, ou les axiomes de Gawedski [62] selon lesquels une théorie conforme correspond à un modèle covariant sous l'action de transformations conformes. L'article [132] propose une approche concise mais néanmoins détaillée de ces différentes notions de théorie conforme des champs, tandis le lecteur qui s'en ressent le courage pourra consulter le livre de référence [56] pour sa part fort peu concis.

1.1.2.2 Symétrie conforme et théorie de Liouville

La théorie de Liouville est supposée dans la littérature physique être une théorie conforme des champs : la méthode du bootstrap conforme permet donc en théorie de calculer les fonctions de corrélation qui lui sont associées. Nous expliquons ici quelle forme cette procédure prend dans le cas de la théorie de Liouville sur la sphère de Riemann $\mathbb{C} \cup \{\infty\}$.

Il est communément admis en physique que comprendre la théorie de Liouville revient à calculer toutes ses fonctions de corrélation d'opérateurs vertex. Ces derniers sont des fonctionnelles du champ de Liouville Φ , dépendant d'un point $z \in \mathbb{C}$ ainsi que d'un

poids $\alpha \in \mathbb{C}$ et prenant formellement la forme de $V_\alpha(z)[\Phi] = e^{\alpha\Phi(z)}$. Les fonctions de corrélation d'opérateurs vertex sont de cette manière définies en utilisant l'intégrale de chemin via

$$\langle \prod_{k=1}^N V_{\alpha_k}(z_k) \rangle_{\gamma, \mu} := \frac{1}{\mathcal{Z}} \int \prod_{k=1}^N e^{\alpha_k \phi(z_k)} e^{-S_L(\phi)} D\phi$$

où N est un entier supérieur ou égal à 3. La connaissance des fonctions de corrélation permet de déterminer la loi du champ de Liouville Φ en cela qu'elle permet le calcul de toutes les quantités

$$\langle e^{-\int_{\mathbb{S}^2} f(x) \Phi(x) v_g(dx)} \rangle_{\gamma, \mu}, \quad f \in L^2(\mathbb{S}^2),$$

celles-ci formant, d'un point de vue probabiliste, l'analogue de la transformée de Laplace d'une variable aléatoire réelle X , où le paramètre λ dans $\mathbb{E}[e^{-\lambda X}]$ serait remplacé par une fonction $f : \mathbb{S}^2 \rightarrow \mathbb{R}$.

Afin de calculer ces fonctions de corrélation, la première étape dans la procédure du bootstrap conforme est de déterminer les constantes de structure de la théorie, qui sont ici les fonctions de corrélation avec $N = 3$ opérateurs vertex. Leur calcul est rendu possible par l'existence de contraintes liées à la symétrie conforme. Celles-ci peuvent se manifester par certaines identités – les identités de Ward – exprimant la covariance conforme des fonctions de corrélation, mais également de manière plus élaborée en étudiant la théorie des représentations de l'algèbre de Virasoro, donnant ainsi lieu à des identités supplémentaires. Cette approche est au cœur de la méthode de preuve proposée par Teschner du calcul des structures de constantes de Liouville [131], implémentée dans un cadre mathématique rigoureux dans [84], retrouvant ainsi la formule DOZZ [45, 141] prédite en physique.

L'étape suivante dans le calcul des fonctions de corrélation est désormais de comprendre comment mettre en place une procédure récursive pour les déterminer à partir des constantes de structure. La méthode du bootstrap conforme s'appuie sur la connaissance de certains objets appelés *blocs conformes* $\mathcal{F}_P$ – qui sont universels dans le sens où ils dépendent uniquement de la théorie par sa *charge centrale* et *poids conformes* Δ_α – et la donnée du spectre de la théorie de Liouville, la demi-droite $Q + i\mathbb{R}^+$. La formule du bootstrap conforme stipule alors que le calcul de fonctions de corrélation à $N = 4$ points se ramène à celui pour $N = 3$ via

$$\begin{aligned} & \langle V_{\alpha_1}(0) V_{\alpha_2}(z) V_{\alpha_3}(1) V_{\alpha_4}(\infty) \rangle_{\gamma, \mu} = \\ & \frac{1}{8\pi} \int_{\mathbb{R}^+} C_{\gamma, \mu}^{\text{DOZZ}}(\alpha_1, \alpha_2, Q - iP) C_{\gamma, \mu}^{\text{DOZZ}}(Q + iP, \alpha_3, \alpha_4) |z|^{2(\Delta_{Q+iP} - \Delta_{\alpha_1} - \Delta_{\alpha_2})} |\mathcal{F}_P(z)|^2 dP \end{aligned} \quad (1.1.6)$$

où $C_{\gamma, \mu}^{\text{DOZZ}}(\alpha_1, \alpha_2, \alpha_3)$ fait référence à la formule DOZZ décrivant les constantes de structure de la théorie de Liouville. Prouver rigoureusement un tel résultat est assez exigeant mais cela a néanmoins été effectué dans l'article [66]. Le calcul de fonctions de corrélation avec $N \geq 5$ s'effectue de la même manière en ramenant le calcul à celui de fonctions de corrélation avec un nombre inférieur d'insertions, ce qui permet d'exprimer une fonction de corrélation générale en terme de la formule DOZZ et blocs conformes comme démontré dans l'article [67].

1.1.3 . De la théorie de Liouville vers les théories de Toda

Suite à l'article de BPZ de 1984 [12] et la mise en place du bootstrap conforme, une question se pose alors naturellement : que se passe-t-il lorsque l'algèbre de symétrie contient strictement l'algèbre de Virasoro ? En d'autres termes, est-ce que les mêmes techniques restent valides lorsque la symétrie du modèle est étendue au-delà de la symétrie conforme ? Zamolodchikov propose en 1985 [143] un cadre destiné à étendre cette procédure à des modèles possédant une symétrie supérieure, la *W-symétrie* ou *symétrie de spin supérieur*. Ces symétries sont codées par des *W-algèbres*, des algèbres vertex contenant l'algèbre de Virasoro en tant que sous-algèbre. Ces objets sont particulièrement étudiées de point de vue de la théorie des représentations des algèbres vertex [6].

Les théories de Toda, une famille de théories conformes des champs indexée par les algèbres de Lie semi-simples et complexes $\mathfrak{g}$, fournissent des extensions naturelles de la théorie de Liouville dans ce cadre. En effet la théorie de Liouville est en fait l'exemple le plus simple de théorie de Toda, dans le sens où elle correspond au choix d'algèbre de Lie $\mathfrak{g} = \mathfrak{sl}_2$, et dans ce cas l'algèbre de symétrie est simplement l'algèbre de Virasoro. Toutefois pour $\mathfrak{g}$ générique ces algèbres de symétrie contiennent strictement l'algèbre de Virasoro, et pour cette raison les théories de Toda sont loin d'être complètement comprises, en dépit d'avoir initié une quantité considérable de travaux dans la littérature physique. Ces théories sont de fait intéressantes dans la perspective de la *W-symétrie* [26], mais également pour leurs liens avec d'autres théories quantiques des champs (et notamment celles avec symétrie de Kac-Moody, voir [9, 8] pour leurs liens avec les modèles de Wess-Zumino-Witten), certains modèles de physique statistique (en particulier le modèle d'Ising [1] dont un traitement journalistique est donné dans [23]), ou encore au sein de la correspondance AGT [4, 39, 104]. Dans cette sous-section nous nous attacherons à expliquer comment ces théories sont définies dans la littérature physique.

1.1.3.1 Les théories de Toda et l'intégrale de chemin

De la même manière que les théories de Liouville, les théories de Toda se définissent par l'intermédiaire d'une intégrale de chemin. Plus précisément les théories de Toda définissent une fonction aléatoire, le *champ de Toda* Φ prenant ses valeurs sur une surface riemannienne (Σ, g) et à valeur dans un espace euclidien $\mathfrak{a} \simeq \mathbb{R}^r$ ($r = 1$ dans le cas de Liouville). Cet espace euclidien est équipé d'un produit scalaire $\langle \cdot, \cdot \rangle$ et d'une base spéciale, composée de *racines simples* $(e_1, \dots, e_r)$, tous deux hérités de la structure d'algèbre de Lie sous-jacente. Cette base est représentée en 1.1 ci-dessous dans le cas de $\mathfrak{g} = \mathfrak{sl}_3$, pour lequel $\mathfrak{a} \simeq \mathbb{R}^2$ est équipée d'une base (e_1, e_2) vérifiant $\langle e_i, e_j \rangle = 2$ pour $i = j$ tandis que $\langle e_i, e_j \rangle = -1$ dès que $i \neq j$.

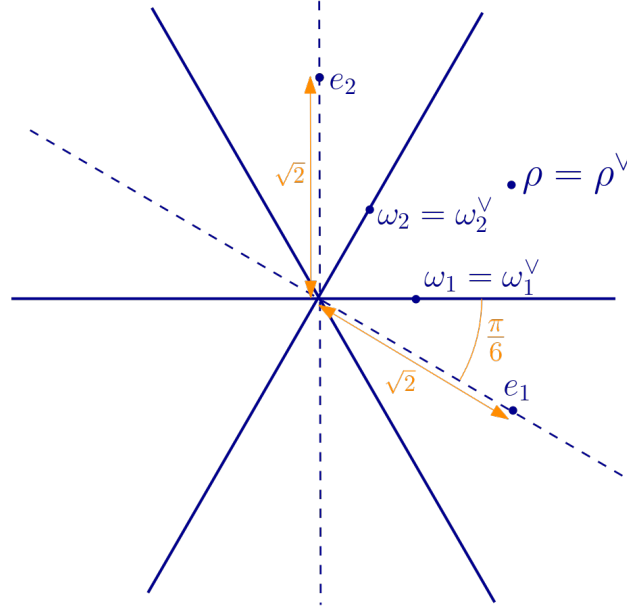


Figure 1.1: Les racines simples associées à $\mathfrak{g} = \mathfrak{sl}_3$

Étant donnée une métrique riemannienne g sur Σ , l'intégrale de chemin définissant le champ de Toda prend alors la forme de¹ :

$$\langle F(\Phi) \rangle_{T,g} := \frac{1}{\mathcal{Z}} \int_{\mathcal{F}} F(\phi) e^{-S_{T,g}(\phi,g)} D\phi \quad (1.1.7)$$

où $D\phi$ correspond comme précédemment à une “mesure uniforme” sur un espace $\mathcal{F}$ de fonctions définies sur Σ et à valeurs dans $\mathfrak{a}$, et où $S_{T,g}$ est l'action de Toda

$$S_{T,g}(\phi, g) := \frac{1}{4\pi} \int_{\Sigma} \left(\langle \partial_g \phi(x), \partial_g \phi(x) \rangle_g + R_g \langle Q, \phi(x) \rangle + 4\pi \sum_{i=1}^r \mu_i e^{\gamma \langle e_i, \phi(x) \rangle} \right) v_g(dx). \quad (1.1.8)$$

Dans cette expression la métrique riemannienne g a courbure scalaire R_g , gradient ∂_g et forme volume v_g , tandis que $\langle \cdot, \cdot \rangle_g$ est le produit scalaire sur l'espace tangent associé aux fonctions différentiables $\Sigma \rightarrow \mathfrak{a}$. La constante de couplage γ est à valeurs dans $(0, \sqrt{2})$ ² tandis que la charge de fond Q est à valeurs dans $\mathfrak{a}$. Les constantes cosmologiques μ_i , $1 \leq i \leq r$ sont à valeurs dans $\mathbb{R}$. Afin de s'assurer que le modèle est bien covariant conforme, la charge de fond et la constante de couplage doivent être liées par la relation

$$Q = \gamma \rho + \frac{2}{\gamma} \rho^\vee \quad (1.1.9)$$

où ρ et $\rho^\vee$ sont deux vecteurs spéciaux dans $\mathfrak{a}$, que nous définirons au cours du Chapitre 3 (voir l'illustration 1.1 pour $\mathfrak{g} = \mathfrak{sl}_3$). La théorie classique associée aux théories de Toda

¹Nous n'incluons pas la dépendance en γ dans les notations introduites ici afin de les garder concises.

²La constante de couplage diffère de celle de Liouville par un facteur multiplicatif de $\sqrt{2}$ (dû au fait que les racines simples “longues” ont norme $\sqrt{2}$), expliquant pourquoi elle prend ces valeurs dans $(0, \sqrt{2})$ et non plus dans $(0, 2)$.

est également d'origine géométrique et prend son fondement dans l'étude des surfaces minimales [21] et de la W -géométrie [90, 63].

En analogie avec la théorie de Liouville, les opérateurs vertex sont des fonctionnelles du champ de Toda dépendant d'un point $z \in \Sigma$, mais le poids α est désormais un élément de l'espace euclidien $\mathfrak{a}$ (ou plutôt de sa complexification $\mathfrak{h} = \mathfrak{a} \oplus i\mathfrak{a}$). Elles sont définies par l'expression $V_\alpha(z)[\Phi] = e^{\langle \alpha, \Phi(z) \rangle}$, si bien que leurs fonctions de corrélation prennent la forme de

$$\left\langle \prod_{k=1}^N V_{\alpha_k}(z_k) \right\rangle_{T,g} := \int_{\mathcal{F}} \prod_{k=1}^N e^{\langle \alpha_k, \phi(z_k) \rangle} e^{-S_{T,\mathfrak{g}}(\phi,g)} D\phi. \quad (1.1.10)$$

Tout comme pour la théorie de Liouville, calculer ces fonctions de corrélation est un des buts principaux de l'étude des théories de Toda.

1.1.3.2 Le bootstrap conforme pour les théories de Toda

Les théories de Toda possèdent plus de symétries que la théorie de Liouville, mais du fait que le champ de Toda est à valeurs dans un espace euclidien et non plus simplement à valeurs réelles leur étude est en fait plus compliquée et comprendre comment implémenter la procédure du bootstrap conforme n'est pas un problème résolu à l'heure actuelle, et ce même du point de vue de la physique. Certaines fonctions de corrélation en ont été calculées [52, 55], mais fournir une expression pour toutes les structures de constante reste pour l'heure actuelle une question ouverte. Nous pouvons néanmoins citer les articles [101, 74] où de telles formules sont exprimées – sans toutefois être explicitement calculées – en termes de quantités apparaissant dans le cadre de la correspondance AGT, ainsi que l'article [38] où la méthode récursive au cœur du bootstrap conforme est esquissée.

Au cours de cette thèse une attention particulière sera consacrée à la théorie de Toda associée à l'algèbre de Lie $\mathfrak{g} = \mathfrak{sl}_3$, qui est le cas le plus simple pour lequel l'algèbre de symétrie contient strictement l'algèbre de Virasoro. Pour cette théorie de Toda la méthode du bootstrap ressemble à celle développée pour la théorie de Liouville mais est un peu plus exigeante. Une première conséquence des contraintes posées par les symétries du modèle est le calcul d'une famille de fonctions de corrélation à trois points $\langle V_{\alpha_1}(0)V_{\alpha_2}(1)V_{\alpha_3}(\infty) \rangle$. Une différence majeure avec la théorie de Liouville est que la méthode du bootstrap conforme ne permet pour l'instant de calculer de telles fonctions de corrélation que sous les hypothèses que l'un des opérateurs vertex est un *champ semi-dégénéré*, ce qui signifie que le poids correspondant appartient à un sous-ensemble de $\mathfrak{a}$. Sous cette hypothèse ces fonctions de corrélation à trois points sont décrites par la formule de Fateev-Litvinov [52], notée ici $C_\gamma^{\text{FL}}(\alpha_1, \alpha_2^*, \alpha_3)$ ³, qui généralise la formule DOZZ.

À partir de l'expression de cette famille de fonctions de corrélation à trois points, la question se pose alors de la forme que prendrait la procédure récursive centrale dans le bootstrap conforme, et si une formule similaire à celle pour la théorie de Liouville existe. Un travail en cours mené conjointement avec Colin Guillarmou [32] tend à prouver que c'est effectivement le cas en proposant une formule similaire à l'équation (1.1.6). À notre

³Le signe “*” indique que l'opérateur vertex correspondant est semi-dégénéré.

connaissance il s'agit de la première implémentation de cette procédure pour la théorie de Toda considérée.

1.1.4 . Organisation de la thèse

Un des résultats principaux présentés dans ce manuscrit est le calcul rigoureux d'une famille fonctions de corrélation à trois points dans la théorie de Toda associée à $\mathfrak{sl}_3$, retrouvant ainsi la formule de Fateev-Litvinov [52]. Pour ce faire nous proposons une définition probabiliste des fonctions de corrélation associées aux théories de Toda et nous appuyons sur les propriétés des objets probabilistes ainsi définis possèdent, en relation avec celles attendues en physique.

Dans un premier temps nous établirons le cadre général dans lequel l'interprétation mathématique des théories de Toda se placent, cadre comprenant notamment des notions de base sur les algèbres de Lie simples et complexes, de géométrie conforme ainsi que les objets probabilistes que nous considérerons dans la suite. C'est le contenu du chapitre 3.

Le chapitre suivant, Chapitre 4, sera dédié à la définition mathématique des fonctions de corrélation des théories de Toda basée sur l'interprétation probabiliste de l'intégrale de chemin (1.1.5). Certaines propriétés basiques mais néanmoins cruciales des objets ainsi construits seront présentées également au cours de ce chapitre, dont le contenu sera détaillé en amont dans la section 1.2.

À partir du chapitre 5 nous commencerons à nous focaliser sur l'étude de la théorie de Toda associée à $\mathfrak{g} = \mathfrak{sl}_3$, et dans un premier temps de ses symétries. Pour ce faire nous prouverons que certaines identités encodant ces symétries sont vérifiées sur le modèle probabiliste construit précédemment. Au cours du chapitre 6 nous proposerons un détour hors du cadre de la théorie des champs et détaillerons certaines implications somme toute assez inattendues de la symétrie W sur des objets purement probabilistes. Pour ce faire nous établirons un lien entre le principe de réflexion en théorie des probabilités et la notion de réflexion en théorie conforme des champs (de Toda). Le contenu de ces deux chapitres est esquissé en section 1.3.

En s'appuyant sur les résultats établis au cours des chapitres précédents, nous prouverons dans le chapitre 7 qu'une famille de fonctions de corrélation à trois points pour la théorie de Toda associée à $\mathfrak{sl}_3$ et définies par des expressions probabilistes sont bien décrites par la formule de Fateev-Litvinov. Pour ce faire nous traduirons dans un langage mathématiquement rigoureux un certain nombre de techniques utilisées en physique, comme nous l'explicitons dans la section 1.4. Enfin nous présenterons dans le chapitre 8 un travail en cours avec Guillarmou où est explicitée la manière de conduire la procédure récursive du bootstrap conforme pour la théorie de Toda associée à $\mathfrak{sl}_3$. Nous décrivons notamment comment l'étude de l'*Hamiltonien de Toda* permet de mettre en place le bootstrap conforme dans le cadre probabiliste.

1.2 . Une approche probabiliste des théories de Toda

Ayant présenté les théories de Toda dans la forme sous laquelle elles sont étudiées dans la littérature physique, nous pouvons désormais apporter un regard mathématique sur cette famille de modèles et en premier lieu décrire comment les fonctions de corréla-

tion qui leur sont associées peuvent être rigoureusement définies. Nous nous concentrerons dans ce manuscrit sur la théorie de Toda sur la sphère de dimension deux $\mathbb{S}^2$, identifiée à la sphère de Riemann $\mathbb{C} \cup \{\infty\}$. Cette section fournit un résumé du chapitre 4 en se basant sur les notions introduites dans le chapitre 3. Les résultats présentés sont issus de l'article [34] rédigé conjointement avec Rhodes et Vargas.

1.2.1 . Interprétation probabiliste de l'intégrale de chemin

L'intégrale de chemin (1.1.5) définissant de manière formelle les théories de Toda ne fait pas vraiment sens mathématiquement parlant. Il convient donc dans un premier temps de donner un sens rigoureux à cette écriture et définir les fonctions de corrélation d'opérateurs vertex en tant qu'objets mathématiques bien définis. Pour ce faire nous nous appuyerons sur un cadre probabiliste lui-même basé sur des champs libres gaussiens et la théorie du chaos multiplicatif gaussien, que nous définissons Section 3.2.

1.2.1.1 Définition des fonctions de corrélation

Nous expliquerons au cours de la section 4.1 qu'il est en effet possible d'interpréter le terme quadratique apparaissant dans l'action de Toda (1.1.8) par analogie avec un vecteur Gaussien. De fait une fois inséré dans l'intégrale de chemin (1.1.5) le terme quadratique

$$e^{-\frac{1}{2}\langle \phi, (\frac{-\Delta}{2\pi})\phi \rangle_{L^2}} D\phi \quad \text{avec} \quad \langle f, h \rangle_{L^2} := \int_{\Sigma} \langle f(x), h(x) \rangle_{\mathfrak{g}}(dx) \quad \text{pour} \quad f, h : \Sigma \rightarrow \mathfrak{a}$$

est réminiscent de la densité

$$((2\pi)^N \det(A))^{-\frac{1}{2}} e^{-\frac{1}{2}\langle x, A^{-1}x \rangle} dx_1 \cdots dx_N$$

d'un vecteur gaussien $X = (X_1, \dots, X_N)$ de matrice de covariance A . Il est donc naturel de penser au champ de Toda en terme d'un "vecteur gaussien infini-dimensionnel" – et plus précisément une fonction aléatoire gaussienne sur $\mathbb{C}$ à valeurs dans $\mathfrak{a}$ – décrit par une matrice de covariance de la forme :

$$\mathbb{E}[\langle u, \mathbf{X}^g(x) \rangle \langle v, \mathbf{X}^g(y) \rangle] = \langle u, v \rangle G_g(x, y)$$

où G_g est la fonction de Green de l'opérateur de Laplace-Beltrami Δ_g sur (Σ, g) , correspondant formellement à l'inverse de cet opérateur. Toutefois comme la fonction de Green diverge le long de la diagonale $x = y$ ce champ n'est pas une véritable fonction mais plutôt une distribution aléatoire : il s'agit d'un *champ libre gaussien*. Dans l'étude de cette fonction généralisée il est alors standard de considérer dans un premier temps une version régularisée $(\mathbf{X}_\varepsilon^g)_{\varepsilon>0}$ du champ $\mathbf{X}^g$ – par exemple par convolution $\mathbf{X}_\varepsilon^g := \mathbf{X}^g * \rho_\varepsilon$, voir Section 3.2. Les champs ainsi définis sont des fonctions lisses, ce qui permet de donner un sens à nombre d'expressions (notamment l'exponentielle du champ libre comme nous l'étudierons dans ce manuscrit), tout l'enjeu étant de comprendre comment peut s'effectuer le passage à la limite quand $\varepsilon \rightarrow 0$. Cette procédure est au cœur de la définition du *chaos multiplicatif gaussien*. Avant de poursuivre soulignons que le champ libre gaussien a acquis une place prépondérante dans de nombreux domaines des probabilités et notamment la géométrie aléatoire en dimension deux, en cela qu'il joue un rôle similaire – de par entre autres par son universalité – à celui du mouvement

brownien où la variable de temps serait remplacée par une variable deux-dimensionnelle. Nous renvoyons par exemple aux articles [125, 112] où des propriétés et applications de cet objet fondamental sont décrites. Il est important à ce stade de souligner que le champ considéré ici est à valeurs dans l'espace euclidien $\mathfrak{a}$ et en cela diffère de celui considéré dans [40] pour définir le champ de Liouville.

En se basant sur cette interprétation du terme quadratique de l'action de Toda l'intégrale de chemin peut désormais se définir en termes mathématiques, en identifiant pour toutes fonctions F convenables et métriques riemanniennes g :

$$\begin{aligned} \langle F(\Phi) \rangle_{\mathfrak{g},g} &:= \frac{1}{\mathcal{Z}_g} \int_{\mathfrak{a}} \mathbb{E} \left[F \left(\mathbf{X}^g + \frac{Q}{2} \ln g + \mathbf{c} \right) \right. \\ &\quad \left. \exp \left(-\frac{1}{4\pi} \int_{\mathbb{C}} \left(R_g \langle Q, \mathbf{X}^g + \mathbf{c} \rangle + 4\pi \sum_{i=1}^r \mu_i e^{\gamma \langle e_i, \mathbf{X}^g + \frac{Q}{2} \ln g + \mathbf{c} \rangle} \right) dv_g \right) \right] d\mathbf{c}, \end{aligned} \quad (1.2.1)$$

où $\mathcal{Z}_g$ est défini comme un déterminant régularisé, et où le terme supplémentaire $\frac{Q}{2} \ln g$ apparaissant dans la définition du champ de Toda est déjà présent dans la théorie de Liouville et est nécessaire afin d'obtenir un modèle invariant conforme. Il est important de souligner que rendre compte dans le terme d'action de l'exponentielle du champ libre gaussien n'est pas forcément chose aisée puisque $\mathbf{X}^g$ est une distribution et non une véritable fonction. Afin de donner un sens à cette expression il est nécessaire dans un premier temps de considérer la version régularisée $(\mathbf{X}_\varepsilon^g)_{\varepsilon>0}$ du champ : l'expression $e^{\langle \gamma e_i, \mathbf{X}_\varepsilon^g(x) \rangle}$ est alors bien définie ponctuellement mais diverge quand $\varepsilon \rightarrow 0$. Pour pallier ce problème il convient de renormaliser cette variable aléatoire par son espérance : sous l'hypothèse que $\langle \gamma e_i, \gamma e_i \rangle < 4$, la limite $\lim_{\varepsilon \rightarrow 0} \frac{e^{\langle \gamma e_i, \mathbf{X}_\varepsilon^g(x) \rangle}}{\mathbb{E} [e^{\langle \gamma e_i, \mathbf{X}_\varepsilon^g(x) \rangle}]} dv_g(x)$ définit une mesure aléatoire sur $\mathbb{C}$. Elle est appelée *mesure de chaos multiplicatif gaussien*. Nous renvoyons à la Section 3.2 pour plus de détails sur la construction de cet objet. Initialement introduite par Kahane en 1985 [76], la théorie du chaos multiplicatif gaussien a connu récemment un renouveau d'intérêt en partie du fait de ses nombreuses applications, que ce soit dans le domaine de la finance [48], pour son rôle dans la modélisation de la turbulence [81, 105, 92] ou encore en lien avec des modèles de matrices aléatoires [15, 36] mais aussi certaines propriétés de la fonction ζ de Riemann [120]. La richesse de cette théorie la rend intéressante également en elle-même, et a initié de nombreux champs d'étude – analyse multifractale [16], définition du chaos multiplicatif imaginaire [85, 75], intégrabilité... Nous renvoyons vers [115] et les références s'y trouvant pour une description plus complète de cette théorie et ses applications.

Les fonctions de corrélation d'opérateurs vertex sont alors définies en prenant formellement comme fonctionnelle $F(\Phi) = \prod_{k=1}^N e^{\langle \alpha_k, \Phi(z_k) \rangle}$. Toutefois comme précédemment du fait de l'absence de régularité du champ il convient de régulariser et renormaliser cette expression afin d'obtenir un objet bien défini. Plus précisément, la définition probabiliste des fonctions de corrélation (pourvu qu'elle fasse sens) prend la forme de la limite

$$\begin{aligned} \langle \prod_{k=1}^N V_{\alpha_k}(z_k) \rangle_{\mathfrak{g},g} &:= \lim_{\varepsilon \rightarrow 0} \frac{1}{\mathcal{Z}_g} \int_{\mathfrak{a}} \mathbb{E} \left[\prod_{k=1}^N \frac{e^{\langle \alpha_k, \mathbf{X}_\varepsilon^g(z_k) + \frac{Q}{2} \ln g + \mathbf{c} \rangle}}{g(z_k)^{\frac{\langle \alpha_k, \alpha_k \rangle}{4}} \mathbb{E} [e^{\langle \alpha_k, \mathbf{X}_\varepsilon^g(z_k) \rangle}]} \right. \\ &\quad \left. \exp \left(-\frac{1}{4\pi} \int_{\mathbb{C}} \left(R_g \langle Q, \mathbf{X}^g + \mathbf{c} \rangle + 4\pi \sum_{i=1}^r \mu_i e^{\gamma \langle e_i, \mathbf{X}^g + \frac{Q}{2} \ln g + \mathbf{c} \rangle} \right) dv_g \right) \right] d\mathbf{c}. \end{aligned} \quad (1.2.2)$$

Cette expression peut se reformuler d'une manière plus exploitable en utilisant le théorème de Girsanov (ou Cameron-Martin). En effectuant cette procédure le résultat fondamental suivant montre que les fonctions de corrélation peuvent être ainsi définies en tant qu'objets mathématiques rigoureux :

Théorème 1.2.1. *Soit $\mathfrak{g}$ une algèbre de Lie simple, complexe et de dimension finie. Sous l'hypothèse que la constante de couplage satisfait $\gamma \in (0, \sqrt{2})$, l'existence et la non-trivialité des fonctions de corrélation $\langle V_{\alpha_1}(z_1) \cdots V_{\alpha_N}(z_N) \rangle_{\mathfrak{g},g}$ ne dépend pas de la métrique g au sein de la classe conforme de $\hat{g}$. De plus :*

1. (Bornes de Seiberg) *Les fonctions de corrélation $\langle V_{\alpha_1}(z_1) \cdots V_{\alpha_N}(z_N) \rangle_{\mathfrak{g},g}$ existent et sont non triviales si et seulement si les deux hypothèses suivantes sont vérifiées :*

$$\text{Pour tout } 1 \leq i \leq r, \quad s_i > 0 \quad \text{et} \quad \langle \alpha_k - Q, e_i \rangle < 0 \quad \text{pour tout } 1 \leq k \leq N$$

$$\text{où} \quad s_i := \frac{\langle \sum_{j=1}^N \alpha_j - 2Q, \omega_i^\vee \rangle}{\gamma}.$$

(1.2.3)

2. (Représentation probabiliste) *Dans le cas particulier où $g = \hat{g}$ les fonctions de corrélation prennent la forme de*

$$\begin{aligned} & \langle V_{\alpha_1}(z_1) \cdots V_{\alpha_N}(z_N) \rangle_{\mathfrak{g},\hat{g}} \\ &= \left(\prod_{i=1}^r \frac{\Gamma(s_i) \mu_i^{-s_i}}{\gamma} \right) \prod_{1 \leq k < l \leq N} |z_k - z_l|^{-\langle \alpha_k, \alpha_l \rangle} \mathbb{E} \left[\prod_{i=1}^r Z_{(z,\alpha)}^{\gamma e_i}(\mathbb{C})^{-s_i} \right]. \end{aligned} \quad (1.2.4)$$

où les variables aléatoires apparaissant dans l'espérance correspondent à des mesures de chaos multiplicatif gaussien, formellement définies par

$$Z_{(z,\alpha)}^{\gamma e_i}(dx) = \frac{\hat{g}(x)^{-\frac{1}{4} \langle \gamma e_i, \sum_{k=1}^N \alpha_k \rangle}}{\prod_{k=1}^N |x - z_k|^{\langle \alpha_k, \gamma e_i \rangle}} e^{\langle \gamma e_i, \mathbf{X}^g(x) + \frac{Q}{2} \ln \hat{g}(x) \rangle} dx.$$

La métrique particulière $\hat{g} := \frac{4}{(1+|z|^2)^2} |d^2 z|$ que nous avons considérée dans cet énoncé vient de l'identification de la sphère $\mathbb{S}^2$ (équipée de sa métrique usuelle) avec la sphère de Riemann (équipée avec $\hat{g}$) par projection stéréographique. Les vecteurs $(\omega_i^\vee)_{1 \leq i \leq r}$ forment la base de $\mathfrak{a}$ duale à celle des racines simples : $\langle e_i, \omega_j^\vee \rangle = \delta_{i,j}$. Plus de détails concernant cet énoncé sont donnés Section 4.1 (voir aussi Figure 1.1).

Grâce à ce résultat nous pouvons construire les fonctions de corrélation des théories de Toda pourvu que les bornes de Seiberg (1.2.3) soient satisfaites. L'existence de telles bornes sur les poids est déjà présente dans l'étude de la théorie de Liouville [123, 40].

1.2.1.2 Covariance conforme des fonctions de corrélation

Avant d'étudier des propriétés plus élaborées de ces fonctions de corrélation probabilistes, il convient d'abord de s'assurer qu'elles vérifient les hypothèses de base de la covariance conforme. Pour cela nous démontrons que l'expression probabiliste des fonctions de

corrélation proposée dans le Théorème 1.2.1 satisfait des propriétés de base liés à la symétrie conforme : la *covariance conforme* des fonctions de corrélation et l'existence d'une *anomalie de Weyl* sous changement conforme de la métrique g sur $\mathbb{C} \cup \{\infty\}$. Ces hypothèses correspondent à une partie de l'axiomatique proposée par Gawedski [62] pour la théorie conforme des champs.

La première de ces deux propriétés correspond au comportement des fonctions de corrélation sous des transformations préservant la structure conforme de $\mathbb{C} \cup \{\infty\}$. Plus précisément si l'on agit sur les insertions $z_1, \dots, z_N$ entrant dans la définition des fonctions de corrélation par l'intermédiaire de *transformations de Möbius*, les fonctions de corrélation devraient être perturbées d'une manière explicite, décrite dans l'équation (1.2.5) plus bas. L'anomalie de Weyl montre quant à elle que la dépendance en la métrique riemannienne g équipant la sphère de Riemann est explicite au sein d'une même classe conforme. Plus précisément, l'équation (1.2.6) montre que si g' et g sont deux métriques riemanniennes sur la sphère de Riemann toutes deux conformément équivalentes à $\hat{g}$ alors les fonctions de corrélation qui leur sont associées diffèrent par un facteur multiplicatif explicite dépendant uniquement de la charge centrale de la théorie et non des fonctions de corrélation considérées.

Le résultat suivant montre que les fonctions de corrélation probabilistes satisfont à cette axiomatique dans le sens où :

Théorème 1.2.2. *Sous les hypothèses du Théorème 1.2.1 :*

1. (Covariance conforme) *Soit ψ une transformation de Möbius du plan. Alors*

$$\langle V_{\alpha_1}(\psi(z_1)) \cdots V_{\alpha_N}(\psi(z_N)) \rangle_{\mathfrak{g},g} = \prod_{k=1}^N |\psi'(z_k)|^{-2\Delta_{\alpha_k}} \langle V_{\alpha_1}(z_1) \cdots V_{\alpha_N}(z_N) \rangle_{\mathfrak{g},g} \quad (1.2.5)$$

où les poids conformes sont donnés par $\Delta_{\alpha_j} := \langle \frac{\alpha_j}{2}, Q - \frac{\alpha_j}{2} \rangle$.

2. (Anomalie de Weyl) *Pour une classe de fonctions $\varphi : \mathbb{C} \rightarrow \mathbb{R}$ (précisément $\varphi \in \bar{C}^1(\mathbb{C})$, voir Section 4.2) on a*

$$\langle V_{\alpha_1}(z_1) \cdots V_{\alpha_N}(z_N) \rangle_{\mathfrak{g},e^{\varphi}\hat{g}} = e^{\frac{c_T}{96\pi} S_L(\varphi,\hat{g})} \langle V_{\alpha_1}(z_1) \cdots V_{\alpha_N}(z_N) \rangle_{\mathfrak{g},\hat{g}} \quad (1.2.6)$$

où S_L est la fonctionnelle de Liouville (avec $\Lambda = 0$)

$$S_L(\varphi, \hat{g}) := \int_{\mathbb{C}} (|\partial_{\hat{g}}\varphi|_{\hat{g}}^2 + 2R_{\hat{g}}\varphi) dV_{\hat{g}},$$

et la charge centrale c_T est donnée par $c_T = r + 6\langle Q, Q \rangle$.

Ces propriétés sont en fait des conséquences de celles du champ libre gaussien, soulignant les liens importants entre cet objet probabiliste et la géométrie conforme. Elle seront prouvées Section 4.2. Nous notons que ces caractéristiques sont également présentes dans la formulation probabiliste de la théorie de Liouville [40].

1.2.2 . Quelques propriétés supplémentaires des fonctions de corrélation probabilistes

Les fonctions de corrélation étant désormais des objets mathématiques bien définis et vérifiant certaines hypothèses d'invariance conforme, nous pouvons donc étudier quelques-unes de leurs propriétés plus analytiques. Nous nous appuyerons ici sur les articles [30] et [33], le second rédigé en collaboration avec Huang.

1.2.2.1 Un premier prolongement analytique des fonctions de corrélation

Une hypothèse supplémentaire faite en physique est que les fonctions de corrélation dépendent de manière analytique en leurs poids $\alpha_1, \dots, \alpha_N$. Cet axiome est en fait assez puissant et fondamental dans l'implémentation du bootstrap conforme envisagée par BPZ [12]. Nous nous attacherons donc à vérifier que les fonctions de corrélation probabilistes vérifient cette hypothèse en montrant que l'expression (1.2.4) dépend analytiquement en les poids $(\alpha_k)_{1 \leq k \leq N}$.

En outre nous avons vu précédemment dans le théorème 1.2.1 que la construction probabiliste des fonctions de corrélation plaçait deux restrictions sur les poids qu'il était possible de considérer pour définir les fonctions de corrélation. La première contrainte que $\langle \alpha_k - Q, e_i \rangle < 0$ (autrement dit que $\alpha_k - Q$ appartienne à la *chambre de Weyl* $\mathcal{C}_-$) est nécessaire pour que la représentation probabiliste fasse sens, mais la deuxième qui requiert que $\langle \sum_{k=1}^N \alpha_k - 2Q, \omega_i^\vee \rangle > 0$ peut en fait être dépassée. En effet dans l'équation (1.2.4) les premières singularités apparaissant lorsque $\langle \sum_{k=1}^N \alpha_k - 2Q, \omega_i^\vee \rangle \leq 0$ viennent des pôles sur $-\mathbb{N}$ de la fonction Gamma Γ , tandis que l'espérance est en fait bien définie sous des hypothèses plus générales.

En se basant sur cette observation, nous prouvons que les fonctions de corrélation probabilistes dépendent de manière analytique en les poids $(\alpha_k)_{1 \leq k \leq N}$ et qu'elles admettent également un prolongement analytique au-delà des bornes de Seiberg (1.2.3) toujours défini de manière probabiliste :

Théorème 1.2.3. Soient $z_1, \dots, z_N \in \mathbb{C}$ distincts et considérons le sous ensemble de $(Q + \mathcal{C}_-)^N$ défini par

$$\mathcal{B}_N := \left\{ (\alpha_1, \dots, \alpha_N) \in (Q + \mathcal{C}_-)^N \right. \\ \left. \text{tels que } -s_i < \frac{4}{\gamma^2 \langle e_i, e_i \rangle} \wedge \min_{k=1, \dots, N} \frac{1}{\gamma} \langle Q - \alpha_k, e_i^\vee \rangle \text{ pour tout } 1 \leq i \leq r \right\}. \quad (1.2.7)$$

L'application considérée Équation (1.2.4) :

$$\alpha \mapsto \left(\prod_{i=1}^r \frac{\Gamma(s_i) \mu_i^{-s_i}}{\gamma} \right) \prod_{1 \leq k < l \leq N} |z_k - z_l|^{-\langle \alpha_k, \alpha_l \rangle} \mathbb{E} \left[\prod_{i=1}^r Z_{(z, \alpha)}^{\gamma e_i} (\mathbb{C})^{-s_i} \right] \quad (1.2.8)$$

est méromorphe dans un voisinage complexe de $\mathcal{B}_N \subset (\mathbb{C}^r)^N$, avec pôles dans

$$\mathcal{P}_N^1 := \{ \alpha \in \mathcal{B}_N, \quad s_i = 0 \text{ pour un certain } 1 \leq i \leq r \}.$$

La preuve de ce résultat se base sur certaines propriétés du chaos multiplicatif gaussien et plus précisément sur les conditions sous lesquelles ses moments existent. L'ensemble $\mathcal{B}_N$ en fait l'ensemble optimal pour lequel l'espérance fait sens.

Nous expliquerons plus tard que nous pouvons dépasser en outre cette représentation en étudiant plus en détail la queue de distribution des mesures de chaos multiplicatif gaussien considérées.

1.2.2.2 Quelques propriétés analytiques des fonctions de corrélation

Dans la perspective d'étudier les symétries des théories de Toda nous devons calculer les dérivées de ses fonctions de corrélation. Afin de s'assurer que cela est effectivement possible nous décrirons certaines caractéristiques de celles-ci dans l'optique de démontrer le résultat de régularité suivant :

Proposition 1.2.1. *Soient $z_1, \dots, z_N \in \mathbb{C}$ distincts. Sous l'hypothèse que les poids $(\alpha_0, \dots, \alpha_N)$ appartiennent à $\mathcal{B}_{N+1}$, la fonction $z_0 \mapsto \langle \prod_{k=0}^N V_{\alpha_k}(z_k) \rangle$ est de classe C^2 sur $\mathbb{C} \setminus \{z_1, \dots, z_N\}$.*

La section 4.4 sera consacrée à ces aspects plus techniques des fonctions de corrélation probabilistes. Nous étudierons en particulier certaines *estimées de fusion* ainsi que l'implémentation de l'*intégration par partie gaussienne* dans ce cadre.

1.3 . Étude des symétries des théories de Toda

La construction des fonctions de corrélation probabilistes et les propriétés qu'elles satisfont représentent le point de départ d'une étude mathématique rigoureuse des symétries se trouvant au cœur des théories de Toda. Une des manifestations de ces symétries est l'existence d'identités de Ward satisfaites par les fonctions de corrélation, que nous prouvons rigoureusement pour la théorie de Toda associée à l'algèbre de Lie $\mathfrak{g} = \mathfrak{sl}_3$ et sur la sphère de Riemann $\mathbb{C} \cup \{\infty\}$. Nous nous baserons sur l'article sus-mentionné [33]. Nous expliquerons également comment différents principes de réflexion (dans le cadre des probabilités ou de la théorie des champs) peuvent être mis en perspective et avoir des conséquences inattendues, notamment par la description d'une décomposition de chemins browniens ainsi que de la queue de distribution de mesures de chaos multiplicatif gaussiens corrélés. Ces résultats sont issus de [31].

1.3.1 . Symétrie W et identités de Ward

En théorie conforme des champs, la symétrie des modèles se manifeste notamment par l'existence d'une certaine observable du champ appelée *tenseur énergie-impulsion*. Les identités de Ward du modèle, encodant les symétries locales de celui-ci, montrent qu'insérer ce tenseur dans une fonction de corrélation peut être décrit de manière totalement explicite en termes de dérivées de cette fonction de corrélation.

Un raisonnement similaire peut être appliqué de la même manière pour des modèles possédant un degré plus élevé de symétrie. Associés à ces symétries supplémentaires seront des *tenseurs de spin supérieur*, dont l'insertion dans des fonctions de corrélation conduit à des identités supplémentaires mais plus complexes à exploiter.

Nous détaillons ici ces identités de Ward et expliquons comment elles peuvent se vérifier sur notre modèle probabiliste.

1.3.1.1 Symétrie conforme et tenseur énergie-impulsion

Une conséquence cruciale de la méthode développée par BPZ [12] est de réduire l'étude des symétries du modèle à celle de l'algèbre de Virasoro, l'algèbre de symétrie des théories conformes des champs en dimension deux. Au cours de cette étude la considération d'un *courant holomorphe* de spin 2, le tenseur énergie-impulsion dénoté $\mathbf{T}$, émerge de manière naturelle. En effet ce tenseur admet formellement une expansion en séries de Laurent qui prend la forme

$$\mathbf{T}(z_0) = \sum_{n \in \mathbb{Z}} \frac{\mathbf{L}_n(z)}{(z_0 - z)^{n+2}} \quad (1.3.1)$$

autour d'un point $z \in \mathbb{C}$, et où les modes $\mathbf{L}_n$ correspondent aux générateurs de l'algèbre de Virasoro, dont les relations de commutation sont décrites par

$$[\mathbf{L}_n, \mathbf{L}_m] = (n - m)\mathbf{L}_{n+m} + \frac{c}{12}(n - 1)n(n + 1)\delta_{n+m,0}\text{Id}, \quad (1.3.2)$$

avec c la charge centrale de la théorie conforme des champs considérée.

Dans la théorie de Liouville ce tenseur peut s'obtenir (formellement toujours) en faisant varier le fonctions de corrélation par rapport à la métrique g (non plus dans une même classe conforme) et admet ainsi une expression alternative en tant que fonctionnelle du champ de Liouville Φ . Plus précisément le tenseur énergie-impulsion $\mathbf{T}$ peut être défini par l'expression

$$\left\langle \mathbf{T}(z_0) \prod_{k=1}^N V_{\alpha_k}(z_k) \right\rangle_g := 4\pi \frac{\delta}{\delta g^{zz}(z_0)} \left\langle \prod_{k=1}^N V_{\alpha_k}(z_k) \right\rangle_g \quad (1.3.3)$$

où la structure métrique varie autour du point $z_0 \in \mathbb{C}$. Une conséquence de cette définition est que le tenseur énergie-impulsion admet une expression explicite en tant que fonctionnelle du champ de Liouville :

$$\mathbf{T}(z)[\Phi] = Q\partial_z^2\Phi(z) - (\partial_z\Phi(z))^2. \quad (1.3.4)$$

Une justification mathématique de cette heuristique se trouve dans les articles [82, 107].

Une des propriétés fondamentales de ce tenseur est son *développement de produit d'opérateurs* avec les opérateurs vertex, qui est formellement décrite par l'asymptotique

$$\mathbf{T}(z_0)V_{\alpha}(z) = \frac{\Delta(\alpha)V_{\alpha}(z)}{(z_0 - z)^2} + \frac{\partial_z V_{\alpha}(z)}{z_0 - z} + \text{termes holomorphes} \quad (1.3.5)$$

quand $z_0 \rightarrow z$. Cela implique en particulier que les fonctions de corrélation d'opérateur vertex sont solutions de l'identité de Ward :

$$\left\langle \mathbf{T}(z_0) \prod_{k=1}^N V_{\alpha_k}(z_k) \right\rangle = \sum_{l=1}^N \left(\frac{\Delta_{\alpha_l}}{(z_0 - z_l)^2} + \frac{\partial_{z_l}}{z_0 - z_l} \right) \left\langle \prod_{k=1}^N V_{\alpha_k}(z_k) \right\rangle. \quad (1.3.6)$$

La dimension conforme de l'opérateur vertex V_α est son *poids conforme* Δ_α qui est donné par $\Delta_\alpha := \frac{\alpha}{2} (Q - \frac{\alpha}{2})$. Dans cette expression la dérivée est une dérivée complexe – et ce sera le cas dans le reste du manuscrit – c'est-à-dire que $\partial_z f(x, y) = \frac{1}{2} (\partial_x - i\partial_y) f(x, y)$ tandis que $\partial_{\bar{z}}$ est définie de la même manière par $\partial_{\bar{z}} f(x, y) = \frac{1}{2} (\partial_x + i\partial_y) f(x, y)$. Dans le cadre de la théorie de Liouville cette identité de Ward a été rigoureusement prouvée dans [83] en se basant sur la définition probabiliste de ces objets présentée dans [40].

Combinée avec l'holomorphie à l'infini de ce tenseur, ce qui signifie que $\mathbf{T}(z) \sim \frac{1}{z^4}$ pour $z \rightarrow \infty$ et qui est un fait en général axiomatique, l'identité de Ward implique que les fonctions de corrélation satisfont trois *identités de Ward globales* :

$$\sum_{l=1}^N (z_l^n \partial_{z_l} + n z_l^{n-1} \Delta_{\alpha_l}) \langle \prod_{k=1}^N V_{\alpha_k}(z_k) \rangle = 0$$

pour $0 \leq n \leq 2$, traduisant le fait que les fonctions de corrélation sont covariantes conformes dans le sens où pour n'importe quelle transformation de Möbius ψ du plan,

$$\langle \prod_{k=1}^N V_{\alpha_k}(\psi(z_k)) \rangle = \prod_{l=1}^N |\psi'(z_l)|^{-2\Delta_{\alpha_l}} \langle \prod_{k=1}^N V_{\alpha_k}(z_k) \rangle.$$

1.3.1.2 Symétrie W et courants de spin supérieur

Pour des modèles possédant une symétrie supérieure comme c'est le cas pour les théories de Toda, l'algèbre de symétrie contient strictement l'algèbre de Virasoro. Ces extensions furent introduites peu de temps après l'article fondateur de BPZ [12] par Zamolodchikov [143] avant que des modèles possédant ces symétries ne soient mis au jour [54, 53].

Le même raisonnement basé sur des développements de produit d'opérateurs entre des courants holomorphes et des opérateurs vertex reste valide dans ce cadre, une spécificité étant que ce niveau supérieur de symétrie s'accompagne également de courants holomorphes supplémentaires : les *courants de spin supérieur* $\mathbf{W}^{(i)}$ (pour $3 \leq i \leq r+1$).

Dans cette thèse nous étudions plus spécifiquement la théorie de Toda associée à $\mathfrak{g} = \mathfrak{sl}_3$, auquel cas il y a un unique courant de spin 3 $\mathbf{W} := \mathbf{W}^{(3)}$ en plus du tenseur énergie-impulsion. Ce courant admet une expansion en série de Laurent de la forme

$$\mathbf{W}(z_0) = \sum_{n \in \mathbb{Z}} \frac{\mathbf{W}_n(z)}{(z_0 - z)^{n+3}}. \quad (1.3.7)$$

L'algèbre W_3 est alors l'algèbre vertex engendrée par la famille des $(\mathbf{L}_n, \mathbf{W}_m)_{n,m \in \mathbb{Z}}$ dont les relations de commutation sont données par

$$[\mathbf{L}_m, \mathbf{W}_n] = (2m - n) \mathbf{W}_{m+n}, \quad (1.3.8)$$

tandis que celle entre les modes W est plus compliquée et en fait bilinéaire en les $(\mathbf{L}_n)_{n \in \mathbb{Z}}$ (voir [25, Équation (2.1)]). En particulier l'algèbre W_3 n'est pas une algèbre de Lie. Nous ne nous appuyons pas ici sur cette expansion du courant mais le définirons comme une fonctionnelle du champ, dépendant d'un point $z \in \mathbb{C}$ et définie de manière analogue au tenseur énergie-impulsion. Toutefois et contrairement à celui-ci, il n'est pas compris à l'heure actuelle s'il est possible de décrire ce courant de spin supérieur d'une manière

analogue à l'équation (1.3.3) et ce problème est pour l'heure une question importante liée à l'interprétation géométrique de la W -symétrie (voir par exemple [135, Subsection 8.2] ou [57, Chapter 18.2]).

Le développement de produit d'opérateurs WV émerge lorsqu'un courant de spin supérieur est inséré dans une fonction de corrélation. Axiomatique en physique, ce développement est décrit par

$$\mathbf{W}(z_0)V_\alpha(z) = \frac{w(\alpha)V_\alpha(z)}{(z_0 - z)^3} + \frac{\mathbf{W}_{-1}V_\alpha(z)}{(z_0 - z)^2} + \frac{\mathbf{W}_{-2}V_\alpha(z)}{z_0 - z} + \text{termes holomorphes} \quad (1.3.9)$$

où $w(\alpha) \in \mathbb{C}$ est le *nombre quantique* attaché à $\mathbf{W}$, tandis que les $\mathbf{W}_{-i}V_\alpha(z)$ sont les *champs descendants*. Ces champs sont supposés locaux dans le sens où ils ne devraient dépendre que du poids α ainsi que du champ de Toda et de ses dérivées au point z . L'équation (1.3.9) peut se reformuler au sein d'une fonction de corrélation sous la forme de l'égalité

$$\langle \mathbf{W}(z_0) \prod_{k=1}^N V_{\alpha_k}(z_k) \rangle = \sum_{l=1}^N \left(\frac{w(\alpha_l)}{(z_0 - z_l)^3} + \frac{\mathbf{W}_{-1}^{(l)}}{(z_0 - z_l)^2} + \frac{\mathbf{W}_{-2}^{(l)}}{z_0 - z_l} \right) \langle \prod_{k=1}^N V_{\alpha_k}(z_k) \rangle, \quad (1.3.10)$$

où nous avons introduit la notation

$$\mathbf{W}_{-i}^{(l)} \langle \prod_{k=1}^N V_{\alpha_k}(z_k) \rangle \text{ pour } \langle \mathbf{W}_{-i}V_{\alpha_l}(z_l) \prod_{k \neq l} V_{\alpha_k}(z_k) \rangle.$$

Cette égalité est l'*identité de Ward locale de spin trois* et est propre à la symétrie W du modèle. Il est important de remarquer à ce stade qu'à la différence des identités de Ward associées au tenseur énergie-impulsion, les quantités correspondant aux champs descendants dans l'équation (1.3.10) ne sont pas des dérivées des fonctions de corrélation à proprement parler, ce qui ajoute un niveau de complexité important à l'étude des théories de Toda. Par exemple en général l'expression $\mathbf{W}_{-1}^{(l)} \langle \prod_{k=1}^N V_{\alpha_k}(z_k) \rangle$ ne s'exprime pas en terme de dérivées de la fonction de corrélation $\langle \prod_{k=1}^N V_{\alpha_k}(z_k) \rangle$ mais est donnée par

$$\mathbf{W}_{-1}^{(l)} \langle \prod_{k=1}^N V_{\alpha_k}(z_k) \rangle = \langle \langle u_{\alpha_l}, \partial \Phi(z_l) \rangle \prod_{k=1}^N V_{\alpha_k}(z_k) \rangle$$

où u_{α_l} est un vecteur de $\mathfrak{a}$.

Des *identités de Ward globales de spin trois* peuvent être déduites de cette identité en utilisant l'asymptotique formelle du tenseur quand z_0 diverge vers ∞ , donnée par $\mathbf{W}(z_0) \simeq \frac{1}{z_0^6}$. Elles sont données par les cinq égalités

$$\sum_{l=1}^N \left(z_l^n \mathbf{W}_{-2}^{(l)} + n z_l^{n-1} \mathbf{W}_{-1}^{(l)} + \frac{n(n-1)}{2} z_l^{n-2} w(\alpha_l) \right) \langle \prod_{k=1}^N V_{\alpha_k}(z_k) \rangle = 0 \quad (1.3.11)$$

pour $0 \leq n \leq 4$. Soulignons que contrairement aux identités de Ward globales associées au tenseur énergie-impulsion, il n'est pas déterminé à l'heure actuelle si les identités de Ward globales peuvent être obtenues directement en exploitant une certaine forme de covariance des fonctions de corrélation, ce qui les rend d'autant plus centrales dans la compréhension des symétries du modèle.

1.3.1.3 Identités de Ward pour les fonctions de corrélation probabilistes

Au cours du chapitre 5 nous utiliserons le cadre probabiliste construit précédemment pour donner un sens aux quantités introduites plus haut dans le cas où $\mathfrak{g} = \mathfrak{sl}_3$. Plus précisément nous exploiterons le fait que le tenseur de spin supérieur peut être représenté, de la même manière que l'est le tenseur énergie-impulsion, comme une fonctionnelle du champ de Toda via

$$\begin{aligned} \mathbf{W}(z)[\Phi] := & q^2 \langle \omega_2 - \omega_1, \partial^3 \Phi(z) \rangle \\ & + 2q (\langle e_1, \partial^2 \Phi(z) \rangle \langle \omega_1, \partial \Phi(z) \rangle - \langle e_2, \partial^2 \Phi(z) \rangle \langle \omega_2, \partial \Phi(z) \rangle) \\ & + 8 \langle \omega_1, \partial \Phi(z) \rangle \langle \omega_2 - \omega_1, \partial \Phi(z) \rangle \langle \omega_3, \partial \Phi(z) \rangle \end{aligned} \quad (1.3.12)$$

où $q := \gamma + \frac{2}{\gamma}$. Comme précédemment le manque de régularité du champ de Toda nécessite de considérer une procédure de régularisation pour donner un sens à cet objet.

En nous appuyant sur la définition du champ de Toda proposée équation (1.2.1) en prenant la fonctionnelle $F(\Phi)$ égale à $\mathbf{W}(z)[\Phi]$, nous montrerons que la limite suivante

$$\langle \mathbf{W}(z_0) \prod_{k=1}^N V_{\alpha_k}(z_k) \rangle := \lim_{\varepsilon \rightarrow 0} \langle \mathbf{W}_\varepsilon(z_0) \prod_{k=1}^N V_{\alpha_k, \varepsilon}(z_k) \rangle$$

est bien définie, où les quantités dénotées avec un indice ε sont des régularisations des fonctionnelles correspondantes et où le membre de droite est défini de manière probabiliste. De la même manière une procédure de régularisation permet de définir les quantités $\mathbf{W}_{-i}^{(l)} \langle \prod_{k=1}^N V_{\alpha_k}(z_k) \rangle$ en termes probabilistes. Ces objets définis nous prouvent que les identités de Ward sont valides dans l'environnement probabiliste en montrant l'énoncé suivant :

Théorème 1.3.1. *Supposons que $\mathfrak{g} = \mathfrak{sl}_3$. Sous les hypothèses du théorème 2.2.3 l'identité de Ward de spin trois est valide :*

$$\langle \mathbf{W}(z_0) \prod_{k=1}^N V_{\alpha_k}(z_k) \rangle = \sum_{l=1}^N \left(\frac{w(\alpha_l)}{(z_0 - z_l)^3} + \frac{\mathbf{W}_{-1}^{(l)}}{(z_0 - z_l)^2} + \frac{\mathbf{W}_{-2}^{(l)}}{z_0 - z_l} \right) \langle \prod_{k=1}^N V_{\alpha_k}(z_k) \rangle, \quad (1.3.13)$$

où le nombre quantique $w(\alpha)$ est explicite. Un résultat similaire s'applique au tenseur énergie-impulsion.

Nous prouvons également que les identités de Ward globales sont valides :

Théorème 1.3.2. *Sous les hypothèses du Théorème 1.3.1, pour $0 \leq n \leq 4$:*

$$\sum_{l=1}^N \left(z_l^n \mathbf{W}_{-2}^{(l)} + n z_l^{n-1} \mathbf{W}_{-1}^{(l)} + \frac{n(n-1)}{2} z_l^{n-2} w(\alpha_l) \right) \langle \prod_{k=1}^N V_{\alpha_k}(z_k) \rangle = 0. \quad (1.3.14)$$

Cela est une confirmation supplémentaire que les fonctions de corrélation probabilistes fournissent la bonne manière de donner un sens aux objets considérés en physique. Ces identités sont cruciales dans la compréhension des symétries des théories conformes des champs de Toda et seront au cœur de la preuve de la formule de Fateev-Litvinov.

La preuve de l'identité de Ward décrite par l'équation 1.3.13 s'appuie très fortement sur la structure gaussienne sous-jacente dans la définition probabiliste des théories de Toda. En effet un des outils principaux permettant d'établir la validité du Theorème 1.3.1 est l'*intégration par parties gaussiennes*, correspondant en quelque sorte au calcul de Malliavin pour le champ de Toda et qui est en fait formellement vérifiable sur l'intégrale de chemin.

1.3.2 . Symétrie W et un principe de réflexion

Une condition nécessaire pour que les fonctions de corrélation probabilistes soient bien définies est de supposer que les poids appartiennent au sous-ensemble $Q + \mathcal{C}_-$ de $\mathfrak{a}$ défini par la condition que $\langle \alpha - Q, e_i \rangle < 0$ pour tout $1 \leq i \leq r$. Afin de pallier cette limitation il est supposé en physique qu'il existe une famille de transformations s de $\mathfrak{a}$ envoyant $\mathcal{C}_-$ sur $\mathfrak{a}$ tout entier ainsi que des scalaires $R_s(\alpha)$ tels que les opérateurs vertex V_α et $V_{Q+s(\alpha-Q)}$ soient reliés entre eux par la relation

$$V_\alpha = R_s(\alpha) V_{Q+s(\alpha-Q)}.$$

Ces coefficients sont appelés *coefficients de réflexion de Toda*, tandis que l'ensemble de telles transformations forme en fait un groupe de réflexion appelé le *groupe de Weyl* de $\mathfrak{g}$. Cette hypothèse est axiomatique en physique et vient du fait que les poids et nombres quantiques Δ_α et $w(\alpha)$ sont laissés inchangés par de telles transformations. Ces transformations sont illustrées plus-bas en 1.2 où est également représentée la chambre de Weyl $\mathcal{C}_-$ dans le cas où $\mathfrak{g} = \mathfrak{sl}_3$.

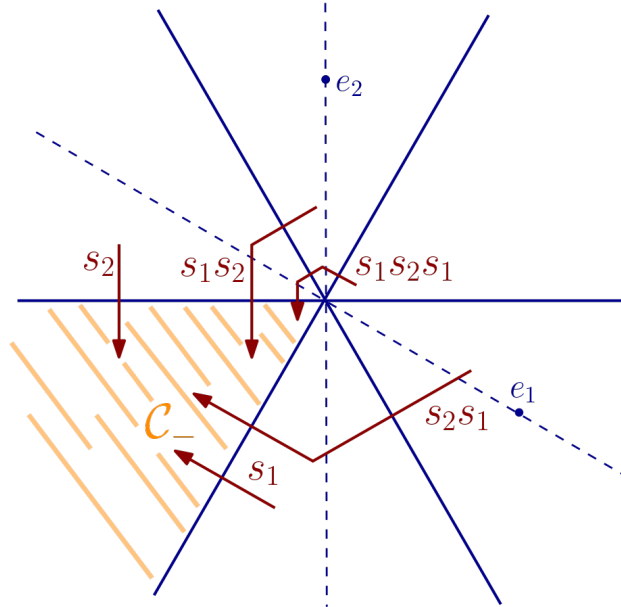


Figure 1.2: Le groupe de Weyl associé à $\mathfrak{g} = \mathfrak{sl}_3$

Dans le cas de la théorie de Liouville, ce groupe de réflexion est constitué uniquement de deux éléments Id and $-\text{Id}$, si bien que la relation de réflexion prend la forme $V_\alpha = R(\alpha) V_{2Q-\alpha}$ où $R(\alpha)$ est le coefficient de réflexion de Liouville.

Inspirés par l'existence d'une telle relation nous explorerons ici certaines connections entre ce phénomène de réflexion et le principe de réflexion en probabilité. Nous mon-

trons en particulier que les coefficients de réflexion apparaissent dans la queue de distribution de variables aléatoires liées à des mesures de chaos multiplicatif gaussien. Le chapitre 6 détaillera ces résultats, établis dans [31]. Nous ne justifierons pas dans ce chapitre la relation de réflexion $V_\alpha = R_s(\alpha)V_{Q+s(\alpha-Q)}$ mais le feront dans le cas où $\mathfrak{g} = \mathfrak{sl}_3$ pour une famille de fonctions de corrélation à trois points au cours du chapitre 7.

1.3.2.1 Principe de réflexion et une décomposition de chemin brownien

Dans un célèbre article [138], Williams en 1974 décrit une décomposition de chemin pour le mouvement brownien et plus généralement pour toute une classe de diffusions unidimensionnelles. Dans sa formulation la plus élémentaire, le processus considéré est un mouvement brownien avec coefficient de dérive positif ν (nous notons $(B_t^\nu)_{t \geq 0} = (B_t + \nu t)_{t \geq 0}$ un tel processus), cette décomposition peut se formuler de la manière suivante : conditionnellement en la valeur du minimum global $\mathbf{M} := \inf_{t \geq 0} B_t^\nu$ du processus, la loi de B^ν (sachant $\mathbf{M}$) n'est plus un processus de Markov mais peut être réalisée en mettant bout à bout deux processus de Markov. Plus spécifiquement, le premier processus a la loi de $B^{-\nu}$ jusqu'à atteindre $\mathbf{M}$, puis le second a la loi du processus de diffusion B^ν qui est celle de B^ν partant de $\mathbf{M}$ et conditionné à rester au-dessus de $\mathbf{M}$.

Suite à sa découverte par Williams, cette décomposition de chemin a été l'objet d'une étude approfondie au sein de la communauté probabiliste et a été à l'origine de nombreux résultats fondamentaux, tels le théorème de Pitman [110]. Étendre cette décomposition pour différentes classes de processus a conduit également au développement d'un champ de recherche [17, 35, 78, 27]. Nous renvoyons vers [87] pour plus d'informations en ce sens. Toutefois il ne semblerait pas exister de formulation de ce résultat pour des diffusions en n'importe quelle dimension.

En utilisant une généralisation du principe de réflexion, nous prouverons que la notion de groupe de réflexion permet d'étendre cette décomposition de chemin à toute une classe de diffusions en dimension quelconque. En utilisant les notations introduites dans le cadre des théories de Toda, les variables aléatoires $\mathbf{M}_i := \inf_{t \geq 0} \langle B_t^\nu, e_i \rangle$ jouent ici le rôle de minimums du mouvement brownien avec coefficient de dérive $\nu \in \mathcal{C}$ B^ν , et peuvent s'interpréter simplement en notant que le point $\mathbf{M} := \sum_{i=1}^r M_i \omega_i^\vee$ correspond au sommet de la plus petite chambre de Weyl contenant l'ensemble de la trajectoire du processus B^ν . Dans ce cadre général la décomposition contiendra $r + 1$ composantes en lieu et place de 2 dans la décomposition de chemin de Williams et prendra la forme suivante :

Théorème 1.3.3. *Soit B^ν un mouvement brownien avec coefficient de dérive $\nu \in \mathcal{C}$ évoluant dans une espace euclidien sur lequel agit un groupe de réflexion de rang r . Alors existent $r + 1$ processus de diffusion $\mathbf{X}^1, \dots, \mathbf{X}^{r+1}$ pour lesquels :*

1. *Tirons les $\mathbf{M}_i$ selon leur loi marginale et définissons un processus $\mathbf{X}$ comme la jonction des processus suivants :*
2.
 - *Tout d'abord un processus de diffusion $\mathbf{X}^1$ indépendant des $(\mathbf{M}_i)_{1 \leq i \leq r}$ lancé depuis l'origine et jusqu'à atteindre $\mathbf{M} + \partial\mathcal{C}$, par exemple en $z_1 \in \mathbf{M} + \partial\mathcal{C}_1$.*

- Un deuxième processus indépendant X^2 partant de z_1 jusqu'à toucher $M + \partial\mathcal{C}$.
- Définir ainsi une famille $(X^1, \dots, X^r)$ de processus. Lorsque X^r touche le bord $M + \partial\mathcal{C}$, lancer un processus final X^{r+1} .

3. Alors le processus X a la loi de B^ν .

Considérer une telle notion de minimum est tout à fait naturel du point de vue de l'étude probabiliste des théories de Toda. Cette notion de minimum est également adoptée par exemple dans l'étude du mouvement brownien dans un cône comme c'est le cas dans l'article [89], où le processus $(B_t^\nu - M_t)_{t \geq 0}$ est considéré avec $M_t = \sum_{i=1}^r \omega_i^\vee \inf_{s \leq t} \langle B_s^\nu, e_i \rangle$ en lieu et place de $(B_t^\nu - M_{+\infty})_{t \geq 0}$ dans la décomposition que nous proposons. Nous tenions à souligner que nous trouvons tout à fait remarquable que des considérations *a priori* liées à l'étude de la théorie conforme des champs aient en fait des implications purement probabilistes.

Les notations apparaissant dans ce théorème seront établies dans la section 3.1, mais la preuve et l'énoncé de ce résultat seront établies section 6.1. Dans le cas où le groupe de réflexion considéré correspond à A_2 (ou $\mathfrak{sl}_3$) nous reproduisons ici une illustration de cette décomposition de chemin.

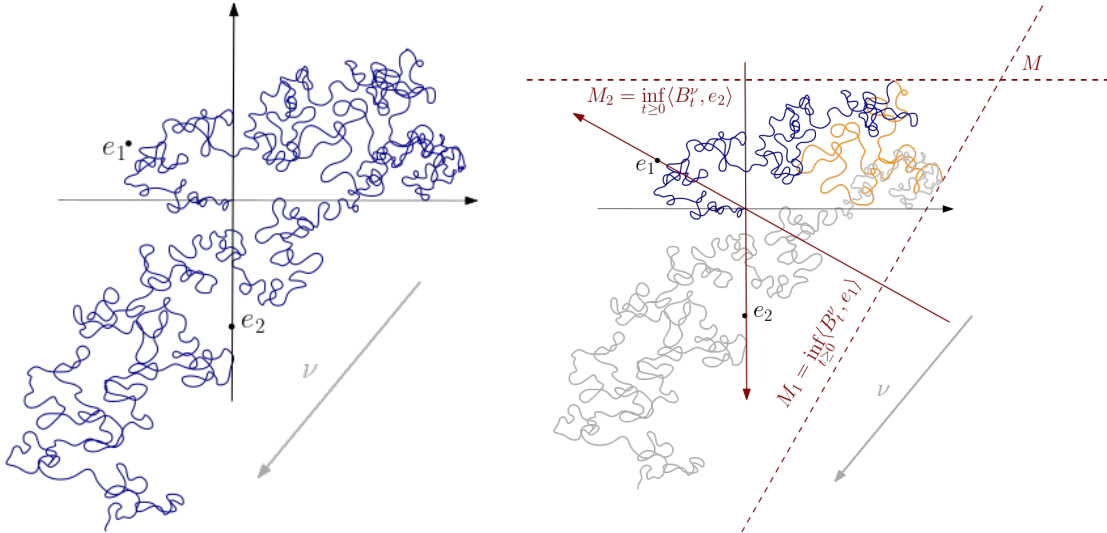


Figure 1.3: Mouvement brownien Figure 1.4: Décomposition de planaire de coefficient de dérive ν chemin associé à A_2

Au cours de la section suivante, section 6.2, nous étudierons certaines propriétés du processus défini par le théorème 1.3.3. Plus précisément nous étudierons la loi de ce processus lorsque son point d'entrée diverge, ce qui correspond à conditionner le processus N^ν à avoir un minimum M tel que $\langle M, e_i \rangle \rightarrow -\infty$ pour tout $1 \leq i \leq r$. Nous verrons que dans cette asymptotique le processus conditionné ressemblera vraiment à la jonction de $r + 1$ mouvements browniens avec terme de dérive et conditionnés à rester dans un sous-domaine de α .

1.3.2.2 Coefficients de réflexion et queue de distribution de mesures de chaos multiplicatif gaussien

Grâce à la décomposition de chemin décrite dans la section 6.1, nous décrirons au cours des sections 6.3 et 6.4 quelques-unes de ses conséquences sur le chaos multiplicatif gaussien ainsi que pour les fonctions de corrélation de Toda. Plus précisément nous montrerons que conditionner des mesures de chaos multiplicatifs gaussiens corrélés à être larges revient à conditionner un mouvement brownien sur $\mathfrak{a}$ avec terme de dérive à ce que son minimum $\mathbf{M}$ satisfasse $\langle \mathbf{M}, e_i \rangle \rightarrow -\infty$ pour tout $1 \leq i \leq r$, si bien que nous pourrions utiliser la décomposition de chemin de 1.3.3. Ce qui nous permettra par conséquent de décrire la queue de distribution de ces mesures mais également de quantifier certaines asymptotiques des fonctions de corrélation de Toda.

Pour ce faire nous utiliserons le fait que si $\mathbf{X}$ est un champ libre gaussien sur le disque $\mathbb{D}$ dont le noyau de covariance est donné par

$$\mathbb{E}[\langle u, \mathbf{X}(x) \rangle \langle v, \mathbf{X}(y) \rangle] = \langle u, v \rangle \ln \frac{1}{|x - y|}$$

alors le processus $(\mathbf{X}_t)_{t \geq 0}$ défini en moyennant le champ libre sur des cercles de rayons e^{-t}

$$\mathbf{X}_t := \frac{1}{2\pi} \int_0^{2\pi} \mathbf{X}(e^{-t+i\theta}) d\theta$$

est un mouvement brownien sur $\mathfrak{a}$. Ce qui implique que les r mesures de chaos multiplicatifs gaussiens corrélés formellement définies par

$$I_i(\alpha) = \int_{\mathbb{D}} |x|^{-\gamma \langle \alpha, e_i \rangle} e^{\langle \gamma e_i, \mathbf{X}(x) \rangle} (d^2 x) \quad \text{for } i = 1, \dots, r$$

admettent une autre expression sous la forme de

$$I_i(\alpha) = \int_0^{+\infty} e^{\langle \gamma e_i, \mathbf{X}_t + (\alpha - Q)t \rangle} Z_t^i dt$$

où $Z_t^i dt$ est une mesure aléatoire sur $\mathbb{R}^+$ indépendante de $(\mathbf{X}_t)_{t \geq 0}$, provenant de la partie angulaire du champ libre. Plus de détails sont donnés en section 3.3.

Nous montrerons dans la Section 6.3 que la probabilité que les intégrales $I_i(\alpha)$ soient grandes est gouverné par la probabilité que le minimum du processus $\left(-\mathbf{X}_t - \frac{\alpha - Q}{\gamma} t\right)_{t \geq 0}$ (qui n'est rien d'autre qu'un mouvement brownien avec coefficient de dérive $\frac{Q - \alpha}{\gamma} \in \mathcal{C}$) soit faible. Ce faisant nous pouvons nous ramener à l'étude des objets considérés dans le Théorème 1.3.3 et obtenir le résultat suivant :

Théorème 1.3.4. *Pour $\alpha \in Q + \mathcal{C}_-$ suffisamment proche de Q , il existe un réel $R_s(\alpha)$ tel que*

$$\mathbb{E} \left[\prod_{k=1}^r \left(\exp \left(-e^{\gamma \langle c, e_{i_k} \rangle} I_{i_k}(\alpha) \right) - 1 \right) \right] \sim e^{\langle s(\alpha - Q) + Q - \alpha, c \rangle} R_s(\alpha) \quad (1.3.15)$$

lorsque $\langle c, e_i \rangle \rightarrow -\infty$ pour tous $1 \leq i \leq r$ selon une certaine asymptotique décrite en section 6.3. $R_s(\alpha)$ est un coefficient de réflexion de Toda et est égal à

$$R_s(\alpha) = \epsilon(s) \frac{A(s(\alpha - Q))}{A(\alpha - Q)}, \quad \text{où}$$

$$A(\alpha) = \prod_{i=1}^r \left(\mu_i \pi l \left(\frac{\gamma^2 \langle e_i, e_i \rangle}{4} \right) \right)^{\frac{\langle \alpha, \omega_i^\vee \rangle}{\gamma}} \prod_{e \in \Phi^+} \Gamma \left(1 - \frac{\gamma}{2} \langle \alpha, e \rangle \right) \Gamma \left(1 - \frac{1}{\gamma} \langle \alpha, e^\vee \rangle \right). \quad (1.3.16)$$

Dans cette équation $\Phi^+ := \{e_1, e_2, \rho\}$ est l'ensemble des racines positives tandis que $\epsilon(s)$ est la signature de la permutation associée à s (voir Section 3.1).

L'expression de ces coefficients de réflexion est en accord avec celle proposée en physique pour les coefficients de réflexion associés aux théories de Toda [3, 2, 51], soulignant encore un peu plus les liens profonds entre le principe de réflexion en théories des probabilités et le phénomène de réflexion en théorie conforme des champs. Dans le cas spécifique où $r = 1$ nous retrouvons le résultat [11, Proposition 4.1], où cette asymptotique permet de décrire la matrice de scattering associée à l'Hamiltonien de Liouville.

1.3.2.3 Queue de distribution du chaos multiplicatif gaussien et prolongement analytique des fonctions de corrélation

Puisque les fonctions de corrélation probabilistes sont définies en termes de mesures de chaos multiplicatif gaussien corrélés, l'énoncé précédent permet de décrire les propriétés de l'espérance apparaissant dans l'équation (1.2.1) dans l'asymptotique où $\langle c, e_i \rangle \rightarrow -\infty$. Cette asymptotique sera décrite en Proposition 6.4.2 sous une forme similaire à celle apparaissant dans l'énoncé du Théorème 1.3.4 en terme de coefficients de réflexion.

En se basant sur cette asymptotique nous proposerons au cours de la section 6.4 un prolongement analytique des fonctions de corrélation au-delà des conditions imposées par le théorème 1.2.1. Dans cette perspective nous montrerons que :

Théorème 1.3.5. Pour $\mathfrak{g} = \mathfrak{sl}_3$, considérons $z_1, \dots, z_N \in \mathbb{C}$ distincts et définissons un sous-ensemble de $(Q + \mathcal{C}_-)^N$ par

$$\mathcal{A}_N := \left\{ (\alpha_1, \dots, \alpha_N) \in (Q + \mathcal{C}_-)^N \text{ tel que pour } i = 1, 2, \quad \left\langle \sum_{k=1}^N \alpha_k - 2Q, \omega_i \right\rangle > -\gamma \right.$$

$$\left. \text{et pour tout } 1 \leq k \leq N, \quad \min_{i=1,2} \langle \alpha_k - Q, e_i \rangle < -\gamma \right\}. \quad (1.3.17)$$

Alors les fonctions de corrélation admettent une représentation probabiliste pour $\alpha \in \mathcal{A}_N$ que nous notons toujours par $\langle \prod_{k=1}^N V_{\alpha_k}(z_k) \rangle$. Cette extension est telle que l'application $\alpha \mapsto \langle \prod_{k=1}^N V_{\alpha_k}(z_k) \rangle$ est méromorphe dans un voisinage complexe de $\mathcal{A}_N$.

La condition imposée sur les poids correspond à l'hypothèse sous laquelle il est possible de décrire les asymptotiques dans la proposition 6.4.2. Les pôles et résidus de cette fonction méromorphe sur $\mathcal{A}_N$ sont aisément calculables. En se basant sur le raisonnement développé pour prouver ce résultat il est possible de représenter probabilistiquement la formule DOZZ pour un ensemble de valeurs des poids plus étendu comparé aux résultats de [84]. Nous renvoyons à [30, Corollary 1.3] pour plus de détails concernant cette extension du domaine de représentation probabiliste de la formule DOZZ.

1.4 . Calcul d'une famille de fonctions de corrélation dans la théorie de Toda associée à $\mathfrak{sl}_3$

Dans la dernière partie de ce manuscrit nous rassemblerons les éléments établis au cours des chapitres précédents dans la perspective de prouver un des résultats les plus importants de cette thèse : une preuve rigoureuse de la formule de Fateev-Litvinov pour une famille de fonctions de corrélation à trois points de la théorie de Toda associée à $\mathfrak{sl}_3$.

Implémenter la procédure du bootstrap dans ce cadre permet en effet de calculer des fonctions de corrélation à trois-points sous l'hypothèse que l'un des trois opérateurs vertex considérés est un *champ semi-dégénéré*, ce qui signifie que le poids correspondant α^* est de la forme $\alpha^* = \kappa\omega_i$ où $\kappa \in \mathbb{R}$ et pour $i = 1$ ou $i = 2$. Si une telle hypothèse n'est plus valide les fonctions de corrélation à trois-points ne sont pas encore connues (il existe cependant une proposition [101, 74] basée sur la correspondance AGT mais toujours non explicite).

En combinant des méthodes inspirées de la physique avec d'autres plus probabilistes nous prouverons Chapitre 7 que la formule de Fateev-Litvinov est vérifiée pour ces fonctions de corrélation à trois points définies de manière probabiliste:

Théorème 1.4.1. *Supposons que la constante de couplage satisfait $\gamma \in [1, \sqrt{2})$ et que $V_{\alpha_2^*}$ est un champ semi-dégénéré. Alors dès que $(\alpha_1, \alpha_2^*, \alpha_3)$ appartient à $\mathcal{A}_3$*

$$C_\gamma(\alpha_1, \alpha_2^*, \alpha_3) = C_\gamma^{\text{FL}}(\alpha_1, \alpha_2^*, \alpha_3) \quad (1.4.1)$$

où le terme de gauche est une fonction de corrélation probabiliste.

Ce résultat est prouvé dans l'article [30]. La formule de Fateev-Litvinov [52] décrivant ces fonctions de corrélation à trois-points est définie en termes de fonctions spéciales, Υ and l :

$$C_\gamma^{\text{FL}}(\alpha_1, \alpha_2^*, \alpha_3) = \left(\pi \mu l \left(\frac{\gamma^2}{2} \right) \left(\frac{\gamma}{\sqrt{2}} \right)^{2-\gamma^2} \right)^{\frac{\langle 2Q - \alpha_1 - \alpha_2^* - \alpha_3, \rho \rangle}{\gamma}} \frac{\Upsilon'(0)^2 \Upsilon(\kappa) \prod_{e \in \Phi^+} \Upsilon(\langle Q - \alpha_1, e \rangle) \Upsilon(\langle Q - \alpha_3, e \rangle)}{\prod_{1 \leq j, k \leq 3} \Upsilon \left(\frac{\kappa}{3} + \langle \alpha_1 - Q, h_j \rangle + \langle \alpha_3 - Q, h_k \rangle \right)} \quad (1.4.2)$$

où $\alpha_2^* = \kappa\omega_2$. Plus de détails seront apportés Section 7.3. Cette formule est réminiscente de la formule de DOZZ qui peut en fait être retrouvée à partir de l'équation (1.4.2).

Nous nous attacherons également à expliquer quelle forme prend le bootstrap conforme dans ce cadre. En particulier nous verrons comment le calcul d'une famille de fonctions de corrélation à quatre points peut se réduire à celui des fonctions de corrélation à trois points décrites plus haut, fournissant ainsi l'ingrédient finale de la procédure du bootstrap conforme pour la théorie de Toda associée à $\mathfrak{sl}_3$.

1.4.1 . Fonctions de corrélation à trois points et la formule de Fateev-Litvinov

Afin de prouver notre résultat principal, Théorème 1.4.1, nous devons en fait étudier certaines fonctions de corrélation à quatre-points. La compréhension de ces fonctions à quatre points est rendue possible en ce qu'elles contiennent beaucoup d'informations en lien avec les symétries du modèle. Leur étude permettra en retour de retrouver les fonctions de corrélation à trois points du Théorème 1.4.1 en étudiant certaines de leurs asymptotiques, ce qui dans le langage de la physique revient à calculer des *développements de produits d'opérateurs*.

1.4.1.1 Une équation différentielle hypergéométrique

Les fonctions de corrélation à quatre points que nous considérerons correspondent aux fonctions à trois points de l'énoncé du Théorème 1.4.1, auxquelles sera ajouté un opérateur vertex supplémentaire V_α . Cet opérateur vertex supplémentaire est un *champ complètement dégénéré*, ce qui signifie que le poids α est de la forme $\alpha = -\chi\omega_1$ avec $\chi \in \{\gamma, \frac{2}{\gamma}\}$. La raison conduisant à faire une telle hypothèse sur le poids α provient du fait que la présence de ce champ au sein de fonctions de corrélation révèle des contraintes supplémentaires sur celles-ci. Au cours de la section 7.1 nous montrerons qu'en combinant ces contraintes supplémentaires avec celles provenant de l'hypothèse que $V_{\alpha_2^*}$ est semi-dégénéré mais également avec les identités de Ward globales (1.3.11) nous pouvons calculer ces fonctions de corrélation de manière complètement explicite à un facteur multiplicatif près :

Théorème 1.4.2. *Considérons un champ complètement dégénéré V_α ainsi qu'un champ semi-dégénéré $V_{\alpha_2^*}$. Considérons également des poids α_1 et α_3 tels que $(\alpha, \alpha_1, \alpha_2^*, \alpha_3)$ appartienne à $\mathcal{A}_4$. Sous ces hypothèses*

$$\begin{aligned} \langle V_\alpha(z) V_{\alpha_1}(0) V_{\alpha_2^*}(1) V_{\alpha_3}(\infty) \rangle &= |z|^{-\langle \alpha, \alpha_0 \rangle} |z - 1|^{-\langle \alpha, \alpha_2^* \rangle} \mathcal{H}(z), \quad \text{où} \\ \mathcal{H}(z) &= C_\gamma(\alpha_1 + \alpha, \alpha_2^*, \alpha_3) \left(|\mathcal{H}_0(z)|^2 + \sum_{i=1}^2 A_\gamma^{(i)}(\alpha, \alpha_1, \alpha_2^*, \alpha_3) |\mathcal{H}_i(z)|^2 \right). \end{aligned} \quad (1.4.3)$$

Les constantes $A_\gamma^{(i)}(\alpha, \alpha_1, \alpha_2^, \alpha_3)$, $i = 1, 2$, sont explicites et données par un produit de fonctions Γ , tandis que les blocs $(\mathcal{H}_i)_{0 \leq i \leq 2}$ sont des fonctions hypergéométriques ${}_3F_2$. La seule indéterminée dans cette expression est la fonction de corrélation à trois points $C_\gamma(\alpha_1 + \alpha, \alpha_2^*, \alpha_3)$.*

La preuve de ce résultat est basée sur le fait que de telles fonctions de corrélation à quatre points sont solutions d'une équation hypergéométrique du troisième ordre dans la variable z , communément appelées équation différentielle du type BPZ, et dont les solutions sont connues. Le fait que les coefficients $A_\gamma^{(i)}(\alpha, \alpha_1, \alpha_2^*, \alpha_3)$, $i = 1, 2$, soient

connus vient du fait que les solutions de cette équation admettent plusieurs expansions (pour z autour de 0, 1 ou ∞) qui doivent être cohérentes entre elles, correspondant à l'hypothèse de la *symétrie de croisement* faite en physique. Prouver cet énoncé s'appuie très fortement sur les symétries du modèle – en particulier les identités de Ward globales – mais également sur la théorie des représentations des W -algèbres de par l'existence de ces champs dégénérés.

1.4.1.2 Des fonctions de corrélation à quatre points à celles à trois points

La représentation probabiliste des fonctions de corrélation nous permet de donner une expression alternative des ces fonctions de corrélation à quatre points. En effet en étudiant le comportement des fonctions de corrélation lorsque $z \rightarrow 0$ (ce qui revient en physique à considérer le développement de produit d'opérateurs $V_\alpha(z)V_{\alpha_1}(0)$) nous pouvons décrire d'une manière différente les coefficients apparaissant dans l'expansion (1.4.3). Pour décrire cette asymptotique nous nous appuyons notamment sur certaines propriétés du chaos multiplicatif gaussien, en particulier sa queue de distribution décrite précédemment, et repose presque intégralement sur des considérations liées à celle-ci.

En combinant l'approche probabiliste pour ces fonctions de corrélation avec la forme *a priori* de celles-ci données par le théorème 1.4.2 nous montrerons Section 7.2 que :

Théorème 1.4.3. *Sous les hypothèses du théorème 1.4.2,*

$$\langle V_\alpha(z)V_{\alpha_1}(0)V_{\alpha_2^*}(1)V_{\alpha_3}(\infty) \rangle = |z|^{-\langle \alpha, \alpha_0 \rangle} |z-1|^{-\langle \alpha, \alpha_2^* \rangle} \mathcal{H}(z), \quad \text{où} \\ \mathcal{H}(z) = \sum_{i=0}^2 B_\gamma^{(i)}(\alpha_1, \chi) C_\gamma \left(\alpha_1 - \chi \left(\omega_1 - \sum_{j=1}^i e_j \right), \alpha_2^*, \alpha_3 \right) |\mathcal{H}_i(z)|^2. \quad (1.4.4)$$

Les coefficients $B_\gamma^{(i)}(\alpha_1, \chi)$ admettent une expression explicite en termes de la fonction Γ , tandis que les fonctions de corrélation à trois points qui y apparaissent sont étendues lorsque $\alpha_1 - \chi h_{i+1}$ est hors de $Q + \mathcal{C}_-$ par la relation

$$C_\gamma(\alpha_1, \alpha_2, \alpha_3) := R_s(\alpha_1) C_\gamma(Q + s(\alpha_1 - Q), \alpha_2, \alpha_3) \\ \text{où } s \text{ est telle que } s(\alpha_1 - Q) \in \mathcal{C}_-. \quad (1.4.5)$$

Une conséquence de cet énoncé est une justification de la relation de réflexion $V_\alpha = R_s(\alpha)V_{Q+s(\alpha-Q)}$ lorsque celle-ci est insérée dans les fonctions de corrélation considérées Équation (1.4.5).

Par contraste avec la preuve du théorème 1.4.2 qui s'appuie sur les contraintes imposées par la symétrie, la preuve de ce résultat est bien plus probabiliste en cela qu'elle s'appuie principalement sur l'étude de chaos multiplicatif gaussien corrélés – bien qu'elle nécessite de connaître la forme *a priori* de ces fonctions de corrélation.

1.4.1.3 Conclusion de la preuve du théorème 1.4.1

Nous pouvons désormais comparer les deux expansions obtenues à partir des deux théorèmes 1.4.2 et 1.4.3. Les coefficients qui apparaissent en face des fonctions hy-

pergéométriques dans ces expansions sont donnés par

$$C_\gamma(\alpha_1 - \chi\omega_1, \alpha_2^*, \alpha_3) A_\gamma^{(i)}(-\chi\omega_1, \alpha_1, \alpha_2^*, \alpha_3) \quad \text{pour le Théorème 1.4.2 ou}$$

$$C_\gamma\left(\alpha_1 - \chi\left(\omega_1 - \sum_{j=1}^i e_j\right), \alpha_2^*, \alpha_3\right) B^{(i)}(\alpha_1, \chi) \quad \text{pour le Théorème 1.4.3.}$$

Les deux expressions proposées devant être égales nous pouvons en déduire que les fonctions de corrélation à trois points sont solutions des deux équations suivantes,

$$\frac{C_\gamma\left(\alpha_1 - \chi\left(\omega_1 - \sum_{j=1}^i e_j\right), \alpha_2^*, \alpha_3\right)}{C_\gamma(\alpha_1 - \chi\omega_1, \alpha_2^*, \alpha_3)} = \frac{A_\gamma^{(i)}(-\chi\omega_1, \alpha_1, \alpha_2^*, \alpha_3)}{B^{(i)}(\alpha_1, \chi)} \quad (1.4.6)$$

valides pour $i = 1, 2$ et $\chi \in \{\gamma, \frac{2}{\gamma}\}$, et dès que la fonction de corrélation

$$\langle V_{-\chi\omega_1}(z) V_{\alpha_1}(0) V_{\alpha_2^*}(1) V_{\alpha_3}(\infty) \rangle$$

fait sens. Cette hypothèse dans le cas où $\chi = \frac{2}{\gamma}$ impose la borne $\gamma > 1$ et correspond au fait que l'équation (1.4.1) n'est prouvée que lorsque $\gamma \in [1, \sqrt{2})$. Nous verrons Section 7.3 que cette paire d'équations caractérise, à une constante multiplicative près, l'expression proposée dans [52] pour les fonctions de corrélation à trois points de Toda. Ce facteur multiplicatif peut être évalué, ce qui montre ainsi que le théorème 1.4.1 est valide.

1.4.2 . Perspectives : fonctions de corrélation à quatre points et bootstrap conforme

Le calcul d'une famille de fonctions de corrélation à trois points représente le point de départ dans l'implémentation du bootstrap conforme et représente en quelque sorte l'étape d'initialisation dans la procédure récursive liée au bootstrap conforme. Dans un travail en cours avec Guillarmou [32] nous avons pour objectif de montrer que celle-ci permet de donner une expression pour les fonctions de corrélation contenant plus d'insertions. Pour ce faire nous souhaitons montrer un résultat qui prendrait la forme suivante :

Théorème 1.4.4 (En préparation). *Supposons que les opérateurs vertex $V_{\alpha_2^*}$ et $V_{\alpha_3^*}$ sont tous deux des champs semi-dégénérés. Sous l'hypothèse supplémentaire que pour $i = 1, 2$:*

$$\begin{aligned} \langle \alpha_k - Q, e_i \rangle &< 0 \quad \text{pour tout } 1 \leq k \leq 4, \quad \text{avec} \\ \langle \alpha_1 + \alpha_2 - Q, \omega_i \rangle &> 0 \quad \text{et} \quad \langle \alpha_3 + \alpha_4 - Q, \omega_i \rangle > 0, \end{aligned} \quad (1.4.7)$$

on a :

$$\begin{aligned} &\langle V_{\alpha_1}(0) V_{\alpha_2^*}(z) V_{\alpha_3^*}(1) V_{\alpha_4}(\infty) \rangle \\ &= \frac{1}{24\pi^2} \int_{\mathbb{R}^2} C_\gamma(\alpha_1, \alpha_2^*, Q - iP) C_\gamma(Q + iP, \alpha_3^*, \alpha_4) |z|^{2(\Delta_{Q+iP} - \Delta_{\alpha_1} - \Delta_{\alpha_2})} |\mathcal{F}_P^T(z)|^2 dP \end{aligned} \quad (1.4.8)$$

où les blocs conformes $\mathcal{F}_P^T$ ne dépendent que de la charge centrale de la théorie ainsi que des quantités Δ_{α_k} et $w(\alpha_k)$ pour $1 \leq k \leq 4$.

Il convient de noter que tandis que le spectre dans la théorie de Liouville est donné par la demi-droite $Q + i\mathbb{R}_+$, celui-ci est compris dans $Q + i\mathbb{R}^2$ pour la théorie de Toda associée à $\mathfrak{sl}_3$. Nous devrions montrer également un tel résultat en ne supposant plus que deux des champs sont semi-dégénérés. Dans ce cas plus général les fonctions de corrélation à trois points apparaissant plus haut sont définies par prolongement analytique de l'expression probabiliste données pour elles, tandis que les blocs conformes possèdent une structure bien plus complexe dans ce cas-là.

Les blocs conformes $\mathcal{F}_P^T$ qui apparaissent dans la formule du bootstrap possèdent un contenu très riche en cela qu'ils interviennent dans de nombreux cadres. Tout d'abord dans le cadre de la théorie des représentations puisque leur expression (au moins au niveau formel) fait intervenir des quantités fondamentales comme le déterminant de Shapovalov. Ces blocs sont également présents dans la formulation générale de la conjecture AGT [4, 140], où ils sont reliés à la fonction de partition de Nekrasov, une quantité fondamentale dans l'étude de certaines théories de jauge en dimension 4. De manière plus géométrique ces blocs conformes ont des liens avec la quantification d'espace de modules de connexions plates [38].

La preuve d'un tel résultat repose sur la considération de l'*Hamiltonien de Toda*, un opérateur auto-adjoint agissant sur un espace L^2 de fonctions, comme c'est également le cas pour la formule du bootstrap (1.1.6) pour Liouville [66]. Son étude fait appel à de nombreuses notions, de la théorie des représentations des W -algèbres à celle du scattering dans des espaces symétriques. Nous ne fournirons pas la preuve intégrale de ce résultat dans ce manuscrit, d'une part par souci de concision et d'autre part afin de ne pas faire appel à des notions trop disparates, mais également car il s'agit pour certaines parties d'un travail en cours ; nous nous attacherons à décrire le cadre probabiliste au sein duquel une telle formule peut être montrée en donnant les étapes les plus importantes de sa preuve. Nous expliquerons notamment comment il est possible de construire une représentation de la W -algèbre associée à $\mathfrak{sl}_3$ – la représentation de Segal-Sugawara – en nous basant dans un premier temps sur la théorie du champ libre sous-jacente à la théorie de Toda. La théorie du champ libre est cruciale en cela qu'elle contient bon nombre de méthodes qui seront réemployées par la suite dans l'étude de la théorie avec terme d'interaction. Nous expliquerons également comment donner un sens au développement formel du tenseur $\mathbf{W}$ introduit précédemment

$$\mathbf{W}(z_0) = \sum_{n \in \mathbb{Z}} \frac{\mathbf{W}_n(z)}{(z_0 - z)^{n+3}}$$

et proposerons une approche dynamique de l'Hamiltonien de Toda qui permettrait, combinée avec une étude fine du scattering induit par cet opérateur, d'accéder à sa résolution spectrale. C'est le contenu du chapitre 8.

2 - Introduction - Anglais

Toda conformal field theories form a family of two-dimensional quantum field theories indexed by semisimple and complex Lie algebras. One of their remarkable features is that they enjoy, in addition to conformal invariance, an enhanced level of symmetry referred to as *W*– or *higher-spin symmetry*. The simplest instance of a Toda theory is *Liouville conformal field theory*, whose study has initiated major breakthroughs in the understanding of two-dimensional random geometry, both from the mathematics and physics perspectives.

Despite being thoroughly studied in the physics literature, Toda theories are far from being fully understood and computing their *correlation functions of Vertex Operators*—one of the main quantities of interest—remains an open question at the time being.

This thesis is dedicated to providing a mathematical study of Toda conformal field theories. To do so we propose to make sense of them in the realm of probability theory by constructing a probabilistic framework allowing to define their correlation functions of Vertex Operators. This definition of the probabilistic correlation functions is based on two key probabilistic objects: Gaussian free fields and Gaussian multiplicative chaos.

In this preliminary chapter we introduce Toda conformal field theories and describe the way we propose to understand them in a mathematical perspective. We will shed light on the most significant results of the present manuscript and in particular show that we can rigorously compute a certain family of correlation functions of the Toda theory associated to the $\mathfrak{g} = \mathfrak{sl}_3$ Lie algebra.

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2.1 . Toda conformal field theories

Before actually diving into the study of Toda conformal field theories, we first need to answer a very basic question: what are Toda conformal field theories? This introductory section is dedicated to providing a proper introduction of this family of two-dimensional conformal field theories. But before actually explaining what are Toda conformal field theories we will focus on a special case of Toda theory which has drawn a lot of attention in both the physics and mathematics community over the past few decades: Liouville theory.

2.1.1 . Liouville conformal field theory

Providing a definition to the notion of random surface in the context of two-dimensional quantum gravity has been a seminal topic since the pioneering work of Polyakov [111] in 1981. In this groundbreaking article were laid the foundations of the *Liouville conformal field theory* (Liouville theory in the sequel), which may be understood as a canonical way of picking at random a geometry on a Riemann surface Σ with fixed topology [123]. Liouville theory is now considered to be an essential feature in the understanding of non-critical string theory and two-dimensional quantum gravity, and has been thoroughly studied both in the physics and mathematics community. For more details on two-dimensional string theory and its interplays with Liouville theory we refer for instance to [79], while [44] highlights some of its connections with quantum gravity.

Liouville theory has besides many interplays with other models which enjoy the same level of symmetry, should it be with models of statistical physics at criticality in a random environment via the Knizhnik-Polyakov-Zamolodchikov [80] relation, matrix theory [142, 137, 24] or random fractals [80, 50, 117]. We will not review here all the possible applications of Liouville theory since this is absolutely not the purpose of this manuscript, but we can refer to the review [103] where some applications as well as extensions of Liouville theory are being detailed. A general overview of the different perspectives that coincide in the physics literature to make sense of Liouville theory can be found as well in [131].

2.1.1.1 Liouville theory and the path integral

The approach we will rely on in this manuscript to make sense of Liouville theory is its definition via a *path integral*. This method provides a natural (but often non-rigorous) way of defining a quantum field theory starting from a classical field theory, by in some sense providing a way of quantizing the principle of least action.

To be more specific, let us consider the case of Liouville theory on the two-dimensional sphere $\mathbb{S}^2$. The classical theory is concerned with the uniformization problem, or put differently the question of finding a conformal metric with constant (negative) curvature. In order to formulate this problem in a quantitative way let us recall the transformation rules under conformal changes of the metric for the two-dimensional Laplace-Beltrami operator Δ_g and Ricci scalar curvature R_g : if g is any Riemannian metric on $\mathbb{S}^2$ and $g' = e^{2\varphi}g$ is a metric conformally equivalent to g , then the quantities Δ_g, R_g and $\Delta_{g'}, R_{g'}$ are related by

$$\Delta_{g'} = e^{-2\varphi} \Delta_g \quad \text{and} \quad R_{g'} e^{2\varphi} = -2\Delta_g \varphi + R_g. \quad (2.1.1)$$

In particular saying that the conformal metric $g' = e^{2\Phi}g$ has constant scalar curvature -2Λ is tantamount to having the relation

$$-2\Delta_g\Phi + R_g + 2\Lambda e^{2\Phi} = 0. \quad (2.1.2)$$

This is the so-called *Liouville equation*. The weak formulation of the above problem then amounts to saying that Φ is a critical point of the *Liouville action*

$$S_L^0(\phi, g) := \frac{1}{4\pi} \int_{\mathbb{S}^2} \left(\langle \partial_g \phi(x), \partial_g \phi(x) \rangle_g + R_g \phi(x) + \Lambda e^{2\phi(x)} \right) v_g(dx). \quad (2.1.3)$$

Note that in the case of the sphere if we assume Λ to be positive then Gauss-Bonnet theorem provides a topological obstruction preventing such a classical field to exist. To remedy this issue one needs to assume the field to be non-smooth, and a standard way to do so is to assume that it has conical singularities: for some $(z_k)_{1 \leq k \leq N}$ in $\mathbb{S}^2$, $\Phi(z) \sim -\alpha_k \ln |z - z_k|$ around z_k , with $\alpha_k < 2$. Under the assumption that $\sum_{k=1}^N \alpha_k > 4$ (which in particular implies that $N \geq 3$) it is shown for instance in [134] that the above problem does admit a solution.

In the quantum field theory the field is no longer deterministic but rather fluctuates around this critical point. In order to make this quantitative one first introduces a coupling constant $\gamma > 0$ which somehow characterizes the level of randomness of the model and consider a quantized version of the Liouville action (2.1.3):

$$S_L(\phi, g) := \frac{1}{4\pi} \int_{\mathbb{S}^2} \left(\langle \partial_g \phi(x), \partial_g \phi(x) \rangle_g + Q R_g \phi(x) + \Lambda e^{\gamma \phi(x)} \right) v_g(dx) \quad (2.1.4)$$

where $Q = \frac{\gamma}{2} + \frac{2}{\gamma}$ is the so-called *background charge*. The quantum field of Liouville theory is then sampled according to a measure defined on a functional space $\mathcal{F} \subset L^2(\mathbb{S}^2 \rightarrow \mathbb{R})$ by formally setting for suitable $F : \mathcal{F} \rightarrow \mathbb{R}$

$$\langle F(\Phi) \rangle_{\gamma, \mu} := \frac{1}{\mathcal{Z}} \int_{\mathcal{F}} F(\phi) e^{-S_L(\phi, g)} D\phi. \quad (2.1.5)$$

The measure element that appears on the right-hand side refers to the putative “uniform measure” on $\mathcal{F}$, which is actually not well-defined from a mathematical viewpoint. The partition function $\mathcal{Z}$ corresponds to the quantity $\mathcal{Z} = \int_{\mathcal{F}} e^{-S_L(\phi, g)} D\phi$ so that Equation (2.1.5) formally defines the law of a random function. Of course this is purely heuristic and does not really make sense from a mathematical perspective. However we will explain in this manuscript how such expressions can be interpreted using a probabilistic framework.

A standard instance of the translation of a path integral in a probabilistic language is given by the Wiener measure $W(dw)$, viewed here as a probability measure on the space $C_0^0(\mathbb{R}^+, \mathbb{R})$ of continuous functions w from $\mathbb{R}^+$ to $\mathbb{R}$ with $w(0) = 0$. The path integral that formally defines this measure takes the form of

$$W(dw) = \frac{1}{\mathcal{Z}} e^{-S(w)} dw, \quad \text{where} \quad S(w) := \frac{1}{2} \int_{\mathbb{R}^+} w'(t)^2 dt.$$

By writing $S(w) = \frac{1}{2} \int_{\mathbb{R}^+} w''(t)w(t)dt$ we can make an analogy with a Gaussian measure that would satisfy, for suitable f, g ,

$$\int_{C_0^0(\mathbb{R}^+, \mathbb{R})} \left(\int_{\mathbb{R}^+} f''(t)w(t)dt \right) \left(\int_{\mathbb{R}^+} g''(t)w(t)dt \right) W(dw) = \left(\int_{\mathbb{R}^+} f''(t)g(t)dt \right).$$

By taking $f(u) = u \wedge s$ and $g(u) = u \wedge t$ we recover the well-know property of the Brownian motion:

$$\int_{C_0^0(\mathbb{R}^+, \mathbb{R})} w(s)w(t)dtW(dw) = s \wedge t.$$

The path integral formalism is also similar to that of Gibbs measures that appears for instance in statistical physics, in which the action functional would play the role of the Hamiltonian while the integral over a functional space would be replaced by a sum over all possible configurations.

The path integral approach is natural in many ways in the prospect of defining models in quantum field theory. One motivation for this definition is that it provides a standard manner of quantizing a classical theory in that the measure on functions thus defined tends to concentrate on fields for which the action is small. This becomes all the more true as the coupling constant gets small: in the limit $\gamma \rightarrow 0$ the measures formally concentrates on the classical fields of the theory, that is the minimums of the action. This can be thought of as the counterpart of Laplace's method for a function of one real variable where instead of a one-dimensional variable we consider a functional over an infinite-dimensional space. Coming back to the example of the Brownian motion the semi-classical limit corresponds to taking $\gamma \rightarrow 0$ in expressions of the form

$$\frac{1}{Z_\gamma} \int_{C_0^0(\mathbb{R}^+, \mathbb{R})} F(w) e^{-\frac{1}{\gamma^2} S(w)} dw.$$

The latter defining the law of a Brownian motion with variance γ it is easily seen that the classical field is the constant function $w = 0$, which does indeed correspond to the minimum over $C_0^0(\mathbb{R}^+, \mathbb{R})$ of the action S .

The classical field of the theory should be recovered from the quantum field theory in the semi-classical limit, that is when the level of randomness is taken to zero. This corresponds here to looking at the $\gamma \rightarrow 0$ limit of the path integral. It was proved in [86] that it was indeed the case based on a probabilistic representation of the path integral.

2.1.1.2 Liouville theory in the mathematics literature

Following these developments, providing a rigorous framework for Liouville theory has been a challenge for mathematicians and led to major important breakthroughs in the understanding of two-dimensional random geometry, as exemplified by the diversity of methods employed to make sense of the notion of “canonical random surface”.

In analogy with the construction of the Brownian motion as the scaling limit of random walks, a natural way of defining such a random surface would be to consider its discrete counterpart and then study its scaling limit in an appropriate sense. The Brownian sphere introduced in [93] provides the first proposal for such an object, and was later shown [88, 96] to agree with the scaling limit of certain random planar maps. This construction was later generalized to other surfaces, see [18, 19].

On a similar perspective but distinct in its motivations and means, another framework for two-dimensional random geometry has been designed in [49], where a notion of so-called *quantum surface* has been proposed directly in the continuum. One of the key properties of the random surface thus defined is its relationships with three key objects: *Gaussian free fields*, *Schramm-Loewner evolutions* [121] and *continuum random trees*.

First defined as a Riemann surface equipped with a volume form, it was later shown to admit a well-defined metric structure thanks to the construction of the so-called *Liouville Quantum Gravity metric* [42, 47, 70]. For more background on this mathematical approach of Liouville theory we refer to [69] as well as [43] and the references therein.

The program developed by David-Guillarmou-Kupiainen-Rhodes-Vargas and initiated in 2014 [40, 68] is the closest to the framework we will develop in this thesis. Building on a probabilistic definition of the correlation functions of Liouville theory, they built a bridge between the physics and mathematics community by recovering several predictions made by physicists, for instance by proving that Ward identities, that encode certain symmetry assumptions related to conformal covariance made in the physics literature, were indeed valid [83]. One of their main achievements is a mathematical proof of the celebrated DOZZ formula [84], predicted by physicists [45, 141] in the 90's, as well as a rigorous derivation of the conformal bootstrap procedure [66], one of the key inputs in the study of two-dimensional conformal field theory in the physics literature. On a similar perspective was provided [67] a rigorous derivation of Segal's axioms—a functorial definition proposed in 1987 by Segal for two-dimensional conformal field theory—within the framework of Liouville theory.

These different proposals being all aimed at defining what a “canonical random surface” looks like, it should not come as a surprise that there are strong connections between these objects. Interplays between Liouville theory and random planar maps led to numerous breakthroughs, all the more since it was shown that the Brownian sphere actually corresponds to a specific quantum surface—see for instance [97, 98, 99, 71] for such a connection. These links allowed to derive many properties, among them the description of the scaling limit (in some sense) of a certain class of random planar maps in terms of Liouville Quantum Gravity decorated by conformal loop ensembles. The notion of random surface as presented in [49] and the one introduced in [40] also coincide on specific instances as proved in [7, 28] where two such notions were proved to be equivalent. Building on this intrinsic connection between Liouville and conformal loop ensembles [126, 128, 100], the imaginary DOZZ formula was shown by Ang and Sun [5] to describe certain conformal loop ensembles observables. We redirect the curious reader to [127] for an informal review of these different notions of random surface.

2.1.2 . Two-dimensional conformal field theories

In the definition of Liouville theory by David-Guillarmou-Kupiainen-Rhodes-Vargas, a key input is the fact that the model thus defined is actually a two-dimensional conformal field theory. Before going any further we need to provide some background on this notion and why it such an property.

2.1.2.1 What is conformal field theory ?

Shortly after the pioneering article of Polyakov [111], Belavin, Polyakov and Zamolodchikov (BPZ) presented in their 1984 seminal work [12] a systematic procedure to solve models which like Liouville theory possess certain conformal symmetries, now referred to as *two-dimensional conformal field theories*. The main input of their method was to exploit the constraints imposed by conformal symmetry through the study of the algebra of its generators, the *Virasoro algebra*. To do so one translates properties inherited

from representation theory of the Virasoro algebra into actual constraints imposed on the model. These constraints in turn completely determine (up to the so-called structure constants and spectrum) the main quantities of interest, namely the correlation functions of certain special operators, thanks to a recursive procedure dubbed *conformal bootstrap*. We will explain below this procedure in the case of Liouville theory, and implement (part of) it in this manuscript.

There are still different ways of defining a conformal field theory depending on the viewpoint one wants to adopt. For instance an algebraic approach to conformal field theory was developed shortly after the BPZ paper by means of the notion of Vertex Operator Algebra [22, 59], while the study of line bundles over moduli spaces of complex curves initiated in [61] can be thought of as its geometric counterpart. See also [58] for a review of the interplays between conformal field theory and the geometric Langlands program. Closer to our language are the gluing axioms of Segal [122] as well as the axiomatic developed by Gawedski's [62] which views a conformal field theory as a model covariant under conformal transformations. For a detailed but concise account on the notion of two-dimensional conformal field theory one might want to have a look at the review [132], while for a less concise reference the brave reader may consult the "Yellow Book" [56].

2.1.2.2 Some implications of the conformal symmetry for Liouville theory

Liouville theory is assumed in the physics literature to be a conformal field theory, so that the conformal bootstrap method should allow to compute the correlation functions of the theory. Let us explain in detail how it goes for Liouville theory on the Riemann sphere $\mathbb{C} \cup \{\infty\}$.

In order to understand the theory one needs to compute all correlation functions of Vertex Operators. Vertex Operators are functionals of the Liouville field Φ formally defined by taking $V_\alpha(z)[\Phi] = e^{\alpha\Phi(z)}$ where $\alpha \in \mathbb{C}$ is a weight and $z \in \mathbb{C}$ a point on the sphere. The correlation functions of Vertex Operators are then defined using the path integral by setting

$$\left\langle \prod_{k=1}^N V_{\alpha_k}(z_k) \right\rangle_{\gamma, \mu} := \frac{1}{\mathcal{Z}} \int \prod_{k=1}^N e^{\alpha_k \phi(z_k)} e^{-S_L(\phi)} D\phi$$

for some integer $N \geq 3$. From a probabilistic viewpoint, the correlation functions are reminiscent of the Laplace transform of a real-valued random variable X , where functions $f : \mathbb{S}^2 \rightarrow \mathbb{R}$ would play a similar role as the parameter λ in the expression $\mathbb{E}[e^{-\lambda X}]$. Namely the knowledge of all the correlation functions allows to compute quantities of the form

$$\left\langle e^{-\int_{\mathbb{S}^2} f(x) \Phi(x) \nu_g(dx)} \right\rangle_{\gamma, \mu}, \quad f \in L^2(\mathbb{S}^2),$$

which in turn would characterize the law of the random field Φ .

In order to compute such correlation functions, the first step is to determine the structure constants of the theory —these correspond to taking $N = 3$ in the above expression. To do so one exploits the fact that conformal invariance imposes constraints on the correlation functions: these are conformally covariant in the sense that so-called

Ward identities (which we explain below) hold true. Additional constraints arise thanks to assumptions related to representation theory of the Virasoro algebra. This approach is at the heart of the derivation of these structure constants in [131], later rigorously implemented in [84], which are described by the DOZZ formula [45, 141].

Having determined the structure constants of the theory, one then needs to implement a recursive procedure to compute all correlation functions from these basic ones. The conformal bootstrap relies on the knowledge of universal quantities known as *conformal blocks* $\mathcal{F}_P$ —universal in that they only depend on the conformal field theory considered via its *central charge* and conformal weights Δ_α —and the spectrum of Liouville theory $Q + i\mathbb{R}^+$. The recursive procedure at the heart of the conformal bootstrap in particular implies that the computation of four-point correlation functions can be reduced to that of the structure constants via

$$\begin{aligned} & \langle V_{\alpha_1}(0)V_{\alpha_2}(z)V_{\alpha_3}(1)V_{\alpha_4}(\infty) \rangle_{\gamma,\mu} \\ &= \frac{1}{8\pi} \int_{\mathbb{R}^+} C_{\gamma,\mu}^{\text{DOZZ}}(\alpha_1, \alpha_2, Q - iP) C_{\gamma,\mu}^{\text{DOZZ}}(Q + iP, \alpha_3, \alpha_4) |z|^{2(\Delta_{Q+iP} - \Delta_{\alpha_1} - \Delta_{\alpha_2})} |\mathcal{F}_P(z)|^2 dP \end{aligned}$$

where $C_{\gamma,\mu}^{\text{DOZZ}}(\alpha_1, \alpha_2, \alpha_3)$ stands for the DOZZ formula describing the structure constants. Proving rigorously such a statement is highly involved but has nonetheless been achieved in [66]. Computations of correlation functions with a higher number of insertions can be done in the same fashion, which allows to express correlation functions with an arbitrary number of insertions in terms of the DOZZ formula and conformal blocks as shown in [67].

2.1.3 . From Liouville to Toda theories

A natural question which appeared shortly after the 1984 article of BPZ [12] was: what happens when the algebra of symmetry *strictly contains* the Virasoro algebra? In other words, do the same techniques apply when Virasoro symmetry is extended to feature an additional level of symmetry? Inspired by the BPZ seminal work, Zamolodchikov proposed in 1985 [143] a framework designed to extend this machinery to models that enjoy, in addition to conformal invariance, an enhanced level of symmetry. These additional symmetries, called *higher-spin* or *W-symmetries*, are encoded by *W-algebras*, which are vertex algebras that contain the Virasoro algebra as a subalgebra.

Toda theories, a family of two-dimensional conformal field theories indexed by semisimple and complex Lie algebras $\mathfrak{g}$, provide natural extensions of Liouville theory within this setting. Indeed Liouville theory is actually the simplest case of a Toda theory, in that it actually corresponds to the choice of $\mathfrak{g} = \mathfrak{sl}_2$ for the underlying Lie algebra. However for generic $\mathfrak{g}$ the algebras of symmetry of these two-dimensional conformal field theories are no longer given by the Virasoro algebra but rather by *W-algebras*, which makes their study particularly interesting from the point of view of representation theory of *W-algebras* (more on this topic can be found e.g. in [6]), but also from the perspective of *W-symmetry* (in this respect we refer to the review [26] and the references therein) and for their links with two-dimensional Quantum Field Theories with Kac-Moody symmetry (see for instance [9, 8] for their interplays with Wess-Zumino-Witten models). However and unlike Liouville theory, Toda theories are still far from being completely understood, despite having initiated a huge amount of work in the physics literature, all the more thanks to their numerous links with other models. Such connections arise

for instance with certain four-dimensional gauge theories —since they are the general setting for the AGT correspondence [4] (see also [39, 104] for a mathematical take on this correspondence)— and models of statistical physics (for instance the reference [23] provides a journalistic survey on the connection, first unveiled by Zamolodchikov [1], between the Ising Model in a Magnetic Field at criticality and a certain Toda theory associated to the exceptional Lie algebra E_8). We will explain in this subsection how these theories are defined and studied in the physics literature.

2.1.3.1 Toda theories and the path integral

Like Liouville theory, one specificity of this family of conformal field theories is that they can be defined using a path integral approach, thanks to which they admit a probabilistic representation that allows their mathematical study. Namely Toda theories provide a way of picking at random a function from a Riemannian surface (Σ, g) to an Euclidean space $\mathfrak{a} \simeq \mathbb{R}^r$, the *Toda field* Φ . This Euclidean space comes equipped with a scalar product $\langle \cdot, \cdot \rangle$ as well as a special basis of so-called *simple roots* $(e_1, \dots, e_r)$, both inherited from the underlying Lie algebra structure. We have depicted in Figure 2.1 below the basis associated to $\mathfrak{g} = \mathfrak{sl}_3$, in which case $\mathfrak{a} \simeq \mathbb{R}^2$ equipped with a basis (e_1, e_2) such that $\langle e_i, e_j \rangle = 2$ if $i = j$ while $\langle e_i, e_j \rangle = -1$ if $i \neq j$.

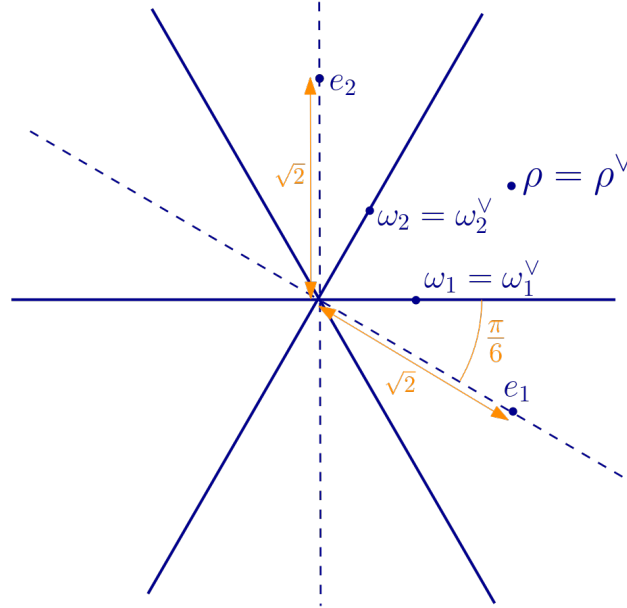


Figure 2.1: The simple roots associated to $\mathfrak{g} = \mathfrak{sl}_3$

For a given Riemannian metric g over Σ , the path integral defines heuristically the law of the Toda field via

$$\langle F(\Phi) \rangle_{T,g} := \frac{1}{Z} \int_{\mathcal{F}} F(\phi) e^{-S_{T,g}(\phi,g)} D\phi \quad (2.1.6)$$

where $D\phi$ should stand for a “uniform measure” over a subspace $\mathcal{F}$ of the space of square

integrable $\mathfrak{a}$ -valued maps defined on Σ and $S_{T,\mathfrak{g}}$ is the Toda action given by

$$S_{T,\mathfrak{g}}(\phi, g) := \frac{1}{4\pi} \int_{\Sigma} \left(\langle \partial_g \phi(x), \partial_g \phi(x) \rangle_g + R_g \langle Q, \phi(x) \rangle + 4\pi \sum_{i=1}^r \mu_i e^{\gamma \langle e_i, \phi(x) \rangle} \right) v_g(dx). \quad (2.1.7)$$

The Riemannian metric g has associated scalar curvature R_g , gradient ∂_g and volume form v_g , while $\langle \cdot, \cdot \rangle_g$ denotes the scalar product associated to the tangent space of $\mathfrak{a}$ -valued functions defined on Σ . The notation $\gamma \in (0, \sqrt{2})$ ¹ stands for the coupling constant and $Q \in \mathfrak{a}$ is the background charge; the constants μ_i , $1 \leq i \leq r$, are positive and referred to as the cosmological constants. In order to ensure conformal symmetry, the background charge is related to the coupling constant via the relation $Q := \gamma \rho + \frac{2}{\gamma} \rho^\vee$ where ρ and $\rho^\vee$ are special vectors in $\mathfrak{a}$. We will provide additional details on the objects involved in Chapter 3 below. Let us emphasize that one recovers Liouville theory when $\mathfrak{g}$ is the Lie algebra $\mathfrak{sl}_2$ (of 2×2 complex matrices with vanishing trace, in which case $r = 1$). Like Liouville theory, the classical field theory associated to Toda theories come from a geometric background and emerge for instance in the study of W -geometry [63] where the Toda equations (for $\mathfrak{g} = \mathfrak{sl}_n$) are associated to the question of finding certain holomorphic embeddings from $\mathbb{CP}^1$ to $\mathbb{CP}^n$ as noticed in [90], and describe minimal surfaces [21].

Within this framework, Vertex Operators are functionals of the Toda field that depend on an insertion point $z \in \Sigma$ as well as a weight $\alpha \in \mathfrak{a}$, and are formally defined by setting $V_\alpha(z)[\Phi] = e^{\langle \alpha, \Phi(z) \rangle}$. The correlation functions then take the form²

$$\langle \prod_{k=1}^N V_{\alpha_k}(z_k) \rangle_{T,g} := \int_{\mathcal{F}} \prod_{k=1}^N e^{\langle \alpha_k, \phi(z_k) \rangle} e^{-S_{T,\mathfrak{g}}(\phi,g)} D\phi. \quad (2.1.8)$$

In the same fashion as in the study of Liouville theory, understanding Toda theories boils down to the computation of the correlation functions.

2.1.3.2 The conformal bootstrap method for Toda theories

As already mentioned, Toda theories are conformal field theories but they enjoy in addition an higher level of symmetry. However due to the fact that the Toda field is no longer real-valued but has rather values in a Euclidean space, this does not mean that Toda theories are more constrained than Liouville theory. It is actually the contrary: implementing the conformal bootstrap procedure is far from being completely understood, even in the physics literature. Certain correlation functions of the theories can be computed [52, 55], but providing a general process to do so for any correlation functions remains an open question up to now despite the diversity of approaches having been proposed to remedy this issue. For instance in [101, 74] the three-point correlation functions are expressed in terms of quantities that arise in the AGT correspondence—but still remain non explicit,

¹The range of values $(0, \sqrt{2})$ for the coupling constant γ only differs from the one commonly encountered in Liouville theory by a conventional matter. Namely the coupling constant from Liouville and the one of Toda are related by $\gamma \leftrightarrow \sqrt{2}\gamma$. This convention accounts for the fact that some elements of the basis have squared norm 2.

²In order to keep the notations concise we do not include the parameter γ in the coming definitions.

while in [38] a general conformal bootstrap procedure is sketched in relation with the quantization of moduli spaces of flat connections.

Indeed W -symmetry together with the constraints it implies still allow to understand certain properties of the model being studied. To settle the ideas, let us explain how to implement the conformal bootstrap procedure in the case of the Toda theory associated to the Lie algebra $\mathfrak{g} = \mathfrak{sl}_3$ and for a special class of correlation functions. The picture is quite similar to that of Liouville but slightly more involved: the first step is to exploit the symmetries of the model to compute a family of three-point correlation functions $\langle V_{\alpha_1}(0)V_{\alpha_2}(1)V_{\alpha_3}(\infty) \rangle$. However unlike in Liouville theory, this does not allow to find an explicit expression for any three-point correlation function but only under the assumption that one of the Vertex Operator is a *semi-degenerate field*, by which we mean that the corresponding weight belongs to some subset of $\mathfrak{a}$. Under this assumption such correlation functions are described by the Fateev-Litvinov formula [52], which we denote by $C_{\gamma}^{\text{FL}}(\alpha_1, \alpha_2^*, \alpha_3)$ and that generalizes the DOZZ formula for Liouville. With the sign “*” we indicate that the corresponding Vertex Operator is semi-degenerate.

Having computed this family of three-point correlation functions one then wonders how to compute correlation functions with a higher number of points, and whether a formula similar to the one for Liouville holds true. To the best of our knowledge such a procedure has not been written down in the literature, however work in progress in collaboration with Guillarmou [32] tends to suggest that a conformal bootstrap formula would hold true for certain four-point correlation functions.

2.1.4 . Overview of the thesis

In this manuscript one of our main achievements is a rigorous derivation of the Fateev-Litvinov formula for the $\mathfrak{sl}_3$ Toda theory based on a mathematical construction of the correlation functions of Vertex Operators. Proving such a statement relies on several steps, combining inspiration from the method implemented in the physics literature with probabilistic reasonings.

We will start by introducing the basic notions necessary to make sense of Toda conformal field theories in Chapter 3, where we will provide some reminders on semisimple and complex Lie algebras as well as conformal geometry, but also present the probabilistic tools we will work with in the rest of the document.

The following chapter, Chapter 4, will be dedicated to proposing a probabilistic definition for the Toda correlation functions by making sense of the path integral (2.1.5). We will shed light on some of their basic properties that will prove to be key in the subsequent chapters. The main results of this chapter are explained in Section 2.2 below.

Starting from Chapter 5 we will focus on the case of the $\mathfrak{g} = \mathfrak{sl}_3$ Toda theory. At first we will be concerned with understanding certain aspects of the symmetry enjoyed by this model, and for this purpose will prove that Ward identities, which in some sense encode this symmetry, hold true on our probabilistic model. In Chapter 6 we will also make a small detour and study some unexpected implications of W -symmetry on purely probabilistic objects by making a connection between the reflection principle in probability theory and the notion of reflection in (Toda) conformal field theory. The plan of these two chapters will be sketched in Section 2.3.

Relying on all the notions introduced in the aforementioned chapters, we will finally

provide in Chapter 7 a rigorous proof of the Fateev-Litvinov formula for a family of three-point correlation functions associated to the $\mathfrak{sl}_3$ Toda theory. To do so we will translate into a mathematical language several of the techniques used in the physics literature such as *operator product expansions*, a use of the *reflection coefficients* or existence of *BPZ-type differential equations*. Section 2.4 will provide a high-level picture of these methods and explain the main steps leading to our main result. To conclude we will present some work in progress concerning the conformal bootstrap procedure for the $\mathfrak{sl}_3$ Toda theory in Chapter 8. We will explain how the study of the so-called *Toda Hamiltonian* allows to carry the conformal bootstrap procedure within our probabilistic setting.

2.2 . A probabilistic take on Toda conformal field theories

Now that Toda conformal field theories have been properly introduced, let us explain the contribution of the present manuscript to the mathematical understanding of these models. To start with we would like to sketch briefly how we can mathematically make sense of the correlation functions of Vertex Operators of the Toda conformal field theories on the two-dimensional sphere $\mathbb{S}^2$, which we identify with the Riemann sphere $\mathbb{C} \cup \{\infty\}$. This summarizes the content of Chapter 4, building on basic notions presented in Chapter 3. This chapter is based on an article written in collaboration with Rhodes and Vargas [34].

2.2.1 . A probabilistic interpretation of the path integral

The path integral (2.1.5) that formally defines Toda theories does not make sense *per se*. Our first task is therefore to give a rigorous meaning to this equation and construct the correlation functions of Vertex Operators as well-defined mathematical objects. To do so we will consider a probabilistic framework that involves the consideration of Gaussian free fields and the theory of Gaussian multiplicative chaos introduced in Section 3.2.

2.2.1.1 Definition of Toda correlation functions

As we will explain in Section 4.1, we can indeed interpret in the path integral (2.1.5) the part stemming from the quadratic term that appears in the Toda field action (2.1.7) in terms of the probability measure with respect to a Gaussian free field $\mathbf{X}^g$ over $\mathbb{C}$ and with values in $\mathfrak{a}$, that is $\mathbf{X}^g : \mathbb{C} \rightarrow \mathfrak{a}$. Indeed the quadratic term

$$e^{-\frac{1}{2}\langle\phi, (\frac{-\Delta}{2\pi})\phi\rangle_{L^2}} D\phi \quad \text{with} \quad \langle f, h \rangle_{L^2} := \int_{\Sigma} \langle f(x), h(x) \rangle_{\mathfrak{g}} dx \quad \text{for} \quad f, h : \Sigma \rightarrow \mathfrak{a}$$

is reminiscent of the density function

$$((2\pi)^N \det(A))^{-\frac{1}{2}} e^{-\frac{1}{2}\langle x, A^{-1}x \rangle} dx_1 \cdots dx_N$$

of a Gaussian vector $X = (X_1, \dots, X_N)$ with covariance matrix A so that one can interpret the Toda field in terms of a “infinite-dimensional Gaussian vector” described by

its covariance kernel:

$$\mathbb{E}[\langle u, \mathbf{X}^g(x) \rangle \langle v, \mathbf{X}^g(y) \rangle] = \langle u, v \rangle G_g(x, y)$$

where G_g is the Green function of the Laplace-Beltrami operator Δ_g on (Σ, g) , and formally corresponds to the inverse of this operator. Because the Green function diverges over the diagonal $x = y$ this field is not a proper function but rather a random distribution. When studying this random generalised function it is standard to regularize it in order to work with a smooth function. For this purpose one considers a family of approximations $(\mathbf{X}_\varepsilon^g)_{\varepsilon > 0}$ of the field $\mathbf{X}^g$, for instance defined by convolution based on a smooth mollifier ρ_ε via $\mathbf{X}_\varepsilon^g := \mathbf{X}^g * \rho_\varepsilon$ (more details are provided Section 3.2). This allows to make sense of several expressions involving the Gaussian free field, such as its exponential—that will play a key role in this manuscript, provided that one understands how such expressions behave in the limit where $\varepsilon \rightarrow 0$. This is the procedure at the heart of the construction of *Gaussian multiplicative chaos measures*. *Before moving on let us highlight that Gaussian free fields have become a key object in a wide range of topics and especially in two-dimensional random geometry in that it plays a role similar to that of a Brownian motion where the two-dimensional space variable replaces the time variable. We redirect for instance to the reviews [125, 112] for additional details on these objects.* The probabilistic definition of Liouville theory initiated in [40] is based on a similar interpretation and differs from the present one in that the underlying field has now values in an Euclidean space $\mathfrak{a}$ (of dimension greater than two for $\mathfrak{g} \neq \mathfrak{sl}_2$) rather than $\mathbb{R}$.

Interpreting the quadratic term in such a way we can translate the path integral into mathematical terms. Namely we make sense of the path integral that defines Toda conformal field theories in a probabilistic way by making the identification for any suitable maps F and metrics g :

$$\begin{aligned} \langle F(\Phi) \rangle_{\mathfrak{g}, g} := & \frac{1}{\mathcal{Z}_g} \int_{\mathfrak{a}} \mathbb{E} \left[F \left(\mathbf{X}^g + \frac{Q}{2} \ln g + \mathbf{c} \right) \right. \\ & \left. \exp \left(-\frac{1}{4\pi} \int_{\mathbb{C}} \left(R_g \langle Q, \mathbf{X}^g + \mathbf{c} \rangle + 4\pi \sum_{i=1}^r \mu_i e^{\gamma \langle e_i, \mathbf{X}^g + \frac{Q}{2} \ln g + \mathbf{c} \rangle} \right) dv_g \right) \right] d\mathbf{c}, \end{aligned} \quad (2.2.1)$$

where $\mathcal{Z}_g$ is defined as a regularized determinant. In order to make sense of the exponential of the Gaussian free field one needs to rely on a regularization procedure and define them as random measure called *Gaussian multiplicative chaos measures*. The presence of the term $\frac{Q}{2} \ln g$ in the definition of the Toda field is reminiscent of its counterpart in Liouville conformal field theory, and accounts for the conformal invariance of the theory. It is important to shed light on the fact that making sense of the exponential of the free field present in the action functions is far from straightforward due to the fact that the field $\mathbf{X}^g$ is not a well-defined function but rather lives in a distributional space. To address this issue we first have to come back to the regularized version $(\mathbf{X}_\varepsilon^g)_{\varepsilon > 0}$ of the field so that the quantity $e^{\langle \gamma e_i, \mathbf{X}_\varepsilon^g(x) \rangle}$ does indeed make sense, the problem being that as $\varepsilon \rightarrow 0$ the expression will diverge. To remedy this issue we renormalize the exponential by its expectation: the result is then that under the assumption that $\langle \gamma e_i, \gamma e_i \rangle < 4$, the limit $\lim_{\varepsilon \rightarrow 0} \frac{e^{\langle \gamma e_i, \mathbf{X}_\varepsilon^g(x) \rangle}}{\mathbb{E}[e^{\langle \gamma e_i, \mathbf{X}_\varepsilon^g(x) \rangle}]} dv_g(x)$ defines a random Radon measure over $\mathbb{C}$. Such a measure

is called a *Gaussian Multiplicative Chaos measure*. In Section 3.2 we provide more details on the construction of such an object. Initially introduced by Kahane in 1985 [76], there has been renewed interest over the past decade in the study of the theory of Gaussian Multiplicative Chaos. Indeed it was shown to have many applications, should they concern financial models [48] or for its role in the modelling of turbulence [81, 105, 92], but also thanks to its links with some random matrix theories [15, 36] as well as with certain properties of Riemann ζ function [120]. The theory is also of special interest *per se* and has been studied from a wide range of perspective —multifractal analysis [16], definition of imaginary Gaussian multiplicative chaos measures [85, 75], integrability... The reader may consult the review [115] and the references therein for a more complete description of this theory and its applications.

The correlation functions of Vertex Operators are then defined by making sense of the above expression when one takes $F(\Phi) = \prod_{k=1}^N e^{\langle \alpha_k, \Phi(z_k) \rangle}$. However and in the same fashion as with the definition of the Gaussian Multiplicative Chaos measures, making sense of such expressions requires to work first with a regularized version and then investigate how the limit can be understood. More precisely, the probabilistic definition of the Toda correlation functions takes the form of the limit, provided it makes sense

$$\begin{aligned} \langle \prod_{k=1}^N V_{\alpha_k}(z_k) \rangle_{\mathfrak{g},g} &:= \lim_{\varepsilon \rightarrow 0} \frac{1}{Z_g} \int_{\mathfrak{a}} \mathbb{E} \left[\prod_{k=1}^N \frac{e^{\langle \alpha_k, \mathbf{X}_\varepsilon^g(z_k) + \frac{Q}{2} \ln g(z_k) + \mathbf{c} \rangle}}{g(z_k)^{\frac{\langle \alpha_k, \alpha_k \rangle}{4}}} \mathbb{E} \left[e^{\langle \alpha_k, \mathbf{X}_\varepsilon^g(z_k) \rangle} \right] \right. \\ &\quad \left. \exp \left(-\frac{1}{4\pi} \int_{\mathbb{C}} \left(R_g \langle Q, \mathbf{X}^g + \mathbf{c} \rangle + 4\pi \sum_{i=1}^r \mu_i e^{\gamma \langle e_i, \mathbf{X}^g + \frac{Q}{2} \ln g + \mathbf{c} \rangle} \right) dv_g \right) \right] d\mathbf{c}. \end{aligned} \quad (2.2.2)$$

Using Girsanov (or Cameron-Martin) theorem we can reformulate this expression in a more tractable way. Following this approach we are able to show that we can provide a rigorous definition of Toda correlation functions:

Theorem 2.2.1. *Let $\mathfrak{g}$ be any simple and complex finite-dimensional Lie algebra and assume that the coupling constant satisfies $\gamma \in (0, \sqrt{2})$. Then existence and non-triviality of the correlation function $\langle V_{\alpha_1}(z_1) \cdots V_{\alpha_N}(z_N) \rangle_{\mathfrak{g},g}$ do not depend on the background metric g in the conformal class of the spherical metric $\hat{g}$. Furthermore:*

1. **(Seiberg bounds)** *The correlation functions $\langle V_{\alpha_1}(z_1) \cdots V_{\alpha_N}(z_N) \rangle_{\mathfrak{g},g}$ exist and are non trivial if and only if the two following conditions hold:*

For all $1 \leq i \leq r$, $s_i > 0$ and $\langle \alpha_k - Q, e_i \rangle < 0$ for any $1 \leq k \leq N$

where $s_i := \frac{\langle \sum_{j=1}^N \alpha_j - 2Q, \omega_i^\vee \rangle}{\gamma}$.

(2.2.3)

2. **(Probabilistic representation)** *In the particular case where $g = \hat{g}$ is the spherical metric, one gets the following explicit expression for the correla-*

tion function

$$\begin{aligned} & \langle V_{\alpha_1}(z_1) \cdots V_{\alpha_N}(z_N) \rangle_{\mathfrak{g}, \hat{g}} \\ &= \left(\prod_{i=1}^r \frac{\Gamma(s_i) \mu_i^{-s_i}}{\gamma} \right) \prod_{1 \leq k < l \leq N} |z_k - z_l|^{-\langle \alpha_k, \alpha_l \rangle} \mathbb{E} \left[\prod_{i=1}^r Z_{(z, \alpha)}^{\gamma e_i}(\mathbb{C})^{-s_i} \right]. \end{aligned} \quad (2.2.4)$$

where the random variables that appear inside the expectation correspond to Gaussian multiplicative chaos measures, formally defined by

$$Z_{(z, \alpha)}^{\gamma e_i}(dx) = \frac{\hat{g}(x)^{-\frac{1}{4} \langle \gamma e_i, \sum_{k=1}^N \alpha_k \rangle}}{\prod_{k=1}^N |x - z_k|^{\langle \alpha_k, \gamma e_i \rangle}} e^{\langle \gamma e_i, \mathbf{X}^g(x) + \frac{Q}{2} \ln \hat{g}(x) \rangle} dx.$$

The spherical metric $\hat{g} := \frac{4}{(1+|z|^2)^2} |d^2 z|$ that appears in this statement stems from the identification via stereographic projection of the sphere $\mathbb{S}^2$ (equipped with its standard metric) with the Riemann sphere $\mathbb{C} \cup \{\infty\}$ equipped with the Riemannian metric $\hat{g}$. The vectors denoted by $(\omega_i^\vee)_{1 \leq i \leq r}$ form the basis of $\mathfrak{a}$ dual to the basis of simple roots: $\langle e_i, \omega_j^\vee \rangle = \delta_{i,j}$. Additional details on this statement are provided in Section 4.1 (see also Figure 2.1).

Thanks to this result we are able to construct the correlation functions for Toda theories on the sphere as soon as the weights satisfy the Seiberg bounds (2.2.3). The existence of such bounds on the weights already appears in Liouville theory [123, 40].

2.2.1.2 Probabilistic correlation functions and conformal invariance

Having defined Toda correlation functions and before going any further it is natural to check whether they are indeed good candidates for the correlation functions considered in the physics literature. To do so we check that the probabilistic expression proposed in Theorem 2.2.1 satisfy two basic properties related to conformal symmetry of the model: *conformal covariance* of the correlation functions and existence of a *Weyl anomaly* under variation of the metric g on $\mathbb{C} \cup \{\infty\}$.

The first of these two properties is concerned with the behaviour of the correlation functions under transformations preserving the conformal structure of the Riemann sphere $\mathbb{C} \cup \{\infty\}$. Namely this assumption corresponds to the fact that when insertions are moved according to a diffeomorphism of the Riemann sphere —a *Möbius transform* of the plane— the correlation functions are perturbed in a completely explicit way, in the sense of Equation (2.2.5) below. The Weyl anomaly, also referred to as local scale covariance, shows that the dependence in the Riemannian metric g with which the sphere is equipped is completely determined within a given conformal class. Put differently, we show in Equation (2.2.6) below that if g' and g are two Riemannian metrics over the Riemann sphere that both belong to the conformal class of the spherical metric $\hat{g}$ then the associated correlation functions only differ by an explicit multiplicative factor independent of the correlation function considered. Actually this factor only depends on the theory via its central charge. These correspond to basic assumptions required in the physics literature to develop an axiomatic approach towards conformal field theory, see e.g. Gawedski's axioms [62].

In this perspective, the following theorem is an additional indication that the probabilistic correlation functions are consistent with properties expected in the physics:

Theorem 2.2.2. *Under the assumptions of Theorem 2.2.1, the following holds true:*

1. (Conformal covariance) *Let ψ be a Möbius transform of the plane. Then*

$$\langle V_{\alpha_1}(\psi(z_1)) \cdots V_{\alpha_N}(\psi(z_N)) \rangle_{\mathfrak{g},g} = \prod_{k=1}^N |\psi'(z_k)|^{-2\Delta_{\alpha_k}} \langle V_{\alpha_1}(z_1) \cdots V_{\alpha_N}(z_N) \rangle_{\mathfrak{g},g} \quad (2.2.5)$$

where the conformal weights are given by $\Delta_{\alpha_j} := \langle \frac{\alpha_j}{2}, Q - \frac{\alpha_j}{2} \rangle$.

2. (Weyl anomaly) *For suitable φ (namely $\varphi \in \bar{C}^1(\mathbb{C})$, see notations in Section 4.2) we have the following relation*

$$\langle V_{\alpha_1}(z_1) \cdots V_{\alpha_N}(z_N) \rangle_{\mathfrak{g},e^{\varphi}\hat{g}} = e^{\frac{c_T}{96\pi}S_L(\varphi,\hat{g})} \langle V_{\alpha_1}(z_1) \cdots V_{\alpha_N}(z_N) \rangle_{\mathfrak{g},\hat{g}} \quad (2.2.6)$$

where S_L is the Liouville functional (with $\Lambda = 0$)

$$S_L(\varphi, \hat{g}) := \int_{\mathbb{C}} (|\partial_{\hat{g}}\varphi|_{\hat{g}}^2 + 2R_{\hat{g}}\varphi) dV_{\hat{g}},$$

and the central charge c_T is given by $c_T = r + 6\langle Q, Q \rangle$.

These properties are inherited from those of Gaussian free fields, highlighting the strong connections between this probabilistic object and conformal geometry in dimension two. They will be proved in Section 4.2. We note that such features are already present in the probabilistic formulation of Liouville theory [40].

2.2.2 . Some additional properties of the probabilistic correlation functions

The correlations are now well-defined mathematical objects, and we have shown that they satisfy basic assumptions related to conformal invariance. Before actually trying to compute them we will need to understand some of their analytic properties, related to their continuation or their regularity. We will build on the two articles [30] and [33], the second one having been written together with Huang.

2.2.2.1 A first analytic continuation of the correlation functions

An additional assumption usually made in the physics literature is that the correlation functions depend analytically on their weights $\alpha_1, \dots, \alpha_N$. This assumption is actually rather strong and is key in the implementation of the conformal procedure as envisioned by BPZ [12]. In order to provide a rigorous derivation of this method one first needs to ensure that this requirement is met within our probabilistic framework and show that the expression (2.2.4) depends analytically in the weights $(\alpha_k)_{1 \leq k \leq N}$.

Besides, in the construction of the correlation functions proposed there the statement of Theorem 2.2.1 places two restrictions on the range of values for which they can be probabilistically represented. The first assumption that $\langle \alpha_k - Q, e_i \rangle < 0$ (or put differently that $\alpha_k - Q$ belongs to the *Weyl chamber* $\mathcal{C}_-$) is really necessary for the probabilistic representation to make sense, but the second one, that demands that $\langle \sum_{k=1}^N \alpha_k - 2Q, \omega_i^\vee \rangle > 0$ can actually be relaxed. Indeed in Equation (2.2.4) we see

that when $\langle \sum_{k=1}^N \alpha_k - 2Q, \omega_i^\vee \rangle \leq 0$ the singularity comes from the poles at $-\mathbb{N}$ of the Gamma function Γ , but the expectation term still makes sense under more general assumptions.

Based on this observation, we show that the probabilistic correlation functions depend analytically in its weights $(\alpha_k)_{1 \leq k \leq N}$ and that in addition they admit an analytic continuation beyond the Seiberg bounds (2.2.3):

Theorem 2.2.3. *Let $z_1, \dots, z_N \in \mathbb{C}$ be distinct and define a subset of $(Q + \mathcal{C}_-)^N$ by setting*

$$\mathcal{B}_N := \left\{ (\alpha_1, \dots, \alpha_N) \in (Q + \mathcal{C}_-)^N \right. \\ \left. \text{such that } -s_i < \frac{4}{\gamma^2 \langle e_i, e_i \rangle} \wedge \min_{k=1, \dots, N} \frac{1}{\gamma} \langle Q - \alpha_k, e_i^\vee \rangle \text{ for all } 1 \leq i \leq r \right\}. \quad (2.2.7)$$

Then the map defined by Equation (2.2.4):

$$\alpha \mapsto \left(\prod_{i=1}^r \frac{\Gamma(s_i) \mu_i^{-s_i}}{\gamma} \right) \prod_{1 \leq k < l \leq N} |z_k - z_l|^{-\langle \alpha_k, \alpha_l \rangle} \mathbb{E} \left[\prod_{i=1}^r Z_{(\mathbf{z}, \alpha)}^{\gamma e_i}(\mathbb{C})^{-s_i} \right] \quad (2.2.8)$$

is meromorphic in a complex neighbourhood of $\mathcal{B}_N \subset (\mathbb{C}^r)^N$, with poles given by

$$\mathcal{P}_N^1 := \{ \alpha \in \mathcal{B}_N, \quad s_i = 0 \text{ for some } 1 \leq i \leq r \}.$$

To prove this statement we will study properties of Gaussian multiplicative chaos measures and more precisely derive conditions ensuring existence of their moments. As we will see the set $\mathcal{B}_N$ is the optimal one in which the expectation term makes sense.

We will explain later that we can actually go beyond this representation by providing the exact tail expansion of such Gaussian multiplicative chaos measures.

2.2.2.2 Some analytic properties of the correlation functions

In order to study the symmetry enjoyed by Toda theories we will need to compute derivatives of the correlation functions. In order to ensure differentiability of the correlation functions several of their properties will have to be disclosed. Namely we will provide some estimates (including fusion estimates) ensuring that they can be integrated, and then explain how derivatives can be computed using Gaussian integration by parts which in turn will allow to reduce existence of the derivatives to these estimates. Our main result in this perspective is the fact that correlation functions are twice differentiable:

Proposition 2.2.4. *Take $z_1, \dots, z_N \in \mathbb{C}$ distinct and assume that the weights $(\alpha_0, \dots, \alpha_N)$ belong to $\mathcal{B}_{N+1}$. Then $z_0 \mapsto \langle \prod_{k=0}^N V_{\alpha_k}(z_k) \rangle$ is C^2 on the set $\mathbb{C} \setminus \{z_1, \dots, z_N\}$.*

Along Section 4.4 we will review all these technical aspects.

2.3 . A mathematical study of the symmetries of Toda theories

The construction of the correlation functions together with their properties highlighted represent the starting point for a mathematically rigorous study of the symmetries enjoyed by Toda theories. As we will explain below, this symmetry will manifest itself via the existence of Ward identities on the correlation functions. We rigorously prove this statement for the $\mathfrak{g} = \mathfrak{sl}_3$ Toda theory following the aforementioned article [33], and like before for the theory defined on the sphere $\mathbb{S}^2$ that we identify with the Riemann sphere $\mathbb{C} \cup \{\infty\}$. We will also explain how the reflection principle in probability theory can be related to the reflection phenomenon which occurs in the study of Toda theories, which will allow us to describe a Brownian path decomposition as well as a tail expansion of Gaussian multiplicative chaos measures. These results have been proved in [31].

2.3.1 . W-symmetry and Ward identities

In conformal field theory, the symmetry enjoyed by the models can be seen via the existence of a certain observable of the field called *stress-energy tensor*. The effect of inserting this tensor within correlation function is explicit and described in terms of so-called *Ward identities*, which in some sense encode the local symmetries of the theory.

The same picture remains true when the model considered also enjoys an enhanced level of symmetry. Namely the existence of additional symmetries will manifest itself via the existence of additional identities obtained via the insertion of other tensors within correlation functions.

We present here in more details on these Ward identities and explain how they can be rigorously derived within our probabilistic framework.

2.3.1.1 Conformal symmetry and the stress-energy tensor

The method developed by Belavin, Polyakov and Zamolodchikov in [12] for solving two-dimensional conformal field theories is based on a systematic exploitation of the conformal symmetry enjoyed by the underlying model. This is done through the study of the Virasoro algebra, the symmetry algebra of two-dimensional conformal field theories. As a feature of this machinery, the conformal symmetry of the model yields the existence of a *holomorphic current* of spin 2, the stress-energy tensor usually denoted by $\mathbf{T}$. This tensor admits a (formal) expansion in Laurent series which takes the form

$$\mathbf{T}(z_0) = \sum_{n \in \mathbb{Z}} \frac{\mathbf{L}_n(z)}{(z_0 - z)^{n+2}} \quad (2.3.1)$$

around some point $z \in \mathbb{C}$, where the modes $\mathbf{L}_n$ are the generators of the Virasoro algebra, with the commutation relations given by

$$[\mathbf{L}_n, \mathbf{L}_m] = (n - m)\mathbf{L}_{n+m} + \frac{c}{12}(n - 1)n(n + 1)\delta_{n+m,0}\text{Id}, \quad (2.3.2)$$

with c the central charge of the conformal field theory being considered.

In Liouville theory this tensor can (formally) be defined via the variation of the correlation functions with respect to the metric g and thus admits an alternative expression

in terms of the Liouville field Φ . To be more specific one can define the stress-energy tensor by means of:

$$\left\langle \mathbf{T}(z_0) \prod_{k=1}^N V_{\alpha_k}(z_k) \right\rangle_g := 4\pi \frac{\delta}{\delta g^{zz}(z_0)} \left\langle \prod_{k=1}^N V_{\alpha_k}(z_k) \right\rangle_g \quad (2.3.3)$$

where in the above we vary the metric around the point $z_0 \in \mathbb{C}$. It follows from this definition that the stress-energy tensor admits an explicit expression as an observable of the Toda field via

$$\mathbf{T}(z)[\Phi] = Q\partial_z^2\Phi(z) - (\partial_z\Phi(z))^2. \quad (2.3.4)$$

See for instance [82, 107] for additional details on the mathematical justification of the above.

One of the key properties of this tensor is its *operator product expansion* with Vertex Operators, which (again formally) takes the form:

$$\mathbf{T}(z_0)V_\alpha(z) = \frac{\Delta(\alpha)V_\alpha(z)}{(z_0 - z)^2} + \frac{\partial_z V_\alpha(z)}{z_0 - z} + \text{holomorphic terms} \quad (2.3.5)$$

as $z_0 \rightarrow z$. This in particular implies that the correlation functions of Vertex Operators solve the so-called *Ward identity*:

$$\langle \mathbf{T}(z_0) \prod_{k=1}^N V_{\alpha_k}(z_k) \rangle = \sum_{l=1}^N \left(\frac{\Delta_{\alpha_l}}{(z_0 - z_l)^2} + \frac{\partial_{z_l}}{z_0 - z_l} \right) \langle \prod_{k=1}^N V_{\alpha_k}(z_k) \rangle. \quad (2.3.6)$$

Here we have introduced the conformal dimension $\Delta_\alpha := \frac{\alpha}{2} (Q - \frac{\alpha}{2})$ of the Vertex Operator V_α . The derivative should be understood as a complex (*i.e.* Wirtinger) derivative³, that is $\partial_z f(x, y) = \frac{1}{2} (\partial_x - i\partial_y) f(x, y)$; $\partial_{\bar{z}}$ is defined analogously by means of $\partial_{\bar{z}} f(x, y) = \frac{1}{2} (\partial_x + i\partial_y) f(x, y)$. When combined with the holomorphicity at infinity of this tensor, by which is meant that $\mathbf{T}(z) \sim \frac{1}{z^4}$ as $z \rightarrow \infty$ and which is usually axiomatic in two-dimensional conformal field theories, the above identity implies that Liouville correlation functions satisfy three *global Ward identities*:

$$\sum_{l=1}^N (z_l^n \partial_{z_l} + n\Delta_{\alpha_l}) \langle \prod_{k=1}^N V_{\alpha_k}(z_k) \rangle = 0$$

for $0 \leq n \leq 2$, which actually means that they enjoy a property of conformal covariance in the sense that for any Möbius transform ψ of the complex plane,

$$\langle \prod_{k=1}^N V_{\alpha_k}(\psi(z_k)) \rangle = \prod_{l=1}^N |\psi'(z_l)|^{-2\Delta_{\alpha_l}} \langle \prod_{k=1}^N V_{\alpha_k}(z_k) \rangle.$$

The above Ward identity has been rigorously proved in the setting of Liouville theory in [83] based on the probabilistic framework introduced in [40].

³Unless explicitly stated, holomorphic derivatives will be considered throughout the rest of the document.

2.3.1.2 W-symmetry and higher-spin currents

For models with enhanced symmetry such as Toda conformal field theories, the algebra of symmetry is no longer by the Virasoro algebra but rather contains it. These extensions appeared shortly after [12] in a work by Zamolodchikov [143] where the author introduced the notion of W – (or higher-spin) symmetry, based on extensions of the Virasoro algebra called W -algebras. Instances of models with this additional level of symmetry emerged in subsequent works, first in [54] and then in general in [53]. A similar reasoning involving operator product expansions with respect to holomorphic currents should remain valid when the model being studied enjoys higher-spin symmetry in addition to the conformal symmetry. This additional level of symmetry also comes with additional holomorphic currents that contain information related to this higher-spin symmetry: the so-called *higher-spin currents* $\mathbf{W}^{(i)}$ (for $3 \leq i \leq r + 1$).

In this manuscript, we focus on the study of the $\mathfrak{g} = \mathfrak{sl}_3$ Toda theory, in which case there will be one additional holomorphic current of spin three $\mathbf{W} := \mathbf{W}^{(3)}$, which admits the Laurent series expansion

$$\mathbf{W}(z_0) = \sum_{n \in \mathbb{Z}} \frac{\mathbf{W}_n(z)}{(z_0 - z)^{n+3}}. \quad (2.3.7)$$

The W_3 algebra is then a vertex algebra generated by the $(\mathbf{L}_n, \mathbf{W}_m)_{n,m \in \mathbb{Z}}$ and with commutation rules given by

$$[\mathbf{L}_m, \mathbf{W}_n] = (2m - n)\mathbf{W}_{m+n}. \quad (2.3.8)$$

The commutation rules for the $(\mathbf{W}_n)_{n \in \mathbb{Z}}$ is rather complicated and bilinear in the $(\mathbf{L}_n)_{n \in \mathbb{Z}}$ (see [25, Equation (2.1)]). In particular the W_3 algebra is not a Lie algebra. In the present document we will not use this modes representation but rather define the tensor as an observable of the field that depends on a position $z \in \mathbb{C}$ and which can be defined by an expression similar (but more complicated) to the stress-energy tensor. However in contrast with the derivation of the stress-energy tensor as explained in Equation (2.3.3) it is not known at the time being whether it would be possible to describe the higher-spin current in a similar fashion, even if it should take the form of “derivatives in an extra direction”(cf. [135, Subsection 8.2]). This is related to the geometrical interpretation of W -symmetry, which is not really well understood for the time being —as stressed in [57, Chapter 18.2], “it would be interesting to identify the deformation problems related to ... W -algebras”.

The so-called *WV operator product expansion*, axiomatic in the physics literature, is key in the understanding of the higher-spin symmetry and takes the form:

$$\mathbf{W}(z_0)V_\alpha(z) = \frac{w(\alpha)V_\alpha(z)}{(z_0 - z)^3} + \frac{\mathbf{W}_{-1}V_\alpha(z)}{(z_0 - z)^2} + \frac{\mathbf{W}_{-2}V_\alpha(z)}{z_0 - z} + \text{holomorphic terms} \quad (2.3.9)$$

where $w(\alpha) \in \mathbb{C}$ is the *quantum number* associated to $\mathbf{W}$, and the $\mathbf{W}_{-i}V_\alpha(z)$ are the *descendent fields*. These fields are said to be local, in the sense that they should only depend on the weight α and derivatives of the Toda field at the point z . However unlike the descendent fields associated to the stress-energy tensor, they are not actually obtained by differentiating the Vertex Operators. We will explain how they can be defined within our probabilistic framework.

Equation (2.3.9) formally holds in the sense of operators, but can be rephrased in terms of correlation functions as

$$\langle \mathbf{W}(z_0) \prod_{k=1}^N V_{\alpha_k}(z_k) \rangle = \sum_{l=1}^N \left(\frac{w(\alpha_l)}{(z_0 - z_l)^3} + \frac{\mathcal{W}_{-1}^{(l)}}{(z_0 - z_l)^2} + \frac{\mathcal{W}_{-2}^{(l)}}{z_0 - z_l} \right) \langle \prod_{k=1}^N V_{\alpha_k}(z_k) \rangle, \quad (2.3.10)$$

where

$$\mathcal{W}_{-i}^{(l)} \langle \prod_{k=1}^N V_{\alpha_k}(z_k) \rangle \quad \text{stands for} \quad \langle \mathbf{W}_{-i} V_{\alpha_l}(z_l) \prod_{k \neq l} V_{\alpha_k}(z_k) \rangle.$$

This equality is usually referred to as the *local spin-three Ward identity* and is an instance of the higher-spin symmetry enjoyed by the model. We would like to stress here that unlike the Ward identities associated to the stress-energy tensor, the quantities that we have denoted $\mathcal{W}_{-i}^{(l)} \langle \prod_{k=1}^N V_{\alpha_k}(z_k) \rangle$ are not derivatives of the correlation functions properly speaking, but rather correspond to inserting observables within them. For instance generally speaking the expression $\mathcal{W}_{-1}^{(l)} \langle \prod_{k=1}^N V_{\alpha_k}(z_k) \rangle$ cannot be expressed as a derivative of the correlation function $\langle \prod_{k=1}^N V_{\alpha_k}(z_k) \rangle$ but is rather given by

$$\mathcal{W}_{-1}^{(l)} \langle \prod_{k=1}^N V_{\alpha_k}(z_k) \rangle = \langle \langle u_{\alpha_l}, \partial \Phi(z_l) \rangle \prod_{k=1}^N V_{\alpha_k}(z_k) \rangle$$

for some vector $u_{\alpha_l} \in \mathfrak{a}$. This key difference makes the study of Toda theories much more difficult to understand than Liouville theory.

In a similar fashion as for the stress-energy tensor, the asymptotics of this tensor when $z_0 \rightarrow \infty$ allow to write that correlation functions are subject to certain constraints, known as *global spin-three Ward identities*. Namely the assumption that $\mathbf{W}(z_0) \simeq \frac{1}{z_0^6}$ allows to infer the following set of equalities, valid for $0 \leq n \leq 4$:

$$\sum_{l=1}^N \left(z_l^n \mathcal{W}_{-2}^{(l)} + n z_l^{n-1} \mathcal{W}_{-1}^{(l)} + \frac{n(n-1)}{2} z_l^{n-2} w(\alpha_l) \right) \langle \prod_{k=1}^N V_{\alpha_k}(z_k) \rangle = 0. \quad (2.3.11)$$

We stress that unlike the global Ward identities associated to the stress-energy tensor, it is not known at the time being whether the global spin-three Ward identities can be derived in a similar fashion by exploiting properties of the correlation functions related to some covariance. This makes them all the more key in the understanding of the symmetry of the theory. Likewise due to the fact that the quantities that appear in the global Ward identities are not *stricto sensu* derivatives of the correlation functions make these five identities necessary in the prospect of implementing the conformal bootstrap procedure.

2.3.1.3 Ward identities for the probabilistic correlation functions

In Chapter 5 and building on our probabilistic definition of the correlation functions, we will provide a rigorous meaning to the quantities that appear above under the assumption that $\mathfrak{g} = \mathfrak{sl}_3$. Namely thanks to our probabilistic framework and under suitable assumptions we can make sense of the quantity $\langle \mathbf{W}(z_0) \prod_{k=1}^N V_{\alpha_k}(z_k) \rangle$. Namely we use

the fact that in analogy with the stress-energy tensor $\mathbf{T}$, $\mathbf{W}(z)$ can be represented as a functional of the Toda in that it formally takes the form:

$$\begin{aligned}\mathbf{W}(z)[\Phi] := & q^2 \langle \omega_2 - \omega_1, \partial^3 \Phi(z) \rangle \\ & + 2q \left(\langle e_1, \partial^2 \Phi(z) \rangle \langle \omega_1, \partial \Phi(z) \rangle - \langle e_2, \partial^2 \Phi(z) \rangle \langle \omega_2, \partial \Phi(z) \rangle \right) \\ & + 8 \langle \omega_1, \partial \Phi(z) \rangle \langle \omega_2 - \omega_1, \partial \Phi(z) \rangle \langle \omega_3, \partial \Phi(z) \rangle\end{aligned}\quad (2.3.12)$$

where we have set $q := \gamma + \frac{2}{\gamma}$. However as before due to the lack of regularity of the Toda field the latter must be appropriately regularized to be meaningful.

Relying on the definition of the Toda field proposed in Equation (2.2.1) with the map $F(\Phi)$ being taken as $\mathbf{W}(z)[\Phi]$, we can show that the limit of these regularized quantities is well-defined in the sense that the limit

$$\langle \mathbf{W}(z_0) \prod_{k=1}^N V_{\alpha_k}(z_k) \rangle := \lim_{\varepsilon \rightarrow 0} \langle \mathbf{W}_\varepsilon(z_0) \prod_{k=1}^N V_{\alpha_k, \varepsilon}(z_k) \rangle$$

does exist, where the right-hand side admits a probabilistic representation based on Equation (2.2.1). We can define analogously the expression $\mathbf{W}_{-i}^{(l)} \langle \prod_{k=1}^N V_{\alpha_k}(z_k) \rangle$ based on a limiting procedure. We then prove that the Ward identities do indeed hold within our probabilistic framework in the sense that the following statement holds true:

Theorem 2.3.1. *Assume that $\mathfrak{g} = \mathfrak{sl}_3$. Under the assumptions of Theorem 2.2.3 the spin-three Ward identity holds true:*

$$\langle \mathbf{W}(z_0) \prod_{k=1}^N V_{\alpha_k}(z_k) \rangle = \sum_{l=1}^N \left(\frac{w(\alpha_l)}{(z_0 - z_l)^3} + \frac{\mathbf{W}_{-1}^{(l)}}{(z_0 - z_l)^2} + \frac{\mathbf{W}_{-2}^{(l)}}{z_0 - z_l} \right) \langle \prod_{k=1}^N V_{\alpha_k}(z_k) \rangle \quad (2.3.13)$$

with the quantum number $w(\alpha)$ explicit. A similar statement holds for the stress-energy tensor.

On the same perspective we prove that:

Theorem 2.3.2. *Under the assumptions of Theorem 2.3.1, the global Ward identities hold for $0 \leq n \leq 4$:*

$$\sum_{l=1}^N \left(z_l^n \mathbf{W}_{-2}^{(l)} + n z_l^{n-1} \mathbf{W}_{-1}^{(l)} + \frac{n(n-1)}{2} z_l^{n-2} w(\alpha_l) \right) \langle \prod_{k=1}^N V_{\alpha_k}(z_k) \rangle = 0. \quad (2.3.14)$$

These are additional arguments advocating towards the fact that the probabilistic correlation functions are indeed good mathematical objects to consider in order to provide a rigorous meaning to correlation functions from the physics literature. The derivation of these identities is a major step forward in the mathematical understanding of the $\mathfrak{sl}_3$ Toda theory and more specifically of its symmetries. These identities will prove to be crucial in the computation of a family of correlation functions.

The proof of the Ward identity described by Equation 2.3.13 strongly relies on the Gaussian structure at the heart of the probabilistic definition of Toda theories. Indeed one of the key ingredients in the proof of Theorem 2.3.1 is *Gaussian integration by parts*, which in a probabilistic language corresponds to the Malliavin calculus for the Toda field. This property can be seen (at the formal level) from the path integral.

2.3.2 . W-symmetry and a reflection principle

We have seen above that in order for the correlation functions to be well-defined one needs the weights to belong to a subset $Q + \mathcal{C}_-$ of $\mathfrak{a}$ defined by the condition that $\langle \alpha - Q, e_i \rangle < 0$ for all $1 \leq i \leq r$. In order to make sense of the correlation functions when this assumption is no longer satisfied it is assumed in the physics literature that there exist transformations s mapping $\overline{\mathcal{C}_-}$ to the whole $\mathfrak{a}$ as well as scalar numbers $R_s(\alpha)$ such that the Vertex Operators V_α and $V_{Q+s(\alpha-Q)}$ are related one to the other by the equality

$$V_\alpha = R_s(\alpha) V_{Q+s(\alpha-Q)}.$$

These coefficients are called *Toda reflection coefficients*, while the group of such transformations is a reflection group called the *Weyl group* of $\mathfrak{g}$. This assumption is axiomatic in the physics literature and stems from the fact that the conformal and quantum weights Δ_α and $w(\alpha)$ are invariant under such transformations. In Figure 2.2 below we provide a representation of this so-called *Weyl chamber* $\mathcal{C}_-$ together with the associated transformations in the case of $\mathfrak{g} = \mathfrak{sl}_3$.

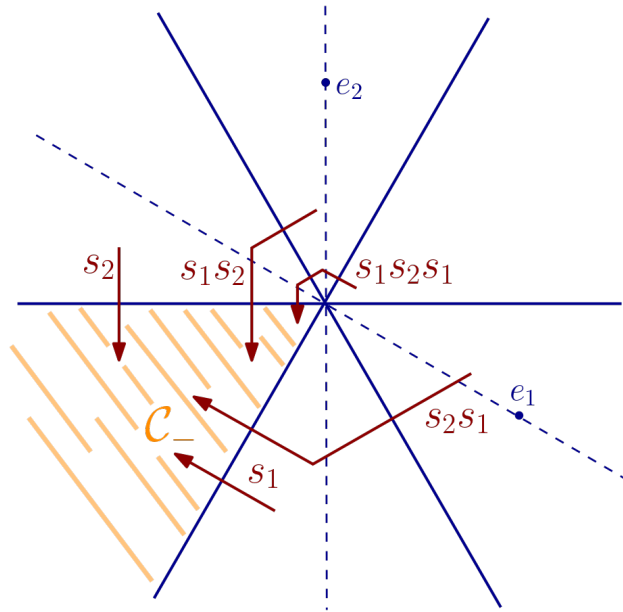


Figure 2.2: The Weyl group associated to $\mathfrak{g} = \mathfrak{sl}_3$

We stress that in the case of Liouville theory, this reflection group is made of only two elements Id and $-\text{Id}$. The reflection relation then reads $V_\alpha = R(\alpha) V_{2Q-\alpha}$ where $R(\alpha)$ is Liouville reflection coefficient.

Inspired by the existence of such a relation we will explore some interplays between this reflection phenomenon and the reflection principle in probability and explain how these reflection coefficients arise in the tail expansion of certain random variables associated to Gaussian multiplicative chaos measures. Chapter 6, that details these results, will mostly follow the article [31]. We will not actually provide a rigorous meaning to the reflection relation $V_\alpha = R_s(\alpha) V_{Q+s(\alpha-Q)}$ yet, but will do so in the case where $\mathfrak{g} = \mathfrak{sl}_3$ and for a family of three-point correlation functions in Chapter 7.

2.3.2.1 Reflection principle and a Brownian path decomposition

In a celebrated article [138], Williams in 1974 described a remarkable path decomposition for Brownian paths and more generally one-dimensional diffusions. In its simplest form where the underlying process is a Brownian motion with positive drift ν (which we denote B^ν), this decomposition can be formulated by saying that, conditionally on the value of the global minimum of the process $\mathbf{M} := \inf_{t \geq 0} B_t^\nu$, the law of B^ν (knowing $\mathbf{M}$) is no longer a Markov process but it can be realized by joining together two Markov processes. Namely, the first process has the law of $B^{-\nu}$ until reaching $\mathbf{M}$, and the second one has the law of the diffusion process B^ν whose law is that of B^ν conditioned on staying above $\mathbf{M}$.

Following its discovery by Williams, this path decomposition has been thoroughly investigated in the probability community and has inspired numerous fundamental statements such as Pitman's celebrated theorem [110]. Extending this decomposition for different classes of processes has been a very active field of research [17, 35, 78, 27] since Williams' 1974 article. See the account (in French) by Le Gall [87] on this topic. However to the best of our knowledge, there is no general formulation of Williams path decomposition for diffusions in any dimensions.

Based on a generalized reflection principle, we will prove that the notion of reflection group allows to extend this path decomposition to any suitable diffusion on a Euclidean space. In the context of Toda theories described above, the random variables $\mathbf{M}_i := \inf_{t \geq 0} \langle B_t^\nu, e_i \rangle$ will play the role of the minimum of the process and the decomposition will feature $r + 1$ components instead of 2 in the path decomposition by Williams. To be more specific we will prove that:

Theorem 2.3.3. *Assume that B^ν is a Brownian motion with drift $\nu \in \mathcal{C}$ that evolves on a Euclidean space on which acts a rank r reflection group. Then there exist $r + 1$ diffusion processes $\mathbf{X}^1, \dots, \mathbf{X}^{r+1}$ such that the following holds true:*

1. *Pick the $\mathbf{M}_i$ according to their marginal laws and define a process $\mathbf{X}$ to be the joining of the following processes:*
 2. • *Start by sampling a diffusion process $\mathbf{X}^1$ started from the origin and independent from $\mathbf{M} = \sum_{i=1}^r M_i \omega_i^\vee$. Run it until it hits $\mathbf{M} + \partial\mathcal{C}$, say at $z_1 \in \mathbf{M} + \partial\mathcal{C}_1$.*
 - *Then run an independent process $\mathbf{X}^2$ started from z_1 upon hitting $\mathbf{M} + \partial\mathcal{C}$.*
 - *Thus define a family of processes $(\mathbf{X}^1, \dots, \mathbf{X}^r)$. When $\mathbf{X}^r$ reaches the boundary of $\mathbf{M} + \partial\mathcal{C}$, sample $\mathbf{X}^{r+1}$.*
3. *Then $\mathbf{X}$ has the law of B^ν , a Brownian motion with drift $\nu \in \mathcal{C}$.*

We would like to mention to that the consideration of such a notion of minimum is very natural from the perspective of the probabilistic study of Toda conformal theories as we will see in the description of the tail expansion of Gaussian multiplicative chaos measures below. This notion of minimum is actually quite common and is used e.g. in the study of Brownian motion in a cone. For instance in the article [89] the process

$(B_t^\nu - M_t)_{t \geq 0}$ where $M_t = \sum_{i=1}^r \omega_i^\vee \inf_{s \leq t} \langle B_s^\nu, e_i \rangle$ is studied, the difference with our setting being that we rather consider the minimum of the whole path, which corresponds with these notations to the process $(B_t^\nu - M_{+\infty})_{t \geq 0}$. We find it remarkable that such notions inherited from the study of symmetries of models of two-dimensional conformal field theory actually have purely probabilistic implications.

All the notations that appear in the above theorem will be made precise in Section 3.1, while the proof and precise statements of this theorem can be found in Section 6.1. For the sake of concreteness we reproduce here an illustration of the corresponding decomposition of a planar Brownian motion in the case where the reflection group considered corresponds to A_2 (or to $\mathfrak{g} = \mathfrak{sl}_3$).

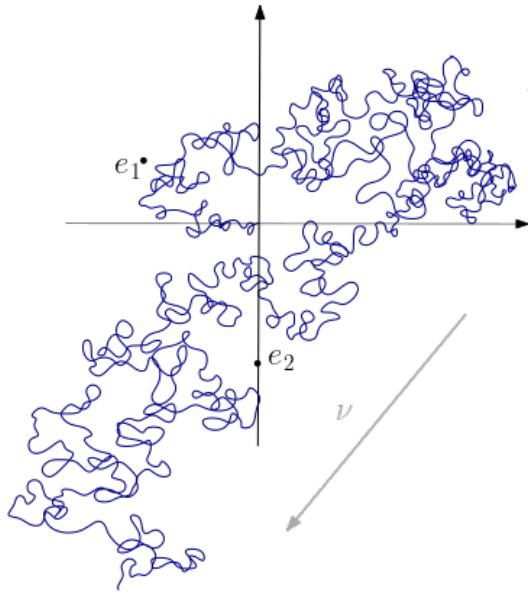


Figure 2.3: Planar Brownian motion with drift ν

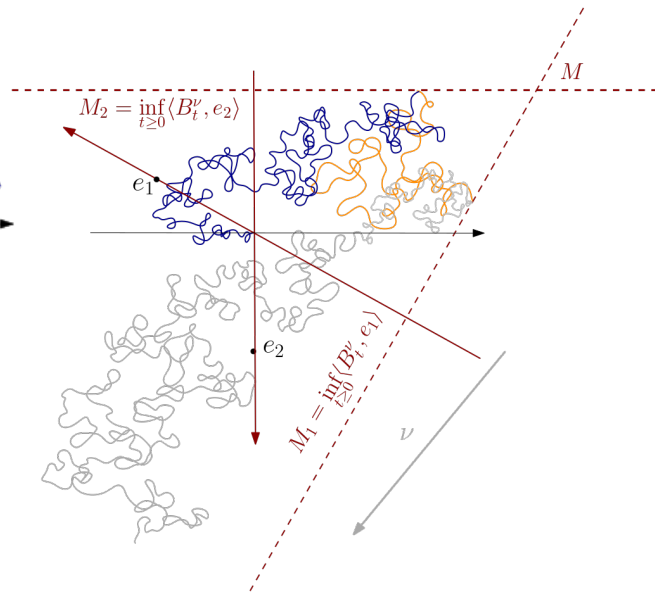


Figure 2.4: Decomposition of the path associated with A_2

In the subsequent section, Section 6.2, we will study some of the properties of the diffusion processes that appear in Theorem 2.3.3. To be more specific we will study the law of this process when its entrance point is taken close to infinity. Put differently we will study the limit where the drifted Brownian motion is conditioned on having a minimum M such that $\langle M, e_i \rangle \rightarrow -\infty$ for all $1 \leq i \leq r$. We will see that in these asymptotics the conditioned process will really look like the joining of $r + 1$ Brownian motions with drift and conditioned on staying in a subdomain of $\mathfrak{a}$.

2.3.2.2 Reflection coefficients and tail expansion of Gaussian multiplicative chaos measures

Based on the path decomposition unveiled in Section 6.1 we will describe in Sections 6.3 and 6.4 some of its implications on Gaussian multiplicative chaos as well as Toda correlation functions. To be more specific the path decomposition from 2.3.3 will be the starting point for the description of the tail expansion of Gaussian multiplicative chaos measures,

which in turn allows to provide a probabilistic representation of Toda reflection coefficients. Indeed we will show that conditioning correlated Gaussian multiplicative chaos measures on being large amounts to conditioning a certain drifted Brownian motion α on having its minimum $\mathbf{M}$ with $\langle \mathbf{M}, e_i \rangle \rightarrow -\infty$ for all $1 \leq i \leq r$.

To make this connection precise let us recall that if $\mathbf{X}$ is a Gaussian free field on the disk $\mathbb{D}$ with covariance given by

$$\mathbb{E}[\langle u, \mathbf{X}(x) \rangle \langle v, \mathbf{X}(y) \rangle] = \langle u, v \rangle \ln \frac{1}{|x - y|}$$

then the process $(\mathbf{X}_t)_{t \geq 0}$ defined by considering its averages on circles centered at the origin and with radii e^{-t}

$$\mathbf{X}_t := \frac{1}{2\pi} \int_0^{2\pi} \mathbf{X}(e^{-t+i\theta}) d\theta$$

is a Brownian motion over α . Therefore the r correlated Gaussian multiplicative chaos measures formally defined by considering

$$I_i(\alpha) = \int_{\mathbb{D}} |x|^{-\gamma \langle \alpha, e_i \rangle} e^{\langle \gamma e_i, \mathbf{X}(x) \rangle} (d^2 x) \quad \text{for } i = 1, \dots, r$$

can be rewritten under the form of

$$I_i(\alpha) = \int_0^{+\infty} e^{\langle \gamma e_i, \mathbf{X}_t + (\alpha - Q)t \rangle} Z_t^i dt$$

for some random measures $Z_t^i dt$ independent of $(\mathbf{X}_t)_{t \geq 0}$ stemming from the angular part of the free field. Additional details will be provided in Section 3.3.

We will show along Section 6.3 that the probability for the integrals $I_i(\alpha)$ to be large is governed by the probability that the minimum of the process $\left(-\mathbf{X}_t - \frac{\alpha - Q}{\gamma} t\right)_{t \geq 0}$ is small. The latter being a Brownian motion with drift $\frac{Q - \alpha}{\gamma} \in \mathcal{C}$ we can use the path decomposition described before and work within this setting. Based on this connection between Gaussian multiplicative chaos measures and drifted Brownian motions over α we are able to prove the following:

Theorem 2.3.4. *For $\alpha \in Q + \mathcal{C}_-$ close enough to Q , there exists a non-zero real number $R_s(\alpha)$ such that*

$$\mathbb{E} \left[\prod_{k=1}^r \left(\exp \left(-e^{\gamma \langle \mathbf{c}, e_{i_k} \rangle} I_{i_k}(\alpha) \right) - 1 \right) \right] \sim e^{\langle s(\alpha - Q) + Q - \alpha, \mathbf{c} \rangle} R_s(\alpha) \quad (2.3.15)$$

as $\langle \mathbf{c}, e_i \rangle \rightarrow -\infty$ for all $1 \leq i \leq r$ following a certain asymptotic which we explain in Section 6.3. $R_s(\alpha)$ is a Toda reflection coefficient, and is equal to

$$R_s(\alpha) = \epsilon(s) \frac{A(s(\alpha - Q))}{A(\alpha - Q)}, \quad \text{where}$$

$$A(\alpha) = \prod_{i=1}^r \left(\mu_i \pi l \left(\frac{\gamma^2 \langle e_i, e_i \rangle}{4} \right) \right)^{\frac{\langle \alpha, \omega_i^\vee \rangle}{\gamma}} \prod_{e \in \Phi^+} \Gamma \left(1 - \frac{\gamma}{2} \langle \alpha, e \rangle \right) \Gamma \left(1 - \frac{1}{\gamma} \langle \alpha, e^\vee \rangle \right). \quad (2.3.16)$$

Here $\Phi^+ := \{e_1, e_2, \rho\}$ is the set of positive roots while $\epsilon(s)$ is the signature of the permutation associated to s (see Section 3.1).

The expression of the reflection coefficients that arise in these asymptotics agree with that of Toda reflection coefficients considered in the physics literature [3, 2, 51], further highlighting the connection between the reflection principle in probability and the reflection phenomenon in conformal field theory. In the special case where $r = 1$ we recover the statement of [11, Proposition 4.1], where this asymptotic expansion allows to describe the scattering matrix associated to the Liouville Hamiltonian.

2.3.2.3 Asymptotics of correlation functions and a further analytic continuation

Because probabilistic correlation functions are defined using correlated Gaussian multiplicative chaos measures, the above statement has implications on the asymptotics of Toda correlation functions, and more precisely one the asymptotic of the expectation term in Equation (2.2.1) as $\langle c, e_i \rangle \rightarrow -\infty$. We will describe them in Proposition 6.4.2, and can be written down in the same fashion as in Theorem 2.3.4 in terms of reflection coefficients—but are slightly more involved.

This asymptotic expansion is at the heart of the analytic continuation of the correlation functions. Indeed one can subtract this asymptotic in the expectation term in Equation (2.2.1) to control the behaviour in the c variable of the integral over α . Thanks to this reasoning we are able to extend the range of definition of the correlation functions as stated below:

Theorem 2.3.5. *Assume that $\mathfrak{g} = \mathfrak{sl}_3$ and let $z_1, \dots, z_N \in \mathbb{C}$ be distinct. Define a subset of $(Q + \mathcal{C}_-)^N$ by setting*

$$\mathcal{A}_N := \left\{ (\alpha_1, \dots, \alpha_N) \in (Q + \mathcal{C}_-)^N \text{ s.t. for } i = 1, 2, \quad \left\langle \sum_{k=1}^N \alpha_k - 2Q, \omega_i \right\rangle > -\gamma \right. \\ \left. \text{and for any } 1 \leq k \leq N, \quad \min_{i=1,2} \langle \alpha_k - Q, e_i \rangle < -\gamma \right\}. \quad (2.3.17)$$

Then the correlation functions admit a probabilistic representation for $\alpha \in \mathcal{A}_N$ still denoted by $\langle \prod_{k=1}^N V_{\alpha_k}(z_k) \rangle$. This extension is such that the map $\alpha \mapsto \langle \prod_{k=1}^N V_{\alpha_k}(z_k) \rangle$ is meromorphic in a complex neighbourhood of $\mathcal{A}_N$.

The condition imposed on the weights correspond to the assumption that the asymptotic expansion of the correlation functions can be described thanks to Proposition 6.4.2. We stress that we can describe explicitly the poles and residues of the above meromorphic function over $\mathcal{A}_N$. We also highlight that as a corollary the reasoning developed to prove this statement allows to extend the range of validity over which a probabilistic representation of the DOZZ formula [84] can be defined. We refer to [30, Corollary 1.3] for additional details on this extension of the range of values for which the probabilistic representation of the DOZZ formula makes sense. We will prove Theorem 2.3.5 in Section 6.4.

2.4 . Computation of a family of correlation functions for the $\mathfrak{sl}_3$ Toda theory

In this final part of the manuscript we will gather all the ingredients disclosed in the previous chapters and combine them to prove one of the main results of this thesis: a rigorous derivation of the Fateev-Litvinov formula for a family of three-point correlation functions associated to the $\mathfrak{sl}_3$ Toda theory.

Implementing the bootstrap machinery allows to derive three-point correlation functions under the assumption that one of the three Vertex Operators being involved is a *semi-degenerate field*, by which is meant that its weight α^* is of the form $\alpha^* = \kappa\omega_i$ with $\kappa \in \mathbb{R}$, and with $i = 1, 2$. Without such an assumption proposing a formula for general three-point correlation functions remains an open question at the time being, even in the physics literature —see nonetheless the proposal in [101, 74] based on the AGT correspondence but still not fully explicit.

By combining methods inspired from the physics with more probabilistic ones we will prove in Chapter 7 that we are able to recover within our probabilistic framework the Fateev-Litvinov formula by showing that the following holds true:

Theorem 2.4.1. *Let the coupling constant satisfy $\gamma \in [1, \sqrt{2})$ and assume that $V_{\alpha_2^*}$ is a semi-degenerate field. Then as soon as $(\alpha_1, \alpha_2^*, \alpha_3)$ belongs to $\mathcal{A}_3$*

$$C_\gamma(\alpha_1, \alpha_2^*, \alpha_3) = C_\gamma^{\text{FL}}(\alpha_1, \alpha_2^*, \alpha_3) \quad (2.4.1)$$

where the left-hand side is a probabilistic three-point correlation function.

This statement is proved in the article [30]. The Fateev-Litvinov formula [52] that describes such three-point correlation functions is defined in terms of the special functions Υ and l and takes the form

$$C_\gamma^{\text{FL}}(\alpha_1, \alpha_2^*, \alpha_3) = \left(\pi \mu l \left(\frac{\gamma^2}{2} \right) \left(\frac{\gamma}{\sqrt{2}} \right)^{2-\gamma^2} \right)^{\frac{\langle 2Q - \alpha_1 - \alpha_2^* - \alpha_3, \rho \rangle}{\gamma}} \frac{\Upsilon'(0)^2 \Upsilon(\kappa) \prod_{e \in \Phi^+} \Upsilon(\langle Q - \alpha_1, e \rangle) \Upsilon(\langle Q - \alpha_3, e \rangle)}{\prod_{1 \leq j, k \leq 3} \Upsilon \left(\frac{\kappa}{3} + \langle \alpha_1 - Q, h_j \rangle + \langle \alpha_3 - Q, h_k \rangle \right)} \quad (2.4.2)$$

where $\alpha_2^* = \kappa\omega_2$. We refer to Section 7.3 for more background. This formula is reminiscent of the DOZZ formula for Liouville three-point correlation functions (and can actually be recovered from Equation (2.4.2)).

We will also explain what the conformal bootstrap procedure looks like in this context. Namely we will see how the computation of a family of four-point correlation functions can be reduced to that of the three-point correlation functions considered above, thus providing the final ingredient in the conformal bootstrap procedure for the $\mathfrak{sl}_3$ Toda theory.

2.4.1 . Three-point correlation functions and the Fateev-Litvinov formula

Let us review here the different steps leading to the proof of Theorem 2.4.1. As we will see, in order to compute the values of the three-point correlation functions considered there we will actually need to study certain four-point correlation functions. Indeed such four-point correlations contain a lot of information related to the symmetries of the model and as we will see, when investigating some of their asymptotics we will be able to recover the expression of the three-point correlation functions proposed in Theorem 2.4.1. The study of these asymptotics represent the mathematical counterpart of the *Operator Product Expansions* that are used in the physics literature.

2.4.1.1 A differential equation for certain four-point correlation functions

The four-point correlation functions we will consider corresponds to the three-point correlation functions studied in Theorem 2.4.1 to which we add another Vertex Operator V_α . This Vertex Operator is a *fully degenerate field*, which means that the weight α is of the form $\alpha = -\chi\omega_1$ with $\chi \in \{\gamma, \frac{2}{\gamma}\}$. The reason for choosing such a field stems from the fact that the presence of this field within correlation functions provides additional constraints on it. In Section 7.1 we will see that by combining these additional constraints with the ones coming from the assumption that $V_{\alpha_2^*}$ is semi-degenerate as well as the global Ward identities (2.3.14) we are able to compute explicitly these four-point correlation functions up to a global multiplicative factor:

Theorem 2.4.2. *Assume that V_α is a fully degenerate field while $V_{\alpha_2^*}$ is semi-degenerate. Further assume that $(\alpha, \alpha_1, \alpha_2^*, \alpha_3)$ belongs to $\mathcal{A}_4$. Then*

$$\langle V_\alpha(z)V_{\alpha_1}(0)V_{\alpha_2^*}(1)V_{\alpha_3}(\infty) \rangle = |z|^{-\langle \alpha, \alpha_0 \rangle} |z-1|^{-\langle \alpha, \alpha_2^* \rangle} \mathcal{H}(z), \quad \text{where} \\ \mathcal{H}(z) = C_\gamma(\alpha_1 + \alpha, \alpha_2^*, \alpha_3) \left(|\mathcal{H}_0(z)|^2 + \sum_{i=1}^2 A_\gamma^{(i)}(\alpha, \alpha_1, \alpha_2^*, \alpha_3) |\mathcal{H}_i(z)|^2 \right). \quad (2.4.3)$$

The constants $A_\gamma^{(i)}(\alpha, \alpha_1, \alpha_2^, \alpha_3)$, $i = 1, 2$, are explicit and given by a product of Γ functions, while the blocks $(\mathcal{H}_i)_{0 \leq i \leq 2}$ are ${}_3F_2$ hypergeometric functions. The only unknown quantity in this expression is the three-point correlation function $C_\gamma(\alpha_1 + \alpha, \alpha_2^*, \alpha_3)$.*

The proof of this statement is based on the fact that such four-point correlation functions are solutions of a hypergeometric differential equation in the z variable of the third order, which we refer to as a BPZ-type differential equation. The condition imposed on the form of the solution $\mathcal{H}$ is a consequence of the so-called *crossing symmetry* assumption made in the physics literature and that corresponds that different possible expansions of the solutions (around $z = 0$ and $z = \infty$) must be consistent with the probabilistic definition of the correlation functions. This statement strongly relies on the existence of an enhanced level of symmetry within the $\mathfrak{sl}_3$ Toda theory in that it is based on the global Ward identities as well as the existence of degenerate fields, which are inputs coming from a representation theoretical consideration of W -algebras.

2.4.1.2 From four-point to three-point correlation functions: Operator Product Expansions and shift equations

On the other hand, a consequence of the probabilistic representation of the correlation functions is that we can provide an alternative expansion of these four-point correlation functions. Indeed it is assumed in the physics literature that the operator product expansion $V_\alpha(z)V_{\alpha_1}(0)$ provides another way of obtaining the expression (2.4.3). Within our probabilistic framework recovering these operator product expansions boils down to the study of the asymptotics of the correlation functions as $z \rightarrow 0$, which in turn involves a careful treatment of the Gaussian multiplicative chaos measures considered.

By studying these asymptotics and with the *a priori* knowledge of the form of the correlation functions from Theorem 2.4.2 we prove in Section 7.2 that:

Theorem 2.4.3. *Under the same assumptions as in Theorem 2.4.2,*

$$\begin{aligned} \langle V_\alpha(z)V_{\alpha_1}(0)V_{\alpha_2^*}(1)V_{\alpha_3}(\infty) \rangle &= |z|^{-\langle \alpha, \alpha_0 \rangle} |z-1|^{-\langle \alpha, \alpha_2^* \rangle} \mathcal{H}(z), \quad \text{where} \\ \mathcal{H}(z) &= \sum_{i=0}^2 B_\gamma^{(i)}(\alpha_1, \chi) C_\gamma \left(\alpha_1 - \chi \left(\omega_1 - \sum_{j=1}^i e_j \right), \alpha_2^*, \alpha_3 \right) |\mathcal{H}_i(z)|^2. \end{aligned} \quad (2.4.4)$$

The coefficients $B_\gamma^{(i)}(\alpha_1, \chi)$ admit an explicit expression in terms of the Γ function, while the three-point correlation functions that appear are extended for $\alpha_1 - \chi h_{i+1}$ outside of $Q + \mathcal{C}_-$ via the relation

$$C_\gamma(\alpha_1, \alpha_2, \alpha_3) := R_s(\alpha_1) C_\gamma(Q + s(\alpha_1 - Q), \alpha_2, \alpha_3) \quad \text{where } s \text{ is such that } s(\alpha_1 - Q) \in \mathcal{C}_-. \quad (2.4.5)$$

From this statement we can infer the reflection relation $V_\alpha = R_s(\alpha) V_{Q+s(\alpha-Q)}$ presented above, as soon as this relation is inserted within the three-point correlation functions considered in Equation (2.4.5).

In contrast with the proof of Theorem 2.4.2 which is based on the constraints imposed by the symmetries of the theory, the proof of this statement strongly relies on certain asymptotic properties of Gaussian multiplicative chaos measures and is therefore much more probabilistic.

2.4.1.3 Conclusion of the proof of Theorem 2.4.1

We can now compare the two expansions from Theorems 2.4.2 and 2.4.3. The coefficients appearing in front of the hypergeometric functions in these expansions are given by either

$$\begin{aligned} &C_\gamma(\alpha_1 - \chi\omega_1, \alpha_2^*, \alpha_3) A_\gamma^{(i)}(-\chi\omega_1, \alpha_1, \alpha_2^*, \alpha_3) \quad \text{for Theorem 2.4.2 or} \\ &C_\gamma \left(\alpha_1 - \chi \left(\omega_1 - \sum_{j=1}^i e_j \right), \alpha_2^*, \alpha_3 \right) B_\gamma^{(i)}(\alpha_1, \chi) \quad \text{for Theorem 2.4.3.} \end{aligned}$$

Because the two expressions proposed for the four-point correlation functions must coincide and using linear independence of the hypergeometric functions considered, we can

infer the following pair of *shift equations* for the three-point correlation functions:

$$\frac{C_\gamma \left(\alpha_1 - \chi \left(\omega_1 - \sum_{j=1}^i e_j \right), \alpha_2^*, \alpha_3 \right)}{C_\gamma(\alpha_1 - \chi \omega_1, \alpha_2^*, \alpha_3)} = \frac{A_\gamma^{(i)}(-\chi \omega_1, \alpha_1, \alpha_2^*, \alpha_3)}{B^{(i)}(\alpha_1, \chi)} \quad (2.4.6)$$

which are valid for $i = 1, 2$ and $\chi \in \{\gamma, \frac{2}{\gamma}\}$, and as soon as the four-point correlation functions

$$\langle V_{-\chi \omega_1}(z) V_{\alpha_1}(0) V_{\alpha_2^*}(1) V_{\alpha_3}(\infty) \rangle$$

make sense. The assumption that such correlation functions are probabilistically well-defined when $\chi = \frac{2}{\gamma}$ stems from the fact that Equation (2.4.1) holds true only under the assumption that $\gamma \in [1, \sqrt{2})$. We will see in Section 7.3 that this set of shift equations characterizes, up to a multiplicative constant, the expression proposed in [52] for Toda three-point correlation functions. One can evaluate the value of this multiplicative constant and thus conclude for the proof of Theorem 2.4.1.

2.4.2 . Four-point correlation functions and the conformal bootstrap

Having computed the value of a family of three-point correlation functions in the $\mathfrak{sl}_3$ Toda theory, the next step in the conformal field theory textbook would be to carry out the recursive procedure at the heart of the conformal bootstrap machinery. So as to explain how such a method can be carried out we need to understand how the computation of a family of four-point correlation functions can be reduced to that of the three-point correlation functions considered above. In a work in progress together with Guillarmou [32] we aim to prove that this method takes the following form:

Theorem 2.4.4 (In preparation). *Assume that the Vertex Operators $V_{\alpha_2^*}$ and $V_{\alpha_3^*}$ are semi-degenerate fields. Further assume that the weights α satisfy the assumption that for $i = 1, 2$:*

$$\begin{aligned} \langle \alpha_k - Q, e_i \rangle &< 0 \quad \text{for all } 1 \leq k \leq 4, \quad \text{with} \\ \langle \alpha_1 + \alpha_2 - Q, \omega_i \rangle &> 0 \quad \text{and} \quad \langle \alpha_3 + \alpha_4 - Q, \omega_i \rangle > 0. \end{aligned} \quad (2.4.7)$$

Then

$$\begin{aligned} &\langle V_{\alpha_1}(0) V_{\alpha_2^*}(z) V_{\alpha_3^*}(1) V_{\alpha_4}(\infty) \rangle \\ &= \frac{1}{24\pi^2} \int_{\mathbb{R}^2} C_\gamma(\alpha_1, \alpha_2^*, Q - iP) C_\gamma(Q + iP, \alpha_3^*, \alpha_4) |z|^{2(\Delta_Q + iP - \Delta_{\alpha_1} - \Delta_{\alpha_2})} |\mathcal{F}_P^T(z)|^2 dP \end{aligned} \quad (2.4.8)$$

where the conformal blocks $\mathcal{F}_P^T$ only depend on the central charge of the theory as well as the conformal and W -quantum weights Δ_{α_k} and $w(\alpha_k)$ for $1 \leq k \leq 4$.

It is worth pointing out that while the spectrum in Liouville is given by the half-line $Q + i\mathbb{R}_+$ in the $\mathfrak{sl}_3$ Toda theory this spectrum would be contained in $Q + i\mathbb{R}^2$. Besides we would like to stress that a similar statement should hold true when we no longer assume two of the Vertex Operators to be semi-degenerate fields. In this general case the three-point correlation functions that appear in the integral over $\mathbb{R}^2$ are defined by analytic

continuation of their probabilistic representation, which is rather involved. Likewise the conformal blocks have a much more complicated structure in this general case.

The conformal blocks $\mathcal{F}_P^T$ that appear in the conformal bootstrap formula have a very valuable content in that they interact with a wide range of topics. Along the proof of Theorem 2.4.4 we will describe them in terms of fundamental quantities such as Shapovalov determinants, thus highlighting their importance in the setting of representation theory. Likewise these conformal blocks arise in the general formulation of the AGT correspondence [4, 140], where they are related to a key object in the study of certain four-dimensional gauge theories: Nekrasov instanton partition function. On a more geometrical perspective these conformal blocks naturally arise in the study of the quantization of the moduli space of flat connections [38].

Like in the derivation of the conformal bootstrap formula for Liouville theory [66], such a formula is shown to correspond to the Plancherel formula of a certain self-adjoint operator called the *Toda Hamiltonian*. This operator acts over a L^2 functional space called the *Hilbert space of the $\mathfrak{sl}_3$ Toda theory*. The understanding of this Hamiltonian requires a wide range of tools, from representation theory of W -algebra to scattering theory. In this manuscript we will only sketch the proof of this statement in order to keep the document concise and mainly based on probability theory, and since certain proofs are still in preparation. Additional details will be provided in the future work [32].

We will mostly describe the probabilistic setting in which such a formula can be proved and highlight the main steps leading to Theorem 2.4.4. Nonetheless we will explain how to construct a representation of the W -algebra associated to the $\mathfrak{sl}_3$ Toda theory based on the free-field theory underlying in the study of the Toda theories: this corresponds to the Segal-Sugawara representation. Understanding the free field theory is fundamental since many of the tools employed to study it can actually be reimplemented in the case of the interacting theory. We will also explain how to give a meaning to the power series expansion of the higher-spin tensor $\mathbf{W}$

$$\mathbf{W}(z_0) = \sum_{n \in \mathbb{Z}} \frac{\mathbf{W}_n(z)}{(z_0 - z)^{n+3}}$$

and propose a dynamical study of the Toda Hamiltonian which would allow to describe its spectral resolution based on tools from scattering theory. This is the content of Chapter 8.

Part I

Une approche probabiliste des théories conformes des champs de Toda

3 - Premières définitions et établissement du cadre

Les théories de Toda telles que définies dans la littérature physique s'appuient sur une définition formelle —par intégrale de chemin— d'une fonction aléatoire. Afin de définir de manière mathématiquement rigoureuse une telle fonction, nous aurons recours à de nombreux outils, qu'ils proviennent de la théorie des algèbres de Lie, de la géométrie conforme ou des probabilités.

Ce chapitre introductif s'attache à mettre en place le cadre nécessaire à une définition mathématique rigoureuse des théories de Toda. Pour ce faire les outils mathématiques fondamentaux entrant dans la formulation de ces théories seront présentés au cours de ce chapitre et fourniront la matière de base de l'étude mathématique des théories de Toda. Dans un premier temps nous exposerons ces notions cruciales puis établirons quelques propriétés basiques liées à celles-ci. Cette première approche devrait permettre de se familiariser avec des notions diverses, entre algèbres de Lie simples et complexes et objets probabilistes avancés tels les champs libres gaussiens, le chaos multiplicatif gaussien ou les processus de diffusion, en passant par des éléments basiques de géométrie conforme.

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3.1 . Some reminders on simple and complex Lie algebras

This section is dedicated to provide the necessary background on Lie algebras needed to make sense of Toda CFTs and their definition via the path integral. We will be very synthetic and provide only notions that will be used in the rest of the manuscript: in the textbook [72] the interested reader may find additional details on the objects exposed here.

3.1.1 . Finite-dimensional simple and complex Lie algebras

A simple Lie algebra $\mathfrak{g}$ is a (non-Abelian) Lie algebra that does not admit any proper, nonzero ideal. When the Lie algebra being considered is finite-dimensional and complex, this assumption is actually rather restrictive in that such Lie algebras are completely classified up to isomorphism. Namely a finite-dimensional, simple and complex Lie algebra is either isomorphic to a classical Lie algebra, that is one of the Lie algebras $(A_n)_{n \geq 1}$ (corresponding to $\mathfrak{sl}_{n+1}$), $(B_n)_{n \geq 2}$ (for $\mathfrak{o}_{2n+1}$), $(C_n)_{n \geq 3}$ ($\mathfrak{sp}_n$) and $(D_n)_{n \geq 4}$ ($\mathfrak{o}_{2n}$), or an exceptional Lie algebra, that is either E_6 , E_7 , E_8 , F_4 or G_2 .

To any finite-dimensional simple and complex Lie algebra $\mathfrak{g}$ is naturally attached an Euclidean space $(\alpha, \langle \cdot, \cdot \rangle)$. This finite-dimensional real vector space is such that the dual $\mathfrak{h}^*$ of the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ can be written under the form $\mathfrak{h}^* = \alpha \oplus i\alpha$, and comes equipped with a (positive definite) scalar product $\langle \cdot, \cdot \rangle$. This scalar product is inherited from the Killing form of $\mathfrak{g}$ in that both are proportional one to the other: the choice of a normalization will be fixed in the next paragraph. This Euclidean space is unique up to isomorphism and can be thought of as $\mathbb{R}^r$ endowed with its standard scalar product, where r is the *rank* of $\mathfrak{g}$. We will denote by $(v_i)_{1 \leq i \leq r}$ any orthonormal basis of α .

However this Euclidean space also comes with a special basis $(e_i)_{1 \leq i \leq r}$ made of so-called *simple roots*. This basis satisfies the property that

$$2 \frac{\langle e_i, e_j \rangle}{\langle e_i, e_i \rangle} = A_{i,j} \quad \text{for all } 1 \leq i, j \leq r \quad (3.1.1)$$

where A is the Cartan matrix of $\mathfrak{g}$. This matrix is explicit when $\mathfrak{g}$ is any of the aforementioned Lie algebras: for instance the A_n Cartan matrix is tridiagonal with 2 on the diagonal and -1 on the entries (i, j) with $|i - j| = 1$. In general the entries of this matrix are integral, equal to 2 on the diagonal and non-positive elsewhere; the matrix is invertible. However the matrix is not always symmetric: when this is the case (for $\mathfrak{g}$ being one of the A_n , B_n or E_n) the Lie algebra is said to be simply-laced.

As usually assumed in the physics literature, we choose to normalize the scalar product $\langle \cdot, \cdot \rangle$ so that the longest roots have squared norm 2, which we will do in the sequel. The renormalization constant used is given by $2h^\vee$, where $h^\vee$ is the so-called *dual Coxeter number*, an explicit positive integer that depends on the underlying Lie algebra.

It is very natural to introduce the basis of the *fundamental weights* $(\omega_i)_{1 \leq i \leq r}$, which is the basis of α^* (which we identify with α in the sequel) dual to that of the simple

roots. Namely it is defined by setting

$$\omega_i := \sum_{l=1}^r (A^{-1})_{i,l} e_l. \quad (3.1.2)$$

They are defined so that (δ_{ij} is the Kronecker symbol)

$$\langle e_i^\vee, \omega_j \rangle = \delta_{ij}, \quad \langle \omega_i, \omega_j \rangle = \sum_{l,l'=1}^r (A^{-1})_{i,l} A_{l,l'} (A^{-1})_{l',j} = (A^{-1})_{i,j} \quad (3.1.3)$$

where $e_i^\vee := 2 \frac{e_i}{\langle e_i, e_i \rangle}$ is the *coroot* associated to e_i . Whence by construction for any two vectors $\alpha, \beta \in \mathfrak{a}$

$$\langle \alpha, \beta \rangle = \sum_{i=1}^r \langle \alpha, \omega_i \rangle \langle \beta, e_i^\vee \rangle. \quad (3.1.4)$$

The *Weyl vector* is a key element of $\mathfrak{a}$ which is defined by

$$\rho := \sum_{i=1}^r \omega_i. \quad (3.1.5)$$

It naturally enjoys the property that $\langle \rho, e_i^\vee \rangle = 1$ for all $1 \leq i \leq r$. We will also consider the Weyl vector associated to the coroots by considering the vector $\rho^\vee = \sum_{i=1}^r \omega_i^\vee$ where the $(\omega_i^\vee)_{1 \leq i \leq r}$ are defined in such a way that $\langle \omega_i^\vee, e_j \rangle = \delta_{i,j}$ for all $1 \leq i, j \leq r$. The squared norm of the Weyl vector can be expressed explicitly in terms of the Lie algebra under consideration via the *Freudenthal-de Vries strange formula* for simple Lie algebras [60, Equation (47.11)]¹

$$|\rho|^2 = \frac{h^\vee \dim \mathfrak{g}}{12}. \quad (3.1.6)$$

Using the explicit values of $h^\vee$ and $\dim \mathfrak{g}$ this quantity is seen to given by

$$\begin{aligned} & \frac{n(n+1)(n+2)}{12} \text{ for } A_n, & \frac{n(2n-1)(2n+1)}{12} \text{ for } B_n, \\ & \frac{n(n+1)(2n+1)}{12} \text{ for } C_n, & \frac{(n-1)n(2n-1)}{6} \text{ for } D_n, \end{aligned} \quad (3.1.7)$$

and 78, $\frac{399}{2}$, 620, 39, $\frac{14}{3}$ for the exceptional Lie algebras E_6 , E_7 , E_8 , F_4 and G_2 . More generally we can explicitly compute the values of $|\rho^\vee|^2$ and $\langle \rho, \rho^\vee \rangle$ in all the cases considered (these quantities arise in the computation of the central charge of the Toda CFTs, see Table (4.2.7) below).

In this manuscript a specific attention will be given to the Toda CFT associated to the Lie algebra $\mathfrak{g} = \mathfrak{sl}_3$ (or A_2). In that case there are two simple roots e_1 and e_2 satisfying

$$\langle e_1, e_1 \rangle = \langle e_2, e_2 \rangle = 2 \quad \text{and} \quad \langle e_1, e_2 \rangle = -1. \quad (3.1.8)$$

¹This equation differs from the one in [60] by a multiplicative factor $2h^\vee$. This is due to our normalization convention for the scalar product $\langle \cdot, \cdot \rangle$ on $\mathfrak{a}^*$.

They can be thought of as vectors in $\mathbb{R}^2$ whose coordinates in its canonical basis are given by $e_1 \left(\sqrt{\frac{3}{2}}; -\frac{1}{\sqrt{2}} \right)$ and $e_2 (0; \sqrt{2})$. The dual basis and the Weyl vector then take the form

$$\omega_1 = \frac{2e_1 + e_2}{3} \quad \text{and} \quad \omega_2 = \frac{e_1 + 2e_2}{3} \quad \text{and} \quad \rho = e_1 + e_2. \quad (3.1.9)$$

The fundamental weights in the first fundamental representation π_1 of $\mathfrak{sl}_3$ with the highest weight ω_1 are defined by setting

$$h_1 := \frac{2e_1 + e_2}{3}, \quad h_2 := \frac{-e_1 + e_2}{3}, \quad h_3 := -\frac{e_1 + 2e_2}{3}. \quad (3.1.10)$$

These quantities are represented in Figure 3.1 below.

3.1.2 . Simple Lie algebras and reflection groups

Reflection groups are naturally associated to simple Lie algebras in that their Weyl group is actually a reflection group. This notion will prove to be crucial in the study of Toda CFTs and will naturally manifest itself via the existence of certain fundamental quantities called the *reflection coefficients*. Before making the connection between reflection groups and simple Lie algebras let us provide some reminders on this notion. We refer the reader to another textbook by Humphreys [73] for a more exhaustive insight on reflection groups.

The general framework is an Euclidean space $\mathbf{V}$ equipped with a scalar product $\langle \cdot, \cdot \rangle$ and associated norm $|x|^2 := \langle x, x \rangle$. On this Euclidean space we consider a **finite**² reflection group W (i.e. the realisation of a Coxeter group), that is a finite subgroup of the general linear group of $\mathbf{V}$ that is generated by finitely many reflections

$$s_\alpha : x \mapsto x - 2 \frac{\langle x, \alpha \rangle}{\langle \alpha, \alpha \rangle} \alpha \quad (3.1.11)$$

for $\alpha \in \mathbf{V}^* \simeq \mathbf{V}$, which are reflections across the hyperplanes $\{x \in \mathbf{V}, \langle x, \alpha \rangle = 0\}$. Associated to it is a *root system* of the reflection group $(W, \mathbf{V})$, defined as a finite subset Φ of $\mathbf{V} \setminus \{0\}$ such that

- (i) the elements of Φ span $\mathbf{V}$,
 - (ii) $\Phi \cap \mathbb{R}\alpha = \{\alpha, -\alpha\}$ for all $\alpha \in \Phi$,
 - (iii) $s_\alpha \Phi = \Phi$ for all $\alpha \in \Phi$.
- (3.1.12)

Its elements are called the *roots* and are such that W is generated by the $(s_\alpha)_{\alpha \in \Phi}$. Of particular significance are the *simple roots* $(e_i)_{1 \leq i \leq r}$ (with $r = \dim \mathbf{V}$) of W , which are linearly independent roots with the additional property that any element of the root system can be written as a linear combination, with coefficients of the same sign, of simple roots³. We further introduce the subset Φ^+ of Φ made of *positive roots*, that is roots of the form

$$\alpha^+ = \sum_{i=1}^r \lambda_i e_i \quad \text{with } \lambda_i \geq 0.$$

²Reflection groups will be implicitly taken finite in what follows.

³Different choices of simple roots are of course possible, but will be related by conjugation under W . Such a simple system always exists [73, Section 1.3]

The hyperplanes orthogonal to the roots divide the space into finitely many connected components, called *Weyl chambers*, on which W acts freely and transitively. We denote by

$$\mathcal{C} := \{x \in \mathbf{V}, \langle x, e_i \rangle > 0 \text{ for all } 1 \leq i \leq r\} \quad (3.1.13)$$

the *fundamental Weyl chamber* and set $\mathcal{C}_- := -\mathcal{C}$ (which is also a Weyl chamber). The boundary $\partial\mathcal{C}$ of this chamber is made of r components $(\partial\mathcal{C}_i)_{1 \leq i \leq r}$ which are defined by

$$\partial\mathcal{C}_i := \{x \in \mathbf{V}, \langle x, e_i \rangle = 0\} \cap \partial\mathcal{C} \quad (3.1.14)$$

and which we refer to as *walls*. We have represented in Figure 3.1 the six Weyl chambers and elements of the reflection group in the case where this reflection group is associated to the Lie algebra $\mathfrak{g} = \mathfrak{sl}_3$.

Any element s of the reflection group can be written as a composition of reflections orthogonal to the simple roots: $s = s_{i_1} \cdots s_{i_k}$. It admits a reduced expression, that is a product of the above form with a minimal number of simple reflections. This number is the *length* of s , $l(s)$, such that $\det(s) = (-1)^{l(s)}$. In the sequel we will denote by $\epsilon(s)$ the above quantity —by analogy with the signature of a group of permutations.

Let us now explain the connection between reflection group and simple Lie algebras. Consider $\mathfrak{g}$ as in the previous subsection and the Euclidean space $\mathfrak{a}$ associated to it together with its basis of simple roots $(e_i)_{1 \leq i \leq r}$. Then the group generated by the reflections orthogonal to the simple roots is a reflection group: it is called the *Weyl group* of $\mathfrak{g}$. The notion of coroot and of fundamental weights remains valid in the general setting of reflection groups and we will use the same notation in this context too. In passing we note that the *Weyl vector* ρ can alternatively be defined as the half-sum of the positive roots:

$$\rho := \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha.$$

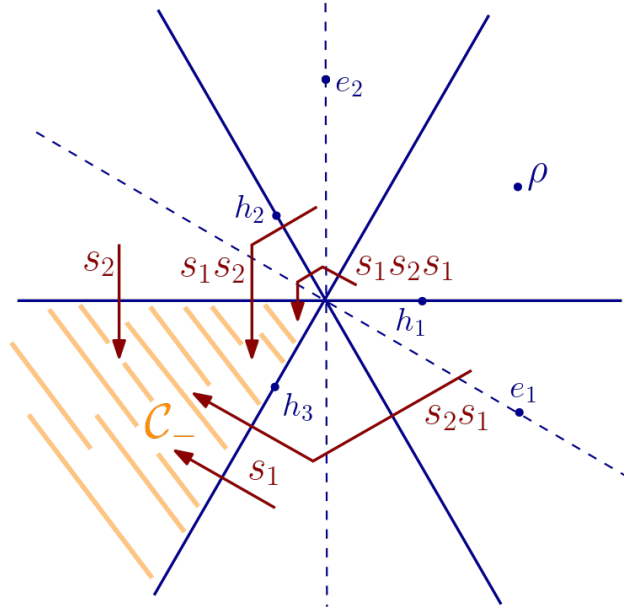


Figure 3.1: The root system and the reflection group associated to $\mathfrak{g} = \mathfrak{sl}_3$

3.2 . The probabilistic framework: Gaussian free fields, Gaussian multiplicative chaos and conditioned Brownian motions

With basic notions on simple and complex Lie algebras at hand, we are now in position to properly make sense of the action functional that enters the path integral formally defining the Toda CFTs. However making sense of the path integral in itself will be more involved and will require the introduction of a probabilistic framework that involves two key probabilistic objects: Gaussian free fields and Gaussian multiplicative chaos. Therefore and before actually providing a mathematically rigorous definition of the Toda CFTs we will present these two notions in the coming section. We will also discuss the notion of conditioned Brownian motion, which will naturally arise in the study of the reflection coefficients associated to Toda CFTs as disclosed in Chapter 6.

3.2.1 . Conformal geometry on the Riemann sphere

The interplays between Gaussian Free Fields (GFFs hereafter) and conformal geometry have proved to be particularly thriving in that GFFs naturally arise in a wide range of topics related to conformal geometry. For instance the quantization of the uniformization problem in even dimensions — related to finding a conformal metric with constant Q -curvature within the conformal class of an even-dimensional Riemannian manifold (M, g) — can be formulated based on a GFF (see e.g. [29, 41] for more on this topic). However in this document we are only concerned with manifolds conformally equivalent to the two-dimensional sphere so we will not enter into much details related to such issues and focus only on the notions relevant for our purpose.

3.2.1.1 Metrics on the Riemann sphere

Indeed throughout this manuscript we will focus on Toda CFTs described by a field ϕ taking values on the two-dimensional sphere $\mathbb{S}^2$, viewed as a Riemannian manifold $(\mathbb{S}^2, g_S)$ with g_S the standard metric on the sphere. The stereographic projection allows to conformally map $(\mathbb{S}^2, g_S)$ to the (compactified) plane (*i.e.* the Riemann sphere) which we view both as $\mathbb{R}^2 \cup \{\infty\}$ and $\mathbb{C} \cup \{\infty\}$ equipped with the “round” metric $\hat{g}$:

$$\hat{g} := \frac{4}{(1 + |z|^2)^2} |dz|^2. \quad (3.2.1)$$

We will also consider differentiable conformal metrics of the form $g = e^\phi \hat{g}$ with $\phi \in \bar{C}^1(\mathbb{R}^2)$ where, for $k \geq 0$, $\bar{C}^k(\mathbb{R}^2)$ stands for the space of functions $\phi : \mathbb{R}^2 \rightarrow \mathbb{R}$ for which both ϕ and $x \mapsto \phi(1/x)$ are k -times differentiable with continuous derivatives. Of special interest will be the metric

$$g_+(z) := |z|_+^{-4} |dz|^2 \quad (3.2.2)$$

with $|z|_+ := \max(|z|, 1)$. As we will see in the study of the conformal bootstrap procedure in Chapter 8 this metric is particularly relevant from the perspective of reflection positivity (also referred to as Osterwalder-Schrader positivity).

We will work in what follows with such metrics g on the plane, for which we will denote by ∂_g the gradient, Δ_g the Laplace-Beltrami operator, $R_g = -\Delta_g \ln \sqrt{\det g}$ the Ricci scalar curvature and v_g the volume form. We identify $\mathbb{R}^2$ and its tangent space, so that if f, h are maps $\mathbb{R}^2 \rightarrow \mathfrak{a}$, we can make sense of the inner products $(f, h)_g$ or $(\partial_g f, \partial_g h)_g$, defined with respect to the metric g , by setting $(\partial_g f(z), \partial_g h(z))_g := (\langle \partial_g f(z), \partial_g h(z) \rangle)_g$ ($|\cdot|_g$ will stand for the associated norm). The associated L^2 scalar product is defined by considering

$$\langle f, h \rangle_g := \int_{\mathbb{R}^2} (f(x), h(x))_g dv_g(x)$$

for f, h smooth and compactly supported. When no index is given, this means that the object has to be understood in terms of the usual Euclidean metric on the plane (*i.e.* ∂ , Δ , R , v and $(\cdot, \cdot)$). Since the stereographic projection is an isometry, we already know that the spherical metric $\hat{g}$ is such that $R_{\hat{g}} = 2$ (its Gaussian curvature is 1) with total mass $v_{\hat{g}}(\mathbb{R}^2) = 4\pi$.

Two metrics g and g' will be said to be conformally equivalent when

$$g = e^\phi g'$$

for some $\phi \in \bar{C}^1(\mathbb{R}^2)$. It is readily seen that as soon as g' is in the conformal class of the spherical metric—that is when $g' = e^\phi \hat{g}$ with $\phi \in \bar{C}^1(\mathbb{R}^2)$ —one has $\int_{\mathbb{R}^2} (|\partial_{g'} \phi|_{g'}^2 + 2R_{g'} \phi) dv_{g'} < \infty$. Furthermore, for $\phi \in \bar{C}^2(\mathbb{R}^2)$, the curvatures of two such metrics are related by the relation

$$R_g = e^{-\phi} (R_{g'} - \Delta_{g'} \phi). \quad (3.2.3)$$

In the sequel and for given metrics g and $h \in \bar{C}^1(\mathbb{R}^2)$, we will denote by $m_g(h)$ the mean value of h in the metric g , that is the quantity

$$m_g(h) := \frac{1}{v_g(\mathbb{R}^2)} \int_{\mathbb{R}^2} h(z) v_g(dz) \quad (3.2.4)$$

and work in the Sobolev space $H^1(\mathbb{R}^2, g)$, which is the closure of $C_c^\infty(\mathbb{R}^2)$ with respect to the Hilbert-norm

$$\int_{\mathbb{R}^2} h(x)^2 v_g(dx) + \int_{\mathbb{R}^2} |\partial_g h(x)|_g^2 v_g(dx). \quad (3.2.5)$$

The continuous dual of $H^1(\mathbb{R}^2, g)$ will be denoted $H^{-1}(\mathbb{R}^2, g)$. At this stage it may be worth noticing that the Dirichlet energy is a conformal invariant, that is to say is independent of the metric within a given conformal class:

$$\int_{\mathbb{R}^2} |\partial_{g'} h(x)|_{g'}^2 v_{g'}(dx) = \int_{\mathbb{R}^2} |\partial_g h(x)|_g^2 v_g(dx). \quad (3.2.6)$$

3.2.1.2 Conformal transformations of the Riemann sphere

Among smooth transformations of the Riemann sphere, there is a family that plays a key role in the framework of conformal geometry: the group of conformal transformations. These are smooth maps $\psi : (\mathbb{C} \cup \{\infty\}, \hat{g}) \rightarrow (\mathbb{C} \cup \{\infty\}, \hat{g})$ such that the pullback of the

metric $\hat{g}$ by ψ is conformally equivalent to $\hat{g}$. Put differently if $g_\psi(z) := |\psi'(z)|^2 g(\psi(z))$, then the two Riemannian metrics $\hat{g}$ and $\hat{g}_\psi$ are conformally equivalent. These transformations are completely classified and form the Möbius group. Its elements are the maps $\psi : \mathbb{C} \rightarrow \mathbb{C}$ of the form

$$\psi(z) = \frac{az + b}{cz + d}$$

where the complex numbers a, b, c, d satisfy $ad - bc \neq 0$.

Among others, Möbius transformations satisfy the following remarkable properties:

- The cross-ratio of four distinct complex points z_1, z_2, z_3, z_4

$$\frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}$$

is invariant under Möbius transformations.

- Given f a meromorphic function over $\mathbb{C}$, its Schwarzian derivative is defined by

$$Sf(z) := \frac{f'''(z)}{f'(z)} - \frac{3}{2} \left(\frac{f''(z)}{f'(z)} \right)^2.$$

Then $Sf = 0$ if and only if f is a Möbius transformation.

3.2.1.3 Green kernels

Given a metric g on the Riemann sphere that is conformally equivalent to the spherical metric $\hat{g}$, we denote by G_g the Green function of the problem

$$\Delta_g u = -2\pi (f - m_g(f)) \quad \text{on } \mathbb{R}^2, \quad \int_{\mathbb{R}^2} u(x) v_g(dx) = 0$$

where f belongs to the space $L^2(\mathbb{R}^2, g)$ and u is in $H^1(\mathbb{R}^2, g)$. Put differently the solution u can be expressed as

$$u = \int_{\mathbb{R}^2} G_g(\cdot, x) f(x) v_g(dx) =: G_g f \quad (3.2.7)$$

with $m_g(G_g(x, \cdot)) = 0$ for all $x \in \mathbb{R}^2$. The kernel G_g actually has an explicit expression given by (see [40, Equation (2.9)])

$$G_g(x, y) = \ln \frac{1}{|x - y|} - m_g \left(\ln \frac{1}{|x - \cdot|} \right) - m_g \left(\ln \frac{1}{|y - \cdot|} \right) + \theta_g \quad (3.2.8)$$

where

$$\theta_g := \frac{1}{v_g(\mathbb{R}^2)^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln \frac{1}{|x - y|} v_g(dx) v_g(dy).$$

For future reference we shed light on the expression of this Green function for the metrics $\hat{g}$ and g_+ :

$$\begin{aligned} G_{\hat{g}}(x, y) &= \ln \frac{1}{|x - y|} - \frac{1}{4} (\ln \hat{g}(x) + \ln \hat{g}(y)) + \ln 2 - \frac{1}{2} \\ G_{g_+}(x, y) &= \ln \frac{1}{|x - y|} + \ln |x|_+ + \ln |y|_+. \end{aligned} \quad (3.2.9)$$

These Green functions naturally arise in the study of conformal geometry on the sphere since they satisfy a remarkable property of covariance under conformal transformations, that is the following fact:

Lemma 3.2.1 (Conformal covariance of Green functions). *Let ψ be a Möbius transform of the Riemann sphere and g be a Riemannian metric conformally equivalent to the spherical one. Then*

$$G_{g_\psi}(x, y) = G_g(\psi(x), \psi(y)). \quad (3.2.10)$$

See [40, Proposition 2.2] for instance for a justification of this equality. In particular this implies that for the spherical metric:

$$G_{\hat{g}}(\psi(x), \psi(y)) = G_{\hat{g}}(x, y) - \frac{1}{4}(\phi(x) + \phi(y)) \quad (3.2.11)$$

where ϕ is such that $e^\phi = \frac{\hat{g}_\psi}{\hat{g}}$.

3.2.2 . Gaussian free fields

3.2.2.1 GFFs on the Riemann sphere

The quantum field that allows to describe Toda CFTs can be seen as a random (generalized) function over the Riemann sphere $(\mathbb{C}, g)$ with values in a Euclidean space $(\mathfrak{a}, \langle \cdot, \cdot \rangle)$. This random function can be properly defined based on a vectorial GFF $\mathbf{X}^g$, which is a Gaussian random distribution whose covariance kernel is given by the Green kernel G_g . Namely, this random distribution formally satisfies the property that for any pair of vectors $u, v \in \mathfrak{a}$ and $x \neq y$ in $\mathbb{C}$:

$$\mathbb{E} [\langle u, \mathbf{X}^g(x) \rangle \langle v, \mathbf{X}^g(y) \rangle] = \langle u, v \rangle G_g(x, y). \quad (3.2.12)$$

Due to the singularity of the Green kernel on the diagonal $x = y$, the latter is not a proper function but rather belongs to a distributional space $H^{-1}(\mathbb{C}, g)$ (see [46, 125] for more details on properties and construction of GFFs). However the GFF $\mathbf{X}^g$ is characterized by the property that there exists a probability space such that for any pair f, h of compactly supported smooth functions $\mathbb{C} \rightarrow \mathfrak{a}$ the random variables $\langle f, \mathbf{X}^g \rangle_g$ and $\langle h, \mathbf{X}^g \rangle_g$ are Gaussian variables over this probability space and their covariance is given by

$$\mathbb{E} [\langle f, \mathbf{X}^g \rangle_g \langle h, \mathbf{X}^g \rangle_g] = \langle G_g f, h \rangle_g. \quad (3.2.13)$$

Hereafter we will denote by $\mathbf{X}$ the GFF associated to the metric g_+ .

The lack of regularity of the GFFs $\mathbf{X}^g$ prevents them from being pointwise defined. A convenient way to overcome this issue is to work instead with a regularization of the field. More specifically let us define a smooth approximation of $\mathbf{X}$ by setting

$$\mathbf{X}_\varepsilon^g := \mathbf{X}^g * \eta_\varepsilon = \int_{\mathbb{C}} \mathbf{X}^g(\cdot - z) \eta_\varepsilon(z) d^2 z \quad (3.2.14)$$

with $\eta_\varepsilon := \frac{1}{\varepsilon^2} \eta(\frac{\cdot}{\varepsilon})$ a smooth and compactly supported mollifier. For the GFF $\mathbf{X}$ we will also consider its circle-averages as a regularization of the field, that is set

$$\mathbf{X}_t(z) := \frac{1}{2\pi} \int_0^{2\pi} \mathbf{X}(z + e^{-t+i\theta}) d\theta \quad (3.2.15)$$

for $t > 0$. They correspond to averaging the GFF $\mathbf{X}$ on the boundary of $B_t(z) := \{x \in \mathbb{C}, |x - z| < e^{-t}\}$.

3.2.2.2 GFFs on the unit disk and its boundary

We will also consider GFFs taking values on the unit disk $\mathbb{D}$ as well as over its boundary $\partial\mathbb{D}$ (which we will also denote by $\mathbb{S}^1$ or $\mathbb{T}$). The first one, $\mathbf{X}_{\mathbb{D}}$, is chosen to have boundary conditions, by which we mean that its covariance kernel is given by

$$\mathbb{E}[\langle u, \mathbf{X}_{\mathbb{D}}(x) \rangle \langle v, \mathbf{X}_{\mathbb{D}}(y) \rangle] = \langle u, v \rangle G_{\mathbb{D}}(x, y), \quad G_{\mathbb{D}}(x, y) := \ln \frac{|1 - x\bar{y}|}{|x - y|} \quad (3.2.16)$$

for any u, v in $\mathfrak{a}$ and $x, y \in \mathbb{D}$, while $\mathbf{X}_{\mathbb{D}}$ is taken to be 0 outside of the disk. Its counterpart on $\mathbb{C} \setminus \bar{\mathbb{D}}$ has the law of $\mathbf{X}_{\mathbb{D}^c} \stackrel{(\text{law})}{=} \mathbf{X}_{\mathbb{D}} \circ \theta$, where $\theta(z) := \frac{1}{\bar{z}}$.

A GFF φ on the circle $\mathbb{S}^1$ admits the expansion

$$\varphi := \sum_{n \in \mathbb{Z}^*} \sum_{i=1}^r \varphi_{n,i} \mathbf{e}_{n,i} \quad (3.2.17)$$

where $\mathbf{f}_{n,j}(e^{i\theta}) := e^{in\theta} \mathbf{v}_j$. For positive n the modes $\varphi_{n,i}$, $i = 1, 2$, are centered, complex, independent Gaussian variables with variance $\frac{1}{2n}$, while for negative n we have $\varphi_{n,i} = \bar{\varphi}_{-n,i}$. For future reference note that the harmonic extension of φ takes the form

$$P\varphi(z) := \sum_{n \geq 1} \sum_{i=1}^2 (\varphi_{n,i} z^n + \bar{\varphi}_{n,i} \bar{z}^n) \mathbf{f}_i. \quad (3.2.18)$$

3.2.2.3 Connection between these GFFs

We can actually relate the different GFFs considered up to now via the following decomposition:

$$\mathbf{X} \stackrel{(\text{law})}{=} P\varphi + \mathbf{X}_{\mathbb{D}} + \mathbf{X}_{\mathbb{D}^c} \quad (3.2.19)$$

where $P\varphi$ denotes the harmonic extension to $\mathbb{C}$ of the GFF $\varphi : \mathbb{S}^1 \rightarrow \mathfrak{a}$, while $\mathbf{X}_{\mathbb{D}}$ and $\mathbf{X}_{\mathbb{D}^c}$ are two independent (vectorial) GFFs as introduced above. Hereafter we will assume that such GFFs live in a probability space denoted $(\Omega, \Sigma, \mathbb{P})$. The above decompositions translate as $\Omega = \Omega_{\mathbb{T}} \times \Omega_{\mathbb{D}} \times \Omega_{\mathbb{D}^c}$ with similar decompositions for Σ and $\mathbb{P}$.

3.2.3 . Gaussian multiplicative chaos

In order to provide a rigorous meaning to the path integral defining Toda CFTs we need to make sense of the exponential of the Toda field Φ . As we will explain in Section 4.1 Φ will exhibit the same lack of regularity as the GFFs $\mathbf{X}^g$ and hence has to be thought of as a random distribution. The theory of Gaussian multiplicative chaos [76, 115] allows to address this issue by giving a rigorous meaning of the quantity $e^{\langle \Phi, \gamma e_i \rangle}$ where γ is a real parameter which we will choose in $(0, \sqrt{2})$.

For this purpose we first consider the regularized GFF $\mathbf{X}_{\varepsilon}^g$: the latter being smooth the quantity $e^{\langle \mathbf{X}_{\varepsilon}^g(z), \gamma e_i \rangle}$ is a well-defined random variable, whose expectation is given by $e^{\frac{1}{2} \mathbb{E}[\langle \mathbf{X}_{\varepsilon}^g(z), \gamma e_i \rangle^2]}$. Because the variance that appears in this exponential is asymptotically equivalent to $\frac{\gamma^2 \langle e_i, e_i \rangle}{2} \ln \frac{1}{\varepsilon}$, it is therefore natural to consider the sequence of appropriately renormalized random measures over $\mathbb{C}$

$$M_{g,\varepsilon}^{\gamma e_i}(dz) := \varepsilon^{\frac{\gamma^2 \langle e_i, e_i \rangle}{2}} e^{\langle \mathbf{X}_{\varepsilon}^g(z) + \frac{Q}{2} \ln g(z), \gamma e_i \rangle} dv(z) \quad (3.2.20)$$

where recall that $Q = \gamma\rho + \frac{2}{\gamma}\rho^\vee$. The reason for incorporating the extra term $\frac{Q}{2} \ln g$ in the definition of the GMC measures will be made clear in the interpretation of Toda path integral, see Section 4.1. It may be worth noting that since, in agreement with [40, Proposition 2.5],

$$\mathbb{E} [\langle X_\varepsilon^{\hat{g}}(z), e_i \rangle^2] = \langle e_i, e_i \rangle \left(-\ln \varepsilon - \frac{1}{2} \ln \hat{g}(z) + \theta_\eta + o(1) \right)$$

as ε goes to 0 and with $\theta_\eta := \int_{\mathbb{C}} \int_{\mathbb{C}} \eta(x) \eta(y) \ln \frac{1}{|x-y|} v(dx) v(dy) + \ln 2 - \frac{1}{2}$, the GMC measure defined above for $g = \hat{g}$ and the limiting measure defined by

$$\lim_{\varepsilon \rightarrow 0} e^{\langle \gamma e_i, X_\varepsilon^{\hat{g}}(z) + \frac{\gamma}{2} \theta_\eta e_i \rangle - \frac{1}{2} \mathbb{E} [\langle \mathbf{X}_\varepsilon^{\hat{g}}(z), \gamma e_i \rangle^2]} dv_g(z)$$

actually define the same random measure.

Then a crucial statement is that the limit as $\varepsilon \rightarrow 0$ of this random measure does exist and defines a random measure referred to as a *Gaussian Multiplicative Chaos measure* (GMC hereafter). More precisely under the assumption that $0 < \gamma < \sqrt{2}$ ⁴ and for $1 \leq i \leq r$, the limit

$$M_g^{\gamma e_i}(dz) := \lim_{\varepsilon \rightarrow 0} M_{g,\varepsilon}^{\gamma e_i}(dz) \quad (3.2.21)$$

holds in probability within the space of Radon measures equipped with the weak topology and defines a non-trivial random measure [14, 124].

3.2.4 . Conditioned Brownian motion

We will explain in Subsection 3.3.2.3 below that GFFs and Brownian motions are naturally connected one to the other. A consequence of this link is that Brownian motions with values in $\mathfrak{a}$ arise in the mathematical study of Toda CFTs, and as we will see in Chapter 6 the strong symmetries related to the reflection group associated to $\mathfrak{a}$ will have strong implications on these Brownian motions. To make explicit these consequences we provide here the necessary notions on conditioning of diffusion processes needed for our purpose. We will restrict our attention here to the special case where the diffusion process being considered is derived from a Brownian motion in $\mathfrak{a}$, though the reasoning remains valid for general diffusion processes. The interested reader may find a more detailed discussion in the article [31] where a more general framework is considered. For more details on the definitions and properties of the objects involved we also refer to the textbooks [118, 119] and [130] where the general theory of diffusions is described in great details.

Throughout this manuscript we will consider a (continuous) diffusion process B with state space $(V, \langle \cdot, \cdot \rangle)$ some Euclidean space, and infinitesimal generator $\mathcal{A}$ of the form

$$\mathcal{A}f(x) = \frac{1}{2} \Delta f(x) + b(x) \cdot \nabla f(x) \quad (3.2.22)$$

with b bounded and Lipschitz. We denote by $p_t(x, y)$ the transition semigroup of B . For $\nu \in \mathcal{C}$ the fundamental Weyl chamber from Equation (3.1.13) we also introduce the

⁴Note that the assumption that $\gamma < \sqrt{2}$ differs from the usual one that $\gamma < 2$. This comes from the fact that the longest roots have length $\sqrt{2}$ rather than 1.

notation

$$p_\nu(t; x, y) := p_t(x, y) \exp \left(\langle \nu, y - x \rangle - \frac{|\nu|^2}{2} t \right) \quad (3.2.23)$$

with $p_t = \frac{1}{(2\pi t)^{\frac{d}{2}}} \exp \left(-\frac{|x-y|^2}{2t} \right)$ the transition probabilities of a standard Brownian motion, and which represent the transition probabilities of a Brownian motion with constant drift $\nu \in \mathcal{C}$.

Conditioning of such processes can be done via Doob's conditioning, based on so-called h -transforms and defined as follows (see e.g. [37, Chapter 11] for additional details and justifications). Let h be a C^2 function that is $\mathcal{A}$ -harmonic on $\mathbf{V}$, and let us introduce the notations $\mathbf{V}_h := \{x \in \mathbf{V}, \quad 0 < h(x) < \infty\}$ and $\mathbf{p}_t^h(x, y)$ for the transition probabilities of the process $\mathbf{B}$ killed upon exiting $\mathbf{V}_h$. Then the Doob h -transform of $\mathbf{B}$ is the continuous Markov process $\mathbf{B}^h$ with transition probabilities given by

$$\begin{aligned} \mathbf{p}_t^h(x, dy) &:= \frac{h(y)}{h(x)} \mathbf{p}_t^h(x, dy) \quad \text{for } x \text{ inside } \mathbf{V}_h \\ &= 0 \quad \text{otherwise.} \end{aligned} \quad (3.2.24)$$

It is a diffusion process with generator

$$\mathcal{A}^h = \mathcal{A} + \nabla \log h \cdot \nabla. \quad (3.2.25)$$

It follows from [37, Remark 11.4] that such a process $\mathbf{B}^h$ started inside $\mathbf{V}_h$ will, almost surely, never hit the boundary of $\mathbf{V}_h$. In the special case where h is given by the probability that the process $\mathbf{B}$, started from x , never exits a domain of the form $\mathbf{M} + \mathcal{C}$ for some $\mathbf{M} \in \mathbf{V}$, the process $\mathbf{Y}$ has the law of $\mathbf{X}$ conditioned to stay inside $\mathbf{M} + \mathcal{C}$ at all times.

Some properties of h -transforms include the fact that the process $h(\mathbf{B}_t)$ is a (local) positive martingale, while the process $\mathbf{B}$ is such that

$$d\mathbb{P}_{\mathbf{B}}|_{\mathcal{F}_t} = \frac{h(\mathbf{B}_t)}{h(\mathbf{B}_0)} d\mathbb{P}_{\mathbf{B}}|_{\mathcal{F}_t}$$

where $(\mathcal{F}_t)_{t \geq 0}$ is the standard filtration of the processes.

3.3 . Basic properties of the probabilistic tools

Having properly introduced the probabilistic tools we will work with in the sequel, in this section we highlight some of their features that will prove to be crucial in the rest of the document. To be more specific we will detail here some additional properties enjoyed by the GFFs and GMC measures considered in this manuscript in the realm of Toda CFTs.

3.3.1 . A toolbox on Gaussian fields

To start with, let us first present some general statements inherited from the Gaussian structure of the objects considered up to now. These are now classical results for which we will not provide proofs but redirect the reader to references where they are carried out.

3.3.1.1 Girsanov theorem and Gaussian integration by parts

The first statement we are interested in –and that we will repeatedly use in the present manuscript– is the celebrated *Girsanov theorem* (or Cameron-Martin) theorem, see e.g. Chapter VIII in [114]:

Theorem A (Girsanov theorem). *Let D be a subdomain of $\mathbb{C}$ on which is defined a family of smooth centered Gaussian fields*

$$(\mathbf{X}(x))_{x \in D} := (X_1(x), \dots, X_{n-1}(x))_{x \in D}.$$

Assume that Z is a Gaussian variable belonging to the L^2 closure of the subspace spanned by $(\mathbf{X}(x))_{x \in D}$. Then, for any bounded functional F over the space of continuous functions one has that

$$\mathbb{E} \left[e^{Z - \frac{\mathbb{E}[Z^2]}{2}} F(\mathbf{X}(x))_{x \in D} \right] = \mathbb{E} [F(\mathbf{X}(x) + \mathbb{E}[Z\mathbf{X}(x)])_{x \in D}].$$

We will apply this statement to the non-regular GFF $\mathbf{X}$ by considering first its regularization and then consider a limiting procedure.

Another property that will be key in our study is *Gaussian integration by parts* and can actually be thought of as a specific instance of Malliavin calculus. For this purpose recall the standard formulation of Gaussian integration by parts for a centered Gaussian vector $(Z, Y_1, \dots, Y_N)$ and f a smooth function on $\mathbb{R}^N$ with bounded derivatives:

$$\mathbb{E} [Z f(Y_1, \dots, Y_N)] = \sum_{k=1}^N \mathbb{E} [Z Y_k] \mathbb{E} [\partial_{Y_k} f(Y_1, \dots, Y_N)]. \quad (3.3.1)$$

From this statement and after regularization and a proper limiting procedure, one can infer the following equality for the GFF $\mathbf{X}$

$$\mathbb{E} [\langle u, \mathbf{X}(z) \rangle e^{\langle \mathbf{X}, f \rangle_{\mathbb{D}}}] = \int_{\mathbb{D}} G(z, x) \langle u, f(x) \rangle \mathbb{E} [e^{\langle \mathbf{X}, f \rangle_{\mathbb{D}}}] dx \quad (3.3.2)$$

for $f \in C_0^\infty(\mathbb{D} \rightarrow \mathfrak{a})$, $z \in \mathbb{D}$ and $u \in \mathfrak{a}$, as well as its counterpart statement for $\mathbf{X}_{\mathbb{D}}$:

$$\mathbb{E}_{\varphi} [\langle u, \mathbf{X}_{\mathbb{D}}(z) \rangle e^{\langle \mathbf{X}, f \rangle_{\mathbb{D}}}] = \int_{\mathbb{D}} G_{\mathbb{D}}(z, x) \langle u, f(x) \rangle \mathbb{E} [e^{\langle \mathbf{X}, f \rangle_{\mathbb{D}}}] dx, \quad (3.3.3)$$

where we have set

$$\langle f, h \rangle_{\mathbb{D}} := \int_{\mathbb{D}} \langle f(z), h(z) \rangle dz.$$

We note that these statements are easily recovered thanks to the Girsanov [A](#)

3.3.1.2 Comparison lemmas

We now provide some comparison lemmas, that allow to turn a comparison of the covariance kernel of two Gaussian fields into a comparison of certain functionals of these Gaussian fields:

Lemma 3.3.1. Let F be some smooth function defined on $(\mathbb{R}^n)^r$ such that F as well as its derivatives up to order 2 have at most polynomial growth at infinity. Also assume that F is such that for $(x^1, \dots, x^r) \in (\mathbb{R}^n)^r$ (where $x^i = (x_1^i, \dots, x_n^i)$) the following inequalities hold:

$$\text{for all } i \neq j \text{ and } k, k' \quad \frac{\partial^2 F}{\partial x_k^i \partial x_{k'}^j} \geq 0.$$

Under these assumptions, if $\mathbf{X} := (X^1, \dots, X^r)$ and $\tilde{\mathbf{X}} := (\tilde{X}^1, \dots, \tilde{X}^r)$ are two centered Gaussian vectors in $(\mathbb{R}^n)^r$ such that for all i , X^i has the same law as $\tilde{X}^i$ while as soon as $i \neq j$

$$\mathbb{E}[X_k^i X_{k'}^j] \leq \mathbb{E}[\tilde{X}_k^i \tilde{X}_{k'}^j]$$

for any k, k' , then the following inequality holds:

$$\mathbb{E}[F(X^1, \dots, X^r)] \leq \mathbb{E}[F(\tilde{X}^1, \dots, \tilde{X}^r)].$$

Proof. For $t \in [0, 1]$, we set $X_t = \sqrt{t}X + \sqrt{1-t}\tilde{X}$, where X and $\tilde{X}$ are chosen to be independent, and denote

$$G(t) := \mathbb{E}[F(X_t)].$$

By using Gaussian integration by parts, we can write that

$$\begin{aligned} G'(t) &= \frac{1}{2} \sum_{i=1}^d \sum_{k=1}^n \mathbb{E} \left[\frac{\partial F}{\partial x_k^i}(X_t) \left(\frac{1}{\sqrt{t}} X_k^i - \frac{1}{\sqrt{1-t}} \tilde{X}_k^i \right) \right] \\ &= \frac{1}{2} \sum_{i=1}^d \sum_{k=1}^n \sum_{i'=1}^d \sum_{k'=1}^n \mathbb{E} \left[\frac{\partial^2 F}{\partial x_k^i \partial x_{k'}^{i'}}(X_t) \right] \mathbb{E} \left[(\sqrt{t} X_{k'}^{i'} + \sqrt{1-t} \tilde{X}_{k'}^{i'}) \left(\frac{1}{\sqrt{t}} X_k^i - \frac{1}{\sqrt{1-t}} \tilde{X}_k^i \right) \right] \\ &= \frac{1}{2} \sum_{i=1}^d \sum_{k=1}^n \sum_{i' \neq i}^d \sum_{k'=1}^n \mathbb{E} \left[\frac{\partial^2 F}{\partial x_k^i \partial x_{k'}^{i'}}(X_t) \right] (\mathbb{E}[X_{k'}^{i'} X_k^i] - \mathbb{E}[\tilde{X}_{k'}^{i'} \tilde{X}_k^i]) \\ &\leq 0. \end{aligned}$$

Therefore $G(1) \leq G(0)$, which proves our claim. $\square$

A straightforward consequence of this statement is the following:

Corollary 3.3.2. Let H be some smooth function defined on $(\mathbb{R}^n)^r$ such that H as well as its derivatives up to order 2 have at most polynomial growth at infinity, and consider any partition $P_1, \dots, P_m$ of the set $\{1, \dots, r\}$. Assume that for $(x_1, \dots, x_r) \in \mathbb{R}_+^r$, the following inequality holds for all $s, s' \in \{1, \dots, m\}$ with $s \neq s'$, all $i \in P_s$ and all $j \in P_{s'}$

$$\frac{\partial^2 H}{\partial x_i \partial x_j} \geq 0.$$

Further assume that $X^1, \dots, X^r$ is a family of continuous centered Gaussian fields respectively defined over domains $D_i \subset \mathbb{R}^n$ (for $i = 1, \dots, r$) such that for all $s, s' \in \{1, \dots, m\}$ with $s \neq s'$, all $i \in P_s$ and all $j \in P_{s'}$:

$$\forall x \in D_i, \forall x' \in D_j, \quad \mathbb{E}[X^i(x) X^j(x')] \leq 0.$$

Let $\tilde{X} = (\tilde{X}^1, \dots, \tilde{X}^r)$ be another family of continuous centered Gaussian fields such that:

- 1) for all $s = 1, \dots, m$, $(\tilde{X}^i)_{i \in P_s}$ has same distribution as $(X^i)_{i \in P_s}$.
- 2) the families $(\tilde{X}^i)_{i \in P_1}, \dots, (\tilde{X}^i)_{i \in P_m}$ are independent.

Eventually, let $f_1, \dots, f_r$ be a family of positive functions each of which respectively defined on D_i . For $i = 1, \dots, r$, we set

$$M^i := \int_{D_i} e^{X^i(x) - \frac{1}{2}\mathbb{E}[X^i(x)^2]} f_i(x) dx \quad \text{and} \quad \tilde{M}^i := \int_{D_i} e^{\tilde{X}^i(x) - \frac{1}{2}\mathbb{E}[\tilde{X}^i(x)^2]} f_i(x) dx.$$

Then the following inequality holds

$$\mathbb{E}[H(M^1, \dots, M^r)] \leq \mathbb{E}[H(\tilde{M}^1, \dots, \tilde{M}^r)].$$

Proof. Up to a discretization of the fields, it suffices to apply Lemma 3.3.1 with

$$F(X^1, \dots, X^r) := H \left(\sum_{k_1} p_{k_1}^1 e^{\gamma X_{k_1}^1 - \frac{\gamma^2}{2} \mathbb{E}[(X_{k_1}^1)^2]}, \dots, \sum_{k_r} p_{k_r}^r e^{\gamma X_{k_r}^r - \frac{\gamma^2}{2} \mathbb{E}[(X_{k_r}^r)^2]} \right)$$

for some nonnegative numbers $p_{k_i}^i$ obtained by discretizing f_i over D_i . $\square$

3.3.2 . Elementary properties of Gaussian free fields

We now would like to shed light on some properties of GFFs and their interplays with other notions, such as conformal geometry and Brownian motions.

3.3.2.1 GFFs and metrics on the sphere

The GFFs introduced in Section 3.2 depend on a choice of underlying metric g on the Riemann sphere. The choice of this metric can be thought of as a requirement that the field $\mathbf{X}^g$ has almost-surely zero average with respect to the metric g , which is a consequence of the fact that $m_g(G_g(x, \cdot)) = 0$ for any $x \in \mathbb{R}^2$. This feature actually carries most of the dependence of the GFF in the metric g in the sense that the following holds true:

Lemma 3.3.3 (Metric-dependence of the GFFs). *Assume that g and g' are two Riemannian metrics over $\mathbb{R}^2$ conformally equivalent to $\hat{g}$. Then*

$$\mathbf{X}^{g'} \stackrel{(law)}{=} \mathbf{X}^g - m_{g'}(\mathbf{X}^g). \quad (3.3.4)$$

Proof. Explicit computations show that the Gaussian field on the right-hand side has covariance kernel given by $G_{g'}$, which is nothing but the covariance kernel of the Gaussian field $\mathbf{X}^{g'}$. The knowledge of the covariance kernel characterizing the underlying Gaussian field the equality in law (3.3.4) follows. $\square$

3.3.2.2 GFFs and Möbius transforms

Since GFFs are defined based on Green functions enjoying remarkable properties under the action of the Möbius group as explained in Lemma 3.2.1, GFFs should in turn behave nicely under conformal transformations. The following statement can be understood as a rigorous formulation of this connection:

Lemma 3.3.4 (Conformal covariance of GFFs). *Let ψ be a Möbius transform of the Riemann sphere and g be a Riemannian metric conformally equivalent to the spherical one. Then*

$$\mathbf{X}^g \circ \psi \stackrel{(law)}{=} \mathbf{X}^{g_\psi}. \quad (3.3.5)$$

Proof. It suffices to note that thanks to Lemma 3.2.1 the two Gaussian fields have same covariance kernel. $\square$

3.3.2.3 GFFs and Brownian motions

Interestingly, Brownian motions naturally arise in the study of the GFFs considered here. The GFFs introduced in Subsection 3.2.2 have strong interplays with Brownian motions, since the latter naturally arise when writing GFFs in polar coordinates. Namely, recall that a regularization of the GFF $\mathbf{X}$ can be defined by means of circle-averages and yield processes $(B_t)_{t \in \mathbb{R}} := (\mathbf{X}_t(z))_{t \in \mathbb{R}}$ for any $z \in \mathbb{C}$. Then we can decompose the GFF $\mathbf{X}$ by means of

$$\mathbf{X}(z + e^{-t+i\theta}) = B_t + Y(t, \theta) \quad (3.3.6)$$

where $Y(t, \theta) := \mathbf{X}(z + e^{-t+i\theta}) - B_t$.

One can then check using the covariance kernel of $\mathbf{X}$ that the process $(B_t)_{t \geq 0}$ thus defined is a Brownian motion in $\mathfrak{a}$ started from the origin, while Y is a Gaussian field with covariance kernel given by

$$\mathbb{E}[\langle u, Y(t, \theta) \rangle \langle v, Y(t', \theta') \rangle] = \langle u, v \rangle \ln \frac{e^{-t} \vee e^{-t'}}{|e^{-t+i\theta} - e^{-t'+i\theta'}|} \quad (3.3.7)$$

for any $u, v \in \mathfrak{a}$. Moreover B and Y are seen to be independent.

3.3.3 . Basic features of GMC measures

We are then interested in the description of some properties of the GMC measures defined above.

3.3.3.1 GMC and Brownian motions

GFFs and Brownian motions being related on to the other there is naturally a counterpart of this connection for GMC measures. Indeed thanks to Equation (3.3.6) above the GMC defined from the GFF $\mathbf{X}$ admits the alternative representation

$$M^{\gamma_{e_i}}(dz) = e^{\gamma \langle B_t - Q_t, e_i \rangle} M_Y^{\gamma_{e_i}}(dt, d\theta), \quad (3.3.8)$$

where M_Y is the GMC measure defined from Y (this random measure is well-defined too via the same arguments used to make sense of the GMC measure associated to $\mathbf{X}$). Put differently, for any bounded map F and $\mathcal{O}$ a measurable subset of $\mathbb{C}$

$$\mathbb{E} \left[F \left(\int_{\mathcal{O}} M^{\gamma_{e_i}}(dz) \right) \right] = \mathbb{E} \left[F \left(\int_{\mathbb{R}} e^{\gamma \langle B_t - Q_t, e_i \rangle} \int_0^{2\pi} \mathbb{1}_{e^{-t+i\theta} \in \mathcal{O}} M_Y^{\gamma_{e_i}}(dt, d\theta) \right) \right].$$

For future purpose we introduce the random measure on $\mathbb{R}$ defined by

$$Z^i(dt) := \int_0^{2\pi} M_Y^{\gamma_{e_i}}(dt, d\theta). \quad (3.3.9)$$

By a slight abuse of notation we will often denote it by $Z_t^i dt$. Note that it is stationary, by which we mean that for any positive h Z_t^i and Z_{t+h}^i have same law.

3.3.3.2 Moments of GMC measures

We now turn our attention to the moments of the GMC measures introduced above, and more precisely on the conditions ensuring their existence. Their properties are now rather well understood and we refer to [115] for an overview as well as a justification of the following statements.

To start with let us consider a one-dimensional GFF $\mathbf{X}^g$ with values in $\mathbb{R}$ and covariance kernel given by G_g . Then the GMC measure formally defined by $e^{\sqrt{2}\gamma\mathbf{X}^g}d\mathbf{v}_g$ is well-defined as long as $\gamma \in [0, \sqrt{2})$, and under this assumption its moments satisfy the following bounds:

Lemma 3.3.5. *For any bounded and non-trivial interval I ,*

$$\mathbb{E} \left[\left(M_g^{\sqrt{2}\gamma}(I) \right)^p \right] < \infty \quad (3.3.10)$$

if and only if $-\infty < p < \frac{2}{\gamma^2}$.

More generally one may wonder what type of singularities can be integrated with respect to the GMC measure $M_g^{\sqrt{2}\gamma}$. The following statement addresses this issue:

Lemma 3.3.6. *For any complex x and positive r , set $B(x, r) := \{z \in \mathbb{C}, |z - x| < r\}$. Then the random variable*

$$\int_{B(x, r)} |z - x|^{-\gamma\alpha} M_g^{\sqrt{2}\gamma}(dz)$$

is almost surely finite if and only if $\alpha < Q = \gamma + \frac{2}{\gamma}$. Under this assumption

$$\mathbb{E} \left[\left(\int_{B(x, r)} |z - x|^{-\gamma\alpha} M_g^{\sqrt{2}\gamma}(dz) \right)^p \right] < \infty \quad (3.3.11)$$

if and only if $-\infty < p < \frac{2}{\gamma^2} \wedge \frac{Q-\alpha}{\gamma}$. On the contrary if $\alpha > Q$ then for any $p < 0$

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E} \left[\left(\int_{B(x, r)} (\varepsilon + |z - x|)^{-\gamma\alpha} M_g^{\sqrt{2}\gamma}(dz) \right)^p \right] = 0$$

3.3.4 . Conditioned Brownian motions and reflection groups

To conclude for this chapter dedicated to outlining the framework we will work within in this manuscript, we study some interplays between Brownian motions evolving over $\mathbf{V}$, Doob's conditioning and reflection groups. In this subsection we assume that the diffusion process B as defined above evolves in $\mathbf{V}$ to which is associated a reflection group $(W, \mathbf{V})$.

3.3.4.1 Brownian motions conditioned to stay positive

In the study of Toda reflection coefficients conducted in Chapter 6, we will need to consider a certain diffusion process defined thanks to the one-dimensional path decomposition by Williams [138]. Namely for positive ν , we introduce the process B^ν started from $x > 0$ to be the junction of:

- a one-dimensional Brownian motion with negative drift $-\nu$, upon hitting 0;
- an independent one-dimensional Brownian motion with positive drift ν , conditioned to stay positive. This process is a diffusion with generator

$$\frac{1}{2} \frac{d^2}{dx^2} + \nu \coth(\nu x) \frac{d}{dx}.$$

Toda reflection coefficients will be defined using the random variable $J_\gamma(-\nu)$, where for positive ν

$$J_\gamma(-\nu) := \int_0^{+\infty} e^{-\gamma \mathcal{B}_t^\nu} Z_t dt. \quad (3.3.12)$$

In the above equation the process $\mathcal{B}^\nu$ is (formally) started from $+\infty$, while Z_t is defined in the same fashion as in Equation (3.3.9) in the case where $r = 1$, by

$$Z(dt) := \int_0^{2\pi} M_Y^\gamma(dt, d\theta)$$

from the field Y with covariance

$$\mathbb{E}[Y(t, \theta)Y(t', \theta')] = \ln \frac{e^{-t} \vee e^{-t'}}{|e^{-t+i\theta} - e^{-t'+i\theta'}|}.$$

A proper way to make sense of the random variable thus defined is via a limiting procedure by setting for any positive, bounded and continuous map $F : \mathbb{R}^+ \rightarrow \mathbb{R}^+$

$$\mathbb{E}[F(J_\gamma(-\nu))] := \lim_{x \rightarrow +\infty} \mathbb{E}\left[F\left(\int_0^{+\infty} e^{-\gamma \mathcal{B}_t^\nu} Z_t dt\right)\right].$$

Alternatively one can use that $J_\gamma(-\nu)$ has same law as

$$\int_{-\infty}^{+\infty} e^{-\gamma \tilde{\mathcal{B}}_t^\nu} Z_t dt$$

where $(\tilde{\mathcal{B}}_t^\nu)_{t \geq 0}$ and $(\tilde{\mathcal{B}}_{-t}^\nu)_{t \geq 0}$ are two independent Brownian motions with positive drift ν and conditioned to stay positive (this follows from the time-reversal property of such a process [138, Theorem 2.5], see for instance [84, Section 3] for more details). We recall the statement of [84, Lemma 3.3] which provides an analog result of Lemma 3.3.5.

Lemma 3.3.7. *For any $\nu \in (-\frac{2}{\gamma}, 0)$ and $-\infty < p < \frac{4}{\gamma^2}$,*

$$\mathbb{E}[J_\gamma(\nu)^p] < \infty. \quad (3.3.13)$$

3.3.4.2 Exit from a Weyl chamber

Further assume that $\mathbf{B}$ has almost surely a minimum in the direction of the simple roots, by which we mean that almost surely, for any $1 \leq i \leq r$ the quantities

$$\mathbf{M}_i := \inf_{t \geq 0} \langle \mathbf{B}_t, e_i \rangle \quad (3.3.14)$$

are finite. Note that this is the case if $\mathbf{B}$ has the law of B^ν , a Brownian motion with constant drift ν inside the Weyl chamber $\mathcal{C}$.

The law of this minimum can be computed by noticing that having $\mathbf{M}_i(\mathbf{B}) \geq m_i$ for all $1 \leq i \leq r$ means that $\mathbf{B}$ stays inside the shifted Weyl chamber $\mathbf{m} + \partial\mathcal{C}$ with $\mathbf{m} = \sum_{i=1}^r m_i \omega_i$. As a consequence

$$\mathbb{P}_x(\forall 1 \leq i \leq r, \mathbf{M}_i(\mathbf{B}) \geq m_i) = h^{\mathbf{m}}(x), \quad (3.3.15)$$

where $h^{\mathbf{m}}$ is a solution to the Dirichlet problem

$$\begin{cases} \mathcal{A}h^{\mathbf{m}} = 0 & \text{in } \mathbf{m} + \mathcal{C} \\ h^{\mathbf{m}} = 0 & \text{on } \mathbf{m} + \partial\mathcal{C} \end{cases} \quad (3.3.16)$$

with the asymptotic $h^{\mathbf{m}}(x) \rightarrow 1$ as $x \rightarrow \infty$ along a ray inside the Weyl chamber. In the case where $\mathbf{B} = B^\nu$, this h -function can be explicitly computed using the symmetries prescribed by the reflection group:

Proposition 3.3.8. *Let us define a map $h : \mathcal{C} \rightarrow [0, 1]$ by setting*

$$h(x) := \sum_{s \in W} \epsilon(s) e^{\langle s\nu - \nu, x \rangle}. \quad (3.3.17)$$

Then the law of the minimums of the drifted Brownian motion B^ν in the direction of the simple roots is described by

$$\mathbb{P}(\mathbf{M}_i(B^\nu) \geq m_i \quad \forall 1 \leq i \leq r) = h(-\mathbf{m}) \mathbb{1}_{-\mathbf{m} \in \mathcal{C}}. \quad (3.3.18)$$

This statement is proved for instance in [20, Section 5]. It implies that for the process B^ν $h^{\mathbf{m}}(x) = h(x - \mathbf{m})$ with h as in Equation (3.3.17).

3.3.4.3 The killed process

In this context it is very natural to introduce the process whose law is that of $\mathbf{B}$ killed when exiting the Weyl chamber $\mathbf{M} + \mathcal{C}$. For this purpose we define

$$\mathbf{p}_t^{\mathbf{M}}(x, y) := \sum_{s \in W^{\mathbf{M}}} \epsilon(s) \mathbf{p}_t(sx, y) \quad (3.3.19)$$

which vanishes on the boundary of $\mathbf{M} + \mathcal{C}$. Here we have considered the reflection group $W^{\mathbf{M}}$ generated by the reflections centered at $\mathbf{M}$, that is

$$s_i(x) = x - \langle x - \mathbf{M}, e_i^\vee \rangle e_i. \quad (3.3.20)$$

For the diffusion processes considered, the reflection principle takes the following form, which generalizes the well-known property of the one-dimensional Brownian motion.

Proposition 3.3.9. *Let $\mathbf{M} \in \mathbf{V}$ and assume that the drift function b is invariant under the action of the reflection group centered at $\mathbf{M}$:*

$$\forall s \in W^{\mathbf{M}}, \quad b \circ s = b. \quad (3.3.21)$$

Then the transition probabilities of the process $\mathbf{X}$ killed when exiting the Weyl chamber $\mathbf{M} + \mathcal{C}$ are given by $\mathbf{p}_t^{\mathbf{M}}(x, y)$.

Proof. For the sake of completeness, we reproduce the argument of [64, Theorem 1] and [65, Theorem 5]. Let us consider all paths from sx to y where s ranges over $W^{\mathbf{M}}$. Then to any path which hits the boundary of a Weyl chamber, we can associate another path obtained by reflecting, across the boundary component being hit, the initial path before having reached the boundary of the Weyl chamber. The assumption made on the process shows that both have same probability to occur, so these probabilities cancel out in the alternating sum (3.3.19). The only remaining term is given by the probability that a path never crosses the boundary of a Weyl chamber, or put differently that the process goes from x to y without exiting the Weyl chamber. $\square$

This formula can also be found e.g. in [20, Equation (5.1)] in the case where the underlying process is a Brownian motion. The assumption made on the process implies that it is reflectable in the sense of [65]. Note that we can actually define b only inside $\mathbf{M} + \mathcal{C}$ and extend it by $b \circ s = b$ over $\mathbf{V}$ to apply the above statement.

3.3.4.4 Some further conditioned processes

Combining all of the above, the process $\mathcal{B}$ whose law is that of B conditioned to stay inside the Weyl chamber $\mathbf{M} + \mathcal{C}$ can be defined to be a diffusion with generator $\mathcal{A}^{h^{\mathbf{M}}}$ where $h^{\mathbf{M}}$ has been described above. Put differently its semigroup is defined by

$$\mathbf{p}_t^{\mathbf{M}}(x, y) := \frac{h^{\mathbf{M}}(y)}{h^{\mathbf{M}}(x)} \mathbf{p}_t^{\mathbf{M}}(x, y). \quad (3.3.22)$$

for any x, y inside $\mathbf{M} + \mathcal{C}$. Here $h^{\mathbf{M}}$ is defined by Equation (3.3.15).

In the sequel we will further assume that the process B is such that for any $\mathbf{m} \in \mathcal{C}_-$, such maps $h^{\mathbf{m}}$ are of C^r regularity over $\mathbf{m} + \mathcal{C}$. In particular, for any $1 \leq k \leq r$ and $i_1, \dots, i_k$ distinct in $\{1, \dots, r\}$, the quantities defined by

$$\partial_{i_1, \dots, i_k} h^{\mathbf{m}} := (-1)^k \partial_{\mathbf{m}_{i_1}} \dots \partial_{\mathbf{m}_{i_k}} h^{\mathbf{m}} \quad (3.3.23)$$

are well-defined in $\mathbf{m} + \mathcal{C}$. These maps satisfy

$$\begin{cases} \mathcal{A} \partial_{i_1, \dots, i_k} h^{\mathbf{m}} = 0 & \text{in } \mathbf{m} + \mathcal{C} \\ \partial_{i_1, \dots, i_k} h^{\mathbf{m}} = 0 & \text{on } \mathbf{m} + \partial \mathcal{C} \setminus (\partial \mathcal{C}_{i_1} \cup \dots \cup \partial \mathcal{C}_{i_k}). \end{cases} \quad (3.3.24)$$

They are also non-negative inside $\mathbf{m} + \mathcal{C}$: indeed note that $\partial_{1, \dots, k} h^{\mathbf{m}}$ is given by

$$\lim_{\varepsilon_1, \dots, \varepsilon_r \rightarrow 0^+} \frac{1}{\varepsilon_1 \dots \varepsilon_k} \mathbb{P}_x \left(\forall 1 \leq i \leq k, m_i \leq \mathbf{M}_i(\mathbf{Y}) \leq m_i + \varepsilon_i; \right. \\ \left. \mathbf{M}_i(\mathbf{Y}) \geq m_i \text{ for } k+1 \leq i \leq r \right). \quad (3.3.25)$$

We also stress that $\partial_{1, \dots, r} h^{\mathbf{M}}$ represents the probability density function of the random variable $\mathbf{M}$. When the process being considered is given by B^ν these maps are completely explicit and can be seen to be positive inside $\mathbf{m} + \mathcal{C}$ but vanish on its boundary.

As we will see, the diffusion processes with generators given by

$$\mathcal{A}^{i_1, \dots, i_k} := \mathcal{A}^{\partial_{i_1, \dots, i_k} h} = \mathcal{A} + \sum_{j=1}^r \left(a_{ij} \frac{\partial_{x_i} \partial_{i_1, \dots, i_k} h}{\partial_{i_1, \dots, i_k} h} \right) \partial_{x_j} \quad (3.3.26)$$

and killed upon hitting the boundary of $\mathbf{M} + \mathcal{C}$ will naturally arise in the study of the interplays between Brownian motions and reflection groups. We have already highlighted above that for any $x \in \mathbf{M} + \mathcal{C}$, these processes will almost surely never reach $\mathbf{M} + \partial\mathcal{C}$ on $\partial\mathcal{C} \setminus (\partial\mathcal{C}_{i_1} \cup \dots \cup \partial\mathcal{C}_{i_k})$. A consequence of our main result below is that such a process will almost surely reach $\mathbf{M} + \partial\mathcal{C}$ at $\partial\mathcal{C}_{i_1} \cup \dots \cup \partial\mathcal{C}_{i_k}$ (and will be killed upon hitting).

4 - Définition probabiliste des fonctions de corrélation

Ayant introduit et étant désormais familiarisés avec les objets de base permettant de comprendre d'un point de vue mathématique les théories de Toda, nous sommes désormais en mesure de définir de manière rigoureuse ces théories en interprétant l'intégrale de chemin en termes probabilistes. Cette formulation des théories de Toda forme le point de départ de notre quête de compréhension de ces modèles.

Pour ce faire nous nous attacherons dans un premier temps à donner un sens à l'intégrale de chemin en nous basant sur les notions présentées lors du chapitre précédent, et plus précisément à décrire les fonctions de corrélation des théories de Toda dans un langage probabiliste. Certaines propriétés de ces fonctions de corrélation seront ensuite étudiées dans la perspective de nous armer en vue de comprendre les théories de Toda.

Ce chapitre se base sur l'article [34], fruit du travail conduit en collaboration avec Rémi Rhodes et Vincent Vargas. Nous évoquerons également certains résultats issus de l'article [33] écrit conjointement avec Yichao Huang ainsi que certains éléments de [30] concernant certaines propriétés plus analytiques des fonctions de corrélation.

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4.1 . A probabilistic take on the path integral

The goal of this section is to make sense of the path integral definition of Toda theories in the realm of probability theory. In order to achieve this translation from the physics language to a probabilistic one, we will need to interpret the formal measure that features the Toda field action $S_{T,\mathfrak{g}}$, which takes the form of

$$S_{T,\mathfrak{g}}(\phi, g) := \frac{1}{4\pi} \int_{\mathbb{R}^2} \left(\langle \partial_g \phi(x), \partial_g \phi(x) \rangle_g + R_g \langle Q, \phi(x) \rangle + 4\pi \sum_{i=1}^r \mu_i e^{\gamma \langle e_i, \phi(x) \rangle} \right) v_g(dx) \quad (4.1.1)$$

for $\phi : \mathbb{R}^2 \rightarrow \mathfrak{a}$ and with g a metric on $\mathbb{R}^2$ conformally equivalent to $\hat{g}$. Recall that Q is the $\mathfrak{a}$ -valued background charge defined from the coupling constant $\gamma > 0$ by

$$Q := \gamma \rho + \frac{2}{\gamma} \rho^\vee, \quad (4.1.2)$$

while the scalar $\mu := (\mu_1 > 0, \dots, \mu_r > 0)$ are the cosmological constants.

With these notations at hand we wish here to make sense of the formal path integral over fields $\phi : \mathbb{R}^2 \rightarrow \mathfrak{a}$:

$$\langle F(\Phi) \rangle_{T,g}^{\mathfrak{a}} := \int_{L^2(\mathbb{R}^2 \rightarrow \mathfrak{a})} F(\phi) e^{-S_{T,\mathfrak{g}}(\phi, g)} D\phi \quad (4.1.3)$$

for any F bounded over $L^2(\mathbb{R}^2 \rightarrow \mathfrak{a})$.

4.1.1 . Probabilistic interpretation of the path integral

The Toda field action can be decomposed as a sum of two terms. Namely we can write it under the form

$$S_{T,\mathfrak{g}}(\phi, g) = \frac{1}{2} \left\langle \phi, \left(\frac{-\Delta_g}{2\pi} \right) \phi \right\rangle_g + V(\phi, g) \quad (4.1.4)$$

where we have set

$$V(\phi, g) := \frac{1}{4\pi} \int_{\mathbb{R}^2} \left(R_g \langle Q, \phi(x) \rangle + 4\pi \sum_{i=1}^r \mu_i e^{\gamma \langle e_i, \phi(x) \rangle} \right) v_g(dx).$$

We will treat separately these two components and start with the first one.

4.1.1.1 Gaussian measure interpretation of the squared gradient term

The first component of the Toda action functional is the quadratic term

$$\frac{1}{2} \left\langle \phi, \left(\frac{-\Delta_g}{2\pi} \right) \phi \right\rangle_g$$

which hints towards the fact that Gaussian random variables will be involved. Indeed the measure formally written as $e^{-\frac{1}{2} \langle \phi, \left(\frac{-\Delta_g}{2\pi} \right) \phi \rangle_g} D\phi$ is reminiscent of the density law

of a Gaussian vector, whose covariance kernel would be given by $\left(\frac{-\Delta_g}{2\pi}\right)^{-1}$. To be more specific, when restricted to the space

$$\Sigma := \{\phi \in H^{-1}(\mathbb{R}^2 \rightarrow \mathfrak{a}, g); \int_{\mathbb{R}^2} \phi(x) \mathbf{v}_g(dx) = 0\}$$

(where $H^{-1}(\mathbb{R}^2 \rightarrow \mathfrak{a}, g)$ is the set of $\mathfrak{a}$ -valued generalized functions with each component in $H^{-1}(\mathbb{R}^2, g)$), the formal measure

$$\det\left(\frac{-\Delta_g}{2\pi}\right)^{\frac{1}{2}} e^{-\frac{1}{2}\langle \phi, \left(\frac{-\Delta_g}{2\pi}\right) \phi \rangle_g} D\phi,$$

can be understood as the density law of a Gaussian field $\mathbf{X}^g$ who would satisfy

$$\mathbb{E} \left[\langle \mathbf{X}^g, \left(\frac{-\Delta_g}{2\pi}\right) f \rangle_g \langle \mathbf{X}^g, \left(\frac{-\Delta_g}{2\pi}\right) h \rangle_g \right] = \langle h, \left(\frac{-\Delta_g}{2\pi}\right) f \rangle_g \quad (4.1.5)$$

for $f, h \in H^1(\mathbb{R}^2 \rightarrow \mathfrak{a}, g)$. This is achieved in particular if $\mathbf{X}^g$ has the law of the GFF introduced in Equation (3.2.12).

However in general we should not restrict our attention to fields with vanishing mean in the metric g . To remedy this issue we need to add an independent constant c to the GFF $\mathbf{X}^g$ —note that this preserves the validity of Equation (4.1.5). And it turns out that this constant has to be sampled uniformly with respect to the Lebesgue measure in order to define a model that satisfies the expectations from the physics literature. Put differently, the quantity $m_g(\Phi)$ should be sampled in $\mathfrak{a}$ according to the measure $d\mathbf{v} := d\mathbf{v}_1 \cdots d\mathbf{v}_r$ where recall that the basis $(\mathbf{v}_i)_{1 \leq i \leq r}$ is orthonormal.

In brief, the proper way to make sense of this formal measure is to make the following identification:

$$\int_{L^2(\mathbb{R}^2 \rightarrow \mathfrak{a})} F(\phi) e^{-\frac{1}{2}\langle \phi, \left(\frac{-\Delta_g}{2\pi}\right) \phi \rangle_g} D\phi := Z(g) \int_{\mathfrak{a}} \mathbb{E}[F(\mathbf{X}^g + c)] dc \quad (4.1.6)$$

where the renormalization constant $Z(g)$ is given by the regularized determinant:

$$Z(g) := \left(\frac{\det\left(\frac{-\Delta_g}{2\pi}\right)}{\text{vol}_g(\mathbb{R}^2)^r} \right)^{-\frac{1}{2}}.$$

To be more specific, the Laplacian $\frac{-\Delta_g}{2\pi}$ acting over $H^1(\mathbb{R}^2 \rightarrow \mathbb{R}, g)$ has positive (apart from the zero eigenvalue) and discrete spectrum $(\lambda_j)_{j \geq 0}$, thanks to which one can define its spectral Zeta function $\zeta(s) := \sum_{j \geq 1} \lambda_j^{-s}$ for $\Re(s) \gg 1$. The associated regularized determinant is then given by

$$\det\left(\frac{-\Delta_g}{2\pi}\right) = \exp(-\partial_s \zeta(s)|_{s=0}).$$

A remarkable property of the partition function $Z(g)$ is its variation under a conformal change of metric. Indeed in the $r = 1$ case it is proved in [108, Equation (1.13)] that

$$\log \frac{\det(-\Delta_{g'})}{\text{vol}_{g'}(\mathbb{R}^2)^r} = \log \frac{\det(-\Delta_g)}{\text{vol}_g(\mathbb{R}^2)^r} + \frac{1}{96\pi} \int_{\mathbb{R}^2} (|\text{d}\varphi|_g^2 + 2R_g \varphi) d\mathbf{v}_g$$

with $g' = e^\varphi g$. For $r > 1$ since Δ acts independently on the r components of a map $\mathbb{R}^2 \rightarrow \mathbb{R}^r$ we see that $\det(-\Delta_{g'}) = \det(-\Delta_g^1)^r$ where Δ^1 denote the Laplace operator acting on $H^1(\mathbb{R}^2 \rightarrow \mathbb{R}, g)$. Therefore

$$\log \frac{\det(-\Delta_{g'})}{\text{vol}_{g'}(\mathbb{R}^2)^r} = \log \frac{\det(-\Delta_g)}{\text{vol}_g(\mathbb{R}^2)^r} + \frac{r}{96\pi} \int_{\mathbb{R}^2} (|\text{d}\varphi|_g^2 + 2R_g \varphi) \text{d}v_g. \quad (4.1.7)$$

As a consequence, up to a global factor, one has

$$Z(e^\varphi \hat{g}) = e^{\frac{r}{96\pi} \int_{\mathbb{R}^2} (|\text{d}\varphi|_{\hat{g}}^2 + 2R_{\hat{g}} \varphi) \text{d}v_g} \quad (4.1.8)$$

within the conformal class of the spherical metric $\hat{g}$.

Eventually we make sense of the formal Gaussian measure by interpreting in the following way:

$$\int_{L^2(\mathbb{R}^2 \rightarrow \mathfrak{a})} F(\phi) e^{-\frac{1}{2} \langle \phi, \left(\frac{-\Delta_g}{2\pi} \right) \phi \rangle_g} D\phi := e^{\frac{r}{96\pi} \int_{\mathbb{R}^2} (|\text{d}\varphi|_{\hat{g}}^2 + 2R_{\hat{g}} \varphi) \text{d}v_g} \int_{\mathfrak{a}} \mathbb{E} [F(\mathbf{X}^g + \mathbf{c})] d\mathbf{c} \quad (4.1.9)$$

for each continuous and bounded functional F on $H^{-1}(\mathbb{R}^2 \rightarrow \mathfrak{a}, g)$ with $g = e^\varphi \hat{g}$ in the conformal class of the spherical metric $\hat{g}$.

4.1.1.2 Gaussian Multiplicative Chaos and the potential term

Thanks to the interpretation of the first term that appears in the Toda action functional, we see that the path integral should be interpreted by means of

$$\int_{L^2(\mathbb{R}^2 \rightarrow \mathfrak{a})} F(\phi) e^{-S_{T, \mathfrak{g}}(\phi, g)} D\phi = e^{\frac{r}{96\pi} \int_{\mathbb{R}^2} (|\text{d}\varphi|_{\hat{g}}^2 + 2R_{\hat{g}} \varphi) \text{d}v_g} \int_{\mathfrak{a}} \mathbb{E} [F(\mathbf{X}^g + \mathbf{c}) e^{-V(\mathbf{X}^g + \mathbf{c}, g)}] d\mathbf{c}.$$

The issue here is that if the field being considered is the GFF $\mathbf{X}^g$, the exponential term that is part of the expression of $V(\mathbf{X}^g + \mathbf{c}, g)$ is not well-defined. However we have seen in Section 3.2 that it was possible to make sense of this quantity thanks to the theory of GMC. Put differently we interpret the corresponding term as

$$\mathbb{E} [F(\mathbf{X}^g + \mathbf{c}) e^{-V(\mathbf{X}^g + \mathbf{c}, g)}] := \mathbb{E} \left[F(\mathbf{X}^g + \mathbf{c}) e^{-\frac{1}{4\pi} \langle Q R_g, \mathbf{X}^g + \mathbf{c} \rangle_g - \sum_{i=1}^r \mu_i e^{\langle \mathbf{c}, \gamma e_i \rangle} M_g^{\gamma e_i}(\mathbb{R}^2)} \right]$$

4.1.1.3 Conclusion

We have now interpreted all the terms that enter the path integral defining Toda CFTs. However defined as such the theory does not satisfy a key property enjoyed by two-dimensional CFTs: Weyl covariance, which is concerned by the behaviour of the theory under a conformal change of metric g . To remedy this issue we need to consider the Toda field as the random distribution

$$\Phi = \mathbf{X}^g + \frac{Q}{2} \ln g + \mathbf{c}.$$

By doing so the model thus defined satisfies the basic properties expected in the physics literature as shown in Theorem 4.2.2 below. To summarize, we interpret the path integral

of Toda CFTs in a probabilistic way by making the identification for any $F \in H^{-1}(\mathbb{R}^2 \rightarrow \mathfrak{a}, g)$:

$$\begin{aligned} & \int_{L^2(\mathbb{R}^2 \rightarrow \mathfrak{a})} F(\phi) e^{-S_{T, \mathfrak{g}}(\phi, g)} D\phi \\ &:= e^{\frac{r}{96\pi} \int_{\mathbb{R}^2} (|d\varphi|_{\hat{g}}^2 + 2R_{\hat{g}}\varphi) dv_g} \int_{\mathfrak{a}} \mathbb{E} \left[F \left(\mathbf{X}^g + \frac{Q}{2} \ln g + \mathbf{c} \right) e^{-\frac{1}{4\pi} \langle QR_g, \mathbf{X}^g + \mathbf{c} \rangle_g - \sum_{i=1}^r \mu_i e^{\langle \mathbf{c}, \gamma_{e_i} \rangle} M_g^{\gamma_{e_i}}(\mathbb{R}^2)} \right] d\mathbf{c}, \end{aligned} \quad (4.1.10)$$

as soon as it makes sense, and where $g = e^\varphi \hat{g}$ is in the conformal class of the spherical metric.

4.1.2 . Probabilistic correlation functions of Vertex Operators

There is a class of functionals F which play a key role in the study of Toda theories. Usually referred to as *Vertex Operators*, computing their correlation functions is often one of the main issue in the study of two-dimensional CFTs. In this subsection we aim to provide a probabilistic definition of these objects, definition which will be the starting point of a mathematically rigorous study of Toda CFTs.

4.1.2.1 Vertex Operators

In Toda CFTs these Vertex Operators formally correspond to taking $F(\Phi) = e^{\langle \alpha, \Phi(z) \rangle}$ for $z \in \mathbb{C}$ and $\alpha \in \mathfrak{a}$. However such functionals are not defined over $H^{-1}(\mathbb{R}^2 \rightarrow \mathfrak{a}, g)$ and for the very same reasons that led us to introduce GMC measures we cannot straightforwardly define these quantities since Φ has the same regularity as the GFF $\mathbf{X}^g$. To overcome this issue we still proceed in the same way and rely on a regularization procedure in order to define their correlations. Namely for $z \in \mathbb{C}$ and $\alpha \in \mathfrak{a}$ let us define the regularized Vertex Operator $V_{\alpha, \varepsilon}^g(z)$ by setting

$$V_{\alpha, \varepsilon}^g(z) := \varepsilon^{\frac{|\alpha|^2}{2}} e^{\langle \alpha, \mathbf{X}_\varepsilon^g(z) \rangle + \frac{Q}{2} \ln g + \mathbf{c} - \frac{\alpha \theta \eta}{2}} \quad (4.1.11)$$

where $X_\varepsilon^g(z)$ is the field regularized as before. Similarly to the GMC measure, when $g = \hat{g}$ this regularized Vertex Operator has same limit when $\varepsilon \rightarrow 0$ as the Wick exponential

$$e^{\langle \alpha, \mathbf{X}_\varepsilon^{\hat{g}}(x) + \mathbf{c} \rangle - \frac{1}{2} \mathbb{E}[\langle \alpha, \mathbf{X}_\varepsilon^{\hat{g}}(x) \rangle^2]} \hat{g}(x)^{\langle \frac{\alpha}{2}, Q - \frac{\alpha}{2} \rangle}. \quad (4.1.12)$$

4.1.2.2 Correlation functions of Vertex Operators

Taking the average of a product of such Vertex Operators with respect to the measure defined via the path integral defines their correlation functions, denoted by $\langle \prod_{k=1}^N V_{\alpha_k}(z_k) \rangle_{\mathfrak{g}, g}$. Combining Equation (4.1.10) with the above shows that as soon as it makes sense, the correlation functions of Vertex Operators are thus seen to be defined as the limits,

provided that it makes sense, of

$$\begin{aligned} & \left\langle \prod_{k=1}^N V_{\alpha_k}(z_k) \right\rangle_{\mathfrak{g},g} := \\ & \lim_{\varepsilon \rightarrow 0} e^{\frac{r}{96\pi} \int_{\mathbb{R}^2} (|d\varphi|_{\hat{g}}^2 + 2R_{\hat{g}}\varphi) dv_g} \int_{\mathfrak{a}} \mathbb{E} \left[\prod_{k=1}^N V_{\alpha_k, \varepsilon}(z_k) e^{-\frac{1}{4\pi} \langle QR_g, \mathbf{X}_{\varepsilon}^g + \mathbf{c} \rangle_g - \sum_{i=1}^r \mu_i e^{\langle \mathbf{c}, \gamma e_i \rangle} M_{g, \varepsilon}^{\gamma e_i}(\mathbb{R}^2)} \right] d\mathbf{c}, \end{aligned} \quad (4.1.13)$$

for $\alpha_1, \dots, \alpha_N$ elements of $\mathfrak{a}$ and $z_1, \dots, z_N$ distinct points in $\mathbb{C}$. The forthcoming sections are dedicated to providing conditions ensuring their existence as well as elementary properties they enjoy.

4.2 . Existence and basic properties of the correlation functions

Having provided the framework in which we aim to study Toda CFTs from a probabilistic perspective, we now focus on the correlation functions of Vertex Operators defined in such a way and to start with investigate the assumptions necessary to ensure their existence. We then proceed to the study of some of their basic properties, which are strong indications that the objects thus defined does indeed agree with expectations from the physics literature.

4.2.1 . Existence of the correlation functions

It is expected in the physics literature that in order to be well-defined, correlation functions must be subject to certain conditions called *Seiberg bounds* [123]. Our goal here is to recover these bounds within our probabilistic setting.

To start with, the form of the correlation functions (4.1.13) is not really convenient when it comes to investigating its convergence as $\varepsilon \rightarrow 0$. The introduction of the random measures

$$Z_{(\mathbf{z}, \boldsymbol{\alpha})}^{\gamma e_i}(dx) := \frac{\hat{g}(x)^{-\frac{1}{4} \langle \gamma e_i, \sum_{k=1}^N \alpha_k \rangle}}{\prod_{k=1}^N |x - z_k|^{\langle \alpha_k, \gamma e_i \rangle}} M_{\hat{g}}^{\gamma e_i}(dx) \quad (4.2.1)$$

associated to $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_N)$ and $\mathbf{z} = (z_1, \dots, z_N)$ allows to write them in a more tractable fashion. Indeed expressing the Vertex operators $V_{\alpha_k, \varepsilon}(z_k)$ as Wick exponentials as done in Equation (4.1.12) allows to interpret them as Girsanov weights that have the effect of shifting the law of the GFF by an additive term. More precisely, it follows from Theorem A that for suitable F

$$\begin{aligned} & \mathbb{E} \left[\prod_{k=1}^N V_{\alpha_k, \varepsilon}^g(z_k) F(\mathbf{X}_{\varepsilon}^{\hat{g}}) \right] \\ &= \prod_{k=1}^N \hat{g}(z_k)^{\Delta_{\alpha_k}} e^{\sum_{k < l} \langle \alpha_k, \alpha_l \rangle G_{\hat{g}, \varepsilon}(z_k, z_l)} e^{\langle \sum_{k=1}^N \alpha_k, \mathbf{c} \rangle} \mathbb{E} \left[F \left(\mathbf{X}_{\varepsilon}^{\hat{g}} + \sum_{k=1}^N \alpha_k G_{\hat{g}, \varepsilon}(\cdot, z_k) \right) \right] \end{aligned} \quad (4.2.2)$$

where $G_{\hat{g},\varepsilon}$, the covariance kernel of $X_{\varepsilon}^{\hat{g}}$, is a mollified version of $G_{\hat{g}}$. The conformal weight Δ_{α} is defined by

$$\Delta_{\alpha} := \left\langle \frac{\alpha}{2}, Q - \frac{\alpha}{2} \right\rangle. \quad (4.2.3)$$

This reformulation is essential to prove Theorem 4.2.1 below.

The existence of the correlation functions is then ensured by certain conditions on the each weight separately and on the quantity

$$s := \frac{\sum_{j=1}^N \alpha_j - 2Q}{\gamma} \quad \text{as well as, for all } i, \quad s_i := \frac{\langle \sum_{j=1}^N \alpha_j - 2Q, \omega_i^{\vee} \rangle}{\gamma} \quad (4.2.4)$$

as the following statement discloses:

Theorem 4.2.1. *Existence and non triviality of the correlation function $\langle V_{\alpha_1}(z_1) \cdots V_{\alpha_N}(z_N) \rangle_{\mathfrak{g},g}$ do not depend on the background metric g in the conformal class of the spherical metric $\hat{g}$. Furthermore:*

1. **(Seiberg bounds)** *The correlation functions $\langle V_{\alpha_1}(z_1) \cdots V_{\alpha_N}(z_N) \rangle_{\mathfrak{g},g}$ defined by Equation (4.1.13) exist and are non trivial if and only if the two following conditions hold:*

$$\text{For all } 1 \leq i \leq r, \quad s_i > 0 \quad \text{and} \quad \langle \alpha_k - Q, e_i \rangle < 0 \quad \text{for any } 1 \leq k \leq N. \quad (4.2.5)$$

2. **(GMC representation)** *In the particular case where $g = \hat{g}$ is the round metric, one gets the following explicit expression for the correlation function*

$$\begin{aligned} & \langle V_{\alpha_1}(z_1) \cdots V_{\alpha_N}(z_N) \rangle_{\mathfrak{g},\hat{g}} \\ &= \left(\prod_{i=1}^r \frac{\Gamma(s_i) \mu_i^{-s_i}}{\gamma} \right) \prod_{1 \leq k < l \leq N} |z_k - z_l|^{-\langle \alpha_k, \alpha_l \rangle} \mathbb{E} \left[\prod_{i=1}^r Z_{(z,\alpha)}^{\gamma e_i}(\mathbb{C})^{-s_i} \right]. \end{aligned} \quad (4.2.6)$$

The condition that the weights must satisfy the assumption that for all $1 \leq i \leq r$, $\langle \alpha_k - Q, e_i \rangle < 0$ corresponds to the requirement that α belongs to the shifted Weyl chamber $Q + \mathcal{C}$, see Figure 3.1.

Proof. The proof follows closely that developed in [40] for the case where $\mathfrak{g} = \mathfrak{sl}_2$, that is when the CFT being studied is Liouville CFT. The main difference lies in the fact that many of the quantities involved are no longer scalar but rather vectors of the Euclidean space $\mathfrak{a}$. Likewise several GMC measures, stemming from the form r different exponential potential that appear in the Toda field action (4.1.1), need to be taken into account.

To prove our statement we rely on Lemma 4.3.3 from Section 4.3 below, which provides sufficient conditions ensuring that the expression (4.2.6) does make sense probabilistically speaking. Also anticipating on the conformal anomaly formula from Theorem 4.2.2 below, let us assume that we work with the spherical metric $\hat{g}$; the general case where g belongs to the conformal class of $\hat{g}$ is a straightforward consequence of the Weyl anomaly stated there. First note that the metric $\hat{g}$ is such that

$$\frac{1}{4\pi} \int_{\mathbb{C}} \langle QR_{\hat{g}}(x), X^{\hat{g}} + c \rangle v_{\hat{g}}(dx) = 2\langle Q, c \rangle$$

since $R_{\hat{g}}$ is constant and $X^{\hat{g}}$ has zero mean value in the metric $\hat{g}$. With the help of Equation (4.2.2) we can rewrite the regularized correlation functions as

$$\prod_{k=1}^N \hat{g}(z_k)^{\Delta_{\alpha_k}} e^{\sum_{k < l} \langle \alpha_k, \alpha_l \rangle G_{\hat{g}, \varepsilon}(z_k, z_l)} \int_{\mathbb{R}^r} e^{\sum_{i=1}^r \gamma s_i c_i} \mathbb{E} \left[e^{-\sum_{i=1}^r \mu_i e^{\gamma c_i} \tilde{Z}_{(\mathbf{z}, \boldsymbol{\alpha}), \varepsilon}^{\gamma e_i}(\mathbb{C})} \right] dc_1 \dots dc_r$$

where $\tilde{Z}_{(\mathbf{z}, \boldsymbol{\alpha}), \varepsilon}^{\gamma e_i}(dx) := e^{\gamma \sum_{k=1}^N \langle \alpha_k, e_i \rangle G_{\hat{g}, \varepsilon}(x, z_j)} M_{\hat{g}, \varepsilon}^{\gamma e_i}(dx)$. Now, if one of the s_i is non-positive, then the whole integral can be lower-bounded by

$$\int_{\mathbb{R}^r} e^{\sum_{i=1}^r \gamma s_i c_i} e^{-\sum_{i=1}^r \mu_i e^{\gamma c_i} M} dc_1 \dots dc_r \mathbb{P} \left(\forall 1 \leq i \leq r, \tilde{Z}_{(\mathbf{z}, \boldsymbol{\alpha}), \varepsilon}^{\gamma e_i}(\mathbb{C}) \leq M \right) = +\infty$$

where $M > 0$ is taken so that $\mathbb{P} \left(\forall 1 \leq i \leq r, \tilde{Z}_{(\mathbf{z}, \boldsymbol{\alpha}), \varepsilon}^{\gamma e_i}(\mathbb{C}) \leq M \right) > 0$ (to see why this is possible note that $G_{\hat{g}, \varepsilon}(x, \cdot)$ is bounded over $\mathbb{C}$ and apply Lemma 4.3.3 with all the α taken equal to zero). Therefore the ε -regularized partition function is infinite if one of the s_i is non-positive. Conversely if these s_i are all positive and using Lemma 4.3.3 we can make the change of variable $y_i = \mu_i e^{\gamma c_i} \tilde{Z}_{(\mathbf{z}, \boldsymbol{\alpha}), \varepsilon}^{\gamma e_i}(\mathbb{C})$ for $1 \leq i \leq r$ in the integral so that we are left with

$$\prod_{k=1}^N \hat{g}(z_k)^{\Delta_{\alpha_k}} e^{\sum_{k < l} \langle \alpha_k, \alpha_l \rangle G_{\hat{g}, \varepsilon}(z_k, z_l)} \int_{(0, \infty)^r} \mathbb{E} \left[\prod_{i=1}^r \frac{\left(\mu_i \tilde{Z}_{(\mathbf{z}, \boldsymbol{\alpha}), \varepsilon}^{\gamma e_i}(\mathbb{C}) \right)^{-s_i}}{\gamma} y_i^{s_i-1} e^{-y_i} \right] dy_1 \dots dy_r.$$

Using Fubini-Tonelli's theorem the latter can be evaluated and is found to be equal to

$$\begin{aligned} & \prod_{k=1}^N \hat{g}(z_k)^{\Delta_{\alpha_k}} e^{\sum_{k < l} \langle \alpha_k, \alpha_l \rangle G_{\hat{g}, \varepsilon}(z_k, z_l)} \mathbb{E} \left[\prod_{i=1}^r \frac{\left(\mu_i \tilde{Z}_{(\mathbf{z}, \boldsymbol{\alpha}), \varepsilon}^{\gamma e_i}(\mathbb{C}) \right)^{-s_i}}{\gamma} \right] \int_{(0, \infty)^r} \prod_{i=1}^r y_i^{s_i-1} e^{-y_i} dy_1 \dots dy_r \\ &= \left(\prod_{i=1}^r \frac{\Gamma(s_i) \mu_i^{-s_i}}{\gamma} \right) \prod_{1 \leq k < l \leq N} |z_k - z_l|^{-\langle \alpha_k, \alpha_l \rangle} \mathbb{E} \left[\prod_{i=1}^r Z_{(\mathbf{z}, \boldsymbol{\alpha}), \varepsilon}^{\gamma e_i}(\mathbb{C})^{-s_i} \right] \end{aligned}$$

by using the explicit expression (3.2.9) of $G_{\hat{g}}$ together with the fact that $\sum_{i=1}^r s_i \langle \alpha_k, \gamma e_i \rangle = \langle \alpha_k, \sum_{l=1}^N \alpha_l - 2Q \rangle$. To conclude it remains to show that:

- If for all $1 \leq i \leq r$, and $1 \leq j \leq N$, $\langle \alpha_j, e_i \rangle < \langle Q, e_i \rangle$, then

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E} \left[\prod_{i=1}^r Z_{(\mathbf{z}, \boldsymbol{\alpha}), \varepsilon}^{\gamma e_i}(\mathbb{C})^{-s_i} \right] = \mathbb{E} \left[\prod_{i=1}^r Z_{(\mathbf{z}, \boldsymbol{\alpha})}^{\gamma e_i}(\mathbb{C})^{-s_i} \right] > 0.$$

- If for some $1 \leq i \leq r$ and $1 \leq k \leq N$, $\langle \alpha_k, e_i \rangle \geq \langle Q, e_i \rangle$, then

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E} \left[\prod_{i=1}^r Z_{(\mathbf{z}, \boldsymbol{\alpha}), \varepsilon}^{\gamma e_i}(\mathbb{C})^{-s_i} \right] = 0.$$

For this let us first assume that for all $1 \leq i \leq r$, and $1 \leq k \leq N$, $\langle \alpha_k, e_i \rangle < \langle Q, e_i \rangle$. Then we know from Lemma 4.3.3 that the family of random variables $\left(\prod_{i=1}^r Z_{(\mathbf{z}, \boldsymbol{\alpha}), \varepsilon}^{\gamma e_i}(\mathbb{C})^{-s_i} \right)_{\varepsilon \geq 0}$

have (uniformly bounded in ε) positive moments of all orders. Thus we can write that

$$\begin{aligned} \mathbb{E} \left[\left| \prod_{i=1}^r Z_{(\mathbf{z}, \boldsymbol{\alpha}), \varepsilon}^{\gamma e_i}(\mathbb{C})^{-s_i} - \prod_{i=1}^r Z_{(\mathbf{z}, \boldsymbol{\alpha})}^{\gamma e_i}(\mathbb{C})^{-s_i} \right| \right] &\leq \mathbb{E} \left[\left| Z_{(\mathbf{z}, \boldsymbol{\alpha}), \varepsilon}^{\gamma e_1}(\mathbb{C})^{-s_1} - Z_{(\mathbf{z}, \boldsymbol{\alpha})}^{\gamma e_1}(\mathbb{C})^{-s_1} \right| \prod_{i=2}^r Z_{(\mathbf{z}, \boldsymbol{\alpha}), \varepsilon}^{\gamma e_i}(\mathbb{C})^{-s_i} \right] \\ &+ \mathbb{E} \left[\left| \prod_{i=2}^r Z_{(\mathbf{z}, \boldsymbol{\alpha}), \varepsilon}^{\gamma e_i}(\mathbb{C})^{-s_i} - \prod_{i=2}^r Z_{(\mathbf{z}, \boldsymbol{\alpha})}^{\gamma e_i}(\mathbb{C})^{-s_i} \right| Z_{(\mathbf{z}, \boldsymbol{\alpha})}^{\gamma e_1}(\mathbb{C})^{-s_1} \right] \\ &\leq \mathbb{E} \left[\left| Z_{(\mathbf{z}, \boldsymbol{\alpha}), \varepsilon}^{\gamma e_1}(\mathbb{C})^{-s_1} - Z_{(\mathbf{z}, \boldsymbol{\alpha})}^{\gamma e_1}(\mathbb{C})^{-s_1} \right|^p \right]^{\frac{1}{p}} \mathbb{E} \left[\prod_{i=2}^r Z_{(\mathbf{z}, \boldsymbol{\alpha}), \varepsilon}^{\gamma e_i}(\mathbb{C})^{-q s_i} \right]^{\frac{1}{q}} \\ &+ \mathbb{E} \left[\left| \prod_{i=2}^r Z_{(\mathbf{z}, \boldsymbol{\alpha}), \varepsilon}^{\gamma e_i}(\mathbb{C})^{-s_i} - \prod_{i=2}^r Z_{(\mathbf{z}, \boldsymbol{\alpha})}^{\gamma e_i}(\mathbb{C})^{-s_i} \right|^p \right]^{\frac{1}{p}} \mathbb{E} \left[Z_{(\mathbf{z}, \boldsymbol{\alpha})}^{\gamma e_1}(\mathbb{C})^{-q s_1} \right]^{\frac{1}{q}} \end{aligned}$$

where we have used Hölder inequality with some $p = \frac{q}{q-1} > 1$. We can thus proceed

by induction on r so that the only point to check is $\lim_{\varepsilon \rightarrow 0} \mathbb{E} \left[\left| Z_{(\mathbf{z}, \boldsymbol{\alpha}), \varepsilon}^{\gamma e_1}(\mathbb{C})^{-s_1} - Z_{(\mathbf{z}, \boldsymbol{\alpha})}^{\gamma e_1}(\mathbb{C})^{-s_1} \right|^p \right]^{\frac{1}{p}} = 0$.

This fact has already been proved by the authors in [40, Lemma 3.3]. For the second bullet point, let us introduce the set $\mathcal{P} := \{i = 1, \dots, r \mid \exists 1 \leq k \leq N, \langle \alpha_k - Q, e_i \rangle \geq 0\}$ and assume that it is non-empty. Then we can write that, for positive $\frac{1}{p_i}$ and $\frac{1}{q}$ summing to one,

$$\mathbb{E} \left[\prod_{i=1}^r Z_{(\mathbf{z}, \boldsymbol{\alpha}), \varepsilon}^{\gamma e_i}(\mathbb{C})^{-s_i} \right] \leq \prod_{i \in \mathcal{P}} \mathbb{E} \left[Z_{(\mathbf{z}, \boldsymbol{\alpha}), \varepsilon}^{\gamma e_i}(\mathbb{C})^{-p_i s_i} \right]^{\frac{1}{p_i}} \mathbb{E} \left[\prod_{i \notin \mathcal{P}} Z_{(\mathbf{z}, \boldsymbol{\alpha}), \varepsilon}^{\gamma e_i}(\mathbb{C})^{-q s_i} \right]^{\frac{1}{q}}.$$

Then we have already seen that the second expectation in the right-hand-side had a finite limit as $\varepsilon \rightarrow 0$ thanks to the results of Lemma 4.3.3. Conversely the statement of Lemma 3.3.6 implies that for any $i \in \mathcal{P}$, $\lim_{\varepsilon \rightarrow 0} \mathbb{E} \left[Z_{(\mathbf{z}, \boldsymbol{\alpha}), \varepsilon}^{\gamma e_i}(\mathbb{C})^{-p_i s_i} \right]^{\frac{1}{p_i}} = 0$. This concludes for the proof of Theorem 4.2.1. $\square$

4.2.2 . Correlation functions and conformal field theory

Toda CFTs enjoy a level of symmetry that includes conformal invariance. For such CFTs it is usually assumed in the physics literature that the correlation functions of Vertex Operators must satisfy a certain set of axioms such as conformal covariance under Möbius transforms as well as the existence of a Weyl anomaly under change of background metric g . See for instance the review by Gawedski [62] where such axioms are being exposed.

The following statement shows that the objects defined using our probabilistic framework satisfy these assumptions, thus confirming that they provide a good translation of their counterparts from the physics literature:

Theorem 4.2.2. *Under the assumptions of Theorem 4.2.1, the following holds true:*

1. (Conformal covariance) *Let ψ be a Möbius transform of the plane. Then*

$$\langle V_{\alpha_1}(\psi(z_1)) \cdots V_{\alpha_N}(\psi(z_N)) \rangle_{\mathfrak{g}, g} = \prod_{k=1}^N |\psi'(z_k)|^{-2\Delta_{\alpha_k}} \langle V_{\alpha_1}(z_1) \cdots V_{\alpha_N}(z_N) \rangle_{\mathfrak{g}, g},$$

where the conformal weights are given by $\Delta_{\alpha_j} := \langle \frac{\alpha_j}{2}, Q - \frac{\alpha_j}{2} \rangle$.

2. (Weyl anomaly) If $\varphi \in \bar{C}^1(\mathbb{R}^2)$ then we have the following relation

$$\langle V_{\alpha_1}(z_1) \cdots V_{\alpha_N}(z_N) \rangle_{\mathfrak{g}, e^{\varphi} \hat{g}} = e^{\frac{c_T}{96\pi} S_L(\varphi, \hat{g})} \langle V_{\alpha_1}(z_1) \cdots V_{\alpha_N}(z_N) \rangle_{\mathfrak{g}, \hat{g}}$$

where S_L is the Liouville functional

$$S_L(\varphi, \hat{g}) := \int_{\mathbb{R}^2} (|\partial_{\hat{g}} \varphi|_{\hat{g}}^2 + 2R_{\hat{g}} \varphi) dv_{\hat{g}},$$

and the central charge c_T is given by $c_T = r + 6|Q|^2$.

This statement shows that the theory thus defined is a CFT in the sense of Gawedski's axioms, with central charge c_T given by:

$\mathfrak{g}$	$c_{T,\mathfrak{g}}$	$\mathfrak{g}$	$c_{T,\mathfrak{g}}$
A_n	$n + \frac{n(n+1)(n+2)}{2}(\gamma + \frac{2}{\gamma})^2$	E_6	$6 + 468(\gamma + \frac{2}{\gamma})^2$
B_n	$n + \gamma^2 \frac{n(2n-1)(2n+1)}{2} + 2n(n+1)(4n-1)$ $+ \frac{4}{\gamma^2} n(n+1)(2n+1)$	E_7	$7 + 1197(\gamma + \frac{2}{\gamma})^2$
C_n	$n + \gamma^2 \frac{n(n+1)(2n+1)}{2} + 2n(n+1)(4n-1)$ $+ \frac{4}{\gamma^2} n(2n-1)(2n+1)$	E_8	$8 + 3720(\gamma + \frac{2}{\gamma})^2$
D_n	$n + (n-1)n(2n-1)(\gamma + \frac{2}{\gamma})^2$	F_4	$4 + 234\gamma^2 + 330 + 468\frac{4}{\gamma^2}$
		G_2	$2 + 28\gamma^2 + 192 + 84\frac{4}{\gamma^2}$

(4.2.7)

Proof. Let us start with the second item and consider $g = e^{\varphi} \hat{g}$ in the conformal class of $\hat{g}$. Then by making the change of variable in the zero mode c given by $c \leftrightarrow c - m_{\hat{g}}(\mathbf{X}^g)$, we can in fact rewrite Equation (4.1.13) by replacing the field $\mathbf{X}^g + c$ by $\mathbf{X}^g - m_{\hat{g}}(\mathbf{X}^g) + c$. Using the fact that $\mathbf{X}^g - m_{\hat{g}}(\mathbf{X}^g)$ and $\mathbf{X}^{\hat{g}}$ have same law in agreement with Lemma 3.3.3, we can therefore assume that the underlying GFF has the law of $\mathbf{X}^{\hat{g}}$. Since $g = e^{\varphi} \hat{g}$, we have that

$$\langle V_{\alpha_1, \varepsilon}(z_1) \cdots V_{\alpha_N, \varepsilon}(z_N) \rangle_g = Z(g) \prod_{k=1}^N e^{\frac{\langle \alpha_k, Q \rangle}{2} \varphi(x_k)} \times \lim_{\varepsilon \rightarrow 0} \int_{\mathfrak{a}} e^{\gamma \langle s, c \rangle} \mathbb{E} \left[\prod_{j=1}^k \tilde{V}_{\alpha_j, \varepsilon}^{\hat{g}}(z_j) e^{-\frac{1}{4\pi} \int_{\mathbb{C}} R_g \langle Q, \mathbf{X}_{\varepsilon}^{\hat{g}} \rangle(x) v_g(dx) - \sum_{i=1}^r \mu_i e^{\gamma c_i} \int_{\mathbb{C}} e^{\gamma^2 \frac{\varphi(x)}{2}} V_{\gamma e_i, \varepsilon}^{\hat{g}}(x) v_g(dx)} \right] dc$$

where this time regularization is done with respect to the round metric, and $\tilde{V}$ is the Vertex Operator without constant mode, that is

$$\tilde{V}_{\alpha, \varepsilon}^{\hat{g}}(z) := \varepsilon^{\frac{|\alpha|^2}{2}} e^{\langle \alpha, \mathbf{X}_{\varepsilon}^{\hat{g}}(z) + \frac{Q}{2} \ln g - \frac{\alpha \theta_n}{2} \rangle}.$$

We will consider the term $e^{-\frac{1}{4\pi} \int_{\mathbb{C}} R_g \langle Q, \mathbf{X}_{\varepsilon}^{\hat{g}} \rangle(x) v_g(dx)}$ as a Girsanov transform. Namely we can use the fact that $R_g(y) v_g(dy) = (-\Delta_{\hat{g}} \varphi(y) + 2) v_{\hat{g}}(dy)$ (at least in the weak sense since $\varphi \in \bar{C}^1(\mathbb{R}^2)$) and the definition of the Green function $G_{\hat{g}}$ to see that for any $\alpha \in \mathfrak{a}$:

$$\mathbb{E} \left[\left(\frac{1}{4\pi} \int_{\mathbb{C}} R_g(y) \langle Q, \mathbf{X}^{\hat{g}} \rangle(y) v_g(dy) \right) \langle \alpha, \mathbf{X}^{\hat{g}} \rangle(x) \right] = \langle Q, \frac{\alpha}{2} \rangle (\varphi(x) - m_{\hat{g}}(\varphi)).$$

The variance of this expression is given by

$$\begin{aligned}
& \mathbb{E} \left[\left(\frac{1}{4\pi} \int_{\mathbb{C}} R_g(x) \langle Q, \mathbf{X}^{\hat{g}} \rangle(x) v_g(dx) \right)^2 \right] \\
&= \frac{1}{16\pi^2} \int_{\mathbb{C} \times \mathbb{C}} R_g(x) R_g(y) \mathbb{E}[\langle Q, \mathbf{X}^{\hat{g}} \rangle(x) \langle Q, \mathbf{X}_{\varepsilon}^{\hat{g}} \rangle(y)] v_g(dx) v_g(dy) \\
&= \frac{1}{16\pi^2} |Q|^2 \int_{\mathbb{C} \times \mathbb{C}} R_g(x) R_g(y) G_{\hat{g}}(x, y) v_g(dx) v_g(dy) \\
&= \frac{1}{16\pi^2} |Q|^2 \int_{\mathbb{C}} R_g(x) \left(\int_{\mathbb{C}} R_g(y) G_{\hat{g}}(x, y) v_g(dy) \right) v_g(dx) \\
&= \frac{1}{8\pi} |Q|^2 \int_{\mathbb{C}} R_g(x) (\varphi(x) - m_{\hat{g}}(\varphi)) v_g(dx) \\
&= \frac{1}{8\pi} |Q|^2 \int_{\mathbb{C}} (-\Delta_{\hat{g}} \varphi(x) + 2) (\varphi(x) - m_{\hat{g}}(\varphi)) v_{\hat{g}}(dx) \\
&= \frac{1}{8\pi} |Q|^2 \int_{\mathbb{C}} |\partial_{\hat{g}} \varphi|^2 v_{\hat{g}}(dx).
\end{aligned}$$

As a consequence we obtain thanks to Theorem A that

$$\mathbb{E} \left[e^{-\frac{1}{4\pi} \int_{\mathbb{C}} R_g \langle Q, \mathbf{X}^{\hat{g}} \rangle(x) v_g(dx)} F(\mathbf{X}^{\hat{g}}) \right] = e^{\frac{1}{16\pi} |Q|^2 \int_{\mathbb{C}} |\partial_{\hat{g}} \varphi|^2 v_{\hat{g}}(dx)} \mathbb{E} \left[F \left(\mathbf{X}^{\hat{g}} + \frac{Q}{2} (\varphi(x) - m_{\hat{g}}(\varphi)) \right) \right].$$

Using like before the change of variable $\mathbf{c} \leftrightarrow \mathbf{c} + \frac{Q}{2} m_{\hat{g}}(\varphi)$ and recollecting terms we end up with

$$\begin{aligned}
\langle V_{\alpha_1}(z_1) \cdots V_{\alpha_N}(z_N) \rangle_g &= Z(g) e^{\frac{|Q|^2}{16\pi} \int_{\mathbb{C}} |\partial_{\hat{g}} \varphi|^2 v_{\hat{g}} \times} \\
&\lim_{\varepsilon \rightarrow 0} \int_{\mathfrak{a}} e^{\gamma \langle \mathbf{s}, \mathbf{c} \rangle} \mathbb{E} \left[\prod_{k=1}^N \tilde{V}_{\alpha_k, \varepsilon}^{\hat{g}}(z_k) e^{-\sum_{i=1}^r \mu_i e^{\gamma c_i} \int_{\mathbb{C}} e^{\varphi(x)} (\gamma^2 \frac{|e_i|^2}{4} - \frac{\gamma \langle Q, e_i \rangle}{2}) \tilde{V}_{\gamma e_i, \varepsilon}(x) v_g(dx)} \right] d\mathbf{c}.
\end{aligned}$$

Since Q is such that $\langle Q, e_i \rangle = \frac{\gamma |e_i|^2}{2} + \frac{2}{\gamma}$ for all $1 \leq i \leq r$, we know that $\frac{\gamma |e_i|^2}{4} - \frac{\gamma \langle Q, e_i \rangle}{2} = -1$, by the change of variable $\mathbf{c} \leftrightarrow \mathbf{c} - \frac{Q}{2} m_{\hat{g}}(\varphi)$ we get that

$$\begin{aligned}
& \langle V_{\alpha_1}(z_1) \cdots V_{\alpha_N}(z_N) \rangle_g = \\
& Z(g) e^{\frac{|Q|^2}{16\pi} \int_{\mathbb{C}} |\partial_{\hat{g}} \varphi|^2 v_{\hat{g}} + \frac{|Q|^2}{4\pi} \int_{\mathbb{C}} \varphi v_{\hat{g}}} \times \lim_{\varepsilon \rightarrow 0} \int_{\mathfrak{a}} \mathbb{E} \left[\prod_{k=1}^N V_{\varepsilon, \alpha_k}^{\hat{g}}(z_k) e^{-\sum_{i=1}^r \mu_i \int_{\mathbb{C}} V_{\gamma e_i, \varepsilon}^{\hat{g}}(x) v_{\hat{g}}(dx)} \right] d\mathbf{c},
\end{aligned}$$

whence the result, by using the expression (4.1.8) for the regularized determinant $Z(g)$.

For the first item, we see that according to the proof of Theorem 4.2.1 and more precisely Equation (4.2.2), the quantity $\langle F(\Phi) \prod_{k=1}^N V_{\alpha_k}(z_k) \rangle$ for F bounded continuous over $H^{-1}(\mathbb{R}^2 \rightarrow \mathfrak{a}, \hat{g})$ is actually given by

$$\begin{aligned}
& Z(g) \prod_{k=1}^N \hat{g}(z_k)^{\Delta_{\alpha_k}} e^{\sum_{k < l} \langle \alpha_k, \alpha_l \rangle G_{\hat{g}}(z_k, z_l)} \\
& \int_{\mathfrak{a}} e^{\gamma \langle \mathbf{s}, \mathbf{c} \rangle} \mathbb{E} \left[F \left(\mathbf{X}^{\hat{g}} + \frac{Q}{2} \ln \hat{g} + \mathbf{c} + \sum_{k=1}^N \alpha_k G_{\hat{g}}(\cdot, z_k) \right) e^{-\sum_{i=1}^r \mu_i e^{\gamma \langle \mathbf{c}, e_i \rangle} Z_{(\mathbf{z}, \alpha)}^{\gamma e_i}(\mathbb{C})} \right] d\mathbf{c}.
\end{aligned}$$

Because ψ is a conformal map we know that the Riemannian metric $\psi^*\hat{g}$ (that we have denoted $\hat{g}_\psi$ before) lies within the conformal class of $\hat{g}$; as a consequence using Lemma 3.3.3 the GFFs $\mathbf{X}^{\hat{g}_\psi} - m_{\hat{g}}(\mathbf{X}^{\hat{g}_\psi})$ and $\mathbf{X}^{\hat{g}}$ have same law. Moreover from Lemma 3.3.4 we know that $\mathbf{X}^{\hat{g}_\psi}$ has same law as $\mathbf{X}^{\hat{g}} \circ \psi$. In a nutshell,

$$\mathbf{X}^{\hat{g}} \circ \psi - m_{\hat{g}}(\mathbf{X}^{\hat{g}} \circ \psi) \stackrel{(\text{law})}{=} \mathbf{X}^{\hat{g}}.$$

Besides we saw in Equation (3.2.11) that $G_{\hat{g}}(\cdot, \psi(z_k)) \circ \psi + \frac{1}{4}(\phi + \phi(z_k)) = G_{\hat{g}}(\cdot, z_k)$ where $e^\phi = \frac{\hat{g}_\psi}{\hat{g}}$. Combining these two assertions yields that the laws of

$$\mathbf{X}^{\hat{g}} + \sum_{k=1}^N \alpha_k G_{\hat{g}}(\cdot, z_k) \quad \text{and} \quad \left(\mathbf{X}^{\hat{g}} + \sum_{k=1}^N \alpha_k G_{\hat{g}}(\cdot, \psi(z_k)) \right) \circ \psi + \frac{1}{4} \sum_{k=1}^N \alpha_k (\phi + \phi(z_k)) - m_{\hat{g}}(\mathbf{X}^{\hat{g}} \circ \psi)$$

are actually the same. Since $\hat{g}_\psi = |\psi'|^2 \hat{g} \circ \psi$ the latter further implies that

$$\Phi_{\hat{g}}^z \stackrel{(\text{law})}{=} \Phi_{\hat{g}}^{\psi z} \circ \psi + Q \ln |\psi'| - \frac{Q}{2} \phi + \frac{1}{4} \sum_{k=1}^N \alpha_k (\phi + \phi(z_k)) - m_{\hat{g}}(\mathbf{X}^{\hat{g}} \circ \psi)$$

where $\Phi_{\hat{g}}^z$ is a shorthand for $\mathbf{X}^{\hat{g}} + \frac{Q}{2} \ln \hat{g} + \sum_{k=1}^N \alpha_k G_{\hat{g}}(\cdot, z_k)$ and $\psi z := (\psi(z_1), \dots, \psi(z_N))$. Therefore $\langle F(\Phi) \prod_{k=1}^N V_{\alpha_k}(z_k) \rangle$ can be put under the form

$$Z(g) \prod_{k=1}^N \hat{g}(z_k)^{\Delta_{\alpha_k}} e^{\sum_{k < l} \langle \alpha_k, \alpha_l \rangle G_{\hat{g}}(z_k, z_l)} \int_{\mathfrak{a}} e^{\gamma \langle \mathbf{s}, \mathbf{c} \rangle} d\mathbf{c} \\ \mathbb{E} \left[F \left(\Phi_{\hat{g}}^{\psi z} \circ \psi + Q \ln |\psi'| + \frac{\gamma \mathbf{s}}{4} \phi + \mathbf{c} + \frac{1}{4} \sum_{k=1}^N \alpha_k \phi(z_k) - m_{\hat{g}}(\mathbf{X}^{\hat{g}} \circ \psi) \right) e^{-\sum_{i=1}^r \mu_i e^{\gamma \langle \mathbf{c}, \mathbf{e}_i \rangle} Z_{(\mathbf{z}, \boldsymbol{\alpha})}^{\gamma \mathbf{e}_i}(\mathbb{C})} \right].$$

This motivates the shift in the zero mode $\mathbf{c} \leftrightarrow \mathbf{c} + m_{\hat{g}}(\mathbf{X}^{\hat{g}}) - \frac{1}{4} \sum_{k=1}^N \alpha_k \phi(z_k) - \frac{\gamma}{4} \mathbf{s} m_{\hat{g}}(\phi)$. After this change of variable we are left with

$$Z(g) \prod_{k=1}^N \hat{g}(z_k)^{\Delta_{\alpha_k}} e^{\sum_{k < l} \langle \alpha_k, \alpha_l \rangle G_{\hat{g}}(z_k, z_l)} e^{-\frac{\gamma}{4} \langle \mathbf{s}, \boldsymbol{\alpha} \rangle} \int_{\mathfrak{a}} e^{\gamma \langle \mathbf{s}, \mathbf{c} \rangle} d\mathbf{c} \\ \mathbb{E} \left[e^{\gamma \langle \mathbf{s}, m_{\hat{g}}(\mathbf{X}^{\hat{g}}) \rangle - m_{\hat{g}}(\phi) \frac{\gamma^2 \langle \mathbf{s}, \mathbf{s} \rangle}{4}} F \left(\Phi_{\hat{g}}^{\psi z} \circ \psi + Q \ln |\psi'| + \frac{1}{4} \mathbf{s} (\phi - m_{\hat{g}}(\phi)) \right) e^{-\sum_{i=1}^r \mu_i e^{\gamma \langle \mathbf{c}, \mathbf{e}_i \rangle} Z_{(\mathbf{z}, \boldsymbol{\alpha})}^{\gamma \mathbf{e}_i}(\mathbb{C})} \right].$$

Collecting up terms using (3.2.11) yields:

$$Z(g) \prod_{k=1}^N \hat{g}_\psi(z_k)^{\Delta_{\alpha_k}} e^{\sum_{k < l} \langle \alpha_k, \alpha_l \rangle G_{\hat{g}}(\psi(z_k), \psi(z_l))} \int_{\mathfrak{a}} e^{\gamma \langle \mathbf{s}, \mathbf{c} \rangle} d\mathbf{c} \\ \mathbb{E} \left[e^{\gamma \langle \mathbf{s}, m_{\hat{g}}(\mathbf{X}^{\hat{g}}) \rangle - m_{\hat{g}}(\phi) \frac{\gamma^2 \langle \mathbf{s}, \mathbf{s} \rangle}{4}} F \left(\Phi_{\hat{g}} \circ \psi + Q \ln |\psi'| + \frac{1}{4} \mathbf{s} (\phi - m_{\hat{g}}(\phi)) \right) e^{-\sum_{i=1}^r \mu_i e^{\gamma \langle \mathbf{c}, \mathbf{e}_i \rangle} Z_{(\mathbf{z}, \boldsymbol{\alpha})}^{\gamma \mathbf{e}_i}(\mathbb{C})} \right].$$

The proof is completed by interpreting the exponential term $e^{\langle \mathbf{s}, m_{\hat{g}}(\mathbf{X}^{\psi^* \hat{g}}) \rangle - \frac{m_{\hat{g}}(\phi) \langle \mathbf{s}, \mathbf{s} \rangle}{4}}$ as a Girsanov transform, whose effect is to shift the law of $\Phi_{\hat{g}} \circ \psi$ by $-\frac{1}{4} \mathbf{s} (\phi - m_{\hat{g}}(\phi))$. $\square$

In the proof of the conformal covariance property we have actually shown a slightly more general result. Indeed we have proved that under the Seiberg bounds and for any

Möbius transform of the plane, the following was true for any continuous and bounded map F on $H^{-1}(\mathbb{R}^2 \rightarrow \mathfrak{a}, \hat{g})$:

$$\langle F(\Phi \circ \psi + Q \ln |\psi'|) \prod_{k=1}^N V_{\alpha_k}(\psi(z_k)) \rangle_g = \prod_{k=1}^N |\psi'(z_k)|^{-2\Delta_{\alpha_k}} \langle F(\Phi) \prod_{k=1}^N V_{\alpha_k}(z_k) \rangle_g. \quad (4.2.8)$$

This statement is usually referred to as the conformal covariance of the Toda field.

4.3 . Correlation functions beyond the Seiberg bounds

The goal of this section is to use the probabilistic definition of the correlation functions given by Equation (4.2.6) to extend the range of values for which they can be defined and thus overcome the obstruction of the Seiberg bounds. We will provide in this manuscript two alternative extensions that will naturally arise in different contexts but we will present the second extension in Section 6.4 since it relies on the tail expansion of GMC measures and is much more involved. These extensions will satisfy the key property that they depend analytically on the weights α viewed as elements of $\mathbb{C}^r$. To be more specific, we will say hereafter that a map $F : \alpha \in \mathbb{C}^r \rightarrow \mathbb{C}$ is analytic when its dependence in $\langle \alpha, u \rangle$ is analytic for any $u \in \mathbb{C}^r$.

In order to provide a first analytic continuation of the correlation functions, note that the explicit expression (4.2.6) allows us to isolate the constraints $s_i > 0$ in the product of Γ functions. This term can obviously be analytically removed. The question is then to determine whether the expectation in (4.2.6) makes sense beyond the range of parameters permitted by the Seiberg bounds, and if this extension is indeed analytic. Our goal in this section is to prove that the following theorem does indeed hold:

Theorem 4.3.1. *Let $z_1, \dots, z_N \in \mathbb{C}$ be distinct and define a subset of $(Q + \mathcal{C}_-)^N$ by setting*

$$\mathcal{B}_N := \left\{ (\alpha_1, \dots, \alpha_N) \in (Q + \mathcal{C}_-)^N \text{ such that } -s_i < \frac{4}{\gamma^2 \langle e_i, e_i \rangle} \wedge \min_{k=1, \dots, N} \frac{1}{\gamma} \langle Q - \alpha_k, e_i^\vee \rangle \text{ for all } 1 \leq i \leq r \right\}. \quad (4.3.1)$$

Then the map defined by Equation (4.2.6):

$$\alpha \mapsto \left(\prod_{i=1}^r \frac{\Gamma(s_i) \mu_i^{-s_i}}{\gamma} \right) \prod_{1 \leq k < l \leq N} |z_k - z_l|^{-\langle \alpha_k, \alpha_l \rangle} \mathbb{E} \left[\prod_{i=1}^r Z_{(z, \alpha)}^{\gamma e_i}(\mathbb{C})^{-s_i} \right] \quad (4.3.2)$$

is meromorphic in a complex neighbourhood of $\mathcal{B}_N$, with poles given by

$$\mathcal{P}_N^1 := \{ \alpha \in \mathcal{B}_N, \quad s_i = 0 \text{ for some } 1 \leq i \leq r \}.$$

To prove this statement we will rely on two probabilistic facts. The first one is that the GMC measures considered depend analytically in the weights for α in a complex neighbourhood of $(Q + \mathcal{C}_-)^N$ as we explain in Lemma 4.3.2. The second point is that moments of GMC measures are finite as soon as the weights α belong to $\mathcal{B}_N$. Combining these two properties will allow to derive Theorem 4.3.1.

4.3.1 . Analyticity of Gaussian multiplicative chaos measures

To start with we wish to prove that the dependence on the weights of the GMC measures being considered in Equation (4.1.13) is actually analytic. Namely our goal is to show the following statement:

Lemma 4.3.2. *For any fixed $c \in \mathfrak{a}$, the map*

$$\alpha \mapsto \mathbb{E} \left[\prod_{k=1}^N V_{\alpha_k}(z_k) e^{-\sum_{i=1}^r \mu_i e^{\gamma \langle c, e_i \rangle} M^{\gamma e_i}(\mathbb{C})} \right] \quad (4.3.3)$$

is holomorphic in a complex neighbourhood of $(Q + \mathcal{C}_-)^N$.

To prove this statement we will consider, without loss of generality, as background metric $g = g_+$ and for positive t the circle average regularization $(\mathbf{X}_t(z))_{t>0}$ of the GFF $\mathbf{X}$ as defined in Equation (3.2.15). As explained along the proof of Theorem 4.2.1, the expectation term can be written as

$$\lim_{t \rightarrow +\infty} \prod_{k=1}^N |z_k|_+^{-4\Delta_{\alpha_k}} F_t(\alpha; c), \quad (4.3.4)$$

with $F_t(\alpha; c) := \mathbb{E} \left[\prod_{k=1}^N e^{\langle \alpha_k, \mathbf{X}_t(z_k) + c \rangle - \frac{\mathbb{E}[\langle \alpha_k, \mathbf{X}_t(z_k) \rangle^2]}{2}} e^{-\sum_{i=1}^r \mu_i e^{\gamma \langle c, e_i \rangle} M^{\gamma e_i}(\mathbb{C}_t)} \right],$

and where $\mathbb{C}_t := \mathbb{C} \setminus (\cup_{k=1}^N B_t(z_k))$.

Proof. From its expression it is readily seen that for any positive t , $\alpha \rightarrow F_t(\alpha; c)$ is holomorphic over $(\mathbb{C}^2)^N$. Moreover, we will show that as $t \rightarrow \infty$ for any compact set K of $(Q + \mathcal{C}_-)^N$ and $\alpha \in K$,

$$|F_{t+1}(\alpha + i\beta; c) - F_t(\alpha + i\beta; c)| \leq C e^{-\eta t} \quad (4.3.5)$$

for some positive C and η uniform over K , as soon as $\beta \in (\mathbb{R}^2)^N$ is taken sufficiently close to 0. Therefore assuming that Equation (4.3.5) holds true, we can conclude that for any such K one can find an open complex neighbourhood of K inside $(\mathbb{C}^2)^N$ over which $(F_t(\alpha; c))_t$ converges uniformly (in α) as $t \rightarrow \infty$. Since the $F_t(\cdot; c)$ are holomorphic for any $t > 0$ this shows that the limit is also holomorphic. In brief proving Equation (4.3.3) boils down to proving that Equation (4.3.5) does indeed hold.

To show that this is the case we rely on the fact that the increments $\mathbf{X}_{t+1} - \mathbf{X}_t$ are independent of the sigma-algebra generated by the $(\mathbf{X}(z))_{z \in \mathbb{C}_t}$, so that

$$F_t(\alpha + i\beta; c) = \mathbb{E} \left[\prod_{k=1}^N e^{\langle \alpha_k + i\beta_k, \mathbf{X}_{t+1}(z_k) \rangle - \frac{\mathbb{E}[\langle \alpha_k + i\beta_k, \mathbf{X}_{t+1}(z_k) \rangle^2]}{2}} e^{-\sum_{i=1}^r \mu_i e^{\gamma \langle c, e_i \rangle} M^{\gamma e_i}(\mathbb{C}_t)} \right].$$

Interpreting the first term as a Girsanov transform we see that the latter is given by

$$\prod_{j < k} e^{\mathbb{E}[\langle \mathbf{X}_t(z_k), \alpha_k \rangle \langle \mathbf{X}_t(z_l), \alpha_l \rangle]} \times$$

$$\mathbb{E} \left[\prod_{k=1}^N e^{\langle i\beta_k, \mathbf{X}_{t+1}(z_k) \rangle + \frac{\mathbb{E}[\langle \beta_k, \mathbf{X}_{t+1}(z_k) \rangle^2]}{2} - i\mathbb{E}[\langle \alpha_k, \mathbf{X}_{t+1}(z_k) \rangle \langle \alpha_k, \mathbf{X}_{t+1}(z_k) \rangle]} e^{-\sum_{i=1}^r \mu_i e^{\gamma \langle c, e_i \rangle} I_{t+1}^i(\mathbb{C}_t)} \right]$$

with $I_t^i(d^2 x_i) = \prod_{k=1}^N e^{\gamma \langle \alpha_k, e_i \rangle \mathbb{E}[\mathbf{X}(x_i), \mathbf{X}_{t+1}(z_k)]} M^{\gamma e_i}(d^2 x_i)$. Therefore

$$\begin{aligned} & |F_{t+1}(\alpha + i\beta; \mathbf{c}) - F_t(\alpha + i\beta; \mathbf{c})| \\ & \leq C e^{\sum_{k=1}^N \frac{|\beta_k|^2}{2} t} \mathbb{E} \left[\left| e^{-\sum_{i=1}^r \mu_i e^{\gamma \langle \mathbf{c}, e_i \rangle} I_{t+1}^i(\mathbb{C}_r)} - e^{-\sum_{i=1}^r \mu_i e^{\gamma \langle \mathbf{c}, e_i \rangle} I_{t+1}^i(\mathbb{C}_{t+1})} \right| \right]. \end{aligned}$$

The expectation term can then be rewritten as

$$\begin{aligned} & \mathbb{E} \left[e^{-\sum_{i=1}^r \mu_i e^{\gamma \langle \mathbf{c}, e_i \rangle} I_{t+1}^i(\mathbb{C}_t)} \left(1 - e^{-\sum_{i=1}^r \mu_i e^{\gamma \langle \mathbf{c}, e_i \rangle} I_{t+1}^i(\mathbb{C}_{t+1} \setminus \mathbb{C}_t)} \right) \right] \\ & = \sum_{j=1}^r \mathbb{E} \left[e^{-\sum_{i=1}^r \mu_i e^{\gamma \langle \mathbf{c}, e_i \rangle} I_{t+1}^i(\mathbb{C}_t)} \left(1 - e^{-\mu_j e^{\gamma \langle \mathbf{c}, e_j \rangle} I_{t+1}^j(\mathbb{C}_{t+1} \setminus \mathbb{C}_t)} \right) \right] \\ & + l.o.t. \end{aligned}$$

where with the notation “l.o.t.” we refer to lower order terms. Now the set $\mathbb{C}_{t+1} \setminus \mathbb{C}_t$ is the disjoint union of annuli $A_k(t)$ centered at z_k and with radii $(e^{-(t+1)}, e^{-t})$, so that

$$I_{t+1}^i(\mathbb{C}_{t+1} \setminus \mathbb{C}_t) = \sum_{k=1}^N I_{t+1}^i(A_k(t)).$$

This means that we can further decompose the above expectation to reduce the problem to that of showing that

$$\mathbb{E} \left[e^{-\sum_{i=1}^r \mu_i e^{\gamma \langle \mathbf{c}, e_i \rangle} I_{t+1}^i(\mathbb{C}_t)} \left(1 - e^{-\mu_1 e^{\gamma \langle \mathbf{c}, e_1 \rangle} I_{t+1}^1(A_k(t))} \right) \right] \leq C e^{-\eta' t}$$

for positive C and η' , uniformly on $1 \leq k \leq N$. Put differently using Hölder inequality as well as the inequality $1 - e^{-x} < x$ for positive x we only need to bound $I_{t+1}^1(A_k(t))$, which has been done along the proof of [84, Theorem 6.1]:

$$\mathbb{P}(I_{t+1}^1(A_k(t)) \geq \varepsilon) \leq \varepsilon^{-m} \mathbb{E} [I_{t+1}^1(A_k(t))^m] \leq C \varepsilon^{-m} e^{-t\theta_m}$$

for any $m > 0$ and with $\theta_m^k = m\gamma \langle Q - \alpha_k, e_1 \rangle - \gamma^2 m^2$. Therefore choosing $\varepsilon = e^{-\frac{\theta_m}{p_1+m} t}$ with $\theta_m = \sup_{1 \leq k \leq N} \theta_m^k$ we have that $\mathbb{P}(I_{t+1}^1(A_k(t)) \geq \varepsilon)^{\frac{1}{p_1}} \leq e^{-\frac{\theta_m}{p_1+m} t}$, so collecting terms we end up with the bound

$$|F_{t+1}(\alpha + i\beta) - F_t(\alpha + i\beta)| \leq C e^{-\eta t}$$

where $\eta = \frac{\theta_m}{p_1+m} - \sum_{k=1}^N \frac{|\beta_k|^2}{2}$. We can conclude provided that $\eta > 0$, that is as soon as β and m are chosen small enough. $\square$

4.3.1.1 Moments of correlated Gaussian multiplicative chaos measures

Since the GMC measures considered have been proved to depend analytically in the weights α in a complex neighbourhood of $(Q + \mathcal{C}_-)^N$ it remains to consider under which assumptions does Equation (4.2.6) make sense. We show in this direction that:

Lemma 4.3.3 (Extended Seiberg bounds). *The bound*

$$\mathbb{E} \left[\prod_{i=1}^r (Z_{(z, \alpha)}^{\gamma e_i}(\mathbb{R}^2))^{-s_i} \right] < \infty \quad (4.3.6)$$

holds if and only if for all $i = 1, \dots, r$ one has

$$-s_i < \frac{4}{\gamma^2 \langle e_i, e_i \rangle} \wedge \min_{k=1, \dots, N} \frac{1}{\gamma} \langle Q - \alpha_k, e_i^\vee \rangle \quad (4.3.7)$$

Proof. To begin with we assume that condition (4.3.7) holds and consider the families of indices

$$\mathcal{P} := \{i = 1, \dots, r \mid s_i \geq 0\} \quad \text{and} \quad \mathcal{N} := \{i = 1, \dots, r \mid s_i < 0\}.$$

We also choose $p > 1$ such that for all $i \in \mathcal{N}$, $-ps_i < \frac{4}{\gamma^2 \langle e_i, e_i \rangle} \wedge \min_{k=1, \dots, N} \frac{1}{\gamma} \langle Q - \alpha_k, e_i^\vee \rangle$ and fix the conjugate exponent $q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. By Hölder inequality we can write that

$$\mathbb{E} \left[\prod_{i=1}^r (Z_{(\mathbf{z}, \boldsymbol{\alpha})}^{\gamma e_i}(\mathbb{R}^2))^{-s_i} \right] \leq \mathbb{E} \left[\prod_{i \in \mathcal{P}} (Z_{(\mathbf{z}, \boldsymbol{\alpha})}^{\gamma e_i}(\mathbb{R}^2))^{-qs_i} \right]^{1/q} \mathbb{E} \left[\prod_{i \in \mathcal{N}} (Z_{(\mathbf{z}, \boldsymbol{\alpha})}^{\gamma e_i}(\mathbb{R}^2))^{-ps_i} \right]^{1/p}.$$

The product running over $i \in \mathcal{P}$ is finite because GMC admits negative moments of all order via Lemma 3.3.6. For the product running over $i \in \mathcal{N}$, note that the relation (3.2.12) shows that the GFFs $\langle \gamma e_i, \mathbf{X}^g \rangle$ and $\langle \gamma e_j, \mathbf{X}^g \rangle$, for $i \neq j$, are negatively correlated since $\langle \gamma e_i, \gamma e_j \rangle \leq 0$ (indeed all off-diagonal elements of A are nonpositive). Hence thanks to Corollary 3.3.2 we can infer that

$$\mathbb{E} \left[\prod_{i \in \mathcal{N}} (Z_{(\mathbf{z}, \boldsymbol{\alpha})}^{\gamma e_i}(\mathbb{R}^2))^{-ps_i} \right] \leq \prod_{i \in \mathcal{N}} \mathbb{E} \left[(Z_{(\mathbf{z}, \boldsymbol{\alpha})}^{\gamma e_i}(\mathbb{R}^2))^{-ps_i} \right]. \quad (4.3.8)$$

Now the GMC measure which appears in the expression of $Z_{(\mathbf{z}, \boldsymbol{\alpha})}^{\gamma e_i}$ is defined from the GFF $\langle e_i, \mathbf{X}^g \rangle$. This GFF has the law of $|e_i| X_0^g$ where we have denoted by X_0^g the real-valued field considered in Lemma 3.3.6. This amounts to replacing the coupling constant $\sqrt{2}\gamma$ by $\gamma_i := \gamma |e_i|$ and the weight of the insertion by $\alpha_k^i := \langle \alpha_k, \frac{e_i}{|e_i|} \rangle$ in the statement of [40, Lemma A.1]. This entails that the corresponding expectation term is finite provided that $-ps_i < \frac{4}{\gamma_i^2} \wedge \min_{k=1, \dots, N} \frac{2}{\gamma_i} (\frac{\gamma_i}{2} + \frac{2}{\gamma_i} - \alpha_k^i)$. The latter can be rewritten under the form

$$-ps_i < \frac{4}{\gamma \langle e_i, e_i \rangle} \wedge \min_{k=1, \dots, N} \frac{1}{\gamma} \left(\gamma + \frac{4}{\gamma \langle e_i, e_i \rangle} - \langle \alpha_k, \frac{2e_i}{\langle e_i, e_i \rangle} \rangle \right).$$

We conclude thanks to the expression of Q that each expectation in the product above is finite thanks to our assumptions on the exponents ps_i , $1 \leq i \leq r$.

Conversely, assume that the expectation (4.3.6) is finite. Then by Corollary 3.3.2 applied to the function

$$H(x_1, \dots, x_r) = - \prod_{i \in \mathcal{N}} x_i^{-s_i} \prod_{i \in \mathcal{P}} x_i^{-s_i},$$

with the partition $(\mathcal{P}, \mathcal{N})$ of $\{1, \dots, r\}$ and to the GFFs $(\langle \gamma e_i, X^g \rangle)_{i=1, \dots, r}$ we deduce that

$$\mathbb{E} \left[\prod_{i=1}^r (Z_{(\mathbf{z}, \boldsymbol{\alpha})}^{\gamma e_i}(\mathbb{R}^2))^{-s_i} \right] \geq \mathbb{E} \left[\prod_{i \in \mathcal{P}} (Z_{(\mathbf{z}, \boldsymbol{\alpha})}^{\gamma e_i}(\mathbb{R}^2))^{-s_i} \right] \mathbb{E} \left[\prod_{i \in \mathcal{N}} (Z_{(\mathbf{z}, \boldsymbol{\alpha})}^{\gamma e_i}(\mathbb{R}^2))^{-s_i} \right].$$

Since the GMC admits negative moments of all order the first expectation in the right-hand side above is a finite constant $C > 0$. This implies that the second expectation is finite too. From now on, we fix $i_0 \in \mathcal{N}$ and $j \in \{1, \dots, N\}$. Without loss of generality and for the sake of simplicity, we may assume that $z_j = 0$. Then we can choose $\delta > 0$ such that $\min_{j' \neq j} |z'_j| > 10 \times \delta$ and we can choose non empty balls $(B_i)_{i \neq i_0, i \in \mathcal{N}}$ all of them at distance at least $10 \times \delta > 0$ from each other and all of them at distance at least $10 \times \delta$ from all the z_j 's. Set $B_{i_0} := B(0, \delta)$. Obviously we have

$$\mathbb{E} \left[\prod_{i \in \mathcal{N}} (Z_{(\mathbf{z}, \boldsymbol{\alpha})}^{\gamma e_i}(\mathbb{R}^2))^{-s_i} \right] \geq \mathbb{E} \left[\prod_{i \in \mathcal{N}} (Z_{(\mathbf{z}, \boldsymbol{\alpha})}^{\gamma e_i}(B_i))^{-s_i} \right].$$

Consider the mean value of the field $Y := \frac{1}{2\pi i} \oint_{|x|=2\delta} \mathbf{X}^g(x) \frac{dx}{x}$. A simple check of covariances shows that the law of the field $\mathbf{X}^g - Y$ is the independent sum of the field $\mathbf{X}_h^g$ —which coincides with $\mathbf{X}^g - Y$ outside of $B(0, 2\delta)$ and corresponds inside $B(0, 2\delta)$ to the harmonic extension (component by component) of the field $\mathbf{X}^g - Y$ restricted to the boundary $\partial B(0, 2\delta)$ —plus the Dirichlet field $\mathbf{X}_D$ defined by its covariance kernel

$$\mathbb{E}[\langle u, \mathbf{X}_D^g(x) \rangle \langle v, \mathbf{X}_D^g(y) \rangle] = \langle u, v \rangle G_D(x, y)$$

and $G_D(x, y)$ stands for the Dirichlet Green function inside $B(0, 2\delta)$. From now on we will write $Z_{(\mathbf{z}, \boldsymbol{\alpha})}^{\langle \gamma e_i, \mathbf{X}^g \rangle}(d^2x)$ instead of $Z_{(\mathbf{z}, \boldsymbol{\alpha})}^{\gamma e_i}(d^2x)$ to indicate in the notations the dependence on the underlying Gaussian field. This means that, generally speaking, we will write $Z_{(\mathbf{z}, \boldsymbol{\alpha})}^{\langle \gamma e_i, \mathbf{X} \rangle}$ for

$$Z_{(\mathbf{z}, \boldsymbol{\alpha})}^{\langle \gamma e_i, \mathbf{X} \rangle}(d^2x) := \lim_{\varepsilon \rightarrow 0} e^{\sum_{j=1}^N \langle \alpha_j, \gamma e_i \rangle G_g(x, z_j)} \varepsilon^{\frac{|\gamma e_i|^2}{2}} e^{\langle \gamma e_i, \mathbf{X}_\varepsilon(x) \rangle} \nu_g(dx)$$

where $\mathbf{X}_\varepsilon$ stands for the ε -regularization of the field $\mathbf{X}$ in the metric g . So we can write

$$\mathbb{E} \left[\prod_{i \in \mathcal{N}} (Z_{(\mathbf{z}, \boldsymbol{\alpha})}^{\langle \gamma e_i, \mathbf{X}^g \rangle}(B_i))^{-s_i} \right] = \mathbb{E} \left[e^{-\sum_{i \in \mathcal{N}} s_i \langle \gamma e_i, Y \rangle} \prod_{i \in \mathcal{N}} (Z_{(\mathbf{z}, \boldsymbol{\alpha})}^{\langle \gamma e_i, \mathbf{X}^g - Y \rangle}(B_i))^{-s_i} \right].$$

We can remove the factor $e^{-\sum_{i \in \mathcal{N}} s_i \langle \gamma e_i, Y \rangle}$ by viewing it as a Girsanov transform. Namely, denote by σ^2 the variance of the centered Gaussian random variable $\sum_{i \in \mathcal{N}} s_i \langle \gamma e_i, Y \rangle$. Then by Girsanov theorem A we can write that weighting the law of $\mathbf{X}^g$ by

$$e^{-\sum_{i \in \mathcal{N}} s_i \langle \gamma e_i, Y \rangle - \frac{\sigma^2}{2}}$$

amounts to shifting $\mathbf{X}^g$ by $-\sum_{j \in \mathcal{N}} s_j \gamma e_j \frac{1}{2\pi i} \oint_{|x|=\delta} G_g(\cdot, x) \frac{dx}{x}$. The values of the variance σ^2 and of the covariance of Y with $\mathbf{X}^g - Y$ are actually irrelevant to conclude. Indeed we only need to know that the variance σ^2 is bounded, and that the covariance of Y with $\mathbf{X}^g(x) - Y$ is uniformly bounded for x inside B_i and for all $i \in \mathcal{N}$. This is readily seen from their definition. This entails the existence of some positive constant $C > 0$ such that

$$\mathbb{E} \left[e^{-\sum_{i \in \mathcal{N}} s_i \langle \gamma e_i, Y \rangle} \prod_{i \in \mathcal{N}} (Z_{(\mathbf{z}, \boldsymbol{\alpha})}^{\langle \gamma e_i, \mathbf{X}^g - Y \rangle}(B_i))^{-s_i} \right] \geq C \mathbb{E} \left[\prod_{i \in \mathcal{N}} (Z_{(\mathbf{z}, \boldsymbol{\alpha})}^{\langle \gamma e_i, \mathbf{X}^g - Y \rangle}(B_i))^{-s_i} \right].$$

Using the decomposition of the law of $\mathbf{X}^g - Y = \mathbf{X}_D + \mathbf{X}_h^g$ and independence of $\mathbf{X}_D$ and $\mathbf{X}_h^g$, we get

$$\begin{aligned} & \mathbb{E} \left[\prod_{i \in \mathcal{N}} (Z_{(\mathbf{z}, \boldsymbol{\alpha})}^{\langle \gamma e_i, \mathbf{X}^g - Y \rangle} (B_i))^{-s_i} \right] \\ & \geq \mathbb{E} \left[(Z_{(\mathbf{z}, \boldsymbol{\alpha})}^{\langle \gamma e_{i_0}, \mathbf{X}_D \rangle} (B_{i_0}))^{-s_{i_0}} \right] \mathbb{E} \left[e^{\min_{x \in B_{i_0}} \langle \gamma e_i, \mathbf{X}_h^g(x) \rangle} \prod_{i \neq i_0, i \in \mathcal{N}} (Z_{(\mathbf{z}, \boldsymbol{\alpha})}^{\langle \gamma e_i, \mathbf{X}^g - Y \rangle} (B_i))^{-s_i} \right]. \end{aligned}$$

This implies that both expectations in the right-hand side are finite (they are obviously nonzero). Finiteness of the first expectation above entails, like before by adapting [40, Lemma A.1], that $-s_{i_0} < \frac{4}{\gamma^2 \langle e_i, e_i \rangle} \wedge \frac{1}{\gamma} \langle Q - \alpha_j, e_i^\vee \rangle$. Since the argument is valid for all $i_0 \in \mathcal{N}$ and all $j \in \{1, \dots, N\}$, this yields the result. $\square$

4.3.2 . Correlation functions beyond the Seiberg bounds and moments of Gaussian multiplicative chaos

We are now in position to conclude for the proof of Theorem 4.3.1. We will also describe how to compute functionals of the Toda field under this generalized assumptions.

4.3.2.1 Proof of Theorem 4.3.1

Let us pick $\alpha_0 \in \mathcal{A}_N^1$ and consider an open neighbourhood of α_0 : $\mathcal{O} \subset K$ a compact subset of $\mathcal{A}_N^1$. Then thanks to Lemma 4.3.3 we know that there exists a positive η such that for any $\alpha \in \mathcal{O}$ we have that

$$\mathbb{E} \left[\prod_{i=1}^r \left(Z_{(\mathbf{z}, \boldsymbol{\alpha})}^{\gamma e_i} (\mathbb{R}^2) \right)^{-s_i + \eta} \right] < \infty.$$

Then we can reproduce the reasoning developed in order to prove Lemma 4.3.2 to infer that this assumption allows to derive that the map

$$\alpha \mapsto \mathbb{E} \left[\prod_{i=1}^r \left(Z_{(\mathbf{z}, \boldsymbol{\alpha})}^{\gamma e_i} (\mathbb{R}^2) \right)^{-s_i} \right]$$

is holomorphic in a complex neighbourhood of $\mathcal{O}$, concluding for the proof of Theorem 4.3.1.

4.3.2.2 Functionals of the Toda field under the assumptions of Theorem 4.3.1

In order to provide an analytic continuation of these correlation functions we have implicitly relied on the following technique based on the analytic continuation of the Gamma function, which assesses that one can extend its integral representation by setting when $s < 0$

$$\Gamma(s) = \int_{\mathbb{R}} e^{sc} \left(\exp(-e^c) - \sum_{k=0}^{N-1} \frac{(-e^c)^k}{k!} \right) dc$$

for $\Re(s) \in (-N, 1 - N)$.

This extension is seen to be analytic via the following reasoning. Let us introduce for $\Re(s) > -1$, the function

$$F(s) := \int_{\mathbb{R}} e^{sc} (\exp(-e^c) - \mathbb{1}_{c < 0}) dc + \frac{1}{s}.$$

Then it is easily seen that the integral is absolutely convergent and hence analytic in the s variable: F is a meromorphic function over the half-plane $\Re(s) > -1$ with a single pole at $s = 0$. However under the assumption that $\Re(s) > 0$ then we see that $F(s) = \int_{\mathbb{R}} e^{sc} \exp(-e^c) dc$ is the standard Gamma function while for $\Re(s) \in (0, -1)$ we have $F(s) = \int_{\mathbb{R}} e^{sc} (\exp(-e^c) - 1) dc$. By uniqueness of the analytic continuation this shows that this extension is the unique analytic extension of the Gamma function for $\Re(s) \in (0, -1)$. More generally we have to consider

$$F(s) := \int_{\mathbb{R}} e^{sc} \left(\exp(-e^c) - \mathbb{1}_{c < 0} \sum_{k=0}^{N-1} \frac{(-e^c)^k}{k!} \right) dc + \sum_{k=0}^{N-1} \frac{(-1)^k}{s+k}.$$

This extension can be generalized within the framework of simple and complex Lie algebras we work with via the equality

$$\prod_{i=1}^r \Gamma(s_i) = \int_{\mathfrak{a}} e^{\langle s, \mathbf{c} \rangle} \prod_{i=1}^r \left(\exp(-e^{\langle \mathbf{c}, e_i \rangle}) - \sum_{k=0}^{N_i-1} \frac{(-e^{\langle \mathbf{c}, e_i \rangle})^k}{k!} \right) d\mathbf{c}$$

for $\Re(s_i) \in (-N_i, 1 - N_i)$ for all $1 \leq i \leq r$, where $s = \sum_{i=1}^r s_i e_i$. This shows that if we were to compute functionals of the Toda field under the assumptions of Theorem 4.3.1, we would be led to computing the following quantities:

$$\begin{aligned} \langle F(\Phi) \mathbf{V} \rangle = \int_{\mathfrak{a}} e^{-2\langle Q, \mathbf{c} \rangle} \mathbb{E} \left[F(\mathbf{X} + \frac{Q}{2} \ln g_+ + \mathbf{c}) \prod_{k=1}^N V_{\alpha_k}(z_k) \times \right. \\ \left. \prod_{i=1}^r \left(e^{-\mu_i e^{\langle \mathbf{c}, \gamma e_i \rangle} M^{\gamma e_i}(\mathbb{C})} - \sum_{k=0}^{N_i-1} \frac{(-\mu_i e^{\langle \mathbf{c}, \gamma e_i \rangle} M^{\gamma e_i}(\mathbb{C}))^k}{k!} \right) \right] d\mathbf{c}. \end{aligned} \quad (4.3.9)$$

Using Equation (4.2.2) we can rewrite the above in a more condensed way:

$$\begin{aligned} \langle F(\Phi) \mathbf{V} \rangle = \left(\prod_{i=1}^r \frac{\Gamma(s_i) \mu_i^{-s_i}}{\gamma} \right) \prod_{1 \leq k < l \leq N} |z_k - z_l|^{-\langle \alpha_k, \alpha_l \rangle} \times \\ \mathbb{E} \left[F \left(\mathbf{X} + \frac{Q}{2} \ln g_+ + \sum_{k=1}^N \alpha_k G_{g_+}(\cdot, z_k) + \mathbf{c} \right) \prod_{i=1}^r Z_{(z, \alpha)}^{\gamma e_i}(\mathbb{C})^{-s_i} \right]. \end{aligned} \quad (4.3.10)$$

Such a reasoning based on the analytic continuation of the Gamma function will be key in the extension of the probabilistic definition of the correlation functions. The additional input will be to incorporate the precise tail expansion of GMC measures to add more terms in the integral over $\mathbf{c}$.

4.4 . Some analytic properties of the correlation functions

Having provided a mathematical representation of the correlation functions of Vertex Operators of Toda CFTs and studied some of their properties in the realm of conformal field theory, we now provide some analytic features enjoyed by these objects. We will focus here on the case where the correlation functions satisfy the assumptions of Theorem 4.3.1 and to start with we consider the issue of differentiability of correlation functions. Our goal is to show the following statement:

Theorem 4.4.1. *Let g be any metric within the conformal class of $\hat{g}$, and assume that the weights $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_N)$ are such that bounds of Theorem 4.3.1 hold. Then, for $z_2, \dots, z_N \in \mathbb{C}$ all distinct, the map $z_1 \mapsto \langle \prod_{k=1}^N V_{\alpha_k}(z_k) \rangle_{g,g}$ is C^2 on the set $\mathbb{C} \setminus \{z_2, \dots, z_N\}$.*

In order to prove such a statement we will rely on Gaussian integration by parts as explained above. However to make sense of it we first need to provide some fusion estimates on the correlation functions which we present now.

To simplify the notations we introduce the shorthands

$$|x|_{+, \varepsilon} := \exp \left(\int_{\mathbb{C}} \int_{\mathbb{C}} G_{g_+}(x - z + z') \eta_{\varepsilon}(z) \eta_{\varepsilon}(z') d^2 z d^2 z' \right) \quad \text{and}$$

$$|x|_{\varepsilon} := \exp \left(\int_{\mathbb{C}} \int_{\mathbb{C}} \ln |x - z + z'| \eta_{\varepsilon}(z) \eta_{\varepsilon}(z') d^2 z d^2 z' \right).$$

Before moving on let us stress that under the assumptions of Theorem 4.3.1 the regularized correlation functions are given by

$$\langle \mathbf{V} \rangle_{\varepsilon} := \left(\prod_{i=1}^r \frac{\Gamma(s_i) \mu_i^{-s_i}}{\gamma} \right) \prod_{1 \leq k < l \leq N} |z_k - z_l|_{\varepsilon}^{-\langle \alpha_k, \alpha_l \rangle} \mathbb{E} \left[\prod_{i=1}^r Z_{(z, \alpha), \varepsilon}^{\gamma e_i}(\mathbb{C})^{-s_i} \right] \quad (4.4.1)$$

with $Z_{(z, \alpha), \varepsilon}^{\gamma e_i}(dx) := \frac{|x|_{+, \varepsilon}^{-\gamma \sum_{k=1}^N \langle \alpha_k, e_i \rangle}}{\prod_{k=1}^N |z_k - x|_{\varepsilon}^{\gamma \langle \alpha_k, e_i \rangle}} M_{\varepsilon}^{\gamma e_i}(dx).$

4.4.1 . Fusion estimates for Toda correlation functions

Fusion estimates correspond to the study of the behaviour of correlation functions when two insertion points collide. As we will see the correlation functions become singular but we are still able to control their rate of divergence as the following statement discloses:

Lemma 4.4.2. *Assume that the $z := (z_1, \dots, z_N)$ are distinct and take $\alpha \in \mathcal{A}_N$. Consider a family of complex vectors $(x_i)_{1 \leq i \leq r} = (x_i^{(1)}, \dots, x_i^{(m_i)})_{1 \leq i \leq r}$.*

1. *For any positive ε there exists a positive constant C_{ε} such that*

$$\left\langle \prod_{i=1}^r \prod_{l=1}^{m_i} V_{\gamma e_i, \varepsilon} \left(x_i^{(l)} \right) \mathbf{V} \right\rangle_{\varepsilon} \leq C_{\varepsilon} \prod_{i=1}^r \prod_{l=1}^{r_i} \left(1 + \left| x_i^{(l)} \right| \right)^{-4}.$$

2. For any $\rho > 0$, if the x_i and z stay in the domain $U_\rho := \left\{ w, \rho < \min_{i \neq j} |w_i - w_j| \right\}$, then the above bound remains true for $\varepsilon = 0$ uniformly in $(x_i)_{1 \leq i \leq r}$ over U_ρ with a constant C that depends only on ρ .
3. Assume that all pairs of points in z are separated by some distance $\rho > 0$ except for one pair (z_1, z_2) , and that all the z 's stay in a compact subset of $\mathbb{C}$. Further assume that $\langle \alpha_1 + \alpha_2 - Q, e_i \rangle < 0$ for all $2 \leq i \leq r$. Then, for any positive η , there exists a positive constant $K = K(\rho)$ such that, uniformly on ε ,

$$\langle \mathbf{V} \rangle_\varepsilon \leq K(\rho) |z_1 - z_2|^{-\langle \alpha_1, \alpha_2 \rangle + \left(\frac{1}{2|e_1|^2} \langle \alpha_1 + \alpha_2 - Q, e_1 \rangle^2 - \eta \right) \mathbb{1}_{\langle \alpha_1 + \alpha_2 - Q, e_1 \rangle > 0}}. \quad (4.4.2)$$

4. In the previous estimate, if $\alpha_1 = \gamma e_i$ for some $1 \leq i \leq r$, then there exists some $\zeta > 0$ such that

$$\langle \mathbf{V} \rangle_\varepsilon \leq K(\rho) |z_1 - z_2|^{-2+\zeta}.$$

5. For some $p > 1$ that depends on γ and $\langle \alpha_0, e_1 \rangle$, the map

$$x_1^{(1)} \mapsto \sup_{\varepsilon \in (0,1)} \left\langle \prod_{i=1}^r \prod_{l=1}^{m_i} V_{\gamma e_i, \varepsilon} \left(x_i^{(l)} \right) \mathbf{V} \right\rangle_\varepsilon$$

is in $L^p(\mathbb{C})$.

Proof. We first note that in the expression of $\left\langle \prod_{i=1}^r \prod_{l=1}^{m_i} V_{\gamma e_i, \varepsilon} \left(x_i^{(l)} \right) \mathbf{V} \right\rangle_\varepsilon$, due to the ε -regularization no singularities come from the prefactor. Moreover, viewed as a function of $x_i^{(l)}$ and for fixed $\varepsilon > 0$, the quantity $\left| x_i^{(l)} - z_k \right|_\varepsilon^{-\gamma \langle e_i, \alpha_k \rangle}$ can be bounded by some constant times $\left| 1 + x_i^{(l)} \right|^{-\gamma \langle e_i, \alpha_k \rangle}$. As a consequence this prefactor together with the constant part can be bounded by

$$C_\varepsilon \prod_{i=1}^r \prod_{l=1}^{m_i} \left| 1 + |x_i^{(l)}| \right|^{-\gamma \langle \sum_k \alpha_k, e_1 \rangle} \prod_{(i,l) \neq (j,m)} \left| x_i^{(l)} - x_j^{(m)} \right|_\varepsilon^{-\frac{\gamma^2}{2} \langle e_i, e_j \rangle}. \quad (4.4.3)$$

For item (1), we need to study the expectation part for large values of the x_i . Inside $Z^{\gamma e_i}_{(z, \alpha), \varepsilon}(\mathbb{C})$ we identify singularities of the form:

$$\prod_{k=1}^N |x - z_k|_{+, \varepsilon}^{-\gamma \langle \alpha_k, e_i \rangle} \prod_{j=1}^r \prod_{l=1}^{m_j} \left| x - x_i^{(l)} \right|_{+, \varepsilon}^{-\gamma \langle e_j, e_i \rangle}.$$

We can factorize out these deterministic factors on x_i "at infinity" from the expectation by writing

$$\prod_{j=1}^r \prod_{l=1}^{m_j} \left| x - x_i^{(l)} \right|_{+, \varepsilon}^{-\gamma \langle e_j, e_i \rangle} = \prod_{j=1}^r \prod_{l=1}^{m_j} \frac{\left| 1 + |x_i^{(l)}| \right|^{\gamma \langle e_j, e_i \rangle}}{\left| x - x_i^{(l)} \right|_{+, \varepsilon}^{\gamma \langle e_j, e_i \rangle}} \prod_{j=1}^r \prod_{l=1}^{m_j} \left| 1 + |x_i^{(l)}| \right|^{-\gamma \langle e_j, e_i \rangle}.$$

Because the first factor above goes to 1 as $x^{(l)} \rightarrow \infty$ it follows that, for some positive constant C the expectation term in Equation (4.2.6) can be bounded by

$$C \prod_{i=1}^r \left(\prod_{j=1}^r \prod_{l=1}^{m_j} |1 + |x_j^{(l)}||^{-\gamma \langle e_i, e_j \rangle} \right)^{-s_i} = C \prod_{i=1}^r \prod_{l=1}^{m_i} |1 + |x_1^{(l)}||^{\langle \gamma e_i, \sum_k \alpha_k - 2Q + \sum_{j=1}^r m_j \gamma e_j \rangle}.$$

Combining with the bound (4.4.3) for the prefactor yields:

$$\begin{aligned} \left\langle \prod_{i=1}^r \prod_{l=1}^{m_i} V_{\gamma e_i, \varepsilon} \left(x_i^{(l)} \right) \mathbf{v}_\varepsilon \right\rangle &\leq C_\varepsilon \prod_{i=1}^r \prod_{l=1}^{m_i} |1 + |x_i^{(l)}||^{\langle \gamma e_i, \sum_{j=1}^r m_j \gamma e_j - 2Q \rangle} \prod_{(i,l) \neq (j,m)} |x_i^{(l)} - x_j^{(m)}|_\varepsilon^{-\frac{\gamma^2}{2} \langle e_i, e_j \rangle} \\ &= C_\varepsilon \prod_{l=1}^{r_1} |1 + |x_1^{(l)}||_\varepsilon^{-4} \prod_{m=1}^{r_2} |1 + |x_2^{(m)}||_\varepsilon^{-4} \prod_{(i,l) \neq (j,m)} \left(\frac{|1 + x_i^{(l)}| |1 + x_j^{(m)}|}{|x_i^{(l)} - x_j^{(m)}|_\varepsilon} \right)^{\frac{\gamma^2}{2} \langle e_i, e_j \rangle}. \end{aligned}$$

The last term on the right-hand side being bounded this finishes the proof of item (1).

For item (2), the same reasoning remains valid and therefore the same bounds are still true when looking at the behaviour near infinity of the correlation functions when we take the $\varepsilon \rightarrow 0$ limit of the correlation functions.

For item (3), we investigate the behaviour when ε goes to zero of the fusion of two insertions, that is when $|z_1 - z_2| \rightarrow 0$ but with all other insertions staying at fixed positive distance at least ρ . Suppose that for both $i = 1, \dots, r$, $\langle \alpha_1 + \alpha_2 - Q, e_i \rangle < 0$. In this case, the expectation term remains bounded when z_1 and z_2 merge (since in that case the expectation remains well-defined according to Lemma 3.3.6) and therefore the behaviour of the correlation functions is governed by the prefactor, of which the singularity is of order $|z_1 - z_2|^{-\langle \alpha_1, \alpha_2 \rangle}$.

The analysis is a bit more subtle when $\langle \alpha_1 + \alpha_2 - Q, e_1 \rangle \geq 0$ but $\langle \alpha_1 + \alpha_2 - Q, e_i \rangle < 0$ for all $2 \leq i \leq r$. However we can adapt the result in the Liouville case by using similar arguments as in the proof of Lemma 3.3.5. Indeed, by combining Hölder's inequality with the fact that the GFFs considered are negatively correlated we can write in the same fashion as before that

$$\begin{aligned} &\mathbb{E} \left[\prod_{i=1}^r \left(\int_{\mathbb{C}} \frac{|x_i|_{+, \varepsilon}^{-\gamma \sum_{k=1}^N \langle \alpha_k, e_i \rangle}}{\prod_{k=1}^N |z_k - x_i|_\varepsilon^{\gamma \langle \alpha_k, e_i \rangle}} M_\varepsilon^{\gamma e_i} (dx_i) \right)^{-s_i} \right] \\ &\leq \prod_{i=1}^r \mathbb{E} \left[\left(\int_{\mathbb{C}} \frac{|x_i|_{+, \varepsilon}^{-\gamma \sum_{k=1}^N \langle \alpha_k, e_i \rangle}}{\prod_{k=1}^N |z_k - x_i|_\varepsilon^{\gamma \langle \alpha_k, e_i \rangle}} M_\varepsilon^{\gamma e_i, \hat{g}} (dx_i) \right)^{-p_i s_i} \right]^{\frac{1}{p_i}} \end{aligned}$$

for some $p_i > 1$ such that the expectations are still well-defined. Now for $2 \leq i \leq r$ the expectations remain bounded when z_1 and z_2 merge, while the behaviour of the expectation with index $i = 1$ is reduced to the Liouville case (the so-called freezing estimate, see [83, Lemma 6.5]): for any $\eta > 0$,

$$\mathbb{E} \left[\left(\int_{\mathbb{C}} \frac{|x|_{+, \varepsilon}^{-\gamma \sum_{k=1}^N \langle \alpha_k, e_1 \rangle}}{\prod_{k=1}^N |z_k - x|_\varepsilon^{\gamma \langle \alpha_k, e_1 \rangle}} M_\varepsilon^{\gamma e_1, \hat{g}} (dx) \right)^{-p_1 s_1} \right] \leq C_\varepsilon |z_1 - z_2|^{\frac{\langle \alpha_1 + \alpha_2 - Q, e_1 \rangle^2}{2|e_1|^2} - \eta}.$$

Therefore we see that if $\langle \alpha_1 + \alpha_2 - Q, e_1 \rangle > 0$ then the two-point fusion estimate is governed by

$$|z_1 - z_2|^{-\langle \alpha_1, \alpha_2 \rangle} |z_1 - z_2|^{-\eta' + \frac{\langle \alpha_1 + \alpha_2 - Q, e_1 \rangle^2}{2|e_1|^2}}. \quad (4.4.4)$$

Item (4) is a direct consequence of the previous bound.

For item (5): the bound of item (3) ensures local integrability near z_1 of $\langle V_{\gamma e_1}(x) \mathbf{V} \rangle$ in $L^p(\mathbb{C})$ for $1 \leq p < \frac{4}{4-\gamma^2}$ if we have $\frac{2}{\gamma} - \gamma \leq \langle \alpha_0, e_1 \rangle < \frac{2}{\gamma} + \gamma$, for $1 \leq p < \frac{2}{\gamma \langle \alpha_0, e_1 \rangle}$ if we assume that $0 < \langle \alpha_0, e_1 \rangle < \frac{2}{\gamma} - \gamma$, and in $L^\infty(\mathbb{C})$ if $\langle \alpha_0, e_1 \rangle \leq 0$. $\square$

Based on this statement we now provide a generalized fusion estimate for which we consider the situation where finitely many points merge within a correlation functions. In our applications, we only need the special case where the weights of the merging Vertex Operators are of the form γe_i , for $i = 1, \dots, r$, and where points merge pairwise. This will be the setup that we investigate in this subsection. We assume that z are distinct and that $\alpha \in \mathcal{A}_N$. Without loss of generality we assume that $\rho := \min_{1 \leq k \leq N} |z_k|$ is positive. We are interested in estimating the following correlation functions with $2p$ extra points:

$$\langle \prod_{l=1}^p V_{\gamma e_{\sigma(l)}}(x_l) V_{\gamma e_{\tau(l)}}(y_l) \mathbf{V} \rangle$$

where $\sigma(l), \tau(l) \in \{1, \dots, r\}$ so the extra weights are of the form γe_i for $1 \leq i \leq r$.

To get a reasonable estimate, we need to first reorder the points $(\mathbf{x}, \mathbf{y})$ in such a way that the collision can only happen pairwise, i.e. between each pair x_i and y_i . In short, we restrict the locations on the points: we assume that x_l, y_l belong to sets A_l, B_l in such a way that any two of these sets are disjoint, except for A_l and B_l with the same indices. More precisely, we introduce domains $\mathbf{A} := A_1 \times \dots \times A_p$ and $\mathbf{B} := B_1 \times \dots \times B_p$ such that:

- For any $1 \leq l \leq p$, A_l and B_l are either annuli or balls, and are contained in $B(0, \frac{\rho}{2})$. This is to avoid collisions between $(\mathbf{x}, \mathbf{y})$ and the given z ;
- There exists some positive distance $d > 0$, for which the $(A_l)_{1 \leq l \leq p}$ are at distance at least d one to the other (and similarly for the $(B_l)_{1 \leq l \leq p}$);
- For any $1 \leq l, m \leq p$, the distance between A_l and B_m is zero if and only if $l = m$.

Then we have the following:

Lemma 4.4.3. *There exist two positive constants C and ζ such that, for any $(\mathbf{x}, \mathbf{y}) \in \mathbf{A} \times \mathbf{B}$:*

$$\langle \prod_{l=1}^p V_{\gamma e_{\sigma(l)}, \varepsilon}(x_l) V_{\gamma e_{\tau(l)}, \varepsilon}(y_l) \mathbf{V}_\varepsilon \rangle \leq C \prod_{l=1}^p |x_l - y_l|^{-2+\zeta} \quad (4.4.5)$$

uniformly in ε . In particular the integral

$$\int_{\mathbf{A}} \int_{\mathbf{B}} \frac{1}{\prod_{l=1}^p (x_l - y_l)} \langle \prod_{i=1}^r V_{\gamma e_i}(\mathbf{x}_i) V_{\gamma e_i}(\mathbf{y}_i) \mathbf{V} \rangle d^2 \mathbf{x} d^2 \mathbf{y} \quad (4.4.6)$$

is absolutely convergent.

Proof. The proof of this claim parallels the one in the Liouville case [106, Lemma 3.1]. The study is slightly more involved in the present case because of the presence of additional GFFs in the construction of the correlation functions, but the basic idea remains the same.

Like before we can separate the prefactor and the expectation term: since the domains are disjoint except for A_l and B_m with $l = m$, the prefactor part is bounded by

$$C \prod_{l=1}^p |x_l - y_l|^{-\gamma^2 \langle e_{\sigma(l)}, e_{\tau(l)} \rangle}$$

where C is some positive constant.

For the expectation term the analysis is slightly more subtle, but one can simplify the problem by noticing that when $\sigma(l) \neq \tau(l)$, there is no singularity in the integral (since the sign in the power is the opposite one). As a consequence, and without loss of generality, we may assume $\sigma(l) = \tau(l) = i$ for $l \in E_i := \{k_i + 1, \dots, k_{i+1}\}$. Therefore the integrand can be bounded by some constant times

$$\mathbb{E} \left[\prod_{i=1}^r \left(\int_{\mathbb{C}} \frac{F_i(w_i)}{\prod_{l \in E_i} |x_l - w_i|_{\varepsilon}^{\gamma^2 |e_i|^2} |y_l - w_i|_{\varepsilon}^{\gamma^2 |e_i|^2}} M_{\varepsilon}^{\gamma e_i}(d^2 w_i) \right)^{-s_i} \right]$$

where the F_i are smooth in a neighbourhood of the x_l, y_l . We are now to distinguish between two cases:

- If $\gamma < \sqrt{\frac{4}{3|e_i|^2}}$, then when x_l and y_l merge the singularity remains integrable since in that case $2|e_i|^2 \gamma^2 < \gamma \langle Q, e_i \rangle$ so the expectation still makes sense according to [34, Lemma 4.1]. Therefore if $\gamma < \sqrt{\frac{4}{3|e_i|^2}}$, the fusion estimate is governed by the prefactor, which scales like $|x_l - y_l|^{-\gamma^2 |e_i|^2}$ with $-\gamma^2 |e_i|^2 > -\frac{4}{3} > -2$.
- If $\sqrt{\frac{4}{3\gamma^2 |e_i|^2}} \leq \gamma < \sqrt{2}$, we claim that, for any positive ζ , it is bounded by

$$\prod_{l=1}^p |x_l - y_l|^{\frac{\left(\frac{3\gamma|e_i|}{2} - \frac{2}{\gamma|e_i|}\right)^2}{2} - \zeta}.$$

For $\gamma < \frac{2}{|e_i|}$, one checks that $-\gamma^2 |e_i|^2 + \frac{\left(\frac{3\gamma|e_i|}{2} - \frac{2}{\gamma|e_i|}\right)^2}{2} > -2$, allowing to conclude since $\gamma < \sqrt{2} \leq \frac{2}{|e_i|}$.

We now prove the last claim following [106, Lemma 3.1]. Like before, we can either use that the exponents s_i are positive or that the GFFs considered are negatively correlated to see that we can separate the different expectations involved. This shows that we only have to bound quantities such as

$$\mathbb{E} \left[\left(\sum_{l=k_i+1}^{k_i} \int_{B(x_l, \frac{d}{4})} \frac{M_{\varepsilon}^{\gamma e_i}(d^2 w_i)}{|x_l - w_i|_{\varepsilon}^{\gamma^2 |e_i|^2} |y_l - w_i|_{\varepsilon}^{\gamma^2 |e_i|^2}} \right)^{-\tilde{s}_i} \right].$$

We can suppose that the GFFs in different balls $B(x_l, \frac{d}{4})$ are independent: this is a classical manipulation of Kahane's convexity inequality, since the covariance of different GFFs in different balls is uniformly bounded from below and above by a global

constant (see the proof of [34, Lemma 4.1]). Now, if $\tilde{s}_i > 0$ we use the elementary inequality, for positive a 's and $s > 0$,

$$\left(\sum_{l=1}^r a_l \right)^{-s} \leq \prod_{l=1}^r a_l^{-s/r}$$

to reduce the problem to the following estimate: for $s > 0$ and any $\zeta > 0$ there exists a positive constant C such that

$$\mathbb{E} \left[\left(\int_{B(x, \frac{d}{4})} \frac{M_\varepsilon^{\gamma e_1}(d^2 w)}{|x - w|_\varepsilon^{2\gamma^2} |y - w|_\varepsilon^{2\gamma^2}} \right)^{-s} \right] \leq C |x - y|^{-\frac{\left(\frac{3\gamma|e_i|}{2} - \frac{2}{\gamma|e_i|} \right)^2}{2} - \zeta}.$$

This corresponds to the freezing estimate [83, Lemma 6.5], which yields the bound $|x - y|^{-\frac{\langle 2\gamma e_i - Q, e_i \rangle^2}{2|e_i|^2} - \zeta}$ for any $\zeta > 0$, that is exactly as claimed. If $-1 < \tilde{s}_i < 0$ we can use concavity of the map $x \mapsto x^{-\tilde{s}_i}$ while for $\tilde{s}_i < -1$ Young's inequality does the job.

For the second item of Lemma 4.4.3 we use the first estimate to see that all one needs to prove is that for positive ζ the quantity

$$\int_A \int_B \frac{1}{\prod_{l=1}^p |x_l - y_l|^{3-\zeta}} d^2 x d^2 y$$

is finite. Since the domains are disjoint we see that it is enough to check that the two-fold integral

$$\int_{B(0, \frac{\rho}{6})} \int_{A(0, \frac{\rho}{6}, \frac{\rho}{3})} \frac{1}{|x - y|^{3-\zeta}} d^2 x d^2 y$$

converges. This can be easily seen by using polar coordinates. $\square$

These fusion estimates will be key in providing regularity properties for Toda correlation functions. As a first application of these technical results we provide a rather useful identity. This identity is derived from the μ -dependence of the correlation functions, usually referred to as a "KPZ-relation" [80].

Lemma 4.4.4 (KPZ identity). *Under the assumptions of Theorem 4.3.1, for any non-negative ε the following equality holds in α :*

$$\left(\sum_{k=1}^N \alpha_k - 2Q \right) \langle \mathbf{V}_\varepsilon \rangle = \gamma \sum_{i=1}^r \mu_i e_i \int_{\mathbb{C}} \langle V_{\gamma e_i, \varepsilon}(x) \mathbf{V}_\varepsilon \rangle_\alpha d^2 x. \quad (4.4.7)$$

Proof. It suffices to check that for all $i = 1, \dots, r$ one has that

$$\left\langle \sum_{k=1}^N \alpha_k - 2Q, \omega_i^\vee \right\rangle \langle \mathbf{V}_\varepsilon \rangle = \mu_i \gamma \int_{\mathbb{C}} \langle V_{\gamma e_i, \varepsilon}(x) \mathbf{V}_\varepsilon \rangle d^2 x.$$

To start with we already know from Equation (4.2.6) that the μ_i -dependence of the correlation functions is explicit:

$$\left\langle \prod_{k=1}^N V_{\alpha_k, \varepsilon}(z_k) \right\rangle_\mu = \mu_i^{-s_i} \left\langle \prod_{k=1}^N V_{\alpha_k, \varepsilon}(z_k) \right\rangle_1.$$

Differentiating with respect to μ_i , we end up with the equality

$$\frac{\partial}{\partial \mu_i} \left\langle \prod_{k=1}^N V_{\alpha_k, \varepsilon}(z_k) \right\rangle_{\mu} = - \frac{\langle \sum_{k=1}^N \alpha_k - 2Q, \omega_i^{\vee} \rangle}{\mu_i \gamma} \left\langle \prod_{k=1}^N V_{\alpha_k, \varepsilon}(z_k) \right\rangle_{\mu}.$$

We can also directly differentiate the regularized correlation functions before performing the change of variable in the zero-mode leading to Equation (4.2.6). Put differently we can use the fact that correlation functions under the assumptions of Theorem 4.3.1 are given by Equation (4.3.9) with $F = 1$ and differentiate there with respect to μ_i . We find that

$$\begin{aligned} \frac{\partial}{\partial \mu_i} \langle \mathbf{V}_{\varepsilon} \rangle_{\mu} = & - \int_{\mathfrak{a}} e^{-2\langle Q, c \rangle} \mathbb{E} \left[\int_{\mathbb{C}} V_{\gamma e_i, \varepsilon}(x) \prod_{k=1}^N V_{\alpha_k, \varepsilon}(z_k) \times \right. \\ & \left. \prod_{j=1}^r \left(e^{-\mu_i e^{\langle c, \gamma e_i \rangle} M_{\varepsilon}^{\gamma e_i}(\mathbb{C})} - \sum_{k=0}^{\tilde{N}_j-1} \frac{(-\mu_i e^{\langle c, \gamma e_i \rangle} M_{\varepsilon}^{\gamma e_i}(\mathbb{C}))^k}{k!} \right) \right] d c d^2 x \end{aligned}$$

where we have used the fact that $e^{\langle \gamma e_i, c \rangle} M_{\varepsilon}^{\gamma e_i}(dx) = V_{\gamma e_i, \varepsilon}(x) d^2 x$ and set $\tilde{N}_j = N_j - \delta_{ij}$. Because $\langle s + \gamma e_i, \omega_j^{\vee} \rangle = s_i + \gamma \delta_{ij}$ we see that the remainder term is expanded to the good order with respect to the asymptotics of the integrand as c diverges. We can therefore use Fubini-Lebesgue to exchange the integrals and integrate over the zero-mode to get

$$\frac{\partial}{\partial \mu_i} \langle \mathbf{V}_{\varepsilon} \rangle_{\mu} = - \int_{\mathbb{C}} \langle V_{\gamma e_i, \varepsilon}(x) \prod_{k=1}^N V_{\alpha_k, \varepsilon}(z_k) \rangle d^2 x.$$

We recover the desired result for positive ε , while the $\varepsilon \rightarrow 0$ limit is taken care of thanks to the last item of Lemma 4.4.2. $\square$

4.4.2 . Gaussian integration by parts

When computing derivatives of the correlation functions we will need to make sense of terms that take the form

$$\left\langle \prod_{l=1}^m \langle \alpha_l, \partial^{p_l} \varphi_{\varepsilon}(z_1) \rangle \mathbf{V}_{\varepsilon} \right\rangle$$

for some positive integer m and p_l and for some elements $\alpha_l \in \mathfrak{a}$. We explain in this subsection how Gaussian integration by parts as disclosed in Equation 3.3.2 can be implemented to make sense of such expressions. For this problem we will also need a regularization of the complex-valued map $x \mapsto \frac{1}{x^p}$ for a positive integer p defined by setting

$$\frac{1}{(x)_{\varepsilon}^p} := \frac{(-1)^p}{(p-1)!} \int_{\mathbb{C}} \int_{\mathbb{C}} \frac{1}{(x - z + z')^p} \partial_z^{p-1} \eta_{\varepsilon}(z) \eta_{\varepsilon}(z') d^2 z d^2 z'.$$

The connection with the map $x \mapsto \frac{1}{x^p}$ is made clear by the following statement.

Lemma 4.4.5. *Let $\varepsilon > 0$ and p be a positive integer, and recall R from the regularization (3.2.14). Then for any $|x| > 4R\varepsilon$ it holds that*

$$\frac{x^p}{(x)_{\varepsilon}^p} - 1 = \left(\frac{\varepsilon}{x} \right)^2 F \left(\frac{\varepsilon}{x} \right)$$

where F is continuous over $\mathbb{C}$ and depends only on the mollifier η of the ε -regularization. In particular, the family of functions $\left(\frac{1}{(x)_\varepsilon^p}\right)_{\varepsilon>0}$ converges uniformly to $\frac{1}{x^p}$ on every compact set of $\mathbb{C} \setminus \{0\}$. Similarly, the quantity

$$\sup_{0 < |x| \leq 4R\varepsilon} \left| \frac{x^p}{(x)_\varepsilon^p} \right|$$

remains bounded uniformly on ε .

Proof of Lemma 4.4.5. First assume that we are given $0 < \varepsilon < \frac{|x|}{4R}$. If $|x + \varepsilon(z_1 - z_2)| \leq \frac{|x|}{2}$, then necessarily either $|z_1| \geq R$ or $|z_2| \geq R$, and this implies that $\eta(z_1)\eta(z_2)$ vanishes since η is compactly supported in the domain $B(0, R)$. Therefore we can apply integration by parts and the change of variables $z_i \leftrightarrow \varepsilon z_i$ to get

$$\frac{1}{(x)_\varepsilon^p} = \int_{|x+\varepsilon(z_1-z_2)| > \frac{|x|}{2}} \frac{1}{(x + \varepsilon(z_1 - z_2))^p} \eta(z_1)\eta(z_2) d^2 z_1 d^2 z_2.$$

As a consequence

$$\frac{x^p}{(x)_\varepsilon^p} - 1 = \int_{|x+\varepsilon(z_1-z_2)| > \frac{|x|}{2}} \frac{x^p - (x + \varepsilon(z_1 - z_2))^p}{(x + \varepsilon(z_1 - z_2))^p} \eta(z_1)\eta(z_2) d^2 z_1 d^2 z_2.$$

We also know that, since we have assumed that $0 < \varepsilon < \frac{|x|}{4R}$, we have the bound $\left|\frac{\varepsilon(z_1 - z_2)}{x}\right| < 1$ on the domain where $\eta(z_1)\eta(z_2)$ does not vanish. As a consequence we can expand the integrand as a power series in the variable $\frac{\varepsilon(z_1 - z_2)}{x}$. Since η is compactly supported, the integral of this power series is absolutely convergent. Therefore, we conclude by noticing that the first two terms in the expansion vanish in the limit (the first one is identically zero; for the second order term we use the $z_1 \leftrightarrow z_2$ symmetry), so that we can factorize by $\left(\frac{\varepsilon}{x}\right)^2$.

Now if we assume that $|x| \leq 4R\varepsilon$ then

$$\frac{x^p}{(x)_\varepsilon^p} = \int_{|x+\varepsilon(z_1-z_2)| > \frac{|x|}{2}} \frac{\left(\frac{x}{\varepsilon}\right)^p}{\left(\frac{x}{\varepsilon} + (z_1 - z_2)\right)^p} \eta(z_1)\eta(z_2) d^2 z_1 d^2 z_2.$$

Since $|x| \leq 4R\varepsilon$, the integrand can be bounded by the quantity 3^{-p} , uniformly on $\varepsilon > 0$. This allows to conclude our proof of Lemma 4.4.5. $\square$

4.4.2.1 A first take on Gaussian integration by parts

We first consider the case where $m = 1$ so that there is only one derivative of the field inserted. The following statement explains how to apply Gaussian integration by parts in our context in order to treat such a term:

Lemma 4.4.6. *Let p be a positive integer and take $u \in \mathfrak{a}$. Then for any positive ε and $\alpha \in \mathcal{B}_N$:*

$$\begin{aligned} & \left\langle \langle \alpha, \partial^p \Phi_\varepsilon(z_0) \rangle \mathbf{V}_\varepsilon \right\rangle \\ &= \frac{(-1)^p (p-1)!}{2} \left(\sum_{k=1}^N \frac{\langle \alpha, \alpha_k \rangle}{(z_0 - z_k)_\varepsilon^p} \langle \mathbf{V}_\varepsilon \rangle - \sum_{i=1}^r \mu_i \int_{\mathbb{C}} \frac{\langle \alpha, \gamma e_i \rangle}{(z_0 - x)_\varepsilon^p} \langle V_{\gamma e_i, \varepsilon}(x) \mathbf{V}_\varepsilon \rangle d^2 x \right). \end{aligned}$$

Proof. According to Equation (4.3.10) we can write that

$$\begin{aligned} \langle \langle \alpha, \partial^p \Phi_\varepsilon(z_0) \rangle \rangle \mathbf{V}_\varepsilon &= \left(\prod_{i=1}^r \frac{\Gamma(s_i) \mu_i^{-s_i}}{\gamma} \right) \prod_{1 \leq k < l \leq N} |z_k - z_l|_\varepsilon^{-\langle \alpha_k, \alpha_l \rangle} \\ \mathbb{E} \left[\langle \alpha, \partial^p \left(\mathbf{X}_\varepsilon + \frac{Q}{2} \ln g_{+, \varepsilon} + \sum_{k=1}^N \alpha_k G_{g_{+, \varepsilon}}(\cdot, z_k) \right) (z_0) \rangle \prod_{i=1}^r Z_{(\mathbf{z}, \alpha), \varepsilon}^{\gamma e_i}(\mathbb{C})^{-s_i} \right] \end{aligned}$$

since when computing derivatives of the Toda field we no longer take into account the zero mode c . Now in agreement with Equation (3.3.2) we can write that

$$\begin{aligned} \mathbb{E} \left[\langle \alpha, \partial^p \mathbf{X}_\varepsilon(z_0) \rangle \prod_{i=1}^r Z_{(\mathbf{z}, \alpha), \varepsilon}^{\gamma e_i}(\mathbb{C})^{-s_i} \right] &= \\ - \sum_{i=1}^r s_i \langle \alpha, \gamma e_i \rangle \int_{\mathbb{C}} \partial_{z_0}^p G_{g_{+, \varepsilon}}(z_0, x) \frac{|x|_{+, \varepsilon}^{-\sum_{k=1}^N \langle \alpha_k, \gamma e_i \rangle}}{\prod_{k=1}^N |x - z_k|^{-\langle \alpha_k, \gamma e_i \rangle}} \mathbb{E} \left[M_\varepsilon^{\gamma e_i}(d^2 x) Z_{(\mathbf{z}, \alpha), \varepsilon}^{\gamma e_i}(\mathbb{C})^{-s_i-1} \prod_{j \neq i} Z_{(\mathbf{z}, \alpha), \varepsilon}^{\gamma e_j}(\mathbb{C})^{-s_j} \right]. \end{aligned}$$

Since $s_i \Gamma(s_i) = \Gamma(s_i + 1)$ while $\langle \mathbf{s} + \gamma e_i, \omega_j^\vee \rangle = s_j + \delta_{i,j}$, recollecting terms we can write that

$$\begin{aligned} \langle \langle \alpha, \partial^p \Phi_\varepsilon(z_0) \rangle \rangle \mathbf{V}_\varepsilon &= \left(\sum_{k=1}^N \langle \alpha, \alpha_k \rangle \partial_{z_0}^p G_{g_{+, \varepsilon}}(z_0, z_k) + \langle \alpha, \frac{Q}{2} \partial_{z_0}^p \ln g_{+, \varepsilon} \rangle(z_0) \right) \langle \mathbf{V}_\varepsilon \rangle \\ &\quad - \sum_{i=1}^r \frac{\mu_i \gamma}{2} \langle \alpha, e_i \rangle \int_{\mathbb{C}} \partial_{z_0}^p G_{g_{+, \varepsilon}}(z_0, x) \langle V_{\gamma e_i, \varepsilon}(x) \mathbf{V}_\varepsilon \rangle d^2 x. \end{aligned}$$

Now from the explicit expression of the Green function $G_{g_{+, \varepsilon}}$ from (3.2.9) we get

$$\begin{aligned} &\langle \langle \alpha, \partial^p \Phi_\varepsilon(z_0) \rangle \rangle \mathbf{V}_\varepsilon \\ &= \frac{(-1)^p (p-1)!}{2} \left(\sum_{k=1}^N \frac{\langle \alpha, \alpha_k \rangle}{(z_0 - z_k)_\varepsilon^p} - \sum_{i=1}^r \mu_i \int_{\mathbb{C}} \frac{\langle \alpha, \gamma e_i \rangle}{(z_0 - x)_\varepsilon^p} \langle V_{\gamma e_i, \varepsilon}(x) \mathbf{V}_\varepsilon \rangle d^2 x \right) \\ &\quad + \frac{1}{4} \partial^p \ln g_{+, \varepsilon}(z_0) \left\langle \alpha, \left(\sum_{k=1}^N \alpha_k - 2Q \right) \langle \mathbf{V}_\varepsilon \rangle - \sum_{i=1}^r \mu_i \gamma e_i \int_{\mathbb{C}} \langle V_{\gamma e_i, \varepsilon}(x) \mathbf{V}_\varepsilon \rangle d^2 x \right\rangle. \end{aligned}$$

Like before the above integrals are indeed well-defined thanks to item (5) of Lemma 4.4.2. Now we can use the KPZ identity (4.4.4) to see that the metric-dependent term—the last line—equals zero. $\square$

4.4.2.2 The general form of Gaussian integration by parts

In the sequel we will need to extend the previous statement to products of derivatives of the field, that is to say make sense of expressions of the form $\langle \prod_{l=1}^m \langle u_l, \partial^{p_l} \Phi_\varepsilon(z_0) \rangle \rangle \mathbf{V}_\varepsilon$ for some $u_l \in \mathfrak{a}$. However we see that as soon as $m > 1$, there will be diverging terms as $\varepsilon \rightarrow 0$ in the above expression due to the divergence of the Green function on the diagonal. To remedy this issue we need to interpret the product as Wick products $: \prod_{l=1}^m \langle u_l, \partial^{p_l} \Phi_\varepsilon(z_0) \rangle :$. These are usually defined recursively via the assumptions that

$$\frac{\partial}{\partial X_i} : X_1 \cdots X_n := : X_1 \cdots X_{i-1} X_{i+1} \cdots X_n : \quad \text{and} \quad \mathbb{E}[: X_1 \cdots X_N :] = 0.$$

For instance we have here that

$$:XY := XY - \mathbb{E}[X] \mathbb{E}[Y] \quad \text{and} \quad :XYZ := XYZ - X\mathbb{E}[YZ] - Y\mathbb{E}[XZ] - Z\mathbb{E}[XY].$$

Our goal is then to understand expressions of the form $\langle \prod_{l=1}^m \langle u_l, \partial^{p_l} \Phi_\varepsilon(z_0) \rangle \mathbf{V}_\varepsilon \rangle$

For this purpose we can apply recursively the reasoning of the proof Proposition 4.4.6 above, which yields the following formula:

$$\begin{aligned} & \frac{2}{(-1)^{p_m} (p_m - 1)!} \left\langle : \prod_{l=1}^m \langle u_l, \partial^{p_l} \Phi_\varepsilon(z_0) \rangle : \mathbf{V}_\varepsilon \right\rangle \\ &= \sum_{k=1}^N \frac{\langle u_m, \alpha_k \rangle}{(z_0 - z_k)_\varepsilon^{p_m}} \left\langle : \prod_{l=1}^{m-1} \langle \beta_l, \partial^{p_l} \Phi_\varepsilon(z_0) \rangle : \mathbf{V}_\varepsilon \right\rangle \\ & \quad - \sum_{i=1}^r \mu_i \int_{\mathbb{C}} \frac{\langle \beta_m, \gamma e_i \rangle}{(z_0 - x)_\varepsilon^p} \left\langle : \prod_{l=1}^{m-1} \langle \beta_l, \partial^{p_l} \Phi_\varepsilon(z_0) \rangle : V_{\gamma e_i, \varepsilon}(x) \mathbf{V}_\varepsilon \right\rangle d^2 x. \end{aligned} \tag{4.4.8}$$

valid for $p_1, \dots, p_r$ positive integers.

4.4.3 . Derivatives of the correlation functions

To conclude for this chapter on basic properties of the probabilistic correlation functions, we provide a study of the regularity of Toda correlation functions. Our goal is to show that, viewed as functions of their insertion points z , they are C^2 :

Proposition 4.4.7. *Take $z_1, \dots, z_N \in \mathbb{C}$ distinct and assume that the weights $\alpha = (\alpha_0, \dots, \alpha_N)$ belong to $\mathcal{B}_{N+1}$. Then $z_0 \mapsto \langle \prod_{k=0}^N V_{\alpha_k}(z_k) \rangle$ is C^2 on the set $\mathbb{C} \setminus \{z_1, \dots, z_N\}$.*

Proof. In the following, we only consider the $\frac{\partial}{\partial z}$ derivative; the calculation for the $\frac{\partial}{\partial \bar{z}}$ derivative is exactly the same. To start with note that $\langle V_{\alpha_0, \varepsilon}(z_0) \mathbf{V}_\varepsilon \rangle$ is differentiable with respect to z_0 , with derivative given by $\langle \langle \alpha_0, \partial \Phi(z_0) \rangle \prod_{k=0}^N V_{\alpha_k}(z_k) \rangle$ so that we can apply the results of Lemma 4.4.6.

First of all, from the results of Lemma 4.4.2 and following the proof of Lemma 4.4.6, we know that $\langle V_{\alpha_0, \varepsilon}(z_0) \mathbf{V}_\varepsilon \rangle$ is differentiable with respect to z_0 , with derivative

$$-\frac{1}{2} \sum_{k=1}^N \frac{\langle \alpha_0, \alpha_k \rangle}{(z_0 - z_k)_\varepsilon} \langle V_{\alpha_0, \varepsilon}(z_0) \mathbf{V}_\varepsilon \rangle + \sum_{i=1}^r \mu_i \int_{\mathbb{C}} \frac{\gamma \langle \alpha_0, e_i \rangle}{2(z_0 - x)_\varepsilon} \langle V_{\gamma e_i, \varepsilon}(x) V_{\alpha_0, \varepsilon}(z_0) \mathbf{V}_\varepsilon \rangle d^2 x.$$

From the asymptotics at infinity in Lemma 4.4.2, the only issue when taking the $\varepsilon \rightarrow 0$ limit comes from the behaviour of the integral term near the singularity $\frac{1}{z_0 - x}$. To address this issue, we fix $\rho > 0$ small enough (so that $|z_k - z_0| > \rho$ for any $1 \leq k \leq N$) and we split the integral between the domains $B(z_0, \rho)$ and its complementary $B(z_0, \rho)^c$, over which the integral does indeed converge for $\varepsilon = 0$. For the integral over $B =$

$B(z_0, \rho)$ we proceed by integration by parts¹. Indeed we can use the fact that

$$\begin{aligned} \partial_x \langle V_{\gamma e_i, \varepsilon}(x) V_{\alpha_0, \varepsilon}(z_0) \mathbf{V}_\varepsilon \rangle &= \left(\frac{\gamma \langle \alpha_0, e_i \rangle}{2(z_0 - x)_\varepsilon} + \sum_{k=1}^N \frac{\gamma \langle \alpha_k, e_i \rangle}{2(z_k - x)_\varepsilon} \right) \langle V_{\gamma e_i, \varepsilon}(x) V_{\alpha_0, \varepsilon}(z_0) \mathbf{V}_\varepsilon \rangle \\ &\quad - \sum_{j=1}^r \mu_j \int_{\mathbb{C}} \frac{\gamma^2 \langle e_j, e_i \rangle}{2(x' - x)_\varepsilon} \langle V_{\gamma e_i, \varepsilon}(x) V_{\gamma e_j, \varepsilon}(x') V_{\alpha_0, \varepsilon}(z_0) \mathbf{V}_\varepsilon \rangle d^2 x' \end{aligned}$$

to rewrite the integral over B under the form

$$\begin{aligned} &\sum_{i=1}^r \mu_i \left[\oint_{\partial B} \langle V_{\gamma e_i, \varepsilon}(\xi) V_{\alpha_0, \varepsilon}(z_0) \mathbf{V}_\varepsilon \rangle \sqrt{-1} d\bar{\xi} - \int_B \sum_{k=1}^N \frac{\gamma \langle \alpha_k, e_i \rangle}{2(z_k - x)_\varepsilon} \langle V_{\gamma e_i, \varepsilon}(x) V_{\alpha_0, \varepsilon}(z_0) \mathbf{V}_\varepsilon \rangle d^2 x \right] \\ &- \sum_{i,j=1}^r \mu_i \mu_j \int_B \int_{\mathbb{C}} \frac{\gamma^2 \langle e_i, e_j \rangle}{2(x_1 - x_2)_\varepsilon} \langle V_{\gamma e_i, \varepsilon}(x_1) V_{\gamma e_j, \varepsilon}(x_2) V_{\alpha_0, \varepsilon}(z_0) \mathbf{V}_\varepsilon \rangle d^2 x_1 d^2 x_2. \end{aligned}$$

The first two terms remain bounded in the $\varepsilon \rightarrow 0$ limit, while by symmetry between the x_1 and x_2 variables, we can rewrite the last term as

$$\begin{aligned} &\sum_{i,j=1}^r \mu_i \mu_j \int_B \int_{B^c} \frac{\gamma^2 \langle e_i, e_j \rangle}{2(x_1 - x_2)_\varepsilon} \langle V_{\gamma e_i, \varepsilon}(x_1) V_{\gamma e_j, \varepsilon}(x_2) V_{\alpha_0, \varepsilon}(z_0) \mathbf{V}_\varepsilon \rangle d^2 x_1 d^2 x_2 \\ &+ \int_{B^2} \frac{\gamma^2}{2(x_1 - x_2)_\varepsilon} \sum_{i,j=1}^r \mu_i \mu_j \langle e_i, e_j \rangle \langle V_{\gamma e_i, \varepsilon}(x_1) V_{\gamma e_j, \varepsilon}(x_2) V_{\alpha_0, \varepsilon}(z_0) \mathbf{V}_\varepsilon \rangle d^2 x_1 d^2 x_2. \end{aligned}$$

Now the last line is identically zero by symmetry between the x_1, x_2 variables, so that for the C^1 regularity it remains to show that when $\varepsilon \rightarrow 0$, the quantity

$$\sum_{i,j=1}^r \mu_i \mu_j \int_A \int_{B(z_0, r)^c} \frac{\gamma^2 \langle e_i, e_j \rangle}{2(x_1 - x_2)_\varepsilon} \langle V_{\gamma e_i, \varepsilon}(x_1) V_{\gamma e_j, \varepsilon}(x_2) V_{\alpha_0, \varepsilon}(z_0) \mathbf{V}_\varepsilon \rangle d^2 x_1 d^2 x_2$$

converges. The only singular terms that may occur in the integrand correspond to the case where $|x_1 - x_2|$ tends to zero. This in turn implies that $|x_1 - z_0|, |x_2 - z_0|$ tend to r , and therefore that both x_1 and x_2 are close to $\partial B(z_0, r)$. From Lemma 4.4.3 we know that the integral is uniformly convergent in ε and thus the $\varepsilon \rightarrow 0$ limit is well defined. This shows that the correlation functions are C^1 .

To prove that the correlation functions are C^2 , we can proceed in the same way. Having proved that the correlation functions were C^1 , the only terms that can potentially diverge when differentiating the expression just obtained correspond to the integrals over the domains B and $B \times \mathbb{C}$. The derivative of the first of these integrals is given by

$$\begin{aligned} &\sum_{i=1}^r \mu_i \int_B \sum_{k=1}^N \frac{\langle \alpha_k, \gamma e_i \rangle}{2(z_k - x)_\varepsilon} \left(\sum_{l=1}^N \frac{\langle \alpha_0, \alpha_l \rangle}{2(z_0 - z_l)_\varepsilon} + \frac{\langle \alpha_0, \gamma e_i \rangle}{2(z_0 - x)_\varepsilon} \right) \langle V_{\gamma e_i, \varepsilon}(x) V_{\alpha_0, \varepsilon}(z_0) \mathbf{V}_\varepsilon \rangle d^2 x \\ &- \sum_{i,j=1}^r \mu_i \mu_j \int_{B \times \mathbb{C}} \sum_{k=1}^N \frac{\langle \alpha_k, \gamma e_i \rangle}{2(z_k - x_1)_\varepsilon} \frac{\langle \alpha_0, \gamma e_j \rangle}{2(z_0 - x_2)_\varepsilon} \langle V_{\gamma e_i, \varepsilon}(x_1) V_{\gamma e_j, \varepsilon}(x_2) V_{\alpha_0, \varepsilon}(z_0) \mathbf{V}_\varepsilon \rangle d^2 x_1 d^2 x_2. \end{aligned}$$

¹That is, we apply Stokes' formula $\oint_{\partial B} f(\xi) g(\xi) \frac{\sqrt{-1} d\bar{\xi}}{2} = \int_B \partial_x f(x) g(x) + \partial_x g(x) f(x) d^2 x$.

This quantity is uniformly bounded in ε . To see why, we can again use integration by parts to see that we need to control the terms

$$\begin{aligned}
& \sum_{i=1}^r \mu_i \oint_{\partial B} \sum_{k=1}^N \frac{\langle \alpha_k, \gamma e_i \rangle}{2(z_k - \xi)_\varepsilon} \langle V_{\gamma e_i, \varepsilon}(\xi) V_{\alpha_0, \varepsilon}(z_0) \mathbf{V}_\varepsilon \rangle \frac{\sqrt{-1} d\bar{\xi}}{2} \\
& + \sum_{i=1}^r \mu_i \int_B \sum_{k=1}^N \frac{\langle \alpha_k, \gamma e_i \rangle}{2(z_k - x)_\varepsilon} \left(\sum_{l=1}^N \frac{\langle \alpha_0, \alpha_l \rangle}{2(z_0 - z_k)_\varepsilon} + \frac{\langle \alpha_l, \gamma e_i \rangle}{2(x - z_l)_\varepsilon} + \frac{1}{2(x - z_k)_\varepsilon} \right) \langle V_{\gamma e_i, \varepsilon}(x) V_{\alpha_0, \varepsilon}(z_0) \mathbf{V}_\varepsilon \rangle d^2 x \\
& - \sum_{i,j=1}^r \mu_i \mu_j \int_B \int_{B^c} \sum_{k=1}^N \frac{\langle \alpha_k, \gamma e_i \rangle}{2(z_k - x_1)_\varepsilon} \times \\
& \quad \left(\frac{\langle \alpha_0, \gamma e_j \rangle}{2(z_0 - x_2)_\varepsilon} + \frac{\langle \gamma e_i, \gamma e_j \rangle}{2(x_1 - x_2)_\varepsilon} \right) \langle V_{\gamma e_i, \varepsilon}(x_1) V_{\gamma e_j, \varepsilon}(x_2) V_{\alpha_0, \varepsilon}(z_0) \mathbf{V}_\varepsilon \rangle d^2 x_1 d^2 x_2 \\
& - \sum_{i,j=1}^r \mu_i \mu_j \int_{B^2} \sum_{k=1}^N \frac{\langle \alpha_k, \gamma e_i \rangle}{2(z_k - x_1)_\varepsilon} \times \\
& \quad \left(\frac{\langle \alpha_0, \gamma e_j \rangle}{2(z_0 - x_2)_\varepsilon} + \frac{\langle \gamma e_i, \gamma e_j \rangle}{2(x_1 - x_2)_\varepsilon} \right) \langle V_{\gamma e_i, \varepsilon}(x_1) V_{\gamma e_j, \varepsilon}(x_2) V_{\alpha_0, \varepsilon}(z_0) \mathbf{V}_\varepsilon \rangle d^2 x_1 d^2 x_2.
\end{aligned}$$

All integrals except for the last one remain bounded as $\varepsilon \rightarrow 0$ thanks to Lemmas 4.4.2 and 4.4.3. To treat it we use again integration by parts to get, up to regular terms,

$$\begin{aligned}
& - \sum_{i,j=1}^r \mu_i \mu_j \int_{B^2} \sum_{k,l=1}^N \frac{\langle \alpha_k, \gamma e_i \rangle}{2(z_k - x_1)_\varepsilon} \frac{\langle \alpha_k, \gamma e_j \rangle}{2(z_k - x_2)_\varepsilon} \langle V_{\gamma e_i, \varepsilon}(x_1) V_{\gamma e_j, \varepsilon}(x_2) V_{\alpha_0, \varepsilon}(z_0) \mathbf{V}_\varepsilon \rangle d^2 x_1 d^2 x_2 \\
& + \sum_{i,j,l=1}^r \mu_i \mu_j \mu_l \int_{B^2} \int_{\mathbb{C}} \sum_{k=1}^N \frac{\langle \alpha_k, \gamma e_i \rangle}{2(z_k - x_1)_\varepsilon} \frac{\langle \gamma e_j, \gamma e_l \rangle}{2(x_3 - x_2)_\varepsilon} \\
& \quad \langle V_{\gamma e_i, \varepsilon}(x_1) V_{\gamma e_j, \varepsilon}(x_2) V_{\gamma e_l, \varepsilon}(x_3) V_{\alpha_0, \varepsilon}(z_0) \mathbf{V}_\varepsilon \rangle d^2 x_1 d^2 x_2 d^2 x_3.
\end{aligned}$$

Using the symmetries in the x_2, x_3 variables, the last integral vanishes on A^3 . This means that this term remains bounded when $\varepsilon \rightarrow 0$ by using Lemma 4.4.3.

To finish up with the proof that the correlation functions are C^2 it remains to take care of the derivative of the term

$$- \sum_{i,j=1}^2 \mu_i \mu_j \int_B \int_{B^c} \frac{\gamma^2 \langle e_i, e_j \rangle}{2(x_1 - x_2)_\varepsilon} \langle V_{\gamma e_i, \varepsilon}(x_1) V_{\gamma e_j, \varepsilon}(x_2) V_{\alpha_0, \varepsilon}(z_0) \mathbf{V}_\varepsilon \rangle d^2 x_1 d^2 x_2.$$

We proceed in the same way by using integration by parts to get rid of the terms containing singularities of the form $\frac{1}{(z_0 - x)_\varepsilon}$. We may follow the same lines as in the previous computation, apart from the fact that the new integration domains we will consider either contain or don't the previous integration variables. We won't write down the details here since they are quite lengthy and not informative, nonetheless we see that in the end it is enough to show that the integrals (here η is some positive number)

$$\int_{B^2} \int_{B^c} \frac{\mathbb{1}_{|x_1 - x_2| > \eta}}{(x_1 - x_3)(x_2 - x_3)} \langle \prod_{k=1}^3 V_{\gamma e_*, \varepsilon}(x_k) V_{\alpha_0, \varepsilon}(z_0) \mathbf{V}_\varepsilon \rangle d^2 x_1 d^2 x_2 d^2 x_3$$

and

$$\int_{B^2} \int_{(B^c)^2} \frac{\mathbb{1}_{|x_1 - x_2| > \eta} \mathbb{1}_{|x_3 - x_4| > \eta}}{(x_1 - x_3)(x_2 - x_4)} \langle \prod_{k=1}^4 V_{\gamma e_*, \varepsilon}(x_k) V_{\alpha_0, \varepsilon}(z_0) \mathbf{V}_\varepsilon \rangle d^2 x_1 d^2 x_2 d^2 x_3 d^2 x_4$$

are absolutely convergent, which follows from Lemma 4.4.3. Therefore the $\frac{\partial^2}{\partial z_0^2}$ derivative of the correlation functions are well-defined. Treating the mixed derivatives $\frac{\partial^2}{\partial z_0 \partial \bar{z}_0}$ leads to the same conclusion. Hence existence of both the $\frac{\partial^2}{\partial z_0^2}$ and $\frac{\partial^2}{\partial z_0 \partial \bar{z}_0}$ derivatives: this implies that the correlation functions are C^2 . $\square$

Part II

Étude des symétries des théories conformes des champs de Toda

5 - Symétrie W et identités de Ward

Les théories de Toda, de par leur définition basée sur une structure d'algèbre de Lie, possèdent certaines symétries qui imposent ainsi des contraintes au modèle qu'elles définissent. Ces symétries sont couramment appelées *symétries W* ou *symétries de spin supérieur*, et sont en quelque sorte codées par des objets algébriques nommés *W -algèbres*. L'existence de ces symétries montre une certaine forme de rigidité des théories de Toda et nous permet de ce fait de comprendre les propriétés de cette famille de modèles.

En effet il est admis dans la littérature physique que ces symétries ont pour conséquence première l'existence de certaines contraintes portant sur les fonctions de corrélation des théories de Toda : ce sont les *identités de Ward*. L'objet de ce chapitre est de démontrer que ces identités sont valables pour la théorie de Toda associée à $\mathfrak{g} = \mathfrak{sl}_3$ dans le cadre du modèle rigoureusement défini au long de la partie I. Prouver que ces identités restent valides pour le modèle probabiliste est une indication supplémentaire que l'objet probabiliste considéré fournit en effet une définition rigoureuse des objets étudiés dans la physique.

Ce chapitre se base en majeure partie sur l'article [33] rédigé en collaboration avec Yichao Huang.

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5.1 . Local Ward identities in the $\mathfrak{sl}_3$ Toda theory

Now that the correlation functions have been properly introduced and that we know some of their basic properties we can study some of their more advanced features in relation with the symmetries enjoyed by Toda theories. We derive in this section the local Ward identities associated to the $\mathfrak{sl}_3$ Toda theory:

$$\langle \mathbf{W}(z_0) \prod_{k=1}^N V_{\alpha_k}(z_k) \rangle = \sum_{l=1}^N \left(\frac{w(\alpha_l)}{(z_0 - z_l)^3} + \frac{\mathbf{W}_{-1}^{(l)}}{(z_0 - z_l)^2} + \frac{\mathbf{W}_{-2}^{(l)}}{z_0 - z_l} \right) \langle \prod_{k=1}^N V_{\alpha_k}(z_k) \rangle.$$

which as explained in the introduction contain a lot of information related to the W -symmetry enjoyed by this model. We will first describe within our probabilistic setting how the different terms that appear in the Ward identities can be rigorously defined and then prove the statement stated as such.

5.1.1 . W -descendants and the local Ward identities

As an introductory subsection we describe here how to make sense of the quantities involved in the local Ward identities based on our probabilistic framework.

5.1.1.1 Definition of the holomorphic currents

As explained in the introduction Toda theories contain in addition to the stress-energy tensor $\mathbf{T}$ a family of higher-spin currents $(\mathbf{W}^{(i)})_{3 \leq i \leq r+1}$. In the cases where $\mathfrak{g} = \mathfrak{sl}_n$ for some $n \geq 2$, these currents are usually defined via the Miura transformation¹

$$\prod_{i=1}^{r+1} \left(\frac{q}{2} \partial + \langle h_i, \partial \Phi \rangle \right) := \sum_{i=0}^r \mathbf{W}^{(r-i)} \left(\frac{q}{2} \partial \right)^i \quad (5.1.1)$$

where the $(h_i)_{1 \leq i \leq r+1}$ are the *fundamental weights in the first fundamental representation* π_1 of $\mathfrak{g}$, $q := \gamma + \frac{2}{\gamma}$ and like before ∂ is a holomorphic derivative.

In the sequel, we focus on the study of the $\mathfrak{g} = \mathfrak{sl}_3$ Toda theory. There will be one additional holomorphic current of spin three $\mathbf{W} := \mathbf{W}^{(3)}$ *a priori* defined via Equation (5.1.1):

$$\frac{q^2}{4} \langle h_1, \partial^3 \Phi \rangle + \frac{q}{2} (\langle h_1, \partial^2 \Phi \rangle \langle h_2 + h_3, \partial \Phi \rangle + \langle h_2, \partial^2 \Phi \rangle \langle h_1, \partial \Phi \rangle) + \langle h_1, \partial \Phi \rangle \langle h_2, \partial \Phi \rangle \langle h_3, \partial \Phi \rangle.$$

As suggested in the physics literature [25, 54, 52], in order to recover the standard commutation relations of the W_3 -algebra and obtain an elegant expression for the quantum number associated to the spin-three tensor $w(\alpha)$, we may choose to shift this current by an additional factor $-\frac{q}{8} \partial \mathbf{T}$ and rescale it. Explicitly, we redefine the spin-three current by setting

$$\begin{aligned} \mathbf{W}(z)[\Phi] := & q^2 \langle h_2, \partial^3 \Phi(z) \rangle - 2q (\langle h_2 - h_1, \partial^2 \Phi(z) \rangle \langle h_1, \partial \Phi(z) \rangle + \langle h_3 - h_2, \partial^2 \Phi(z) \rangle \langle h_3, \partial \Phi(z) \rangle) \\ & - 8 \langle h_1, \partial \Phi(z) \rangle \langle h_2, \partial \Phi(z) \rangle \langle h_3, \partial \Phi(z) \rangle \end{aligned} \quad (5.1.2)$$

¹The interested reader may find details on the role of this transformation in the construction of two-dimensional CFTs having higher-spin symmetry for instance in [53], where the Miura transformation is used to construct representations of W -algebras.

with the h_i being defined in Equation (3.1.10). Similarly the stress-energy tensor will be defined via the expression

$$\mathbf{T}(z)[\Phi] := \langle Q, \partial^2 \Phi(z) \rangle - \langle \partial \Phi(z), \partial \Phi(z) \rangle. \quad (5.1.3)$$

Like before a regularization procedure is needed to make sense of the above since the Toda field Φ is not regular. Moreover and as explain below the statement of Lemma 4.4.6, products of the field must be understood as Wick products to ensure that a limit of the regularized quantities exist. This motivates the introduction of the following regularization of the holomorphic currents $\mathbf{T}$ and $\mathbf{W}$:

$$\mathbf{T}_\varepsilon(z)[\Phi] := \langle Q, \partial^2 \Phi_\varepsilon(z) \rangle - : \langle \partial \Phi_\varepsilon(z), \partial \Phi_\varepsilon(z) \rangle : \quad (5.1.4)$$

where like before : $XY := XY - \mathbb{E}[XY]$, while

$$\begin{aligned} \mathbf{W}_\varepsilon(z)[\Phi] &:= q^2 \langle h_2, \partial^3 \Phi_\varepsilon(z) \rangle \\ &- 2q (: \langle h_2 - h_1, \partial^2 \Phi_\varepsilon(z) \rangle \langle h_1, \partial \Phi_\varepsilon(z) \rangle : + : \langle h_3 - h_2, \partial^2 \Phi_\varepsilon(z) \rangle \langle h_3, \partial \Phi_\varepsilon(z) \rangle :) \\ &- 8 : \langle h_1, \partial \Phi_\varepsilon(z) \rangle \langle h_2, \partial \Phi_\varepsilon(z) \rangle \langle h_3, \partial \Phi_\varepsilon(z) \rangle : \end{aligned} \quad (5.1.5)$$

with : $XYZ := XYZ - X\mathbb{E}[YZ] - Y\mathbb{E}[XZ] - Z\mathbb{E}[XY]$.

In addition to this regularization of the currents we will regularize the integral over $\mathbb{C}$ in order to avoid the singularities that will arise at z_0 . To do so we introduce the notation $\mathbb{C}_t := \mathbb{C} \setminus B_{e^{-t}}(z_0)$ and consider the regularization of the correlation functions defined by setting

$$\begin{aligned} \langle F(\Phi_\varepsilon) \prod_{k=1}^N V_{\alpha_k, \varepsilon}(z_k) \rangle_t &:= \left(\prod_{i=1}^r \frac{\Gamma(s_i) \mu_i^{-s_i}}{\gamma} \right) \prod_{1 \leq k < l \leq N} |z_k - z_l|_\varepsilon^{-\langle \alpha_k, \alpha_l \rangle} \\ &\mathbb{E} \left[F \left(\mathbf{X}_\varepsilon + \frac{Q}{2} \ln g_{+, \varepsilon} + \sum_{k=1}^N \alpha_k G_{g_{+, \varepsilon}}(\cdot, z_k) \right) \prod_{i=1}^r Z_{(\mathbf{z}, \alpha), \varepsilon}^{\gamma e_i}(\mathbb{C}_t)^{-s_i} \right]. \end{aligned} \quad (5.1.6)$$

Then one easily checks that the statement of Lemma 4.4.6 remains true by replacing $\mathbb{C}$ with $\mathbb{C}_t$, and therefore the reasoning leading to the proof of Proposition 4.4.7 too.

We will see that we can make sense of the right-hand side in the local Ward identities by showing that:

Lemma 5.1.1. *As soon as $\alpha \in \mathcal{B}_N$ the following limits exist and are well-defined:*

$$\begin{aligned} \langle \mathbf{T}(z_0) \prod_{k=1}^N V_{\alpha_k}(z_k) \rangle &:= \lim_{t \rightarrow +\infty} \lim_{\varepsilon \rightarrow 0} \langle \mathbf{T}_\varepsilon(z_0) \prod_{k=1}^N V_{\alpha_k, \varepsilon}(z_k) \rangle_t \\ \langle \mathbf{W}(z_0) \prod_{k=1}^N V_{\alpha_k}(z_k) \rangle &:= \lim_{t \rightarrow +\infty} \lim_{\varepsilon \rightarrow 0} \langle \mathbf{W}_\varepsilon(z_0) \prod_{k=1}^N V_{\alpha_k, \varepsilon}(z_k) \rangle_t \end{aligned} \quad (5.1.7)$$

where the regularized quantities are defined using Equation (4.3.10).

5.1.1.2 Definition of the W -descendants

We now turn to the right-hand side of the local Ward identities and define the W -descendants associated to Toda Vertex Operators. To the best of our knowledge, explicit expressions for $\mathbf{W}_{-i}V_\alpha(z)$ remain unknown in the physics literature and should look like “derivatives in an extra direction” of the Vertex operator (cf. [135, Subsection 8.2]), Nevertheless, these can be considered as the building blocks for solving the $\mathfrak{sl}_3$ Toda theories since the T and W descendent states are assumed to span the space of states of Toda theories viewed as meromorphic CFTs (see e.g. [26, Subsection 3.1]) so providing an explicit expression for them is key in this direction.

For this purpose let us introduce the notations

$$B(u, v) := \langle h_2 - h_1, u \rangle \langle h_1, v \rangle + \langle h_3 - h_2, u \rangle \langle h_3, v \rangle \quad \text{and} \quad (5.1.8)$$

$$C(u, v, w) := \langle h_1, u \rangle \langle h_2, v \rangle \langle h_3, w \rangle + \langle h_1, v \rangle \langle h_2, w \rangle \langle h_3, u \rangle + \langle h_1, w \rangle \langle h_2, u \rangle \langle h_3, v \rangle. \quad (5.1.9)$$

We then define W -descendants of the Vertex Operators by setting for $j = 1, 2$

$$\begin{aligned} \mathbf{W}_{-j}V_\alpha(z)[\Phi] &:= \mathbf{W}_{-j}(z, \alpha)V_\alpha(z)[\Phi] \quad \text{with} \\ \mathbf{W}_{-1}(z, \alpha)[\Phi] &:= -qB(\alpha, \partial\Phi(z)) - 2C(\alpha, \alpha, \partial\Phi(z)), \\ \mathbf{W}_{-2}(z, \alpha)[\Phi] &:= q(B(\partial^2\Phi(z), \alpha) - B(\alpha, \partial^2\Phi(z))) \\ &\quad - 2C(\alpha, \alpha, \partial^2\Phi(z)) + 4C(\alpha, \partial\Phi(z), \partial\Phi(z)). \end{aligned} \quad (5.1.10)$$

Their regularized counterparts are defined via

$$\begin{aligned} \mathbf{W}_{-1,\varepsilon}(z, \alpha)[\Phi] &:= -qB(\alpha, \partial\Phi_\varepsilon(z)) - 2C(\alpha, \alpha, \partial\Phi_\varepsilon(z)), \\ \mathbf{W}_{-2,\varepsilon}(z, \alpha)[\Phi] &:= q(B(\partial^2\Phi_\varepsilon(z), \alpha) - B(\alpha, \partial^2\Phi_\varepsilon(z))) \\ &\quad - 2C(\alpha, \alpha, \partial^2\Phi_\varepsilon(z)) + 4 : C(\alpha, \partial\Phi_\varepsilon(z), \partial\Phi_\varepsilon(z)) : . \end{aligned} \quad (5.1.11)$$

The quantum number associated to the higher spin current, formally corresponding to $\mathbf{W}_0V_\alpha$, is defined by:

$$w(\alpha) := \langle \alpha - Q, h_1 \rangle \langle \alpha - Q, h_2 \rangle \langle \alpha - Q, h_3 \rangle. \quad (5.1.12)$$

The right-hand side in the local Ward identities is then defined via the following statement:

Lemma 5.1.2. *As soon as $\alpha \in \mathcal{B}_N$ the following limit exists and is well-defined for $j = 1, 2$ and any $1 \leq l \leq N$:*

$$\mathbf{w}_{-j}^{(l)} \left\langle \prod_{k=1}^N V_{\alpha_k}(z_k) \right\rangle := \lim_{\varepsilon \rightarrow 0} \langle \mathbf{W}_{-j,\varepsilon} V_{\alpha_l,\varepsilon}(z_l) \prod_{k \neq l} V_{\alpha_k,\varepsilon}(z_k) \rangle \quad (5.1.13)$$

where the regularized quantities are defined using Equation (4.3.10).

Note that we can also incorporate the regularization in t in the above without causing any issue since the regularization is made around z_0 which does not appear in the expressions considered.

5.1.1.3 Statement of the local Ward identities

Having defined the quantities that appear in the local Ward identities we are now in position to state them:

Theorem 5.1.3. *Assume that $\mathfrak{g} = \mathfrak{sl}_3$. Under the assumptions of Theorem 4.3.1 the spin-three Ward identity holds true:*

$$\langle \mathbf{W}(z_0) \prod_{k=1}^N V_{\alpha_k}(z_k) \rangle = \sum_{l=1}^N \left(\frac{w(\alpha_l)}{(z_0 - z_l)^3} + \frac{\mathbf{w}_{-1}^{(l)}}{(z_0 - z_l)^2} + \frac{\mathbf{w}_{-2}^{(l)}}{z_0 - z_l} \right) \langle \prod_{k=1}^N V_{\alpha_k}(z_k) \rangle. \quad (5.1.14)$$

The quantities that appear in Equation (5.1.14) have been defined in Lemmas 5.1.1 and 5.1.2 above. A similar statement holds for the stress-energy tensor under the form

$$\langle \mathbf{T}(z_0) \prod_{k=1}^N V_{\alpha_k}(z_k) \rangle = \sum_{l=1}^N \left(\frac{\Delta_{\alpha_l}}{(z_0 - z_l)^2} + \frac{\partial_{z_l}}{z_0 - z_l} \right) \langle \prod_{k=1}^N V_{\alpha_k}(z_k) \rangle. \quad (5.1.15)$$

5.1.2 . Computation of the Ward identity for the stress-energy tensor and strategy of proof

To start with note that since the dependence in the cosmological constant is explicit we can assume that $\mu_1 = \mu_2 = 1$. Then in order to prove these identities, we work with the regularized version of the correlation functions. Then:

- We can use integration by parts from Lemma 4.4.6 to rewrite explicitly the left-hand side $\langle \mathbf{W}_\varepsilon(z_0) \mathbf{V}_\varepsilon \rangle$ in terms of multiple integrals in the x variable containing singularities of the form $x \mapsto \frac{1}{(x-z_0)^p}$. We proceed in the same way with the right-hand side in the Ward identity, which will also yield multiple integrals containing singularities of the form $x \mapsto \frac{1}{(x-z_k)^p}$.
- We transform the right-hand side so that the only singularities appearing within integrals are of the form $\frac{1}{(x-z_0)^p}$. Using identities such as $\frac{1}{(z_0-z_k)(x-z_k)} = \frac{1}{z_0-x} \left(\frac{1}{x-z_k} - \frac{1}{z_0-z_k} \right)$ (that we call “symmetrization” identities) we rewrite terms containing expressions of the form $\frac{1}{x-z_k}$ as derivatives of the correlation functions. We then use integration by parts to transform them into the desired form.
- We take the $\varepsilon \rightarrow 0$ limit of the expression obtained. Since singularities are present only for $x = z_0$, thanks to our regularization in t of the integrals this limit does not cause any real problem.
- We then take the $t \rightarrow +\infty$ limit of the remaining terms. As we will see the limit is given by zero, showing that the difference between the left and right-hand side in Equations (5.1.15) and (5.1.14) vanishes.

5.1.2.1 Computation of the spin-two Ward identity

As a warm-up, let us discuss in detail the proof of the Ward identity for the stress-energy tensor $\mathbf{T}$. We can rewrite the regularized expression for $\mathbf{T}$ from Equation (5.1.4) under the form

$$\langle \mathbf{T}_\varepsilon(z_0) \mathbf{V}_\varepsilon \rangle_t = \langle \langle Q, \partial^2 \Phi_\varepsilon(z_0) \rangle \mathbf{V}_\varepsilon \rangle_t - \sum_{i=1}^2 \langle : \langle e_i, \partial \Phi_\varepsilon(z_0) \rangle \langle \omega_i^\vee, \partial \Phi_\varepsilon(z_0) \rangle : \mathbf{V}_\varepsilon \rangle_t.$$

We can then apply Lemma 4.4.6 on the first term and Equation (4.4.8) for the second one to rewrite the right-hand side under the form

$$\begin{aligned} & \frac{1}{2} \sum_{k=1}^N \frac{\langle Q, \alpha_k \rangle}{(z_0 - z_k)_\varepsilon^2} \langle \mathbf{V}_\varepsilon \rangle_t - \sum_{i=1}^2 \int_{\mathbb{C}_t} \frac{\gamma q}{2(z_0 - x)_\varepsilon^2} \langle V_{\gamma e_i, \varepsilon}(x) \mathbf{V}_\varepsilon \rangle_t d^2 x \\ & - \sum_{k,l=1}^N \sum_{i=1}^2 \frac{\langle e_i, \alpha_k \rangle \langle \omega_i^\vee, \alpha_l \rangle}{4(z_0 - z_k)_\varepsilon (z_0 - z_l)_\varepsilon} \langle \mathbf{V}_\varepsilon \rangle \\ & + \sum_{i=1}^2 \int_{\mathbb{C}_t} \left(\sum_{k=1}^N \frac{\langle \alpha_k, \gamma e_i \rangle}{2(z_0 - z_k)_\varepsilon (z_0 - x)_\varepsilon} + \frac{\langle \gamma e_i, \gamma e_i \rangle}{4(z_0 - x)_\varepsilon (z_0 - x)_\varepsilon} \right) \langle V_{\gamma e_i, \varepsilon}(x) \mathbf{V}_\varepsilon \rangle_t d^2 x \\ & - \sum_{i,j=1}^2 \int_{\mathbb{C}_t^2} \frac{\langle \gamma e_i, \gamma e_j \rangle}{4(z_0 - x_1)_\varepsilon (z_0 - x_2)_\varepsilon} \langle V_{\gamma e_i, \varepsilon}(x_1) V_{\gamma e_j, \varepsilon}(x_2) \mathbf{V}_\varepsilon \rangle_t d^2 x_1 d^2 x_2. \end{aligned}$$

Using the fact that in the last double integral the variables x_1 and x_2 are symmetric we can rewrite the last line under the form

$$\begin{aligned} & - \sum_{i,j=1}^2 \int_{\mathbb{C}_t^2} \frac{\langle \gamma e_i, \gamma e_j \rangle}{2(z_0 - x_1)_\varepsilon (x_1 - x_2)_\varepsilon} \langle V_{\gamma e_i, \varepsilon}(x_1) V_{\gamma e_j, \varepsilon}(x_2) \mathbf{V}_\varepsilon \rangle_t d^2 x_1 d^2 x_2 + \mathfrak{R}_\varepsilon^1, \text{ where} \\ \mathfrak{R}_\varepsilon^1 &= \sum_{i,j=1}^2 \int_{\mathbb{C}_t^2} \frac{\langle \gamma e_i, \gamma e_j \rangle}{4} \langle V_{\gamma e_i, \varepsilon}(x_1) V_{\gamma e_j, \varepsilon}(x_2) \mathbf{V}_\varepsilon \rangle_t d^2 x_1 d^2 x_2 \\ & \quad \left(\frac{1}{(z_0 - x_1)_\varepsilon (x_1 - x_2)_\varepsilon} - \frac{1}{(z_0 - x_2)_\varepsilon (x_1 - x_2)_\varepsilon} - \frac{1}{(z_0 - x_1)_\varepsilon (z_0 - x_2)_\varepsilon} \right). \end{aligned}$$

We now turn to the other side of the expression. Along the same lines as in the proof of Proposition 4.4.7 we end up with

$$\begin{aligned} \sum_{k=1}^N \left(\frac{\Delta_{\alpha_k}}{(z_0 - z_k)^2} + \frac{\partial_{z_k}}{(z_0 - z_k)} \right) \langle \mathbf{V}_\varepsilon \rangle_t &= \left(\sum_{k=1}^N \frac{\Delta_{\alpha_k}}{(z_0 - z_k)^2} - \frac{1}{2} \sum_{k \neq l} \frac{\langle \alpha_k, \alpha_l \rangle}{(z_0 - z_k)(z_k - z_l)_\varepsilon} \right) \langle \mathbf{V}_\varepsilon \rangle_t \\ &+ \sum_{i=1}^2 \int_{\mathbb{C}_t} \sum_{k=1}^N \frac{\langle \alpha_k, \gamma e_i \rangle}{2(z_0 - z_k)(z_k - x)_\varepsilon} \langle V_{\gamma e_i, \varepsilon}(x) \mathbf{V}_\varepsilon \rangle_t d^2 x. \end{aligned}$$

Here the symmetrization step is rather straightforward. For the terms that do not involve

integrals:

$$\begin{aligned} \frac{1}{2} \sum_{k \neq l} \frac{\langle \alpha_k, \alpha_l \rangle}{(z_0 - z_k)(z_k - z_l)_\varepsilon} &= \frac{1}{4} \sum_{k \neq l} \frac{\langle \alpha_k, \alpha_l \rangle}{(z_k - z_l)_\varepsilon} \left(\frac{1}{z_0 - z_k} - \frac{1}{z_0 - z_l} \right) \\ &= \frac{1}{4} \sum_{k \neq l} \frac{\langle \alpha_k, \alpha_l \rangle}{(z_0 - z_k)(z_0 - z_l)} + \frac{1}{4} \sum_{k \neq l} \frac{\langle \alpha_k, \alpha_l \rangle}{(z_0 - z_k)(z_0 - z_l)} \left(\frac{z_k - z_l}{(z_k - z_l)_\varepsilon} - 1 \right). \end{aligned}$$

We then transform the terms that contain one integral as

$$\begin{aligned} &\sum_{i=1}^2 \int_{\mathbb{C}_t} \sum_{k=1}^N \frac{\langle \alpha_k, \gamma e_i \rangle}{2(z_0 - z_k)(z_k - x)_\varepsilon} \langle V_{\gamma e_i, \varepsilon}(x) \mathbf{V}_\varepsilon \rangle_t d^2 x \\ &= \sum_{i=1}^2 \int_{\mathbb{C}_t} \sum_{k=1}^N \left(\frac{\langle \alpha_k, \gamma e_i \rangle}{2(z_0 - x)(z_k - x)_\varepsilon} + \frac{\langle \alpha_k, \gamma e_i \rangle}{2(z_0 - x)(z_0 - z_k)} \right) \langle V_{\gamma e_i, \varepsilon}(x) \mathbf{V}_\varepsilon \rangle_t d^2 x \\ &\quad + \sum_{i=1}^2 \int_{\mathbb{C}_t} \sum_{k=1}^N \frac{\langle \alpha_k, \gamma e_i \rangle}{2(z_0 - x)(z_0 - z_k)} \left(\frac{z_k - x}{(z_k - x)_\varepsilon} - 1 \right) \langle V_{\gamma e_i, \varepsilon}(x) \mathbf{V}_\varepsilon \rangle_t d^2 x. \end{aligned}$$

Hence combining our expressions yields

$$\begin{aligned} &\langle \mathbf{T}_\varepsilon(z_0) \mathbf{V}_\varepsilon \rangle_t - \sum_{k=1}^N \left(\frac{\Delta_{\alpha_k}}{(z_0 - z_k)^2} + \frac{\partial_{z_k}}{(z_0 - z_k)} \right) \langle \mathbf{V}_\varepsilon \rangle_t = \\ &\frac{1}{2} \sum_{k=1}^N \frac{1}{(z_0 - z_k)^2} \left[\langle Q, \alpha_k \rangle \left(\frac{(z_0 - z_k)^2}{(z_0 - z_k)_\varepsilon^2} - 1 \right) - \frac{1}{2} \langle \alpha_k, \alpha_k \rangle \left(\frac{(z_0 - z_k)^2}{(z_0 - z_k)_\varepsilon(z_0 - z_k)_\varepsilon} - 1 \right) \right] \langle \mathbf{V}_\varepsilon \rangle_t \\ &- \frac{1}{4} \sum_{k \neq l} \frac{\langle \alpha_k, \alpha_l \rangle}{(z_0 - z_k)(z_0 - z_l)} \left(\frac{(z_0 - z_k)(z_0 - z_l)}{(z_0 - z_k)_\varepsilon(z_0 - z_l)_\varepsilon} - 1 + \frac{z_k - z_l}{(z_k - z_l)_\varepsilon} - 1 \right) \langle \mathbf{V}_\varepsilon \rangle_t \\ &- \sum_{i=1}^2 \int_{\mathbb{C}_t} \frac{\gamma^2}{2(z_0 - x)^2} \left(1 - \frac{(z_0 - x)^2}{(z_0 - x)_\varepsilon(z_0 - x)_\varepsilon} \right) \langle V_{\gamma e_i, \varepsilon}(x) \mathbf{V}_\varepsilon \rangle_t d^2 x \\ &- \sum_{i=1}^2 \int_{\mathbb{C}_t} \frac{\gamma}{2(z_0 - x)^2} \left(\sum_{k=1}^N \frac{(z_0 - x) \langle \alpha_k, e_i \rangle}{(z_0 - z_k)} \left(\frac{z_k - x}{(z_k - x)_\varepsilon} - 1 \right) \right) \langle V_{\gamma e_i, \varepsilon}(x) \mathbf{V}_\varepsilon \rangle_t d^2 x \\ &- \sum_{i=1}^2 \int_{\mathbb{C}_t} \left(\sum_{k=1}^N \frac{\langle \alpha_k, \gamma e_i \rangle}{2(z_0 - x)(z_k - x)_\varepsilon} - \frac{1}{(z_0 - x)_\varepsilon^2} \right) \langle V_{\gamma e_i, \varepsilon}(x) \mathbf{V}_\varepsilon \rangle_t d^2 x \\ &- \sum_{i,j=1}^2 \int_{\mathbb{C}_t^2} \frac{\langle \gamma e_i, \gamma e_j \rangle}{2(z_0 - x_1)_\varepsilon(x_1 - x_2)_\varepsilon} \langle V_{\gamma e_i, \varepsilon}(x_1) V_{\gamma e_j, \varepsilon}(x_2) \mathbf{V}_\varepsilon \rangle_t d^2 x_1 d^2 x_2 + \mathfrak{R}_\varepsilon^1. \end{aligned}$$

Now for $x \notin \{z_0, \dots, z_N\}$, $\partial_x \langle V_{\gamma e_i, \varepsilon}(x) \mathbf{V}_\varepsilon \rangle$ is given by

$$-\frac{1}{2} \sum_{k=1}^N \frac{\langle \alpha_k, \gamma e_i \rangle}{(x - z_k)_\varepsilon} \langle V_{\gamma e_i, \varepsilon}(x) \mathbf{V}_\varepsilon \rangle_t + \sum_{j=1}^2 \int_{\mathbb{C}_t} \frac{\langle \gamma e_i, \gamma e_j \rangle}{2(x - x_2)_\varepsilon} \langle V_{\gamma e_i, \varepsilon}(x) V_{\gamma e_j, \varepsilon}(x_2) \mathbf{V}_\varepsilon \rangle_t d^2 x_2.$$

This shows that

$$\begin{aligned} & \langle \mathbf{T}_\varepsilon(z_0) \mathbf{V}_\varepsilon \rangle_t - \sum_{k=1}^N \left(\frac{\Delta_{\alpha_k}}{(z_0 - z_k)^2} + \frac{\partial_{z_k}}{(z_0 - z_k)} \right) \langle \mathbf{V}_\varepsilon \rangle_t \\ &= \mathfrak{R}_\varepsilon + \sum_{i=1}^2 \int_{\mathbb{C}_t} \partial_x \left(\frac{1}{(z_0 - x)_\varepsilon} \langle V_{\gamma e_i, \varepsilon}(x) \mathbf{V}_\varepsilon \rangle_t \right) d^2x \end{aligned}$$

where the remainder term is given by

$$\begin{aligned} \mathfrak{R}_\varepsilon := & \frac{1}{2} \sum_{k=1}^N \frac{1}{(z_0 - z_k)^2} \left[\langle Q, \alpha_k \rangle \left(\frac{(z_0 - z_k)^2}{(z_0 - z_k)_\varepsilon^2} - 1 \right) - \frac{1}{2} \langle \alpha_k, \alpha_k \rangle \left(\frac{(z_0 - z_k)^2}{(z_0 - z_k)_\varepsilon (z_0 - z_k)_\varepsilon} - 1 \right) \right] \langle \mathbf{V}_\varepsilon \rangle_t \\ & - \frac{1}{4} \sum_{k \neq l} \frac{\langle \alpha_k, \alpha_l \rangle}{(z_0 - z_k)(z_0 - z_l)} \left(\frac{(z_0 - z_k)(z_0 - z_l)}{(z_0 - z_k)_\varepsilon (z_0 - z_l)_\varepsilon} - 1 + \frac{z_k - z_l}{(z_k - z_l)_\varepsilon} - 1 \right) \langle \mathbf{V}_\varepsilon \rangle_t \\ & - \sum_{i=1}^2 \int_{\mathbb{C}_t} \frac{\gamma^2}{2(z_0 - x)^2} \left(1 - \frac{(z_0 - x)^2}{(z_0 - x)_\varepsilon (z_0 - x)_\varepsilon} \right) \langle V_{\gamma e_i, \varepsilon}(x) \mathbf{V}_\varepsilon \rangle_t d^2x \\ & - \sum_{i=1}^2 \int_{\mathbb{C}_t} \frac{\gamma}{2(z_0 - x)^2} \left(\sum_{k=1}^N \frac{(z_0 - x) \langle \alpha_k, e_i \rangle}{(z_0 - z_k)} \left(\frac{z_k - x}{(z_k - x)_\varepsilon} - 1 \right) \right) \langle V_{\gamma e_i, \varepsilon}(x) \mathbf{V}_\varepsilon \rangle_t d^2x \\ & - \sum_{i=1}^2 \int_{\mathbb{C}_t} \sum_{k=1}^N \frac{\langle \alpha_k, \gamma e_i \rangle}{2(z_0 - x)(z_k - x)_\varepsilon} \left(1 - \frac{z_0 - x}{(z_0 - x)_\varepsilon} \right) \langle V_{\gamma e_i, \varepsilon}(x) \mathbf{V}_\varepsilon \rangle_t d^2x + \mathfrak{R}_\varepsilon^1. \end{aligned}$$

The $\varepsilon \rightarrow 0$ limit of this remainder term is equal to zero via Lemma 4.4.5. Using Green's formula allows to infer that

$$\langle \mathbf{T}(z_0) \mathbf{V} \rangle_t - \sum_{k=1}^N \left(\frac{\Delta_{\alpha_k}}{(z_0 - z_k)^2} + \frac{\partial_{z_k}}{(z_0 - z_k)} \right) \langle \mathbf{V} \rangle_t = \sum_{i=1}^2 \oint_{\partial B_{e-t}(z_0)} \frac{1}{(z_0 - x)} \langle V_{\gamma e_i}(x) \mathbf{V} \rangle_t d\bar{x}.$$

By continuity of the correlation functions as $x \rightarrow z_0$ we see that the latter integral is asymptotically equivalent to

$$\sum_{i=1}^2 \langle V_{\gamma e_i}(z_0) \mathbf{V} \rangle \oint_{\partial B_{e-t}(z_0)} \frac{1}{(z_0 - x)} d\bar{x} = 0.$$

Therefore we can conclude that

$$\lim_{t \rightarrow +\infty} \lim_{\varepsilon \rightarrow 0} \left(\langle \mathbf{T}_\varepsilon(z_0) \mathbf{V}_\varepsilon \rangle_t - \sum_{k=1}^N \left(\frac{\Delta_{\alpha_k}}{(z_0 - z_k)^2} + \frac{\partial_{z_k}}{(z_0 - z_k)} \right) \langle \mathbf{V}_\varepsilon \rangle_t \right) = 0.$$

Since the correlation functions are differentiable in z_0 and all z_k , according to Proposition 4.4.7, we already know that the $t \rightarrow +\infty, \varepsilon \rightarrow 0$ limit of $\sum_{k=1}^N \left(\frac{\Delta_{\alpha_k}}{(z_0 - z_k)^2} + \frac{\partial_{z_k}}{(z_0 - z_k)} \right) \langle \mathbf{V}_\varepsilon \rangle_t$ exists. This entails the existence of $\lim_{t \rightarrow +\infty} \lim_{\varepsilon \rightarrow 0} \langle \mathbf{T}_\varepsilon(z_0) \mathbf{V}_\varepsilon \rangle_t$, thus proving Lemma 5.1.1 for the stress-energy tensor. This also shows that the spin-two Ward identity as stated in Theorem 5.1.3 holds true.

5.1.3 . Computation of the spin-three Ward identity

We proceed in a similar way for the spin-three Ward identity, the main difference being that the algebraic computations need to be done are a bit more involved. We start by considering the regularized version of the probabilistic objects

$$\langle \mathbf{W}_\varepsilon(z_0) \mathbf{V}_\varepsilon \rangle_t = q^2 \langle \langle h_2, \partial^3 \Phi_\varepsilon(z_0) \rangle \mathbf{V}_\varepsilon \rangle_t - 2q \langle \langle h_2 - h_1, \partial^2 \Phi_\varepsilon(z_0) \rangle \langle h_1, \partial \Phi_\varepsilon(z_0) \rangle : \mathbf{V}_\varepsilon \rangle_t - 2q \langle \langle h_3 - h_2, \partial^2 \Phi_\varepsilon(z_0) \rangle \langle h_3, \partial \Phi_\varepsilon(z_0) \rangle : \mathbf{V}_\varepsilon \rangle_t - 8 \langle \langle h_1, \partial \Phi_\varepsilon(z_0) \rangle \langle h_2, \partial \Phi_\varepsilon(z_0) \rangle \langle h_3, \partial \Phi_\varepsilon(z_0) \rangle : \mathbf{V}_\varepsilon \rangle_t.$$

Applying Lemma 4.4.6 and more generally Equation (4.4.8) we see that this quantity is given by

$$\langle \mathbf{W}_\varepsilon(z_0) \mathbf{V}_\varepsilon \rangle_t = q^2 \mathbf{I}_1 + \frac{q}{2} \mathbf{I}_2 + \mathbf{I}_3,$$

where the $(\mathbf{I}_i)_{i=1,2,3}$ correspond to the (rather lengthy) expressions:

$$\begin{aligned} \mathbf{I}_1 &= - \sum_{k=1}^N \frac{\langle h_2, \alpha_k \rangle}{(z_0 - z_k)_\varepsilon^3} \langle \mathbf{V}_\varepsilon \rangle_t + \sum_{i=1}^2 \int_{\mathbb{C}_t} \frac{\langle h_2, \gamma e_i \rangle}{(z_0 - x)_\varepsilon^3} \langle V_{\gamma e_i, \varepsilon}(x) \mathbf{V}_\varepsilon \rangle_t d^2 x, \\ \mathbf{I}_2 &= \sum_{k,l=1}^N \frac{B(\alpha_k, \alpha_l)}{(z_0 - z_k)_\varepsilon^2 (z_0 - z_l)_\varepsilon} \langle \mathbf{V}_\varepsilon \rangle_t \\ &\quad - \sum_{i=1}^2 \int_{\mathbb{C}_t} \left(\sum_{k=1}^N \frac{B(\alpha_k, \gamma e_i)}{(z_0 - z_k)_\varepsilon^2 (z_0 - x)_\varepsilon} + \frac{B(\gamma e_i, \alpha_k)}{(z_0 - x)_\varepsilon^2 (z_0 - z_k)_\varepsilon} + \frac{B(\gamma e_i, \gamma e_i)}{(z_0 - x)_\varepsilon^2 (z_0 - x)_\varepsilon} \right) \langle V_{\gamma e_i, \varepsilon}(x) \mathbf{V}_\varepsilon \rangle_t d^2 x \\ &\quad + \sum_{i,j=1}^2 \int_{\mathbb{C}_t^2} \frac{B(\gamma e_i, \gamma e_j)}{(z_0 - x_1)_\varepsilon^2 (z_0 - x_2)_\varepsilon} \langle V_{\gamma e_i, \varepsilon}(x_1) V_{\gamma e_j, \varepsilon}(x_2) \mathbf{V}_\varepsilon \rangle_t d^2 x_1 d^2 x_2 \quad \text{and} \\ \mathbf{I}_3 &= \sum_{k,l,p=1}^N \frac{\langle h_1, \alpha_k \rangle \langle h_2, \alpha_l \rangle \langle h_3, \alpha_p \rangle}{(z_0 - z_k)_\varepsilon (z_0 - z_l)_\varepsilon (z_0 - z_p)_\varepsilon} \langle \mathbf{V}_\varepsilon \rangle_t \\ &\quad - \sum_{i=1}^2 \int_{\mathbb{C}_t} \left(\sum_{k,l=1}^N \frac{C(\alpha_k, \alpha_l, \gamma e_i)}{(z_0 - z_k)_\varepsilon (z_0 - z_l)_\varepsilon (z_0 - x)_\varepsilon} + \sum_{k=1}^N \frac{C(\alpha_k, \gamma e_i, \gamma e_i)}{(z_0 - z_k)_\varepsilon (z_0 - x)_\varepsilon (z_0 - x)_\varepsilon} \right) \langle V_{\gamma e_i, \varepsilon}(x) \mathbf{V}_\varepsilon \rangle_t d^2 x \\ &\quad + \sum_{i,j=1}^2 \int_{\mathbb{C}_t^2} \left(\sum_{k=1}^N \frac{C(\alpha_k, \gamma e_i, \gamma e_j)}{(z_0 - z_k)_\varepsilon (z_0 - x_1)_\varepsilon (z_0 - x_2)_\varepsilon} + \frac{C(\gamma e_i, \gamma e_i, \gamma e_j)}{(z_0 - x_1)_\varepsilon (z_0 - x_1)_\varepsilon (z_0 - x_2)_\varepsilon} \right) \\ &\quad \quad \langle V_{\gamma e_i, \varepsilon}(x_1) V_{\gamma e_j, \varepsilon}(x_2) \mathbf{V}_\varepsilon \rangle_t d^2 x_1 d^2 x_2 \\ &\quad - \sum_{i,j,f=1}^2 \int_{\mathbb{C}_t^3} \frac{C(\gamma e_i, \gamma e_j, \gamma e_f)}{3(z_0 - x_1)_\varepsilon (z_0 - x_2)_\varepsilon (z_0 - x_3)_\varepsilon} \langle V_{\gamma e_i, \varepsilon}(x_1) V_{\gamma e_j, \varepsilon}(x_2) V_{\gamma e_f, \varepsilon}(x_3) \mathbf{V}_\varepsilon \rangle_t d^2 x_1 d^2 x_2 d^2 x_3. \end{aligned}$$

We proceed in the same way with the right-hand side in (5.1.14). Then we can write that $\mathcal{W}_{-1}^{(k)} \langle \mathbf{V}_\varepsilon \rangle_t = \mathbf{J}_{-1}^{(k)}$ where this notation stands for

$$\begin{aligned} \mathbf{J}_{-1}^{(k)} &= \frac{1}{2} \sum_{l \neq k} \frac{qB(\alpha_k, \alpha_l) + 2C(\alpha_k, \alpha_k, \alpha_l)}{(z_k - z_l)_\varepsilon} \langle \mathbf{V}_\varepsilon \rangle_t \\ &\quad - \frac{\mu}{2} \sum_{i=1}^2 \int_{\mathbb{C}_t} \frac{qB(\alpha_k, \gamma e_i) + 2C(\alpha_k, \alpha_k, \gamma e_i)}{(z_k - x)_\varepsilon} \langle V_{\gamma e_i, \varepsilon}(z) \mathbf{V}_\varepsilon \rangle_t d^2 x. \end{aligned}$$

Likewise we can write that $\mathcal{W}_{-2}^{(k)} \langle \mathbf{V}_\varepsilon \rangle_t = J_{-2}^{(k)}$ with

$$\begin{aligned}
J_{-2}^{(k)} = & \frac{1}{2} \sum_{l \neq k} \frac{q(B(\alpha_l, \alpha_k) - B(\alpha_k, \alpha_l)) + 2C(\alpha_k, \alpha_l - \alpha_k, \alpha_l)}{(z_k - z_l)_\varepsilon^2} \langle \mathbf{V}_\varepsilon \rangle_t \\
& + \sum_{k, l, p \text{ distinct}} \frac{C(\alpha_k, \alpha_l, \alpha_p)}{(z_k - z_l)_\varepsilon (z_k - z_p)_\varepsilon} \langle \mathbf{V}_\varepsilon \rangle_t \\
& - \sum_{i=1}^2 \int_{\mathbb{C}_t} \left(\frac{q(B(\gamma_{e_i}, \alpha_k) - B(\alpha_k, \gamma_{e_i})) + 2C(\alpha_k, \gamma_{e_i} - \alpha_k, \gamma_{e_i})}{2(z_k - x)_\varepsilon^2} \right) \langle V_{\gamma_{e_i}, \varepsilon}(z) \mathbf{V}_\varepsilon \rangle_t d^2 x \\
& - \sum_{i=1}^2 \int_{\mathbb{C}_t} \left(\sum_{l \neq k} \frac{C^\sigma(\alpha_k, \alpha_l, \gamma_{e_i})}{(z_k - z_l)_\varepsilon (z_k - x)_\varepsilon} \right) \langle V_{\gamma_{e_i}, \varepsilon}(z) \mathbf{V}_\varepsilon \rangle_t d^2 x \\
& + \sum_{i, j=1}^2 \int_{\mathbb{C}_t^2} \frac{C(\alpha_k, \gamma_{e_i}, \gamma_{e_j})}{(z_k - x_1)_\varepsilon (z_k - x_2)_\varepsilon} \langle V_{\gamma_{e_i}, \varepsilon}(x_1) V_{\gamma_{e_j}, \varepsilon}(x_2) \mathbf{V}_\varepsilon \rangle_t d^2 x_1 d^2 x_2
\end{aligned}$$

and $C^\sigma(\alpha_k, \alpha_l, e_i) := C(\alpha_k, \alpha_l, e_i) + C(\alpha_k, e_i, \alpha_l)$.

5.1.3.1 First step: symmetrizing

We now turn to the first step of the proof which consists in using symmetrization identities so that in the expression of the terms $J_{-1,2}^{(k)}$ the only singularities that will occur will do so around the distinguished point z_0 . To illustrate this, let us consider the first term $J_{-1}^{(k)}$. Using the “symmetrization identity”

$$\frac{1}{(z-y)^2(y-x)} = \frac{1}{(z-y)^2(z-x)} + \frac{1}{(z-y)(z-x)^2} + \frac{1}{(z-x)^2(y-x)}$$

we can write that $\frac{J_{-1}^{(k)}}{(z_0 - z_k)^2}$ is given by

$$\begin{aligned}
& \sum_{l \neq k} \frac{qB(\alpha_k, \alpha_l) + 2C(\alpha_k, \alpha_k, \alpha_l)}{2(z_0 - z_k)^2(z_0 - z_l)} \langle \mathbf{V}_\varepsilon \rangle_t + \sum_{l \neq k} \frac{qB(\alpha_k, \alpha_l) + 2C(\alpha_k, \alpha_k, \alpha_l)}{2(z_0 - z_k)(z_k - z_l)(z_k - z_l)_\varepsilon} \langle \mathbf{V}_\varepsilon \rangle_t \\
& - \sum_{l \neq k} \frac{qB(\alpha_k, \alpha_l) + 2C(\alpha_k, \alpha_k, \alpha_l)}{2(z_0 - z_l)(z_k - z_l)(z_k - z_l)_\varepsilon} \langle \mathbf{V}_\varepsilon \rangle_t \\
& - \sum_{i=1}^2 \int_{\mathbb{C}_t} \left(\frac{qB(\alpha_k, \gamma_{e_i}) + 2C(\alpha_k, \alpha_k, \gamma_{e_i})}{2(z_0 - z_k)^2(z_0 - x)} + \frac{qB(\alpha_k, \gamma_{e_i}) + 2C(\alpha_k, \alpha_k, \gamma_{e_i})}{2(z_0 - z_k)(z_0 - x)^2} \right) \langle V_{\gamma_{e_i}, \varepsilon}(z) \mathbf{V}_\varepsilon \rangle_t d^2 x \\
& - \sum_{i=1}^2 \int_{\mathbb{C}_t} \frac{qB(\alpha_k, \gamma_{e_i}) + 2C(\alpha_k, \alpha_k, \gamma_{e_i})}{2(z_0 - x)^2(z_k - x)_\varepsilon} \langle V_{\gamma_{e_i}, \varepsilon}(z) \mathbf{V}_\varepsilon \rangle_t d^2 x + \mathfrak{R}_{-1}^\varepsilon
\end{aligned}$$

with

$$\begin{aligned}
\mathfrak{R}_{-1}^\varepsilon := & \sum_{l \neq k} \frac{qB(\alpha_k, \alpha_l) + 2C(\alpha_k, \alpha_k, \alpha_l)}{2(z_0 - z_k)^2} \left(\frac{1}{(z_k - z_l)_\varepsilon} - \frac{1}{z_k - z_l} \right) \langle \mathbf{V}_\varepsilon \rangle_t \\
& - \sum_{i=1}^2 \int_{\mathbb{C}_t} \frac{qB(\alpha_k, \gamma_{e_i}) + 2C(\alpha_k, \alpha_k, \gamma_{e_i})}{2(z_0 - z_k)(z_0 - x)} \left(\frac{1}{z_0 - x} + \frac{1}{z_0 - z_k} \right) \left(\frac{z_k - x}{(z_k - x)_\varepsilon} - 1 \right) \langle V_{\gamma_{e_i}, \varepsilon}(z) \mathbf{V}_\varepsilon \rangle_t d^2 x.
\end{aligned}$$

Let us now turn to $J_{-2}^{(k)}$: its study is more involved but nonetheless the very same method still works. In order to keep things simple we explain how to treat the term involving a two-fold integral. The symmetrization method yields

$$\begin{aligned} & \frac{1}{z_0 - z_k} \sum_{i,j=1}^2 \int_{\mathbb{C}_t^2} \frac{C(\alpha_k, e_i, e_j)}{(z_k - x_1)_\varepsilon (z_k - x_2)_\varepsilon} \langle V_{\gamma e_i, \varepsilon}(x_1) V_{\gamma e_j, \varepsilon}(x_2) \mathbf{V}_\varepsilon \rangle_t d^2 x_1 d^2 x_2 = \\ & \sum_{i,j=1}^2 \int_{\mathbb{C}_t^2} \frac{C(\alpha_k, e_i, e_j)}{(z_0 - z_k)(z_0 - x_1)_\varepsilon (z_0 - x_2)_\varepsilon} \langle V_{\gamma e_i, \varepsilon}(x_1) V_{\gamma e_j, \varepsilon}(x_2) \mathbf{V}_\varepsilon \rangle_t d^2 x_1 d^2 x_2 \\ & + \sum_{i,j=1}^2 \int_{\mathbb{C}_t^2} \frac{C(\alpha_k, e_i, e_j) + C(\alpha_k, e_j, e_i)}{(z_0 - x_1)(x_1 - x_2)_\varepsilon (z_k - x_1)_\varepsilon} \langle V_{\gamma e_i, \varepsilon}(x_1) V_{\gamma e_j, \varepsilon}(x_2) \mathbf{V}_\varepsilon \rangle_t d^2 x_1 d^2 x_2 + \mathfrak{R}_{-2}^\varepsilon, \end{aligned}$$

where $\mathfrak{R}_{-2}^\varepsilon$ is defined by

$$\begin{aligned} & \sum_{i,j=1}^2 \int_{\mathbb{C}_t^2} \frac{C(\alpha_k, e_i, e_j)}{(z_0 - z_k)(z_0 - x_1)(z_0 - x_2)} \left(\frac{x_1 - x_2}{(x_1 - x_2)_\varepsilon} - 1 \right) \langle V_{\gamma e_i, \varepsilon}(x_1) V_{\gamma e_j, \varepsilon}(x_2) \mathbf{V}_\varepsilon \rangle_t d^2 x_1 d^2 x_2 \\ & + \sum_{i,j=1}^2 \int_{\mathbb{C}_t^2} \frac{C(\alpha_k, e_i, e_j)}{(z_0 - z_k)} \left(\frac{1}{(z_k - x_1)_\varepsilon (z_k - x_2)_\varepsilon} \right. \\ & \quad \left. - \frac{1}{(x_1 - x_2)_\varepsilon} \left(\frac{1}{(z_k - x_1)_\varepsilon} - \frac{1}{(z_k - x_2)_\varepsilon} \right) \right) \langle V_{\gamma e_i, \varepsilon}(x_1) V_{\gamma e_j, \varepsilon}(x_2) \mathbf{V}_\varepsilon \rangle_t d^2 x_1 d^2 x_2 \\ & + \sum_{i,j=1}^2 \int_{\mathbb{C}_t^2} \frac{C(\alpha_k, e_i, e_j) + C(\alpha_k, e_j, e_i)}{(z_0 - z_k)(z_0 - x_1)(x_1 - x_2)_\varepsilon} \langle V_{\gamma e_i, \varepsilon}(x_1) V_{\gamma e_j, \varepsilon}(x_2) \mathbf{V}_\varepsilon \rangle_t d^2 x_1 d^2 x_2. \end{aligned}$$

Similarly it will be convenient in the sequel to note that, using the symmetries between the integration variables and since the quantities $C(e_i, e_j, e_f)$ are explicit for $i, j, f \in \{1, 2\}$, elementary but lengthy algebraic computations allow to write

$$\begin{aligned} & \sum_{i,j,f=1}^2 \int_{\mathbb{C}_t^3} \frac{C(e_i, e_j, e_f)}{3(z_0 - x_1)(z_0 - x_2)(z_0 - x_3)} \langle V_{\gamma e_i, \varepsilon}(x_1) V_{\gamma e_j, \varepsilon}(x_2) V_{\gamma e_f, \varepsilon}(x_3) \mathbf{V}_\varepsilon \rangle_t d^2 x_1 d^2 x_2 d^2 x_3 \\ & = - \sum_{i \neq j} \int_{\mathbb{C}_t^3} \frac{\langle h_2, e_i \rangle}{(z_0 - x_1)(x_1 - x_2)_\varepsilon (x_1 - x_3)_\varepsilon} \langle V_{\gamma e_i, \varepsilon}(x_1) V_{\gamma e_i, \varepsilon}(x_2) V_{\gamma e_j, \varepsilon}(x_3) \mathbf{V}_\varepsilon \rangle_t d^2 x_1 d^2 x_2 d^2 x_3 + \mathfrak{R}_{-3}^\varepsilon \end{aligned}$$

where $\mathfrak{R}_{-3}^\varepsilon$ is similar to $\mathfrak{R}_{-2}^\varepsilon$. Collecting all the terms and using the identity based on the explicit expression for B and C

$$q(B(\gamma e_i, \alpha_k) - B(\alpha_k, \gamma e_i)) + 2C(\alpha_k, \gamma e_i - \alpha_k, \gamma e_i) = 4\langle h_2, e_i \rangle \omega_{3-i}(\alpha_k) \left(1 + \frac{1}{2} \langle \alpha_k, \gamma e_i \rangle \right)$$

(where $\omega_{3-i}(\alpha_k)$ is a shorthand for $\langle \omega_{3-i}, \alpha_k \rangle$) we can write that

$$\begin{aligned}
& q^2 I_1 + \frac{q}{2} I_2 + I_3 - \sum_{k=1}^N \left(\frac{w^{(3)}(\alpha_k)}{(z_0 - z_k)^3} + \frac{J_{-1}^{(k)}}{(z_0 - z_k)^2} + \frac{J_{-2}^{(k)}}{z_0 - z_k} \right) = \\
& + \sum_{i=1}^2 \int_{\mathbb{C}_t} \frac{2q \langle h_2, e_i \rangle}{(z_0 - x)^3} \langle V_{\gamma e_i, \varepsilon}(x) \mathbf{V}_\varepsilon \rangle_t d^2 x \\
& + \sum_{i=1}^2 \int_{\mathbb{C}_t} \sum_{k=1}^N \frac{qB(\gamma e_i, \alpha_k) + 2C(\alpha_k, \gamma e_i, \gamma e_i)}{2(z_0 - x)^2(z_k - x)_\varepsilon} \langle V_{\gamma e_i, \varepsilon}(z) \mathbf{V}_\varepsilon \rangle_t d^2 x \\
& + 2 \sum_{i=1}^2 \int_{\mathbb{C}_t} \sum_{k=1}^N \frac{\langle h_2, e_i \rangle \omega_{3-i}(\alpha_k)}{(z_0 - x)(z_k - x)_\varepsilon} \left(\frac{1 + \frac{1}{2} \langle \alpha_k, \gamma e_i \rangle}{(z_k - x)_\varepsilon} + \sum_{l \neq k} \frac{\frac{1}{2} \langle \alpha_l, \gamma e_i \rangle}{(z_l - x)_\varepsilon} \right) \langle V_{\gamma e_i, \varepsilon}(z) \mathbf{V}_\varepsilon \rangle_t d^2 x \\
& + \sum_{i,j=1}^2 \int_{\mathbb{C}_t^2} \frac{qB(\gamma e_i, \gamma e_j) + 2C(\gamma e_i, \gamma e_i, \gamma e_j)}{2(z_0 - x_1)^2(x_1 - x_2)_\varepsilon} \langle V_{\gamma e_i, \varepsilon}(x_1) V_{\gamma e_j, \varepsilon}(x_2) \mathbf{V}_\varepsilon \rangle_t d^2 x_1 d^2 x_2 \\
& - 2 \sum_{i \neq j}^2 \int_{\mathbb{C}_t^2} \frac{\langle h_2, \gamma e_i \rangle}{(z_0 - x_1)(x_1 - x_2)_\varepsilon^2} \langle V_{\gamma e_i, \varepsilon}(x_1) V_{\gamma e_j, \varepsilon}(x_2) \mathbf{V}_\varepsilon \rangle_t d^2 x_1 d^2 x_2 \\
& - \sum_{i,j=1}^2 \int_{\mathbb{C}_t^2} \sum_{k=1}^N \frac{C^\sigma(\alpha_k, \gamma e_i, \gamma e_j)}{(z_0 - x_1)(x_1 - x_2)_\varepsilon(z_k - x_1)_\varepsilon} \langle V_{\gamma e_i, \varepsilon}(x_1) V_{\gamma e_j, \varepsilon}(x_2) \mathbf{V}_\varepsilon \rangle_t d^2 x_1 d^2 x_2 \\
& \sum_{i \neq j} \int_{\mathbb{C}_t^3} \frac{\gamma^3 \langle h_2, e_i \rangle}{(z_0 - x_1)(x_1 - x_2)_\varepsilon(x_2 - x_3)_\varepsilon} \langle V_{\gamma e_i, \varepsilon}(x_1) V_{\gamma e_i, \varepsilon}(x_2) V_{\gamma e_j, \varepsilon}(x_3) \mathbf{V}_\varepsilon \rangle_t d^2 x_1 d^2 x_2 d^2 x_3 \\
& + \mathfrak{R}^\varepsilon
\end{aligned}$$

where $\mathfrak{R}^\varepsilon$ is a combination of the $\mathfrak{R}_{-i}^\varepsilon$, $1 \leq i \leq 3$. At first we are concerned with the $\varepsilon \rightarrow 0$ limit of this remainder. There are two terms for which it is not obvious that they converge to zero (here F is some bounded function):

$$\begin{aligned}
& \int_{\mathbb{C}_t} \frac{F(x)}{(x - z_k)_\varepsilon} \left(\frac{x - z_k}{(x - z_k)_\varepsilon} - 1 \right) \langle V_{\gamma e_i, \varepsilon}(x) \mathbf{V}_\varepsilon \rangle_t d^2 x, \quad \text{and} \\
& \sum_{i,j=1}^2 \int_{\mathbb{C}_t^2} \frac{C(\alpha_k, e_i, e_j)}{(z_0 - z_k)} \langle V_{\gamma e_i, \varepsilon}(x_1) V_{\gamma e_j, \varepsilon}(x_2) \mathbf{V}_\varepsilon \rangle_t d^2 x_1 d^2 x_2 \\
& \left(\frac{1}{(z_k - x_1)_\varepsilon(z_k - x_2)_\varepsilon} - \frac{1}{(x_1 - x_2)_\varepsilon} \left(\frac{1}{(z_k - x_1)_\varepsilon} - \frac{1}{(z_k - x_2)_\varepsilon} \right) \right).
\end{aligned}$$

However making the change of variables $x \leftrightarrow 4R\varepsilon x$ (where recall R from the regularization scheme (3.2.14)) and using the same reasoning as in the proof of Lemma 4.4.5 we see that actually such quantities will vanish in the $\varepsilon \rightarrow 0$ limit. Therefore $\lim_{\varepsilon \rightarrow 0} \mathfrak{R}^\varepsilon = 0$ so we may no longer consider it in subsequent computations.

5.1.3.2 Second step: integrating by parts

We are now in position to address the second step of our proof, that is to use integration by parts in order to remove the singular terms that occur away from z_0 . Indeed we can

notice that

$$\begin{aligned} \partial_x \left(\frac{1}{(z_x - x)_\varepsilon} \langle V_{\gamma e_i, \varepsilon}(x) \mathbf{V}_\varepsilon \rangle_t \right) &= \left(\frac{1 + \frac{1}{2} \langle \alpha_k, \gamma e_i \rangle}{(z_k - x)_\varepsilon^2} + \sum_{l \neq k} \frac{\frac{1}{2} \langle \alpha_l, \gamma e_i \rangle}{(z_k - x)_\varepsilon (z_l - x)_\varepsilon} \right) \langle V_{\gamma e_i, \varepsilon}(x) \mathbf{V}_\varepsilon \rangle_t \\ &\quad + \sum_{j=1}^2 \int_{\mathbb{C}_t} \frac{\langle \gamma e_i, \gamma e_j \rangle}{2(z_k - x)_\varepsilon (x - x_2)_\varepsilon} \langle V_{\gamma e_i, \varepsilon}(x) V_{\gamma e_j, \varepsilon}(x_2) \mathbf{V}_\varepsilon \rangle_t d^2 x_2. \end{aligned}$$

This implies, using integration by parts (thanks to Lemma 4.4.2 this is indeed possible), that

$$\begin{aligned} &2 \sum_{i=1}^2 \int_{\mathbb{C}_t} \sum_{k=1}^N \frac{\langle h_2, e_i \rangle \omega_{3-i}(\alpha_k)}{(z_0 - x)(z_k - x)_\varepsilon} \left(\frac{1 + \frac{1}{2} \langle \alpha_k, \gamma e_i \rangle}{(z_k - x)_\varepsilon} + \sum_{l \neq k} \frac{\frac{1}{2} \langle \alpha_l, \gamma e_i \rangle}{(z_l - x)_\varepsilon} \right) \langle V_{\gamma e_i, \varepsilon}(x) \mathbf{V}_\varepsilon \rangle_t d^2 x \\ &= 2 \sum_{i=1}^2 \oint_{\partial B_{e-t}(z_0)} \sum_{k=1}^N \frac{\langle h_2, e_i \rangle \omega_{3-i}(\alpha_k)}{(z_0 - x)(z_k - x)_\varepsilon} \langle V_{\gamma e_i, \varepsilon}(x) \mathbf{V}_\varepsilon \rangle_t d\bar{x} \\ &\quad - 2 \sum_{i=1}^2 \int_{\mathbb{C}_t} \sum_{k=1}^N \frac{\langle h_2, e_i \rangle \omega_{3-i}(\alpha_k)}{(z_0 - x)^2 (z_k - x)_\varepsilon} \langle V_{\gamma e_i, \varepsilon}(x) \mathbf{V}_\varepsilon \rangle_t d^2 x \\ &\quad - \sum_{i,j=1}^2 \int_{\mathbb{C}_t^2} \sum_{k=1}^N \frac{\langle h_2, e_i \rangle \langle \gamma e_i, \gamma e_j \rangle \omega_{3-i}(\alpha_k)}{(z_0 - x_1)(z_k - x_1)_\varepsilon (x_1 - x_2)_\varepsilon} \langle V_{\gamma e_i, \varepsilon}(x_1) V_{\gamma e_j, \varepsilon}(x_2) \mathbf{V}_\varepsilon \rangle_t d^2 x_1 d^2 x_2. \end{aligned}$$

Since $B(e_i, \alpha_k) + 2C(\alpha_k, e_i, e_i) = 2\langle h_2, e_i \rangle \omega_i(\alpha_k) + \langle h_2, e_i \rangle \omega_{3-i}(\alpha_k) = \langle h_2, e_i \rangle \langle e_i, \alpha_k \rangle$, we can use again integration by parts to get that

$$\begin{aligned} &\sum_{i=1}^2 \int_{\mathbb{C}_t} \sum_{k=1}^N \frac{qB(\gamma e_i, \alpha_k) + 2C(\alpha_k, \gamma e_i, \gamma e_i)}{2(z_0 - x)^2 (z_k - x)_\varepsilon} \langle V_{\gamma e_i, \varepsilon}(x) \mathbf{V}_\varepsilon \rangle_t d^2 x \\ &+ 2 \sum_{i=1}^2 \int_{\mathbb{C}_t} \sum_{k=1}^N \frac{\langle h_2, e_i \rangle \omega_{3-i}(\alpha_k)}{(z_0 - x)(z_k - x)_\varepsilon} \left(\frac{1 + \frac{1}{2} \langle \alpha_k, \gamma e_i \rangle}{(z_k - x)_\varepsilon} + \sum_{l \neq k} \frac{\frac{1}{2} \langle \alpha_l, \gamma e_i \rangle}{(z_l - x)_\varepsilon} \right) \langle V_{\gamma e_i, \varepsilon}(x) \mathbf{V}_\varepsilon \rangle_t d^2 x \end{aligned}$$

is actually equal to

$$\begin{aligned} &\sum_{i=1}^2 \int_{\partial B_{e-t}(z_0)} \frac{\langle h_2, e_i \rangle}{z_0 - x} \left(\frac{q}{z_0 - x} + \sum_{k=1}^N \frac{2\langle \alpha_k, \omega_{3-i} \rangle}{(z_k - x)_\varepsilon} \right) \langle V_{\gamma e_i, \varepsilon}(x) \mathbf{V}_\varepsilon \rangle_t d\bar{x} \\ &- \sum_{i=1}^2 \int_{\mathbb{C}_t} \frac{2q\langle h_2, e_i \rangle}{(z_0 - x)^3} \langle V_{\gamma e_i, \varepsilon}(x) \mathbf{V}_\varepsilon \rangle_t d^2 x \\ &- \sum_{i,j=1}^2 \int_{\mathbb{C}_t^2} \frac{\langle h_2, e_i \rangle \langle \gamma e_i, \gamma e_j \rangle}{(z_0 - x_1)(x_1 - x_2)_\varepsilon} \left(\frac{q}{2(z_0 - x_1)} + \sum_{k=1}^N \frac{\langle \alpha_k, \omega_{3-i} \rangle}{(z_k - x_1)_\varepsilon} \right) \langle V_{\gamma e_i, \varepsilon}(x_1) V_{\gamma e_j, \varepsilon}(x_2) \mathbf{V}_\varepsilon \rangle_t d^2 x_1 d^2 x_2. \end{aligned}$$

As a consequence we see that when taking the $\varepsilon \rightarrow$ limit of the expression

$$q^2 I_1 + \frac{q}{2} I_2 + I_3 - \sum_{k=1}^N \left(\frac{w^{(3)}(\alpha_k)}{(z_0 - z_k)^3} + \frac{J_{-1}^{(k)}}{(z_0 - z_k)^2} + \frac{J_{-2}^{(k)}}{z_0 - z_k} \right)$$

the only remaining terms are the 1-fold integral given by

$$- \sum_{i=1}^2 \int_{\partial B_{e-t}(z_0)} \frac{\langle h_2, e_i \rangle}{z_0 - x} \left(\frac{q}{z_0 - x} + \sum_{k=1}^N \frac{2\langle \alpha_k, \omega_{3-i} \rangle}{(z_k - x)_\varepsilon} \right) \langle V_{\gamma e_i, \varepsilon}(x) \mathbf{V}_\varepsilon \rangle_t d\bar{x}$$

as well as, using the definitions of B and C , the limit of the following 2 and 3-fold integrals:

$$\begin{aligned} & + 2 \sum_{i \neq j} \int_{\mathbb{C}^2} \frac{\langle h_2, \gamma e_i \rangle}{(z_0 - x_1)^2 (x_1 - x_2)_\varepsilon} \langle V_{\gamma e_i, \varepsilon}(x_1) V_{\gamma e_j, \varepsilon}(x_2) \mathbf{V}_\varepsilon \rangle_t d^2 x_1 d^2 x_2 \\ & - 2 \sum_{i \neq j} \int_{\mathbb{C}^2} \frac{\langle h_2, \gamma e_i \rangle}{(z_0 - x_1)(x_1 - x_2)_\varepsilon^2} \langle V_{\gamma e_i, \varepsilon}(x_1) V_{\gamma e_j, \varepsilon}(x_2) \mathbf{V}_\varepsilon \rangle_t d^2 x_1 d^2 x_2 \\ & + \sum_{i \neq j} \int_{\mathbb{C}^2} \frac{\langle h_2, \gamma e_i \rangle}{(z_0 - x_1)(x_1 - x_2)_\varepsilon} \sum_{k=1}^N \frac{\langle \alpha_k, \gamma e_i \rangle}{(z_k - x_1)_\varepsilon} \langle V_{\gamma e_i, \varepsilon}(x_1) V_{\gamma e_j, \varepsilon}(x_2) \mathbf{V}_\varepsilon \rangle_t d^2 x_1 d^2 x_2 \\ & + \sum_{i \neq j} \int_{\mathbb{C}^3} \frac{\gamma^3 \langle h_2, e_i \rangle}{(z_0 - x_1)(x_1 - x_2)_\varepsilon (x_1 - x_3)_\varepsilon} \langle V_{\gamma e_i, \varepsilon}(x_1) V_{\gamma e_i, \varepsilon}(x_2) V_{\gamma e_j, \varepsilon}(x_3) \mathbf{V}_\varepsilon \rangle_t d^2 x_1 d^2 x_2 d^2 x_3. \end{aligned}$$

To treat this remainder term we can use integration by parts in the same way as we have proceeded for the 1-fold integrals. For this we notice that the above quantity is nothing but

$$\begin{aligned} & - 2 \sum_{i \neq j} \int_{\mathbb{C}^2} \partial_{x_1} \left(\frac{\langle h_2, \gamma e_i \rangle}{(z_0 - x_1)(x_1 - x_2)_\varepsilon} \langle V_{\gamma e_i, \varepsilon}(x_1) V_{\gamma e_j, \varepsilon}(x_2) \mathbf{V}_\varepsilon \rangle_t \right) d^2 x_1 d^2 x_2 \\ & = - 2 \sum_{i, j=1}^2 \int_{\partial B_{e-t}(z_0) \times \mathbb{C}^2} \frac{\langle h_2, \gamma e_i \rangle}{(z_0 - x_1)(x_1 - x_2)_\varepsilon} \langle V_{\gamma e_i, \varepsilon}(x_1) V_{\gamma e_j, \varepsilon}(x_2) \mathbf{V}_\varepsilon \rangle_t d\bar{x}_1 d^2 x_2. \end{aligned}$$

5.1.3.3 Last step: taking the limit

Combining all the terms we see that

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \left(\langle \mathbf{W}_\varepsilon(z_0) \prod_{k=1}^N V_{\alpha_k, \varepsilon}(z_k) \rangle_t - \sum_{l=1}^N \left(\frac{w(\alpha_l)}{(z_0 - z_l)^3} + \frac{\mathbf{w}_{-1}^{(l)}}{(z_0 - z_l)^2} + \frac{\mathbf{w}_{-2}^{(l)}}{z_0 - z_l} \right) \langle \prod_{k=1}^N V_{\alpha_k, \varepsilon}(z_k) \rangle_t \right) \\ & = - \sum_{i=1}^2 \int_{\partial B_{e-t}(z_0)} \frac{\langle h_2, e_i \rangle}{z_0 - x} \left(\frac{q}{z_0 - x} + \sum_{k=1}^N \frac{2\langle \alpha_k, \omega_{3-i} \rangle}{(z_k - x)} \right) \langle V_{\gamma e_i}(x) \mathbf{V} \rangle_t d\bar{x} \\ & - 2 \sum_{i, j=1}^2 \int_{\partial B_{e-t}(z_0) \times \mathbb{C}^2} \frac{\langle h_2, \gamma e_i \rangle}{(z_0 - x_1)(x_1 - x_2)_\varepsilon} \langle V_{\gamma e_i}(x_1) V_{\gamma e_j}(x_2) \mathbf{V} \rangle_t d\bar{x}_1 d^2 x_2. \end{aligned}$$

In order to show that the remaining term vanishes as $t \rightarrow +\infty$ we will use regularity of the correlation functions, as stated in Proposition 4.4.7.

Indeed, let us start by considering the first one-fold integral, and consider the most problematic term:

$$\int_{\partial B_{e-t}(z_0)} \frac{1}{(z_0 - x)^2} F_t(x) d\bar{x}$$

with $F_t(x) := \langle V_{\gamma e_i}(x) \mathbf{V} \rangle_t$. Since the correlation function that appears in the integral is C^1 for $x \in \partial B_{e^{-t}}(z_0)$ for t large enough, we can write its Taylor expansion $F_t(e^{-t+i\theta} + z_0) = F_t(z_0) + e^{-t} (e^{i\theta} \partial_x F_t(z_0) + e^{-i\theta} \partial_{\bar{x}} F_t(z_0)) + o(t)$. This implies that the integral is governed by

$$e^t \int_0^{2\pi} (F_t(z_0) + e^{-t} (e^{i\theta} \partial_x F_t(z_0) + e^{-i\theta} \partial_{\bar{x}} F_t(z_0))) e^{-3i\theta} d\theta = 0.$$

As a consequence this term does vanish in the $t \rightarrow +\infty$ limit. The double integral is seen to vanish using the same reasoning, one subtlety here being that we have to consider what happens when x_1 and x_2 get close one to the other. However the same reasoning as the one we have described above combined with the one used in the proof of Proposition 4.4.7 allows to conclude in the same way. This shows that the following is true:

$$\lim_{t \rightarrow +\infty} \lim_{\varepsilon \rightarrow 0} \left(\langle \mathbf{W}_\varepsilon(z_0) \mathbf{V}_\varepsilon \rangle_t - \sum_{l=1}^N \left(\frac{w(\alpha_l)}{(z_0 - z_l)^3} + \frac{\mathcal{W}_{-1,\varepsilon}(z_l, \alpha_l)}{(z_0 - z_l)^2} + \frac{\mathcal{W}_{-2,\varepsilon}(z_l, \alpha_l)}{z_0 - z_l} \right) \langle \mathbf{V}_\varepsilon \rangle_t \right) = 0. \quad (5.1.16)$$

To finish up with the proof of our main result, it remains to show that the limiting quantities

$$\mathcal{W}_{-i}^{(k)} \langle \mathbf{V} \rangle = \lim_{t \rightarrow +\infty} \lim_{\varepsilon \rightarrow 0} \mathcal{W}_{-i}^{(l)} \langle \mathbf{V}_\varepsilon \rangle_t$$

do exist for $i = 1, 2$ and any $1 \leq k \leq N$. Convergence of $\mathcal{W}_{-1}^{(k)} \langle \mathbf{V}_\varepsilon \rangle_t$ follows from the very same argument that allows to prove the differentiability of the correlation functions. Namely we can use integration by parts in the exact same fashion to deduce that quantities of the form

$$\int_{\mathbb{C}} \frac{1}{(z_l - x)_\varepsilon} \langle V_{\gamma e_i, \varepsilon}(x) \mathbf{V}_\varepsilon \rangle d^2 x$$

that arise in Gaussian integration by parts do admit a limit when $\varepsilon \rightarrow 0$. As for $\mathcal{W}_{-2,\varepsilon}(z_l, \alpha_l) \langle \mathbf{V}_\varepsilon \rangle_t$, we recall that

$$q(B(\gamma e_i, \alpha) - B(\alpha, \gamma e_i)) + 2C(\alpha, \gamma e_i - \alpha, \gamma e_i) = 4 \langle h_2, e_i \rangle \omega_{3-i}(\alpha) \left(1 + \frac{1}{2} \langle \alpha, \gamma e_i \rangle \right).$$

As a consequence symmetrization identities allow to rewrite the integrals in $\mathcal{W}_{-2}^{(k)} \langle \mathbf{V}_\varepsilon \rangle$ as

$$\begin{aligned} & - \sum_{i=1}^2 \int_{\mathbb{C}} \sum_{l \neq k} \left(\frac{C^\sigma(\alpha_k, \alpha_l, \gamma e_i)}{(z_k - z_l)_\varepsilon (z_k - x)_\varepsilon} - \frac{\langle h_2, e_i \rangle \omega_{3-i}(\alpha_k) \langle \alpha_l, \gamma e_i \rangle}{(z_l - x)_\varepsilon (z_k - x)_\varepsilon} \right) \langle V_{\gamma e_i, \varepsilon}(x) \mathbf{V}_\varepsilon \rangle d^2 x \\ & + \gamma \int_{\mathbb{C}^2} \frac{\langle \gamma e_1, \alpha_k \rangle}{(z_k - x_1)_\varepsilon (x_1 - x_2)_\varepsilon} + \frac{\langle \gamma e_2, \alpha_k \rangle}{(z_k - x_2)_\varepsilon (x_1 - x_2)_\varepsilon} \langle V_{\gamma e_1, \varepsilon}(x_1) V_{\gamma e_2, \varepsilon}(x_2) \mathbf{V}_\varepsilon \rangle d^2 x_1 d^2 x_2. \end{aligned}$$

We can thus proceed in the same way as in the proof that the correlation functions are C^2 to show that the limit does exist. Therefore the statement of Lemma 5.1.2 does indeed hold.

Combining this fact with Equation (5.1.16) we can infer that Lemma 5.1.1 is true as well, and we can therefore conclude for the proof of the local Ward identities as stated in Theorem 5.1.3.

5.2 . Global Ward identities

The local Ward identity (5.1.14) that we have just proved shows that the probabilistic model thus defined is indeed consistent with the expectations of the physics literature and may be understood as a manifestation of the higher-spin symmetry enjoyed by the $\mathfrak{sl}_3$ Toda theory. In addition to this identity there is a second building block which is fundamental in the study of Toda theories: the existence of global Ward identities. They take the following form:

Theorem 5.2.1. *Assume that $\mathfrak{g} = \mathfrak{sl}_3$. Under the assumptions of Theorem 4.3.1 the global Ward identities hold for $0 \leq n \leq 4$:*

$$\sum_{l=1}^N \left(z_l^n \mathcal{W}_{-2}^{(l)} + n z_l^{n-1} \mathcal{W}_{-1}^{(l)} + \frac{n(n-1)}{2} z_l^{n-2} w(\alpha_l) \right) \left\langle \prod_{k=1}^N V_{\alpha_k}(z_k) \right\rangle = 0 \quad (5.2.1)$$

where the quantities that appear in Equation (5.1.14) have been defined in Lemmas 5.1.1 and 5.1.2 above. Likewise for the stress-energy

$$\sum_{l=1}^N \left(z_l^n \partial_{z_l} + n z_l^{n-1} \Delta_{\alpha_l} \right) \left\langle \prod_{k=1}^N V_{\alpha_k}(z_k) \right\rangle = 0. \quad (5.2.2)$$

The global Ward identities for the stress-energy tensor are direct consequences of the conformal covariance of the correlation functions as explained in Theorem 4.2.2. However the derivation of the global Ward identities for the higher-spin tensor is not as straightforward and is actually one of the main difference between Liouville and Toda CFTs.

5.2.1 . Covariance of the higher-spin current

In order to prove that the Equations (5.2.1) hold we will prove that the higher-spin current is “holomorphic at infinity”, a fact which is axiomatic in the physics literature and by which is meant that

$$\mathbf{W}(z_0) \sim \frac{1}{z_0^6}$$

as $z_0 \rightarrow \infty$. To do so we rely on the fact that $\mathbf{W}$ behaves like a covariant tensor of order three in the following sense.

Proposition 5.2.2. *Under the assumptions of Theorem 4.3.1 and for any Möbius transform ψ of the plane we have*

$$\langle \mathbf{W}(z_0) \prod_{k=1}^N V_{\alpha_k}(z_k) \rangle = \psi'(z_0)^3 \prod_{l=1}^N |\psi'(z_l)|^{2\Delta_{\alpha_l}} \langle \mathbf{W}(\psi(z_0)) \prod_{k=1}^N V_{\alpha_k}(\psi(z_k)) \rangle. \quad (5.2.3)$$

Proof of Proposition 5.2.2. As usual we first need to consider the regularized version of the quantities involved and then take a limit. The expression of $\langle \mathbf{W}_\varepsilon(z_0) \prod_{k=1}^N V_{\alpha_k, \varepsilon}(z_k) \rangle$ is defined thanks to the mollified field $\Phi_\varepsilon = \mathbf{X}_\varepsilon + \frac{Q}{2} \ln g_{+, \varepsilon} + c$. Since the latter is smooth,

its derivatives and the ones of $\Phi_\varepsilon \circ \psi$ are well-defined. They are given by

$$\begin{aligned}\partial(\Phi_\varepsilon \circ \psi) &= \psi' \partial \Phi_\varepsilon \circ \psi \\ \partial^2(\Phi_\varepsilon \circ \psi) &= \psi'' \partial \Phi_\varepsilon \circ \psi + (\psi')^2 \partial^2 \Phi_\varepsilon \circ \psi \\ \partial^3(\Phi_\varepsilon \circ \psi) &= \psi''' \partial \Phi_\varepsilon \circ \psi + 3\psi'' \psi' \partial^2 \Phi_\varepsilon \circ \psi + (\psi')^3 \partial^3 \Phi_\varepsilon \circ \psi.\end{aligned}$$

We can proceed in the same way for the map $\ln |\psi'|$ and yields

$$\partial(\ln |\psi'|) = \frac{\psi''}{2\psi'} \quad \text{and} \quad \partial^2(\ln |\psi'|) = \frac{\psi''' \psi' - (\psi'')^2}{2(\psi')^2}.$$

We can now apply the tensor $\mathbf{W}$ to the field $\Phi_\varepsilon \circ \psi + Q \ln |\psi'|$ instead of Φ_ε , yielding

$$\begin{aligned}\mathbf{W}(\Phi_\varepsilon \circ \psi + Q \ln |\psi'|) &= q^2 \langle h_2, (\psi')^3 \partial^3 \Phi_\varepsilon + 3\psi'' \psi' \partial^2 \Phi_\varepsilon + \psi''' \partial \Phi_\varepsilon \rangle \circ \psi \\ &\quad - 2q \left(\psi' \langle h_2 - h_1, \psi'' \partial \Phi_\varepsilon + (\psi')^2 \partial^2 \Phi_\varepsilon \rangle \langle h_1, \partial \Phi_\varepsilon \rangle + \psi' \langle h_3 - h_2, \psi'' \partial \Phi_\varepsilon + (\psi')^2 \partial^2 \Phi_\varepsilon \rangle \langle h_3, \partial \Phi_\varepsilon \rangle \right. \\ &\quad \left. + q \frac{\psi''}{2\psi'} \langle 3h_2, \psi'' \partial \Phi_\varepsilon + (\psi')^2 \partial^2 \Phi_\varepsilon \rangle + q \frac{\psi''' \psi' - (\psi'')^2}{2\psi'} \langle h_2, \partial \Phi_\varepsilon \rangle \right) \circ \psi \\ &\quad - 8(\psi')^3 \langle h_1, \partial \Phi_\varepsilon \rangle \langle h_2, \partial \Phi_\varepsilon \rangle \langle h_3, \partial \Phi_\varepsilon \rangle \circ \psi - 4q\psi' \psi'' \langle h_2, \partial \Phi_\varepsilon \rangle \langle h_3 - h_1, \partial \Phi_\varepsilon \rangle \circ \psi + 8q^2 \frac{(\psi'')^2}{4\psi'} \langle h_2, \partial \Phi_\varepsilon \rangle \circ \psi.\end{aligned}$$

We can simplify the above and get

$$\begin{aligned}\mathbf{W}(\Phi_\varepsilon \circ \psi + Q \ln |\psi'|) &= q^2 (\psi')^3 \langle h_2, \partial^3 \Phi_\varepsilon \rangle \circ \psi \\ &\quad - 2q (\psi')^3 \left(\langle h_2 - h_1, \partial^2 \Phi_\varepsilon \rangle \langle h_1, \partial \Phi_\varepsilon \rangle + \langle h_3 - h_2, \partial^2 \Phi_\varepsilon \rangle \langle h_3, \partial \Phi_\varepsilon \rangle \right) \circ \psi \\ &\quad - 8(\psi')^3 \langle h_1, \partial \Phi_\varepsilon \rangle \langle h_2, \partial \Phi_\varepsilon \rangle \langle h_3, \partial \Phi_\varepsilon \rangle \circ \psi.\end{aligned}$$

Therefore we end up with the equality

$$\mathbf{W}(\Phi_\varepsilon \circ \psi + Q \ln |\psi'|) = (\psi')^3 \mathbf{W}(\Phi_\varepsilon) \circ \psi.$$

On the other hand we know from Equation (4.2.8) that, provided that the limit exists,

$$\lim_{\varepsilon \rightarrow 0} \langle F(\Phi_\varepsilon) \prod_{k=1}^N V_{\alpha_k, \varepsilon}(z_k) \rangle = \prod_{k=1}^N |\psi'(z_k)|^{2\Delta_{\alpha_k}} \lim_{\varepsilon \rightarrow 0} \langle F(\Phi_\varepsilon \circ \psi + Q \ln |\psi'|) \prod_{k=1}^N V_{\alpha_k, \varepsilon}(\psi(z_k)) \rangle \quad (5.2.4)$$

for any continuous map F on $H^{-1}(\mathbb{C} \rightarrow \mathfrak{a}, g_+)$. As a consequence with $F = \mathbf{W}$, in the $\varepsilon \rightarrow 0$ limit we will be left with

$$\langle \mathbf{W}(z_0) \prod_{k=1}^N V_{\alpha_k}(z_k) \rangle = \psi'(z_0)^3 \prod_{k=1}^N |\psi'(z_k)|^{2\Delta_{\alpha_k}} \langle \mathbf{W}(\psi(z_0)) \prod_{k=1}^N V_{\alpha_k}(\psi(z_k)) \rangle.$$

This is the result we were looking for. $\square$

5.2.2 . Proof of Theorem 5.2.1

We have thus proved that the higher-spin current really behaves like a tensor. We can now apply the statement of Proposition 5.2.2 with the Möbius transform given by $\psi : z \mapsto \frac{1}{z}$. This yields

$$\langle \mathbf{W}(z_0) \prod_{k=1}^N V_{\alpha_k}(z_k) \rangle = -\frac{1}{z_0^6} \prod_{l=1}^N |z_l|^{-4\Delta_{\alpha_l}} \langle \mathbf{W}(\frac{1}{z_0}) \prod_{k=1}^N V_{\alpha_k}(\frac{1}{z_k}) \rangle. \quad (5.2.5)$$

As a consequence the asymptotic behaviour of $\mathbf{W}(z_0)$ as $z_0 \rightarrow \infty$ is given by

$$\langle \mathbf{W}(z_0) \prod_{k=1}^N V_{\alpha_k}(z_k) \rangle \sim -\frac{1}{z_0^6} \prod_{l=1}^N |z_l|^{-4\Delta_{\alpha_l}} \langle \mathbf{W}(0) \prod_{k=1}^N V_{\alpha_k}(\frac{1}{z_k}) \rangle. \quad (5.2.6)$$

This provides a rigorous formulation of the axiom that $W(z_0) \sim \frac{1}{z_0^6}$ as $z_0 \rightarrow \infty$.

We can now combine this asymptotic with the expression of the local Ward identities to infer that, when $z_0 \rightarrow \infty$, the left-hand side in the Ward identity (5.1.14) is therefore asymptotic to $\frac{1}{z_0^6}$. On the other hand the leading term in the right-hand side of Equation (5.1.14) is given by

$$\frac{1}{z_0} \sum_{l=1}^N \mathbf{w}_{-2}^{(l)} \langle \prod_{k=1}^N V_{\alpha_k}(z_k) \rangle.$$

Therefore in order for these two asymptotics to be consistent we need to assume that that the $n = 0$ global Ward identity (5.2.1) holds:

$$\sum_{l=1}^N \mathbf{w}_{-2}^{(l)} \langle \prod_{k=1}^N V_{\alpha_k}(z_k) \rangle = 0.$$

We may proceed in the same way for the other terms that appear in the asymptotic of the right-hand side of Equation (5.2.1). To do so let us write the right-hand side of Equation (5.2.1) as

$$\sum_{l=1}^N \left(\frac{1}{z_0^3} \frac{w(\alpha_l)}{(1 - \frac{z_l}{z_0})^3} + \frac{1}{z_0^2} \frac{\mathbf{w}_{-1}^{(l)}}{(1 - \frac{z_l}{z_0})^2} + \frac{1}{z_0} \frac{\mathbf{w}_{-2}^{(l)}}{1 - \frac{z_l}{z_0}} \right) \langle \prod_{k=1}^N V_{\alpha_k}(z_k) \rangle$$

and expand in powers of $\frac{1}{z_0}$ the terms of the form $\frac{1}{(1 - \frac{z_l}{z_0})^p}$, $p = 1, 2, 3$. Then the asymptotic as $z_0 \rightarrow \infty$ can be expanded as negative powers of z_0 , and corresponding coefficients in this expansion are given by

$$\frac{1}{z_0^p} \sum_{l=1}^N \left(z_l^{p-1} \mathbf{w}_{-2}(z_l, \alpha_l) + (p-1) z_l^{p-2} \mathbf{w}_{-1}(z_l, \alpha_l) + \frac{(p-1)(p-2)}{2} z_l^{p-3} w(\alpha_l) \right) \langle \prod_{k=1}^N V_{\alpha_k}(z_k) \rangle.$$

Therefore to be consistent with the asymptotic prescribed by Equation (5.2.6) we need to have, as soon as $0 \leq n \leq 4$,

$$\sum_{l=1}^N \left(z_l^n \mathbf{w}_{-2}(z_l, \alpha_l) + n z_l^{n-1} \mathbf{w}_{-1}(z_l, \alpha_l) + \frac{n(n-1)}{2} z_l^{n-3} w(\alpha_l) \right) \langle \prod_{k=1}^N V_{\alpha_k}(z_k) \rangle = 0.$$

In other words the global Ward identities (5.2.1) must hold.

Remark 5.2.3. *With a little bit of extra work it is possible to show that the right-hand side in the asymptotic of Equation (5.2.6) is given by*

$$-\frac{1}{8} \sum_{l=1}^N \left(z_l^5 \mathbf{w}_{-2}(z_l, \alpha_l) + 5 z_l^4 \mathbf{w}_{-1}(z_l, \alpha_l) + 10 z_l^3 w(\alpha_l) \right) \langle \prod_{k=1}^N V_{\alpha_k}(z_k) \rangle$$

so we may not learn anything new from the exact value of the leading term in the asymptotic of the higher-spin current.

6 - Quelques implications probabilistes inattendues de la symétrie W

L'existence d'identités de Ward pour les fonctions de corrélation définies lors de la Partie I montre que la symétrie W est intrinsèquement présente dans le modèle probabiliste proposé pour définir de manière rigoureuse les théories de Toda. En effet l'utilisation des propriétés fondamentales des objets probabilistes impliqués nous a permis au cours du chapitre précédent d'obtenir l'existence de telles identités.

Nous continuons dans ce chapitre à explorer les interactions entre probabilités et symétries W. Plus spécifiquement nous faisons le chemin inverse de celui parcouru précédemment et utilisons les symétries des théories de Toda pour en induire certaines propriétés des objets probabilistes entrant dans la définition mathématique des théories de Toda.

En particulier nous étudierons dans quelle mesure le principe de réflexion en probabilités et l'existence de coefficients de réflexion au sein des théories de Toda sont intrinsèquement liés. Pour ce faire nous présenterons une nouvelle décomposition de chemin pour un mouvement brownien évoluant au sein d'un groupe de réflexion, décomposition qui nous permettra de décrire de manière précise les asymptotiques de queue de chaos multiplicatifs gaussiens corrélés.

Les résultats présentés ici sont issus de l'article [31], ainsi que, dans une moindre mesure, l'article [30].

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6.1 . Reflection principle and a Brownian path decomposition

The *reflection principle* is quite ubiquitous in probability theory —the interested reader may consult for instance the article [113] for an account on the historical appearance of the reflection principle in probability theory. Its universality can be seen via the number of statements bearing this name. The one we will focus on hereafter arises in the context of the Wiener process, and states that if we set $T_0 := \inf\{t \geq 0, B_t = 0\}$ to be the hitting time of 0 by a Brownian motion, then for all positive x and y

$$\mathbb{P}_x(B_t \in dy, t < T_0) = \mathbb{P}_x(B_t \in dy) - \mathbb{P}_{-x}(B_t \in dy). \quad (6.1.1)$$

As explained in Proposition 3.3.9, this statement generalizes when B is more generally a drifted Brownian motion that evolves on a reflection group $(\mathbf{V}, W)$.

This section is dedicated to the description of our generalized path decomposition for diffusions in Euclidean spaces, which we state in Theorem 6.1.1 below. In particular we will shed light on the fact that showing this result relies on this fundamental reflection principle.

6.1.1 . A Brownian path decomposition

In a landmark article [138], Williams derived in 1984 a celebrated path decomposition for one-dimensional diffusions. In particular this decomposition allows to provide a meaning to a process conditioned on its minimal value. To do so is used the welding of two independent diffusion processes before and after having hit the prescribed minimal value. Remarkably, the argument used by Williams in the proof of his main statement is based on the reflection principle (6.1.1) for the one-dimensional Brownian motion. The main input of our method is that this reflection principle can be generalized for Brownian motions evolving over a reflection group.

We formulate our path decomposition as follows: assume that B^ν is a Brownian motion with constant drift ν and values in a reflection group $(\mathbf{V}, W)$, and further assume that $\nu \in \mathcal{C}$. Set

$$\mathbf{M} := \sum_{i=1}^r \mathbf{M}_i \omega_i \quad \text{with} \quad \mathbf{M}_i := \inf_{t \geq 0} \langle B_t^\nu, e_i \rangle, \quad (6.1.2)$$

to generalize the minimum of a one-dimensional process. Then the process whose law is that of B^ν conditionally on the value of $\mathbf{M}$ can be realized by joining $r + 1$ diffusion processes, and corresponds to a decomposition of the path of B^ν between times when the successive minimums are attained. The diffusion processes involved are defined using a Doob h -transform. This statement is illustrated by Figures 6.1 and 6.2 below, where we have represented on the left the path of a drifted planar Brownian motion and on the right the decomposition corresponding to the A_2 root system viewed in $\mathbf{V} = \mathbb{R}^2$.

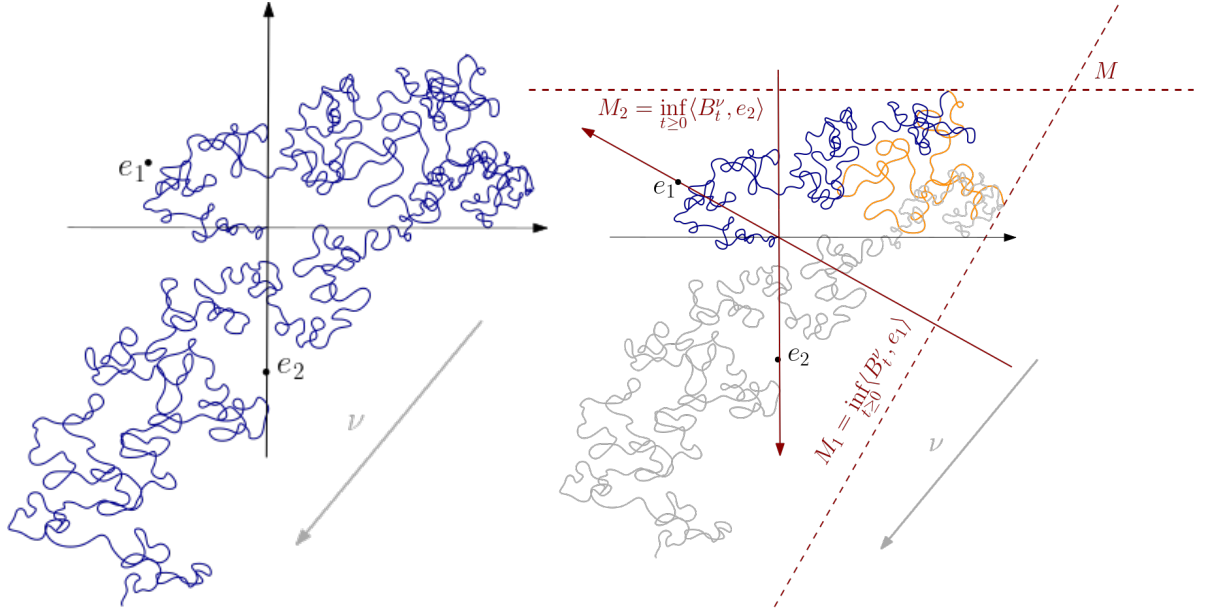


Figure 6.1: Planar Brownian motion with drift ν Figure 6.2: Decomposition of the path associated with A_2

To be more specific, recall the framework introduced in Subsection 3.3.4, and denote by $\mathcal{M}^\nu$ the law of the random variable M from Proposition 3.3.8. Then our Brownian path decomposition takes the following form:

Theorem 6.1.1. *Pick M according to its marginal law $\mathcal{M}^\nu$ and define a process X to be the joining of the following processes:*

- *Start by sampling a diffusion process X^1 started from the origin, with generator $\mathcal{A}^{1,\dots,r}$, and independent from M . Run it until it hits $M + \partial\mathcal{C}$, say at $z_1 \in M + \partial\mathcal{C}_1$.*
- *Then run an independent process X^2 started from z_1 and with generator $\mathcal{A}^{2,\dots,r}$, upon hitting $M + \partial\mathcal{C}$.*
- *Thus define a family of processes $(X^1, \dots, X^r)$. When X^r reaches the boundary of $M + \partial\mathcal{C}$, sample X^{r+1} with generator $\mathcal{A}^\emptyset$.*

Then X has the law of B^ν , a Brownian motion with drift $\nu \in \mathcal{C}$.

In other words, the statement above shows that the law of the process X , conditionally on the value of its minimum M , is the joining of $r + 1$ processes. It should be noted that the process X^{r+1} has the law of X conditioned to stay in the Weyl chamber $M + \mathcal{C}$. One easily checks that this statement is indeed consistent with the one-dimensional result by Williams [138].

In the planar case, Theorem 6.1.1 allows to provide a path decomposition with respect to cones with angles $\frac{\pi}{n}$, $n \geq 1$, that correspond to the Weyl chambers associated to the dihedral groups $\mathcal{D}_n$, $n \geq 1$. This is illustrated by Figure 6.3 below, where we represent the decomposition, associated to $\mathcal{D}_n$ for different values of n , of the path of a Brownian motion with drift ν .

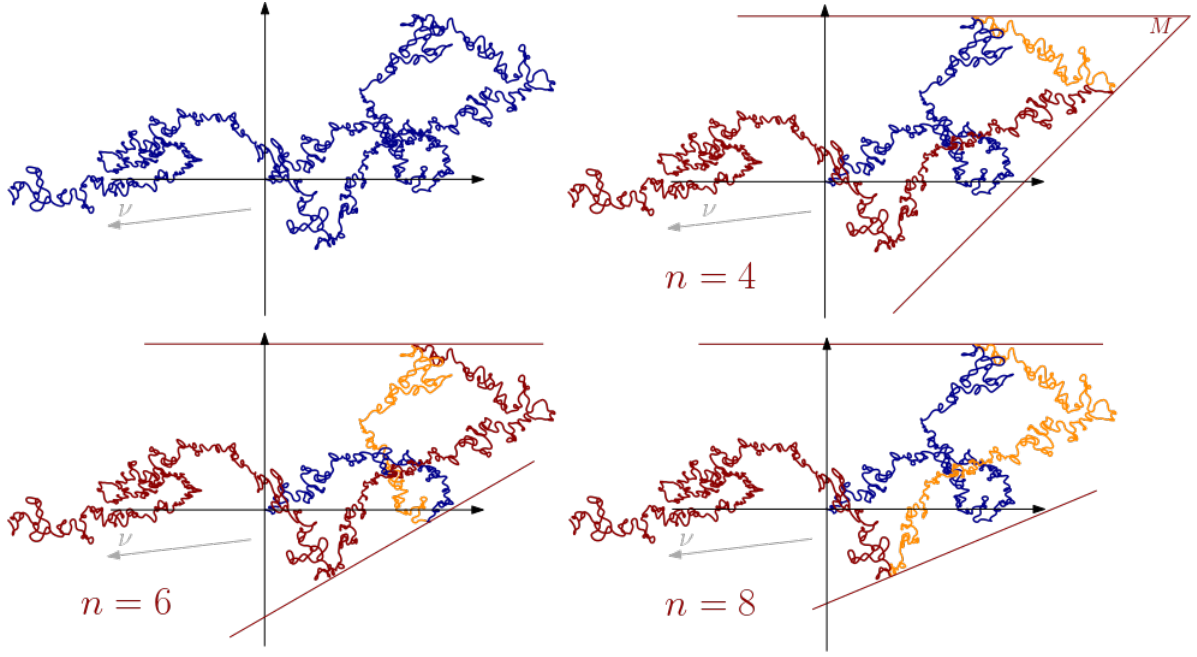


Figure 6.3: Planar Brownian motion with drift ν and its path decompositions associated to $\mathcal{D}_n$, $n = 4, 6, 8$.

Note that while in the one-dimensional case the two diffusion processes are simply a Brownian motion with negative drift and a Brownian motion conditioned to stay positive, in higher dimensions the description of these processes is not as clear. However we will prove in Section 6.2 below that in the asymptotic where $M \rightarrow \infty$ along a ray inside $\mathcal{C}$, the process can essentially be realised by joining drifted Brownian motions that will reflect on the different walls of the Weyl chamber. For instance in the planar case, we show that asymptotically the process looks like a Brownian motion with drift $s_1 s_2 \nu$ (resp. $s_2 s_1 \nu$) until it reaches the boundary of $\mathcal{C}$ on $\partial \mathcal{C}_1$ (resp. $\partial \mathcal{C}_2$), where it will be reflected and behaves like a Brownian motion with drift $s_2 \nu$ (resp. $s_1 \nu$), run until it reaches the boundary of $\mathcal{C}$ on $\partial \mathcal{C}_2$ (resp. $\partial \mathcal{C}_1$). This process will again be reflected and look like a Brownian motion with drift ν conditioned to stay inside $\mathcal{C}$. The initial drift depends on the way $M \rightarrow \infty$ inside $\mathcal{C}$. This asymptotic is depicted in Figure 6.4 below.

Before moving on to the proof of Theorem 6.1.1 let us stress that the result proved in [31] is actually more general than the one presented here, and the proof we present here remains valid in this more general setting.

6.1.2 . Proof of Theorem 6.1.1

The purpose of this subsection is to establish the validity of Theorem 6.1.1. A simple observation that we explain below allows to reduce its proof to the following lemma:

Lemma 6.1.2. *For any $M = \sum_{i=1}^r M_i \omega_i \in \mathbf{V}$ with $m_i \geq 0$ for all $1 \leq i \leq r$, define a process $\mathbf{X}$ as in Theorem 6.1.1. Then for any $0 \leq i \leq r$ and x, y inside $M + \mathcal{C}$,*

$$\mathbb{P}_x(\mathbf{X}_t \in dy, \mathbf{X} \text{ has hit } \partial \mathcal{C} \text{ at } \partial \mathcal{C}_1, \dots, \partial \mathcal{C}_i) = \frac{\partial_{i+1,r} h^M(y)}{\partial_{1,r} h^M(x)} \partial_{1,i} \mathbf{p}_t^M(x, dy) \quad (6.1.3)$$

Of course a similar result also holds when the process has hit different parts of the boundary.

Before proving this claim, let us explain to what extent Theorem 6.1.1 boils down to this statement. Pick any $x, y \in \mathbf{M} + \mathcal{C}$ and some positive time t . Then, provided that Lemma 6.1.2 holds, we can write that

$$\begin{aligned} \partial_{1,r} (h^{\mathbf{M}}(y) \mathbf{p}_t^{\mathbf{M}}(x, dy)) &= \sum_{k=0}^r \sum_{\substack{i_1, \dots, i_k \\ \text{distinct}}} \partial_{1, \dots, r \setminus i_1, \dots, i_k} h^{\mathbf{M}}(y) \partial_{i_1, \dots, i_k} \mathbf{p}_t^{\mathbf{M}}(x, dy) \\ &= \partial_{1,r} h^{\mathbf{M}}(x) \sum_{k=0}^r \sum_{\substack{i_1, \dots, i_k \\ \text{distinct}}} \mathbb{P}_x (\mathbf{X}_t \in dy, \mathbf{X} \text{ has hit } \partial \mathcal{C}_{i_1}, \dots, \partial \mathcal{C}_{i_k}) \\ &= \partial_{1,r} h^{\mathbf{M}}(x) \mathbb{P}_x (\mathbf{X}_t \in dy). \end{aligned}$$

On the other hand, one has the property that

$$\partial_{1,r} [h^{\mathbf{M}}(y) \mathbf{p}_t^{\mathbf{M}}(x, dy)] = \partial_{1,r} h^{\mathbf{M}}(x) \mathbb{P}_x (\mathbf{Y}_t \in dy | \mathbf{M}), \quad (6.1.4)$$

where $\mathbf{Y}$ is the diffusion process with generator $\mathcal{A}$. Indeed, the event that for all $1 \leq i \leq r$, $\mathbf{M}_i(\mathbf{Y}) \geq m_i$ corresponds to the process $\mathbf{Y}$ never hitting $\mathbf{m} + \partial \mathcal{C}$ with $\mathbf{m} = \sum_{i=1}^r m_i \omega_i$. Therefore by Doob's conditioning

$$\mathbb{P}_x (\mathbf{Y}_t \in dy | \forall 1 \leq i \leq r, \mathbf{M}_i \geq m_i) = \frac{h^{\mathbf{m}}(y)}{h^{\mathbf{m}}(x)} \mathbf{p}_t^{\mathbf{m}}(x, dy),$$

so Equation (6.1.4) reduces to

$$\partial_{1,r} [h^{\mathbf{m}}(x) \mathbb{P}_x (\mathbf{Y}_t \in dy | \forall 1 \leq i \leq r, \mathbf{M}_i \geq m_i)] = \partial_{1,r} h^{\mathbf{m}}(x) \mathbb{P}_x (\mathbf{Y}_t \in dy | \forall 1 \leq i \leq r, \mathbf{M}_i = m_i).$$

This follows from the fact that $h^{\mathbf{m}}(x) = \mathbb{P}_x (\forall 1 \leq i \leq r, \mathbf{M}_i(\mathbf{Y}) \geq m_i)$, while $\mathbf{m} \mapsto \partial_{1,r} h^{\mathbf{m}}(x)$ represents the density of $\mathbf{M}$. Eventually this shows that for any $x, y \in \mathbf{M} + \mathcal{C}$ and $t > 0$:

$$\mathbb{P}_x (\mathbf{Y}_t \in dy | \mathbf{M}) = \mathbb{P}_x (\mathbf{X}_t \in dy). \quad (6.1.5)$$

More generally the same reasoning shows that for any times $t_1, \dots, t_k$ and $y_1, \dots, y_k \in \mathbf{V}$

$$\mathbb{P}_x (\mathbf{Y}_{t_l} \in dy_l; 1 \leq l \leq k | \mathbf{M}) = \mathbb{P}_x (\mathbf{X}_{t_l} \in dy_l; 1 \leq l \leq k) \quad (6.1.6)$$

for k a positive integer. This corresponds to the statement of Theorem 6.1.1.

Therefore to prove our main statement on path decomposition it is enough to prove that Lemma 6.1.2 does indeed hold. The remaining part of this subsection is dedicated to proving this statement. Throughout the proof we consider the reflection group centered at M , $W^{\mathbf{M}}$, introduced before. Furthermore in the statement of Lemma 6.1.2 we can always assume that $f_t \equiv 1$, which we will do in what follows.

6.1.2.1 The case $i = 1$

We start by treating the case where the process $\mathbf{X}$ has hit only one part of the boundary of the Weyl chamber. This particular case contains all the ideas used for the general proof.

First of all, note that since $\mathbf{X}^1$ is defined via a Doob's transform from the process $\mathbf{Y}$, we can write that for any $z \in \partial\mathcal{C}$,

$$\mathbb{P}_x(T_{\mathbf{M}+\partial\mathcal{C}}(\mathbf{X}) \in du, \mathbf{X}_u \in dz) = \frac{\partial_{1,r}h^{\mathbf{M}}(z)}{\partial_{1,r}h^{\mathbf{M}}(x)} \mathbb{P}_x(T_{\mathbf{M}+\partial\mathcal{C}}(\mathbf{Y}) \in du, \mathbf{Y}_u \in dz).$$

Whence by independence of the processes appearing in the decomposition of $\mathbf{X}$

$$\begin{aligned} \mathbb{P}_x(\mathbf{X}_t \in dy, \mathbf{X} \text{ has only hit } \mathbf{M} + \partial\mathcal{C}_1) &= \int_0^t \int_{\mathbf{M}+\partial\mathcal{C}_1} \mathbb{P}_x(\mathbf{X}_t \in dy, T_{\mathbf{M}+\partial\mathcal{C}} \in du, \mathbf{X}_u \in dz) \\ &= \int_0^t \int_{\mathbf{M}+\partial\mathcal{C}_1} \frac{\partial_{1,r}h^{\mathbf{M}}(z)}{\partial_{1,r}h^{\mathbf{M}}(x)} \mathbb{P}_x(T_{\mathbf{M}+\partial\mathcal{C}}(\mathbf{Y}) \in du, \mathbf{Y}_u \in dz) \mathbb{P}_z(\mathbf{X}_{t-u}^2 \in dy). \end{aligned}$$

In the above expression in order to make sense of $\mathbb{P}_z(\mathbf{X}_{t-u}^2 \in dy)$ and because the conditioning is made with respect to the process killed when hitting the boundary, we may rely on the following fact:

Lemma 6.1.3. *Let $(z_n)_{n \in \mathbb{N}}$ be a sequence inside the Weyl chamber that converges to $z \in \mathbf{M} + \partial\mathcal{C}_1$. Then*

$$\mathbb{P}_z(\mathbf{X}_t^2 \in dy) := \lim_{n \rightarrow \infty} \mathbb{P}_{z_n}(\mathbf{X}_t^2 \in dy) = \frac{\partial_{2,r}h^{\mathbf{M}}(y)}{\partial_{2,r}h^{\mathbf{M}}(z)} \partial_1 \mathbf{p}_t^{\mathbf{M}}(z, dy). \quad (6.1.7)$$

A similar statement holds with $z \in \mathbf{M} + \partial\mathcal{C}_i$ for any $1 \leq i \leq r$.

Proof. For any such sequence, we know that

$$\mathbb{P}_{z_n}(\mathbf{X}_t^2 \in dy) = \frac{\partial_{2,r}h^{\mathbf{M}}(y)}{\partial_{2,r}h^{\mathbf{M}}(z_n)} \mathbf{p}_t^{\mathbf{M}}(z_n, dy).$$

As z_n approaches $\mathbf{M} + \partial\mathcal{C}_1$ both $\partial_{2,r}h^{\mathbf{M}}(z_n)$ and $\mathbf{p}_t^{\mathbf{M}}(z_n, y)$ get close to zero. However we claim that their Taylor expansions near z are given by:

$$\begin{aligned} \partial_{2,r}h^{\mathbf{M}}(z_n) &= \langle z - z_n, e_1 \rangle \partial_{1,r}h^{\mathbf{M}}(z) + o(|z_n - z|) \\ \mathbf{p}_t^{\mathbf{M}}(z_n, y) &= \langle z - z_n, e_1 \rangle \partial_1 \mathbf{p}_t^{\mathbf{M}}(z, y) + o(|z_n - z|), \end{aligned}$$

where $o(|z_n - z|)$ denotes a quantity which is negligible compared to $|z_n - z|$ as $z_n \rightarrow z$. Of course these expansions imply Equation (6.1.7).

To see why such expansions are valid, let us start by considering the first one and write $z_n = m_1^n \omega_1 + \sum_{i=2}^r \langle z, e_i \rangle \omega_i$ with $m_1^n = m_1 + \varepsilon_n$ converging to $m_1 = \langle z, e_1 \rangle$. Then recalling Equation (3.3.25) we deduce that

$$\begin{aligned} \mathbb{P}_{z_n}(\mathbf{M}_i(\mathbf{Y}) \geq m_i, \text{ for } 1 \leq i \leq r) &= \mathbb{P}_{z_n}(m_1^n \leq \mathbf{M}_i(\mathbf{Y}) \leq m_1^n + \varepsilon_n, \quad \mathbf{M}_i(\mathbf{Y}) \geq m_i, \text{ for } 2 \leq i \leq r) \\ &= \varepsilon_n \partial_1 h^{\mathbf{M}}(z) + o(\varepsilon_n), \end{aligned}$$

and more generally that

$$\begin{aligned} \partial_{2,r}h^{\mathbf{M}}(z_n) &= \langle z - z_n, e_1 \rangle \partial_{\langle z, e_1 \rangle} \partial_{2,r}h^{\mathbf{M}}(z) + o(|z_n - z|) \\ &= \langle z - z_n, e_1 \rangle (-\partial_{\langle M, e_1 \rangle}) \partial_{2,r}h^{\mathbf{M}}(z) + o(|z_n - z|). \end{aligned}$$

Alternatively we could have argued using the maximum principle for $\partial_{2,r}h^{\mathbf{M}}$, which thus admits an extremum on the hyperplane $\langle x, e_1 \rangle = \mathbf{M}_1$ with its gradient normal to this hyperplane.

As for $\mathbf{p}_t^{\mathbf{M}}$, by the reflection principle from Proposition 3.3.9

$$\begin{aligned} \mathbf{p}_t^{\mathbf{M}}(z_n, y) &= \sum_{s \in W^{\mathbf{M}}} \epsilon(s) \mathbf{p}_t(s z_n, y) \\ &= \frac{1}{2} \sum_{s \in W^{\mathbf{M}}} \epsilon(s) \left(\mathbf{p}_t(s z_n, y) - \mathbf{p}_t(s s_1 z_n, y) \right) \\ &= \frac{1}{2} \sum_{s \in W^{\mathbf{M}}} \epsilon(s) \left(\mathbf{p}_t(s z_n, y) - \mathbf{p}_t \left(s(z_n + \langle z_n - z, e_1^\vee \rangle e_1), y \right) \right) \\ &= \langle z - z_n, e_1^\vee \rangle \times \frac{1}{2} \sum_{s \in W^{\mathbf{M}}} \epsilon(s) \langle \nabla_x \mathbf{p}_t, s e_1 \rangle(s z, y) + o(|z_n - z|). \end{aligned}$$

Now for $z \in \mathbf{M} + \partial\mathcal{C}_1$ and $\mathbf{M}'$ close to $\mathbf{M}$ we can write, with s' similar to s but centered at $\mathbf{M}'$:

$$\begin{aligned} \mathbf{p}_t^{\mathbf{M}'}(z, y) &= \sum_{s' \in W^{\mathbf{M}'}} \epsilon(s') \mathbf{p}_t(s' z, y) \\ &= \frac{1}{2} \sum_{s' \in W^{\mathbf{M}'}} \epsilon(s') \left[\mathbf{p}_t(s' z, y) - \mathbf{p}_t(s' s'_1 z, y) \right] \\ &= \frac{1}{2} \sum_{s' \in W^{\mathbf{M}'}} \epsilon(s') \left[\mathbf{p}_t(s' z, y) - \mathbf{p}_t \left(s'(z + \langle \mathbf{M}' - \mathbf{M}, e_1^\vee \rangle e_1), y \right) \right], \quad \text{hence} \\ \mathbf{p}_t^{\mathbf{M}'}(z, y) &= \frac{1}{2} \left(\mathbf{p}_t^{\mathbf{M}'}(z, y) - \mathbf{p}_t^{\mathbf{M}'}(z + \langle \mathbf{M}' - \mathbf{M}, e_1^\vee \rangle e_1, y) \right). \end{aligned} \quad (6.1.8)$$

Therefore taking derivatives with respect to $\langle \mathbf{M}, e_1 \rangle$ of $\mathbf{p}_t^{\mathbf{M}'}(z, y)$ yields that for $z \in \partial\mathcal{C}_1$,

$$\frac{1}{2} \sum_{s \in W^{\mathbf{M}}} \epsilon(s) \langle \nabla_x \mathbf{p}_t, s e_1 \rangle(s z, y) = \frac{\langle e_i, e_i \rangle}{2} \partial_1 \mathbf{p}_u^{\mathbf{M}}(z, y)$$

where the prefactor stems from the fact that $\langle \mathbf{M}' - \mathbf{M}, e_1^\vee \rangle = \frac{2}{\langle e_i, e_i \rangle} \langle \mathbf{M}' - \mathbf{M}, e_1 \rangle$. This shows that $\mathbf{p}_t^{\mathbf{M}}(z_n, y) = \langle z - z_n, e_1 \rangle \partial_1 \mathbf{p}_t^{\mathbf{M}}(z, y) + o(|z_n - z|)$, concluding the proof of Lemma 6.1.3. $\square$

We can now make use of Lemma 6.1.3 in our computations to see that the case $i = 1$ in Lemma 6.1.2 follows from the equality, that no longer depends on the conditioning:

Lemma 6.1.4. *For any $x, y \in \mathbf{M} + \mathcal{C}$ and $t > 0$,*

$$\int_0^t \int_{\mathbf{M} + \partial\mathcal{C}_1} \mathbb{P}_x(T_{\mathbf{M} + \partial\mathcal{C}}(\mathbf{Y}) \in du, \mathbf{Y}_u \in dz) \partial_1 \mathbf{p}_{t-u}^{\mathbf{M}}(z, y) = \partial_1 \mathbf{p}_t^{\mathbf{M}}(x, y). \quad (6.1.9)$$

Proof. Note that for $s' \neq Id$ and $\mathbf{M}'$ sufficiently close to $\mathbf{M}$,

$$\mathbb{P}_x(\mathbf{Y}_t \in d(s'y)) = \int_0^t \int_{\mathbf{M} + \partial\mathcal{C}} \mathbb{P}_x(T_{\mathbf{M} + \partial\mathcal{C}}(\mathbf{Y}) \in du, \mathbf{Y}_u \in dz) \mathbb{P}_z(\mathbf{Y}_{t-u} \in d(s'y)) \quad (6.1.10)$$

since to reach $s'y$ the path has to cross $\mathbf{M} + \partial\mathcal{C}$ (this is true only for $\mathbf{M}'$ and $\mathbf{M}$ sufficiently close). Moreover the quantity that corresponds to $s' = Id$ in the expression of

$$\mathbf{p}_t^{\mathbf{M}'}(x, y) = \sum_{s' \in W^{\mathbf{M}'}} \epsilon(s') p_t(s'x, y)$$

given by the reflection principle 3.3.9 does not depend on $\mathbf{M}'$. As a consequence we can write that

$$\partial_1 \mathbf{p}_t^{\mathbf{M}}(x, y) = \int_0^t \int_{\mathbf{M} + \partial\mathcal{C}} \mathbb{P}_x(T_{\mathbf{M} + \partial\mathcal{C}}(\mathbf{Y}) \in du, \mathbf{Y}_u \in dz) \partial_1 \mathbf{p}_{t-u}^{\mathbf{M}}(z, y).$$

To conclude note that $\partial_1 \mathbf{p}_t^{\mathbf{M}}(z, y)$ vanishes when $z \in \mathbf{M} + \partial\mathcal{C} \setminus \partial\mathcal{C}_1$ —this is a consequence of Equation (6.1.8) above. $\square$

6.1.2.2 The case $i = 2$

The approach remains essentially the same for $i = 2$; in the same spirit as above, we see that

$$\begin{aligned} & \mathbb{P}_x(\mathbf{X}_t \in dy, \mathbf{X} - \mathbf{M} \text{ has hit } \partial\mathcal{C}_1 \text{ before } \partial\mathcal{C}_2) \\ &= \frac{\partial_{3,\dots,r} h^{\mathbf{M}}(y)}{\partial_{1,\dots,r} h^{\mathbf{M}}(x)} \int_0^t \int_{t_1}^t \int_{\mathbf{M} + \partial\mathcal{C}_1} \int_{\mathbf{M} + \partial\mathcal{C}_2} \mathbb{P}_x(T_{\mathbf{M} + \partial\mathcal{C}}(\mathbf{Y}) \in dt_1, \mathbf{Y}_{t_1} \in dz_1) \times \\ & \quad \lim_{z_n^1 \rightarrow z_1} \frac{\mathbb{P}_{z_n^1}(T_{\mathbf{M} + \partial\mathcal{C}}(\mathbf{Y}) \in dt_2, \mathbf{Y}_{t_2} \in dz_2)}{\langle z_1 - z_n^1, e_1 \rangle} \partial_2 \mathbf{p}_{t-t_2}^{\mathbf{M}}(z_2, y) \\ &= \frac{\partial_{3,\dots,r} h^{\mathbf{M}}(y)}{\partial_{1,\dots,r} h^{\mathbf{M}}(x)} \int_0^t \int_{\mathbf{M} + \partial\mathcal{C}_1} \mathbb{P}_x(T_{\mathbf{M} + \partial\mathcal{C}}(\mathbf{Y}) \in dt_1, \mathbf{Y}_{t_1} \in dz_1) \lim_{z_n^1 \rightarrow z_1} \frac{\partial_2 \mathbf{p}_{t-t_1}^{\mathbf{M}}(z_n^1, y)}{\langle z_1 - z_n^1, e_1 \rangle} \\ &= \frac{\partial_{3,\dots,r} h^{\mathbf{M}}(y)}{\partial_{1,\dots,r} h^{\mathbf{M}}(x)} \int_0^t \int_{\mathbf{M} + \partial\mathcal{C}_1} \mathbb{P}_x(T_{\mathbf{M} + \partial\mathcal{C}}(\mathbf{Y}) \in dt_1, \mathbf{Y}_{t_1} \in dz_1) \partial_{1,2} \mathbf{p}_{t-t_1}^{\mathbf{M}}(z_1, y), \end{aligned}$$

where we have used the results from the case $i = 1$, and relied on the fact that $\mathbf{X}^1$ will never come back to $\mathbf{M} + \partial\mathcal{C}_1$. As a consequence the proof boils down to proving the following analog of Lemma 6.1.4:

$$\int_0^t \int_{\mathbf{M} + \partial\mathcal{C}_1 \cup \partial\mathcal{C}_2} \mathbb{P}_x(T_{\partial\mathcal{C}}(\mathbf{Y}) \in dt_1, \mathbf{Y}_{t_1} \in dz_1) \partial_{1,2} \mathbf{p}_{t-t_1}^{\mathbf{M}}(z_1, y) = \partial_{1,2} \mathbf{p}_t^{\mathbf{M}}(x, y).$$

6.1.2.3 The general case

The proof in the general case relies on the very same computations as above. In the end we see that in order to prove that Theorem 6.1.1 all one has to do is to check the validity of the lemma below, which we then recursively apply to get the desired statement.

Lemma 6.1.5. *For any $1 \leq i \leq r$,*

$$\int_0^t \int_{\partial\mathcal{C}_1 \cup \dots \cup \partial\mathcal{C}_i} \mathbb{P}_x(T_{\partial\mathcal{C}}(\mathbf{Y}) \in du, \mathbf{Y}_u \in dz) \partial_{1,\dots,i} \mathbf{p}_{t-u}^{\mathbf{M}}(z, y) = \partial_{1,\dots,i} \mathbf{p}_t^{\mathbf{M}}(x, y).$$

Proof. To start with, recall from Equation (6.1.8) that if $z \in \mathbf{M} + \partial\mathcal{C}_j$ for $j > i$ then

$$\mathbf{p}_t^{\mathbf{M}'}(z, y) = \frac{1}{2} \left(\mathbf{p}_t^{\mathbf{M}'}(z, y) - \mathbf{p}_t^{\mathbf{M}'}(z + \langle \mathbf{M}' - \mathbf{M}, e_j^\vee \rangle e_j, y) \right),$$

and therefore $\partial_{1, \dots, i} \mathbf{p}_{t-t_1}^{\mathbf{M}}(z, y) = 0$. As a consequence we only need to prove that

$$\int_0^t \int_{\partial\mathcal{C}} \mathbb{P}_x(T_{\partial\mathcal{C}}(\mathbf{Y}) \in du, \mathbf{Y}_u \in dz) \partial_{1, \dots, i} \mathbf{p}_{t-u}^{\mathbf{M}}(z, y) = \partial_{1, \dots, i} \mathbf{p}_t^{\mathbf{M}}(x, y).$$

This follows from Equation (6.1.10) along the same lines as in the proof of Lemma 6.1.4. $\square$

This allows to wrap up the proof of Lemma 6.1.2 and therefore of Theorem 6.1.1.

6.2 . On the process started from infinity.

In this section we focus on the case where the process being considered in the statement of Theorem 6.1.1 is a drifted Brownian motion over $\mathbf{V}$. To be more specific we study in this section the behaviour of the drifted Brownian motion B^ν when conditioned on having $\mathbf{M} \rightarrow \infty$ inside $\mathcal{C}_-$. To do so we will provide a detailed analysis of the process $\mathcal{B}^\nu$, $\nu \in \mathcal{C}$, defined from Theorem 6.1.1, in the asymptotic where its starting point x will diverge inside the Weyl chamber $\mathcal{C}$. This process is defined by joining:

- A diffusion process $\mathbf{X}^1$ started from $x \in \mathcal{C}$, with generator $\frac{1}{2}\Delta + \nabla \log \partial_{1, \dots, r} h$ and run until it hits $\partial\mathcal{C}$, say at $z_1 \in \partial\mathcal{C}_1$. Here h is given by Equation (3.3.15).
- Then run an independent process $\mathbf{X}^2$ started from z_1 and with generator $\frac{1}{2}\Delta + \nabla \log \partial_{2, \dots, r} h$, upon hitting $\partial\mathcal{C}$.
- Thus define a family of processes $(\mathbf{X}^1, \dots, \mathbf{X}^r)$. When $\mathbf{X}^r$ reaches the boundary of $\partial\mathcal{C}$, sample $\mathbf{X}^{r+1}$ with generator $\frac{1}{2}\Delta + \nabla \log h$.

Our goal is to describe the behaviour of such a process $\mathcal{B}^\nu$ when $x \rightarrow \infty$ inside $\mathcal{C}$. We will see that, in contrast with the uniqueness of the entrance law from ∞ of the process studied in [10, Section 6], the process will behave in very different ways according to the way x diverges inside $\mathcal{C}$. In what follows we will say that $x \rightarrow \infty$ inside $\mathcal{C}$ when $\langle x, e_i \rangle \rightarrow +\infty$ for all $1 \leq i \leq r$.

Let us describe in details this asymptotic in the case of A_2 . For this purpose, let us introduce for any $0 < \eta < 1$ and a process $\mathbf{Y}$ starting inside $\mathcal{C}$ the events $\mathcal{E}_1^\eta(\mathbf{Y}) := \{\forall t < T_{\partial\mathcal{C}}, \langle \mathbf{Y}, e_2 \rangle > (1 - \eta)\langle \mathbf{Y}_0, e_2 \rangle\}$ and analogously $\mathcal{E}_2^\eta$. We then define a process $\mathbf{Y}^1$ by joining:

1. A Brownian motion with drift $s_1 s_2 \nu$, started from x , and conditioned on $\mathcal{E}_1^\eta(\mathbf{Y}^1)$. This process is run until hitting $\partial\mathcal{C}$ (over $z_1 \in \partial\mathcal{C}_1$).
2. A Brownian motion started from z_1 , with drift $s_2 \nu$ and conditioned not to hit $\partial\mathcal{C}_1$ again. By this we mean a diffusion with drift

$$d_2(y) := \frac{s_2 e^{\langle s_2 \nu, y \rangle} - s_1 s_2 e^{\langle s_1 s_2 \nu, y \rangle}}{e^{\langle s_2 \nu, y \rangle} - e^{\langle s_1 s_2 \nu, y \rangle}} = \frac{s_2 - s_1 s_2 e^{\langle \nu, \rho \rangle \langle y, e_1 \rangle}}{1 - e^{\langle \nu, \rho \rangle \langle y, e_1 \rangle}}.$$

This process will be run upon hitting $\partial\mathcal{C}$ at $z_2 \in \partial\mathcal{C}_2$.

3. A Brownian motion with drift ν , started from z_2 , and conditioned to stay inside $\mathcal{C}$.

Similarly by exchanging e_1 and e_2 we define a process $\mathbf{Y}^2$. As we will see, the process $\mathcal{B}^\nu$ will behave like $\mathbf{Y}^1$ or $\mathbf{Y}^2$ depending on the variable $x^\perp := \langle s_2 s_1 \nu - s_1 s_2 \nu, \mathbf{x} \rangle$. In what follows we consider $(Z_t^1, Z_t^2)_{t \geq 0}$ a pair of bounded, continuous processes independent from $\mathcal{B}^\nu$, $\mathbf{Y}^1$ and $\mathbf{Y}^2$, as well as stationary (in the sense that the law of $(Z_{t+s}^1, Z_{t+s}^2)_{t \geq 0}$ is independent of $s \geq 0$). We further demand that the processes $(Z_t^1)_{t \leq s}$ and $(Z_t^2)_{t \geq h}$ become independent as $h - s \rightarrow +\infty$ by requiring that for any bounded continuous functions f_1, f_2

$$\lim_{h-s \rightarrow +\infty} \mathbb{E} \left[\int_0^s f_1(t) Z_t^1 dt \int_0^{+\infty} f_2(t) Z_{t+h}^2 dt \right] - \mathbb{E} \left[\int_0^s f_1(t) Z_t^1 dt \right] \mathbb{E} \left[\int_0^{+\infty} f_2(t) Z_t^2 dt \right] = 0.$$

With these notations at hand the following statement can be seen as a reformulation of the above heuristic:

Proposition 6.2.1. *For $i = 1, 2$, set for $\mathbf{Y}$ either $\mathcal{B}^\nu$, $\mathbf{Y}^1$ or $\mathbf{Y}^2$*

$$J_i(\mathbf{Y}) := \int_0^{+\infty} e^{-\langle \mathbf{Y}_t, e_i \rangle} Z_t^i dt. \quad (6.2.1)$$

Then for any bounded, continuous function $F : \mathbb{R}^2 \rightarrow \mathbb{R}$, in the limit where $x \rightarrow \infty$ inside $\mathcal{C}$

$$\mathbb{E}_x [F(J_1(\mathcal{B}^\nu), J_2(\mathcal{B}^\nu))] \sim \mathbb{E}_x [F(J_1(\mathbf{Y}^1), J_2(\mathbf{Y}^1))] \quad \text{if } x^\perp \rightarrow -\infty. \quad (6.2.2)$$

Conversely if $x^\perp \rightarrow +\infty$ then $\mathbb{E}_x [F(J_1(\mathcal{B}^\nu), J_2(\mathcal{B}^\nu))] \sim \mathbb{E}_x [F(J_1(\mathbf{Y}^2), J_2(\mathbf{Y}^2))]$.

As a consequence of the proof of Proposition 6.2.1 we will see that if $x^\perp \rightarrow -\infty$, then the processes $\mathcal{B}^\nu$ and $\mathbf{Y}^1$ have asymptotically the same behaviour by showing that their drift functions are asymptotically close in the sense that for any positive η ,

$$\lim_{\substack{x \rightarrow \infty \\ x^\perp \rightarrow -\infty}} \mathbb{P}_x \left(\sup_{0 \leq t \leq T_{\partial \mathcal{C}}} |\nabla \log \partial_{1,2} h(\mathcal{B}^\nu) - s_1 s_2 \nu| \leq C e^{-\eta \langle \nu, e_1 \rangle \langle \mathbf{x}, e_2 \rangle} \right) = 1$$

and likewise for the two other components of the path. On the contrary if $\langle s_2 s_1 \nu - s_1 s_2 \nu, \mathbf{x} \rangle \rightarrow +\infty$ then the process will behave like $\mathbf{Y}^2$. In the case where $\langle s_1 s_2 \nu - s_2 s_1 \nu, \mathbf{x} \rangle$ remains bounded, then the law of the process $\mathcal{B}^\nu$ will be essentially realized by determining which boundary the process hits first and then running one of the two above processes accordingly. These different behaviours are illustrated in Figure 6.4. We stress that in the limit where $x \rightarrow \infty$ the events $\mathcal{E}_1(\mathbf{Y}^1)$ and $\mathcal{E}_2(\mathbf{Y}^2)$ have probability asymptotically 1 so the conditioning becomes unnecessary.

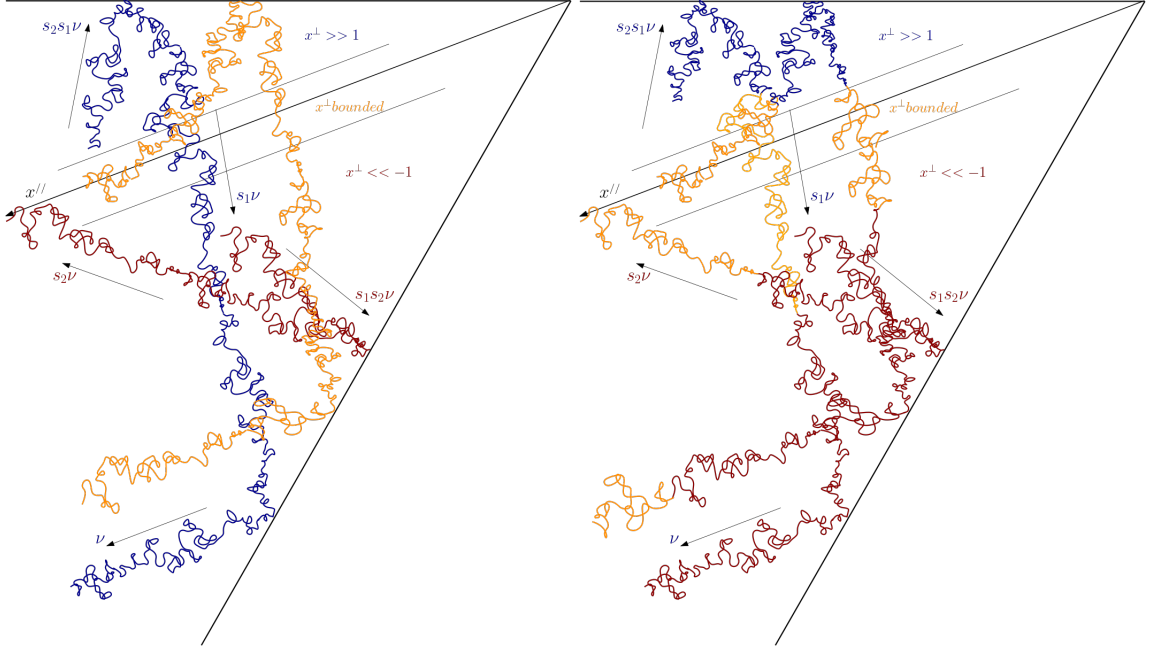


Figure 6.4: The three different behaviours of the process $\mathcal{B}^\nu$ when $x \rightarrow +\infty$. Figure 6.5: Components of the path depending on whether $\langle \mathcal{B}^\nu, e_i \rangle$ is close to zero or large.

In the general case a similar asymptotic behaviour for the process does hold, but takes more effort to be properly stated. For this purpose, let us consider $(Z^1, \dots, Z^r)$ as above with the additional property that Z^i and Z^j are independent as soon as $\langle e_i, e_j \rangle = 0$. We also introduce the subset $W_{1,\dots,r}$ of W defined as the set of the $s \in W$ that admit a reduced expression containing all the reflections $s_1, \dots, s_r$ and set for $i = 1, \dots, r$ the notation

$$J_i(\mu) := \int_0^{+\infty} e^{-B_t^\mu} Z_t^i dt \quad (6.2.3)$$

for $\mu \in \mathbb{R}$ positive, where B^μ is the process started from $+\infty$ whose law is realized by joining a Brownian motion with negative drift $-\mu$ and variance $\langle e_i, e_i \rangle$ upon hitting the origin and a Brownian motion with positive drift μ and variance $\langle e_i, e_i \rangle$ conditioned to stay positive after. An alternative way of writing this integral is to use that $J_i(\mu)$ has same law as

$$\int_{-\infty}^{+\infty} e^{-B_t^\mu} Z_t^i dt$$

where B^μ is a two-sided Brownian motion with positive drift μ and conditioned to stay positive. For more details see Subsection 3.3.4.1.

Proposition 6.2.2. *Set $s = s_1 \cdots s_r$ and assume that $x \rightarrow \infty$ inside $\mathcal{C}$ in the asymptotic where $\langle s\nu - s'\nu, x \rangle \rightarrow +\infty$ for all $s' \neq s \in W_{1,\dots,r}$. Then for $F_1, \dots, F_r$ bounded continuous over $\mathbb{R}^+$*

$$\lim_{x \rightarrow \infty} \mathbb{E}_x \left[\prod_{i=1}^r F_i(J_i(\mathcal{B}^\nu)) \right] = \prod_{i=1}^r \mathbb{E} \left[F_i \left(J_i(\langle s_{i+1} \cdots s_r \nu, e_i \rangle) \right) \right]. \quad (6.2.4)$$

A similar statement does hold when different asymptotics are considered, that is when s is an element of W_1 of the form $s_{\sigma 1} \cdots s_{\sigma r}$ for some permutation σ of $\{1, \dots, r\}$. Proposition 6.2.2 provides an alternative formulation of the result described above for the A_2 case. Proving these two statements is key in the derivation of the asymptotics of Toda Vertex Operators and class one Whittaker functions. In order to prove these statements it will be convenient to start with the A_2 case to settle the ideas and then proceed to the general proof and explain how the arguments developed can be adapted in the general framework. Namely in Subsections 6.2.2 and 6.2.3 we provide a detailed study of the process when the reflection group being considered is associated to A_2 and prove Proposition 6.2.2 under this assumption. The general proof of Proposition 6.2.2 will be carried in Subsection 6.2.4 below.

For future purpose we introduce for $s \in W_{1, \dots, r}$ the shorthand

$$\lambda_s := \prod_{i=1}^r \langle s\nu - \nu, \omega_i^\vee \rangle.$$

6.2.1 . Location of the first hitting point of $\partial\mathcal{C}$

Before actually describing the different components of the path as explained above, we start by showing that with high probability in the asymptotic considered, we can assume that the process stays inside a certain subdomain of $\mathcal{C}$. This fact is itself a consequence of the following Proposition, that describes the location of the first hitting point of the boundary of $\mathcal{C}$ by the process $\mathcal{B}^\nu$:

Proposition 6.2.3. *Assume that $\mathbf{x} \in \mathcal{C}$ and take any $y \in \mathbf{V}$ such that $\langle y, e_i \rangle < \langle \mathbf{x}, e_i \rangle$ for all $1 \leq i \leq r$. Then for all $z \in y + \partial\mathcal{C}_1$*

$$\mathbb{P}_{\mathbf{x}}(T_{y+\partial\mathcal{C}} \in dt; \mathcal{B}_t^\nu \in dz) = \frac{1}{2} \frac{U(z)}{U(\mathbf{x})} \sum_{s \in W} \epsilon(s) \frac{\langle s(\mathbf{x} - y), e_1^* \rangle}{t(2\pi t)^{\frac{r}{2}}} e^{-\frac{|s(\mathbf{x}-y)+y-z|^2}{2t} - \frac{|\nu|^2 t}{2}} dt dz, \quad (6.2.5)$$

where recall that $e_1^* = \frac{e_1}{\sqrt{\langle e_1, e_1 \rangle}}$ and with

$$U(z) := \sum_{s \in W_{1, \dots, r}} \epsilon(s) \lambda_s e^{\langle s\nu, z \rangle}.$$

Proof. Since the process $\mathcal{B}^\nu$ is defined using a Doob transform

$$\mathbb{P}_{\mathbf{x}}(T_{y+\partial\mathcal{C}} \in dt; \mathcal{B}_t^\nu \in dz) = \frac{\partial_{1, \dots, r} h^{\mathbf{M}}(z)}{\partial_{1, \dots, r} h^{\mathbf{M}}(\mathbf{x})} e^{\langle \nu, z - \mathbf{x} \rangle - \frac{|\nu|^2 t}{2}} \mathbb{P}_{\mathbf{x}}(T_{y+\partial\mathcal{C}} \in dt; B_t \in dz)$$

where B is a Brownian motion over $\mathbf{V}$. Direct computations show that

$$\frac{\partial_{1, \dots, r} h^{\mathbf{M}}(z)}{\partial_{1, \dots, r} h^{\mathbf{M}}(\mathbf{x})} e^{\langle \nu, z - \mathbf{x} \rangle} = \frac{U(z)}{U(\mathbf{x})}$$

while thanks to the reasoning presented in the proof of Proposition 3.3.9, adapted by considering paths from $s\mathbf{x}$ to z for $s \in W^y$ with $\langle s(\mathbf{x} - y), e_1 \rangle > 0$ and that do not cross the hyperplane $\langle \cdot - y, e_1 \rangle = 0$,

$$\mathbb{P}_{\mathbf{x}}(T_{y+\partial\mathcal{C}} \in dt; B_t \in dz) = \sum_{\substack{s \in W \\ \langle s(\mathbf{x}-y), e_1 \rangle > 0}} \epsilon(s) \mathbb{P}_{s(\mathbf{x}-y)+y}(T_{\langle y, e_1 \rangle}(\langle B, e_1 \rangle) \in dt; B_t \in dz).$$

Now these probabilities are well-known and given by

$$\mathbb{P}_x(T_0(\langle B, e_1 \rangle) \in dt; B_t \in dz) = \frac{\langle x, e_1^* \rangle}{t(2\pi t)^{\frac{r}{2}}} e^{-\frac{|x-z|^2}{2t}} dt dz.$$

Using the fact that for all $s \in W$, $\epsilon(s_1 s) \langle s_1 s x, e_1 \rangle = \epsilon(s) \langle s x, e_1 \rangle$, the latter sum can be rewritten in a more elegant fashion as

$$\frac{1}{2} \sum_{s \in W} \epsilon(s) \frac{\langle s(x-y), e_1^* \rangle}{t(2\pi t)^{\frac{r}{2}}} e^{-\frac{|s(x-y)+y-z|^2}{2t} - \frac{|\nu|^2 t}{2}} dt dz.$$

□

Thanks to this proposition we can deduce that the Brownian trajectory will stay within a given region, with high probability in the asymptotic where $x \rightarrow \infty$ inside $\mathcal{C}$.

6.2.1.1 The A_2 case

To start with, given a positive $\eta < 1$ one can split $\mathcal{C}$ between the domains $\mathcal{U}_1$, $\mathcal{U}_2$ and $\mathcal{C} \setminus \mathcal{U}$ where for $(i, j) \in \{(1, 2), (2, 1)\}$,

$$\mathcal{U}_i := \{z \in \mathcal{C} \text{ s.t. } \langle z, e_j \rangle > (1 - \eta) \langle x, e_j \rangle\} \quad \text{and} \quad \mathcal{U} := \mathcal{U}_1 \cup \mathcal{U}_2.$$

The events $\mathcal{E}_i$, $i = 1, 2$ are then the events that the process $\mathcal{B}^\nu$ stays inside $\mathcal{U}_i$ until it hits the boundary of $\mathcal{C}$.

Proposition 6.2.4. *Assume that $x \rightarrow \infty$ inside $\mathcal{C}$. Then*

$$U(x) \mathbb{P}_x(\mathcal{E}_1) - \lambda_{s_1 s_2} e^{\langle s_1 s_2 \nu, x \rangle} = \mathcal{O}(e^{\langle s_1 s_2 \nu, x \rangle - \eta \langle x, e_2 \rangle \langle \nu, e_1 \rangle}). \quad (6.2.6)$$

Proof. First note that the event $\mathcal{E}_1$ simply means that the process $\mathcal{B}^\nu$ first hits $y + \partial \mathcal{C}$ over its component $y + \partial \mathcal{C}_1$, where $y = (1 - \eta) \langle x, e_2 \rangle \omega_2$. Therefore thanks to Proposition 6.2.3 and its proof we can write that

$$\begin{aligned} U(x) \mathbb{P}_x(\mathcal{E}_i) &= \sum_{w \in W_{1,2}} \epsilon(w) \lambda_w \mathbb{E}_x \left[e^{\langle w \nu, B_{T_{y+\partial \mathcal{C}}} \rangle - \frac{\langle w \nu \rangle^2 T_{y+\partial \mathcal{C}}}{2}} \mathbb{1}_{T_{y+\partial \mathcal{C}} = T_{y+\partial \mathcal{C}_1}} \right] \\ &= \lambda_{s_1 s_2} e^{\langle s_1 s_2 \nu, x \rangle} - \mathfrak{R}_{s_1 s_2} + \mathfrak{R}_{s_2 s_1} + \mathfrak{R}_{s_1 s_2 s_1} \end{aligned}$$

where we have used that $W_{1,2}$ is in that case reduced to the elements $s_1 s_2$, $s_2 s_1$ and $s_1 s_2 s_1$, and where we have set

$$\begin{aligned} \mathfrak{R}_{s_1 s_2} &:= \lambda_{s_1 s_2} \mathbb{E}_x \left[e^{\langle s_1 s_2 \nu, B_{T_{y+\partial \mathcal{C}}} \rangle - \frac{\langle s_1 s_2 \nu \rangle^2 T_{y+\partial \mathcal{C}}}{2}} \mathbb{1}_{T_{y+\partial \mathcal{C}} = T_{y+\partial \mathcal{C}_2}} \right], \\ \mathfrak{R}_w &:= \epsilon(w) \lambda_w \mathbb{E}_x \left[e^{\langle w \nu, B_{T_{y+\partial \mathcal{C}}} \rangle - \frac{\langle w \nu \rangle^2 T_{y+\partial \mathcal{C}}}{2}} \mathbb{1}_{T_{y+\partial \mathcal{C}} = T_{y+\partial \mathcal{C}_1}} \right]. \end{aligned}$$

Let us start by considering the first remainder term $\mathfrak{R}_{s_1 s_2}$. In virtue of Proposition 6.2.3

$$\mathfrak{R}_{s_1 s_2} = e^{\langle x, s_1 s_2 \nu \rangle} \sum_{s \in W} \epsilon(s) \int_{\mathbb{R}_+ \times \partial \mathcal{C}_2} \frac{\langle s(x-y), e_2^* \rangle}{2\pi t^2} e^{\langle s(x-y)+y-x, z \rangle} e^{-\frac{|x-y-z+s_1 s_2 \nu t|^2}{2t}} dt dz.$$

These integrals are maximal for $s \in \{Id, s_2\}$ so it suffices to bound them in this case. Then the behaviour of the integral is governed by the minimum of the function $F : \partial\mathcal{C}_2 \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ given by

$$F(z, t) := \frac{|\mathbf{x} - y - z + s_1 s_2 \nu t|^2}{2t}.$$

We need to distinguish between two possibilities: first of all if $\langle \mathbf{x} - y, \omega_1 \rangle \langle \nu, e_1 \rangle > \langle \mathbf{x} - y, e_2 \rangle \langle \nu, \omega_2 \rangle$, then the minimum of F is attained at $t_0 = \frac{\langle \mathbf{x} - y, e_2 \rangle}{\langle \nu, e_1 \rangle}$ and $z_0 = \langle \mathbf{x} - y + s_1 s_2 \nu t_0, \omega_1 \rangle \omega_1$, where F is found to be equal to $\langle \mathbf{x} - y, e_2 \rangle \langle \nu, e_1 \rangle$. If $\langle \mathbf{x} - y, \omega_1 \rangle \langle \nu, e_1 \rangle > \langle \mathbf{x} - y, e_2 \rangle \langle \nu, \omega_2 \rangle$ then this minimum is reached at $t_0 = \frac{|\mathbf{x} - y|}{|\nu|}$ and $z_0 = 0$, and $F(z_0, t_0) = |\mathbf{x} - y| |\nu| + \langle \mathbf{x} - y, s_1 s_2 \nu \rangle$, which in the worst-case scenario where $s_1 s_2 s_1 \nu$ and $y - \mathbf{x}$ are colinear, can be bounded below by $\langle \mathbf{x} - y, e_2 \rangle \langle \nu, e_1 \rangle$. In both cases we have the bound

$$F(z_0, t_0) \geq \langle \mathbf{x} - y, e_2 \rangle \langle \nu, e_1 \rangle.$$

By Laplace's method we can therefore estimate the whole integral and check that

$$\int_{\mathbb{R}_+ \times \partial\mathcal{C}_2} \frac{\langle \mathbf{x} - y, e_2^* \rangle}{2\pi t^2} e^{-F(z, t)} dt dz \leq C e^{-F(z_0, t_0)}$$

where C is uniformly bounded as $\mathbf{x} \rightarrow \infty$. Therefore the term $\mathfrak{R}_{s_1 s_2}$ is indeed a $\mathcal{O}(e^{\langle \mathbf{x}, s_1 s_2 \nu \rangle - \eta \langle \mathbf{x}, e_2 \rangle \langle \nu, e_1 \rangle})$, which is as desired.

We can now turn to the other remainder terms, and follow the approach just developed for $\mathfrak{R}_{s_1 s_2}$. Namely we need to evaluate the minimum of the map $F : \partial\mathcal{C}_1 \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ defined by setting

$$F(z, t) := \frac{|\mathbf{x} - y - z + w \nu t|^2}{2t}$$

for $w = s_2 s_1$ and $w = s_1 s_2 s_1$. Like before we have $F(z_0, t_0) \geq \langle \mathbf{x} - y, e_1 \rangle \langle \nu, e_2 \rangle$ for $w = s_2 s_1$, but it can occur that $F(z_0, t_0) = 0$ depending on the values of $\mathbf{x}$ and ν . Nevertheless we can bound the integrals by a $\mathcal{O}(e^{-F(z_0, t_0)})$ by Laplace's method, which yields the upper bound on the remainders

$$\mathfrak{R}_{s_2 s_1} \leq C e^{\langle \mathbf{x}, s_1 s_2 s_1 \nu \rangle} \quad \text{and} \quad \mathfrak{R}_{s_1 s_2 s_1} \leq C e^{\langle \mathbf{x}, s_1 s_2 s_1 \nu \rangle}$$

where we have used that $\langle \mathbf{x}, s_2 s_1 \nu \rangle - \langle \mathbf{x} - y, e_1 \rangle \langle \nu, e_2 \rangle = \langle \mathbf{x}, s_1 s_2 s_1 \nu \rangle$. All together we see that

$$\mathfrak{R}_{s_1 s_2} + \mathfrak{R}_{s_2 s_1} + \mathfrak{R}_{s_1 s_2 s_1} = \mathcal{O}(e^{\langle s_1 s_2 \nu, \mathbf{x} \rangle + \eta \langle \mathbf{x}, e_2 \rangle \langle \nu, e_1 \rangle}).$$

□

Thanks to this Proposition we see that, with high probability, the process $\mathcal{B}^\nu$ will stay in the domain $\mathcal{U}_1 \cup \mathcal{U}_2$. Put differently, in the asymptotic where $\mathbf{x} \rightarrow \infty$ inside $\mathcal{C}$, at any time at most one component of the process along the simple roots (e_1, e_2) will not be very positive.

6.2.1.2 The general case

Up to reordering the simple roots $e_1, \dots, e_r$ we consider $s = s_1 \cdots s_r$ in what follows. Let us denote by i_0 the first index such that s_{i_0} does not commute with all the s_j for

$j \leq i_0$ (note that s_a and s_b commute means that $\langle e_a, e_b \rangle = 0$, and also that $s_a e_b = e_b$). For a positive $\eta > 0$, let us introduce the subset of $\mathcal{C}$

$$\mathcal{U}_s := \{z \in \mathcal{C} \text{ s.t. } \forall i_0 \leq j \leq r, \quad \langle z, e_j \rangle > (1 - \eta) \langle \mathbf{x}, e_j \rangle\},$$

and the corresponding event $\mathcal{E}_s$ that the process $\mathbf{B}^\nu$ stays inside $\mathcal{U}_s$ until it hits the boundary of $\mathcal{C}$. In particular under $\mathcal{E}_s$ the process will hit $\partial\mathcal{C}$ over its boundary component $\partial\mathcal{C}_s := \bigcup_{j=1}^{i_0-1} \partial\mathcal{C}_j$.

Proposition 6.2.5. *Assume that $\mathbf{x} \rightarrow \infty$ inside $\mathcal{C}$ in the asymptotic where $\langle s\nu - s'\nu, \mathbf{x} \rangle \rightarrow +\infty$ for all $s' \neq s \in W_{1,\dots,r}$. Then*

$$\lim_{\mathbf{x} \rightarrow \infty} \mathbb{P}_{\mathbf{x}}(\mathcal{E}_s) = 1. \quad (6.2.7)$$

Proof. Like before thanks to Proposition 6.2.3 we can write

$$U(\mathbf{x})\mathbb{P}_{\mathbf{x}}(\mathcal{E}_s) = \epsilon(s)\lambda_s e^{\langle s\nu, \mathbf{x} \rangle} + \sum_{w \in W_{1,\dots,r}} \epsilon(w)\lambda_w \mathfrak{R}_w$$

where, with y such that $\langle y, e_j \rangle = 0$ for $j < i_0$ and $\langle y, e_j \rangle = (1 - \eta) \langle \mathbf{x}, e_j \rangle$ for $i_0 \leq j \leq r$,

$$\begin{aligned} \mathfrak{R}_s &:= e^{\langle \mathbf{x}, s\nu \rangle} \sum_{\tau \in W} \epsilon(\tau) \int_{\mathbb{R}_+ \times \partial\mathcal{C} \setminus \partial\mathcal{C}_s} \frac{\langle \tau(\mathbf{x} - y), e_n^* \rangle}{2\pi t^2} e^{\langle \tau(\mathbf{x} - y) + y - \mathbf{x}, z \rangle} e^{-\frac{|\mathbf{x} - y - z + s\nu t|^2}{2t}} dt dz, \\ \mathfrak{R}_w &:= e^{\langle \mathbf{x}, w\nu \rangle} \sum_{\tau \in W} \epsilon(\tau) \int_{\mathbb{R}_+ \times \partial\mathcal{C}_s} \frac{\langle \tau(\mathbf{x} - y), e_n^* \rangle}{2\pi t^2} e^{\langle \tau(\mathbf{x} - y) + y - \mathbf{x}, z \rangle} e^{-\frac{|\mathbf{x} - y - z + w\nu t|^2}{2t}} dt dz \quad w \neq s, \end{aligned} \quad (6.2.8)$$

where e_n^* denotes the (normalized) simple root normal orthogonal to the boundary component where z lies. Therefore the proof of Proposition 6.2.5 boils down to studying these integrals.

To start with, we note that for $z \in \partial\mathcal{C}$ we have $e^{\langle \tau(\mathbf{x} - y) + y - \mathbf{x}, z \rangle} \leq 1$ so the asymptotic is governed by the integrals with $\tau = I_d$. Then by Laplace's method the behaviour of the integrals that appear in $\mathfrak{R}_w$, $w \neq s$, is governed by the minimum of the map $F : \partial\mathcal{C}_s \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ given by

$$F(z, t) := \frac{|\mathbf{x} - y - z + w\nu t|^2}{2t}.$$

This allows to show that the remainders $\mathfrak{R}_w$ for $w \neq s$ are lower-order terms in the asymptotic considered, since $e^{\langle w\nu - s\nu, \mathbf{x} \rangle} \rightarrow 0$.

Likewise the term $\mathfrak{R}_s$ is also a lower-order term in the asymptotic considered, and for this by Laplace's method it is enough to prove that the map $F : \partial\mathcal{C} \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ with $w = s$ attains its minimum at some $z_0 \in \partial\mathcal{C}_s$ and $t_0 > 0$ with $F(z_0, t_0) = 0$. Now because $s\nu \notin \mathcal{C}$ we know that this amounts to saying that the half-line $t \mapsto \mathbf{x} + s\nu t$ crosses $y + \partial\mathcal{C}$ on $y + \partial\mathcal{C}_s$, which follows from the fact that for $j \geq i_0$ we have $\langle s\nu, e_j \rangle > 0$. To see why, note that $\langle s\nu, e_j \rangle \leq 0$ implies that $s_j s \notin W_{1,\dots,r}$ since otherwise we would have

$$\langle s\nu - s_j s\nu, \mathbf{x} \rangle = \langle s\nu, e_j^\vee \rangle \langle \mathbf{x}, e_j^\vee \rangle \leq 0,$$

which contradicts our assumptions on the asymptotic of $\mathbf{x}$. Moreover explicit computations show that $\langle s_j s_1 \cdots s_r \nu, \omega_j^\vee \rangle > 0$ as soon as $j \geq i_0$. As a consequence having $s_j s \notin W_{1,\dots,r}$ implies that $j \leq i_0$. Therefore having $\langle s\nu, e_j \rangle \leq 0$ implies that $j \leq i_0$. $\square$

6.2.2 . On the decomposition of the path: the A_2 case

With Proposition 6.2.4 at hand we are now in position to describe the path of the process $\mathcal{B}^\nu$ itself in the asymptotic where the starting point of the process x diverges inside the Weyl chamber $\mathcal{C}$, and to start with we will focus on the A_2 case. As explained above, it is very natural to consider the subsets $\mathcal{U}_1$ and $\mathcal{U}_2$ of $\mathcal{C}$, and associated events $\mathcal{E}_1$ and $\mathcal{E}_2$. Because thanks to Proposition 6.2.4 we know that in the asymptotic where $x \rightarrow \infty$ we have that $\mathbb{P}_x(\mathcal{E}_1) + \mathbb{P}_x(\mathcal{E}_2) = 1 - \mathcal{O}(e^{-\langle \lambda, x \rangle})$ for some $\lambda \in \mathcal{C}$, we can look at the process $\mathcal{B}^\nu$ conditioned on one of the events $\mathcal{E}_i$ and retain all the necessary information. Without loss of generality we will consider the process $\mathcal{B}^\nu$ conditioned on the event $\mathcal{E}_1$.

In order to prove Proposition 6.2.1 we will look at each component of the path separately, according to the moments where the process contribute or not to the integrals J_i as described in Figure 6.5. To be more specific we will introduce stopping times T_1, T'_1 and T_2 such that:

- the portion in blue in Figure 6.5 corresponds to the path of the process in the time interval (T_1, T'_1) ;
- the component in red there is associated to the time interval $(T_2, +\infty)$;
- the portion in orange corresponds to $(0, T_1) \cup (T'_1, T_2)$.

6.2.2.1 The process before $T_{\partial\mathcal{C}}$

Recall that before hitting $\partial\mathcal{C}$ for the first time, the process $\mathcal{B}^\nu$ has a drift given by

$$\nabla \log \partial_{1,2} h(z) = \sum_{w \in W_{1,2}} w \frac{\epsilon(w) \lambda_w e^{\langle w\nu, z \rangle}}{U(z)}, \quad U(z) = \sum_{w \in W_{1,2}} \epsilon(w) \lambda_w e^{\langle w\nu, z \rangle}.$$

The latter can be rewritten under the form

$$s_1 s_2 \nu + (s_2 s_1 \nu - s_1 s_2 \nu) \frac{\lambda_{s_2 s_1} e^{\langle s_2 s_1 \nu, z \rangle}}{\sum_{w \in W_{1,2}} \epsilon(w) \lambda_w e^{\langle w\nu, z \rangle}} + (s_1 s_2 \nu - s_1 s_2 s_1 \nu) \frac{\lambda_{s_1 s_2 s_1} e^{\langle s_1 s_2 s_1 \nu, z \rangle}}{\sum_{w \in W_{1,2}} \epsilon(w) \lambda_w e^{\langle w\nu, z \rangle}}.$$

Under the assumption that the process stays within the domain $\mathcal{U}_1$, this drift is equal to

$$s_1 s_2 \nu + \mathfrak{R}_1(z), \quad \text{with} \quad |\mathfrak{R}_1(z)| \leq C e^{\langle \nu, e_2 \rangle \langle z, e_1 \rangle - (1-\eta) \langle \nu, e_1 \rangle \langle x, e_2 \rangle}$$

where C only depends on ν .

Now for $0 < \eta < \frac{1}{2}$ let us consider T_1 to be the first time where the e_1 component of the process reaches $\eta \langle x, e_2 \rangle$, that is

$$T_1 := \inf_{t \geq 0} \langle \mathcal{B}_t^\nu, e_1 \rangle < \eta \langle x, e_2 \rangle \quad (\text{which is finite almost surely}),$$

$$\text{and also set } T_0 := \inf_{t \geq T_1} \langle \mathcal{B}_t^\nu, e_1 \rangle > (1 - 2\eta) \frac{\langle \nu, e_1 \rangle}{\langle \nu, e_2 \rangle} \langle x, e_2 \rangle.$$

The following lemma explains that as soon as the e_1 component has reached the level $\eta \langle x, e_2 \rangle$ (that is for $t \geq T_1$), it will likely stay small until $\mathcal{B}^\nu$ hits the boundary of the Weyl chamber $\mathcal{C}$:

Lemma 6.2.6. For $\eta > 0$ small enough, as $x \rightarrow \infty$ inside $\mathcal{C}$

$$\lim_{x \rightarrow \infty} \mathbb{P}_x(T_{\partial\mathcal{C}} < T_0) = 1. \quad (6.2.9)$$

Proof. Between T_1 and T_0 the e_1 component of the drift of $\mathcal{B}^\nu$ is given by $-\langle \nu, \rho \rangle + \langle \mathfrak{R}_1(z), e_1 \rangle$ with $|\langle \mathfrak{R}_1(z), e_1 \rangle| \leq C'e^{-\eta\langle x, e_2 \rangle}$. As a consequence the probability that $T_0 < T_{\partial\mathcal{C}}$ is smaller than the probability that a Brownian motion with drift $-\langle \nu, \rho \rangle + C'e^{-\eta\langle x, e_2 \rangle}$ (and variance $\langle e_1, e_1 \rangle$) reaches $\left((1-2\eta)\frac{\langle \nu, e_1 \rangle}{\langle \nu, e_2 \rangle} - \eta\right)\langle x, e_2 \rangle$ before reaching $-\eta\langle x, e_2 \rangle$, which is readily seen to converge to 0 as $\langle x, e_2 \rangle \rightarrow +\infty$ provided that η is chosen small enough. $\square$

Therefore we see that between T_1 and $T_{\partial\mathcal{C}}$, with high probability (under the event $\mathcal{E}_1$) the drift of the process $\mathcal{B}^\nu$ will be given by that of $\mathbf{Y}^1$ plus a remainder term $\mathfrak{R}_1$ uniformly bounded by $Ce^{-\eta\langle x, e_2 \rangle}$.

6.2.2.2 The process between $T_{\partial\mathcal{C}}$ and $T_{\partial\mathcal{C}_2}$

After having hit the boundary of $\mathcal{C}$, the process $\mathcal{B}^\nu$ will have a drift given by

$$\nabla \log \partial_2 h^{\mathbf{M}}(z) = \sum_{s \in W_2} \frac{\epsilon(s) \lambda_s e^{\langle s\nu, z \rangle}}{U_2(z)} s\nu, \quad U_2(z) = \sum_{s \in W_2} \epsilon(s) \lambda_s e^{\langle s\nu, z \rangle}.$$

Like before this drift admits the expression

$$\nabla \log \partial_2 h^{\mathbf{M}}(z) = \frac{s_2\nu - s_1 s_2 \nu e^{\langle s_1 s_2 \nu - s_2 \nu, z \rangle}}{1 - e^{\langle s_1 s_2 \nu - s_2 \nu, z \rangle}} + \mathfrak{R}_2(z) \quad (6.2.10)$$

where the remainder term can be bounded as $|\mathfrak{R}_2(z)| \leq Ce^{-\langle \nu, e_1 \rangle \langle z, \rho \rangle}$. With high probability this remainder is negligible between $T_{\partial\mathcal{C}}$ and $T_{\partial\mathcal{C}_2}$. Indeed, we already know that at time $T_{\partial\mathcal{C}}$ under the event $\mathcal{E}_1^\eta$ we have $\langle \mathcal{B}^\nu, e_2 \rangle > (1-\eta)\langle x, e_2 \rangle$, and therefore that $|\mathfrak{R}_2(\mathcal{B}^\nu)| \leq Ce^{-\eta\langle \nu, e_1 \rangle \langle x, e_2 \rangle}$ between $T_{\partial\mathcal{C}}$ and T_2 , where

$$T_2 := \inf_{t \geq 0} \langle \mathcal{B}_t^\nu, e_2 \rangle < \eta\langle x, e_2 \rangle. \text{ Likewise if } T_1' := \sup_{t \leq T_{\partial\mathcal{C}_2}} \langle \mathcal{B}_t^\nu, e_1 \rangle > \eta\langle x, e_2 \rangle$$

then for t between T_1' and $T_{\partial\mathcal{C}_2}$, $|\mathfrak{R}_2(\mathcal{B}^\nu)| \leq Ce^{-\eta\langle \nu, e_1 \rangle \langle x, e_2 \rangle}$. Therefore for such times the process will behave like $\mathbf{Y}^1$, and all we need to check is that with probability $1 - o(1)$ we have $T_1' < T_2$:

Lemma 6.2.7. For $\eta > 0$ small enough, as $x \rightarrow \infty$ inside $\mathcal{C}$

$$\lim_{x \rightarrow \infty} \mathbb{P}_x(T_1' < T_2 | \mathcal{E}_1^\eta) = 1. \quad (6.2.11)$$

Proof. First assume that $\mathfrak{R}_2 = 0$. In that case the e_1 component of $\mathcal{B}^\nu$ has the law of a Brownian motion with positive drift $\langle \nu, \rho \rangle$ and variance $\langle e_1, e_1 \rangle = 2$ conditioned to stay positive, while the e_2 component of the path is an independent Brownian motion with negative drift lower bounded by $-\langle \nu, \rho \rangle$ and variance $\langle e_2, e_2 \rangle = 2$. As a consequence by the time-reversal property of drifted Brownian motion [138] $T_1' - T_{\partial\mathcal{C}}$ has same law as $\tau_1' := \inf_{t \geq 0} \sqrt{2}B_t + \langle \nu, \rho \rangle t > \eta\langle x, e_2 \rangle$ where B has the law of a standard one-dimensional

Brownian motion started from 0. The law of τ'_1 is well known [77, Equation (5.12)] and given by

$$\mathbb{P}(\tau'_1 \in dt) = \frac{\langle \nu, \rho \rangle}{\sqrt{4\pi t^3}} e^{-\frac{(\eta\langle \mathbf{x}, e_2 \rangle - \langle \nu, \rho \rangle t)^2}{4t}},$$

which concentrates around $\eta \frac{\langle \mathbf{x}, e_2 \rangle}{\langle \nu, \rho \rangle}$ as the latter diverges. Likewise we see that with probability $1 - o(1)$ we have $\tau_2 - T_{\partial\mathcal{C}} > (1 - 3\eta) \frac{\langle \mathbf{x}, e_2 \rangle}{\langle \nu, \rho \rangle}$, so by taking η small enough so that $\frac{1-3\eta}{\eta} > 2$ we see that Equation (6.2.11) does indeed hold for $\mathfrak{R}_2 = 0$.

Now it is readily seen that for $T_{\partial\mathcal{C}} \leq t \leq T'_1 \wedge T_2$ we have $|\mathfrak{R}_2(\mathbf{B}^\nu)| \leq C e^{-\eta\langle \nu, e_1 \rangle \langle \mathbf{x}, e_2 \rangle}$. Therefore for any positive ε and with η small enough like above, we can assume that $(1-\varepsilon)T'_1 < \tau'_1 < 2\tau_2 < (2+\varepsilon)T_2$ with asymptotic probability 1, showing Equation (6.2.11) in the general case. $\square$

Therefore between $T_{\partial\mathcal{C}}$ and $T_{\partial\mathcal{C}_2}$ the process will behave like $\mathbf{Y}^1$, in the sense that $\mathbf{B}^\nu$ will have a drift given by that of $\mathbf{Y}^1$ plus a remainder term $\mathfrak{R}_2$, where $|\mathfrak{R}_2| \leq C e^{-\langle \nu, e_1 \rangle \langle \mathbf{x}, e_2 \rangle}$.

6.2.2.3 The process after $T_{\partial\mathcal{C}_2}$

The third component of the process, *i.e.* after having hit both components of $\partial\mathcal{C}$, has the law of B^ν conditioned to stay inside $\partial\mathcal{C}$. This means that its drift is equal to

$$\nabla \log h^{\mathbf{M}}(z) = \sum_{s \in W_2} \frac{\epsilon(s) \lambda_s e^{\langle s\nu, z \rangle}}{U_3(z)} s\nu, \quad U_3(z) = \sum_{s \in W_2} \epsilon(s) \lambda_s e^{\langle s\nu, z \rangle},$$

which coincides with the drift of the process $\mathbf{Y}^1$.

For future reference and in order to prove Proposition 6.2.2 we stress that

$$\nabla \log h^{\mathbf{M}}(z) = \frac{\nu - s_2 \nu e^{\langle s_2 \nu - \nu, z \rangle}}{1 - e^{\langle s_2 \nu - \nu, z \rangle}} + \mathfrak{R}_3(z), \quad (6.2.12)$$

where $|\mathfrak{R}_3(z)| \leq C e^{-\langle \nu, e_1 \rangle \langle z, e_1 \rangle}$. Like before we can use the explicit expression of the drift to show that with probability asymptotically 1 the e_1 component of the process will never come back to $\eta\langle \mathbf{x}, e_2 \rangle$ after $T_{\partial\mathcal{C}}$, so that the e_2 component of $\mathbf{B}^\nu$ after $T_{\partial\mathcal{C}_2}$ has the law of a Brownian motion with positive drift $\langle \nu, e_2 \rangle$ and variance 2 conditioned to stay positive.

6.2.3 . Proof of Propositions 6.2.1 and 6.2.2 for A_2

We are now in position to prove Propositions 6.2.1 and 6.2.2 under the assumption that the root system being considered is associated to A_2 . Let us decompose the integrals J_i , $i = 1, 2$, as

$$J_i = \int_0^{T_1} e^{-\langle \mathbf{B}_t^\nu, e_i \rangle} Z_t^i dt + \int_{T_1}^{+\infty} e^{-\langle \mathbf{B}_t^\nu, e_i \rangle} Z_t^i dt.$$

As illustrated in Figure 6.5 the integrals over the subset $[0, T_1]$ will become negligible in the $x \rightarrow \infty$ limit. Indeed, in this region the e_1 and e_2 components of the process are bounded below by $(1 - 2\eta)\langle \mathbf{x}, e_2 \rangle$, so that

$$\int_0^{T_1} e^{-\langle \mathbf{B}_t^\nu, e_i \rangle} Z_t^i dt \leq e^{(1-3\eta)\langle \mathbf{x}, e_2 \rangle} \int_0^{T_1} e^{-(\langle \mathbf{B}_t^\nu, e_i \rangle - (1-3\eta)\langle \mathbf{x}, e_2 \rangle)} Z_t^i dt$$

where the integral on the right-hand side is uniformly bounded in $\mathbf{x}$. As a consequence for any bounded continuous function $F : \mathbb{R}^2 \rightarrow \mathbb{R}$

$$\mathbb{E}_{\mathbf{x}} [F(J_1, J_2) | \mathcal{E}_1] = \mathbb{E}_{\mathbf{x}} \left[F \left(\int_{T_1}^{+\infty} e^{-\langle \mathbf{B}_t^\nu, e_1 \rangle} Z_t^1 dt, \int_{T_1}^{+\infty} e^{-\langle \mathbf{B}_t^\nu, e_2 \rangle} Z_t^2 dt \right) | \mathcal{E}_1 \right] + o(1).$$

By the Markov property of $\mathbf{B}^\nu$, we know that the process $(\mathbf{B}_t^\nu)_{t \geq T_1}$ only depends on $(\mathbf{B}_t^\nu)_{t < T_1}$ via the location of $\mathbf{B}_{T_1}^\nu$. Therefore the latter can be rewritten as

$$\int_{u+\partial\mathcal{C}_2} \mathbb{P}_{\mathbf{x}} (\mathbf{B}_{T_1}^\nu \in dz) \mathbb{E}_z \left[F \left(\int_0^{+\infty} e^{-\langle \mathbf{B}_t^\nu, e_1 \rangle} Z_{t+T_1}^1 dt, \int_0^{+\infty} e^{-\langle \mathbf{B}_t^\nu, e_2 \rangle} Z_{t+T_1}^2 dt \right) | \mathcal{E}_1 \right]$$

where $u = u(\mathbf{x})$ is such that $\langle u, e_2 \rangle = (1 - \eta) \langle \mathbf{x}, e_2 \rangle$ and $\langle u, e_1 \rangle = \eta \langle \mathbf{x}, e_1 \rangle$.

Now we have seen that after T_1 the drift of the process $\mathbf{B}^\nu$ and that of $\mathbf{Y}^1$ only differ by a remainder term $\mathfrak{R}(x)$ which becomes negligible in the $\mathbf{x} \rightarrow \infty$ limit. Using continuity of F together with a comparison result such as [77, Chapter 5-Proposition 2.18] to compare the processes $\mathbf{B}^\nu$ and $\mathbf{Y}^1$ we see that the latter is asymptotically equivalent to

$$\int_{u+\partial\mathcal{C}_2} \mathbb{P}_{\mathbf{x}} (\mathbf{B}_{T_1}^\nu \in dz) \mathbb{E}_z \left[F \left(\int_0^{+\infty} e^{-\langle \mathbf{Y}_t^1, e_1 \rangle} Z_t^1 dt, \int_0^{+\infty} e^{-\langle \mathbf{Y}_t^1, e_2 \rangle} Z_t^2 dt \right) | \mathcal{E}_1 \right]$$

where we have used stationarity of the process (Z^1, Z^2) . To conclude for the proof of Proposition 6.2.4 it remains to check that the expectation that appears in the integral becomes independent of z in the limit where $\mathbf{x}$ diverges inside $\mathcal{C}$. More precisely we prove that

Lemma 6.2.8. *As $\mathbf{x} \rightarrow \infty$ inside $\mathcal{C}$, for any $z \in \partial\mathcal{C}_2$*

$$\lim_{\mathbf{x} \rightarrow \infty} \mathbb{E}_{u(\mathbf{x})+z} \left[F \left(\int_0^{+\infty} e^{-\langle \mathbf{Y}_t^1, e_1 \rangle} Z_t^1 dt, \int_0^{+\infty} e^{-\langle \mathbf{Y}_t^1, e_2 \rangle} Z_t^2 dt \right) \right] = \mathbb{E} [F(J(\langle \nu, \rho \rangle), J'(\langle \nu, e_2 \rangle))]$$

where J and J' are independent and with law described by Equation (6.2.3).

Assuming for now that such a statement does indeed hold, we see recovering terms that

$$\mathbb{E}_{\mathbf{x}} [F(J_1, J_2) | \mathcal{E}_1] \sim \mathbb{E} [F(J(\langle \nu, \rho \rangle), J'(\langle \nu, e_2 \rangle))],$$

and $\mathbb{E}_{\mathbf{x}} [F(J_1, J_2) | \mathcal{E}_1] \sim \mathbb{E}_{\mathbf{x}} [F(J_1(\mathbf{Y}^1), J_2(\mathbf{Y}^1))]$. As a consequence using Proposition 6.2.4 we can write that

$$\mathbb{E}_{\mathbf{x}} [F(J_1, J_2)] = \mathbb{P}_{\mathbf{x}}(\mathcal{E}_1) \mathbb{E} [F(J(\langle \nu, \rho \rangle), J'(\langle \nu, e_2 \rangle))] + \mathbb{P}_{\mathbf{x}}(\mathcal{E}_2) \mathbb{E} [F(J(\langle \nu, \rho \rangle), J'(\langle \nu, e_2 \rangle))] + o(1),$$

where besides $\mathbb{P}_{\mathbf{x}}(\mathcal{E}_1) \rightarrow 1$ and $\mathbb{P}_{\mathbf{x}}(\mathcal{E}_2) \rightarrow 0$ in the asymptotic where $\langle s_1 s_2 \nu - s_2 s_1 \nu, \mathbf{x} \rangle \rightarrow +\infty$. Therefore in this asymptotic we both have that

$$\mathbb{E}_{\mathbf{x}} [F(J_1, J_2)] \sim \mathbb{E} [F(J(\langle \nu, \rho \rangle), J'(\langle \nu, e_2 \rangle))] \quad \text{and} \quad \mathbb{E}_{\mathbf{x}} [F(J_1, J_2)] \sim \mathbb{E} [F(J_2(\mathbf{Y}^1), J_2(\mathbf{Y}^1))],$$

concluding for the proof of Proposition 6.2.1 and 6.2.2 for the A_2 case.

Proof of Lemma 6.2.8. Let us split the integrals involved as

$$\int_0^{T'_1} e^{-\langle \mathbf{Y}_t^1, e_1 \rangle} Z_t^1 dt + \int_{T'_1}^{+\infty} e^{-\langle \mathbf{Y}_t^1, e_1 \rangle} Z_t^1 dt, \quad \int_0^{T_2} e^{-\langle \mathbf{Y}_t^1, e_2 \rangle} Z_t^2 dt + \int_{T_2}^{+\infty} e^{-\langle \mathbf{Y}_t^1, e_2 \rangle} Z_t^2 dt.$$

Like before the integrals $\int_{T'_1}^{+\infty} e^{-\langle \mathbf{Y}_t^1, e_1 \rangle} Z_t^1 dt$ and $\int_0^{T_2} e^{-\langle \mathbf{Y}_t^1, e_2 \rangle} Z_t^2 dt$ will become negligible in the $x \rightarrow \infty$ limit, whence by the Markov property for $\mathbf{Y}^1$

$$\begin{aligned} & \mathbb{E}_z \left[F \left(\int_0^{+\infty} e^{-\langle \mathbf{Y}_t^1, e_1 \rangle} Z_t^1 dt, \int_0^{+\infty} e^{-\langle \mathbf{Y}_t^1, e_2 \rangle} Z_t^2 dt \right) \right] \\ &= \int_{v+\partial\mathcal{C}_1} \mathbb{P}_z (\mathbf{Y}_{T_2}^1 \in dz') \mathbb{E}_{z,z'} \left[F \left(\int_0^{T'_1} e^{-\langle \mathbf{Y}_t^1, e_1 \rangle} Z_t^1 dt, \int_0^{+\infty} e^{-\langle \tilde{\mathbf{Y}}_t^1, e_2 \rangle} Z_{t+T_2}^2 dt \right) \right] + o(1) \end{aligned}$$

where $v = \eta\langle x, e_2 \rangle \omega_2$ and under $\mathbb{E}_{z,z'}$ the processes $\mathbf{Y}^1, \tilde{\mathbf{Y}}^1$ are independent and started respectively from z and z' . We have seen before that after T_2 , the e_1 component of the process $\mathbf{Y}^1$ was with high probability very large and therefore that the e_2 component of the process for $t > T_2$ behaves like a one-dimensional Brownian motion with negative drift $-\langle \nu, e_2 \rangle$ and variance 2 upon hitting the origin and positive drift $\langle \nu, e_2 \rangle$ conditioned to stay positive after having hit 0, and only depends of z' through $\langle z', e_2 \rangle = \eta\langle x, e_2 \rangle$. The same applies for the e_1 component of the process before time T'_1 . Moreover we have already seen that $T_2 - T'_1 \rightarrow \infty$ almost surely, whence the processes $(Z_t^1)_{t \leq T_1}$ and $(Z_t^2)_{t \geq T_2}$ decorrelate as $x \rightarrow \infty$. As a consequence and based on [77, Chapter 5-Proposition 2.18] we see that by stationarity of Z

$$\begin{aligned} & \mathbb{E}_z \left[F \left(\int_0^{+\infty} e^{-\langle \mathbf{Y}_t^1, e_1 \rangle} Z_t^1 dt, \int_0^{+\infty} e^{-\langle \mathbf{Y}_t^1, e_2 \rangle} Z_t^2 dt \right) \right] \\ & \sim \int_{v+\partial\mathcal{C}_1} \mathbb{P}_z (\mathbf{Y}_{T_2}^1 \in dz') \mathbb{E}_{\eta\langle x, e_2 \rangle, \eta\langle x, e_2 \rangle} \left[F \left(\int_0^{T'_1} e^{-B_t^{(\nu, \rho)}} Z_t^1 dt, \int_0^{+\infty} e^{-\tilde{B}_t^{(\nu, e_2)}} Z_t^2 dt \right) \right] + o(1) \end{aligned}$$

which is independent of z and converges to $\mathbb{E}[F(J(\langle \nu, \rho \rangle), J'(\langle \nu, e_2 \rangle))]$ as desired. $\square$

6.2.4 . The general case

In the general case the main ideas remain unchanged and the arguments are the same. Therefore in this subsection we mostly shed light on the results that differ from the study of the A_2 case. We will consider $s = s_1 \cdots s_r$ in the sequel.

6.2.4.1 Decomposition of the path

In the same way as above, we see that the we are able to provide another decomposition of the path, based on the analogs of times T_1, T'_1, T_2 . The main difference will be that in the general case we have to split s as

$$s = (s_1 \cdots s_{i_0-1})(s_{i_0} \cdots s_{i_1-1}) \cdots (s_{i_{p-1}} \cdots s_r) =: w_0 \cdots w_p$$

where for all $0 \leq k \leq p$ and $i_{k-1} \leq j, l \leq i_k - 1$, the reflections s_j and s_l commute (via the convention $i_{-1} = 1$ and $i_p = r + 1$). In particular we have $se_j = -w_1 \cdots w_p e_j$ for all $j < i_0$.

Namely, we already know from Proposition 6.2.5 that the event $\mathcal{E}_s$ that the process stays inside the domain $\mathcal{U}_s$ has probability $1 - o(1)$. Therefore before reaching $\partial\mathcal{C}$ the process has a drift

$$\nabla \log \partial_{1,\dots,r} h^{\mathbf{M}}(z) = \sum_{w \in W_{1,\dots,r}} \frac{\epsilon(w) \lambda_w e^{\langle w\nu, z \rangle}}{U(z)} w\nu, \quad U(z) = \sum_{w \in W_{1,\dots,r}} \epsilon(w) \lambda_w e^{\langle w\nu, z \rangle},$$

which can be put under the form $s\nu + \mathfrak{R}_1(z)$ where $|\mathfrak{R}_1(z)| \leq C e^{-\eta \langle x, u \rangle}$ for any $x \in \mathcal{U}_s$ and where u is some vector inside $\mathcal{C}$, up to an event with probability asymptotically 0. The e_i component of this drift is negative and given by $-\langle w_1 \cdots w_p \nu, e_j \rangle$ for $j < i_0$, but positive equal to $\langle w_0 \cdots w_p \nu, e_j \rangle$ for $j \geq i_0$. We stress that before $T_{\partial\mathcal{C}}$ the e_j components of the path are bounded below for $j \geq i_0$, but this is not the case for $j < i_0$.

After having hit $\partial\mathcal{C}$ (say over $\partial\mathcal{C}_1$) for the first time, the drift of the process will be given by $\nabla \log \partial_{2,\dots,r} h^{\mathbf{M}}$. Denoting by $\mathbf{B}_{T_{\partial\mathcal{C}}}^\nu$ the location of the process when first hitting $\partial\mathcal{C}$ it is readily seen that it satisfies $\langle s_2 \cdots s_r \nu - s' \nu, \mathbf{B}_{T_{\partial\mathcal{C}}}^\nu \rangle \rightarrow +\infty$ for all $s' \in W_{2,\dots,r}$ as $x \rightarrow \infty$ with probability tending to 1. Indeed let us set r_1 to be equal to s_1 if $s' \notin W_{1,\dots,r}$ and Id if $s' \in W_{1,\dots,r}$, so that $r_i s' \in W_{1,\dots,r}$. Then we can write that

$$\langle s_2 \cdots s_r \nu - s' \nu, \mathbf{B}_{T_{\partial\mathcal{C}}}^\nu \rangle = \langle s_1 \cdots s_r \nu - r_1 s' \nu, \mathbf{B}_{T_{\partial\mathcal{C}}}^\nu \rangle,$$

since $s_1 \mathbf{B}_{T_{\partial\mathcal{C}}}^\nu = \mathbf{B}_{T_{\partial\mathcal{C}}}^\nu$. We can further write

$$\langle s_2 \cdots s_r \nu - s' \nu, \mathbf{B}_{T_{\partial\mathcal{C}}}^\nu \rangle = \langle s\nu - r_1 s' \nu, \mathbf{x} \rangle + \langle s\nu - r_1 s' \nu, \mathbf{B}_{T_{\partial\mathcal{C}}}^\nu - \mathbf{x} \rangle,$$

where the first term diverges to $+\infty$ since $r_1 s' \in W_{1,\dots,r}$, while the second one is positive (since $\mathbf{B}_{T_{\partial\mathcal{C}}}^\nu - \mathbf{x}$ stays between $s\nu t_0 \pm C\sqrt{t_0}^{1+\varepsilon}$ with probability $1 - o(1)$).

Therefore we can proceed in the same way after $T_{\partial\mathcal{C}}$: up to an event of probability $1 - o(1)$, before hitting $\partial\mathcal{C}$ for the second time the process will have a drift given by

$$\nabla \log \partial_{2,\dots,r} h^{\mathbf{M}}(z) = \frac{s_2 \cdots s_r \nu - s_1 \cdots s_r \nu e^{-\langle s_2 \cdots s_r \nu, e_1 \rangle \langle z, \nu \rangle}}{1 - e^{-\langle s_2 \cdots s_r \nu, e_1 \rangle \langle z, \nu \rangle}} + \mathfrak{R}_2(z)$$

with $|\mathfrak{R}_2(z)| \leq C e^{-\eta \langle x, u \rangle}$ close to $\mathbf{B}_{T_{\partial\mathcal{C}}}^\nu$. Now for $j < i_0$ the e_j component of the drift is negative and given by $\langle s_2 \cdots s_r \nu, e_j \rangle < 0$ but is still positive for $j \geq i_0$ as soon as $i_0 > 2$. Therefore before hitting $\partial\mathcal{C}$ for the second time and under the assumption that $i_0 > 2$ we see that the event that $\langle \mathbf{B}^\nu, e_j \rangle > \eta \langle x, e_j \rangle$ for $j \geq i_0$ has probability asymptotically 1.

A similar scheme will be carried out until the process has hit all the boundary components $\partial\mathcal{C}_j$ for $j < i_0$. After this event there will be a time T'_1 such that after T'_1 the e_j components of the path will be bounded below by $\eta \langle x, e_j \rangle$ for $j < i_0$. In addition the time $T_2 = \inf_{t \geq 0} \{ \langle \mathbf{B}_t^\nu, e_j \rangle < \eta \langle x, e_j \rangle \text{ for some } j \geq i_0 \}$ will be such that $T_2 - T'_1 \rightarrow +\infty$, and T_2 the drift of the process is given by $w_1 \cdots w_p \nu$. The components of the drift corresponding to $j \geq i_1$ or $j < i_0$ will remain bounded below but that corresponding to $i_0 \leq j < i_1$ will not be necessary bounded below. The process will then hit all the components of $\partial\mathcal{C}$ of the form $\partial\mathcal{C}_j$ for $i_0 \leq j < i_1$, and after having hit all these parts of $\partial\mathcal{C}$ there will be a time T'_2 such that after T'_2 the components associated to the roots e_j for $i_0 \leq j < i_1$ are bounded below, and $T_3 - T'_2 \rightarrow +\infty$ with $T_3 = \inf_{t \geq 0} \{ \langle \mathbf{B}_t^\nu, e_j \rangle < \eta \langle x, e_j \rangle \text{ for some } j \geq i_1 \}$. The same scheme will be repeated until the

process has hit all the boundary components; after that the process will have drift ν conditioned to stay inside $\mathcal{C}$.

In a nutshell, we see that the path can be divided between times $T_1 < T'_1 < T_2 < \dots < T_p < T'_p = +\infty$ such that

- Between T_k and T'_k , the process will contribute only to the integrals J_j where j ranges over $\{i_{k-1}, \dots, i_k - 1\}$, since the e_j components of the process for $j \notin \{i_{k-1}, \dots, i_k - 1\}$ will be bounded below by some constants of the form $\eta\langle \mathbf{x}, e_j \rangle$. Between T'_k and T_{k+1} all the e_j components of the path are bounded below by $\eta\langle \mathbf{x}, e_j \rangle$
- For such $j \in \{i_{k-1}, \dots, i_k - 1\}$, the e_j component of the process $\mathcal{B}^\nu$ between T_k and T'_k will behave like a Brownian motion with negative drift $-\langle w_{k+1} \dots w_p \nu, e_i \rangle$ and variance $\langle e_i, e_i \rangle$ upon hitting the origin, where it will be reflected and will have the law of a one-dimensional Brownian motion with positive drift $\langle w_{k+1} \dots w_p \nu, e_i \rangle$ and conditioned to stay positive. The e_j components for $j \notin \{i_{k-1}, \dots, i_k - 1\}$ will have a positive drift.
- With probability tending to 1 as $\mathbf{x} \rightarrow \infty$, all the time increments $T'_k - T_k$ and $T_{k+1} - T'_k$ will diverge to $+\infty$.

6.2.4.2 Proof of Proposition 6.2.2

We are now ready to address the proof of Proposition 6.2.2. Based on the above decomposition for the path, we can split the integrals involved, for $1 \leq j \leq r$, as:

$$J_j = \int_{T_k}^{T'_k} e^{-\langle \mathcal{B}_t^\nu, e_j \rangle} Z_t^j dt + \int_{[0, T_k) \cup (T'_k, +\infty)} e^{-\langle \mathcal{B}_t^\nu, e_j \rangle} Z_t^j dt$$

where k is such that $i_{k-1} \leq j \leq i_k - 1$. Like in the rank two case the second integral will vanish in the limit since for $t \notin [T_{k-1}, T'_k]$ the e_j component of the process can be bounded below by $\eta\langle \mathbf{x}, e_j \rangle$.

Now between time T_k and T'_k , the e_j components of the process $\mathcal{B}^\nu$ for $i_{k-1} \leq j \leq i_k - 1$ can be approximated by a Brownian motion with drift $-\langle w_{k+1} \dots w_p, e_j \rangle$ and variance $\langle e_j, e_j \rangle$ (joined with its reflection on the origin). Moreover since for such j, l we have $\langle e_j, e_l \rangle = 0$ we know that these Brownian motions are independent. Likewise, because $T_{k+1} - T'_k$ diverges to $+\infty$ with probability asymptotically 1, we can use the Markov property in the same way as in the A_2 case to see that the integrals that run over distinct intervals decorrelate in the limit. With these two decorrelations at hand we see that in the end for any bounded continuous functions $F_j : \mathbb{R} \rightarrow \mathbb{R}$

$$\mathbb{E}_{\mathbf{x}} \left[\prod_{j=1}^r F_j(J_j) \right] \sim \prod_{j=1}^r \mathbb{E}_{x_j} \left[F_j \left(\int_0^{+\infty} e^{-B_t^j} Z_t^j dt \right) \right],$$

where x_j is some positive number that diverges to $+\infty$ as $\mathbf{x} \rightarrow \infty$, and where B^j is a Brownian motion with variance $\langle e_j, e_j \rangle$, and negative drift $-\langle w_{k+1} \dots w_p, e_j \rangle$ upon

hitting the origin and positive drift $\langle w_{k+1} \cdots w_p, e_j \rangle$ conditioned to stay positive after. The latter does indeed converge towards

$$\prod_{j=1}^r \mathbb{E}_{+\infty} \left[F_j \left(\int_0^{+\infty} e^{-B_t^j} Z_t^j dt \right) \right],$$

concluding the proof of Proposition 6.2.2.

6.3 . Tail expansion of correlated Gaussian multiplicative chaos measures and reflection coefficients

In the two previous sections we have studied some consequences of the reflection principle in the realm of probability theory which led us to the derivation of a Brownian path decomposition based on a process whose properties have been investigated. Based on these features, in this section we make explicit the connection between this reflection principle and the one present in Toda theories, and which manifests itself via the existence of *reflection coefficients*. Namely we show that Toda reflection coefficients arise in the tail expansion of GMC measures as the following statements disclose.

Theorem 6.3.1. *Assume that $\alpha - Q \in \mathcal{C}_-$ satisfies $\langle \hat{s}\alpha - \alpha, \omega_i^\vee \rangle < \gamma$ for all $1 \leq i \leq r$ and $s \in W_{1,\dots,r}$. Further assume that $s \in W_{1,\dots,r}$ with length r is such that $\langle \hat{s}\alpha - \hat{s}'\alpha, \mathbf{c} \rangle \rightarrow +\infty$ for all $s' \neq s \in W_{1,\dots,r}$. Then there exists a non-zero real number $R_s(\alpha)$ such that*

$$\mathbb{E} \left[\prod_{k=1}^p \left(\exp \left(-e^{\gamma \langle \mathbf{c}, e_{i_k} \rangle} I_{i_k}(\alpha) \right) - 1 \right) \right] \sim e^{\langle \hat{s}\alpha - \alpha, \mathbf{c} \rangle} R_s(\alpha). \quad (6.3.1)$$

$R_s(\alpha)$ is a Toda reflection coefficient, and is equal to

$$R_s(\alpha) = \epsilon(s) \frac{A(s(\alpha - Q))}{A(\alpha - Q)}, \quad \text{where}$$

$$A(\alpha) = \prod_{i=1}^r \left(\mu_i \pi l \left(\frac{\gamma^2 \langle e_i, e_i \rangle}{4} \right) \right)^{\frac{\langle \alpha, \omega_i^\vee \rangle}{\gamma}} \prod_{e \in \Phi^+} \Gamma \left(1 - \frac{\gamma}{2} \langle \alpha, e \rangle \right) \Gamma \left(1 - \frac{1}{\gamma} \langle \alpha, e^\vee \rangle \right). \quad (6.3.2)$$

Theorem 6.3.2. *Under the same assumptions as in Theorem 6.3.1,*

$$\mathbb{P} \left(\int_{\mathbb{D}} |x|^{-\gamma \langle \alpha, e_{i_k} \rangle} M^{\gamma e_i}(d^2x) > e^{-\gamma \langle \mathbf{c}, e_{i_k} \rangle}, \quad k = 1, \dots, p \right) \sim \overline{R}_s(\alpha) e^{\langle \hat{s}\alpha - \alpha, \mathbf{c} \rangle} \quad (6.3.3)$$

where the unit volume reflection coefficient $\overline{R}_s(\alpha)$ takes the form of

$$\overline{R}_s(\alpha) := \epsilon(s) \prod_{i=1}^r \Gamma \left(1 - \frac{1}{\gamma} \langle \hat{s}\alpha - \alpha, \omega_i^\vee \rangle \right)^{-1} R_s(\alpha). \quad (6.3.4)$$

In the above we have considered the subset $W_{1,\dots,r}$ of W defined as the set of the $s \in W$ that admit a reduced expression containing all the reflections $s_1, \dots, s_r$, and set

$$I_i(\alpha) := \int_{\mathbb{D}} |x|^{-\gamma \langle \alpha, e_i \rangle} M^{\gamma e_i}(d^2 x). \quad (6.3.5)$$

This extends the analog statement of [84, Section 7] which corresponds to the case $r = 1$ —see also the works [116, 139] for more general results in the $r = 1$ case. We mention that in [31] we prove additional results for class-one Whittaker functions and show that we can define reflection coefficients for them.

Note that these statements allow to make sense of Toda reflection coefficients when $s = s_{\sigma 1} \cdots s_{\sigma r}$ for σ a permutation of $\{1, \dots, r\}$. The value of these reflection coefficients is in agreement with predictions from the physics literature [3, 2, 51]. The feature that these Toda reflection coefficients arise in asymptotic expansions somewhat appears in [3, 2]. We stress the remarkable property that Liouville reflection coefficients $R_L(\alpha)$ (see Equation (6.3.8) below) can be recovered from Toda reflection coefficients via the identification

$$R_{s_i, \gamma}(\alpha) = R_{L, \tilde{\gamma}} \left(\langle \alpha, e_i \rangle \sqrt{\frac{\langle e_i, e_i \rangle}{2}} \right), \quad \tilde{\gamma} = \sqrt{\frac{2}{\langle e_i, e_i \rangle}} \gamma$$

where with these notations we have stressed the dependence in the coupling constant. Moreover Toda reflection coefficients can be computed recursively thanks to the noticeable identity, valid for any s and τ in W :

$$R_{s\tau}(\alpha) = R_s(\tau\alpha)R_\tau(\alpha),$$

which in particular allows to reduce the computation of Toda reflection coefficients to that of Liouville. Using this relation one checks that in the cases we consider we have an alternative representation of the Toda reflection coefficients:

$$R_s(\alpha) = \epsilon(s) \prod_{i=1}^r \left(\mu_i \pi l \left(\frac{\gamma^2 \langle e_i, e_i \rangle}{4} \right) \right)^{\frac{\langle \hat{s}\alpha - \alpha, \omega_i^\vee \rangle}{\gamma}} \frac{\Gamma \left(1 - \frac{\gamma}{2} \langle \hat{s}\alpha - \alpha, \omega_i \rangle \right) \Gamma \left(1 - \frac{1}{\gamma} \langle \hat{s}\alpha - \alpha, \omega_i^\vee \rangle \right)}{\Gamma \left(1 + \frac{\gamma}{2} \langle \hat{s}\alpha - \alpha, \omega_i \rangle \right) \Gamma \left(1 + \frac{1}{\gamma} \langle \hat{s}\alpha - \alpha, \omega_i^\vee \rangle \right)}. \quad (6.3.6)$$

The rest of this Section is dedicated to proving such asymptotic expansions. For future convenience we set $e_i^* := \frac{e_i}{\sqrt{\langle e_i, e_i \rangle}}$.

6.3.1 . A warm-up: Liouville reflection coefficients

Before actually proving Theorems 6.3.1 and 6.3.2 we focus on the rank one case, that is Liouville reflection coefficients. We will build on some of the ideas implemented in this case to address the general problem which will involve additional reasonings related to the more complex behaviour of the path decomposition unveiled in the previous section and the presence of several correlated GMC measures.

6.3.1.1 Probabilistic Liouville reflection coefficient

In the $\mathfrak{sl}_2$ Toda CFT, *i.e.* Liouville theory, the reflection coefficient also arises in the tail expansion of the GMC measures considered. Indeed it follows from the proof of [84, Lemma 7.1] (see also [11, Proposition 4.1] where it is proved that such an expansion allows to compute a scattering coefficient associated to the Liouville Hamiltonian introduced in [66]) that, for $\langle \alpha - Q, e_1^\vee \rangle < 0$ sufficiently small,

$$\mathbb{E} [\exp(-e^{\gamma c} I(\alpha))] = 1 + e^{2(Q-\alpha)c} R_L(\alpha) + l.o.t.$$

in the asymptotic where $c \rightarrow -\infty$, and with R_L given by the reflection coefficient of Liouville theory. The reflection coefficient that arises in such an expansion admits a probabilistic expression involving the random variable $J_\gamma(\nu)$ introduced in Subsection 3.3.4.1.

$$R_L(\alpha) = \mu_L^{\frac{Q-\alpha}{\gamma}} \frac{Q-\alpha}{\gamma} \Gamma\left(\frac{\alpha-Q}{\gamma}\right) \mathbb{E} \left[J_{\sqrt{2}\gamma} \left(\frac{\alpha-Q}{\sqrt{2}} \right)^{\frac{Q-\alpha}{\gamma}} \right] \quad (6.3.7)$$

where the $\sqrt{2}$ terms come from the fact that the simple roots are not normalized, and with μ_L Liouville cosmological constant. One of the achievements of the aforementioned article is the evaluation of this reflection coefficient, which is in agreement with predictions from the physics literature. Namely [84, Theorem 3.5] shows that

$$R_L(\alpha) = - \left(\pi \mu_L l \left(\frac{\gamma^2}{2} \right) \right)^{\frac{Q-\alpha}{\gamma}} \frac{\Gamma\left(\frac{\alpha-Q}{\gamma}\right) \Gamma\left(\frac{\gamma}{2}(\alpha-Q)\right)}{\Gamma\left(\frac{Q-\alpha}{\gamma}\right) \Gamma\left(\frac{\gamma}{2}(Q-\alpha)\right)} \quad (6.3.8)$$

for $\alpha - Q \in (-\frac{2}{\gamma}, 0)$, where l is the special function

$$l(x) := \frac{\Gamma(x)}{\Gamma(1-x)}.$$

In particular this allows to evaluate the expectation term as

$$\mathbb{E} \left[J_\gamma(\nu)^{-\frac{2\nu}{\gamma}} \right] = \left(\pi l \left(\frac{\gamma^2}{4} \right) \right)^{-\frac{2\nu}{\gamma}} \frac{\Gamma\left(1 + \frac{\gamma}{2}\nu\right)}{\Gamma\left(1 - \frac{2}{\gamma}\nu\right) \Gamma\left(1 - \frac{\gamma}{2}\nu\right)}. \quad (6.3.9)$$

for $\nu \in (-\frac{2}{\gamma}, 0)$ and $\gamma \in (0, 2)$.

6.3.1.2 Tail expansion of a GMC measure

In the Liouville case there is only one GMC measure that is involved in the definition of the correlation functions. In that case the statement of Theorem 6.3.1 reduces to the following Proposition:

Proposition 6.3.3. *Assume that $\langle \alpha - Q, e_i^\vee \rangle \in (-\gamma, 0)$. Then for any positive $\eta < 1$,*

$$\mathbb{E} [\exp(-e^{\gamma \langle c, e_i \rangle} I_i)] = 1 + R_{s_i}(\alpha) e^{\langle \hat{s}_i \alpha - \alpha, c \rangle} + \mathcal{O}(e^{(1-\eta)\gamma \langle c, e_i \rangle}). \quad (6.3.10)$$

Proof. Using the radial-angular decomposition from Equation (3.3.8) we can rewrite I_i as

$$I_i = \int_{\mathbb{D}} |x|^{-\gamma\langle \alpha, e_i \rangle} M^{\gamma e_i} (d^2x) = \int_0^\infty e^{\gamma\langle B_t^\nu, e_i \rangle} Z_t^i dt.$$

Since only a one-dimensional Brownian motion $\langle B^\nu, e_i \rangle$, with variance $\langle e_i, e_i \rangle$ and drift $\frac{\langle \nu, e_i \rangle}{\langle e_i, e_i \rangle} = \frac{1}{2} \langle \nu, e_i^\vee \rangle$, is involved in the computation of $\mathbb{E} [\exp(-e^{\gamma\langle c, e_i \rangle} I_i) - 1]$, we can readily use the one-dimensional path decomposition by Williams for the process $\langle B^\nu, e_i \rangle$ to rewrite the latter as $I_i = e^{\gamma M} J_i$, with the random variable M defined by $M := \sup_{t \geq 0} \langle B^\nu, e_i \rangle$ and where we have introduced the notation

$$J_i = \int_0^{+\infty} e^{\gamma\langle \mathcal{B}^\nu, e_i \rangle} Z_t^i dt.$$

Here $\langle \mathcal{B}^\nu, e_i \rangle$ has the law of the process described in Subsection 3.3.4 (up to an adjustment of the variance and drift) and is started from $-M$, with M independent from $\langle \mathcal{B}^\nu, e_i \rangle$ and with law given by $d\mathbb{P}(M) = \langle \nu, e_i^\vee \rangle e^{\langle \nu, e_i^\vee \rangle M} \mathbf{1}_{M>0}$. Using this decomposition yields

$$\mathbb{E} [\exp(-e^{\gamma\langle c, e_i \rangle} I_i) - 1] = \int_0^{+\infty} \langle \nu, e_i^\vee \rangle e^{\langle \nu, e_i^\vee \rangle M} \mathbb{E}_{-M} [1 - \exp(-e^{\gamma\langle c + M e_i^\vee, e_i \rangle} J_i)] dM$$

with $\mathcal{B}^\nu$ is started from $-M$. By making the change of variable $M \leftrightarrow M + \langle c, e_i \rangle$ we end up with

$$\mathbb{E} [\exp(-e^{\gamma\langle c, e_i \rangle} I_i) - 1] = e^{-\langle \nu, e_i^\vee \rangle \langle c, e_i \rangle} R_{s_i}(\alpha; c)$$

with

$$R_{s_i}(\alpha; c) := \int_{-\langle c, e_i \rangle}^{+\infty} \langle \nu, e_i^\vee \rangle e^{\langle \nu, e_i^\vee \rangle M} \mathbb{E}_{-M+\langle c, e_i \rangle} [1 - \exp(-e^{\gamma M} J_i)] dM. \quad (6.3.11)$$

To evaluate this term, let us split the integral in Equation (6.3.11) as

$$\int_{\langle c, e_i \rangle}^{\langle (1-\eta)c, e_i \rangle} + \int_{\langle (1-\eta)c, e_i \rangle}^{+\infty} \langle \nu, e_i^\vee \rangle e^{\langle \nu, e_i^\vee \rangle M} \mathbb{E}_{-M+\langle c, e_i \rangle} [1 - \exp(-e^{\gamma M} J_i)] dM.$$

In the first integral above we can bound the expectation term as

$$\mathbb{E}_{-M+\langle c, e_i \rangle} [1 - \exp(-e^{\gamma M} J_i)] \leq e^{\gamma M} \mathbb{E}_{-M+\langle c, e_i \rangle} [J_i].$$

Note that thanks to Lemma 3.3.7 the expectation on the right-hand side is uniformly bounded in M . This implies that

$$\begin{aligned} & \int_{\langle c, e_i \rangle}^{\langle (1-\eta)c, e_i \rangle} \langle \nu, e_i^\vee \rangle e^{\langle \nu, e_i^\vee \rangle M} \mathbb{E}_{-M+\langle c, e_i \rangle} [1 - \exp(-e^{\gamma M} J_i)] dM \\ & \leq \int_{\langle c, e_i \rangle}^{\langle (1-\eta)c, e_i \rangle} \langle \nu, e_i^\vee \rangle e^{(\gamma + \langle \nu, e_i^\vee \rangle) M} \mathbb{E}_{-M+\langle c, e_i \rangle} [J_i] dM \\ & \leq C e^{(\gamma + \langle \nu, e_i^\vee \rangle) \langle (1-\eta)c, e_i \rangle} \end{aligned}$$

for some positive constant C .

Let us now turn to the other integral. It is readily seen that since $-M + \langle \mathbf{c}, e_i \rangle$ will diverge to $-\infty$, the integral J_i will converge to

$$\int_0^\infty e^{\gamma \langle \mathcal{B}_t^\nu, e_i \rangle} Z_t^i dt \stackrel{(\text{law})}{=} J_{\gamma_i}(\langle \nu, e_i^* \rangle)$$

where $\gamma_i^2 = \langle e_i, e_i \rangle \gamma^2$. This allows us to write $\mathbb{E}_{-M + \langle \mathbf{c}, e_i \rangle} [1 - \exp(-e^{\gamma M} J_i)]$ under the form

$$\mathbb{E}_{-M + \langle \mathbf{c}, e_i \rangle} [1 - \exp(-e^{\gamma M} J_i)] = \mathbb{E} [1 - \exp(-e^{\gamma M} J_{\gamma_i}(\langle \nu, e_i^* \rangle))] + \Re(M - \langle \mathbf{c}, e_i \rangle).$$

Using the same estimates as above we see that

$$\begin{aligned} & \int_{\langle (1-\eta)\mathbf{c}, e_i \rangle}^{+\infty} \langle \nu, e_i \rangle e^{\langle \nu, e_i \rangle M} \mathbb{E}_{-M + \langle \mathbf{c}, e_i \rangle} [1 - \exp(-e^{\gamma M} J_i)] dM \\ &= \int_{-\infty}^{+\infty} \langle \nu, e_i^\vee \rangle e^{\langle \nu, e_i^\vee \rangle M} \mathbb{E} [1 - \exp(-e^{\gamma M} J_{\gamma_i}(\langle \nu, e_i^* \rangle))] dM \\ &+ \int_{\langle (1-\eta)\mathbf{c}, e_i \rangle}^{+\infty} \langle \nu, e_i^\vee \rangle e^{\langle \nu, e_i^\vee \rangle M} \Re(M - \langle \mathbf{c}, e_i \rangle) dM + \mathcal{O}\left(e^{(\gamma + \langle \nu, e_i^\vee \rangle) \langle (1-\eta)\mathbf{c}, e_i \rangle}\right). \end{aligned}$$

The integral that appears in the second line can be evaluated and is found to be equal to

$$-\frac{\langle \nu, e_i^\vee \rangle}{\gamma} \Gamma\left(\frac{\langle \nu, e_i^\vee \rangle}{\gamma}\right) \mathbb{E} \left[J_{\gamma_i}(\langle \nu, e_i^* \rangle)^{-\frac{\langle \nu, e_i^\vee \rangle}{\gamma}} \right].$$

Using Equation (6.3.9) and noting that $\frac{\langle \nu, e_i^\vee \rangle}{\gamma} = 2 \frac{\langle \nu, e_i^* \rangle}{\gamma_i}$ we recover the expression (6.3.2) given for the reflection coefficient, provided that $\langle \nu, e_i^* \rangle \in (-\frac{2}{\gamma_i}, 0)$ (which does hold via our assumptions) and using the identity $z\Gamma(z) = \Gamma(1+z)$

$$R_{s_i}(\alpha) = -\left(\pi \mu l \left(\frac{\gamma^2}{2}\right)\right)^{\frac{\langle Q - \alpha, e_i^\vee \rangle}{\gamma}} \frac{\Gamma\left(1 + \frac{\langle \alpha - Q, e_i^\vee \rangle}{\gamma}\right) \Gamma\left(1 + \frac{\gamma}{2} \langle \alpha - Q, e_i \rangle\right)}{\Gamma\left(1 + \frac{\langle Q - \alpha, e_i^\vee \rangle}{\gamma}\right) \Gamma\left(1 + \frac{\gamma}{2} \langle Q - \alpha, e_i \rangle\right)}.$$

As a consequence we need to focus on the integral of the remainder term. We will rely on the fact that since $M \geq \langle (1-\eta)\mathbf{c}, e_i \rangle$, the quantity $x := -M + \langle \mathbf{c}, e_i \rangle$ (the starting point of the process $\mathcal{B}^\nu$) will diverge to $-\infty$. Now for any negative $x' > x$, we can rewrite J_i under the form

$$J_i = \int_0^{T_{x'}} + \int_{T_{x'}}^\infty e^{\gamma \mathcal{B}_t^{\langle \nu, e_i \rangle}} Z_t^i dt$$

where $T_{x'}$ is the first time when the process $\mathcal{B}^{\langle \nu, e_i \rangle}$ reaches x' . Then by the Markov property the process $(\mathcal{B}_t^{\langle \nu, e_i \rangle})_{t \geq T_{x'}}$ has the law of $\mathcal{B}^\nu$ started from x' and is independent of $(\mathcal{B}_t^{\langle \nu, e_i \rangle})_{0 \leq t \leq T_{x'}}$. Therefore $E_{x'} [\exp(-e^{\gamma M} J_i)] - \mathbb{E}_x [\exp(-e^{\gamma M} J_i)]$ is equal to

$$\begin{aligned} & \mathbb{E}_x \left[\exp\left(-e^{\gamma M} \int_{T_{x'}}^\infty e^{\gamma \mathcal{B}_t^{\langle \nu, e_i \rangle}} Z_t^i dt\right) \left(1 - \exp\left(-e^{\gamma M} \int_0^{T_{x'}} e^{\gamma \mathcal{B}_t^{\langle \nu, e_i \rangle}} Z_t^i dt\right)\right) \right] \\ & \leq e^{\gamma M} \mathbb{E}_x \left[\int_0^{T_{x'}} e^{\gamma \mathcal{B}_t^{\langle \nu, e_i \rangle}} Z_t^i dt \exp\left(-e^{\gamma M} \int_{T_{x'}}^\infty e^{\gamma \mathcal{B}_t^{\langle \nu, e_i \rangle}} Z_t^i dt\right) \right] \\ & \leq C e^{\gamma(M+x')} \end{aligned}$$

where C is uniformly bounded in M, x' by Lemma 3.3.7. Indeed the term that appears in the exponential can be bounded by 1 while

$$\mathbb{E}_x \left[\int_0^{T_{x'}} e^{\gamma \mathcal{B}_t^{\langle \nu, e_i \rangle}} Z_t^i dt \right] = e^{\gamma x'} \mathbb{E}_0 \left[\int_{-L_{x-x'}}^0 e^{\gamma \mathcal{B}_{-t}^{\langle \nu, e_i \rangle}} Z_t^i dt \right]$$

where $L_{x-x'}$ is the last hitting-time of $x - x'$ (this follows from the time-reversal property of the process $\mathcal{B}^{\langle \nu, e_i \rangle}$, see [138, Theorem 2.5]). By letting $x \rightarrow -\infty$, we end up with

$$|\Re(M - \langle \mathbf{c}, e_i \rangle)| \leq C e^{\gamma \langle \mathbf{c}, e_i \rangle}$$

provided that $\lim_{x \rightarrow -\infty} \mathbb{E}_x [\exp(-e^{\gamma M} J_i)] = \mathbb{E} \left[\exp \left(-e^{\gamma M} J_{\gamma_i} \left(\frac{\langle \alpha - Q, e_i^\vee \rangle}{2\gamma} \right) \right) \right]$. This follows from the proof of [84, Lemma 7.1]. All together this implies that the integral of this remainder term can be bounded by

$$C' e^{(\gamma + (1-\eta) \langle \nu, e_i^\vee \rangle) \langle \mathbf{c}, e_i \rangle}.$$

Finally we have proved the desired result: for any positive η ,

$$R_{s_i}(\alpha; \mathbf{c}) = R_L(\langle \alpha, e_i \rangle) + \mathcal{O} \left(e^{(1-\eta)(\gamma + \langle \nu, e_i^\vee \rangle) \langle \mathbf{c}, e_i \rangle} \right).$$

□

6.3.2 . Probabilistic Toda reflection coefficients: proof of Theorem 6.3.1

Building on the reasoning developed above, we now turn to the proof of Theorem 6.3.1 for the general case where the element of the Weyl group being considered is not necessarily assumed to be an elementary reflection. Without loss of generality we assume that $s = s_1 \cdots s_r$ and $\langle \hat{s}\alpha - \hat{s}'\alpha, \mathbf{c} \rangle \rightarrow +\infty$ for all $s' \neq s \in W_{1 \dots r}$. Recall that we investigate the asymptotics of

$$\mathbb{E} \left[\prod_{i=1}^p (\exp(-e^{\gamma \langle \mathbf{c}, e_i \rangle} I_i) - 1) \right]$$

in the limit where $\mathbf{c} \rightarrow \infty$ inside $\mathcal{C}$.

Our strategy is to rewrite the integrals involved using the radial-angular decomposition from (3.3.8) as

$$I_i = \int_{\mathbb{D}} |x|^{-\gamma \langle \alpha, e_i \rangle} M^{\gamma e_i}(d^2 x) = \int_0^\infty e^{\gamma \langle B_t^\nu, e_i \rangle} Z_t^i dt$$

for $i = 1, \dots, r$ and with $\nu = \alpha - Q$. The probability for these integrals to be large is governed by the maximums of the processes $(\langle B_t^\nu, e_i \rangle)_{t \geq 0}$ so that the path decomposition of Theorem 6.1.1 naturally applies within this setting. To be more specific we can rewrite the random variables I_i under the form

$$I_i = e^{\gamma \langle \mathbf{M}, e_i \rangle} \int_0^\infty e^{\gamma \langle \mathcal{B}_t^\nu, e_i \rangle} Z_t^i dt =: e^{\gamma \langle \mathbf{M}, e_i \rangle} J_i$$

where $\mathbf{M}$ has its law described by

$$\mathbb{P}(\mathbf{M}_i \leq \mathbf{m}_i \quad \forall 1 \leq i \leq r) = h(\mathbf{m}) \mathbb{1}_{\mathbf{m} \in \mathcal{C}},$$

while the independent diffusion process $\mathcal{B}^\nu$ that appears there is defined by joining

- A diffusion process $\mathbf{X}^1$ started from $-\mathbf{M} \in \mathcal{C}_-$, with generator $\frac{1}{2}\Delta + \nabla \log \partial_{1,\dots,r} h$ and run until it hits $\partial\mathcal{C}_-$, say at $z_1 \in \partial\mathcal{C}_1$.
- Then run an independent process $\mathbf{X}^2$ started from z_1 and with generator $\frac{1}{2}\Delta + \nabla \log \partial_{2,\dots,r} h$, upon hitting $\partial\mathcal{C}_-$.
- Thus define a family of processes $(\mathbf{X}^1, \dots, \mathbf{X}^r)$. When $\mathbf{X}^r$ reaches the boundary of $\partial\mathcal{C}_-$, sample $\mathbf{X}^{r+1}$ with generator $\frac{1}{2}\Delta + \nabla \log h$.

In the above h is defined in Equation (3.3.17). By doing so we reformulate our problem by writing that

$$\mathbb{E} \left[\prod_{i=1}^r (\exp(-e^{\gamma\langle \mathbf{c}, e_i \rangle} I_i) - 1) \right] = \int_{\mathcal{C}} d\mathbb{P}(\mathbf{M}) \mathbb{E}_{-\mathbf{M}} \left[\prod_{i=1}^r (\exp(-e^{\gamma\langle \mathbf{c} + \mathbf{M}, e_i \rangle} J_i) - 1) \right]$$

where $d\mathbb{P}(\mathbf{M})$ is the density of the random variable $\mathbf{M}$ while with the notation $\mathbb{E}_{-\mathbf{M}}$ we indicate that the process $\mathbf{B}^\nu$ that enters the definition of the $(J_i)_{1 \leq i \leq r}$ is started from $-\mathbf{M}$.

6.3.2.1 The case of length 2

As a warm-up we first consider the case where s has length 2, that is to say only two integrals are involved. Then we can work in $\mathbf{V} \simeq \mathbb{R}^2$ in which lie the two roots (e_1, e_2) (up to reordering the roots we assume that $(i, j) = (1, 2)$ to simplify the notations). Since $\langle e_1, e_2 \rangle = 0$ implies that the components e_1, e_2 of B^ν are independent so the expectations factorize, this case reduces to the case of length one. Therefore we may assume that it is not the case. In doing so we can write

$$\begin{aligned} & \mathbb{E} [(\exp(-e^{\gamma\langle \mathbf{c}, e_1 \rangle} I_1) - 1) (\exp(-e^{\gamma\langle \mathbf{c}, e_2 \rangle} I_2) - 1)] \\ &= \int_{\mathcal{C}} d\mathbb{P}(\mathbf{M}) \mathbb{E}_{-\mathbf{M}} [(\exp(-e^{\gamma\langle \mathbf{c} + \mathbf{M}, e_1 \rangle} J_1) - 1) (\exp(-e^{\gamma\langle \mathbf{c} + \mathbf{M}, e_2 \rangle} J_2) - 1)]. \end{aligned}$$

In the above equation $d\mathbb{P}(\mathbf{M})$ is the density of the random variable $\mathbf{M}$, given by

$$d\mathbb{P}(\mathbf{M}) = \partial_{1,2} h(-\mathbf{M}) d\mathbf{M} = \sum_{s \in W} \epsilon(s) \langle \hat{s}\alpha - \alpha, \omega_1^\vee \rangle \langle \hat{s}\alpha - \alpha, \omega_2^\vee \rangle e^{\langle \alpha - \hat{s}\alpha, \mathbf{M} \rangle} d\mathbf{M}.$$

Here the coefficients $\langle \hat{s}\alpha - \alpha, \omega_1^\vee \rangle \langle \hat{s}\alpha - \alpha, \omega_2^\vee \rangle$ vanish when $s \in \{Id, s_1, s_2\}$, so the sum actually ranges over $s \in W_{1,2}$. Hence a change in variable $\mathbf{M} \leftrightarrow \mathbf{M} - \mathbf{c}$ yields

$$\mathbb{E} [(\exp(-e^{\gamma\langle \mathbf{c}, e_1 \rangle} I_1) - 1) (\exp(-e^{\gamma\langle \mathbf{c}, e_2 \rangle} I_2) - 1)] = \sum_{s \in W_{1,2}} e^{\langle \hat{s}\alpha - \alpha, \mathbf{c} \rangle} R_s(\alpha; \mathbf{c})$$

where we have introduced

$$\begin{aligned} R_s(\alpha; \mathbf{c}) &:= \epsilon(s) \int_{\mathcal{C} + \mathbf{c}} \langle \hat{s}\alpha - \alpha, \omega_1^\vee \rangle \langle \hat{s}\alpha - \alpha, \omega_2^\vee \rangle e^{\langle \alpha - \hat{s}\alpha, \mathbf{M} \rangle} \\ &\quad \mathbb{E}_{\mathbf{c} - \mathbf{M}} [(\exp(-e^{\gamma\langle \mathbf{M}, e_1 \rangle} J_1) - 1) (\exp(-e^{\gamma\langle \mathbf{M}, e_2 \rangle} J_2) - 1)] d\mathbf{M}. \end{aligned} \tag{6.3.12}$$

However and unlike the case where only one integral was involved, there is one subtle issue that needs to be taken care of here. Indeed in the asymptotic where the starting point of the process $\mathcal{B}^\nu$ from Theorem 6.1.1 will diverge, it is far from clear how the quantities J_i will behave. In Section 6.2 we have provided a refined study of the process $\mathcal{B}^\nu$ in this regime, and a consequence of Proposition 6.2.5 is in particular that as $\mathbf{x} \rightarrow \infty$ inside $\mathcal{C}_-$ with $\langle s_1 s_2 (\alpha - Q) - s_2 s_1 (\alpha - Q), \mathbf{x} \rangle \rightarrow +\infty$,

$$\lim_{\mathbf{x} \rightarrow \infty} \mathbb{E}_{\mathbf{x}} [F(J_1)G(J_2)] = \mathbb{E} \left[F \left(J_{\gamma_1} (\langle s_2 (\alpha - Q), e_1^* \rangle) \right) \right] \mathbb{E} \left[G \left(J_{\gamma_2} (\langle \alpha - Q, e_2^* \rangle) \right) \right] \quad (6.3.13)$$

for F, G be bounded continuous over $\mathbb{R}^+$ (the reasoning conducted in Section 6.2 still works for Z as considered here). This allows to provide the desired result:

Lemma 6.3.4. *Under the assumptions of Theorem 6.3.1 $R_s(\alpha; \mathbf{c})$ converges to a well-defined limit. When $s = s_1 s_2$ this limit is equal to $R_s(\alpha)$.*

Proof. Let us pick some positive η and split the integral between the domains $\mathcal{C} + (1 - \eta)\mathbf{c}$ and $\mathcal{C}_\eta := \mathcal{C} + \mathbf{c} \setminus (\mathcal{C}_\eta + (1 - \eta)\mathbf{c})$.

The integral over the domain $\mathcal{C}_\eta$ will be negligible. Indeed for any $\mathbf{M} \in \mathcal{C}_\eta$ there is some $i \in \{1, 2\}$ such that $\langle \mathbf{M}, e_i \rangle < (1 - \eta)\langle \mathbf{c}, e_i \rangle$ (say $i = 1$); as a consequence over this domain the expectation term is of order at most $e^{\gamma(1-\eta)\langle \mathbf{c}, e_i \rangle}$:

$$\begin{aligned} & \mathbb{E}_{\mathbf{c}-\mathbf{M}} \left[\left(\exp \left(-e^{\gamma \langle \mathbf{M}, e_1 \rangle} J_1 \right) - 1 \right) \left(\exp \left(-e^{\gamma \langle \mathbf{M}, e_2 \rangle} J_2 \right) - 1 \right) \right] \\ & \leq \gamma e^{\gamma(1-\eta)\langle \mathbf{c}, e_1 \rangle} \mathbb{E}_{\mathbf{c}-\mathbf{M}} \left[J_1 \exp \left(-e^{\gamma(1-\eta)\langle \mathbf{c}, e_1 \rangle} J_1 \right) \left(1 - \exp \left(-e^{\gamma \langle \mathbf{M}, e_2 \rangle} J_2 \right) \right) \right] \\ & \leq \gamma e^{\gamma(1-\eta)\langle \mathbf{c}, e_1 \rangle} \mathbb{E}_{\mathbf{c}-\mathbf{M}} \left[J_1 \left(1 - \exp \left(-e^{\gamma \langle \mathbf{M}, e_2 \rangle} J_2 \right) \right) \right]. \end{aligned}$$

This implies that

$$\begin{aligned} & \int_{\langle \mathbf{c}, e_1 \rangle < \langle \mathbf{M}, e_1 \rangle < (1-\eta)\langle \mathbf{c}, e_1 \rangle} e^{\langle \alpha - \hat{s}\alpha, \mathbf{M} \rangle} \mathbb{E}_{\mathbf{c}-\mathbf{M}} \left[\left(\exp \left(-e^{\gamma \langle \mathbf{M}, e_1 \rangle} J_1 \right) - 1 \right) \left(\exp \left(-e^{\gamma \langle \mathbf{M}, e_2 \rangle} J_2 \right) - 1 \right) \right] d\mathbf{M} \\ & \leq \gamma e^{\gamma(1-\eta)\langle \mathbf{c}, e_1 \rangle} \int_{\langle \mathbf{c}, e_1 \rangle < \langle \mathbf{M}, e_1 \rangle < (1-\eta)\langle \mathbf{c}, e_1 \rangle} e^{\langle \alpha - \hat{s}\alpha, \mathbf{M} \rangle} \mathbb{E}_{\mathbf{c}-\mathbf{M}} \left[J_1 \left(1 - \exp \left(-e^{\gamma \langle \mathbf{M}, e_2 \rangle} J_2 \right) \right) \right] d\mathbf{M} \\ & \leq C e^{\gamma(1-\eta)\langle \mathbf{c}, e_1 \rangle} e^{\langle \mathbf{c}, \alpha - \hat{s}\alpha \rangle} \end{aligned}$$

for some positive constant C . Choosing η small enough this term vanishes as $\langle \mathbf{c}, e_1 \rangle$ and $\langle \mathbf{c}, e_2 \rangle$ go to $-\infty$ under the assumptions of Theorem 6.3.1. Therefore the integral over the whole domain $\mathcal{C}_\eta$ becomes negligible in the limit.

Therefore the integral over the other domain $\mathcal{C} + (1 - \eta)\mathbf{c}$ will be the contributing one. Indeed this set has been defined so that inside $\mathcal{C} + (1 - \eta)\mathbf{c}$, the starting point of the process will diverge to ∞ inside $\mathcal{C}_-$, so that we are in the setting of Proposition 6.2.1:

$$\begin{aligned} & \mathbb{E}_{\mathbf{c}-\mathbf{M}} \left[\left(\exp \left(-e^{\gamma \langle \mathbf{M}, e_1 \rangle} J_1 \right) - 1 \right) \left(\exp \left(-e^{\gamma \langle \mathbf{M}, e_2 \rangle} J_2 \right) - 1 \right) \right] \text{ will converge pointwise to} \\ & \mathbb{E} \left[\exp \left(-e^{\gamma \langle \mathbf{M}, e_1 \rangle} J_{\gamma_1} (\langle s_2 (\alpha - Q), e_1^* \rangle) \right) - 1 \right] \mathbb{E} \left[\exp \left(-e^{\gamma \langle \mathbf{M}, e_2 \rangle} J_{\gamma_2} (\langle \alpha - Q, e_2^* \rangle) \right) - 1 \right]. \end{aligned}$$

By dominated convergence this implies that

$$\int_{\mathcal{C} + (1-\eta)\mathbf{c}} e^{\langle \alpha - \hat{s}\alpha, \mathbf{M} \rangle} \mathbb{E}_{\mathbf{c}-\mathbf{M}} \left[\left(\exp \left(-e^{\gamma \langle \mathbf{M}, e_1 \rangle} J_1 \right) - 1 \right) \left(\exp \left(-e^{\gamma \langle \mathbf{M}, e_2 \rangle} J_2 \right) - 1 \right) \right] d\mathbf{M}$$

converges to the quantity

$$\int_{\mathbf{V}} e^{\langle \alpha - \hat{s}\alpha, \omega_1^\vee \rangle \langle \mathbf{M}, e_1 \rangle} \mathbb{E} \left[\left(\exp \left(-e^{\gamma \langle \mathbf{M}, e_1 \rangle} J_{\gamma_1} (\langle s_2(\alpha - Q), e_1^* \rangle) \right) - 1 \right) \right] \times \\ e^{\langle \alpha - \hat{s}\alpha, \omega_2^\vee \rangle \langle \mathbf{M}, e_2 \rangle} \mathbb{E} \left[\left(\exp \left(-e^{\gamma \langle \mathbf{M}, e_2 \rangle} J_{\gamma_2} (\langle \alpha - Q, e_2^* \rangle) \right) - 1 \right) \right] d\mathbf{M}.$$

The latter is nothing but

$$\frac{1}{\gamma} \Gamma \left(\frac{1}{\gamma} \langle \alpha - \hat{s}\alpha, \omega_1^\vee \rangle \right) \mathbb{E} \left[J_{\gamma_1} (\langle s_2(\alpha - Q), e_1^* \rangle)^{\frac{\langle \hat{s}\alpha - \alpha, \omega_1^\vee \rangle}{\gamma}} \right] \times \\ \frac{1}{\gamma} \Gamma \left(\frac{1}{\gamma} \langle \alpha - \hat{s}\alpha, \omega_2^\vee \rangle \right) \mathbb{E} \left[J_{\gamma_2} (\langle \alpha - Q, e_2^* \rangle)^{\frac{\langle \hat{s}\alpha - \alpha, \omega_2^\vee \rangle}{\gamma}} \right].$$

Now if we choose $s = s_1 s_2$, we can write that $\frac{\langle s_2(\alpha - Q), e_1^* \rangle}{\gamma_1} = \frac{\langle s_2(\alpha - Q), e_1^\vee \rangle}{2\gamma} = \frac{\langle \alpha - \hat{s}\alpha, \omega_1^\vee \rangle}{2\gamma}$ as well as $\frac{\langle \alpha - Q, e_2^* \rangle}{\gamma_2} = \frac{\langle \alpha - Q, e_2^\vee \rangle}{2\gamma} = \frac{\langle \alpha - \hat{s}\alpha, \omega_2^\vee \rangle}{2\gamma}$. As a consequence we can evaluate the above quantity thanks to Equation (6.3.9):

$$\left(\pi l \left(\frac{\gamma_1^2}{4} \right) \right)^{\frac{\langle \hat{s}\alpha - \alpha, \omega_1^\vee \rangle}{\gamma}} \frac{1}{\gamma} \frac{\Gamma \left(\frac{1}{\gamma} \langle \alpha - \hat{s}\alpha, \omega_1^\vee \rangle \right) \Gamma \left(1 + \frac{\gamma}{2} \langle \alpha - \hat{s}\alpha, \omega_1^\vee \rangle \right)}{\Gamma \left(1 + \frac{1}{\gamma} \langle \hat{s}\alpha - \alpha, \omega_1^\vee \rangle \right) \Gamma \left(1 + \frac{\gamma}{2} \langle \hat{s}\alpha - \alpha, \omega_1^\vee \rangle \right)} \times \\ \left(\pi l \left(\frac{\gamma_2^2}{4} \right) \right)^{\frac{\langle \hat{s}\alpha - \alpha, \omega_2^\vee \rangle}{\gamma}} \frac{1}{\gamma} \frac{\Gamma \left(\frac{1}{\gamma} \langle \alpha - \hat{s}\alpha, \omega_2^\vee \rangle \right) \Gamma \left(1 + \frac{\gamma}{2} \langle \alpha - \hat{s}\alpha, \omega_2^\vee \rangle \right)}{\Gamma \left(1 + \frac{1}{\gamma} \langle \hat{s}\alpha - \alpha, \omega_2^\vee \rangle \right) \Gamma \left(1 + \frac{\gamma}{2} \langle \hat{s}\alpha - \alpha, \omega_2^\vee \rangle \right)}.$$

Therefore, collecting up terms, we see that

$$\int_{\mathcal{C}_\eta} + \int_{\mathcal{C} + (1-\eta)\mathcal{C}} e^{\langle \alpha - \hat{s}\alpha, \mathbf{M} \rangle} \mathbb{E}_{\mathbf{C}-\mathbf{M}} \left[\left(\exp \left(-e^{\gamma \langle \mathbf{M}, e_1 \rangle} J_1 \right) - 1 \right) \left(\exp \left(-e^{\gamma \langle \mathbf{M}, e_2 \rangle} J_2 \right) - 1 \right) \right] d\mathbf{M}$$

does converge, and for $s = s_1 s_2$ the limit of $R_s(\alpha; \mathbf{c})$ will be given by $R_s(\alpha)$. $\square$

This allows to conclude for the proof of Theorem 6.3.1 in the case where $p = 2$.

6.3.2.2 The general case

Without loss of generality we assume that $s = s_1 \cdots s_r$ and $\langle \hat{s}\alpha - \hat{s}'\alpha, \mathbf{c} \rangle \rightarrow +\infty$ for all $s' \neq s \in W_{1, \dots, r}$. Then along the same lines as above we get

$$\mathbb{E} \left[\prod_{k=1}^r \left(\exp \left(-e^{\gamma \langle \mathbf{c}, e_{i_k} \rangle} I_{i_k} \right) - 1 \right) \right] \\ = \sum_{s'} e^{\langle \hat{s}'\alpha - \alpha, \mathbf{c} \rangle} \epsilon(s') \int_{\mathcal{C} + \mathbf{c}} \prod_{i=1}^r \langle \hat{s}'\alpha - \alpha, \omega_i^\vee \rangle e^{\langle \alpha - \hat{s}'\alpha, \mathbf{M} \rangle} \mathbb{E}_{\mathbf{C}-\mathbf{M}} \left[\prod_{k=1}^p \left(\exp \left(-e^{\gamma \langle \mathbf{c}, e_{i_k} \rangle} I_{i_k} \right) - 1 \right) \right]$$

where the sum ranges over the elements $s' \in W$ whose reduced expression contain $s_1, \dots, s_r$, that is $W_{1, \dots, r}$. Indeed we can write $\langle \hat{s}'\alpha - \alpha, \omega_i^\vee \rangle = \langle \alpha - Q, s'^{-1} \omega_i^\vee - \omega_i^\vee \rangle$ where $s'^{-1} \omega_i^\vee - \omega_i^\vee = 0$ if this is not the case (indeed $s'_i \omega_j^\vee = 0$ for $j \neq i$). Using the same estimates as before, we see that up to a term which vanishes in the limit, the integral

can be reduced to an integral over $\mathcal{C} + (1 - \eta)\mathbf{c}$ where $\eta > 0$ is small enough. Now recall from Section 6.2 that in the limit where $\mathbf{x} \rightarrow \infty$ inside $\mathcal{C}$ with $\langle \hat{s}\alpha - \hat{s}'\alpha, \mathbf{x} \rangle \rightarrow +\infty$ for all $s' \neq s \in W_{1,\dots,r}$,

$$\mathbb{E}_{\mathbf{x}} \left[\prod_{i=1}^r F_i(J_i) \right] \text{ converges to } \prod_{i=1}^r \mathbb{E} \left[F_i \left(J_{\gamma_i} \left(\langle s_{i+1} \cdots s_r(\alpha - Q), e_i^* \rangle \right) \right) \right] \quad (6.3.14)$$

for $F_1, \dots, F_r$ bounded continuous over $\mathbb{R}^+$. With Equation (6.3.14) at hand we see that

$$\begin{aligned} & \lim_{\mathbf{c} \rightarrow \infty} \int_{\mathcal{C} + \mathbf{c}} \prod_{i=1}^r \langle \hat{s}'\alpha - \alpha, \omega_i^\vee \rangle e^{\langle \alpha - \hat{s}'\alpha, \mathbf{M} \rangle} \mathbb{E}_{\mathbf{c} - \mathbf{M}} \left[\prod_{k=1}^p \left(\exp \left(-e^{\gamma \langle \mathbf{c}, e_{i_k} \rangle} I_{i_k} \right) - 1 \right) \right] \\ &= \prod_{i=1}^r \Gamma \left(1 + \frac{1}{\gamma} \langle \hat{s}'\alpha - \alpha, \omega_i^\vee \rangle \right) \mathbb{E} \left[J_{\gamma_i} \left(\langle s_{i+1} \cdots s_r(\alpha - Q), e_i^* \rangle \right)^{\frac{1}{\gamma} \langle \hat{s}'\alpha - \alpha, \omega_i^\vee \rangle} \right]. \end{aligned}$$

Now in the case where $s' = s$, we can use the fact that $s_j \omega_i^\vee = \omega_i^\vee$ for all $j \neq i$ while $s_i \omega_i^\vee = \omega_i^\vee - e_i^\vee$ to see that the exponent is actually equal to

$$\begin{aligned} \frac{1}{\gamma} \langle s\alpha - \alpha, \omega_i^\vee \rangle &= \frac{1}{\gamma} \langle \alpha - Q, s_r \cdots s_1 \omega_i^\vee - \omega_i^\vee \rangle \\ &= \frac{1}{\gamma} \langle \alpha - Q, s_r \cdots s_{i+1} e_i^\vee \rangle = \frac{1}{\gamma_i} \langle s_{i+1} \cdots s_r(\alpha - Q), e_i^* \rangle. \end{aligned}$$

This allows to evaluate the result in that case via Equation (6.3.9):

$$\prod_{i=1}^r \left(\pi l \left(\frac{\gamma_i^2}{4} \right) \right)^{\frac{\langle \hat{s}\alpha - \alpha, \omega_i^\vee \rangle}{\gamma}} \frac{\Gamma \left(1 + \frac{1}{\gamma} \langle \hat{s}\alpha - \alpha, \omega_i^\vee \rangle \right) \Gamma \left(1 + \frac{\gamma}{2} \langle \hat{s}\alpha - \alpha, \omega_i \rangle \right)}{\Gamma \left(1 + \frac{1}{\gamma} \langle \alpha - \hat{s}\alpha, \omega_i^\vee \rangle \right) \Gamma \left(1 + \frac{\gamma}{2} \langle \alpha - \hat{s}\alpha, \omega_i \rangle \right)}.$$

Moreover under our assumptions we know that the higher order term in the expansion of

$$\sum_{s' \in W_{1,\dots,r}} e^{\langle \hat{s}'\alpha - \alpha, \mathbf{c} \rangle} \epsilon(s') \int_{\mathcal{C} + \mathbf{c}} \prod_{i=1}^r \langle \hat{s}'\alpha - \alpha, \omega_i^\vee \rangle e^{\langle \alpha - \hat{s}'\alpha, \mathbf{M} \rangle} \mathbb{E}_{\mathbf{c} - \mathbf{M}} \left[\prod_{k=1}^p \left(\exp \left(-e^{\gamma \langle \mathbf{c}, e_{i_k} \rangle} I_{i_k} \right) - 1 \right) \right]$$

will correspond to $s' = s$. The proof of Theorem 6.3.1 is thus complete.

6.3.2.3 Proof of Theorem 6.3.2

The proof of the tail expansion of the GMC measures follows the very same lines as the reasoning presented above so we will be brief. One only needs to replace the terms of the form $\mathbb{E} \left[\prod_{i=1}^p \left(\exp \left(-e^{\gamma \langle \mathbf{c}, e_{i_k} \rangle} I_{i_k} \right) - 1 \right) \right]$ by $\mathbb{P} \left(I_i > e^{-\gamma \langle \mathbf{c}, e_{i_k} \rangle}, 1 \leq i \leq p \right)$. By doing so we will end up with expression of the type

$$\begin{aligned} \mathbb{P} \left(I_i > e^{-\gamma \langle \mathbf{c}, e_{i_k} \rangle}, 1 \leq i \leq p \right) &= \int_{\mathcal{C}} d\mathbb{P}(\mathbf{M}) \mathbb{P}_{-\mathbf{M}} \left(J_i > e^{-\gamma \langle \mathbf{c} + \mathbf{M}, e_{i_k} \rangle}, 1 \leq i \leq p \right) \\ &= \sum_{s \in W_{1,\dots,p}} e^{\langle \hat{s}\alpha - \alpha, \mathbf{c} \rangle} \overline{R_s}(\alpha; \mathbf{c}) + u \end{aligned}$$

with u a lower order term and $\overline{R}_s(\alpha; \mathbf{c})$ defined as

$$\epsilon(s) \int_{\mathcal{C}+\mathbf{c}} \prod_{i=1}^p \langle \hat{s}\alpha - \alpha, \omega_i^\vee \rangle e^{\langle \alpha - \hat{s}\alpha, \mathbf{M} \rangle} \mathbb{P}_{\mathbf{c}-\mathbf{M}}(J_i > e^{-\gamma \langle \mathbf{M}, e_i \rangle}, 1 \leq i \leq p) d\mathbf{M}.$$

Thanks to Proposition 6.2.2 above we can prove in the same way as before that $\overline{R}_s(\alpha; \mathbf{c})$ converges and its limit is given by $\overline{R}_s(\alpha)$ when s is as in the statement of Theorem 6.3.2. This wraps up the proof of Theorem 6.3.2.

6.4 . Asymptotics and analytic extension of the correlation functions.

In the previous section we have explained how reflection coefficients associated to Toda CFTs naturally arise in the tail expansion of correlated GMC measures. We can further implement this connection to provide an asymptotic expansion of the expectation term that appears in the probabilistic definition of the correlation functions from (4.1.13). As a consequence we will see that we will be able to extend the range of values for which the probabilistic representation of the correlation makes sense. Namely we show that:

Theorem 6.4.1. *Assume that $\mathfrak{g} = \mathfrak{sl}_3$ and let $z_1, \dots, z_N \in \mathbb{C}$ be distinct. Define a subset of $(Q + \mathcal{C}_-)^N$ by setting*

$$\begin{aligned} \mathcal{A}_N := \left\{ (\alpha_1, \dots, \alpha_N) \in (Q + \mathcal{C}_-)^N \text{ s.t. for } i = 1, 2, \quad \left\langle \sum_{k=1}^N \alpha_k - 2Q, \omega_i \right\rangle > -\gamma \right. \\ \left. \text{and for any } 1 \leq k \leq N, \quad \min_{i=1,2} \langle \alpha_k - Q, e_i \rangle < -\gamma \right\}. \end{aligned} \quad (6.4.1)$$

Then the correlation functions admit a probabilistic representation for $\alpha \in \mathcal{A}_N$ still denoted by $\langle \prod_{k=1}^N V_{\alpha_k}(z_k) \rangle$. This extension is such that the map $\alpha \mapsto \langle \prod_{k=1}^N V_{\alpha_k}(z_k) \rangle$ is meromorphic in a complex neighbourhood of $\mathcal{A}_N$, with poles given by

$$\begin{aligned} \mathcal{P}_N := \{ \alpha \in \mathcal{A}_N, \quad \langle \alpha_k - Q, e \rangle \in -\gamma \mathbb{N}^* \cup -\frac{2}{\gamma} \mathbb{N}^* \text{ for some } 1 \leq k \leq N \text{ and } e \in \Phi^+ \} \\ \bigcup \{ \alpha \in \mathcal{A}_N, \quad \langle s(w), \omega_i \rangle = 0 \text{ for some } w : \{1, \dots, N\} \rightarrow W \text{ and } 1 \leq i \leq r \}. \end{aligned} \quad (6.4.2)$$

Here we have used the notation for any map $w : \{1, \dots, N\} \rightarrow W$ and $\alpha \in (\mathbb{C}^2)^N$:

$$s(w) := \sum_{k=1}^N \hat{w}(k) \alpha_k - 2Q.$$

The probabilistic correlation functions can be defined under the assumptions of Theorem 6.4.1 by considering expressions of the form

$$\left\langle V_{\alpha_1}(z_1) \cdots V_{\alpha_N}(z_N) \right\rangle = \int_{\mathfrak{a}} e^{-2\langle Q, \mathbf{c} \rangle} \mathbb{E} \left[\prod_{k=1}^N V_{\alpha_k}(z_k) e^{-\sum_{i=1}^r \mu_i e^{\gamma \langle \mathbf{c}, e_i \rangle} M^{\gamma e_i}(\mathbb{C})} - \mathcal{R}_{(z, \alpha)}(\mathbf{c}) \right] d\mathbf{c} \quad (6.4.3)$$

where the remainder term $\mathcal{R}_{(z,\alpha)}(c)$ is defined in Equation (6.4.5) below in terms of Toda reflection coefficients. This expression coincides with Equation (4.2.6) when α belongs in addition to the set $\mathcal{B}_N$.

The assumption that $\mathfrak{g} = \mathfrak{sl}_3$ made in the statement of Theorem 6.4.1 is actually not necessary in that the reasoning developed in this section can be easily adapted to other choices of underlying Lie algebra based on a straightforward generalization of Proposition 6.4.2 below. We stick to the case where $\mathfrak{g} = \mathfrak{sl}_3$ hereafter so as to keep the computations understandable and since this the case to which we have dedicated the most attention via the derivation of the Ward identities and of the Fateev-Litvinov formula.

6.4.1 . Asymptotics of the correlation functions

In order to provide an analytic extension of the correlation functions beyond the bounds prescribed by Theorem 4.3.1 we first need to describe the asymptotics of the expectation term that enters the correlation functions when the zero-mode c diverges. This is the purpose of the following statement:

Proposition 6.4.2. *For any $z_1, \dots, z_N \in \mathbb{C}$ distinct there exists a positive ξ such that the map*

$$e^{\xi|c|} e^{-2\langle Q, c \rangle} \mathbb{E} \left[\prod_{k=1}^N V_{\alpha_k}(z_k) e^{-\sum_{i=1}^2 \mu_i e^{\langle \gamma e_i, c \rangle} M^{\gamma e_i}(\mathbb{C})} - \mathcal{R}_{(z,\alpha)}(c) \right] \quad (6.4.4)$$

is uniformly bounded in c , where the remainder term is defined by setting:

$$\begin{aligned} \mathcal{R}_{(z,\alpha)}(c) &:= \mathcal{R}_{\alpha}^1(c) + \mathcal{R}_{\alpha}^2(c) + \mathcal{R}_{\alpha}^{1,2}(c), \quad \text{with} \\ \mathcal{R}_{\alpha}^1(c) &:= \sum_{w: \{1, \dots, N\} \rightarrow \{Id, s_2\}} \mathbb{1}_{\langle s(w), \omega_1 \rangle < 0} \prod_{k=1}^N R_{w(k)}(\alpha_k) V_{\hat{w}(k)\alpha_k}(z_k) e^{-\mu_1 e^{\langle \gamma, e_1 \rangle} M^{\gamma e_1}(\mathbb{C})} \\ &\quad - \sum_{w: \{1, \dots, N\} \rightarrow \{Id, s_1, s_2, s_1 s_2\}} \mathbb{1}_{\langle s(w), \omega_1 \rangle < 0} \prod_{k=1}^N R_{w(k)}(\alpha_k) V_{\hat{w}(k)\alpha_k}(z_k), \\ \mathcal{R}_{\alpha}^{1,2}(c) &:= \sum_{w: \{1, \dots, N\} \rightarrow W} \mathbb{1}_{\max_{j=1,2} \langle s(w), \omega_j \rangle < 0} \prod_{k=1}^N R_{w(k)}(\alpha_k) V_{\hat{w}(k)\alpha_k}(z_k). \end{aligned} \quad (6.4.5)$$

In the rest of this subsection we prove this statement.

6.4.1.1 Notations and preliminary remarks

We start by introducing some notations to simplify the coming computations. For V a subset of the Weyl group W we set

$$\mathcal{R}_V \prod_{k=1}^N V_{\alpha_k}(z_k) := \prod_{k=1}^N \left(\sum_{s \in V} R_s(\alpha_k) V_{\hat{s}\alpha_k}(z_k) \right)$$

and also define $\mathcal{R}_{V_1}\mathcal{R}_{V_2} := \mathcal{R}_{V_1V_2}$ where $V_1V_2 = \{v_1v_2, v_i \in V_i \text{ for } i = 1, 2\}$. We further introduce for $\alpha \in (Q + \mathcal{C}_-)^N$

$$\mathbf{R}_\alpha(\mathbf{c}) := \mathbf{R}_\alpha^1(\mathbf{c}) + \mathbf{R}_\alpha^2(\mathbf{c}) + \mathbf{R}_\alpha^{1,2}(\mathbf{c}), \quad \text{with}$$

$$\mathbf{R}_\alpha^1(\mathbf{c}) := \mathbb{1}_{\langle \mathbf{c}, e_2 \rangle < 0} \left(e^{-\mu_1 e^{\gamma \langle \mathbf{c}, e_1 \rangle} M^{\gamma e_1}(\mathbb{C})} - \mathbb{1}_{\langle \mathbf{c}, e_1 \rangle < 0} \mathcal{R}_{Id, s_1} \right) \mathcal{R}_{Id, s_2} \prod_{k=1}^N V_{\alpha_k}(z_k), \quad (6.4.6)$$

$$\mathbf{R}_\alpha^{1,2}(\mathbf{c}) := \mathbb{1}_{\max_{i=1,2} \langle \mathbf{c}, e_i \rangle < 0} \mathcal{R}_W \prod_{k=1}^N V_{\alpha_k}(z_k).$$

To lighten the notations we denote in this subsection $I^i(\mathbb{C}) := Z_{(\mathbf{z}, \alpha)}^{\gamma e_i}(\mathbb{C})$. Note that since the random variables $I^i(\mathbb{C})$ have negative moments of any order, we know that for any positive $R > 0$

$$e^{R \langle \mathbf{c}, e_1 \rangle} \mathbb{E} \left[\prod_{k=1}^N e^{-\sum_{i=1}^2 \mu_i e^{\gamma \langle \mathbf{c}, e_i \rangle} I^i(\mathbb{C})} \right] \rightarrow 0 \quad \text{as } \langle \mathbf{c}, e_1 \rangle \rightarrow +\infty.$$

Of course the same applies in the region where $\langle \mathbf{c}, e_2 \rangle \rightarrow +\infty$, while this remains true for $R > 0$ small enough for the terms containing reflection coefficients due to the assumptions made on the parameter s . Therefore we focus on what happens as $\langle \mathbf{c}, e_1 \rangle \rightarrow -\infty$ with $\langle \mathbf{c}, e_2 \rangle$ bounded below, and when both $\langle \mathbf{c}, e_1 \rangle$ and $\langle \mathbf{c}, e_2 \rangle$ diverge to $-\infty$. To do so our strategy is to show that in these asymptotics the integrals that appear in the expectation term concentrate around the singular points $x = z_k$. Put differently we will see that the behaviour of the expectation term is governed by the asymptotics of

$$\mathbb{E} \left[\prod_{k=1}^N e^{-\sum_{i=1}^2 \mu_i e^{\gamma \langle \mathbf{c}, e_i \rangle} I^i(B_{r_k}(z_k))} \right]$$

for some well-chosen r_k .

6.4.1.2 The case where $\langle \mathbf{c}, e_2 \rangle$ is bounded below

We first consider the regime where $\langle \mathbf{c}, e_2 \rangle$ is bounded below. Our goal is to prove that there exists a positive $\xi > 0$ such that, as $\langle \mathbf{c}, e_1 \rangle \rightarrow -\infty$ with $\langle \mathbf{c}, e_2 \rangle$ bounded below:

$$e^{-2 \langle Q, \mathbf{c} \rangle} \mathbb{E} \left[e^{-\mu_2 e^{\gamma \langle \mathbf{c}, e_2 \rangle} M^{\gamma e_2}(\mathbb{C})} \left(e^{-\mu_1 e^{\gamma \langle \mathbf{c}, e_1 \rangle} M^{\gamma e_1}(\mathbb{C})} - \mathcal{R}_{Id, s_1} \right) \prod_{k=1}^N V_{\alpha_k}(z_k) \right] = \mathcal{O}(e^{\xi \langle \mathbf{c}, e_1 \rangle}). \quad (6.4.7)$$

To start with we choose $\varepsilon > 0$ small enough and consider for $\mathbf{c} \in \mathbb{R}^2$ the radii $r_k = r_k(\mathbf{c})$ such that

$$(1+\varepsilon) \left(\frac{\langle \mathbf{c}, e_1 \rangle_-}{2 \langle \alpha_k - Q, \rho \rangle} \vee \frac{\langle \mathbf{c}, e_1 \rangle_-}{\langle Q - \alpha_k, e_1 \rangle - 2\gamma} \mathbb{1}_{\langle \alpha_k - Q, e_1 \rangle > -\gamma} \right) \leq r_k \leq (1-\varepsilon) \frac{\langle \mathbf{c}, e_1 \rangle_-}{\langle \alpha_k - Q, e_1 \rangle}$$

and assume that $\langle \mathbf{c}, e_1 \rangle$ is negative enough so that the balls $B_{r_k}(z_k)$ remain disjoint. Note that the above bounds can be satisfied as soon as $\langle \alpha_k - Q, e_1 \rangle \neq -\gamma$ (which we assumed to hold) and for ε small enough. Then the integral $M^{\gamma e_1}(\mathbb{C})$ can be decomposed

as $M^{\gamma e_1}(\mathbb{C}) = M^{\gamma e_1}(\mathbb{C}_r) + \sum_{k=1}^N M^{\gamma e_1}(B_{r_k}(z_k))$ with $\mathbb{C}_r := \mathbb{C} \setminus (\cup_{k=1}^N B_{r_k}(z_k))$. This allows to write that for any bounded continuous map F over $H^{-1}(\mathbb{C} \rightarrow \mathbb{R}^2, g)$

$$\begin{aligned} & e^{-2\langle Q, \mathbf{c} \rangle} \mathbb{E} \left[\prod_{k=1}^N V_{\alpha_k}(z_k) e^{-\mu_1 e^{\gamma \langle \mathbf{c}, e_1 \rangle} M^{\gamma e_1}(\mathbb{C})} F(\mathbf{X}) \right] \\ &= \prod_{j < k} |z_j - z_k|^{-\langle \alpha_k, \alpha_j \rangle} e^{\langle \mathbf{s}, \mathbf{c} \rangle} \mathbb{E} \left[e^{-\mu_1 e^{\gamma \langle \mathbf{c}, e_1 \rangle} I^1(\mathbb{C})} G(\mathbf{X}) \right], \end{aligned}$$

where $G(\mathbf{X}) := F\left(\mathbf{X} + \sum_{k=1}^N \alpha_k G(z_k, \cdot)\right)$ and with

$$\begin{aligned} \mathbb{E} \left[e^{-\mu_1 e^{\gamma \langle \mathbf{c}, e_1 \rangle} I^1(\mathbb{C})} G(\mathbf{X}) \right] &= \mathbb{E} \left[\prod_{k=1}^N e^{-\mu_1 e^{\gamma \langle \mathbf{c}, e_1 \rangle} I^1(B_{r_k}(z_k))} G(\mathbf{X}) \right] \\ &+ \mathbb{E} \left[\left(e^{-\mu_1 e^{\gamma \langle \mathbf{c}, e_1 \rangle} I^1(\mathbb{C}_r)} - 1 \right) \prod_{k=1}^N e^{-\mu_1 e^{\gamma \langle \mathbf{c}, e_1 \rangle} I^1(B_{r_k}(z_k))} G(\mathbf{X}) \right]. \end{aligned}$$

Let us start by considering the first expectation term, which we rewrite under the form

$$\sum_{\mathcal{U} \subset \{1, \dots, N\}} \mathbb{E} \left[\prod_{k \in \mathcal{U}} \left(e^{-\mu_1 e^{\gamma \langle \mathbf{c}, e_1 \rangle} I^1(B_{r_k}(z_k))} - 1 \right) G(\mathbf{X}) \right]$$

where the sum ranges over subsets $\mathcal{U}$ of $\{1, \dots, N\}$. In the case where $\langle \alpha_1, e_1 \rangle < \frac{2}{\gamma}$, then the term

$$\mathbb{E} \left[I^1(B_{r_1}(z_1)) \prod_{k \neq 1 \in \mathcal{U}} \left(e^{-\mu_1 e^{\gamma \langle \mathbf{c}, e_1 \rangle} I^1(B_{r_k}(z_k))} - 1 \right) G(\mathbf{X}) \right]$$

is well defined thanks to Lemma 3.3.6, which allows to write that

$$\sum_{\mathcal{U} \subset \{1, \dots, N\}} \mathbb{E} \left[\left(e^{-\mu_1 e^{\gamma \langle \mathbf{c}, e_1 \rangle} I^1(B_{r_1}(z_1))} - 1 \right) \prod_{k \neq 1 \in \mathcal{U}} \left(e^{-\mu_1 e^{\gamma \langle \mathbf{c}, e_1 \rangle} I^1(B_{r_k}(z_k))} - 1 \right) G(\mathbf{X}) \right] = \mathcal{O}(e^{\gamma \langle \mathbf{c}, e_1 \rangle}).$$

When $\langle \alpha_k - Q, e_1 \rangle > -\gamma$ for some $k \in \mathcal{U}$, the analysis is more subtle. Namely one needs to use the radial-angular decomposition (3.3.6) around each insertion z_k to put the integrals involved under the form

$$\begin{aligned} I^1(B_{r_k}(z_k)) &= \int_{r_k}^{+\infty} e^{\gamma \langle B_t^k, e_1 \rangle} \int_0^{2\pi} F_1^k(t, \theta) M_{\mathbf{Y}^k}^{\gamma e_1}(dt, d\theta) \\ &= e^{\gamma \langle B_{r_k}^k, e_1 \rangle} \int_0^{+\infty} e^{\gamma \tilde{B}_t^k} \int_0^{2\pi} F_1^k(t + r_k, \theta) M_{\mathbf{Y}^k}^{\gamma e_1}(dt + r_k, d\theta) \end{aligned}$$

where $B_t^k := \mathbf{X}_t(z_k) + (\alpha_k - Q)t$ and $\mathbf{Y}^k(t, \theta) := \mathbf{X}(z_k + e^{-t+i\theta}) - \mathbf{X}_t(z_k)$ have the law of the pair described in Equation (3.3.6), while $F_1^k(t, \theta) := \prod_{l \neq k} \left(\frac{|z_k + e^{-t+i\theta}|_+}{|z_k - z_l + e^{-t+i\theta}|} \right)^{\gamma \langle \alpha_k, e_1 \rangle}$.

In the last equation $\tilde{B}^k$ is a one-dimensional Brownian motion with drift $\nu_k := \langle \alpha_k -$

$Q, e_1\rangle$ and variance 2, started from the origin and independent of the sigma algebra generated by the $(\mathbf{X}(z))_{z \notin B_{r_k}(z_k)}$. We can apply Williams path decomposition (see Subsection 3.3.4.1) to this Brownian motion, which allows to write that

$$I^1(B_{r_k}(z_k)) = e^{\gamma\langle B_{r_k}^k + \mathbf{M}_k, e_1 \rangle} \int_0^{+\infty} e^{\gamma \mathcal{B}_t^k} \int_0^{2\pi} F_1^k(t + r_k, \theta) M_{\mathbf{Y}^k}^{\gamma e_1}(dt + r_k, d\theta)$$

where $\mathcal{B}^k$ is the one-dimensional process described in Subsection 3.3.4.1, started from $-\langle \mathbf{M}_k, e_1 \rangle$ which is an exponential variable of parameter $-\nu_k$ independent of everything. Therefore, with $q = |\mathcal{U}|$ and for any G_r that only depends on $(\mathbf{X}_z)_{z \in \mathbb{C}_r}$,

$$\begin{aligned} & \mathbb{E} \left[\prod_{k \in \mathcal{U}} \left(e^{-\mu_1 e^{\gamma\langle \mathbf{c}, e_1 \rangle} I^1(B_{r_k}(z_k))} - 1 \right) G_r(\mathbf{X}) \right] \\ &= \int_{(0, +\infty)^q} \prod_{k \in \mathcal{U}} (-\nu_k) e^{\nu_k \langle \mathbf{M}_k, e_1 \rangle} d\langle \mathbf{M}_k, e_1 \rangle \mathbb{E} \left[\prod_{k=1}^p \left(e^{-\mu_1 e^{\gamma\langle \mathbf{c} + B_{r_k}^k + \mathbf{M}_k, e_1 \rangle} J_k^r(-\langle \mathbf{M}_k, e_1 \rangle)} - 1 \right) G_r(\mathbf{X}) \right] \end{aligned}$$

$$\text{with } J_k^r(-\langle \mathbf{M}_k, e_1 \rangle) := \int_0^{+\infty} e^{\gamma \mathcal{B}_t^k} \int_0^{2\pi} F_1^k(t + r_k, \theta) M_{\mathbf{Y}^k}^{\gamma e_1}(dt + r_k, d\theta)$$

and where we use the notation $J_k^r(-\langle \mathbf{M}_k, e_1 \rangle)$ to stress that the process $\mathcal{B}^k$ is started from $-\langle \mathbf{M}_k, e_1 \rangle$. We can now make a change of variable $\langle \mathbf{M}_k, e_1 \rangle \leftrightarrow \langle \mathbf{M}_k + \mathbf{c} + B_{r_k}^k, e_1 \rangle$ to end up with

$$\begin{aligned} & \mathbb{E} \left[\int_{\mathbb{R}^q} \prod_{k \in \mathcal{U}} (-\nu_k) e^{\nu_k \langle \mathbf{M}_k - \mathbf{c} - B_{r_k}^k, e_1 \rangle} d\langle \mathbf{M}_k, e_1 \rangle \right. \\ & \quad \left. \left(e^{-\mu_1 e^{\gamma\langle \mathbf{M}_k, e_1 \rangle} J_k^r(\langle \mathbf{c} + B_{r_k}^k - \mathbf{M}_k, e_1 \rangle)} - 1 \right) \mathbb{1}_{\langle \mathbf{M}_k - \mathbf{c} - B_{r_k}^k, e_1 \rangle > 0} G_r(\mathbf{X}) \right]. \end{aligned}$$

We can interpret the exponential term $\prod_{k \in \mathcal{U}} e^{-\nu_k \langle B_{r_k}^k, e_1 \rangle}$ as a Girsanov transform. Namely recalling that $B_{r_k}^k = \mathbf{X}_{r_k}(z_k)$, we see that

$$\begin{aligned} & - \sum_{k \in \mathcal{U}} \nu_k \langle B_{r_k}^k, e_1 \rangle = \sum_{k \in \mathcal{U}} \langle Q - \alpha_k, e_1 \rangle \langle \mathbf{X}_{r_k}(z_k), e_1 \rangle - \langle Q - \alpha_k, e_1 \rangle^2 r_k \\ &= \sum_{k \in \mathcal{U}} \langle Q - \alpha_k, e_1 \rangle \langle \mathbf{X}_{r_k}(z_k), e_1 \rangle - \frac{\mathbb{E} \left[\left(\sum_{k \in \mathcal{U}} \langle Q - \alpha_k, e_1 \rangle \langle \mathbf{X}_{r_k}(z_k), e_1 \rangle \right)^2 \right]}{2} \\ &+ \sum_{k < l \in \mathcal{U}} \langle Q - \alpha_k, e_1 \rangle \langle Q - \alpha_l, e_1 \rangle G'_r(z_k, z_l), \quad \text{where} \end{aligned}$$

$$G'_r(z_k, z_l) := \frac{1}{2\pi} \oint_{\partial B_{r_l}(z_l)} G_{r_k}(z_k, w) \frac{dw}{w} \quad \text{and} \quad G_{r_k}(z_k, \cdot) := \frac{1}{2\pi} \oint_{\partial B_{r_k}(z_k)} G(w, \cdot) \frac{dw}{w}.$$

Note that if we take $\langle \mathbf{c}, e_1 \rangle$ large enough, then $G'_{r_k}(z_k, z_l) = G(z_k, z_l)$ for $k \neq l$. Likewise explicit computations show that $G'_r(z_k, z_k) = r + 2 \ln |z_k|_+$ for r large enough. Therefore without loss of generality we can assume that both assumptions hold in the sequel.

Now in virtue of Theorem A this exponential term has the effect of shifting the law of $\mathbf{X}$ by $\sum_k \langle Q - \alpha_k, e_1 \rangle G_{r_k}(z_k, \cdot) e_1$, and in particular shifts the law of B_r^k by

$$\lambda_k(r_k) := \langle Q - \alpha_k, e_1 \rangle (r_k + 2 \ln |z_k|_+) e_1 + \sum_{l \neq k} \langle Q - \alpha_k, e_1 \rangle G(z_k, z_l) e_1.$$

This shows that the above expectation term is equal to

$$= \prod_{k \neq l \in \mathcal{U}} e^{\langle Q - \alpha_k, e_1 \rangle \langle Q - \alpha_l, e_1 \rangle G(z_k, z_l)} \prod_{k \in \mathcal{U}} (-\nu_k) e^{-\nu_k \langle c, e_1 \rangle} \int_{\mathbb{R}^q} \prod_{k \in \mathcal{U}} (-\nu_k) e^{\nu_k \langle \mathbf{M}_k, e_1 \rangle} d\langle \mathbf{M}_k, e_1 \rangle$$

$$\mathbb{E} \left[\left(e^{-\mu_1 e^{\gamma \langle \mathbf{M}_k, e_1 \rangle} J_k^r(\langle \lambda_k(r_k) - \mathbf{M}_k, e_1 \rangle)} - 1 \right) \mathbb{1}_{\langle \mathbf{M}_k - \lambda_k(r_k), e_1 \rangle > 0} G_r \left(\mathbf{X} + \sum_{k \in \mathcal{U}} \langle Q - \alpha_k, e_1 \rangle G(z_k, \cdot) e_1 \right) \right]$$

where on the last line we have used that for $z \in \mathbb{C}_r$, $G_{r_k}(z_k, z) = G(z_k, z)$ for all $1 \leq k \leq N$, and with

$$\lambda_k(r_k) := \mathbf{c} + B_{r_k}^k + \langle Q - \alpha_k, e_1 \rangle (r_k + 2 \ln |z_k|_+) e_1 + \sum_{l \neq k} \langle Q - \alpha_k, e_1 \rangle G(z_k, z_l) e_1.$$

Now we have assumed that for some positive ε we have $r_{k,1} < (1 - \varepsilon) \frac{\langle c, e_1 \rangle}{\langle \alpha_k - Q, e_1 \rangle}$ for all $1 \leq k \leq N$. As a consequence we see that $\langle \lambda_k(r), e_1 \rangle \rightarrow -\infty$ almost surely, so that along the same lines as in the proof of [31, Proposition 4.10] the latter will be asymptotically equivalent to

$$\prod_{k \neq l \in \mathcal{U}} e^{\langle Q - \alpha_k, e_1 \rangle \langle Q - \alpha_l, e_1 \rangle G(z_k, z_l)} \prod_{k \in \mathcal{U}} R_{s_1}(\alpha_k) e^{\langle Q - \alpha_k, e_1 \rangle \langle c, e_1 \rangle} \mathbb{E} \left[G_r \left(\mathbf{X} + \sum_{k \in \mathcal{U}} \langle Q - \alpha_k, e_1 \rangle G(z_k, \cdot) e_1 \right) \right].$$

More precisely the reasoning developed in the proof of Proposition 6.3.3, based on the Markov property for the process entering the definition of J_k^r (see also the proof of Equation (6.4.8) below), allows to write that

$$\mathbb{E} \left[\prod_{k \in \mathcal{U}} \left(e^{-\mu_1 e^{\gamma \langle c, e_1 \rangle} I^1(B_r(z_k))} - 1 \right) G_r(\mathbf{X}) \right] =$$

$$\prod_{k \neq l \in \mathcal{U}} e^{\langle Q - \alpha_k, e_1 \rangle \langle Q - \alpha_l, e_1 \rangle G(z_k, z_l)} \prod_{k \in \mathcal{U}} R_{s_1}(\alpha_k) e^{\langle Q - \alpha_k, e_1 \rangle \langle c, e_1 \rangle} \mathbb{E} \left[G_r \left(\mathbf{X} + \sum_{k \in \mathcal{U}} \langle Q - \alpha_k, e_1 \rangle G(z_k, \cdot) e_1 \right) \right]$$

$$+ \mathcal{O} \left(e^{(1-\eta) \langle c, \gamma e_1 \rangle} \right)$$

as soon as $|\mathcal{U}| \geq 1$.

Therefore recollecting terms yields

$$e^{-2 \langle Q, c \rangle} \mathbb{E} \left[\prod_{k=1}^N V_{\alpha_k}(z_k) e^{-\mu_1 e^{\gamma \langle c, e_1 \rangle} M^{\gamma e_1}(B_{r_k}(z_k))} F_r(\mathbf{X}) \right]$$

$$= \sum_{\mathcal{U} \subset \{1, \dots, N\}} e^{-2 \langle Q, c \rangle} \mathbb{E} \left[\prod_{k \notin \mathcal{U}} V_{\alpha_k}(z_k) \prod_{k \in \mathcal{U}} R_{s_1}(\alpha_k) V_{\hat{s}_1 \alpha_k}(z_k) F_r(\mathbf{X}) \right] + \mathcal{O} \left(e^{\langle s, c \rangle + (1-\eta) \langle c, \gamma e_1 \rangle} \right)$$

$$= e^{-2 \langle Q, c \rangle} \mathbb{E} \left[\prod_{k=1}^N (V_{\alpha_k}(z_k) + R_{s_1}(\alpha_k) V_{\hat{s}_1 \alpha_k}(z_k)) F_r(\mathbf{X}) \right] + \mathcal{O} \left(e^{\langle s, c \rangle + (1-\eta) \langle c, \gamma e_1 \rangle} \right).$$

Under the assumption that $\langle s, \omega_1 \rangle > -\gamma$ (which we assumed to hold since $\alpha \in \mathcal{A}_N$) the remainder term is as desired, a term with asymptotic bounded by a $\mathcal{O} \left(e^{\xi \langle c, e_1 \rangle} \right)$ for

some positive ξ as $\langle \mathbf{c}, e_1 \rangle \rightarrow -\infty$ with $\langle \mathbf{c}, e_2 \rangle$ bounded below. By choosing $F_r(\mathbf{X}) = M^{\gamma e_2}(\mathbb{C}_r)$ we have therefore proved that

$$e^{-2\langle Q, \mathbf{c} \rangle} \mathbb{E} \left[\left(e^{-\mu_1 e^{\gamma \langle \mathbf{c}, e_1 \rangle} M^{\gamma e_1}(\mathbb{C} \setminus \mathbb{C}_r)} - \mathcal{R}_{Id, s_1} \right) \prod_{k=1}^N V_{\alpha_k}(z_k) e^{-\mu_2 e^{\gamma \langle \mathbf{c}, e_2 \rangle} M^{\gamma e_2}(\mathbb{C}_r)} \right]$$

is a $\mathcal{O}(e^{\xi \langle \mathbf{c}, e_1 \rangle})$ in this asymptotic. As a consequence it remains to control the terms

$$e^{-2\langle Q, \mathbf{c} \rangle} \mathbb{E} \left[\prod_{k=1}^N V_{\alpha_k}(z_k) \left(e^{\mu_1 e^{\gamma \langle \mathbf{c}, e_1 \rangle} M^{\gamma e_1}(\mathbb{C}_r)} - 1 \right) e^{-\sum_{i=1}^2 \mu_i e^{\gamma \langle \mathbf{c}, e_i \rangle} M^{\gamma e_i}(\mathbb{C})} \right] \quad \text{and} \\ e^{-2\langle Q, \mathbf{c} \rangle} \mathbb{E} \left[\left(e^{-\mu_1 e^{\gamma \langle \mathbf{c}, e_1 \rangle} M^{\gamma e_1}(\mathbb{C} \setminus \mathbb{C}_r)} - \mathcal{R}_{Id, s_1} \right) \left(e^{\mu_2 e^{\gamma \langle \mathbf{c}, e_2 \rangle} M^{\gamma e_2}(\mathbb{C} \setminus \mathbb{C}_r)} - 1 \right) \prod_{k=1}^N V_{\alpha_k}(z_k) \right].$$

We start with the second term. Along the same lines as above it suffices to show that

$$\mathbb{E} \left[\left(e^{-\mu_1 e^{\gamma \langle \mathbf{c}, e_1 \rangle} I^1(B_{r_1}(z_1))} - 1 - R_{s_1}(\alpha_1) V_{\hat{s}_1 \alpha_1 - \alpha_1}(z_1) \right) \left(e^{\mu_2 e^{\gamma \langle \mathbf{c}, e_2 \rangle} I^2(B_{r_1}(z_1))} - 1 \right) \right]$$

is a $\mathcal{O}(e^{(1-\eta)\gamma \langle \mathbf{c}, e_1 \rangle})$. Now like before we see that the law of the planar, drifted Brownian motion $X_{t+r_1}(z_1) - \mathbf{X}_{r_1}(z_1) + (\alpha_1 - Q)t$ can be realized by sampling $\langle \mathbf{M}_1, e_1 \rangle$ according to its marginal law and then sampling the process $\mathcal{B}^1$, whose e_1 component is the process $\mathcal{B}_t^1$ used above (and described in Subsection 3.3.4.1) while its ω_2 component is an independent Brownian motion $\tilde{B}^1$ with drift $\langle \alpha_1 - Q, \omega_2 \rangle$ and variance $|\omega_2|^2 = \frac{2}{3}$. This shows that its e_2 component is the independent sum of $-\frac{1}{2}\mathcal{B}^1$ and $\frac{3}{2}\tilde{B}^1$, so that

$$I^2(B_{r_1}(z_1)) = e^{\gamma(\langle \mathbf{X}_{r_1}(z_1) + (\alpha_0 - Q)r_1, e_2 \rangle)} \int_0^{+\infty} e^{\gamma(-\frac{1}{2}\mathcal{B}^1 + \frac{3}{2}\tilde{B}_t^1)} \int_0^{2\pi} F_2^1(t+r_1, \theta) M_{\mathbf{Y}^1}^{\gamma e_2}(dt+r_1, d\theta).$$

The Girsanov term $e^{\langle Q - \alpha_1, e_1 \rangle B_{r_1}^1}$ has the effect of changing I^2 into $\hat{I}^2 = e^{\langle \alpha_0 - Q, e_1 \rangle r_1} I^2$. Thanks to our assumption that $r_1 \geq (1+\varepsilon) \frac{\langle \mathbf{c}, e_1 \rangle}{2\langle \alpha_0 - Q, \rho \rangle}$, which ensures that $\langle \alpha_0 - Q, \rho \rangle r_1 + \frac{\langle \mathbf{M}_1, e_1 \rangle}{2} \rightarrow -\infty$ almost surely for $\langle \mathbf{M}_1, e_1 \rangle = -\langle \lambda_1(r_1), e_1 \rangle$, we see that for $\xi > 0$ small enough

$$e^{\xi r_1} \times e^{\gamma(\langle \mathbf{X}_1(z_1), e_2 \rangle + \langle \alpha_0 - Q, \rho \rangle r_1)} \int_0^{+\infty} e^{\gamma(-\frac{1}{2}\mathcal{B}^1 + \frac{3}{2}\tilde{B}_t^1)} \int_0^{2\pi} F_2^1(t+r_1, \theta) M_{\mathbf{Y}^1}^{\gamma e_2}(dt+r_1, d\theta) \rightarrow 0$$

as $\langle \mathbf{c}, e_1 \rangle \rightarrow -\infty$, almost surely. Therefore

$$\mathbb{E} \left[\left(e^{-\mu_1 e^{\gamma \langle \mathbf{c}, e_1 \rangle} I^1(B_{r_1}(z_1))} - 1 - R_{s_1}(\alpha_1) V_{\hat{s}_1 \alpha_1 - \alpha_1}(z_1) \right) \left(e^{\mu_2 e^{\gamma \langle \mathbf{c}, e_2 \rangle} I^2(B_{r_1}(z_1))} - 1 \right) \right]$$

is a lower order term compared to

$$\mathbb{E} \left[\left(e^{-\mu_1 e^{\gamma \langle \mathbf{c}, e_1 \rangle} I^1(B_{r_1}(z_1))} - 1 - R_{s_1}(\alpha_1) V_{\hat{s}_1 \alpha_1 - \alpha_1}(z_1) \right) e^{\mu_2 e^{\gamma \langle \mathbf{c}, e_2 \rangle} I^2(\mathbb{C} \setminus B_{r_1}(z_1))} \right]$$

which was already shown to be a $\mathcal{O}(e^{(1-\eta)\gamma \langle \mathbf{c}, e_1 \rangle})$. As a consequence to show that Equation (6.4.7) does indeed hold it only remains to prove that

$$e^{-2\langle Q, \mathbf{c} \rangle} \mathbb{E} \left[\prod_{k=1}^N V_{\alpha_k}(z_k) \left(e^{\mu_1 e^{\gamma \langle \mathbf{c}, e_1 \rangle} M^{\gamma e_1}(\mathbb{C}_r)} - 1 \right) e^{-\sum_{i=1}^2 \mu_i e^{\gamma \langle \mathbf{c}, e_i \rangle} M^{\gamma e_i}(\mathbb{C})} \right] = \mathcal{O}(e^{\gamma \langle \mathbf{c}, e_1 \rangle}).$$

For this we can write that

$$\begin{aligned}\mathbb{E} \left[I^1(\mathbb{C}_r) e^{-\sum_{i=1}^2 \mu_i e^{\gamma \langle \mathbf{c}, e_i \rangle} I^i(\mathbb{C})} \right] &= \int_{\mathbb{C}_r} \prod_{k=1}^N \left(\frac{|x|_+}{|z_k - x|} \right)^{\gamma \langle \alpha_k, e_1 \rangle} \mathbb{E} \left[V_{\gamma e_1}(x) e^{-\sum_{i=1}^2 \mu_i e^{\gamma \langle \mathbf{c}, e_i \rangle} I^i(\mathbb{C})} \right] d^2 x \\ &= \int_{\mathbb{C}_r} \prod_{k=1}^N \left(\frac{|x|_+}{|z_k - x|} \right)^{\gamma \langle \alpha_k, e_1 \rangle} \mathbb{E} \left[e^{-\sum_{i=1}^2 \mu_i e^{\gamma \langle \mathbf{c}, e_i \rangle} \tilde{I}^i(x)} \right] d^2 x, \\ \text{where } \tilde{I}^i(x) &= \int_{\mathbb{C}} \prod_{k=1}^N \left(\frac{|y|_+}{|z_k - y|} \right)^{\gamma \langle \alpha_k, e_1 \rangle} \left(\frac{|y|_+}{|y - x|} \right)^{2\gamma^2} M^{\gamma e_1}(d^2 y).\end{aligned}$$

When $r \rightarrow +\infty$, the singularities around $x = z_k$ are integrable if $\langle \alpha_k, e_1 \rangle < \frac{2}{\gamma}$, so that in that case the remainder term is a $\mathcal{O}(e^{\gamma \langle \mathbf{c}, e_1 \rangle})$. On the contrary if $\langle \alpha_k, e_1 \rangle > \frac{2}{\gamma}$ then around $x = z_k$ (say in an annulus $A_k(r)$ centered at z_k and radii e^{-r_k} and $2e^{-r_k}$) we can make the change of variable $x \leftrightarrow z_k + e^{-r_k}(x - z_k)$ to see that

$$\begin{aligned}\mathbb{E} \left[I^1(A_k(r)) e^{-\sum_{i=1}^2 \mu_i e^{\gamma \langle \mathbf{c}, e_i \rangle} I^i(\mathbb{C})} \right] \\ \sim e^{-r_k(2-\gamma \langle \alpha_k, e_1 \rangle)} \int_{A_k(1)} \prod_{k=1}^N \left(\frac{|z_k|_+}{|z_k - x|} \right)^{\gamma \langle \alpha_k, e_1 \rangle} \mathbb{E} \left[e^{-\sum_{i=1}^2 \mu_i e^{\gamma \langle \mathbf{c}, e_i \rangle} \tilde{I}^i(z_k + e^{-r_k}(x - z_k))} \right] d^2 x.\end{aligned}$$

Now when $r_k \rightarrow +\infty$, we can use the fusion asymptotics (4.4.2) to see that for any positive ε :

$$\mathbb{E} \left[e^{-\sum_{i=1}^2 \mu_i e^{\gamma \langle \mathbf{c}, e_i \rangle} \tilde{I}^i(z_k + e^{-r_k}(x - z_k))} \right] = o \left(e^{-r_k \left(\frac{\langle \alpha_k + \gamma e_1 - Q, e_1 \rangle^2}{4} - \varepsilon \right)} \right).$$

Therefore we end up with the bound

$$\mathbb{E} \left[\left(e^{\mu_1 e^{\gamma \langle \mathbf{c}, e_1 \rangle} I^1(\mathbb{C}_r)} - 1 \right) e^{-\sum_{i=1}^2 \mu_i e^{\gamma \langle \mathbf{c}, e_i \rangle} M^{\gamma e_i}(\mathbb{C})} \right] \leq C e^{\gamma \langle \mathbf{c}, e_1 \rangle} e^{-r_k \left(2 - \gamma \langle \alpha_k, e_1 \rangle + \frac{\langle \alpha_k + \gamma e_1 - Q, e_1 \rangle^2}{4} - \varepsilon \right)}$$

which shows that this term is a lower order term too since the exponent $2 - \gamma \langle \alpha_k, e_1 \rangle + \frac{\langle \alpha_k + \gamma e_1 - Q, e_1 \rangle^2}{4} = \frac{\langle \alpha_k - Q, e_1 \rangle^2}{4}$ is positive.

This concludes for the proof of Equation (6.4.7).

6.4.1.3 The case where $c \rightarrow \infty$ inside $\mathcal{C}_-$

We can proceed in a similar fashion for the asymptotic where both $\langle \mathbf{c}, e_1 \rangle$ and $\langle \mathbf{c}, e_2 \rangle$ diverge to $-\infty$. In that case we aim to prove that

$$\begin{aligned}\mathbb{E} \left[\prod_{k=1}^N V_{\alpha_k}(z_k) e^{-\sum_{i=1}^2 \mu_i e^{\gamma \langle \mathbf{c}, e_i \rangle} M^{\gamma e_i}(\mathbb{C})} \right] \\ = \mathbb{E} \left[\prod_{k=1}^N \left(\sum_{s \in W} R_s(\alpha_k) V_{s\alpha_k}(z_k) \right) \right] + \mathcal{O} \left(\sum_{i=1}^2 e^{(1-\eta)\gamma \langle \mathbf{c}, e_i \rangle} \right).\end{aligned}\tag{6.4.8}$$

In order to prove Equation (6.4.8) we first pick r_k like before and write

$$\mathbb{E} \left[\prod_{k=1}^N e^{-\sum_{i=1}^2 \mu_i e^{\gamma \langle \mathbf{c}, e_i \rangle} I^i(\mathbb{C})} \right] = \sum_{\mathcal{U}_i \subset \{1, \dots, N\}} \mathbb{E} \left[\prod_{i=1}^2 e^{-\mu_i e^{\gamma \langle \mathbf{c}, e_i \rangle} I^i(\mathbb{C}_r)} \prod_{k \in \mathcal{U}_i} \left(e^{-\mu_i e^{\gamma \langle \mathbf{c}, e_i \rangle} I^i(B_{r_k}(z_k))} - 1 \right) \right],$$

which we can further put under the form

$$\sum_{\substack{\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3 \subset \{1, \dots, N\} \\ \text{disjoint}}} \mathbb{E} \left[\prod_{i=1}^2 e^{-\mu_i e^{\gamma \langle \mathbf{c}, e_i \rangle} I^i(\mathbb{C}_r)} \prod_{k_i \in \mathcal{U}_i} \left(e^{-\mu_i e^{\gamma \langle \mathbf{c}, e_i \rangle} I^i(B_{r_k}(z_{k_i}))} - 1 \right) \prod_{k_3 \in \mathcal{U}_3} \left(e^{-\mu_i e^{\gamma \langle \mathbf{c}, e_i \rangle} I^i(B_{r_k}(z_{k_3}))} - 1 \right) \right].$$

We now distinguish on whether $k \in \mathcal{U}_1, \mathcal{U}_2$ or $\mathcal{U}_3$. If k belongs to $\mathcal{U}_i$ for $i = 1$ or $i = 2$ then the only integral surrounding the singular point $z = k$ is given by $I^i(B_{r_k}(z_k))$. This term can be processed along the same lines as above by applying Williams path decomposition to the one-dimensional Brownian motion $\langle \mathbf{X}_{t+r}(z_k) - \mathbf{X}_r(z_k) + \langle \alpha_k - Q \rangle t, e_i \rangle$. Hence

$$\begin{aligned} \mathbb{E} \left[\prod_{k=1}^N e^{-\sum_{i=1}^2 \mu_i e^{\gamma \langle \mathbf{c}, e_i \rangle} I^i(\mathbb{C})} \right] = \\ \sum_{\substack{\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3 \subset \{1, \dots, N\} \\ \text{disjoint}}} \mathbb{E} \left[\prod_{i=1}^2 e^{-\mu_i e^{\gamma \langle \mathbf{c}, e_i \rangle} I^i(\mathbb{C}_r)} \prod_{k_i \in \mathcal{U}_i} R_{s_i}(\hat{s}_i \alpha_{k_i}) V_{\hat{s}_i \alpha_{k_i} - \alpha_{k_i}}(z_{k_i}) \prod_{k_3 \in \mathcal{U}_3} \left(e^{-\mu_i e^{\gamma \langle \mathbf{c}, e_i \rangle} I^i(B_{r_k}(z_{k_3}))} - 1 \right) \right] \\ + \mathcal{O} \left(\sum_{i=1}^2 e^{(1-\eta)\gamma \langle \mathbf{c}, e_i \rangle} \right). \end{aligned}$$

For terms that correspond to $k_3 \in \mathcal{U}_3$, because of our assumptions on the weights we know that $\langle s\nu_k - \nu_k, \omega_i \rangle < -\gamma$ for $i = 1$ or $i = 2$; without loss of generality we can assume that $\langle s_1 s_2 \nu_k - \nu_k, \omega_1 \rangle < -\gamma$. The asymptotic of the corresponding term is then governed by

$$-\mu_1 e^{\gamma \langle \mathbf{c}, e_1 \rangle} \mathbb{E} \left[I^1(B_{r_k}(z_{k_3})) \prod_{i=1}^2 e^{-\mu_i e^{\gamma \langle \mathbf{c}, e_i \rangle} I^i(\mathbb{C}_r)} \prod_{k_i \in \mathcal{U}_i} R_{s_i}(\hat{s}_i \alpha_{k_i}) V_{\hat{s}_i \alpha_{k_i} - \alpha_{k_i}}(z_{k_i}) \right]$$

which is a $\mathcal{O}(e^{\gamma \langle \mathbf{c}, e_1 \rangle})$ via the same analysis as before. Thanks to the fact that for $s \notin \{Id, s_1, s_2\}$, $V_{\hat{s}\alpha_k - \alpha_k}$ is a $\mathcal{O}(\sum_{i=1}^2 e^{\gamma \langle \mathbf{c}, e_i \rangle})$, this shows that Equation (6.4.8) does indeed hold.

6.4.1.4 Recollecting terms

Recall that we have defined $E_\alpha(\mathbf{c})$ via the expression

$$E_\alpha(\mathbf{c}) = e^{-2\langle Q, \mathbf{c} \rangle} \mathbb{E} \left[\prod_{k=1}^N V_{\alpha_k}(z_k) e^{-\sum_{i=1}^2 \mu_i e^{\gamma \langle \mathbf{c}, e_i \rangle} M^{\gamma e_i}(\mathbb{C})} - \mathbf{R}_\alpha^1(\mathbf{c}) - \mathbf{R}_\alpha^2(\mathbf{c}) - \mathbf{R}_\alpha^W(\mathbf{c}) \right],$$

where the terms that appear are defined in Equation (6.4.6).

With the notations introduced there, another way of formulating Equation (6.4.7) is to write that as $\langle \mathbf{c}, e_1 \rangle \rightarrow -\infty$ with $\langle \mathbf{c}, e_2 \rangle$ bounded below

$$e^{-2\langle Q, \mathbf{c} \rangle} \mathbb{E} \left[\left(e^{-\mu_1 e^{\gamma \langle \mathbf{c}, e_1 \rangle} M^{\gamma e_1}(\mathbb{C})} - \mathcal{R}_{Id, s_1} \right) e^{-\mu_2 e^{\gamma \langle \mathbf{c}, e_2 \rangle} M^{\gamma e_2}(\mathbb{C})} \prod_{k=1}^N V_{\alpha_k}(z_k) \right] = \mathcal{O}(e^{\xi \langle \mathbf{c}, e_1 \rangle}),$$

while the same reasoning shows that in this regime too $e^{-2\langle Q, c \rangle} \mathbb{E} [\mathbf{R}_\alpha^1(c)] = \mathcal{O}(e^{\xi\langle c, e_1 \rangle})$. As a consequence in this asymptotic we can write that

$$E_\alpha(c) = e^{-2\langle Q, c \rangle} \mathbb{1}_{\langle c, e_2 \rangle < 0} \mathbb{E} \left[\mathcal{R}_{Id, s_1, s_2, s_2 s_1} \prod_{k=1}^N V_{\alpha_k}(z_k) - \mathbf{R}_\alpha^{1,2}(c) \right] + \mathcal{O}(e^{\xi\langle c, e_1 \rangle}).$$

Now from its explicit expression we readily see that the above expectation term is asymptotically equivalent to

$$\max_{1 \leq k \leq N} e^{-2\langle Q, c \rangle} \mathbb{E} \left[R_{s_1 s_2}(\alpha_k) V_{\hat{s}_1 \hat{s}_2 \alpha_k}(z_k) \prod_{l \neq k} V_{\alpha_l}(z_l) \right]$$

which scales like $\max_{1 \leq k \leq N} e^{\langle s + \hat{s}_1 \hat{s}_2 \alpha_k - \alpha_k, c \rangle}$. As a consequence this remainder term is integrable as soon as $\max_{1 \leq k \leq N} \langle s + \hat{s}_1 \hat{s}_2 \alpha_k - \alpha_k, \omega_1 \rangle > 0$, which follows from the fact that $\langle s, \omega_1 \rangle > -\gamma$ while $\langle \hat{s}_1 \hat{s}_2 \alpha_k - \alpha_k, \omega_1 \rangle > \gamma$ because of our assumptions on α for it to belong to $\mathcal{A}_N$. This shows that $e^{\xi|c|} E_\alpha(c)$ remains bounded over the domain where $\langle c, e_2 \rangle$ is bounded from below. Of course the same applies for $\langle c, e_1 \rangle$ bounded below. Likewise if $c \rightarrow \infty$ inside $\mathcal{C}_-$ the same reasoning shows that

$$e^{-2\langle Q, c \rangle} \mathbb{E} [\mathbf{R}_\alpha^1(c) + \mathbf{R}_\alpha^2(c)] = \mathcal{O}(e^{\xi\langle c, e_1 \rangle}) + \mathcal{O}(e^{\xi\langle c, e_2 \rangle}).$$

Now we have also proved that as soon as $\langle s + \gamma e_i, c \rangle \leq -\xi|c|$,

$$e^{-2\langle Q, c \rangle} \mathbb{E} \left[\prod_{k=1}^N V_{\alpha_k}(z_k) e^{-\sum_{i=1}^2 \mu_i e^{\gamma\langle c, e_i \rangle} M^{\gamma e_i}(\mathbb{C})} - \mathbf{R}_\alpha^{1,2}(c) \right] = \mathcal{O}(e^{-\xi|c|}).$$

This hypothesis follows from the assumption that $\langle s, \omega_i \rangle > -\gamma$ made in the definition of $\mathcal{A}_N$. Therefore we see that for $\xi > 0$ small enough, $e^{\xi|c|} E_\alpha(c)$ remains bounded when $c \rightarrow \infty$ inside $\mathcal{C}_-$. All in all, this shows that as desired,

$$|E_\alpha(c)| \leq C^{-\xi|c|}.$$

This concludes for the proof of Proposition 6.4.2.

6.4.2 . A further analytic continuation of the correlation functions

Based on the asymptotics of the correlation functions described in Proposition 6.4.2, we are now in position to address the issue of providing an analytic continuation of the correlation functions beyond the bounds prescribed by Theorem 4.3.1.

6.4.2.1 Reducing the proof

To settle the ideas, let us first consider the subset $\mathcal{A}_N^{1;0}$ of $\mathcal{A}_N$ defined by assuming that for $2 \leq k \leq N$ and $i = 1, 2$, $\langle \alpha_k - Q, e_i \rangle < -\gamma$, and that the weight α_1 is such that $\langle \alpha_1 - Q, e_1 \rangle > -\gamma$ while $\langle \alpha_1 - Q, e_2 \rangle < -\gamma$. In that case we have that $\mathcal{R}_\alpha(c)$ is given by

$$\mathcal{R}_\alpha(c) = \mathbb{1}_{\langle s, \omega_1 \rangle < 0} \prod_{k=2}^N V_{\alpha_k}(z_k) (V_{\alpha_1}(z_1) + \mathbb{1}_{\langle s, \omega_1 \rangle < \langle \alpha_k - Q, e_1 \rangle} R_{s_1}(\alpha_1) V_{\hat{s}_1 \alpha_1}(z_1)) e^{-\mu_2 e^{\gamma\langle c, e_2 \rangle} M^{\gamma e_2}(\mathbb{C})}.$$

Now let us introduce

$$\mathbf{R}_\alpha(\mathbf{c}) := \mathbb{1}_{\langle \mathbf{c}, e_1 \rangle < 0} \prod_{k=2}^N V_{\alpha_k}(z_k) (V_{\alpha_1}(z_1) + R_{s_1}(\alpha_1) V_{\hat{s}_1 \alpha_1}(z_1)) e^{-\mu_2 e^{\gamma \langle \mathbf{c}, e_2 \rangle} M^{\gamma e_2}(\mathbb{C})}.$$

We aim to prove that the map $F : (\mathbb{C}^2)^N \rightarrow \mathbb{R}$ defined by setting

$$\begin{aligned} F(\alpha) &:= \int_{\mathbb{R}^2} e^{-2\langle Q, \mathbf{c} \rangle} \mathbb{E} \left[\prod_{k=1}^N V_{\alpha_k}(z_k) e^{-\sum_{i=1}^2 \mu_i e^{\gamma \langle \mathbf{c}, e_i \rangle} M^{\gamma e_i}(\mathbb{C})} - \mathbf{R}_\alpha(\mathbf{c}) \right] d\mathbf{c} \\ &+ \frac{1}{\langle \mathbf{s}, \omega_1 \rangle} \int_{\mathbb{R}} \prod_{j < k} |z_j - z_k|^{-\langle \alpha_j, \alpha_k \rangle} e^{-\langle \mathbf{s}, \omega_2 \rangle c_2} \mathbb{E} \left[e^{-\mu_2 e^{\gamma c_2} I^2(\mathbb{C})} \right] dc_2 \\ &+ \frac{R_{s_1}(\alpha_1)}{\langle \mathbf{s} + \hat{s}_1 \alpha_1 - \alpha_1, \omega_1 \rangle} \int_{\mathbb{R}} \prod_{j < k} |z_j - z_k|^{-\langle \hat{\alpha}_j, \hat{\alpha}_k \rangle} e^{\langle \hat{\mathbf{s}}, \omega_2 \rangle c_2} \mathbb{E} \left[e^{-\mu_2 e^{\gamma c_1} \hat{I}^2(\mathbb{C})} \right] dc_2 \end{aligned} \quad (6.4.9)$$

is analytic in a complex neighbourhood of $\mathcal{A}_N^{1;0}$, where the quantities denoted with a “ $\wedge$ ” sign are defined by replacing α_1 with $\hat{s}_1 \alpha_1$.

Indeed it is readily seen that in the case where $\langle \mathbf{s}, \omega_i \rangle > 0$ for $i = 1, 2$ this map is given by

$$\int_{\mathbb{R}^2} e^{-2\langle Q, \mathbf{c} \rangle} \mathbb{E} \left[\prod_{k=1}^N V_{\alpha_k}(z_k) e^{-\sum_{i=1}^2 \mu_i e^{\gamma \langle \mathbf{c}, e_i \rangle} M^{\gamma e_i}(\mathbb{C})} \right] d\mathbf{c},$$

so that F defines an analytic continuation of the correlation functions over $\mathcal{A}_N^{1;0}$. Likewise, F is seen to coincide with

$$\int_{\mathbb{R}^2} e^{-2\langle Q, \mathbf{c} \rangle} \mathbb{E} \left[\prod_{k=1}^N V_{\alpha_k}(z_k) e^{-\sum_{i=1}^2 \mu_i e^{\gamma \langle \mathbf{c}, e_i \rangle} M^{\gamma e_i}(\mathbb{C})} - \mathcal{R}_\alpha(\mathbf{c}) \right] d\mathbf{c}$$

under the assumptions that $\langle \mathbf{s}, \omega_2 \rangle > 0$ with either $0 > \langle \mathbf{s}, \omega_1 \rangle > \langle \alpha_1 - Q, e_1 \rangle$ or $\langle \alpha_1 - Q, e_1 \rangle > \langle \mathbf{s}, \omega_1 \rangle > -\gamma$.

This shows that the map defined by Equation (6.4.3) is indeed analytic over an open complex neighbourhood of $\mathcal{A}_N^{1;0}$. Furthermore as soon as every integral term is holomorphic over $\mathcal{A}_N^{1;0}$, we see that the poles of F are given by $\langle \mathbf{s}, \omega_1 \rangle = 0$ and $\langle \mathbf{s} + \hat{s}_1 \alpha_1 - \alpha_1, \omega_1 \rangle = 0$. Therefore this claim allows to conclude for the proof of Theorem 6.4.1 in this very case. In general, the same reasoning still works. Namely recalling the notations of Equation (6.4.6) we may consider the map F defined by

$$F(\alpha) := \int_{\mathbb{R}^2} E_\alpha(\mathbf{c}) d\mathbf{c} + \sum_{i=1}^2 \int_{\mathbb{R}} E_\alpha^i(c_i) dc_i + E_\alpha^{1,2}, \quad \text{where we have set} \quad (6.4.10)$$

$$E_\alpha(\mathbf{c}) = e^{-2\langle Q, \mathbf{c} \rangle} \mathbb{E} \left[\prod_{k=1}^N V_{\alpha_k}(z_k) e^{-\sum_{i=1}^2 \mu_i e^{\gamma \langle \mathbf{c}, e_i \rangle} M^{\gamma e_i}(\mathbb{C})} - \mathbf{R}_\alpha(\mathbf{c}) \right],$$

$$\begin{aligned} E_\alpha^1(\langle \mathbf{c}, e_1 \rangle) &= \sum_{w: \{1, \dots, N\} \rightarrow \{Id, s_2\}} \frac{e^{-\langle \mathbf{s}(w), \omega_2 \rangle \langle \mathbf{c}, e_2 \rangle}}{\langle \mathbf{s}(w), \omega_2 \rangle} \\ &e^{-2\langle Q, \mathbf{c} \rangle} \mathbb{E} \left[\left(e^{-\mu_1 e^{\gamma \langle \mathbf{c}, e_1 \rangle} M^{\gamma e_1}(\mathbb{C})} - \mathbb{1}_{\langle \mathbf{c}, e_1 \rangle < 0} (1 + \mathcal{R}_{s_1}) \right) \prod_{k=1}^N R_{w(k)}(\alpha_k) V_{\hat{w}(k)\alpha_k}(z_k) \right] \end{aligned}$$

which indeed depends only on $\mathbf{c}$ through $\langle \mathbf{c}, e_1 \rangle$, and

$$E_{\alpha}^{1,2} = \sum_{w:\{1,\dots,N\}\rightarrow W} \frac{e^{-\langle \mathbf{s}(w), \mathbf{c} \rangle}}{\langle \mathbf{s}(w), \omega_1 \rangle \langle \mathbf{s}(w), \omega_2 \rangle} e^{-2\langle Q, \mathbf{c} \rangle} \mathbb{E} \left[\prod_{k=1}^N R_{w(k)}(\alpha_k) V_{\hat{w}(k)\alpha_k}(z_k) \right] \\ - \sum_{i=1}^2 \sum_{w:\{1,\dots,N\}\rightarrow \{Id, s_i\}} \frac{e^{-\langle \mathbf{s}(w), \mathbf{c} \rangle}}{\langle \mathbf{s}(w), \omega_i \rangle} e^{-2\langle Q, \mathbf{c} \rangle} \mathbb{E} \left[\prod_{k=1}^N R_{w(k)}(\alpha_k) V_{\hat{w}(k)\alpha_k}(z_k) \right]$$

which is independent of $\mathbf{c}$. The reason for introducing such a map F follows from the observation that F is seen to coincide with

$$\int_{\mathbb{R}^2} e^{-2\langle Q, \mathbf{c} \rangle} \mathbb{E} \left[\prod_{k=1}^N V_{\alpha_k}(z_k) e^{-\sum_{i=1}^2 \mu_i e^{\gamma \langle \mathbf{c}, e_i \rangle} M^{\gamma e_i}(\mathbb{C})} - \mathcal{R}_{\alpha}(\mathbf{c}) \right] d\mathbf{c}$$

as soon as every term makes sense. Therefore proving Theorem 6.4.1 under its most general assumptions boils down to showing that the following holds true:

Lemma 6.4.3. *Let us denote by $\mathcal{A}_N^0$ the subset of $\mathcal{A}_N$ defined by the condition that $\langle \alpha_k - Q, e \rangle \notin -\gamma\mathbb{N}^* \cup -\frac{2}{\gamma}\mathbb{N}^*$ for all $1 \leq k \leq N$ and $e \in \Phi^+$. Then the following map is holomorphic in a complex neighbourhood of $\mathcal{A}_N^0$:*

$$G(\alpha) := \int_{\mathbb{R}^2} e^{-2\langle Q, \mathbf{c} \rangle} \mathbb{E} \left[\prod_{k=1}^N V_{\alpha_k}(z_k) e^{-\sum_{i=1}^2 \mu_i e^{\gamma \langle \mathbf{c}, e_i \rangle} M^{\gamma e_i}(\mathbb{C})} - \mathbf{R}_{\alpha}(\mathbf{c}) \right] d\mathbf{c}. \quad (6.4.11)$$

Indeed proving that terms of the form $\int_{\mathbb{R}} E_{\alpha}^i(c_i) dc_i$ are also holomorphic follows from the very same arguments. This shows that as soon as $\langle \alpha_k - Q, e \rangle \notin -\gamma\mathbb{N}^* \cup -\frac{2}{\gamma}\mathbb{N}^*$ for all $1 \leq k \leq N$ and $e \in \Phi^+$, the poles of the map $F(\alpha)$ defined by Equation (6.4.10) are given by the $\langle \mathbf{s}(w), \omega_i \rangle = 0$ for some $s \in W$ and $i \in \{1, 2\}$. When $\langle \alpha_k - Q, e \rangle \in -\gamma\mathbb{N}^* \cup -\frac{2}{\gamma}\mathbb{N}^*$ for some $1 \leq k \leq N$ and $e \in \Phi^+$ then the reflection coefficients have a pole (which may still be removable in some cases) so that F does too. This shows that the statement of Theorem 6.4.1 holds true as soon as Lemma 6.4.3 does.

6.4.2.2 Proof of Lemma 6.4.3

Let us consider $\mathcal{O}$ an open subset of $\mathcal{A}_N \setminus \mathcal{P}_N$ contained in K a compact subset of $\mathcal{A}_N \setminus \mathbb{P}_N$. We have seen in Lemma 4.3.2 that the expression $E_{\alpha}(\mathbf{c})$ was holomorphic in a complex neighbourhood K' of K for any fixed $\mathbf{c}$. Moreover thanks to Proposition 6.4.2 we know that the family of holomorphic functions $(E_{\alpha}(\mathbf{c})e^{\xi|\mathbf{c}|})_{\mathbf{c}}$ is uniformly bounded in $\mathbf{c}$ as soon as $\alpha \in \mathcal{O}$. It is also readily seen to depend continuously in $\mathbf{c}$. This implies that in a small complex neighbourhood $\mathcal{O}' \subset K'$ of $\mathcal{O}$ the family of maps $(E_{\alpha}(\mathbf{c})e^{\xi|\mathbf{c}|})_{\mathbf{c} \in \mathbb{R}^2}$ remains bounded uniformly in $\mathbf{c}$. In particular we see that for $\alpha \in \mathcal{O}'$ the map $\mathbf{c} \mapsto E_{\alpha}(\mathbf{c})$ is absolutely integrable over $\mathbb{R}^2$. As a consequence if we take $\alpha \in \mathcal{O}'$ and Γ any closed and piecewise C^1 curve (say a triangle) surrounding $\langle \alpha_1, e_1 \rangle$ on which $(\beta\omega_1 + \langle \alpha, e_2 \rangle \omega_2, \alpha_2, \dots, \alpha_N)$ stays inside $\mathcal{O}'$ if $\beta \in \Gamma$ we see that

$$\int_{\Gamma} \left(\int_{\mathbb{R}^2} E_{(\beta\omega_1 + \langle \alpha, e_2 \rangle \omega_2, \alpha_2, \dots, \alpha_N)}(\mathbf{c}) d\mathbf{c} \right) d\beta \\ = \int_{\mathbb{R}^2} \left(\int_{\Gamma} E_{(\beta\omega_1 + \langle \alpha, e_2 \rangle \omega_2, \alpha_2, \dots, \alpha_N)}(\mathbf{c}) d\beta \right) d\mathbf{c}$$

using Fubini-Tonelli theorem.

Now for fixed $\mathbf{c}$ the map $\beta \mapsto E_{(\beta\omega_1 + \langle \alpha, e_2 \rangle \omega_2, \alpha_2, \dots, \alpha_N)}(\mathbf{c})$ is holomorphic; therefore the integral over Γ vanishes. This shows that for any such Γ

$$\int_{\Gamma} \left(\int_{\mathbb{R}^2} E_{(\beta\omega_1 + \langle \alpha, e_2 \rangle \omega_2, \alpha_2, \dots, \alpha_N)}(\mathbf{c}) d\mathbf{c} \right) d\beta = 0.$$

By Morera's theorem this implies that $\beta \mapsto \int_{\mathbb{R}^2} E_{(\beta\omega_1 + \langle \alpha, e_2 \rangle \omega_2, \alpha_2, \dots, \alpha_N)}(\mathbf{c}) d\mathbf{c}$ is holomorphic in a neighbourhood of $\langle \alpha_1, e_1 \rangle$. The same reasoning being of course valid if we replace $\langle \alpha_1, e_1 \rangle$ by any of the $\langle \alpha_k, e_i \rangle$ for $1 \leq k \leq N$ and $i = 1, 2$ we see that the map considered in Lemma 6.4.3 is indeed holomorphic in a complex neighbourhood of $\mathcal{A}_N$. This wraps up the proof of Lemma 6.4.3 and therefore of Theorem 6.4.1.

Part III

Évaluation d'une famille de fonctions de corrélations pour la théorie de Toda associée à $\mathfrak{sl}_3$

7 - Fonctions de corrélation à trois points : la formule de Fateev-Litvinov

Il est communément admis dans la littérature physique que la connaissance des fonctions de corrélation d'une théorie conforme des champs permet de comprendre —du moins en grande partie— celle-ci. Pour ce faire il existe une procédure très puissante basée sur une procédure récursive : le *bootstrap conforme*. Cette technique s'appuie sur la connaissance d'une famille de fonctions de corrélation de base, les *constantes de structure*, ainsi que d'autres données permettant d'implémenter le bootstrap conforme. Dans ce chapitre nous évaluons ces constantes de structure en montrant que l'expression probabiliste considérée ici pour ces fonctions de corrélation fondamentales coïncide avec celle proposée dans la littérature physique par Fateev-Litvinov [52], généralisant ainsi la preuve de la formule DOZZ [84].

Les résultats évoqués ici sont issus de l'article [30].

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7.1 . Four-point correlation functions and BPZ-type differential equations

In Chapter 5 we have shed light on the presence of W -symmetry within the probabilistic framework proposed to make sense of Toda CFTs. Based on the constraints unveiled there, our goal in this section is to prove that these imply that a family of four-point correlation functions can actually be computed completely explicitly up to an explicit multiplicative constant, constant given by a three-point correlation function. For this purpose, let us introduce for $\alpha = -\chi\omega_1$ with $\chi \in \{\gamma, \frac{2}{\gamma}\}$ and $\alpha_2^* = \kappa\omega_2$ - the notations

$$\begin{aligned} A_i &:= \frac{\chi}{2} \langle \alpha_2^* + \alpha_2^* + \alpha - Q, h_1 \rangle + \frac{\chi}{2} \langle \alpha_3 - Q, h_i \rangle \\ B_i &:= 1 + \frac{\chi}{2} \langle \alpha_2^* - Q, h_1 - h_{i+1} \rangle. \end{aligned} \quad (7.1.1)$$

To these quantities one can associate the ${}_3F_2$ hypergeometric function

$$\mathcal{H}_0(z) := {}_3F_2 \left(\begin{matrix} A_1, A_2, A_3 \\ B_1, B_2 \end{matrix} \middle| z \right) = \sum_{n \in \mathbb{N}} \frac{(A_1)_n (A_2)_n (A_3)_n}{(B_1)_n (B_2)_n} \frac{z^n}{n!}, \quad (7.1.2)$$

where $(a)_n$ denotes the Pochhammer symbol $(a)_n = (a)(a+1) \cdots (a+n-1)$. Such a function is well-defined over the unit disc $\mathbb{D}$, and is a solution of an hypergeometric differential equation of order three:

$$\left[z (A_1 + z\partial_z) (A_2 + z\partial_z) (A_3 + z\partial_z) - (B_1 - 1 + z\partial_z) (B_2 - 1 + z\partial_z) z\partial_z \right] \mathcal{H} = 0. \quad (7.1.3)$$

Other complex-valued solutions of Equation (7.1.3) are given by

$$\mathcal{H}_1(z) := z^{1-B_1} {}_3F_2 \left(\begin{matrix} 1 - B_1 + A_1, 1 - B_1 + A_2, 1 - B_1 + A_3 \\ 2 - B_1, 1 - B_1 + B_2 \end{matrix} \middle| z \right) \quad \text{and} \quad (7.1.4)$$

$$\mathcal{H}_2(z) := z^{1-B_2} {}_3F_2 \left(\begin{matrix} 1 - B_2 + A_1, 1 - B_2 + A_2, 1 - B_2 + A_3 \\ 1 - B_2 + B_1, 2 - B_2 \end{matrix} \middle| z \right) \quad (7.1.5)$$

where we have set a branch cut for the logarithm to be the negative real axis $(-\infty, 0]$. The hypergeometric functions admit an analytic continuation outside of the interval $(1, \infty)$ that we will work with in the sequel and denote in the same way.

With all these objects at hand we are in position to state the main result of the present section:

Theorem 7.1.1. *Assume that $\alpha = -\chi\omega_1$ with $\chi \in \{\gamma, \frac{2}{\gamma}\}$ and that $\alpha_2^* = \kappa\omega_2$ for $\kappa < q$. Further assume that α_1, α_3 are such that $(\alpha, \alpha_1, \alpha_2^*, \alpha_3)$ belongs to $\mathcal{A}_4$. Then*

$$\begin{aligned} \langle V_\alpha(z) V_{\alpha_2^*}(0) V_{\alpha_2^*}(1) V_{\alpha_3}(\infty) \rangle &= |z|^{\chi \langle h_1, \alpha_2^* \rangle} |z-1|^{\chi \langle h_1, \alpha_2^* \rangle} \mathcal{H}(z), \quad \text{where} \\ \mathcal{H}(z) &= C_\gamma(\alpha_1 + \alpha, \alpha_2^*, \alpha_3) \left(|\mathcal{H}_0(z)|^2 + \sum_{i=1}^2 A_\gamma^{(i)}(\alpha, \alpha_1, \alpha_2^*, \alpha_3) |\mathcal{H}_i(z)|^2 \right). \end{aligned} \quad (7.1.6)$$

The constants $A_\gamma^{(i)}(\alpha, \alpha_1, \alpha_2^*, \alpha_3)$, $i = 1, 2$, are given by

$$A_\gamma^{(i)}(\alpha, \alpha_1, \alpha_2^*, \alpha_3) := \frac{\prod_{j=1}^3 l(A_j) l(B_i - A_j) l(1 + B_1 + B_2 - 2B_i)}{l(B_1) l(B_2) l(B_i - 1)}. \quad (7.1.7)$$

The quantities that appear above are defined using the special function

$$l(z) = \frac{\Gamma(z)}{\Gamma(1-z)}.$$

In order to prove such a statement we will first show that the global Ward identities as well as the assumptions that the correlation functions considered contain degenerate fields, together combined, imply that the four-point correlations considered are solution of the hypergeometric equation (7.1.3). Based on this fact we will exploit the probabilistic representation and more specifically the fact that the objects considered are real-valued to infer the form of the solutions prescribed by Equation (7.1.6).

7.1.1 . A BPZ equation for four-point correlation functions

In order to prove that a hypergeometric differential equation holds true for the four-point correlations considered in Theorem 7.1.1, we will exploit the global Ward identities from Theorem 5.2.1 but we will also rely on the fact that some of the Vertex Operators considered are degenerate fields.

7.1.1.1 A first application to degenerate fields at level one

To start with let us investigate the existence of (partially) degenerate fields at the level one. These correspond to Vertex Operators V_α for which the weight α takes a special value, which allows to write down the quantity $\mathcal{W}_{-1}$ in terms of derivatives of the correlation functions.

As explained in Chapter 5, the first W -descendent can be expressed in terms of the Toda field via

$$\mathcal{W}_{-1}^{(1)}\langle \mathbf{V} \rangle = \lim_{\varepsilon \rightarrow 0} \langle \mathbf{W}_{-1,\varepsilon} \mathbf{V}_\varepsilon \rangle$$

where we have set in agreement with the results proved there

$$\mathbf{W}_{-1,\varepsilon}(z, \alpha) = -qB(\alpha, \partial\Phi_\varepsilon(z)) - 2C(\alpha, \alpha, \partial\Phi_\varepsilon(z)).$$

Our goal here is to express $\mathcal{W}_{-1}$ in terms of derivatives of the field, and to do so we may introduce the first Virasoro descendent of V_{α_1} by setting

$$\mathcal{L}_{-1}^{(1)}\langle \mathbf{V} \rangle := \partial_{z_1}\langle \mathbf{V} \rangle.$$

The latter can be rewritten based on the reasoning developed in the proof of Proposition 4.4.7 as

$$\partial_{z_1}\langle \mathbf{V}_\varepsilon \rangle = \left\langle \langle \alpha_1, \partial\Phi_\varepsilon(z_1) \rangle \mathbf{V}_\varepsilon \right\rangle.$$

Hence the Virasoro descendent $\mathcal{L}_{-1}^{(1)}\langle \mathbf{V} \rangle$ can be represented as the limit

$$\lim_{\varepsilon \rightarrow 0} \left\langle \langle \alpha_1, \partial\Phi_\varepsilon(z_1) \rangle \mathbf{V}_\varepsilon \right\rangle.$$

As a consequence we can express the first W -descendent in terms of the first Virasoro descendent when $\mathbf{W}_{-1,\varepsilon}(z, \alpha)$ and $\langle \alpha, \partial\Phi_\varepsilon(z) \rangle$ are proportional. Therefore we will say that the primary field is degenerate at the level one when there exists a real κ for which

$$\mathbf{W}_{-1,\varepsilon}(z, \alpha) = \kappa \langle \alpha, \partial\Phi_\varepsilon(z) \rangle.$$

Put differently we need to find α such that for all $u \in \mathfrak{h}^*$,

$$-qB(\alpha, u) - 2C(\alpha, \alpha, u) = \kappa \langle \alpha, u \rangle.$$

Taking respectively $u = e_1$ and $u = e_2$ and setting $\alpha := \alpha^1 \omega_1 + \alpha^2 \omega_2$ we see that we must have

$$\kappa \alpha^1 = q \alpha^1 - 2 \alpha^1 \frac{2\alpha^2 + \alpha^1}{3} \quad \text{and} \quad \kappa \alpha^2 = -q \alpha^2 + 2 \alpha^2 \frac{2\alpha^1 + \alpha^2}{3}$$

which implies that either $\alpha^1 \alpha^2 = 0$ or $q = \alpha^1 + \alpha^2$.

Proposition 7.1.2. *Degenerate fields at level one are given by the V_α with α of the form $\chi \omega_1$ or $\chi \omega_2$ for $\chi < q$, or $\lambda \omega_1 + (q - \lambda) \omega_2$ for some $0 < \lambda < q$. In that case we have the relation*

$$\mathcal{W}_{-1}^{(1)} \langle V_\alpha(z_1) \prod_{k=2}^N V_{\alpha_k}(z_k) \rangle = \frac{3w(\alpha)}{2\Delta_\alpha} \partial_{z_1} \langle V_\alpha(z_1) \prod_{k=2}^N V_{\alpha_k}(z_k) \rangle, \quad (7.1.8)$$

valid as soon as the z are distinct and the α satisfies the bounds of Theorem 4.3.1.

7.1.1.2 Degenerate fields at the levels two and three

We may proceed in the same way for other descendants at the levels two and three. Vertex Operators that satisfy similar requirements as above but for descendants at the second or third level will be called *degenerate fields at the second or third level*. These are also sometimes referred to as *fully degenerate fields* by contrast with degenerate fields at the first level only. In order to keep notations simple we will omit the Wick product convention in this subsection: for instance quantities such as $\langle \partial \Phi(z), \partial \Phi(z) \rangle$ are to be understood as : $\langle \partial \Phi(z), \partial \Phi(z) \rangle$:.

We start by considering the Virasoro descendants of order two. We define them by the expressions

$$L_{-(1,1)}(z, \alpha) := \langle \alpha, \partial^2 \Phi(z) \rangle + \langle \alpha, \partial \Phi(z) \rangle^2, \quad (7.1.9)$$

$$L_{-2}(z, \alpha) := \langle Q + \alpha, \partial^2 \Phi(z) \rangle - \langle \partial \Phi(z), \partial \Phi(z) \rangle. \quad (7.1.10)$$

By doing so it is immediate to see that

$$\langle L_{-(1,1)}(z_1, \alpha_1) \mathbf{V} \rangle = \partial_{z_1}^2 \langle \mathbf{V} \rangle = \left(\mathcal{L}_{-1}^{(1)} \right)^2 \langle \mathbf{V} \rangle.$$

The effect of the L_{-2} descendent on the correlation functions is less obvious; however conformal invariance allows to claim the following statement.

Lemma 7.1.3. *Set $\mathcal{L}_{-2}^{(1)} \langle \mathbf{V} \rangle := \lim_{\varepsilon \rightarrow 0} \langle L_{-2, \varepsilon}(z, \alpha) V_{\alpha, \varepsilon}(z) \mathbf{V}_\varepsilon \rangle$. Then*

$$\mathcal{L}_{-2}^{(1)} \langle V_\alpha(z) \mathbf{V} \rangle = \sum_{k=2}^N \left(\frac{\partial_{z_k}}{z - z_k} + \frac{\Delta_{\alpha_k}}{(z - z_k)^2} \right) \langle V_\alpha(z) \mathbf{V} \rangle. \quad (7.1.11)$$

Proof. At the regularized level, the right-hand side divided by the non-zero quantity $\langle V_{\alpha,\varepsilon}(z)\mathbf{V}_\varepsilon \rangle$ is given by

$$\begin{aligned}
&= \sum_{k \neq l} \frac{\langle \alpha_k, \alpha_l \rangle}{2(z - z_k)(z_l - z_k)_\varepsilon} + \sum_{k=1}^N \frac{\langle \alpha_k, \alpha \rangle}{2(z - z_k)(z - z_k)_\varepsilon} + \frac{\Delta_{\alpha_k}}{(z - z_k)^2} + \text{I} \\
&= \sum_{k \neq l} \frac{\langle \alpha_k, \alpha_l \rangle}{4(z_l - z_k)} \left(\frac{1}{z - z_k} - \frac{1}{z - z_l} \right) + \sum_{k=1}^N \frac{\langle \alpha_k, Q + \alpha \rangle}{2(z - z_k)^2} - \frac{\langle \alpha_k, \alpha_k \rangle}{4(z - z_k)^2} + o_\varepsilon(1) + \text{I} \\
&= - \sum_{k \neq l} \frac{\langle \alpha_k, \alpha_l \rangle}{4(z - z_k)(z - z_l)} + \sum_{k=1}^N \frac{\langle \alpha_k, Q + \alpha \rangle}{2(z - z_k)^2} - \frac{\langle \alpha_k, \alpha_k \rangle}{4(z - z_k)^2} + o_\varepsilon(1) + \text{I} \\
&= - \sum_{k,l=1}^N \frac{\langle \alpha_k, \alpha_l \rangle}{4(z - z_k)(z - z_l)} + \sum_{k=1}^N \frac{\langle \alpha_k, Q + \alpha \rangle}{2(z - z_k)^2} + o_\varepsilon(1) + \text{I},
\end{aligned}$$

where $\langle V_\alpha(z)\mathbf{V}_\varepsilon \rangle \text{I}$ is equal to:

$$\sum_{i=1}^2 \int_{\mathbb{C}} \sum_{k=1}^N \frac{\langle \alpha_k, \gamma e_i \rangle}{4(z - z_k)(z_k - x)_\varepsilon} \langle V_{\gamma e_i, \varepsilon}(x) V_{\alpha, \varepsilon}(z) \mathbf{V}_\varepsilon \rangle d^2 x.$$

Using the same reasoning as in Section 5.1 involving symmetrization and Stokes' formula, the latter is equal (up to terms that vanish in the $\varepsilon \rightarrow 0$ limit) to

$$\begin{aligned}
&\sum_{i=1}^2 \int_{\mathbb{C}} \left(\sum_{k=1}^N \frac{\langle \alpha_k, \gamma e_i \rangle}{4(z - z_k)(z - x)_\varepsilon} - \frac{\langle \alpha, \gamma e_i \rangle + 2}{2(z - x)_\varepsilon^2} \right) \langle V_{\gamma e_i, \varepsilon}(x) V_{\alpha, \varepsilon}(z) \mathbf{V}_\varepsilon \rangle d^2 x \\
&- \sum_{i,j=1}^2 \int_{\mathbb{C}^2} \frac{\langle \gamma e_i, \gamma e_j \rangle}{4(z - x_1)_\varepsilon(z - x_2)_\varepsilon} \langle V_{\gamma e_i, \varepsilon}(x_1) V_{\gamma e_j, \varepsilon}(x_2) V_{\alpha, \varepsilon}(z) \mathbf{V}_\varepsilon \rangle d^2 x_1 d^2 x_2.
\end{aligned}$$

This expression coincides with the left-hand side in the $\varepsilon \rightarrow 0$ limit by applying Gaussian integration by parts in the same spirit as in Section 5.1, concluding the proof:

$$\begin{aligned}
\langle \mathbf{L}_{-2, \varepsilon}(z, \alpha) V_{\alpha, \varepsilon}(z) \mathbf{V}_\varepsilon \rangle &= \sum_{k=1}^N \frac{\langle Q + \alpha, \alpha_k \rangle}{2(z - z_k)_\varepsilon^2} \langle V_{\alpha, \varepsilon}(z) \mathbf{V}_\varepsilon \rangle - \sum_{i=1}^2 \int_{\mathbb{C}} \frac{\langle Q + \alpha, \gamma e_i \rangle}{2(z - x)_\varepsilon^2} \langle V_{\gamma e_i, \varepsilon}(x) V_{\alpha, \varepsilon}(z) \mathbf{V}_\varepsilon \rangle d^2 x \\
&- \sum_{k,l=1}^N \frac{\langle \alpha_k, \alpha_l \rangle}{4(z - z_k)_\varepsilon(z - z_l)_\varepsilon} \langle V_{\alpha, \varepsilon}(z) \mathbf{V}_\varepsilon \rangle \\
&+ \sum_{i=1}^2 \int_{\mathbb{C}} \left(\sum_{k=1}^N \frac{\langle \alpha_k, \gamma e_i \rangle}{4(z - z_k)_\varepsilon(z - x)_\varepsilon} + \frac{\langle \gamma e_i, \gamma e_i \rangle}{4(z - x)_\varepsilon(z - x)_\varepsilon} \right) \langle V_{\gamma e_i, \varepsilon}(x) V_{\alpha, \varepsilon}(z) \mathbf{V}_\varepsilon \rangle d^2 x \\
&- \sum_{i,j=1}^2 \int_{\mathbb{C}^2} \frac{\langle \gamma e_i, \gamma e_j \rangle}{4(z - x_1)_\varepsilon(z - x_2)_\varepsilon} \langle V_{\gamma e_i, \varepsilon}(x_1) V_{\gamma e_j, \varepsilon}(x_2) V_{\alpha, \varepsilon}(z) \mathbf{V}_\varepsilon \rangle d^2 x_1 d^2 x_2.
\end{aligned}$$

□

Thanks to the expression of the Virasoro descendants at the level two, we can readily search for degenerate fields at the level two. As before, we think of them as Vertex Operators for which the W -descendent at the level two, $\mathbf{W}_{-2}(z, \alpha)$, can be expressed

as a linear combination of Virasoro descendants at the level two. Put differently, these are α such that there exist κ_1, κ_2 real numbers with

$$\kappa_1 \mathbf{L}_{-(1,1)} + \kappa_2 \mathbf{L}_{-2} = \mathbf{W}_{-2}.$$

The latter implies that $\alpha = \alpha_1 \omega_1 + \alpha_2 \omega_2$ is a solution of the following set of equations

$$\begin{cases} \kappa_1 \alpha_1^2 - 2\kappa_2 &= 2 \frac{2\alpha_2 + \alpha_1}{3} \\ \kappa_1 \alpha_2^2 - 2\kappa_2 &= -2 \frac{2\alpha_1 + \alpha_2}{3} \\ \kappa_1 \alpha_1 \alpha_2 + \kappa_2 &= 2 \frac{\alpha_1 - \alpha_2}{3} \\ \kappa_1 \alpha_1 + \kappa_2 (q + \alpha_1) &= -(q + \alpha_1) \frac{2\alpha_2 + \alpha_1}{3} \\ \kappa_1 \alpha_2 + \kappa_2 (q + \alpha_2) &= (q - \alpha_2) \frac{2\alpha_1 + \alpha_2}{3}. \end{cases}$$

Explicit computations show the following.

Proposition 7.1.4. *Degenerate fields at the level two are given by the Vertex Operators V_α with α of the form $-\chi \omega_i$ where χ is either $\frac{2}{\gamma}$ or γ and $i \in \{1, 2\}$. In that case*

$$\begin{aligned} \mathbf{W}_{-2}(\cdot, \alpha) &= -\frac{4}{\chi} \mathbf{L}_{-(1,1)}(\cdot, \alpha) - \frac{4\chi}{3} \mathbf{L}_{-2}(\cdot, \alpha), \text{ hence} \\ \mathcal{W}_{-2}^{(1)}(V_\alpha(z_1) \prod_{k=2}^N V_{\alpha_k}(z_k)) &= -\left(\frac{4}{\chi} (\mathcal{L}_{-1}^{(1)})^2 + \frac{4\chi}{3} \mathcal{L}_{-2}^{(1)} \right) \langle V_\alpha(z_1) \prod_{k=2}^N V_{\alpha_k}(z_k) \rangle. \end{aligned} \quad (7.1.12)$$

A similar reasoning remains true when we turn to descendants at the level three; nevertheless calculations are slightly more involved. Like before, we shall first introduce the three Virasoro descendants at the third level by setting

$$\begin{aligned} \mathbf{L}_{-(1,1,1)}(z, \alpha) &:= \langle \alpha, \partial^3 \Phi(z) \rangle + 3 \langle \alpha, \partial \Phi(z) \rangle \langle \alpha, \partial^2 \Phi(z) \rangle + \langle \alpha, \partial \Phi(z) \rangle^3 \\ \mathbf{L}_{-(1,2)}(z, \alpha) &:= \langle Q + \alpha, \partial^3 \Phi(z) \rangle + \langle Q + \alpha, \partial^2 \Phi(z) \rangle \langle \alpha, \partial \Phi(z) \rangle \\ &\quad - 2 \langle \partial^2 \Phi(z), \partial \Phi(z) \rangle - \langle \partial \Phi(z), \partial \Phi(z) \rangle \langle \alpha, \partial \Phi(z) \rangle. \end{aligned} \quad (7.1.13)$$

These are such that (provided that the objects exist)

$$\begin{aligned} \langle \mathbf{L}_{-(1,1,1)}(z_1, \alpha_1) \mathbf{V} \rangle &= (\mathcal{L}_{-1}^{(1)})^3 \langle \mathbf{V} \rangle \\ \langle \mathbf{L}_{-(1,2)}(z_1, \alpha_1) \mathbf{V} \rangle &= \mathcal{L}_{-1}^{(1)} \mathcal{L}_{-2}^{(1)} \langle \mathbf{V} \rangle. \end{aligned} \quad (7.1.14)$$

We also introduce

$$\mathbf{L}_{-3}(z, \alpha) := \langle Q + \frac{\alpha}{2}, \partial^3 \Phi(z) \rangle - 2 \langle \partial^2 \Phi(z), \partial \Phi(z) \rangle \quad (7.1.15)$$

which is has been defined in order to satisfy the below equation:

$$\langle \mathbf{L}_{-3}(z_1, \alpha_1) \mathbf{V} \rangle = \sum_{k=2}^N \left(\frac{\partial_{z_k}}{(z_1 - z_k)^2} + \frac{2\Delta_{\alpha_k}}{(z_1 - z_k)^3} \right) \langle \mathbf{V} \rangle. \quad (7.1.16)$$

To see that we may proceed like in the proof of Lemma 7.1.3 and the very same arguments still apply. Nonetheless to motivate this claim simply note that

$$\begin{aligned}
& \sum_{k=1}^N \left(\frac{\partial_{z_k}}{(z - z_k)^2} + \frac{2\Delta_{\alpha_k}}{(z - z_k)^3} \right) \langle V_\alpha(z) \mathbf{V} \rangle \\
&= \sum_{k \neq l} \frac{\langle \alpha_k, \alpha_l \rangle}{2(z - z_k)^2(z_l - z_k)} + \sum_{k=1}^N \frac{\langle \alpha_k, \alpha \rangle}{2(z - z_k)^3} + \frac{2\Delta_{\alpha_k}}{(z - z_k)^3} + \text{integral terms} \\
&= \sum_{k \neq l} \frac{\langle \alpha_k, \alpha_l \rangle}{4(z_l - z_k)} \left(\frac{1}{(z - z_k)^2} - \frac{1}{(z - z_l)^2} \right) + \sum_{k=1}^N \frac{\langle \alpha_k, Q + \frac{\alpha}{2} \rangle}{(z - z_k)^3} - \frac{\langle \alpha_k, \alpha_k \rangle}{2(z - z_k)^3} + \text{integral terms} \\
&= - \sum_{k \neq l} \frac{\langle \alpha_k, \alpha_l \rangle}{2(z - z_k)^2(z - z_l)} + \sum_{k=1}^N \frac{\langle \alpha_k, Q + \frac{\alpha}{2} \rangle}{(z - z_k)^3} - \frac{\langle \alpha_k, \alpha_k \rangle}{2(z - z_k)^3} + \text{integral terms} \\
&= - \sum_{k,l=1}^N \frac{\langle \alpha_k, \alpha_l \rangle}{2(z - z_k)^2(z - z_l)} + \sum_{k=1}^N \frac{\langle \alpha_k, Q + \frac{\alpha}{2} \rangle}{(z - z_k)^3} + \text{integral terms} \\
&= \left\langle -2\langle \partial^2 \Phi(z), \partial \Phi(z) \rangle V_\alpha(z) \mathbf{V} \right\rangle + \left\langle \left\langle Q + \frac{\alpha}{2}, \partial^3 \Phi(z) \right\rangle V_\alpha(z) \mathbf{V} \right\rangle.
\end{aligned}$$

We now turn to the W -descendants at the third level. With a reasoning similar to the one of Lemma 7.1.3 we end up with:

Lemma 7.1.5. *Let us set*

$$\begin{aligned}
\mathbf{W}_{-3}(z, \alpha) &:= q^2 \langle h_2, \partial^3 \Phi(z) \rangle + \frac{q}{2} (2B(\partial^3 \Phi(z), \alpha) - B(\alpha, \partial^3 \Phi(z))) - C(\partial^3 \Phi(z), \alpha, \alpha) \\
&\quad - 2qB(\partial^2 \Phi(z), \partial \Phi(z)) + 4C(\partial^2 \Phi(z), \partial \Phi(z), \alpha) + 4C(\partial \Phi(z), \partial^2 \Phi(z), \alpha) \\
&\quad - 8\langle h_1, \partial \Phi(z) \rangle \langle h_2, \partial \Phi(z) \rangle \langle h_3, \partial \Phi(z) \rangle.
\end{aligned} \tag{7.1.17}$$

Then

$$\lim_{\varepsilon \rightarrow 0} \langle \mathbf{W}_{-3, \varepsilon}(z_1, \alpha_1) \mathbf{V}_\varepsilon \rangle = \sum_{k=2}^N \left(\frac{\mathcal{W}_{-2}^{(k)}}{z_1 - z_k} + \frac{\mathcal{W}_{-1}^{(k)}}{(z_1 - z_k)^2} + \frac{w(\alpha_k)}{(z_1 - z_k)^3} \right) \langle \mathbf{V} \rangle. \tag{7.1.18}$$

Proof. Computations parallel the ones made in Section 5.1. The right-hand side in (7.1.18), when divided by $\langle V_\alpha(z) \mathbf{V} \rangle_\varepsilon$, can be expanded as

$$\begin{aligned}
& \sum_{k,l,p \text{ distinct}} \frac{C(\alpha_k, \alpha_l, \alpha_p)}{(z - z_k)(z_k - z_l)_\varepsilon(z_k - z_p)_\varepsilon} + \sum_{k \neq l} \frac{C^\sigma(\alpha, \alpha_k, \alpha_l)}{(z - z_k)(z_k - z)_\varepsilon(z_k - z_l)_\varepsilon} \\
&+ \sum_{k \neq l} \frac{q(B(\alpha_l, \alpha_k) - B(\alpha_k, \alpha_l)) + 2C(\alpha_k, \alpha_l - \alpha_k, \alpha_l)}{2(z - z_k)(z_k - z_l)_\varepsilon^2} + \frac{qB(\alpha_k, \alpha_l) + 2C(\alpha_k, \alpha_k, \alpha_l)}{2(z - z_k)^2(z_k - z_l)_\varepsilon} \\
&+ \sum_{k=1}^N \frac{q(B(\alpha, \alpha_k) - B(\alpha_k, \alpha)) + 2C(\alpha_k, \alpha - \alpha_k, \alpha)}{2(z - z_k)(z - z_k)_\varepsilon^2} + \frac{qB(\alpha_k, \alpha) + 2C(\alpha_k, \alpha_k, \alpha)}{2(z - z_k)^2(z_k - z)_\varepsilon} + \frac{w(\alpha_k)}{(z - z_k)^3} \\
&+ \text{integral terms.}
\end{aligned}$$

The first quantity can be easily dealt with by recursive application of symmetrization identities:

$$\begin{aligned} \sum_{k,l,p \text{ distinct}} \frac{C(\alpha_k, \alpha_l, \alpha_p)}{(z - z_k)(z_k - z_l)(z_k - z_p)} &= \frac{1}{3} \sum_{k,l,p \text{ distinct}} \frac{C(\alpha_k, \alpha_l, \alpha_p)}{(z - z_k)(z - z_l)(z - z_p)} \\ &= \sum_{k,l,p} \frac{\langle \alpha_k, h_1 \rangle \langle \alpha_l, h_2 \rangle \langle \alpha_p, h_3 \rangle}{(z - z_k)(z - z_l)(z - z_p)} - \sum_{k,l} \frac{C(\alpha_k, \alpha_k, \alpha_l)}{(z - z_k)^2(z - z_l)} - \sum_{k=1}^N \frac{\langle \alpha_k, h_1 \rangle \langle \alpha_k, h_2 \rangle \langle \alpha_k, h_3 \rangle}{(z - z_k)^3}. \end{aligned}$$

Similarly we can write that

$$\begin{aligned} &\sum_{k \neq l} \frac{qB(\alpha_k, \alpha_l) + 2C(\alpha_k, \alpha_k, \alpha_l)}{2(z - z_k)^2(z_k - z_l)} + \frac{q(B(\alpha_l, \alpha_k) - B(\alpha_k, \alpha_l)) + 2C(\alpha_k, \alpha_l - \alpha_k, \alpha_l)}{2(z - z_k)(z_k - z_l)^2} \\ &= \sum_{k \neq l} \frac{qB(\alpha_k, \alpha_l) + 2C(\alpha_k, \alpha_k, \alpha_l)}{2(z - z_k)^2(z_k - z_l)} + \frac{q(B(\alpha_l, \alpha_k) - B(\alpha_k, \alpha_l)) + 2C(\alpha_k, \alpha_l - \alpha_k, \alpha_l)}{4(z - z_k)(z - z_l)(z_k - z_l)} \\ &= \sum_{k \neq l} \frac{qB(\alpha_k, \alpha_l) + 2C(\alpha_k, \alpha_k, \alpha_l)}{2(z - z_k)^2(z - z_l)} \\ &= \sum_{k,l} \frac{qB(\alpha_k, \alpha_l) + 2C(\alpha_k, \alpha_k, \alpha_l)}{2(z - z_k)^2(z - z_l)} - \sum_{k=1}^N \frac{qB(\alpha_k, \alpha_k) + 2C(\alpha_k, \alpha_k, \alpha_k)}{2(z - z_k)^3}. \end{aligned}$$

Therefore the renormalized right-hand side in (7.1.18) is actually equal to

$$\begin{aligned} &\sum_{k,l,p} \frac{\langle \alpha_k, h_1 \rangle \langle \alpha_l, h_2 \rangle \langle \alpha_p, h_3 \rangle}{(z - z_k)(z - z_l)(z - z_p)} + \sum_{k,l} \frac{qB(\alpha_k, \alpha_l) - 2C^\sigma(\alpha, \alpha_k, \alpha_l)}{2(z - z_k)^2(z - z_l)} \\ &+ \sum_{k=1}^N \frac{-2q^2 \langle \alpha_k, h_2 \rangle + q(B(\alpha, \alpha_k) - 2B(\alpha_k, \alpha)) + 2C(\alpha_k, \alpha, \alpha)}{2(z - z_k)^3} + o_\varepsilon(1) + \text{integral terms}. \end{aligned} \quad (7.1.19)$$

It remains to treat the integral terms. This is done with a treatment similar to the one we have used in Section 5.1, the only difference being that there is an extra Vertex Operator within the correlation function. Nevertheless along the same lines we get (up to a factor $\langle V_{\alpha,\varepsilon}(z) V_\varepsilon \rangle$ and a $o_\varepsilon(1)$ quantity) a sum of the terms

$$\begin{aligned} &-\sum_{i=1}^2 \int_{\mathbb{C}} \left(\frac{-2q \langle e_i, h_2 \rangle}{(z - x)_\varepsilon^3} + \sum_{k=1}^N \left(\frac{qB(\alpha_k, \gamma e_i) + 2C(\alpha_k, \alpha_k, \gamma e_i)}{2(z - z_k)^2(z - x)_\varepsilon} + \frac{qB(\gamma e_i, \alpha_k) + 2C(\gamma e_i, \gamma e_i, \alpha_k)}{2(z - z_k)(z - x)_\varepsilon^2} \right) \right. \\ &\quad \left. + \sum_{k,l=1}^N \frac{C(\alpha_k, \alpha_l, \gamma e_i)}{(z - z_k)(z - z_l)(z - x)_\varepsilon} \right) \langle V_{\gamma e_i, \varepsilon}(x) V_{\alpha, \varepsilon}(z) \mathbf{V}_\varepsilon \rangle d^2 x \\ &+ \sum_{i,j=1}^2 \int_{\mathbb{C}^2} \left(\sum_{k=1}^N \frac{C(\gamma e_i, \gamma e_j, \alpha_k)}{(z - z_k)(z - x_1)_\varepsilon(z - x_2)_\varepsilon} + \frac{qB(\gamma e_i, \gamma e_j) + C(\gamma e_i, \gamma e_i, \gamma e_j)}{2(z - x_1)_\varepsilon^2(z - x_2)_\varepsilon} \right) \\ &\quad \langle V_{\gamma e_i, \varepsilon}(x_1) V_{\gamma e_j, \varepsilon}(x_2) V_{\alpha, \varepsilon}(z) \mathbf{V}_\varepsilon \rangle d^2 x_1 d^2 x_2 \\ &- \sum_{i,j,f=1}^2 \int_{\mathbb{C}^3} \frac{\langle \gamma e_i, h_1 \rangle \langle \gamma e_j, h_2 \rangle \langle \gamma e_f, h_3 \rangle}{(z - x_1)_\varepsilon(z - x_2)_\varepsilon(z - x_3)_\varepsilon} \langle V_{\gamma e_i, \varepsilon}(x_1) V_{\gamma e_j, \varepsilon}(x_2) V_{\gamma e_f, \varepsilon}(x_3) V_{\alpha, \varepsilon}(z) \mathbf{V}_\varepsilon \rangle d^2 x_1 d^2 x_2 d^2 x_3 \end{aligned}$$

coming from the reasoning developed in Section 5.1, and additional ones given by

$$\begin{aligned}
& - \sum_{i=1}^2 \langle h_2, \gamma e_i \rangle \int_{\mathbb{C}} \langle V_{\gamma e_i, \varepsilon}(x) V_{\alpha, \varepsilon}(z) \mathbf{V}_{\varepsilon} \rangle \left(\frac{\langle 2\gamma Q + (2 - 3\gamma^2) \omega_{3-i}, \alpha \rangle + 2\langle \gamma e_i, \alpha \rangle \langle \omega_{3-i}, \alpha \rangle}{2(z-x)_{\varepsilon}^3} \right. \\
& \quad \left. + \sum_{k=1}^N \frac{\langle \alpha, \gamma e_i \rangle \omega_{3-i}(\alpha_k) + \langle \alpha_k, \gamma e_i \rangle \omega_{3-i}(\alpha)}{(z-z_k)^2(z-x)_{\varepsilon}} + \frac{\langle \alpha, \gamma e_i \rangle \omega_{3-i}(\alpha_k) + \langle \alpha_k, \gamma e_i \rangle \omega_{3-i}(\alpha)}{(z-z_k)(z-x)_{\varepsilon}^2} \right) d^2x \\
& - \sum_{i,j=1}^2 \int_{\mathbb{C}^2} \frac{C^{\sigma}(\alpha, \gamma e_i, \gamma e_j)}{(z-x_1)_{\varepsilon}^2(z-x_2)_{\varepsilon}} \langle V_{\gamma e_i, \varepsilon}(x_1) V_{\gamma e_j, \varepsilon}(x_2) V_{\alpha, \varepsilon}(z) \mathbf{V}_{\varepsilon} \rangle_{\delta} d^2x_1 d^2x_2
\end{aligned}$$

that account for the extra Vertex operator in the correlation function and which are obtained by recursive integration by parts. Therefore using Gaussian integration by parts as well as the explicit expression of B and C we see that the expression (7.1.19) coincides with $\langle \mathbf{W}_{-3, \varepsilon}(z, \alpha) V_{\alpha, \varepsilon}(z) \mathbf{V}_{\varepsilon} \rangle$, up to a term that vanishes in the $\varepsilon \rightarrow 0$ limit. $\square$

We are now in position to address the question of finding degenerate fields at the level three.

Proposition 7.1.6. *Degenerate fields at the levels two and three are given by the V_{α} with α of the form $-\chi\omega_1$ or $-\chi\omega_2$ with $\chi \in \{\gamma, \frac{2}{\gamma}\}$. In that case both relations*

$$\mathbf{W}_{-2} = -\frac{4}{\chi} \mathbf{L}_{-(1,1)} - \frac{4\chi}{3} \mathbf{L}_{-2} \quad \text{and} \quad (7.1.20)$$

$$\mathbf{W}_{-3} = -\left(\frac{\chi}{3} + \frac{2}{\chi}\right) \mathbf{L}_{-3} + \frac{4}{\chi} \mathbf{L}_{-(1,2)} + \frac{8}{\chi^3} \mathbf{L}_{-(1,1,1)}, \quad (7.1.21)$$

are valid as soon as the z are distinct and that α satisfies the bounds of Theorem 4.3.1.

As a consequence correlation functions of the form $\langle V_{-\chi\omega_1}(z) \mathbf{V} \rangle$ are solutions of the following BPZ-type identity:

$$\begin{aligned}
& \left[\frac{8}{\chi^3} \partial_z^3 + \frac{4}{\chi} \partial_z \sum_{k=1}^N \left(\frac{\partial_{z_k}}{z-z_k} + \frac{\Delta_{\alpha_k}}{(z-z_k)^2} \right) + \left(\frac{\chi}{3} + \frac{2}{\chi} \right) \sum_{k=1}^N \left(\frac{\partial_{z_k}}{(z-z_k)^2} + \frac{2\Delta_{\alpha_k}}{(z-z_k)^3} \right) \right. \\
& \quad \left. - \sum_{k=1}^N \left(\frac{\mathbf{W}_{-2}^{(k)}}{z-z_k} + \frac{\mathbf{W}_{-1}^{(k)}}{(z-z_k)^2} + \frac{w(\alpha_k)}{(z-z_k)^3} \right) \right] \langle V_{-\chi\omega_1}(z) \mathbf{V} \rangle = 0. \quad (7.1.22)
\end{aligned}$$

Proof. Equation (7.1.22) simply corresponds to inserting the equality in Equation (7.1.21) within a correlation function and using Equations (7.1.14), (7.1.16) and (7.1.18). Proving Equation (7.1.21) follows from calculations very similar to those we have done for degenerate fields at the levels one and two. $\square$

Note that this in particular implies that the correlation function considered is C^3 in the z variable for $z \in \mathbb{C} \setminus \{z_1, \dots, z_N\}$.

7.1.2 . Implications on a four-point correlation function

The expression of the BPZ-type equation (7.1.22) is not always tractable when it comes to deriving exact expressions for the correlation functions because the W -descendants do not correspond to derivatives of the Vertex Operators. However when considering a small number of Vertex Operators some cancellations occur thanks to the global Ward identities from Theorem 5.2.1, allowing to express the differential equation in several variables (7.1.22) as a differential equation in only one variable.

7.1.2.1 A first differential equation

Indeed let us consider a four-point correlation function with one point at infinity $\langle V_\alpha(z)V_{\alpha_1}(0)V_{\alpha_2}(1)V_{\alpha_3}(\infty) \rangle$, defined by the limit

$$\langle V_\alpha(z)V_{\alpha_1}(0)V_{\alpha_2}(1)V_{\alpha_3}(\infty) \rangle := \lim_{z' \rightarrow \infty} |z'|^{4\Delta_{\alpha_3}} \langle V_\alpha(z)V_{\alpha_1}(0)V_{\alpha_2}(1)V_{\alpha_3}(z') \rangle. \quad (7.1.23)$$

This limit is non-zero and admits the following representation:

$$\langle V_\alpha(z)V_{\alpha_1}(0)V_{\alpha_2}(1)V_{\alpha_3}(\infty) \rangle = |z|^{-\langle \alpha_1, \alpha \rangle} |z-1|^{-\langle \alpha_2, \alpha \rangle} \mathcal{H}(z)$$

where $\mathcal{H}(z)$ is equal to, with $\alpha_0 = \alpha$ and $(z_0, z_1, z_2) = (z, 0, 1)$:

$$\left(\prod_{i=1}^2 \frac{\Gamma(s_i) \mu_i^{-s_i}}{\gamma} \right) \mathbb{E} \left[\prod_{i=1}^2 \left(\int_{\mathbb{C}} \frac{|y_i|^{-\gamma \sum_{k=0}^3 \langle \alpha_k, e_i \rangle}}{\prod_{k=0}^2 |z_k - y_i|^{\gamma \langle \alpha_k, e_i \rangle}} M^{\gamma e_i}(d^2 y_i) \right)^{-s_i} \right].$$

Then by using the global Ward identities we can express the different quantities $\mathcal{W}_{-i}^{(k)}$ for $k = 1, 2, 3$ and $i = 1, 2$ in terms of $\mathcal{W}_{-i}$ associated to V_α , $i = 1, 2$, and $\mathcal{W}_{-1}^{(2)}$. Heuristically this is due to the fact that there are six such quantities linked by five linearly independent constraints. Inverting this system of equations is a bit tedious so we will omit the computations in the present document. After algebraic manipulations we find the following:

$$\begin{aligned} \mathcal{W}_{-3} \langle V_\alpha(z) \mathbf{V} \rangle &= \left[- \left(\frac{2}{z} + \frac{1}{z-1} \right) \mathcal{W}_{-2} - \left(\frac{1}{z^2} + \frac{2}{z(z-1)} \right) \mathcal{W}_{-1} + \frac{\mathcal{W}_{-1}^{(2)}}{(z(z-1))^2} \right. \\ &\quad \left. - \frac{w_\alpha + w_3}{z^2(z-1)} - \frac{w_1}{z^3(z-1)} + \frac{w_2}{z(z-1)^2} \left(\frac{1}{z} + \frac{1}{z-1} \right) \right] \langle V_\alpha(z) \mathbf{V} \rangle. \end{aligned} \quad (7.1.24)$$

In the special case where the Vertex Operator V_α is fully degenerate, we can use Propositions 7.1.2 and 7.1.6 to rewrite the left-hand side using Virasoro descendants and get the equality:

$$\begin{aligned} &\left[\frac{8}{\chi^3} \partial_z^3 + \frac{4}{\chi} \partial_z \mathcal{L}_{-2} - \left(\frac{\chi}{3} + \frac{2}{\chi} \right) \mathcal{L}_{-3} \right] \langle V_{-\chi \omega_1}(z) \mathbf{V} \rangle \\ &= \left[\left(\frac{2}{z} + \frac{1}{z-1} \right) \left(\frac{4}{\chi} \partial_z^2 + \frac{4\chi}{3} \mathcal{L}_{-2} \right) - \left(\frac{1}{z^2} + \frac{2}{z(z-1)} \right) \left(\frac{5\chi}{3} + \frac{2}{\chi} \right) \partial_z + \frac{\mathcal{W}_{-1}^{(2)}}{(z(z-1))^2} \right. \\ &\quad \left. - \frac{w + w_3}{z^2(z-1)} - \frac{w_1}{z^3(z-1)} + \frac{w_2}{z(z-1)^2} \left(\frac{1}{z} + \frac{1}{z-1} \right) \right] \langle V_{-\chi \omega_1}(z) \mathbf{V} \rangle. \end{aligned} \quad (7.1.25)$$

Now we proceed in the same way for the Virasoro descendants. Using the three global Ward identities given by conformal covariance of the correlation function of Vertex Operators in Theorem 5.1.3 we end up with

$$\begin{aligned}\mathcal{L}_{-3}\langle V_{-\chi\omega_1}(z)\mathbf{V}\rangle &= \left[\frac{3z^2 - 3z + 1}{(z(z-1))^2} \partial_z + \frac{\Delta - \Delta_3}{z(z-1)} \left(\frac{1}{z} + \frac{1}{z-1} \right) \right. \\ &\quad \left. + \frac{\Delta_1}{z^2(z-1)} \left(\frac{2}{z} + \frac{1}{z-1} \right) - \frac{\Delta_2}{z(z-1)^2} \left(\frac{1}{z} + \frac{2}{z-1} \right) \right] \langle V_{-\chi\omega_1}(z)\mathbf{V}\rangle \\ \mathcal{L}_{-2}\langle V_{-\chi\omega_1}(z)\mathbf{V}\rangle &= \left[\frac{2z-1}{z(1-z)} \partial_z + \frac{\Delta_3 - \Delta}{z(z-1)} - \frac{\Delta_1}{z^2(z-1)} + \frac{\Delta_2}{z(z-1)^2} \right] \langle V_{-\chi\omega_1}(z)\mathbf{V}\rangle.\end{aligned}\tag{7.1.26}$$

Combining the two last equations, some elementary but lengthy algebraic manipulations show that we have the equality

$$\mathcal{D}_0(z)\langle V_{-\chi\omega_1}(z)V_{\alpha_1}(0)V_{\alpha_2}(1)V_{\alpha_3}(\infty)\rangle = \frac{\mathcal{W}_{-1}^{(2)}}{z-1}\langle V_{-\chi\omega_1}(z)V_{\alpha_1}(0)V_{\alpha_2}(1)V_{\alpha_3}(\infty)\rangle\tag{7.1.27}$$

where on the left-hand side we have considered the differential operator

$$\begin{aligned}\mathcal{D}_0(z) &:= \frac{8z^2(z-1)}{\chi^3} \partial_z^3 + \frac{4}{\chi} z(3-5z) \partial_z^2 \\ &+ \left[\frac{2}{\chi} \left(\frac{4z^2 - 5z + 2}{z-1} + 2z(\Delta_3 - \Delta) - 2\Delta_1 + 2\Delta_2 \frac{z}{z-1} \right) + \chi \frac{12z^2 - 15z + 4}{z-1} \right] \partial_z \\ &+ \frac{2}{\chi} \left((1-2z) \frac{\Delta_3 - \Delta}{z-1} + (3z-2) \frac{\Delta_1}{z(z-1)} - (3z-1) \frac{\Delta_2}{(z-1)^2} \right) \\ &+ \frac{\chi}{3} \left((7-10z) \frac{\Delta_3 - \Delta}{z-1} + (9z-6) \frac{\Delta_1}{z(z-1)} + (7-9z) \frac{\Delta_2}{(z-1)^2} \right) \\ &+ w + w_3 + \frac{w_1}{z} - \frac{w_2}{(z-1)^2} (2z-1).\end{aligned}\tag{7.1.28}$$

7.1.2.2 Analytic extension of the differential equation

Equation (7.1.27) is valid as soon as the weights satisfy the assumptions of Theorem 4.3.1. Our first task is to extend the range of validity of the latter to the whole range of values prescribed by Theorem 6.4.1. The reason why we need this extension is that we will need to consider later on correlation functions defined for weights that belong to $\mathcal{A}_4$ but not to $\mathcal{B}_4$.

For this purpose we will consider a weak formulation of the above problem since correlation functions may not be differentiable if the weights no longer belong to $\mathcal{B}_4$. However the last quantity involving a W_{-1} descendent is not directly a differential operator applied to the correlation functions if the Vertex Operator V_{α_2} is not a semi-degenerate field. To remedy this issue we will be a bit astute and rely on the fact that we can actually express the W_{-1} descendent as a differential operator but acting on a larger space. To be more specific we note that

$$\mathbf{W}_{-1}(z, \alpha) = (q - 2\langle \alpha, \omega_2 \rangle) \langle \alpha, e_1 \rangle \langle \partial \Phi(z), \omega_1 \rangle - (q - 2\langle \alpha, \omega_1 \rangle) \langle \alpha, e_2 \rangle \langle \partial \Phi(z), \omega_2 \rangle$$

so that we can formally write that

$$\mathcal{W}_{-1}\langle V_\alpha(z)\mathbf{V}\rangle = \left[(q - 2\langle\alpha, \omega_2\rangle) \partial_{x_1} - (q - 2\langle\alpha, \omega_1\rangle) \partial_{x_2} \right] \langle V_{\langle\alpha, e_1\rangle\omega_1}(x_1) V_{\langle\alpha, e_2\rangle\omega_2}(x_2) \mathbf{V} \rangle$$

evaluated at $x_1 = x_2 = z$. However due to the fact that correlation functions are defined based on a regularization procedure this is no longer true and we need to be more precise. This is done by considering the regularized correlation functions. Namely at the regularized level we can write that

$$\begin{aligned} \mathcal{W}_{-1}\langle V_{\alpha, \varepsilon}(z) \mathbf{V}_\varepsilon \rangle &= \hat{g}(z)^{\Delta_\alpha} \langle \mathbf{W}_{-1, \varepsilon}(z, \alpha) e^{\langle\alpha, \mathbf{X}_\varepsilon^{\hat{g}}(z) + \mathbf{c}\rangle - \frac{1}{2} \mathbb{E}[\langle\alpha, \mathbf{X}_\varepsilon^{\hat{g}}(z)\rangle^2]} V_\varepsilon \rangle \\ &= \hat{g}(z)^{\Delta_{\alpha_1}} \left[(q - 2\langle\alpha, \omega_2\rangle) \partial_{x_1} - (q - 2\langle\alpha, \omega_1\rangle) \partial_{x_2} \right] \Big|_{x_1=x_2=z} \\ &\quad \langle e^{\langle\alpha, e_1\rangle\langle\mathbf{X}_\varepsilon^{\hat{g}}(x_1) + \mathbf{c}, \omega_1\rangle} e^{\langle\alpha, e_2\rangle\langle\mathbf{X}_\varepsilon^{\hat{g}}(x_2) + \mathbf{c}, \omega_2\rangle} e^{-\frac{1}{2} \mathbb{E}[\langle\alpha, \mathbf{X}_\varepsilon^{\hat{g}}(z)\rangle^2]} V_\varepsilon \rangle \end{aligned}$$

up to metric-dependent term that vanish thanks to the KPZ identity 4.4.4 in the same fashion as in the proof of Lemma 4.4.6. Now we note that $\mathbb{E}[\langle\alpha, e_2\rangle\langle\mathbf{X}_\varepsilon^{\hat{g}}(z), \omega_2\rangle^2]$ is given by

$$\mathbb{E} \left[(\langle\alpha, e_1\rangle\langle\mathbf{X}_\varepsilon^{\hat{g}}(z), \omega_1\rangle)^2 \right] + \mathbb{E} \left[(\langle\alpha, e_2\rangle\langle\mathbf{X}_\varepsilon^{\hat{g}}(z), \omega_2\rangle)^2 \right] + 2\mathbb{E} \left[\langle\alpha, e_1\rangle\langle\alpha, e_2\rangle\langle\mathbf{X}_\varepsilon^{\hat{g}}(z), \omega_1\rangle\langle\mathbf{X}_\varepsilon^{\hat{g}}(z), \omega_2\rangle \right].$$

Recollecting terms this shows that

$$\begin{aligned} \mathcal{W}_{-1}\langle V_{\alpha, \varepsilon}(z) \mathbf{V}_\varepsilon \rangle &= \mathcal{T} \Big|_{x_1=x_2=z} \left(|x_1 - x_2|^{\langle\alpha, e_1\rangle\langle\alpha, e_2\rangle\langle\omega_1, \omega_2\rangle} \langle V_{\langle\alpha, e_1\rangle\omega_1}(x_1) V_{\langle\alpha, e_2\rangle\omega_2}(x_2) \mathbf{V} \rangle \right) \\ &\quad \text{with } \mathcal{T} := (q - 2\langle\alpha, \omega_2\rangle) \partial_{x_1} - (q - 2\langle\alpha, \omega_1\rangle) \partial_{x_2}. \end{aligned} \tag{7.1.29}$$

Note that the explicit expression (4.2.6) shows that the limit $\lim_{x_1=x_2=z} |x_1 - x_2|^{\langle\alpha, e_1\rangle\langle\alpha, e_2\rangle\langle\omega_1, \omega_2\rangle} \langle V_{\langle\alpha, e_1\rangle\omega_1}(x_1) V_{\langle\alpha, e_2\rangle\omega_2}(x_2) \mathbf{V} \rangle$ exists and is given by $\langle V_\alpha(z) \mathbf{V} \rangle$.

We can now define the weak formulation of the above problem that will allow us to extend the range of validity of the above differential equation. Namely let us denote

$$F_\alpha(z, x_1, x_2) := |x_1 - x_2|^{\langle\alpha, e_1\rangle\langle\alpha, e_2\rangle\langle\omega_1, \omega_2\rangle} \langle V_{\langle\alpha, e_1\rangle\omega_1}(x_1) V_{\langle\alpha, e_2\rangle\omega_2}(x_2) \mathbf{V} \rangle$$

and consider test functions $\phi : \mathbb{C}^3 \rightarrow \mathbb{R}$ that are smooth, bounded and compactly supported, with $\phi(z, x_1, x_2) = 0$ if $|z - x_i| < \delta$ or $|z| < \delta$ for some positive δ . Then for any such function the quantity

$$\int_{\mathbb{C}^3} \mathcal{D}_0^*(z) \phi(z, x_1, x_2) F_\alpha(z, x_1, x_2) d^2 z d^2 x_1 d^2 x_2$$

with $\mathcal{D}_0^*(z)$ the adjoint operator of $\mathcal{D}_0(z)$ can analytically continued over the whole $\mathcal{A}_N$. Likewise we can extend analytically over $\mathcal{A}_N$ the quantity

$$\int_{\mathbb{C}^3} \mathcal{T} \phi(z, x_1, x_2) F_\alpha(z, x_1, x_2) d^2 z d^2 x_1 d^2 x_2.$$

If we now consider $\phi = \phi_\varepsilon$ to be a sequence of such test functions with $\phi_\varepsilon(z, x_1, x_2) = 0$ as soon as $|x_1 - 1| > \varepsilon$ or $|x_2 - 1| > \varepsilon$, we see that Equation (7.1.27) shows that as soon as $\alpha \in \mathcal{B}_4$,

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathbb{C}^3} (\mathcal{D}_0^*(z) + \mathcal{T}) \phi_\varepsilon(z, x_1, x_2) F_\alpha(z, x_1, x_2) d^2 z d^2 x_1 d^2 x_2 = 0. \quad (7.1.30)$$

Now the sequence of integral is meromorphic in $\alpha \in \mathcal{A}_4$ since the integral are absolutely convergent. Moreover the limit is also uniformly convergent in a complex neighbourhood of $\mathcal{A}_4$ since we avoid the singularities when points may merge. This shows that for any $\alpha \in \mathcal{A}_4$ Equation (7.1.30) remains valid too.

7.1.2.3 An hypergeometric differential equation

Let us now further assume that the Vertex Operator $V_{\alpha_2}(1)$ is semi-degenerate, i.e. that $\alpha_2 = \alpha_2^* = \kappa\omega_2$ for some real κ ¹. In that case the differential operator $\mathcal{T}$ is made of only one term:

$$\mathcal{T} = (q - 2\langle\omega_1, \alpha_2^*\rangle)\partial_{x_2}.$$

Using the fact that the correlation functions are conformally invariant, which implies that (at least in the weak sense) they satisfy the global Ward identities for the stress-energy tensor from Theorem 5.2.1 we can write that in the weak sense

$$\begin{aligned} & \mathcal{W}_{-1}^{(2)} \langle V_{-\chi\omega_1}(z) V_{\alpha_1}(0) V_{\alpha_2^*}(1) V_{\alpha_3}(\infty) \rangle \\ &= (q - 2\langle\omega_1, \alpha_2^*\rangle) \left(z\partial_z + \sum_{k=0}^3 \Delta_k - 2\Delta_3 \right) \langle V_{-\chi\omega_1}(z) V_{\alpha_1}(0) V_{\alpha_2^*}(1) V_{\alpha_3}(\infty) \rangle. \end{aligned}$$

This shows that the four-point correlations we consider are weak solutions of the following differential equation in the z -variable:

$$\begin{aligned} & \mathcal{D}(z) \langle V_{-\chi\omega_1}(z) V_{\alpha_1}(0) V_{\alpha_2^*}(1) V_{\alpha_3}(\infty) \rangle = 0 \quad \text{with} \\ & \mathcal{D}(z) = \mathcal{D}_0(z) + (q - 2\langle\omega_1, \alpha_2^*\rangle) \frac{z\partial_z + \sum_{k=0}^3 \Delta_k - 2\Delta_3}{1 - z}. \end{aligned} \quad (7.1.31)$$

Then some (lengthy) algebraic manipulations show that we can write

$$\langle V_{-\chi\omega_1}(z) V_{\alpha_1}(0) V_{\kappa\omega_2}(1) V_{\alpha_3}(\infty) \rangle = |z|^{\chi\langle h_1, \alpha_1 \rangle} |z - 1|^{\frac{\chi\kappa}{3}} \mathcal{H}(z)$$

with $\mathcal{H}$ solution of the hypergeometric differential equation of order three

$$\left[z(A_1 + z\partial_z)(A_2 + z\partial_z)(A_3 + z\partial_z) - (B_1 - 1 + z\partial_z)(B_2 - 1 + z\partial_z)z\partial_z \right] \mathcal{H} = 0. \quad (7.1.32)$$

In the above equation we have set

$$\begin{aligned} A_i &:= \frac{\chi}{2} \langle \alpha_1 + \kappa\omega_2 - \chi\omega_1 - Q, h_i \rangle + \frac{\chi}{2} \langle \alpha_3 - Q, h_i \rangle \\ B_i &:= 1 + \frac{\chi}{2} \langle \alpha_1 - Q, h_1 - h_{i+1} \rangle. \end{aligned} \quad (7.1.33)$$

¹We stress that in that case for four-point correlation functions to be well-defined we really need the assumption that $\alpha \in \mathcal{A}_4$.

The fact that $\mathcal{H}$ is a (at least distributional) solution of Equation (7.1.32) allows to claim that $\mathcal{H}$ is actually a real analytic function via a standard elliptic regularity argument. Indeed, we can apply the differential operator $\partial_{\bar{z}}^3$ to Equation (7.1.32). By doing so we see that $\mathcal{H}$, viewed as a function of two real variables, is a solution of a partial differential equation $P\mathcal{H} = 0$ with analytic coefficients on $\mathbb{R}^2 \setminus \{(0, 0); (1, 0)\}$, and whose term of highest degree is given by $z^3(z-1)\Delta^3 (z = x + iy)$ where Δ is the standard Laplace operator $\Delta f(x, y) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$. In particular $\mathcal{H}$ is seen to be a solution of $P\mathcal{H} = 0$ where P is an analytic hypoelliptic operator on $\mathbb{R}^2 \setminus \{(0, 0); (1, 0)\}$. This implies that $\mathcal{H}$ is real analytic on $\mathbb{C} \setminus \{0, 1\}$.

7.1.3 . Proof of Theorem 7.1.1

We have seen above that under the assumption that $\alpha \in \mathcal{A}_4$, the four-point correlation functions considered in Theorem 7.1.1 are solutions of an hypergeometric differential equation of the third order. This completely determines them up to a global constant, thanks to the following:

Proposition 7.1.7. *Assume that for any two distinct elements U, V in the set $\{0, B_1, B_2, A_1, A_2, A_3\}$, the quantity $U - V$ is a non-integer real number. Then real-valued solutions of the hypergeometric differential equation of the third order (7.1.3) in $\mathbb{C} \setminus \{0, 1\}$ are of the form*

$$\mathcal{H}(z) = \mathcal{H}(0) \left(|\mathcal{H}_0(z)|^2 + \sum_{i=1}^2 A_{\gamma}^{(i)}(\alpha, \alpha_1, \alpha_2^*, \alpha_3) |\mathcal{H}_i(z)|^2 \right). \quad (7.1.34)$$

As a consequence in order to prove that Theorem 7.1.1 does indeed hold we only need to check that $\mathcal{H}$ defined via $\langle V_{-\chi\omega_1}(z)V_{\alpha_1}(0)V_{\kappa\omega_2}(1)V_{\alpha_3}(\infty) \rangle = |z|^{\chi\langle h_1, \alpha_1 \rangle} |z-1|^{\frac{\chi\kappa}{3}} \mathcal{H}(z)$ is real-valued, satisfies $\mathcal{H}(0) = C_{\gamma}(\alpha_1 - \chi\omega_1, \kappa\omega_2, \alpha_3)$ and that the coefficients A_i , $1 \leq i \leq 3$ and B_j , $j = 1, 2$ satisfy the assumptions of Proposition 7.1.7.

The first point is straightforward, while the second one follows from the probabilistic representation of the correlation functions, which allows to evaluate

$$\mathcal{H}(0) = C_{\gamma}(\alpha_1 - \chi\omega_1, \alpha_2^*, \alpha_3).$$

Therefore the result holds true as soon as the coefficients A and B meet the requirements of Proposition 7.1.7. Because both $\mathcal{H}$ and the right-hand side in Equation (7.1.6) depend analytically on the weights α via Theorem 6.4.1, the statement extends to the whole range of values of Theorem 7.1.1.

Before actually proving Proposition 7.1.7, we start with the general description of the set of real-valued solutions of Equation (7.1.3) on the subset $\mathbb{C} \setminus \{(-\infty, 0] \cup [1, +\infty)\}$ of $\mathbb{C} \setminus \{0, 1\}$.

Lemma 7.1.8. *Real-valued solutions of Equation (7.1.3) on $\mathbb{C} \setminus \{(-\infty, 0] \cup [1, +\infty)\}$ are linear combinations of*

$$|F_i(z)|^2 \text{ for } i = 0, 1, 2; \quad \Re(F_i(z)F_j(\bar{z})) \text{ for } 0 \leq i \neq j \leq 2; \quad \Im(F_i(z)F_j(\bar{z})) \text{ for } 0 \leq i \neq j \leq 2.$$

Proof. The functions proposed in the statement being real-valued solutions of Equation (7.1.3), it is enough to bound the dimension of the set of such functions by nine.

Now the statement of [33, Theorem 1.3] also implies that such a function $\mathcal{H}$ is actually real analytic; therefore it suffices to bound this dimension on some open subset of $\mathbb{C} \setminus \{0, 1\}$.

To do so we view $\mathcal{H}$ as a function of two real variables x, y with x (resp. y) being the real (resp. imaginary) part of z . Then, by taking the real and imaginary parts of Equation (7.1.3), $\mathcal{H}$ is a solution of

$$\partial_{xxx}\mathcal{H} - 3\partial_{xyy}\mathcal{H} + \text{lower derivatives} = 0 \quad \partial_{yyy}\mathcal{H} - 3\partial_{xxy}\mathcal{H} + \text{lower derivatives} = 0.$$

The characteristic equation $\tau^3 - 3\tau\xi^2 = 0$ being of the hyperbolic type we see that $\mathcal{H}$, viewed as a function in the variables x, y , is a solution of a pair of hyperbolic partial differential equations of the third order. As a consequence it is completely determined in a complex neighbourhood $\mathcal{O}$ of -1 by the data of $u_0(x) := \mathcal{H}(x, 0)$, $u_1(x) := \partial_y \mathcal{H}(x, 0)$ and $u_2(x) := \partial_{yy} \mathcal{H}(x, 0)$. Using the explicit expression of the differential operator (7.1.3) these real valued functions are solutions of

$$\begin{aligned} & \left[x(A_1 + x\partial)(A_2 + x\partial)(A_3 + x\partial) - (B_1 - 1 + x\partial)(B_2 - 1 + x\partial)x\partial \right] u_0 \\ &= 3x^3(x-1)u'_2 + x^2(x(A_1 + A_2 + A_3 + 3) - (B_1 + B_2 + 1))u_2, \end{aligned}$$

which corresponds to taking the real part of (7.1.3), and, by taking ∂_y derivatives of the previous expression combined with the imaginary part of (7.1.3),

$$\begin{aligned} & \left[x(A_1 + x\partial)(A_2 + x\partial)(A_3 + x\partial) - (B_1 - 1 + x\partial)(B_2 - 1 + x\partial)x\partial \right] u_1 \\ &= 3x^3(x-1)u'_3 + x^2(x(A_1 + A_2 + A_3 + 3) - (B_1 + B_2 + 1))u_3, \end{aligned}$$

$$\begin{aligned} & \left[x(A_1 + x\partial)(A_2 + x\partial)(A_3 + x\partial) - (B_1 - 1 + x\partial)(B_2 - 1 + x\partial)x\partial \right] u_2 \\ &= 3x^3(x-1)u'_4 + x^2(x(A_1 + A_2 + A_3 + 3) - (B_1 + B_2 + 1))u_4, \end{aligned}$$

$$\begin{aligned} \text{with } u_3 &= 3u_1^{(2)} + 2 \frac{x(A_1 + A_2 + A_3 + 3) - (B_1 + B_2 + 1)}{x(x-1)} u'_1 \\ &+ \frac{x(1 + A_1 + A_2 + A_3 + A_1A_2 + A_1A_3 + A_2A_3) - B_1B_2}{x^2(x-1)} u_1, \end{aligned}$$

$$\begin{aligned} \text{and } u_4 &= 3u_2^{(2)} + 2 \frac{x(A_1 + A_2 + A_3 + 3) - (B_1 + B_2 + 1)}{x(x-1)} u'_2 \\ &+ \frac{x(1 + A_1 + A_2 + A_3 + A_1A_2 + A_1A_3 + A_2A_3) - B_1B_2}{x^2(x-1)} u_2. \end{aligned}$$

Therefore u_1 and u_2 both live in a three-dimensional space; similarly u_0 lives in a three-dimensional space determined by u_2 . This implies that $\mathcal{H}$ lives in a set of dimension at most nine as expected. $\square$

We are now in position to address the proof of Proposition 7.1.7.

Proof of Proposition 7.1.7. We rely on the fact that any solution $\mathcal{H}$, along with its derivatives, must be regular when crossing the real axis. This fact will imply that, among the solutions of Equation (7.1.3) on $\mathbb{C} \setminus \{(-\infty, 0] \cup [1, +\infty)\}$ the only ones that meet this requirement will be those of the form $|F_i(z)|^2$ for $i = 0, 1, 2$. In addition to ensure

this condition one must further assume that $\mathcal{H}$ has the form prescribed by Proposition 7.1.7.

To see why, let us first consider what happens when a solution crosses the negative real axis $(-\infty, 0)$. Because the hypergeometric functions ${}_3F_2$ are continuous over $\mathbb{C} \setminus (1, +\infty)$, the lack of continuity of the solutions across the negative real axis would come from the fractional power of z . Indeed, let us take r to be some positive real number for which the hypergeometric functions evaluated at $z = -r$ are non-zero. Since we have defined the branch cut of the logarithm to be on $(-\infty, 0)$, we can write that for $z_{\pm, \varepsilon} := re^{\pm i(\pi - \varepsilon)}$, $F_0(z_{+, \varepsilon})F_1(z_{+, \varepsilon}) - F_0(z_{-, \varepsilon})F_1(z_{-, \varepsilon})$ will converge as $\varepsilon \rightarrow 0$ to $(e^{i(1-B_1)\pi} - e^{-i(1-B_1)\pi})$ up to a non-zero (real) multiplicative constant. Since we have assumed B_1 not to be an integer, this means that $\Im(F_0(z)F_1(\bar{z}))$ will not be continuous across $(-\infty, 0)$. Similarly, when considering the partial derivatives of $(x, y) \mapsto F_0(x + iy)F_1(x - iy)$ we see that, expanding around $z = 0$,

$$\partial_y F_0(z)F_1(\bar{z}) = \bar{z}^{1-B_1} \partial_y (F_0(z)\bar{z}^{B_1-1}F_1(\bar{z})) + i(B_1-1)\frac{1}{\bar{z}}F_0(z)F_1(\bar{z}) = i(B_1-1)\bar{z}^{-B_1} + o(\bar{z}^{-B_1}).$$

Because of the branch cut on the negative real axis we see again that for $\Re(F_0(z)F_1(\bar{z}))$ to be continuous we must assume B_1 to be an integer, which is not. As a consequence for solutions of Equation (7.1.3) to be continuous along with their derivatives we must rule out (under the assumptions that $B_1, B_2, B_1 - B_2$ are not integers) those of the form $\Re(F_i(z)F_j(\bar{z}))$ for $0 \leq i \neq j \leq 2$ or $\Im(F_i(z)F_j(\bar{z}))$ for $0 \leq i \neq j \leq 2$. Since the three hypergeometric functions are linearly independent a linear combination of such solutions will not be continuous either.

The remaining solutions $|F_i(z)|^2$ are continuous along with their derivatives across the negative real axis; therefore the next step is the investigation of continuity across $(1, \infty)$. To make this explicit let us consider an alternative basis of solutions defined around $z = \infty$. These are defined via the expressions for $i = 1, 2, 3$ (and the convention that $A_k = A_{k \bmod 3}$):

$$G_i(z) := (-z)^{A_i} {}_3F_2 \left(\begin{matrix} A_i, 1 + A_i - B_1, 1 + A_i - B_2 \\ 1 + A_i - A_{i-1}, 1 + A_i - A_{i+1} \end{matrix} \middle| \frac{1}{z} \right). \quad (7.1.35)$$

Again the hypergeometric functions can be continued to analytic functions over $\mathbb{C} \setminus [0, 1]$. Both basis are related thanks to the remarkable equality (see [133] or [129, Theorem 1]) valid outside of the real axis:

$$\frac{\Gamma(A_1)\Gamma(A_2)\Gamma(A_3)}{\Gamma(B_1)\Gamma(B_2)} {}_3F_2 \left(\begin{matrix} A_1, A_2, A_3 \\ B_1, B_2 \end{matrix} \middle| z \right) = \sum_{i=1}^3 \frac{\Gamma(A_i)\Gamma(A_{i+1} - A_i)\Gamma(A_{i-1} - A_i)}{\Gamma(B_1 - A_i)\Gamma(B_2 - A_i)} G_i(z). \quad (7.1.36)$$

Whence the general solution $\mathcal{H}(z) = \lambda_0 |F_0(z)|^2 + \lambda_1 |F_1(z)|^2 + \lambda_2 |F_2(z)|^2$ admits the expansion

$$\mathcal{H}(z) = \sum_{i=0}^2 \alpha_i |G_i(z)|^2 + \beta_3 \Re(G_0(z)G_1(\bar{z})) + \beta_0 \Re(G_1(z)G_2(\bar{z})) + \beta_1 \Re(G_2(z)G_0(\bar{z}))$$

where the coefficients β_i are given by $\frac{\Gamma(A_i - A_{i-1})\Gamma(A_i - A_{i+1})\Gamma(A_{i-1} - A_{i+1})\Gamma(A_{i+1} - A_{i-1})l(B_1)l(B_2)}{\Gamma(B_1 - A_{i+1})\Gamma(B_2 - A_{i+1})\Gamma(A_{i+1})\Gamma(B_1 - A_{i-1})\Gamma(B_2 - A_{i-1})\Gamma(A_{i-1})l(A_i)} \times$

$$\begin{aligned} & \lambda_0 \frac{\Gamma(B_1)\Gamma(1-B_1)\Gamma(B_2)\Gamma(1-B_2)}{\Gamma(A_i)\Gamma(1-A_i)} \\ & + \lambda_1 \frac{\Gamma(B_1-1)\Gamma(2-B_1)\Gamma(B_2-B_1)\Gamma(1-B_1+B_2)}{\Gamma(B_1-A_i)\Gamma(1-B_1+A_i)} \frac{\prod_{j=1}^3 l(A_j)l(B_1-A_j)}{l(B_1)l(B_2)} \frac{l(1-B_1+B_2)}{l(B_1-1)} \\ & + \lambda_2 \frac{\Gamma(B_2-1)\Gamma(2-B_2)\Gamma(B_1-B_2)\Gamma(1-B_2+B_1)}{\Gamma(B_2-A_i)\Gamma(1-B_2+A_i)} \frac{\prod_{j=1}^3 l(A_j)l(B_2-A_j)}{l(B_1)l(B_2)} \frac{l(1-B_2+B_1)}{l(B_2-1)}. \end{aligned}$$

This can be further reduced (up to a global multiplicative factor) to the form

$$\lambda_0 \sin(\pi A_i) \sin(\pi(B_1 - B_2)) - \widetilde{\lambda}_1 \sin(\pi(B_1 - A_i)) \sin(\pi B_2) + \widetilde{\lambda}_2 \sin(\pi(B_2 - A_i)) \sin(\pi B_1)$$

with $\widetilde{\lambda}_i := \lambda_i A_i^{(i)}(\alpha, \alpha_1, \alpha_2^*, \alpha_3)$, and where the identity $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$, valid for $z \notin \mathbb{Z}$, has been used. Since we have assumed the $A_j - A_i$ not to be integers for $j \neq i$ along the same lines as above continuity across the $(1, \infty)$ axis of (derivatives of) $\mathcal{H}$ implies that the β_i are zero. After a little algebra this implies that either $\lambda_0 = \widetilde{\lambda}_1 = \widetilde{\lambda}_2$ or the factor in front of the λ vanishes, which won't occur since it is assumed that none of the A_i and $B_j - A_i$ are integers. Whence the desired relations between the λ : this wraps up the proof of Proposition 7.1.7. $\square$

7.2 . Four-point correlation functions and Operator Product Expansions

We have proved in the previous section that certain four-point correlation functions in the $\mathfrak{sl}_3$ Toda theory can be expressed as a sum of hypergeometric functions, and to do so we relied on the fact that they are solutions of a BPZ-type differential equation. Our goal here is to provide an alternative way of deriving such an expansion, based on the probabilistic representation of these correlation functions. More precisely this expansion will be obtained using *Operator Product Expansions* (OPEs hereafter), which are based on an asymptotic expansion of the correlation functions when two insertion points collide (here $z \rightarrow 0$). This alternative expansion will allow to obtain shift equations for three-point correlation functions, which as we will see in Section 7.3, fully characterize these correlation functions.

As a byproduct of these OPEs we provide a rigorous meaning to the reflection relation between Vertex Operators:

$$V_\alpha = R_s(\alpha) V_{s\alpha}$$

which we prove to hold for all elements s of the Weyl group W and within certain three-point correlation functions. To make this statement precise we introduce for fixed $\alpha_2^*, \alpha_3 \in \mathbb{R}^2$ the set $\mathcal{U}(\alpha_2^*, \alpha_3)$ defined via

$$\mathcal{U}(\alpha_2^*, \alpha_3) := \{ \alpha_1 \in \mathbb{R}^2, (\hat{s}\alpha_1, \alpha_2^*, \alpha_3) \in \mathcal{A}_3 \text{ for some } s \in W \}. \quad (7.2.1)$$

The following statement provides a rigorous meaning to the reflection relation:

Proposition 7.2.1. Given $\alpha_2^*, \alpha_3 \in \mathbb{R}^2$, extend the function $C_\gamma(\alpha_1, \alpha_2^*, \alpha_3)$ over $\mathcal{U}(\alpha_2^*, \alpha_3)$ by setting

$$C_\gamma(\alpha_1, \alpha_2^*, \alpha_3) := R_s(\alpha_1)C_\gamma(\hat{s}\alpha_1, \alpha_2^*, \alpha_3) \text{ where } s \in W \text{ is such that } s(\alpha_1 - Q) \in \mathcal{C}_-. \quad (7.2.2)$$

Then the map thus defined is analytic in a complex neighbourhood of $\mathcal{U}(\alpha_2^*, \alpha_3)$.

Thanks to this statement we can define probabilistic three-point correlation functions when the weights are not longer inside $(Q + \mathcal{C}_-)^N$. We still denote by $C_\gamma(\alpha_1, \alpha_2^*, \alpha_3)$ this extension. As we will see this extension naturally arises in the alternative expansion proposed for the four-point correlation functions of Theorem 7.1.1:

Theorem 7.2.2. Under the same assumptions as in Theorem 7.1.1,

$$\begin{aligned} \langle V_\alpha(z)V_{\alpha_1}(0)V_{\alpha_2^*}(1)V_{\alpha_3}(\infty) \rangle &= |z|^{\chi(h_1, \alpha_1)} |z-1|^{\frac{\chi\kappa}{3}} \mathcal{H}(z), \quad \text{where} \\ \mathcal{H}(z) &= \sum_{i=0}^2 B_\gamma^{(i)}(\alpha_1, \chi) C_\gamma(\alpha_1 - \chi h_{i+1}, \alpha_2^*, \alpha_3) |\mathcal{H}_i(z)|^2 \end{aligned} \quad (7.2.3)$$

as soon as $\alpha_1 - \chi h_i \in \mathcal{U}(\alpha_2^*, \alpha_3)$ for $1 \leq i \leq 3$. The coefficients $B_\gamma^{(i)}(\alpha_1, \chi)$ admit the explicit expression

$$B_\gamma^{(i)}(\alpha_1, \chi) = \prod_{j=1}^i \left(\pi \mu l \left(\frac{\gamma^2}{2} \right) \right)^{\frac{\chi}{\gamma}} \left(\frac{\chi^2}{2} \right)^2 \frac{l(\frac{\chi}{2} \langle \alpha_1 - Q, h_j - h_{i+1} \rangle)}{l(1 + \frac{\chi^2}{2} + \frac{\chi}{2} \langle \alpha_1 - Q, h_j - h_{i+1} \rangle)}. \quad (7.2.4)$$

7.2.1 . Method of proof

Let us briefly recall the framework introduced in the proof of Theorem 6.4.1. We have seen there that the four-point correlation functions we consider could be defined using Equation (6.4.10) via

$$\langle V_{\alpha_1}(0)V_{-\chi h_1}(z)V_{\alpha_2^*}(1)V_{\alpha_3}(\infty) \rangle = \int_{\mathbb{R}^2} E_\alpha(\mathbf{c}) d\mathbf{c} + \sum_{i=1}^2 \int_{\mathbb{R}} E_\alpha^i(c_i) dc_i + E_\alpha^{1,2},$$

where the quantities $E_\alpha(\mathbf{c})$ are expectation terms. In order to highlight the dependency in z of these quantities, let us introduce the notation

$$\begin{aligned} \Phi_{\alpha_1, \alpha}(z; \mathbf{c}) &:= e^{\langle \mathbf{s}, \mathbf{c} \rangle} \mathbb{E} \left[\exp \left(- \sum_{i=1}^2 \mu_i e^{\langle \gamma e_i, \mathbf{c} \rangle} I_{\alpha_1, \alpha}^i(z) \right) \right], \quad \text{with} \\ I_{\alpha_1, \alpha}^i(z) &:= \int_{\mathbb{C}} \frac{|x-z|^{-\gamma \langle \alpha, e_i \rangle}}{|x|^{\gamma \langle \alpha_1, e_i \rangle}} F_i(x) M^{\gamma e_i}(d^2x) \text{ and } F_i(x) := \frac{|x|_+^{\gamma \langle \alpha + \alpha_1 + \alpha_2^* + \alpha_3, e_i \rangle}}{|x-1|^{\gamma \langle \alpha_2^*, e_i \rangle}} \end{aligned} \quad (7.2.5)$$

so that

$$E_\alpha(\mathbf{c}) = |z|^{\chi \langle \alpha_1, h_1 \rangle} |z-1|^{\chi \langle h_1, \alpha_2^* \rangle} (\Phi_{\alpha_1, \alpha}(z; \mathbf{c}) - \mathfrak{R}_{\alpha_1, \alpha}(z; \mathbf{c})),$$

where the remainder term is given using Equation (6.4.6) by

$$\mathfrak{R}_{\alpha_1, \alpha}(z; \mathbf{c}) := |z|^{-\chi \langle \alpha_1, h_1 \rangle} |z-1|^{-\chi \langle h_1, \alpha_2^* \rangle} e^{-2 \langle Q, \mathbf{c} \rangle} \mathbb{E} [\mathbf{R}_\alpha(\mathbf{c})].$$

Likewise we put $E_\alpha^i(c_1)$ under the form

$$E_\alpha^i(c_i) = |z|^{\langle \alpha_1, h_1 \rangle} |z - 1|^{\langle h_1, \alpha_2^* \rangle} (\Phi_{\alpha_1, \alpha}^i(z; c_1) - \mathfrak{R}_{\alpha_1, \alpha}^i(z; c_1)) \quad \text{where}$$

$$|z|^{\langle \alpha_1, h_1 \rangle} |z - 1|^{\langle h_1, \alpha_2^* \rangle} \Phi_{\alpha_1, \alpha}^1(z; \langle \mathbf{c}, e_1 \rangle) := \sum_{w: \{0,1,\infty\} \rightarrow \{Id, s_2\}} \frac{e^{-\langle \mathbf{s}(w), \omega_2 \rangle \langle \mathbf{c}, e_2 \rangle}}{\langle \mathbf{s}(w), \omega_2 \rangle} e^{-2\langle Q, \mathbf{c} \rangle} \mathbb{E} \left[V_{-\chi h_1}(z) \prod_{x \in \{0,1,\infty\}} R_{w(x)}(\alpha_x) V_{\tilde{w}(x)\alpha_x}(x) e^{-\mu_1 e^{\gamma \langle \mathbf{c}, e_1 \rangle} M^{\gamma e_1}(\mathbb{C})} \right].$$

Finally we set $\Phi_{\alpha_1, \alpha}^{1,2}(z) := |z|^{-\chi \langle \alpha_1, h_1 \rangle} |z - 1|^{-\chi \langle h_1, \alpha_2^* \rangle} E_\alpha^{1,2}$. Note that reflection terms corresponding to the fully degenerate field $V_{-\chi h_1}$ do not show up in the expression of $\mathbf{R}_\alpha(\mathbf{c})$ since $\langle -\chi h_1 - Q, e_i \rangle < -\gamma$ for $i = 1, 2$. With these notations at hand we can write down the four-point correlation functions considered in Theorems 7.1.1 and 7.2.2 via

$$\mathcal{H}(z) = \int_{\mathbb{R}^2} (\Phi_{\alpha_1, \alpha}(z; \mathbf{c}) - \mathfrak{R}_{\alpha_1, \alpha}(z; \mathbf{c})) d\mathbf{c} + \sum_{i=1}^2 \int_{\mathbb{R}} (\Phi_{\alpha_1, \alpha}^i(z; c_i) - \mathfrak{R}_{\alpha_1, \alpha}^i(z; c_i)) dc_i + \Phi_{\alpha_1, \alpha}^{1,2}(z).$$

Now a consequence of Theorem 7.1.1 is that these four-point correlation functions have an explicit expansion as $z \rightarrow 0$. For instance in the case where $2(1 - B_2) < 1$ we can write that $\mathcal{H}(z)$ is given around $z = 0$ by

$$C_\gamma(\alpha_1 - \chi h_1, \alpha_2^*, \alpha_3) \left(1 + A_\gamma^{(1)}(-\chi h_1, \alpha_1, \alpha_2^*, \alpha_3) |z|^{\langle Q - \alpha_1, e_1 \rangle} + A_\gamma^{(2)}(-\chi h_1, \alpha_1, \alpha_2^*, \alpha_3) |z|^{\langle Q - \alpha_1, \rho \rangle} \right)$$

up to lower order terms. The three-point correlation functions that appear in this expansion are defined analogously to $\mathcal{H}(z)$. Namely we will write that

$$C_\gamma(\alpha_1, \alpha_2^*, \alpha_3) = \int_{\mathbb{R}^2} (\Phi_{\alpha_1}(\mathbf{c}) - \mathfrak{R}_{\alpha_1}(\mathbf{c})) d\mathbf{c} + \sum_{i=1}^2 \int_{\mathbb{R}} (\Phi_{\alpha_1}^i(c_1) - \mathfrak{R}_{\alpha_1}^i(c_1)) dc_i + \Phi_{\alpha_1}^{1,2}(\mathbf{c})$$

where the quantities that appear in this expression are defined like above.

In order to prove Theorem 7.2.2 we will study the asymptotic of terms of the form $\Phi_{\alpha_1, \alpha}(z; \mathbf{c})$ and $\mathfrak{R}_{\alpha_1, \alpha}(z; \mathbf{c})$ around $z = 0$ and prove that we end up with an expansion of the correlation functions similar to that from Theorem 7.1.1 but with the coefficients given by the ones in Equation (7.2.3). Theorem 7.2.2 then follows by identifying these coefficients with the ones from Equation (7.1.6).

To do so we first show that Equation (7.2.3) holds true under certain different set of assumptions on the weights α depending on the values of χ , $\langle \alpha_1, e_1 \rangle$ and $\langle \alpha_1, e_2 \rangle$, and then use the results of the previous Section 7.1 to infer that the extension defined by Equation (7.2.2) is analytic. We will then recover the whole range of values for α prescribed by Theorem 7.2.2 using analyticity of the correlation functions (Theorem 6.4.1). Throughout the rest of this Section we assume (up to shifting $\mathbf{c}$ by $\sum_i \frac{\ln \mu_i}{\gamma} \omega_i$) that $\mu_i = 1$ for $i = 1, 2$.

7.2.2 . The case where $\chi = \gamma$, $\langle \alpha_1, e_1 \rangle < \frac{2}{\gamma}$ and $\langle \alpha_1, e_2 \rangle > \frac{2}{\gamma}$

To start with we consider in this subsection the case where the weight α_1 is such that where $\langle \alpha_1, e_1 \rangle < \frac{2}{\gamma}$ and $\langle \alpha_1, e_2 \rangle > \frac{2}{\gamma}$, with $\chi = \gamma$. Our goal is then to prove that under these assumptions the following holds true:

Lemma 7.2.3. *Assume that $\langle \alpha_1, e_1 \rangle < \frac{2}{\gamma}$ and $\frac{2}{\gamma} < \langle \alpha_1, e_2 \rangle < q$. Then*

$$\begin{aligned} \mathcal{H}(z) = & C_\gamma(\alpha_1 - \gamma h_1, \alpha_2^*, \alpha_3) |\mathcal{H}_0(z)|^2 + B_\gamma^{(1)}(\alpha_1, \gamma) C_\gamma(\alpha_1 - \gamma h_2, \alpha_2^*, \alpha_3) |\mathcal{H}_1(z)|^2 \\ & + B_\gamma^{(2)}(\alpha_1, \gamma) R_{s_2}(\alpha_1 - \gamma h_3) C_\gamma(\hat{s}_2(\alpha_1 - \gamma h_3), \alpha_2^*, \alpha_3) |\mathcal{H}_2(z)|^2 \end{aligned} \quad (7.2.6)$$

as soon as $(\alpha_1, -\gamma h_1, \alpha_2^*, \alpha_3) \in \mathcal{A}_4$ with $\langle s, \omega_2 \rangle > 0$.

Proof. In order to prove this statement we will first study the asymptotics of the expression $\mathcal{H}(z)$ when $z \rightarrow 0$ under some additional assumptions on the weight α_1 . Namely we wish to prove that there exists a positive ε such that if $\langle \alpha_1, e_1 \rangle > \frac{2}{\gamma} - \varepsilon$ and $\langle \alpha_1, e_2 \rangle > q - \varepsilon$ then as $z \rightarrow 0$

$$\begin{aligned} \mathcal{H}(z) = & C_\gamma(\alpha_1 - \gamma h_1, \alpha_2^*, \alpha_3) + |z|^{\gamma(Q-\alpha_1, e_1)} B_\gamma^{(1)}(\alpha_1, \gamma) C_\gamma(\alpha_1 - \gamma h_2, \alpha_2^*, \alpha_3) \\ & + |z|^{\gamma(Q-\alpha_1, \rho)} B_\gamma^{(2)}(\alpha_1, \gamma) R_{s_2}(\alpha_1 - \gamma h_3) C_\gamma(\hat{s}_2(\alpha_1 - \gamma h_3), \alpha_2^*, \alpha_3) + l.o.t. \end{aligned}$$

if $\gamma < 1$, while if $1 < \gamma < \sqrt{2}$:

$$\begin{aligned} \mathcal{H}(z) = & C_\gamma(\alpha_1 - \gamma h_1, \alpha_2^*, \alpha_3) + zC + \bar{z}\bar{C} + |z|^{\gamma(Q-\alpha_1, e_1)} B_\gamma^{(1)}(\alpha_1, \gamma) C_\gamma(\alpha_1 - \gamma h_2, \alpha_2^*, \alpha_3) \\ & + |z|^{\gamma(Q-\alpha_1, \rho)} B_\gamma^{(2)}(\alpha_1, \gamma) R_{s_2}(\alpha_1 - \gamma h_3) C_\gamma(\hat{s}_2(\alpha_1 - \gamma h_3), \alpha_2^*, \alpha_3) + l.o.t. \end{aligned}$$

where C is some complex constant.

7.2.2.1 The first expectation term

So as to prove such an expansion for $\mathcal{H}$ we start by providing an expansion of the term $\Phi_{\alpha_1, -\gamma h_1}(z; \mathbf{c})$ that enters the expression of $\mathcal{H}$ and defined via Equation (7.2.5). For this purpose around $z = 0$ we can write $I_{\alpha_1, \alpha_1}^1(z)$ under the form

$$I_{\alpha_1, -\gamma h_1}^1(z) = I_{\alpha_1 - \gamma h_1, 0}^1 + \delta I(z), \quad \delta I(z) := \int_{\mathbb{C}} \frac{|x - z|^{\gamma^2} - |x|^{\gamma^2}}{|x|^{\gamma\langle \alpha_1, e_1 \rangle}} F_1(x) (d^2x).$$

Note that the integrals $I_{\alpha_1 - \gamma h_1}^i$ are well-defined since $\langle \alpha_1 - \gamma h_1 - Q, e_i \rangle \leq \langle \alpha_1 - Q, e_i \rangle < 0$; also note that $I_{\alpha_1, -\gamma h_1}^2(z) = I_{\alpha_1 - \gamma h_1}^2$. This allows to write

$$\begin{aligned} \Phi_{\alpha_1, -\gamma h_1}(z; \mathbf{c}) &= e^{\langle \mathbf{s}, \mathbf{c} \rangle} \mathbb{E} \left[e^{-e^{\gamma\langle \mathbf{c}, e_1 \rangle} \delta I(z)} \prod_{i=1}^2 e^{-e^{\gamma\langle \mathbf{c}, e_i \rangle} I_{\alpha_1 - \gamma h_1}^i} \right] \\ &= \Phi_{\alpha_1 - \gamma h_1}(\mathbf{c}) + e^{\langle \mathbf{s}, \mathbf{c} \rangle} \mathbb{E} \left[\left(\exp \left(-e^{\gamma\langle \mathbf{c}, e_1 \rangle} \delta I(z) \right) - 1 \right) \prod_{i=1}^2 e^{-e^{\gamma\langle \mathbf{c}, e_i \rangle} I_{\alpha_1 - \gamma h_1}^i} \right]. \end{aligned}$$

We focus on the expectation term. To this end, we rewrite

$$\begin{aligned} \exp \left(-e^{\gamma\langle \mathbf{c}, e_1 \rangle} \delta I(z) \right) - 1 &= -e^{\gamma\langle \mathbf{c}, e_1 \rangle} \delta I(z) \int_0^1 \exp \left(-te^{\gamma\langle \mathbf{c}, e_1 \rangle} \delta I(z) \right) dt \\ &= -e^{\gamma\langle \mathbf{c}, e_1 \rangle} \int_{\mathbb{C}} \frac{|x - z|^{\gamma^2} - |x|^{\gamma^2}}{|x|^{\gamma\langle \alpha_1, e_1 \rangle}} F_1(x) M^{\gamma e_1}(d^2x) \int_0^1 \exp \left(-te^{\gamma\langle \mathbf{c}, e_1 \rangle} \delta I(z) \right) dt. \end{aligned}$$

Moreover by the Girsanov (Cameron-Martin) theorem [A](#) we can write that

$$\begin{aligned}
& \mathbb{E} \left[M^{\gamma e_i}(d^2x) \exp \left(-te^{\gamma \langle c, e_1 \rangle} \delta I(z) \right) \prod_{i=1}^2 e^{-e^{\gamma \langle c, e_i \rangle} I_{\alpha_1 - \gamma h_1}^i} \right] \\
&= \mathbb{E} \left[\exp \left(-te^{\gamma \langle c, e_1 \rangle} \delta J(z, x) \right) \prod_{i=1}^2 e^{-e^{\gamma \langle c, e_i \rangle} J_{\alpha_1 - \gamma h_1}^i(x)} \right] d^2x, \quad \text{with} \\
& \delta J(z, x) = \int_{\mathbb{C}} \frac{|y - z|^{\gamma^2} - |y|^{\gamma^2}}{|y|^{\gamma \langle \alpha_1, e_1 \rangle} |y - x|^{2\gamma^2}} \tilde{F}_1(y) M^{\gamma e_1}(d^2y), \quad \tilde{F}_i(y) := F_i(y) |y|_+^{\gamma^2 \langle e_1, e_i \rangle}, \quad \text{and} \\
& J_{\alpha_1 - \gamma h_1}^i(x) = \int_{\mathbb{C}} \frac{|y - x|^{-\gamma^2 \langle e_1, e_i \rangle}}{|y|^{\gamma \langle \alpha_1 - \gamma h_1, e_i \rangle}} \tilde{F}_i(y) M^{\gamma e_i}(d^2y).
\end{aligned}$$

Therefore $\Phi_{\alpha_1, -\gamma h_1}(z; c) - \Phi_{\alpha_1, -\gamma h_1}(c)$ is equal to

$$\begin{aligned}
\delta \Phi(z) &:= -e^{\langle s + \gamma e_1, c \rangle} \int_{\mathbb{C}} \frac{|x - z|^{\gamma^2} - |x|^{\gamma^2}}{|x|^{\gamma \langle \alpha_1, e_1 \rangle}} F_1(x) \times \\
& \quad \mathbb{E} \left[\int_0^1 e^{-te^{\gamma \langle c, e_1 \rangle} \delta J(z, x)} dt \prod_{i=1}^2 e^{-e^{\gamma \langle c, e_i \rangle} J_{\alpha_1 - \gamma h_1}^i(x)} \right] d^2x.
\end{aligned}$$

The case $\gamma < 1$. If we assume that $\gamma < 1$, then we under the additional assumption that $\gamma + \frac{1}{\gamma} < \langle \alpha_1, e_1 \rangle < \frac{2}{\gamma}$ we can make the change of variable $x \leftrightarrow zx$ in the integral to end up with

$$\begin{aligned}
\delta \Phi(z) &= -e^{\langle s + \gamma e_1, c \rangle} |z|^{\gamma \langle Q - \alpha_1, e_1 \rangle} \int_{\mathbb{C}} \frac{|x - 1|^{\gamma^2} - |x|^{\gamma^2}}{|x|^{\gamma \langle \alpha_1, e_1 \rangle}} \times \\
& \quad \mathbb{E} \left[F_1(zx) \int_0^1 e^{-te^{\gamma \langle c, e_1 \rangle} \delta J(z, zx)} dt \prod_{i=1}^2 e^{-e^{\gamma \langle c, e_i \rangle} J_{\alpha_1 - \gamma h_1}^i(zx)} \right] d^2x.
\end{aligned}$$

Therefore, as $z \rightarrow 0$ and thus $\delta J(z, zx) \rightarrow 0$ while $J_{\alpha_1 - \gamma h_1}^i(zx) \rightarrow I_{\alpha_1 - \gamma h_2}^i$, the latter will be asymptotically equivalent to

$$B_{\gamma}^{(1)}(\alpha_1, \gamma) |z|^{\gamma \langle Q - \alpha_1, e_1 \rangle} \Phi_{\alpha_1 - \gamma h_2}(c), \quad B_{\gamma}^{(1)}(\alpha_1, \gamma) = \int_{\mathbb{C}} \frac{|x|^{\gamma^2} - |x - 1|^{\gamma^2}}{|x|^{\gamma \langle \alpha_1, e_1 \rangle}} d^2x. \quad (7.2.7)$$

The factor $B_{\gamma}^{(1)}(\alpha_1, \gamma)$ is evaluated in Lemma [7.2.4](#) below, whose assumptions are satisfied as soon as $\gamma + \frac{1}{\gamma} < \langle \alpha_1, e_1 \rangle < q$. It is found there to be equal to the factor from Theorem [7.2.2](#), so that we have obtained the first two terms in the expansion. However in order to get the third term in the expansion we have to be more precise. For this purpose, let us decompose $J_{\alpha_1 - \gamma h_1}^2(zx)$ as

$$J_{\alpha_1 - \gamma h_1, B_r}^2(zx) + J_{\alpha_1 - \gamma h_1, \mathbb{C}_r}^2(zx), \quad \text{where} \quad J_{\alpha_1 - \gamma h_1, B_r}^2(zx) := \int_{B_r} \frac{|y - zx|^{\gamma^2}}{|y|^{\gamma \langle \alpha_1, e_2 \rangle}} \tilde{F}_2(y) M^{\gamma e_2}(d^2y)$$

and $B_r = B_{e-r}(0)$ with $r = -(1 + \varepsilon) \ln |zx|$ for some positive ε small enough. Then

$$\begin{aligned} \delta\Phi(z) &= B_\gamma^{(1)}(\alpha_1, \gamma) |z|^{\gamma\langle Q-\alpha_1, e_1 \rangle} \Phi_{\alpha_1-\gamma h_2}(c) + \delta\Phi^1(z) + \mathfrak{R}(z), \quad \text{where} \\ \delta\Phi^1(z) &:= -e^{\langle s+\gamma e_1, c \rangle} |z|^{\gamma\langle Q-\alpha_1, e_1 \rangle} \int_{\mathbb{C}} \frac{|x-1|^{\gamma^2} - |x|^{\gamma^2}}{|x|^{\gamma\langle \alpha_1, e_1 \rangle}} \times \\ &\quad \mathbb{E} \left[\left(e^{-e^{\gamma\langle c, e_2 \rangle} J_{\alpha_1-\gamma h_1, B_r}^2(zx)} - 1 \right) e^{-\sum_{i=1}^2 e^{\gamma\langle c, e_i \rangle} I_{\alpha_1-\gamma h_2}^i} \right] d^2x \quad \text{and} \\ \mathfrak{R}(z) &:= -e^{\langle s+\gamma e_1, c \rangle} \int_{\mathbb{C}} \frac{|x-z|^{\gamma^2} - |x|^{\gamma^2}}{|x|^{\gamma\langle \alpha_1, e_1 \rangle}} \mathbb{E} \left[e^{-e^{\gamma\langle c, e_2 \rangle} J_{\alpha_1-\gamma h_1, B_r}^2(x)} e^{-\sum_{i=1}^2 e^{\gamma\langle c, e_i \rangle} I_{\alpha_1-\gamma h_2}^i} \right. \\ &\quad \left. \left(F_1(x) \int_0^1 e^{-e^{\gamma\langle c, e_1 \rangle} (t\delta J(z,x) + \delta J_{\alpha_1-\gamma h_1}^i(x))} dt e^{-e^{\gamma\langle c, e_2 \rangle} J_{\alpha_1-\gamma h_1, C_r}^2(x)} - 1 \right) \right] d^2x. \end{aligned}$$

where we have denoted

$$\delta J_{\alpha_1-\gamma h_1}^i(zx) = \int_{\mathbb{C}} \frac{|y-zx|^{-\gamma^2\langle e_1, e_i \rangle} - |y|^{-\gamma^2\langle e_1, e_i \rangle}}{|y|^{\gamma\langle \alpha_1-\gamma h_1, e_i \rangle}} \tilde{F}_i(y) M^{\gamma e_i}(d^2y), \quad \tilde{F}_i(y) = F_i(y) |y|_+^{\gamma^2\langle e_1, e_i \rangle}.$$

The second remainder term $\mathfrak{R}(z)$ is a lower order term in the asymptotic studied. To be more specific it is a $o\left(|z|^{\gamma\langle Q-\alpha_1, \rho \rangle}\right)$ as soon as $\langle \alpha_1, e_2 \rangle$ is close enough to q . To see why, we first show that terms of the form

$$\begin{aligned} &\int_{\mathbb{C}} \frac{|x-z|^{\gamma^2} - |x|^{\gamma^2}}{|x|^{\gamma\langle \alpha_1, e_1 \rangle}} \mathbb{E} \left[\delta J_{\alpha_1-\gamma h_1}^1(x) \prod_{i=1}^2 e^{-e^{\gamma\langle c, e_i \rangle} I_{\alpha_1-\gamma h_2}^i} \right] d^2x \quad \text{and} \\ &\int_{\mathbb{C}} \frac{|x-z|^{\gamma^2} - |x|^{\gamma^2}}{|x|^{\gamma\langle \alpha_1, e_1 \rangle}} \mathbb{E} \left[\delta J(z, x) \prod_{i=1}^2 e^{-e^{\gamma\langle c, e_i \rangle} I_{\alpha_1-\gamma h_2}^i} \right] d^2x \end{aligned}$$

are $o\left(|z|^{\gamma\langle Q-\alpha_1, \rho \rangle}\right)$. To this end let us write that

$$\begin{aligned} \mathbb{E} \left[\delta J_{\alpha_1-\gamma h_1}^1(x) \prod_{i=1}^2 e^{-e^{\gamma\langle c, e_i \rangle} I_{\alpha_1-\gamma h_2}^i} \right] &= \int_{\mathbb{C}} \frac{|y-x|^{-2\gamma^2} - |y|^{-2\gamma^2}}{|y|^{\gamma\langle \alpha_1-\gamma h_1, e_1 \rangle}} \mathbb{E} \left[\prod_{i=1}^2 e^{-e^{\gamma\langle c, e_i \rangle} \tilde{I}_{\alpha_1-\gamma h_2}^i(y)} \right] \tilde{F}_1(y) d^2y \\ \text{where } \tilde{I}_{\alpha_1-\gamma h_2}^i(y) &= \int_{\mathbb{C}} \frac{|w-y|^{-\gamma^2\langle e_1, e_i \rangle}}{|w|^{\gamma\langle \alpha_1-\gamma h_2, e_i \rangle}} \tilde{F}_i(w) |w|_+^{\gamma^2\langle e_1, e_i \rangle} M^{\gamma e_i}(d^2w). \end{aligned}$$

Here we need to distinguish between two cases according to the value of $\langle \alpha_1, e_1 \rangle$. In the case where $\xi := \langle \alpha_1 - \gamma h_2 + \gamma e_1 - Q, e_1 \rangle$ is negative, then the integrals $\tilde{I}_{\alpha_1-\gamma h_2}^i(0)$ do make sense. However if $\xi \geq 0$ then the integral $\tilde{I}_{\alpha_1-\gamma h_2}^1(0)$ is not well-defined. Still we can proceed along the same lines as in the proof of the fusion asymptotics in Lemma 4.4.2 to see that in that case the leading term in the $y \rightarrow 0$ asymptotic will be given by

$$\mathbb{E} \left[\prod_{i=1}^2 e^{-e^{\gamma\langle c, e_i \rangle} J_{\alpha_1-\gamma h_1}^i(y)} \right] = o\left(|y|^{\frac{\xi^2}{4} - \eta}\right)$$

for any positive η . As a consequence we see that as $z \rightarrow 0$,

$$\mathbb{E} \left[\delta J_{\alpha_1-\gamma h_1}^1(zx) \prod_{i=1}^2 e^{-e^{\gamma\langle c, e_i \rangle} I_{\alpha_1-\gamma h_2}^i} \right] = o\left(|zx|^{2-\gamma^2-\gamma\langle \alpha_1, e_1 \rangle + \frac{\xi^2}{4} \mathbb{1}_{\xi \geq 0} - \eta}\right).$$

Therefore we end up with the bound

$$\int_{\mathbb{C}} \frac{|x-z|^{\gamma^2} - |x|^{\gamma^2}}{|x|^{\gamma\langle\alpha_1, e_1\rangle}} \mathbb{E} \left[\left(e^{-e^{\gamma\langle c, e_1\rangle} \delta J_{\alpha_1 - \gamma h_1}^1(x)} - 1 \right) \prod_{i=1}^2 e^{-e^{\gamma\langle c, e_i\rangle} I_{\alpha_1 - \gamma h_2}^i} \right] d^2x \\ \leq C |z|^{\gamma\langle Q - \alpha_1, e_1\rangle + \frac{\langle\alpha_1 - \gamma h_2 - Q, e_1\rangle^2}{4}}.$$

This in particular implies that if we take $\langle\alpha_1, e_2\rangle$ arbitrarily close to q then this term is a $o\left(|z|^{\gamma\langle Q - \alpha_1, \rho\rangle}\right)$ (since $\langle Q - \alpha_1, \rho\rangle = \langle Q - \alpha_1, e_1\rangle + \langle Q - \alpha_1, e_2\rangle$), hence a lower order term in the asymptotic studied. The same applies for

$$\int_{\mathbb{C}} \frac{|x-z|^{\gamma^2} - |x|^{\gamma^2}}{|x|^{\gamma\langle\alpha_1, e_1\rangle}} \mathbb{E} \left[\left(\int_0^1 e^{-te^{\gamma\langle c, e_1\rangle} \delta J(z, x)} dt - 1 \right) \prod_{i=1}^2 e^{-e^{\gamma\langle c, e_i\rangle} I_{\alpha_1 - \gamma h_2}^i} \right] d^2x,$$

which is also seen to be a $o\left(|z|^{\gamma\langle Q - \alpha_1, \rho\rangle}\right)$. Likewise, fusion asymptotics imply that

$$\mathbb{E} \left[M^{\gamma e_2}(d^2x) e^{-\sum_{i=1}^2 e^{\gamma\langle c, e_i\rangle} I_{\alpha_1 - \gamma h_2}^i} \right] = o \left(|x|^{\frac{(\langle\alpha_1, e_2\rangle - \frac{2}{\gamma})^2}{4} - \eta} \right) d^2x$$

so that

$$\mathbb{E} \left[\delta J_{\alpha_1 - \gamma h_1, C_r}^2(zx) e^{-\sum_{i=1}^2 e^{\gamma\langle c, e_i\rangle} I_{\alpha_1 - \gamma h_2}^i} \right] = o \left(|zx|^{\gamma^2} e^{-r \left(2 - \gamma\langle\alpha_1, e_2\rangle + \frac{(\langle\alpha_1, e_2\rangle - \frac{2}{\gamma})^2}{4} - \eta \right)} \right).$$

As a consequence the associated remainder term is at most a

$$o \left(|z|^{\gamma\langle Q - \alpha_1, e_1\rangle + \gamma^2 + (1+\varepsilon) \left(2 - \gamma\langle\alpha_1, e_2\rangle + \frac{(\langle\alpha_1, e_2\rangle - \frac{2}{\gamma})^2}{4} - \eta \right)} \right).$$

This exponent can be rewritten more conveniently as

$$\gamma\langle Q - \alpha_1, \rho\rangle + \varepsilon(2 - \gamma\langle\alpha_1, e_2\rangle) + (1 + \varepsilon) \left(\frac{(\langle\alpha_1, e_2\rangle - \frac{2}{\gamma})^2}{4} - \eta \right)$$

which is strictly larger than $\gamma\langle Q - \alpha_1, \rho\rangle$ provided that ε is chosen small enough. Eventually collecting up terms we see that $\Re(z) = o\left(|z|^{\gamma\langle Q - \alpha_1, \rho\rangle}\right)$ as expected.

Let us now turn to $\delta\Phi^1(z)$. We wish to prove that

$$\delta\Phi^1(z) = |z|^{\gamma\langle Q - \alpha_1, \rho\rangle} B^{(1)}(\hat{s}_2\alpha_1, \gamma) R_{s_2}(\alpha_1) \Phi_{\hat{s}_2\alpha_1 - \gamma h_2}(c) + l.o.t. \quad (7.2.8)$$

For this we use the radial-angular decomposition to rewrite $J_{\alpha_1 - \gamma h_1, B_r}^2(zx)$ as

$$e^{\gamma\langle \mathbf{X}_r(0) + (\alpha_1 - \frac{\gamma}{1+\varepsilon} h_2 - Q)r, e_2\rangle} \int_0^{+\infty} e^{\gamma B_t^\nu} \int_0^{2\pi} \left| 1 - |zx|^\varepsilon e^{-t+i\theta} \right|^{\gamma^2} \tilde{F}_2(e^{-t-r+i\theta}) M^{\gamma e_2}(dt + r, d\theta) \\ = e^{\gamma\langle \mathbf{X}_r(0) + (\alpha_1 - \frac{\gamma}{1+\varepsilon} h_2 - Q)r + \mathbf{M}, e_2\rangle} \int_0^{+\infty} e^{\gamma B_t^\nu} \int_0^{2\pi} \left| 1 - |zx|^\varepsilon e^{-t+i\theta} \right|^{\gamma^2} \tilde{F}_2(e^{-t-r+i\theta}) M^{\gamma e_2}(dt + r, d\theta)$$

using Williams path decomposition, and where $\nu = \langle \alpha_1, e_2 \rangle$. As a consequence we can follow the same lines as in the proof of Proposition 6.4.2 to see that

$$\begin{aligned} \delta\Phi^1(z) &= -e^{\langle s+\gamma e_1, c \rangle} |z|^{\gamma\langle Q-\alpha_1, e_1 \rangle} \int_{\mathbb{C}} \frac{|x-1|^{\gamma^2} - |x|^{\gamma^2}}{|x|^{\gamma\langle \alpha_1, e_1 \rangle}} d^2x \times \\ &\mathbb{E} \left[\int_{\langle c+\lambda_r-\frac{\gamma h_2}{1+\varepsilon}r, e_2 \rangle}^{+\infty} (-\nu) e^{\nu\langle \mathbf{M}-\lambda_r-c-\frac{\gamma h_2}{1+\varepsilon}r, e_2 \rangle} d\langle \mathbf{M}, e_2 \rangle \left(e^{-e^{\gamma\langle \mathbf{M}, e_2 \rangle} J_r} - 1 \right) e^{-\sum_{i=1}^2 e^{\gamma\langle c, e_i \rangle} I_{\alpha_1-\gamma h_2}^i} \right] \\ &= -e^{\langle s+\gamma e_1+\hat{s}_2\alpha_1-\alpha_1, c \rangle} |z|^{\gamma\langle Q-\alpha_1, \rho \rangle} \int_{\mathbb{C}} \frac{|x-1|^{\gamma^2} - |x|^{\gamma^2}}{|x|^{\gamma\langle \alpha_1, e_1 \rangle + \gamma\langle \alpha_1-Q, e_2 \rangle}} d^2x \times \\ &\mathbb{E} \left[e^{-\nu\langle \lambda_r, e_2 \rangle} \int_{\langle c+\lambda_r-\frac{\gamma h_2}{1+\varepsilon}r, e_2 \rangle}^{+\infty} (-\nu) e^{\nu\langle \mathbf{M}, e_2 \rangle} d\langle \mathbf{M}, e_2 \rangle \left(e^{-e^{\gamma\langle \mathbf{M}, e_2 \rangle} J_r} - 1 \right) e^{-\sum_{i=1}^2 e^{\gamma\langle c, e_i \rangle} I_{\alpha_1-\gamma h_2}^i} \right] \end{aligned}$$

where we have set $\lambda_r := \mathbf{X}_r(0) + (\alpha_1 - Q)r$ and

$$J_r = \int_0^{+\infty} e^{\gamma \mathcal{B}_t^\nu} \int_0^{2\pi} \left| 1 - |zx|^\varepsilon e^{-t+i\theta} \right|^{\gamma^2} \tilde{F}_2(e^{-t-r+i\theta}) M^{\gamma e_2}(dt + r, d\theta)$$

with $\mathcal{B}$ started from $\langle c + \lambda_r - \frac{\gamma h_2}{1+\varepsilon}r, e_2 \rangle - \langle \mathbf{M}, e_2 \rangle$. The exponential term $e^{-\nu\langle \lambda_r, e_2 \rangle}$ is a Girsanov transform whose effect is to shift the law of $\mathbf{X}$ by $G_r(\cdot, 0)\langle Q - \alpha_k, e_2 \rangle e_2$, which as $r \rightarrow 0$ converges towards $G(\cdot, 0)(\hat{s}_2\alpha_1 - \alpha_1)$. Therefore proceeding in the same way as in the proof of Proposition 6.4.2 we see that

$$\delta\Phi^1(z) = |z|^{\gamma\langle Q-\alpha_1, e_1 \rangle} \int_{\mathbb{C}} \frac{|x|^{\gamma^2} - |x-1|^{\gamma^2}}{|x|^{\gamma\langle \hat{s}_2\alpha_1, e_1 \rangle}} d^2x R_{s_2}(\alpha_1) e^{\langle s+\gamma e_1, c \rangle} \mathbb{E} \left[e^{-\sum_{i=1}^2 e^{\gamma\langle c, e_i \rangle} I_{\hat{s}_2\alpha_1-\gamma h_2}^i} \right] + l.o.t.$$

which is nothing but Equation (7.2.8), where the integral over $\mathbb{C}$ is well-defined at least for $\langle \alpha_1 - Q, e_2 \rangle$ close enough to 0 and evaluated using Lemma 7.2.4 below, thanks to which it is found to be equal to $B_\gamma^{(1)}(\hat{s}_2\alpha_1, \gamma)$.

Now from the explicit expressions of the quantities $B^{(i)}$ and R_{s_2} one can deduce that

$$B_\gamma^{(1)}(\hat{s}_2\alpha_1, \gamma) R_{s_2}(\alpha_1) = B_\gamma^{(2)}(\alpha_1, \gamma) R_{s_2}(\alpha_1 - \gamma h_3).$$

As a consequence we can infer the following expansion for $\Phi_{\alpha_1, -\gamma h_1}(z; c)$ as $z \rightarrow 0$:

$$\begin{aligned} \Phi_{\alpha_1, -\gamma h_1}(z; c) &= \Phi_{\alpha_1-\gamma h_1}(c) + |z|^{2(1-B_1)} B_\gamma^{(1)}(\alpha_1, \gamma) \Phi_{\alpha_1-\gamma h_2}(c) \\ &+ |z|^{2(1-B_2)} B_\gamma^{(2)}(\alpha_1, \gamma) R_{s_2}(\alpha_1 - \gamma h_3) \Phi_{\hat{s}_2(\alpha_1-\gamma h_3)}(c) + l.o.t. \end{aligned} \quad (7.2.9)$$

The case $\gamma > 1$. We proceed in the same way for $\gamma > 1$. Our assumptions on the weight α_1 are then that $\gamma < \langle \alpha_1, e_1 \rangle < \frac{2}{\gamma}$ and $q - \varepsilon < \langle \alpha_1, e_2 \rangle < q$. The only difference with the above case is that we write $\delta\Phi(z)$ as

$$\begin{aligned} &\frac{\gamma^2}{2} e^{\langle s+\gamma e_1, c \rangle} \int_{\mathbb{C}} \frac{\frac{z}{x} + \frac{\bar{z}}{\bar{x}}}{|x|^{\gamma\langle \alpha_1-\gamma h_1, e_1 \rangle}} F_1(x) \mathbb{E} \left[\int_0^1 e^{-te^{\gamma\langle c, e_1 \rangle} \delta J(z, x)} dt \prod_{i=1}^2 e^{-e^{\gamma\langle c, e_i \rangle} J_{\alpha_1-\gamma h_1}^i(x)} \right] d^2x \\ &- |z|^{\gamma\langle Q-\alpha_1, e_1 \rangle} e^{\langle s+\gamma e_1, c \rangle} \int_{\mathbb{C}} \frac{\left| 1 - \frac{1}{x} \right|^{\gamma^2} - 1 - \frac{\gamma^2}{2} \left(\frac{1}{x} + \frac{1}{\bar{x}} \right)}{|x|^{\gamma\langle \alpha_1-\gamma h_1, e_1 \rangle}} F_1(zx) \times \\ &\mathbb{E} \left[\int_0^1 e^{-te^{\gamma\langle c, e_1 \rangle} \delta J(z, zx)} dt \prod_{i=1}^2 e^{-e^{\gamma\langle c, e_i \rangle} J_{\alpha_1-\gamma h_1}^i(zx)} \right] d^2x. \end{aligned}$$

Then we can proceed in the very same fashion as above to see that the term on the last two lines is equal to

$$|z|^{2(1-B_1)} B_\gamma^{(1)}(\alpha_1, \gamma) \Phi_{\alpha_1-\gamma h_2}(\mathbf{c}) + |z|^{2(1-B_2)} B_\gamma^{(2)}(\alpha_1, \gamma) R_{s_2}(\alpha_1 - \gamma h_3) \Phi_{\hat{s}_2(\alpha_1-\gamma h_3)}(\mathbf{c}) + l.o.t.$$

The only difference lies in the evaluation of the integrals that arise, but they are still found to be given by $B_\gamma^{(1)}(\alpha_1, \gamma)$ in agreement with Lemma 7.2.4. This means that all we have to prove is that the term on the first line is of the form $Cz + \bar{C}\bar{z} + o(|z|^{\gamma(Q-\alpha_1, \rho)})$. To see why this is indeed the case note that

$$C := e^{\langle \mathbf{s} + \gamma e_1, \mathbf{c} \rangle} \int_{\mathbb{C}} \frac{F_1(x)}{x |x|^{\gamma \langle \alpha_1 - \gamma h_1, e_1 \rangle}} \mathbb{E} \left[\prod_{i=1}^2 e^{-e^{\gamma \langle \mathbf{c}, e_i \rangle} J_{\alpha_1 - \gamma h_1}^i(x)} \right] d^2 x$$

is well-defined since the singularity around $x = 0$ is integrable thanks to the fact that $\langle \alpha_1, e_1 \rangle < \frac{2}{\gamma} < \gamma + \frac{1}{\gamma}$. As a consequence the corresponding term in the expansion of $\delta\Phi(z)$ is given by

$$\begin{aligned} & \frac{\gamma^2}{2} (Cz + \bar{C}\bar{z}) \\ & + e^{\langle \mathbf{s} + \gamma e_1, \mathbf{c} \rangle} \int_{\mathbb{C}} \frac{\frac{\gamma^2}{2} \left(\frac{z}{x} + \frac{\bar{z}}{\bar{x}} \right)}{|x|^{\gamma \langle \alpha_1 - \gamma h_1, e_1 \rangle}} F_1(x) \mathbb{E} \left[\int_0^1 \left(e^{-te^{\gamma \langle \mathbf{c}, e_1 \rangle} \delta J(z, x)} - 1 \right) dt \prod_{i=1}^2 e^{-e^{\gamma \langle \mathbf{c}, e_i \rangle} J_{\alpha_1 - \gamma h_1}^i(x)} \right] d^2 x. \end{aligned}$$

To conclude it remains to check that

$$\int_{\mathbb{C}} \frac{\frac{\gamma^2}{2} \left(\frac{z}{x} + \frac{\bar{z}}{\bar{x}} \right)}{|x|^{\gamma \langle \alpha_1 - \gamma h_1, e_1 \rangle}} F_1(x) \mathbb{E} \left[\delta J(z, x) \prod_{i=1}^2 e^{-e^{\gamma \langle \mathbf{c}, e_i \rangle} J_{\alpha_1 - \gamma h_1}^i(x)} \right] d^2 x$$

is a lower order term, which can be seen via the same reasoning:

$$\delta J(z, x) \sim \int_{\mathbb{C}} \frac{\frac{\gamma^2}{2} \left(\frac{z}{y} + \frac{\bar{z}}{\bar{y}} \right)}{|y|^{\gamma \langle \alpha_1 - \gamma h_1, e_1 \rangle} |x - y|^{2\gamma^2}} \tilde{F}_1(y) M^{\gamma e_1}(d^2 y)$$

with the integral

$$\int_{\mathbb{D} \times \mathbb{D}} \frac{1}{x |x|^{\gamma \langle \alpha_1 - \gamma h_1, e_1 \rangle} y |y|^{\gamma \langle \alpha_1 - \gamma h_1, e_1 \rangle} |x - y|^{2\gamma^2}} d^2 y$$

absolutely integrable. This implies that this term is at most of order z^2 , hence is a $o(|z|^{\gamma(Q-\alpha_1, \rho)})$ as soon as $\langle \alpha_1, e_2 \rangle$ is close enough to q since $\gamma^2 < 2$. This shows that under the assumptions that $\gamma > 1$ with $\gamma < \langle \alpha_1, e_1 \rangle < \frac{2}{\gamma}$ and $q - \varepsilon < \langle \alpha_1, e_2 \rangle < q$ then

$$\begin{aligned} \Phi_{\alpha_1, -\gamma h_1}(z; \mathbf{c}) &= \Phi_{\alpha_1 - \gamma h_1}(\mathbf{c}) + Cz + \bar{C}\bar{z} + |z|^{2(1-B_1)} B_\gamma^{(1)}(\alpha_1, \gamma) \Phi_{\alpha_1 - \gamma h_2}(\mathbf{c}) \\ &+ |z|^{2(1-B_2)} B_\gamma^{(2)}(\alpha_1, \gamma) R_{s_2}(\alpha_1 - \gamma h_3) \Phi_{\hat{s}_2(\alpha_1 - \gamma h_3)}(\mathbf{c}) + l.o.t. \end{aligned} \quad (7.2.10)$$

7.2.2.2 The reflection terms

Having described the asymptotic expansion of $\Phi_{\alpha_1, -\gamma h_1}(z; \mathbf{c})$, we now turn to the other terms in the expression of $\mathcal{H}$.

To start with, let us focus on the term $\mathbb{E} [\mathbf{R}_\alpha^2(c)]$. The dependence in z of this term is completely explicit and we have for some constant C :

$$\begin{aligned} & |z|^{-\gamma\langle h_1, \alpha_1 \rangle} |z - 1|^{-\gamma\langle h_1, \alpha_2^* \rangle} \mathbb{E} [\mathbf{R}_\alpha^2(c)] = \\ & \mathbb{1}_{\langle c, e_1 \rangle < 0} \mathbb{E} \left[e^{-\mu_2 e^{\gamma\langle c, e_2 \rangle} M^{\gamma e_2}(\mathbb{C})} \left(V_{\alpha_1 - \gamma h_1}(0) + |z|^{\gamma(Q - \alpha_1, e_1)} R_{s_1}(\alpha_1) V_{\hat{s}_1 \alpha_1 - \gamma h_1}(0) \right) \mathcal{R}_{Id, s_1} V_{\alpha_2^*}(1) V_{\alpha_3}(\infty) \right] \\ & - \mathbb{1}_{\max_i \langle c, e_i \rangle < 0} \mathbb{E} \left[(V_{\alpha_1 - \gamma h_1}(0) + R_{s_2}(\alpha_1) V_{\hat{s}_2 \alpha_1 - \gamma h_1}(0)) \mathcal{R}_{Id, s_1, s_2, s_2 s_1} V_{\alpha_2^*}(1) V_{\alpha_3}(\infty) \right] \\ & - \mathbb{1}_{\max_i \langle c, e_i \rangle < 0} |z|^{\gamma(Q - \alpha_1, e_1)} \times \\ & \quad \mathbb{E} \left[(R_{s_1}(\alpha_1) V_{\hat{s}_1 \alpha_1 - \gamma h_1}(0) + R_{s_2 s_1}(\alpha_1) V_{\hat{s}_2 \hat{s}_1 \alpha_1 - \gamma h_1}(0)) \mathcal{R}_{Id, s_1, s_2, s_2 s_1} V_{\alpha_2^*}(1) V_{\alpha_3}(\infty) \right] \\ & + Cz + \bar{C}\bar{z} + l.o.t. \end{aligned}$$

Now we can use the fact that $R_{s_2}(\alpha_1) = R_{s_2}(\alpha_1 - \gamma h_1)$ (which follows from $\langle h_1, e_2 \rangle = 0$) as well as the equality

$$R_{s_1}(\alpha_1) = B_\gamma^{(1)}(\alpha_1, \gamma) R_{s_1}(\alpha_1 - \gamma h_2)$$

to rewrite the latter as

$$\begin{aligned} & \mathbb{1}_{\langle c, e_1 \rangle < 0} \mathbb{E} \left[\left(e^{-\mu_2 e^{\gamma\langle c, e_2 \rangle} M^{\gamma e_2}(\mathbb{C})} - \mathbb{1}_{\langle c, e_2 \rangle < 0} \mathcal{R}_{Id, s_2} \right) V_{\alpha_1 - \gamma h_1}(0) \mathcal{R}_{Id, s_1} V_{\alpha_2^*}(1) V_{\alpha_3}(\infty) \right] \\ & + |z|^{\gamma(Q - \alpha_1, e_1)} B_\gamma^{(1)}(\alpha_1, \gamma) \times \\ & \mathbb{1}_{\langle c, e_1 \rangle < 0} \mathbb{E} \left[\left(e^{-\mu_2 e^{\gamma\langle c, e_2 \rangle} M^{\gamma e_2}(\mathbb{C})} - \mathbb{1}_{\langle c, e_2 \rangle < 0} \mathcal{R}_{Id, s_2} \right) R_{s_1}(\alpha_1 - \gamma h_2) V_{\hat{s}_1(\alpha_1 - \gamma h_2)}(0) \mathcal{R}_{Id, s_1} V_{\alpha_2^*}(1) V_{\alpha_3}(\infty) \right] \\ & + Cz + \bar{C}\bar{z} + l.o.t. \end{aligned}$$

Therefore we see that

$$\begin{aligned} \mathbb{E} [\mathbf{R}_\alpha^2(c)] &= \mathbb{E} [\mathbf{R}_{\alpha - \gamma h_1}^2(c)] + |z|^{\gamma(Q - \alpha_1, e_1)} B_\gamma^{(1)}(\alpha_1, \gamma) \mathbb{E} [\mathbf{R}_{\alpha - \gamma h_2}^2(c)] + Cz + \bar{C}\bar{z} + l.o.t. \\ & - \mathbb{1}_{\langle c, e_1 \rangle < 0} \mathbb{E} \left[\left(e^{-\mu_2 e^{\gamma\langle c, e_2 \rangle} M^{\gamma e_2}(\mathbb{C})} - \mathbb{1}_{\langle c, e_2 \rangle < 0} \mathcal{R}_{Id, s_2} \right) \mathcal{R}_{s_1} V_{\alpha_1 - \gamma h_1}(0) \mathcal{R}_{Id, s_1} V_{\alpha_2^*}(1) V_{\alpha_3}(\infty) \right] \\ & - |z|^{\gamma(Q - \alpha_1, e_1)} B_\gamma^{(1)}(\alpha_1, \gamma) \mathbb{1}_{\langle c, e_1 \rangle < 0} \times \\ & \quad \mathbb{E} \left[\left(e^{-\mu_2 e^{\gamma\langle c, e_2 \rangle} M^{\gamma e_2}(\mathbb{C})} - \mathbb{1}_{\langle c, e_2 \rangle < 0} \mathcal{R}_{Id, s_2} \right) V_{\alpha_1 - \gamma h_2}(0) \mathcal{R}_{Id, s_1} V_{\alpha_2^*}(1) V_{\alpha_3}(\infty) \right]. \end{aligned}$$

We now want to discard the two terms that appear between the second and fourth lines, and for this purpose we want to show that they are integrable in the variable c . To see why this is true we use that for any $w : \{0, 1, \infty\} \rightarrow \{Id, s_1, s_2, s_2 s_1\}$ such that $w(0) \in \{Id, s_2\}$:

$$\langle \hat{w}(0) \hat{s}_1(\alpha_1 - \gamma h_1) + \hat{w}(1) \alpha_2^* + \hat{w}(\infty) \alpha_3 - 2Q, \omega_1 \rangle \geq \langle \hat{s}_1(\alpha_1 - \gamma h_1) + \alpha_2^* + \alpha_3 - 2Q, \omega_1 \rangle,$$

the latter being given by $\langle s, \omega_1 \rangle + \langle Q - \alpha_1, e_1 \rangle$. Since $\langle \alpha_1, e_1 \rangle < \frac{2}{\gamma}$ and $\langle s, \omega_1 \rangle > -\gamma$ we see that

$$\langle \hat{w}(0) \hat{s}_1(\alpha_1 - \gamma h_1) + \hat{w}(1) \alpha_2^* + \hat{w}(\infty) \alpha_3 - 2Q, \omega_1 \rangle > 0.$$

As a consequence the term in the second line is integrable over $\mathbb{R}$ with respect to $\langle c, e_1 \rangle$. The same reasoning shows that the term on the third and fourth lines is integrable too, since $\langle \alpha_1 - \gamma h_2 + \alpha_2^* + \alpha_3 - 2Q, \omega_1 \rangle = \langle s, \omega_1 \rangle + \gamma$. Now by construction of the analytic continuation of the correlation functions we see that this term will be compensated by other terms in $\mathfrak{R}_{\alpha_1, -\gamma h_1}^1(0; c)$. More precisely, in a similar fashion as

above we can expand

$$\begin{aligned}
E_{\alpha}^2(c) &= E_{\alpha_1 - \gamma h_1}^2(c) + |z|^{\gamma(Q - \alpha_1, e_1)} B_{\gamma}^{(1)}(\alpha_1, \gamma) E_{\alpha - \gamma h_2}^2(c) + Cz + \bar{C}\bar{z} + l.o.t. \\
&- \sum_{w: \{1, \infty\} \rightarrow \{Id, s_1\}} \frac{e^{-\langle s'(w), \omega_1 \rangle \langle c, e_1 \rangle}}{\langle s'(w), \omega_1 \rangle} \mathbb{E} \left[\left(e^{-\mu_2 e^{\gamma \langle c, e_2 \rangle} M^{\gamma e_2}(\mathbb{C})} - \mathbb{1}_{\langle c, e_2 \rangle < 0} \mathcal{R}_{Id, s_2} \right) \right. \\
&\quad \left. R_{s_1}(\alpha_1 - \gamma h_1) V_{\hat{s}_1(\alpha_1 - \gamma h_1)}(0) \mathcal{R}_{Id, s_1} R_{w(1)}(\alpha_2^*) V_{\hat{w}(1)\alpha_2^*}(1) R_{w(\infty)}(\alpha_3) V_{\hat{w}(\infty)\alpha_3}(\infty) \right] \\
&- |z|^{\gamma(Q - \alpha_1, e_1)} B_{\gamma}^{(1)}(\alpha_1, \gamma) \mathbb{1}_{\langle c, e_1 \rangle < 0} \sum_{w: \{1, \infty\} \rightarrow \{Id, s_1\}} \frac{e^{-\langle \tilde{s}(w), \omega_1 \rangle \langle c, e_1 \rangle}}{\langle \tilde{s}(w), \omega_1 \rangle} \times \\
&\quad \mathbb{E} \left[\left(e^{-\mu_2 e^{\gamma \langle c, e_2 \rangle} M^{\gamma e_2}(\mathbb{C})} - \mathbb{1}_{\langle c, e_2 \rangle < 0} \mathcal{R}_{Id, s_2} \right) V_{\alpha_1 - \gamma h_2}(0) \mathcal{R}_{Id, s_1} V_{\alpha_2^*}(1) V_{\alpha_3}(\infty) \right]
\end{aligned}$$

where $s'(w) = \hat{s}_1(\alpha_1 - \gamma h_1) + \hat{w}(1)\alpha_2^* + \hat{w}(\infty)\alpha_3 - 2Q$ and $\tilde{s}(w) = \alpha_1 - \gamma h_2 + \hat{w}(1)\alpha_2^* + \hat{w}(\infty)\alpha_3 - 2Q$. Explicit computations show that the terms on the last three lines are given by the integral with respect to $\langle c, e_1 \rangle$ of the remainder terms from $\mathbb{E} [\mathbf{R}_{\alpha}^2(c)]$, so that both terms will actually compensate and vanish in the expansion.

We can proceed in the same way for the other terms. For $\mathbb{E} [\mathbf{R}_{\alpha}^1(c)]$ we have

$$\begin{aligned}
&e^{-2\langle Q, c \rangle} \mathbb{E} [\mathcal{R}_{Id, s_1} \mathcal{R}_{Id, s_2} V_{\alpha_1}(0) V_{-\gamma h_1}(z) V_{\alpha_2^*}(1) V_{\alpha_3}(\infty)] \\
&= \mathbb{1}_{\max_i \langle c, e_i \rangle < 0} \mathbb{E} [(V_{\alpha_1 - \gamma h_1}(0) + R_{s_2}(\alpha_1) V_{\hat{s}_2 \alpha_1 - \gamma h_1}(0)) \mathcal{R}_{Id, s_1, s_2, s_1 s_2} V_{\alpha_2^*}(1) V_{\alpha_3}(\infty)] \\
&+ \mathbb{1}_{\max_i \langle c, e_i \rangle < 0} |z|^{\gamma(Q - \alpha_1, e_1)} \mathbb{E} [(R_{s_1}(\alpha_1) V_{\hat{s}_1 \alpha_1 - \gamma h_1}(0)) \mathcal{R}_{Id, s_1, s_2, s_1 s_2} V_{\alpha_2^*}(1) V_{\alpha_3}(\infty)] \\
&+ \mathbb{1}_{\max_i \langle c, e_i \rangle < 0} |z|^{\gamma(Q - \alpha_1, \rho)} \mathbb{E} [R_{s_1 s_2}(\alpha_1) V_{\hat{s}_1 \hat{s}_2 \alpha_1 - \gamma h_1}(0) \mathcal{R}_{Id, s_1, s_2, s_1 s_2} V_{\alpha_2^*}(1) V_{\alpha_3}(\infty)] \\
&+ Cz + \bar{C}\bar{z} + l.o.t.
\end{aligned}$$

while by the same reasoning as the one developed for $\Phi_{\alpha_1, \alpha}(z; c)$, we have the equality

$$\begin{aligned}
&\mathbb{E} [\mathcal{R}_{Id, s_2} V_{\alpha_1}(0) V_{-\gamma h_1}(z) V_{\alpha_2^*}(1) V_{\alpha_3}(\infty) e^{-\mu_1 e^{\gamma \langle c, e_1 \rangle} M^{\gamma e_1}(\mathbb{C})}] \\
&= \mathbb{E} [V_{\alpha_1 - \gamma h_1}(0) \mathcal{R}_{Id, s_2} V_{\alpha_2^*}(1) V_{\alpha_3}(\infty) e^{-\mu_1 e^{\gamma \langle c, e_1 \rangle} M^{\gamma e_1}(\mathbb{C})}] \\
&+ |z|^{\gamma(Q - \alpha_1, e_1)} B_{\gamma}^{(1)}(\alpha_1, \gamma) \mathbb{E} [V_{\alpha_1 - \gamma h_2}(0) \mathcal{R}_{Id, s_2} V_{\alpha_2^*}(1) V_{\alpha_3}(\infty) e^{-\mu_1 e^{\gamma \langle c, e_1 \rangle} M^{\gamma e_1}(\mathbb{C})}] \\
&+ \mathbb{E} [R_{s_2}(\alpha_1) V_{\hat{s}_2 \alpha_1 - \gamma h_1}(0) \mathcal{R}_{Id, s_2} V_{\alpha_2^*}(1) V_{\alpha_3}(\infty) e^{-\mu_1 e^{\gamma \langle c, e_1 \rangle} M^{\gamma e_1}(\mathbb{C})}] \\
&+ |z|^{\gamma(Q - \hat{s}_2 \alpha_1, e_1)} B_{\gamma}^{(1)}(\hat{s}_2 \alpha_1, \gamma) \mathbb{E} [R_{s_2}(\alpha_1) V_{\hat{s}_2 \alpha_1 - \gamma h_2}(0) \mathcal{R}_{Id, s_2} V_{\alpha_2^*}(1) V_{\alpha_3}(\infty) e^{-\mu_1 e^{\gamma \langle c, e_1 \rangle} M^{\gamma e_1}(\mathbb{C})}] \\
&+ Cz + \bar{C}\bar{z} + o(|z|^{\gamma(Q - \alpha_1, \rho)}).
\end{aligned}$$

Since $\langle \hat{s}_2 \alpha_1 - Q, e_1 \rangle = \langle \alpha_1 - Q, \rho \rangle$ we infer from the above reasoning that

$$\begin{aligned}
&\mathbb{E} [\mathcal{R}_{Id, s_2} V_{\alpha_1 - \gamma h_1}(0) V_{\alpha_2^*}(1) V_{\alpha_3}(\infty) e^{-\mu_1 e^{\gamma \langle c, e_1 \rangle} M^{\gamma e_1}(\mathbb{C})}] \\
&+ |z|^{\gamma(Q - \alpha_1, e_1)} B_{\gamma}^{(1)}(\alpha_1, \gamma) \mathbb{E} [V_{\alpha_1 - \gamma h_2}(0) \mathcal{R}_{Id, s_2} V_{\alpha_2^*}(1) V_{\alpha_3}(\infty) e^{-\mu_1 e^{\gamma \langle c, e_1 \rangle} M^{\gamma e_1}(\mathbb{C})}] \\
&+ |z|^{\gamma(Q - \alpha_1, \rho)} B_{\gamma}^{(2)}(\alpha_1, \gamma) \mathbb{E} [R_{s_2}(\alpha_1 - \gamma h_3) V_{\hat{s}_2(\alpha_1 - \gamma h_3)}(0) \mathcal{R}_{Id, s_2} V_{\alpha_2^*}(1) V_{\alpha_3}(\infty) e^{-\mu_1 e^{\gamma \langle c, e_1 \rangle} M^{\gamma e_1}(\mathbb{C})}] \\
&+ Cz + \bar{C}\bar{z} + o(|z|^{\gamma(Q - \alpha_1, \rho)}).
\end{aligned}$$

Put differently we can write that

$$\begin{aligned}
\mathbb{E} [\mathbf{R}_\alpha^1(\mathbf{c})] &= \mathbb{E} [\mathbf{R}_{\alpha_1-\gamma h_1}^1(\mathbf{c})] + |z|^{\gamma(Q-\alpha_1, e_1)} B_\gamma^{(1)}(\alpha_1, \gamma) \mathbb{E} [(\mathbf{R}_{\alpha_1-\gamma h_2}^1(\mathbf{c}))] \\
&\quad + |z|^{\gamma(Q-\alpha_1, \rho)} B_\gamma^{(2)}(\alpha_1, \gamma) R_{s_2}(\alpha_1 - \gamma h_3) \mathbb{E} [\mathbf{R}_{\hat{s}_2(\alpha_1-\gamma h_3)}^1(\mathbf{c})] + Cz + \bar{C}\bar{z} + l.o.t. \\
&+ \mathbb{1}_{\max_i \langle \mathbf{c}, e_i \rangle < 0} \mathbb{E} [\mathcal{R}_{s_1, s_1 s_2} V_{\alpha_1-\gamma h_1}(0) \mathcal{R}_{Id, s_1} V_{\alpha_2^*}(1) V_{\alpha_3}(\infty)] \\
&\quad - |z|^{\gamma(Q-\alpha_1, e_1)} B_\gamma^{(1)}(\alpha_1, \gamma) \mathbb{1}_{\langle \mathbf{c}, e_2 \rangle < 0} \times \\
&\quad \left(\mathbb{E} \left[e^{-\mu_1 e^{\gamma \langle \mathbf{c}, e_1 \rangle} M^{\gamma e_1}(\mathbb{C})} \mathcal{R}_{s_2} V_{\alpha_1-\gamma h_2}(0) \mathcal{R}_{Id, s_2} V_{\alpha_2^*}(1) V_{\alpha_3}(\infty) \right] \right. \\
&\quad \left. - \mathbb{1}_{\langle \mathbf{c}, e_1 \rangle < 0} \mathbb{E} [\mathcal{R}_{Id, s_2, s_1 s_2} V_{\alpha_1-\gamma h_2}(0) \mathcal{R}_{Id, s_1} V_{\alpha_2^*}(1) V_{\alpha_3}(\infty)] \right) \\
&\quad - |z|^{\gamma(Q-\alpha_1, \rho)} B_\gamma^{(2)}(\alpha_1, \gamma) \mathbb{1}_{\langle \mathbf{c}, e_2 \rangle < 0} \times \\
&\quad \left(\mathbb{E} \left[\left(e^{-\mu_1 e^{\gamma \langle \mathbf{c}, e_1 \rangle} M^{\gamma e_1}(\mathbb{C})} - \mathbb{1}_{\langle \mathbf{c}, e_1 \rangle < 0} \mathcal{R}_{Id, s_1} \right) V_{\alpha_1-\gamma h_3}(0) \mathcal{R}_{Id, s_2} V_{\alpha_2^*}(1) V_{\alpha_3}(\infty) \right] \right. \\
&\quad \left. - \mathbb{1}_{\max_i \langle \mathbf{c}, e_i \rangle < 0} \mathbb{E} [\mathcal{R}_{Id, s_1, s_2} V_{\alpha_1-\gamma h_3}(0) \mathcal{R}_{Id, s_2} V_{\alpha_2^*}(1) V_{\alpha_3}(\infty)] \right).
\end{aligned}$$

In analogy with $\mathbb{E} [\mathbf{R}_\alpha^1(\mathbf{c})]$ we see that the remainder terms are integrable with respect to $\mathbf{c}$ and will be compensated by another remainder term from $\mathfrak{R}_\alpha^1(z; \mathbf{c})$ and $\mathfrak{R}_\alpha^{1,2}(z; \mathbf{c})$. Namely the first remainder term is integrable in $\langle \mathbf{c}, e_1 \rangle$ (since $\langle \hat{s}_1(\alpha_1 - \gamma h_1) + \alpha_2^* + \alpha_3 - 2Q, \omega_1 \rangle > 0$) and in $\langle \mathbf{c}, e_2 \rangle$ as soon as $\langle \mathbf{s}, \omega_2 \rangle > 0$ (which we assumed to hold) and will be compensated by a term in $\mathfrak{R}_\alpha^{1,2}(z; \mathbf{c})$. The second remainder term is integrable with respect to $\langle \mathbf{c}, e_2 \rangle$ and $\langle \mathbf{c}, e_1 \rangle$ too and will be compensated by terms in $\mathfrak{R}_\alpha^1(z; \mathbf{c})$ and $\mathfrak{R}_\alpha^{1,2}(z; \mathbf{c})$, while the penultimate one is integrable with respect to $\langle \mathbf{c}, e_2 \rangle$ and will correspond to a term in $\mathfrak{R}_\alpha^1(z; \mathbf{c})$. Eventually the last term is also integrable over $\mathbb{R}^2$ and will be annihilated by a term in $\mathfrak{R}_\alpha^{1,2}(z; \mathbf{c})$.

Finally we can expand $\mathbb{E} [\mathbf{R}_\alpha^{1,2}(\mathbf{c})]$ in the same way, which yields

$$\begin{aligned}
\mathbb{E} [\mathbf{R}_\alpha^{1,2}(\mathbf{c})] &= \mathbb{E} [\mathbf{R}_{\alpha_1-\gamma h_1}^{1,2}(\mathbf{c})] + |z|^{\gamma(Q-\alpha_1, e_1)} B_\gamma^{(1)}(\alpha_1, \gamma) \mathbb{E} [(\mathbf{R}_{\alpha_1-\gamma h_2}^{1,2}(\mathbf{c}))] \\
&\quad + |z|^{\gamma(Q-\alpha_1, \rho)} B_\gamma^{(2)}(\alpha_1, \gamma) R_{s_2}(\alpha_1 - \gamma h_3) \mathbb{E} [\mathbf{R}_{\hat{s}_2(\alpha_1-\gamma h_3)}^{1,2}(\mathbf{c})] + Cz + \bar{C}\bar{z} + l.o.t. \\
&+ \mathbb{1}_{\max_i \langle \mathbf{c}, e_i \rangle < 0} \mathbb{E} [\mathcal{R}_{s_1, s_1 s_2, s_2 s_1, s_1 s_2 s_1} V_{\alpha_1-\gamma h_1}(0) \mathcal{R}_{Id, s_1} V_{\alpha_2^*}(1) V_{\alpha_3}(\infty)] \\
&\quad + |z|^{\gamma(Q-\alpha_1, e_1)} B_\gamma^{(1)}(\alpha_1, \gamma) \mathbb{1}_{\max_i \langle \mathbf{c}, e_i \rangle < 0} \mathbb{E} [\mathcal{R}_{Id, s_2, s_1 s_2, s_1 s_2 s_1} V_{\alpha_1-\gamma h_2}(0) \mathcal{R}_{Id, s_2} V_{\alpha_2^*}(1) V_{\alpha_3}(\infty)] \\
&\quad + |z|^{\gamma(Q-\alpha_1, \rho)} B_\gamma^{(2)}(\alpha_1, \gamma) \mathbb{1}_{\max_i \langle \mathbf{c}, e_i \rangle < 0} \times \\
&\quad R_{s_2}(\alpha_1 - \gamma h_3) \mathbb{E} [\mathcal{R}_{Id, s_2, s_1 s_2, s_1 s_2 s_1} V_{\hat{s}_2(\alpha_1-\gamma h_3)}(0) \mathcal{R}_{Id, s_1} V_{\alpha_2^*}(1) V_{\alpha_3}(\infty)].
\end{aligned}$$

Putting everything together shows that

$$\begin{aligned}
\mathfrak{R}_{\alpha_1, \alpha}(z; \mathbf{c}) &= \mathfrak{R}_{\alpha_1-\gamma h_1}(\mathbf{c}) + |z|^{\gamma(Q-\alpha_1, e_1)} B_\gamma^{(1)}(\alpha_1, \gamma) \mathfrak{R}_{\alpha_1-\gamma h_2}(\mathbf{c}) \\
&\quad + |z|^{\gamma(Q-\alpha_1, \rho)} B_\gamma^{(2)}(\alpha_1, \gamma) R_{s_2}(\alpha_1 - \gamma h_3) \mathfrak{R}_{\hat{s}_2(\alpha_1-\gamma h_3)}(\mathbf{c}) + Cz + \bar{C}\bar{z} + o(|z|^{\gamma(Q-\alpha_1, \rho)}) \\
&\quad + R(z; \mathbf{c}),
\end{aligned}$$

where $R(z; \mathbf{c})$ is not a lower order term but will be compensated by terms appearing in the expansions of $\mathfrak{R}_{\alpha_1, \alpha}^i(z; \mathbf{c})$, $i = 1, 2$, and $\mathfrak{R}_{\alpha_1, \alpha}^{1,2}(z; \mathbf{c})$.

7.2.2.3 Finishing up the proof of Lemma 7.2.3

All in all we see that we can write an expansion of $\mathcal{H}(z)$ which takes the form

$$\begin{aligned} \mathcal{H}(z) = & \int_{\mathbb{R}^2} (\Phi_{\alpha_1 - \gamma h_1}(\mathbf{c}) - \Re_{\alpha_1 - \gamma h_1}(\mathbf{c})) d\mathbf{c} + \sum_{i=1}^2 \int_{\mathbb{R}} (\Phi_{\alpha_1 - \gamma h_1}^1(c_i) - \Re_{\alpha_1 - \gamma h_1}^i(c_i)) dc_i + \Phi_{\alpha_1 - \gamma h_1}^{1,2} \\ & + |z|^{\gamma \langle Q - \alpha_1, e_1 \rangle} B_{\gamma}^{(1)}(\alpha_1, \gamma) \times \\ & \int_{\mathbb{R}^2} (\Phi_{\alpha_1 - \gamma h_2}(\mathbf{c}) - \Re_{\alpha_1 - \gamma h_2}(\mathbf{c})) d\mathbf{c} + \sum_{i=1}^2 \int_{\mathbb{R}} (\Phi_{\alpha_1 - \gamma h_2}^1(c_i) - \Re_{\alpha_1 - \gamma h_2}^i(c_i)) dc_i + \Phi_{\alpha_1 - \gamma h_2}^{1,2} \\ & + |z|^{\gamma \langle Q - \alpha_1, \rho \rangle} B_{\gamma}^{(2)}(\alpha_1, \gamma) R_{s_2}(\alpha_1 - \gamma h_3) \times \\ & \int_{\mathbb{R}^2} (\Phi_{\hat{s}_2(\alpha_1 - \gamma h_3)}(\mathbf{c}) - \Re_{\hat{s}_2(\alpha_1 - \gamma h_3)}(\mathbf{c})) d\mathbf{c} + \sum_{i=1}^2 \int_{\mathbb{R}} (\Phi_{\hat{s}_2(\alpha_1 - \gamma h_3)}^1(c_i) - \Re_{\hat{s}_2(\alpha_1 - \gamma h_3)}^i(c_i)) dc_i + \Phi_{\hat{s}_2(\alpha_1 - \gamma h_3)}^{1,2} \\ & + Cz + \bar{C}\bar{z} + o(|z|^{\gamma \langle Q - \alpha_1, \rho \rangle}) \end{aligned}$$

as soon as the lower order terms which appear in the expansions of $\Phi_{\alpha_1, -\gamma h_1}(z; \mathbf{c})$ and the $\Re_{\alpha_1, -\gamma h_1}(z; \mathbf{c})$ are integrable in $\mathbf{c}$, uniformly in z in a neighbourhood of the origin.

To see why this is indeed the case, let us for instance consider the remainder term $\Re(z)$ arising in the expansion of $\Phi_{\alpha_1, -\gamma h_1}(z; \mathbf{c})$:

$$\begin{aligned} \Re(z) = & e^{\langle \mathbf{s} + \gamma e_1, \mathbf{c} \rangle} \int_{\mathbb{C}} \frac{|x - z|^{\gamma^2} - |x|^{\gamma^2}}{|x|^{\gamma \langle \alpha_1, e_1 \rangle}} \mathbb{E} \left[e^{-e^{\gamma \langle \mathbf{c}, e_2 \rangle} J_{\alpha_1 - \gamma h_1, B_r}^2(x)} e^{-\sum_{i=1}^2 e^{\gamma \langle \mathbf{c}, e_i \rangle} I_{\alpha_1 - \gamma h_2}^i} \right. \\ & \left. \left(F_1(x) \int_0^1 e^{-e^{\gamma \langle \mathbf{c}, e_1 \rangle} (t \delta J(z, x) + \delta J_{\alpha_1 - \gamma h_1}^i(x))} dt e^{-e^{\gamma \langle \mathbf{c}, e_2 \rangle} \delta J_{\alpha_1 - \gamma h_1, \mathbb{C}_r}^2(x)} - 1 \right) \right] d^2 x. \end{aligned}$$

Then it is readily seen that this term is at most $\mathcal{O}(e^{\langle \mathbf{s} + \gamma e_1, \mathbf{c} \rangle})$ as $\mathbf{c} \rightarrow \infty$ inside $\mathcal{C}_-$, and therefore is integrable in this region thanks to our assumption on $\mathbf{s}$. Now it is also easily seen to be integrable when $\langle \mathbf{c}, e_1 \rangle \rightarrow +\infty$. As a consequence this remainder term is indeed integrable. We can proceed in the same ways for the other remainder terms that arise in the expansion of $\Phi_{\alpha_1, -\gamma h_1}(z; \mathbf{c})$ and of $\Re_{\alpha_1, -\gamma h_1}^1(z; \mathbf{c})$ too. As for $\Re_{\alpha_1, -\gamma h_1}(z; \mathbf{c})$ the dependence in $\mathbf{c}$ of the remainder terms is completely explicit and readily seen to be uniformly integrable too.

From this we infer that the desired asymptotic expansion for $\mathcal{H}$ does indeed hold provided that α_1 is taken so that $\langle \alpha_1, e_1 \rangle > \frac{2}{\gamma} - \varepsilon$ and $\langle \alpha_1, e_2 \rangle > q - \varepsilon$, where ε is positive and small enough. As a consequence we can identify the coefficients that appear in this expansion with the ones stemming from the statement of Theorem 7.1.1 in this very case. This shows that we have the equalities

$$\begin{aligned} \frac{C_{\gamma}(\alpha_1 - \gamma h_2, \alpha_2^*, \alpha_3)}{A_{\gamma}^{(1)}(-\gamma h_1, \alpha_1, \alpha_2^*, \alpha_3)} &= \frac{C_{\gamma}(\alpha_1 - \gamma h_1, \alpha_2^*, \alpha_3)}{B_{\gamma}^{(1)}(\alpha_1, \gamma)} \quad \text{and} \\ \frac{R_{s_2}(\hat{s}_2(\alpha_1 - \gamma h_3)) C_{\gamma}(\hat{s}_2(\alpha_1 - \gamma h_3), \alpha_2^*, \alpha_3)}{A_{\gamma}^{(2)}(-\gamma h_1, \alpha_1, \alpha_2^*, \alpha_3)} &= \frac{C_{\gamma}(\alpha_1 - \gamma h_1, \alpha_2^*, \alpha_3)}{B_{\gamma}^{(2)}(\alpha_1, \gamma)}. \end{aligned}$$

All the quantities that appear in the above expression are analytic in the weight α_1 as soon as the probabilistic representation makes sense. Uniqueness of the analytic continuation allows to extend the validity of this equality to the whole range of values prescribed by Lemma 7.2.3.

7.2.2.4 Evaluation of some integrals

To finish up with the proof of Lemma 7.2.3 it only remains to check that the expression of the coefficients $B_\gamma^{(i)}$ does coincide with that given by the integrals encountered above:

Lemma 7.2.4. *Let $a < 2$ and $b > -2$ be two real numbers and set*

$$r_{a,b}(x) := |x|^b \left(\mathbb{1}_{a-b < 2} + \mathbb{1}_{a-b < 1} \frac{b}{2} \left(\frac{1}{x} + \frac{1}{\bar{x}} \right) \right).$$

Then as soon as $a - b \in (0, +\infty) \setminus \{1, 2\}$:

$$\int_{\mathbb{C}} \frac{|x-1|^b - r_{a,b}(x)}{|x|^a} d^2x = \pi \frac{l(-1 + \frac{a-b}{2})}{l(-\frac{b}{2})l(\frac{a}{2})}. \quad (7.2.11)$$

Proof. The integral (7.2.11) is the analytic continuation in the a, b variables of the integral

$$\int_{\mathbb{C}} \frac{|x-1|^b}{|x|^a} d^2x,$$

from the region $\Re(a-b) > 2$ to the region $2 > \Re(a-b) > 1$, the above integral being given by the expected result $\pi \frac{l(-1 + \frac{a-b}{2})}{l(-\frac{b}{2})l(\frac{a}{2})}$ (see [109, p. 504]). To see why, set

$$F(a, b) := \int_{\mathbb{C}} \frac{|x-1|^b - |x|_+^b}{|x|^a} d^2x - \frac{2\pi}{2+b-a}.$$

Then F is analytic in the domain $\Re(a, -b) < 2, \Re(a-b) > 1, 2+b-a \neq 0$. Furthermore over the subdomain where $\Re(a-b) > 2$, it is equal to

$$\int_{\mathbb{C}} |x|^{a-b} |x-1|^b d^2x = \pi \frac{l(-1 + \frac{a-b}{2})}{l(-\frac{b}{2})l(\frac{a}{2})}.$$

As a consequence by uniqueness of the analytic continuation $F(a, b)$ is also equal to $\pi \frac{l(-1 + \frac{a-b}{2})}{l(-\frac{b}{2})l(\frac{a}{2})}$ in the subdomain $2 > \Re(a-b) > 1$. But in that case $F(a, b)$ is found to be equal to

$$\int_{\mathbb{C}} \frac{|x-1|^b - |x|_+^b}{|x|^a} d^2x.$$

More generally the same argument shows that

$$\int_{\mathbb{C}} \frac{|x-1|^b - r_{a,b}(x)}{|x|^a} d^2x$$

is the analytic continuation of

$$\int_{\mathbb{C}} \frac{|x-1|^b}{|x|^a} d^2x,$$

which allows to conclude that Equation (7.2.11) does indeed hold. $\square$

With all these different pieces now put together, we can wrap up the proof of Lemma 7.2.3. $\square$

7.2.3 . The case where $\chi = \gamma$, $\langle \alpha_1, e_1 \rangle > \frac{2}{\gamma}$ and $\langle \alpha_1, e_2 \rangle < \frac{2}{\gamma}$

The case we consider next arises when $\langle \alpha_1, e_1 \rangle > \frac{2}{\gamma}$ and $\langle \alpha_1, e_2 \rangle < \frac{2}{\gamma}$, but still with $\chi = \gamma$. Based on a similar reasoning as the one developed above we can obtain an expansion that closely resembles that of Lemma 7.2.3:

Lemma 7.2.5. *Under the assumptions that $q - \varepsilon < \langle \alpha_1, e_1 \rangle < q$ and $\frac{2}{\gamma} - \varepsilon < \langle \alpha_1, e_2 \rangle < \frac{2}{\gamma}$:*

$$\begin{aligned} \mathcal{H}(z) = & C_\gamma(\alpha_1 - \gamma h_1, \alpha_2^*, \alpha_3) |\mathcal{H}_0(z)|^2 \\ & + B_\gamma^{(1)}(\alpha_1, \gamma) R_{s_1}(\hat{s}_1(\alpha_1 - \gamma h_2)) C_\gamma(\hat{s}_1(\alpha_1 - \gamma h_2), \alpha_2^*, \alpha_3) |\mathcal{H}_1(z)|^2 \\ & + B_\gamma^{(2)}(\alpha_1, \gamma) C_\gamma(\alpha_1 - \gamma h_3, \alpha_2^*, \alpha_3) |\mathcal{H}_2(z)|^2 \end{aligned} \quad (7.2.12)$$

provided that $(\alpha_1, -\gamma h_1, \alpha_2^*, \alpha_3) \in \mathcal{A}_4$ with $\langle s, \omega_2 \rangle > 0$.

Proof. The arguments developed to prove this statement are similar to the ones presented along the proof of Lemma 7.2.3 but are slightly more subtle and require additional care. Namely the main difference with the previous case is that the random variable $\delta I(z)$ introduced above no longer has a finite L^1 moment because of the singularity at $x = 0$ in the integrand. In order to understand what happens around this singular point this time we write that

$$\delta I(z) = \delta I_r(z) + \delta J_r(z), \quad \text{with} \quad \delta J_r(z) := \int_{B_r} \frac{|x - z|^{\gamma^2} - |x|^{\gamma^2}}{|x|^{\gamma(\alpha_1, e_1)}} F_1(x) M^{\gamma e_1}(d^2 x_1)$$

where $B_r = B(0, e^{-r})$ for some $r = r(z)$ such that $|z|e^r \rightarrow +\infty$ to be fixed later on. The asymptotic of the expectation term is governed by $\delta J_r(z)$ rather than $\delta I_r(z)$. To see why let us write that

$$\begin{aligned} \Phi_{\alpha_1, -\gamma h_1}(z; \mathbf{c}) &= \Phi_{\alpha_1 - \gamma h_1}(\mathbf{c}) + \sum_{i=1}^3 \Phi^i(z; \mathbf{c}) \quad \text{where} \\ \Phi^1(z; \mathbf{c}) &:= e^{\langle \mathbf{s}, \mathbf{c} \rangle} \mathbb{E} \left[\left(\exp \left(-e^{\gamma \langle \mathbf{c}, e_1 \rangle} \delta J_r(z) \right) - 1 \right) e^{-\sum_{i=1}^2 e^{\gamma \langle \mathbf{c}, e_i \rangle} I_{\alpha_1 - \frac{2}{\gamma} h_1}^i} \right] \\ \Phi^2(z; \mathbf{c}) &:= e^{\langle \mathbf{s}, \mathbf{c} \rangle} \mathbb{E} \left[\left(\exp \left(-e^{\gamma \langle \mathbf{c}, e_1 \rangle} \delta I_r(z) \right) - 1 \right) e^{-\sum_{i=1}^2 e^{\gamma \langle \mathbf{c}, e_i \rangle} I_{\alpha_1 - \frac{2}{\gamma} h_1}^i} \right] \\ \Phi^3(z; \mathbf{c}) &:= e^{\langle \mathbf{s}, \mathbf{c} \rangle} \mathbb{E} \left[\left(\exp \left(-e^{\gamma \langle \mathbf{c}, e_1 \rangle} \delta I_r(z) \right) - 1 \right) \left(\exp \left(-e^{\gamma \langle \mathbf{c}, e_1 \rangle} \delta J_r(z) \right) - 1 \right) e^{-\sum_{i=1}^2 e^{\gamma \langle \mathbf{c}, e_i \rangle} I_{\alpha_1 - \frac{2}{\gamma} h_1}^i} \right] \end{aligned}$$

and study the behaviour of these three terms.

Let us start with Φ^1 . This term can be treated in the same fashion as the last order term that arises in the asymptotic of Lemma 7.2.3. Namely we can use the radial-angular decomposition (3.3.6) for the one-dimensional Brownian motion with drift $\nu = \langle \alpha_1 - Q, e_1 \rangle \langle \mathbf{X}_{t+r}(0) - \mathbf{X}_r(0), e_1 \rangle + \nu t$ to write that

$$\begin{aligned} \delta J_r(z) &= e^{\gamma(\lambda_r' + \mathbf{M}_1)} J_r(z; -\mathbf{M}_1), \quad \text{with} \\ J_r(z; -\mathbf{M}_1) &:= \int_0^{+\infty} e^{\gamma B_t^\nu} \int_0^{2\pi} \left(\left| \frac{e^{-t-r+i\theta}}{z} - 1 \right|^{\gamma^2} - \left| \frac{e^{-t-r}}{z} \right|^{\gamma^2} \right) F_1(e^{-t-r+i\theta}) M_Y^{\gamma e_1}(dt + r, d\theta). \end{aligned}$$

Here $\lambda'_r := \langle \mathbf{X}_r(0) + (\alpha_1 - Q)r + \gamma h_1 \ln |z|, e_1 \rangle$ and $(B'_t)_{t \geq 0}$ is independent of $(\mathbf{X}(x))_{|x| > e^{-r}}$ and started from $-\mathbf{M}_1$, sampled according to its marginal law. Proceeding along the same lines as above we see that, as soon as $\frac{e^{-r}}{z} \rightarrow 0$ as $z \rightarrow 0$ (which we assumed to hold)

$$\Phi^1(z; \mathbf{c}) = |z|^{\gamma(Q-\alpha_1, e_1)} R_{s_1}(\alpha_1) \Phi_{\hat{s}_1 \alpha_1 - \gamma h_1}(\mathbf{c}) + l.o.t.$$

Now from their expressions we know that $R_{s_1}(\alpha_1) = B_\gamma^{(1)}(\alpha_1, \gamma) R_{s_1}(\hat{s}_1(\alpha_1 - \gamma h_2))$ and that $\hat{s}_1 \alpha_1 - \gamma h_1 = \hat{s}_1(\alpha_1 - \gamma h_2)$. As a consequence we end up with the asymptotic:

$$\Phi^1(\mathbf{c}) = |z|^{\gamma(Q-\alpha_1, e_1)} R_{s_1}(\hat{s}_1(\alpha_1 - \gamma h_2)) \Phi_{\hat{s}_1(\alpha_1 - \gamma h_2)}(\mathbf{c}) + \mathfrak{R}^1(z)$$

where $\mathfrak{R}^1(z)$ is a $o(|z|^{\gamma(Q-\alpha_1, e_1)})$. We will provide a more precise description of $\mathfrak{R}^1$ below.

Let us now turn to Φ^2 . Since the integral now avoids the singular point $x = 0$ we can proceed in the same way as for the first term in the expansion of Lemma 7.2.3 but by replacing $\delta I(z)$ by $\delta I_r(z)$:

$$\begin{aligned} & \mathbb{E} \left[\left(\exp \left(-e^{\gamma \langle \mathbf{c}, e_1 \rangle} \delta I_r(z) \right) - 1 \right) e^{-\sum_{i=1}^2 e^{\gamma \langle \mathbf{c}, e_i \rangle} I_{\alpha_1 - \frac{2}{\gamma} h_1, 0}^i} \right] \\ &= -e^{\langle s + \gamma e_1, \mathbf{c} \rangle} \int_{\mathbb{C} \setminus B_r} \frac{|x - z|^{\gamma^2} - |x|^{\gamma^2}}{|x|^{\gamma \langle \alpha_1, e_1 \rangle}} F_1(x) \times \\ & \quad \mathbb{E} \left[\int_0^1 e^{-te^{\gamma \langle \mathbf{c}, e_1 \rangle} \delta J(z, x)} dt \prod_{i=1}^2 e^{-e^{\gamma \langle \mathbf{c}, e_i \rangle} J_{\alpha_1 - \gamma h_1}^i(x)} \right] d^2 x. \end{aligned}$$

The next step is to write the expectation term as

$$\begin{aligned} & \mathbb{E} \left[\int_0^1 e^{-te^{\gamma \langle \mathbf{c}, e_1 \rangle} \delta J(z, x)} dt e^{-e^{\gamma \langle \mathbf{c}, e_1 \rangle} J_{\alpha_1 - \gamma h_1}^1(x)} e^{-e^{\gamma \langle \mathbf{c}, e_2 \rangle} I_{\alpha_1 - \gamma h_2}^2} \right] d^2 x \\ &= e^{\gamma \langle \mathbf{c}, e_2 \rangle} \mathbb{E} \left[\delta J_{\alpha_1 - \gamma h_1}^2(x) \int_0^1 e^{-te^{\gamma \langle \mathbf{c}, e_1 \rangle} \delta J(z, x)} dt e^{-e^{\gamma \langle \mathbf{c}, e_1 \rangle} J_{\alpha_1 - \gamma h_1}^1(x)} e^{-e^{\gamma \langle \mathbf{c}, e_2 \rangle} I_{\alpha_1 - \gamma h_2}^2} \right] d^2 x + l.o.t. \end{aligned}$$

Along the same lines as in the proof of Lemma 7.2.3 we infer that as $z \rightarrow 0$, the term that appears on the second line is such that

$$\begin{aligned} & e^{\langle s + \gamma \rho, \mathbf{c} \rangle} \int_{\mathbb{C} \setminus B_r} \frac{|x - z|^{\gamma^2} - |x|^{\gamma^2}}{|x|^{\gamma \langle \alpha_1, e_1 \rangle}} F_1(x) \times \\ & \quad \mathbb{E} \left[\delta J_{\alpha_1 - \gamma h_1}^2(x) \int_0^1 e^{-te^{\gamma \langle \mathbf{c}, e_1 \rangle} \delta J(z, x)} dt e^{-e^{\gamma \langle \mathbf{c}, e_1 \rangle} J_{\alpha_1 - \gamma h_1}^1(x)} e^{-e^{\gamma \langle \mathbf{c}, e_2 \rangle} I_{\alpha_1 - \gamma h_2}^2} \right] d^2 x \\ & \sim |z|^{\gamma(Q-\alpha_1, \rho)} \Phi_{\alpha_1 - \gamma h_3}(\mathbf{c}) \int_{\mathbb{C} \setminus B_r} \frac{|x - 1|^{\gamma^2} - |x|^{\gamma^2}}{|x|^{\gamma \langle (\alpha_1, e_1 + e_2) - q \rangle}} d^2 x \int_{\mathbb{C}} \frac{|y - 1|^{\gamma^2} - |y|^{\gamma^2}}{|y|^{\gamma \langle \alpha_1, e_2 \rangle}} d^2 y. \end{aligned}$$

The last integrals are well-defined as $r \rightarrow +\infty$, and can be evaluated using Lemma 7.2.4 above, thanks to which we can deduce that this term is asymptotically equivalent to

$$|z|^{\gamma(Q-\alpha_1, \rho)} B_\gamma^{(2)}(\alpha_1, \gamma) \Phi_{\alpha_1 - \gamma h_3}(\mathbf{c})$$

as soon as $r \rightarrow +\infty$ when $z \rightarrow 0$. Therefore we can write that

$$\Phi^2(z; \mathbf{c}) = |z|^{\gamma(Q-\alpha_1, \rho)} B_\gamma^{(2)}(\alpha_1, \gamma) \Phi_{\alpha_1 - \gamma h_3}(\mathbf{c}) + \mathfrak{R}^2(z),$$

where the remainder term is given by

$$\mathfrak{R}_2(z) = -e^{\langle s+\gamma e_1, c \rangle} \int_{\mathbb{C} \setminus B_r} \frac{|x-z|^{\gamma^2} - |x|^{\gamma^2}}{|x|^{\gamma \langle \alpha_1, e_1 \rangle}} F_1(x) \times \\ \mathbb{E} \left[\int_0^1 e^{-te^{\gamma \langle c, e_1 \rangle} \delta J(z, x)} dt e^{-e^{\gamma \langle c, e_1 \rangle} J_{\alpha_1 - \gamma h_1}^1(x)} e^{-e^{\gamma \langle c, e_2 \rangle} I_{\alpha_1 - \gamma h_2}^2(x)} \right] d^2 x$$

up to a $o\left(|z|^{\gamma \langle Q - \alpha_1, \rho \rangle}\right)$.

7.2.3.1 The reflection terms

The reflection terms required to make sense of the four-point correlation functions can be processed in the very same way as in the proof of Lemma 7.2.3 so we omit the computations of the asymptotic of these terms. We stress that conducting these explicit but tedious computations show that the condition for the reflection terms involved to be integrable is the same as in Lemma 7.2.3, that is $\langle s, \omega_2 \rangle > 0$.

7.2.3.2 The remainder terms

It remains to consider the remainder term $\mathfrak{R}(z) := \mathfrak{R}^1(z) + \mathfrak{R}^2(z) + \Phi^3(z)$ that arise in the above expansions. We already know thanks to Theorem 7.1.1 that the *a priori* expansion of $\mathfrak{R}(z)$ only features terms which are (infinite) polynomials in $z, \bar{z}$ multiplied by $|z|^{\gamma \langle Q - \alpha_1, h_1 - h_i \rangle}$ for $i = 1, 2, 3$ as soon as $\mathcal{H}$ makes sense. Put differently

$$\mathfrak{R}(z) = \sum_{i=0}^2 |z|^{\gamma \langle Q - \alpha_1, \sum_{j=1}^i e_j \rangle} P_i(z, \bar{z}; \alpha_1) \quad (7.2.13)$$

where $P_i(z, \bar{z}; \alpha_1)$ is a power series in $z, \bar{z}$, provided that $\mathcal{H}$ is well-defined and under the assumption that $\langle \alpha_1, e_2 \rangle < \frac{2}{\gamma}$.

Moreover the explicit expression of $\mathfrak{R}(z)$ shows, in agreement with Theorem 6.4.1, that the dependence in $\langle \alpha_1, e_2 \rangle$ of $\mathfrak{R}(z)$ is actually analytic in a complex neighbourhood of $(\frac{2}{\gamma} - \varepsilon, q)$. For instance let us consider the first remainder term $\mathfrak{R}^1(z)$, defined by

$$\mathfrak{R}^1(z) = \Phi^1(c) - |z|^{\gamma \langle Q - \alpha_1, e_1 \rangle} B_\gamma^{(1)}(\alpha_1, \gamma) R_{s_1}(\hat{s}_1(\alpha_1 - \gamma h_2)) \Phi_{\hat{s}_1(\alpha_1 - \gamma h_2)}(c).$$

Then the probabilistic representation of $\Phi_{\hat{s}_1(\alpha_1 - \gamma h_2)}$ makes sense for $\langle \hat{s}_1(\alpha_1 - \gamma h_2) - Q, e_i \rangle < 0$ for $i = 1, 2$, and in particular makes sense for α_1 close to Q . Likewise the term

$$\Phi^1(c) = e^{\langle s, c \rangle} \mathbb{E} \left[\left(\exp \left(-e^{\gamma \langle c, e_1 \rangle} \delta J_r(z) \right) - 1 \right) e^{-\sum_{i=1}^2 e^{\gamma \langle c, e_i \rangle} I_{\alpha_1 - \gamma h_1}^i} \right]$$

is perfectly well-defined when α_1 is taken close to Q . The reasoning developed along the proof of Theorem 6.4.1 shows that in that case both terms are analytic in $\langle \alpha_1, e_2 \rangle$. We can proceed in the same way for the other remainder terms, which are probabilistically speaking perfectly well-defined for α_1 close to Q . This shows that $\mathfrak{R}$ depends analytically in $\langle \alpha_1, e_2 \rangle$ in a complex neighbourhood of $(\frac{2}{\gamma} - \varepsilon, q)$. In particular this allows to extend the validity of Equation (7.2.13) for $\langle \alpha_1, e_2 \rangle$ in $(\frac{2}{\gamma} - \varepsilon, q)$.

Now we can use the fusion asymptotics (4.4.2) to infer that

$$\mathfrak{R}^2(z) = o \left(|z|^{\gamma \langle Q - \alpha_1, e_1 \rangle + \frac{(\langle \alpha_1, e_1 \rangle - \frac{2}{\gamma})^2}{4} - \varepsilon} \right). \quad (7.2.14)$$

To see why we can reproduce the argument developed to show that Lemma 7.2.3 holds true. Namely

$$\begin{aligned} & \int_{\mathbb{C} \setminus B_r} \frac{|x-z|^{\gamma^2} - |x|^{\gamma^2}}{|x|^{\gamma\langle\alpha_1, e_1\rangle}} F_1(x) \times \\ & \mathbb{E} \left[\int_0^1 e^{-te^{\gamma\langle c, e_1\rangle} \delta J(z, x)} dt e^{-e^{\gamma\langle c, e_1\rangle} J_{\alpha_1 - \gamma h_1}^1(x)} e^{-e^{\gamma\langle c, e_2\rangle} I_{\alpha_1 - \gamma h_2}^2} \right] d^2x \\ & \sim |z|^{\gamma^2} e^{-r(2-\gamma\langle\alpha_1, e_1\rangle)} \int_{\mathbb{C} \setminus B_0} |x|^{-\gamma\langle\alpha_1, e_1\rangle} F_1(xe^{-r}) \times \\ & \mathbb{E} \left[\int_0^1 e^{-te^{\gamma\langle c, e_1\rangle} \delta J(z, e^{-r}x)} dt e^{-e^{\gamma\langle c, e_1\rangle} J_{\alpha_1 - \gamma h_1}^1(e^{-r}x)} e^{-e^{\gamma\langle c, e_2\rangle} I_{\alpha_1 - \gamma h_2}^2} \right] d^2x \end{aligned}$$

where we can use Equation (4.4.2) to see that the expectation term is at most a $o\left(e^{-r\left(\frac{(\langle\alpha_1, e_1\rangle - \frac{2}{\gamma})^2}{4} - \varepsilon\right)}\right)$. Since r is chosen so that $\frac{e^{-r}}{z} \rightarrow 0$ as $z \rightarrow 0$ we recover Equation (7.2.14). This shows that $\mathfrak{R}^2(z)$ is a $o\left(|z|^{\gamma(Q-\alpha_1, \rho)}\right)$ as soon as $\langle\alpha_1, e_2\rangle$ is taken close enough to q and $\langle\alpha_1, e_1\rangle$ different from $\frac{2}{\gamma}$. The same applies to the other terms $\mathfrak{R}^1(z)$ and $\Phi^3(z; c)$. This is readily seen for Φ^3 which is a lower order term compared to Φ^2 , while for $\mathfrak{R}^1$ we can write that

$$\begin{aligned} & \mathbb{E} \left[\left(\exp \left(-e^{\gamma\langle c, e_1\rangle} \delta J_r(z) \right) - 1 \right) e^{-\sum_{i=1}^2 e^{\gamma\langle c, e_i\rangle} I_{\alpha_1 - \gamma h_1}^i} \right] \\ & = \int_0^{+\infty} (-\nu) e^{\nu \mathbf{M}_1} \mathbb{E} \left[\left(\exp \left(-e^{\gamma(\langle c, e_1\rangle + \lambda'_r + \mathbf{M}_1)} J_r(z; -\mathbf{M}_1) \right) - 1 \right) e^{-\sum_{i=1}^2 e^{\gamma\langle c, e_i\rangle} I_{\alpha_1 - \gamma h_1}^i} \right] d\mathbf{M}_1 \\ & = |z|^{\gamma(Q-\alpha_1, e_1)} e^{\langle Q-\alpha_1, e_1\rangle \langle c, e_1\rangle} \times \\ & \mathbb{E} \left[e^{-\nu \mathbf{X}_r(0) - \nu^2 r} \int_{\langle c, e_1\rangle + \lambda'_r}^{+\infty} (-\nu) e^{\nu \mathbf{M}_1} \left(e^{-e^{\gamma \mathbf{M}_1} J_r(z; \langle c, e_1\rangle + \lambda'_r - \mathbf{M}_1)} - 1 \right) e^{-\sum_{i=1}^2 e^{\gamma\langle c, e_i\rangle} I_{\alpha_1 - \gamma h_1}^i} \right] d\mathbf{M}_1. \end{aligned}$$

In the latter note that $I_{\alpha_1 - \frac{2}{\gamma} h_1}^i$ does depend on $\mathbf{M}_1$ and J since it involves the sigma-algebra generated by the $(\mathbf{X}(x))_{|x| \leq e^{-r}}$. However we can control the integral over B_r that enters $I_{\alpha_1 - \frac{2}{\gamma} h_1}^i$ in the same fashion as in the proof of Lemma 7.2.3. As a consequence and in order to keep the proof concise we can assume that $I_{\alpha_1 - \gamma h_1}^i$ is independent of $\mathbf{M}_1$ and J . Now the first exponential term entering the expectation is a Girsanov transform, thanks to which we end up with

$$\begin{aligned} & = |z|^{\gamma(Q-\alpha_1, e_1)} e^{\langle Q-\alpha_1, e_1\rangle \langle c, e_1\rangle} \times \\ & \mathbb{E} \left[\int_{\langle c, e_1\rangle + \hat{\lambda}_r}^{+\infty} (-\nu) e^{\nu \mathbf{M}_1} \left(e^{-e^{\gamma \mathbf{M}_1} J_r(z; \langle c, e_1\rangle + \hat{\lambda}_r - \mathbf{M}_1)} - 1 \right) e^{-\sum_{i=1}^2 e^{\gamma\langle c, e_i\rangle} I_{\hat{s}_1 \alpha_1 - \gamma h_1}^i} \right] d\mathbf{M}_1. \end{aligned}$$

where $\hat{\lambda}_r = \langle \mathbf{X}_r(0) + (Q - \alpha_1)r + \gamma h_1 \ln |z|, e_1 \rangle$. Therefore we can write that

$$\begin{aligned} \mathfrak{R}^1(z) & = |z|^{\gamma(Q-\alpha_1, e_1)} e^{\langle Q-\alpha_1, e_1\rangle \langle c, e_1\rangle} \times \\ & \mathbb{E} \left[\int_{-\infty}^{\langle c, e_1\rangle + \hat{\lambda}_r} (-\nu) e^{\nu \mathbf{M}_1} \left(e^{-e^{\gamma \mathbf{M}_1} J(-\infty)} - 1 \right) e^{-\sum_{i=1}^2 e^{\gamma\langle c, e_i\rangle} I_{\hat{s}_1 \alpha_1 - \gamma h_1}^i} \right] d\mathbf{M}_1 \\ & + \mathbb{E} \left[\int_{\langle c, e_1\rangle + \hat{\lambda}_r}^{+\infty} (-\nu) e^{\nu \mathbf{M}_1} \left(e^{-e^{\gamma \mathbf{M}_1} J_r(z; \langle c, e_1\rangle + \hat{\lambda}_r - \mathbf{M}_1)} - e^{-e^{\gamma \mathbf{M}_1} J(-\infty)} \right) e^{-\sum_{i=1}^2 e^{\gamma\langle c, e_i\rangle} I_{\hat{s}_1 \alpha_1 - \gamma h_1}^i} \right] d\mathbf{M}_1 \end{aligned}$$

up to lower order terms, where $J(-\infty) := \lim_{z \rightarrow 0} J_r(z; \langle \mathbf{c}, e_1 \rangle + \hat{\lambda}_r - \mathbf{M}_1)$. The first expectation term is asymptotically equivalent to

$$\mathbb{E} \left[J(-\infty) \int_{-\infty}^{\langle \mathbf{c}, e_1 \rangle + \hat{\lambda}_r} \nu e^{(\nu + \gamma) \mathbf{M}_1} e^{-\sum_{i=1}^2 e^{\gamma \langle \mathbf{c}, e_i \rangle} I_{\hat{s}_1 \alpha_1 - \gamma h_1}^i} \right] d\mathbf{M}_1$$

which is at most a $\mathcal{O}(|z|^{\gamma^2})$ while the second one can also be seen to be a $o(|z|^\varepsilon)$ for some positive ε that only depends on $\langle \alpha_1, e_1 \rangle$. Indeed one can use the Markov property of the process $\mathcal{B}$ in the same fashion as in the proof of Proposition 6.3.3 to see that the second expectation term is governed by a term of same order as

$$\mathbb{E} \left[e^{\gamma \langle \mathbf{c}, e_1 \rangle + \hat{\lambda}_r} \int_{\langle \mathbf{c}, e_1 \rangle + \hat{\lambda}_r}^{+\infty} e^{(\nu + \gamma) \mathbf{M}_1} e^{-e^{\gamma \mathbf{M}_1} J(-\infty)} e^{-\sum_{i=1}^2 e^{\gamma \langle \mathbf{c}, e_i \rangle} I_{\hat{s}_1 \alpha_1 - \gamma h_1}^i} \right] d\mathbf{M}_1$$

which is as desired. In particular for $\langle Q - \alpha_1, e_2 \rangle$ small enough we see that the corresponding term in the expansion of $\mathfrak{R}$ is a $o(|z|^{\gamma \langle Q - \alpha_1, \rho \rangle})$.

Recollecting terms allows to claim that for $\langle \alpha_1, e_2 \rangle$ close to q

$$\mathfrak{R}(z) = o(|z|^{\gamma \langle Q - \alpha_1, \rho \rangle}).$$

As a consequence we can infer that the polynomials $P_i(z, \bar{z}; \alpha_1)$ are such that $P_i(0, 0; \alpha_1) = 0$ for $i = 1, 2, 3$ under this assumption. Now this coefficient is known to depend analytically in $\langle \alpha_1, e_2 \rangle$, so that we can deduce that $P_i(0, 0; \alpha_1) = 0$ as soon as $\mathfrak{R}$ makes sense. This allows to discard the remainder term $\mathfrak{R}(z)$ in the identification of the coefficients arising in the expansion of $\mathcal{H}$ from either Theorem 7.1.1 and Lemma 7.2.5.

To sum things up, we have proved that Φ has the following expansion:

$$\begin{aligned} \Phi_{\alpha_1, -\gamma h_1}(z; \mathbf{c}) &= \Phi_{\alpha_1 - \gamma h_1}(\mathbf{c}) + Az + \bar{A}\bar{z} \\ &+ |z|^{\gamma \langle Q - \alpha_1, e_1 \rangle} B_\gamma^{(1)}(\alpha_1, \gamma) R_{s_1}(\hat{s}_1(\alpha_1 - \gamma h_2)) \Phi_{\hat{s}_1(\alpha_1 - \gamma h_2)}(\mathbf{c}) (Bz + \bar{B}\bar{z}) \\ &+ |z|^{\gamma \langle Q - \alpha_1, \rho \rangle} B_\gamma^{(2)}(\alpha_1, \gamma) \Phi_{\alpha_1 - \gamma h_3}(\mathbf{c}) + l.o.t. \end{aligned}$$

We can conclude for the proof of Lemma 7.2.5 in the same fashion as we did for Lemma 7.2.3 by identification with the coefficients arising in the expansion of Theorem 7.1.1. $\square$

7.2.4 . The case where $\chi = \frac{2}{\gamma}$

Eventually we treat the case where χ is equal to $\frac{2}{\gamma}$. Under this assumption the expansion features two reflection terms, as the following statement discloses:

Lemma 7.2.6. *Assume that $\chi = \frac{2}{\gamma}$. Then*

$$\begin{aligned} \mathcal{H}(z) &= C_\gamma(\alpha_1 - \frac{2}{\gamma} \gamma h_1, \alpha_2^*, \alpha_3) |\mathcal{H}_0(z)|^2 \\ &+ B_\gamma^{(1)}(\alpha_1, \frac{2}{\gamma}) R_{s_1}(\hat{s}_1(\alpha_1 - \frac{2}{\gamma} h_2)) C_\gamma(\hat{s}_1(\alpha_1 - \frac{2}{\gamma} h_2), \alpha_2^*, \alpha_3) |\mathcal{H}_1(z)|^2 \\ &+ B_\gamma^{(2)}(\alpha_1, \frac{2}{\gamma}) R_{s_1 s_2}(\hat{s}_1 \hat{s}_2(\alpha_1 - \frac{2}{\gamma} h_3)) C_\gamma(\hat{s}_1 \hat{s}_2(\alpha_1 - \frac{2}{\gamma} h_3), \alpha_2^*, \alpha_3) |\mathcal{H}_2(z)|^2 \end{aligned} \tag{7.2.15}$$

as soon as $(\alpha_1, -\frac{2}{\gamma}h_1, \alpha_2^*, \alpha_3) \in \mathcal{A}_4$.

Proof. The terms associated to the reflection coefficients can be dealt with in the same way as in the proof of Lemma 7.2.3 so that we will omit them in what follows and focus on the expansion of $\Phi_{\alpha_1, -\frac{2}{\gamma}h_1}(z; \mathbf{c})$. To start with we make the assumption that α_1 is close to Q , for a meaning of close that will be made precise during the proof. We start by picking $r = r(z) > 0$ such that $|z|e^r \rightarrow +\infty$ as $z \rightarrow 0$, to be fixed later on, and split the integrals involved as

$$I_{\alpha_1, -\frac{2}{\gamma}h_1}^i(z) = I_r^i(z) + J_r^i(z), \quad \text{with} \quad J_r^i(z) := \int_{B_r} \frac{|x_i - z|^{\frac{2}{\gamma}\langle h_1, e_i \rangle}}{|x_i|^{\gamma\langle \alpha_1, e_i \rangle}} F_i(x) M^{\gamma e_i}(d^2 x_i)$$

where $B_r = B(0, e^{-r})$. By doing so we can write that

$$\begin{aligned} \Phi_{\alpha_1, -\frac{2}{\gamma}h_1}(z; \mathbf{c}) &= \Phi_{\alpha_1 - \frac{2}{\gamma}h_1}(\mathbf{c}) + \Phi^1(z; \mathbf{c}) + \Phi^2(z; \mathbf{c}) + \Phi^3(z; \mathbf{c}), \quad \text{where} \\ \Phi^1(z; \mathbf{c}) &:= e^{\langle \mathbf{s}, \mathbf{c} \rangle} \mathbb{E} \left[e^{-\sum_{i=1}^2 e^{\gamma\langle \mathbf{c}, e_i \rangle} I_r^i(0)} \left(e^{-e^{\gamma\langle \mathbf{c}, e_1 \rangle} J_r^1(z)} - 1 \right) \right] \\ \Phi^2(z; \mathbf{c}) &:= e^{\langle \mathbf{s}, \mathbf{c} \rangle} \mathbb{E} \left[e^{-\sum_{i=1}^2 e^{\gamma\langle \mathbf{c}, e_i \rangle} I_r^i(0)} \left(e^{-e^{\gamma\langle \mathbf{c}, e_1 \rangle} J_r^1(z)} - 1 \right) \left(e^{-e^{\gamma\langle \mathbf{c}, e_2 \rangle} J_r^2(z)} - 1 \right) \right] \\ \Phi^3(z; \mathbf{c}) &:= e^{\langle \mathbf{s}, \mathbf{c} \rangle} \mathbb{E} \left[e^{-e^{\gamma\langle \mathbf{c}, e_1 \rangle} I_{\alpha_1, -\frac{2}{\gamma}h_1}^1(z)} e^{-e^{\gamma\langle \mathbf{c}, e_2 \rangle} I_{\alpha_1, -\frac{2}{\gamma}h_1}^2(z)} \left(1 - e^{-e^{\gamma\langle \mathbf{c}, e_i \rangle} (I_r^1(0) - I_r^1(z))} \right) \right] \\ &\quad + e^{\langle \mathbf{s}, \mathbf{c} \rangle} \mathbb{E} \left[e^{-e^{\gamma\langle \mathbf{c}, e_1 \rangle} I_r^1(0)} e^{-e^{\gamma\langle \mathbf{c}, e_2 \rangle} I_{\alpha_1, -\frac{2}{\gamma}h_1}^2(z)} \left(1 - e^{-e^{\gamma\langle \mathbf{c}, e_i \rangle} J_r^1(0)} \right) \right]. \end{aligned}$$

As we will see, the expansion of Φ^1 allows to infer the equality

$$R_{s_1}(\alpha_1) C_\gamma \left(\hat{s}_1 \alpha_1 - \frac{2}{\gamma} h_1, \alpha_2^*, \alpha_3 \right) = A_\gamma^{(1)} \left(-\frac{2}{\gamma} h_1, \alpha_1, \alpha_2^*, \alpha_3 \right) C_\gamma \left(\alpha_1 - \frac{2}{\gamma} h_1, \alpha_2^*, \alpha_3 \right)$$

while the expansion of Φ^2 yields

$$R_{s_1 s_2}(\alpha_1) C_\gamma \left(\hat{s}_1 \hat{s}_2 \alpha_1 - \frac{2}{\gamma} h_1, \alpha_2^*, \alpha_3 \right) = A_\gamma^{(2)} \left(-\frac{2}{\gamma} h_1, \alpha_1, \alpha_2^*, \alpha_3 \right) C_\gamma \left(\alpha_1 - \frac{2}{\gamma} h_1, \alpha_2^*, \alpha_3 \right)$$

Since the coefficients that appear in these expansions satisfy the properties that

$$R_{s_1}(\alpha_1) = B_\gamma^{(1)}(\alpha_1, \frac{2}{\gamma}) R_{s_1} \left(\alpha_1 - \frac{2}{\gamma} h_2 \right), \quad R_{s_1 s_2}(\alpha_1) = B_\gamma^{(2)}(\alpha_1, \frac{2}{\gamma}) R_{s_1 s_2} \left(\alpha_1 - \frac{2}{\gamma} h_3 \right),$$

the statement of Lemma 7.2.6 follows from these equalities along the same lines as in the proof of Lemma 7.2.3.

7.2.4.1 The leading terms in the expansion

We start by treating the first term in the expansion of $\Phi_{\alpha_1, -\frac{2}{\gamma}h_1}(z; \mathbf{c})$, given by Φ^1 . The very same reasoning as the one conducted along the proof of Lemmas 7.2.3 and 7.2.5 still applies and shows that this term is asymptotically equivalent to

$$|z|^{\frac{2}{\gamma}\langle Q - \alpha_1, e_1 \rangle} R_{s_1}(\alpha_1) \Phi_{\hat{s}_1 \alpha_1 - \frac{2}{\gamma}h_1}(\mathbf{c}).$$

To see why note that the only change to make compared to the proof of Lemma 7.2.5 is to consider $\lambda_r + \frac{2}{\gamma}h_1 \ln |z|$ instead of $\lambda_r + \gamma h_1 \ln |z|$ hereafter. With this new notation at hand we again use the radial-angular decomposition (3.3.6) to write that

$$J_r^1(z) = e^{\gamma(\lambda_r + \mathbf{M}_1)} \int_0^{+\infty} e^{\gamma \langle \mathcal{B}_t^\nu, e_1 \rangle} \int_0^{2\pi} \left| \frac{e^{-t-r+i\theta}}{z} - 1 \right|^2 F_1(e^{-t-r+i\theta}) M_Y^{\gamma e_1}(dt + r, d\theta).$$

The asymptotic is thus described in the same fashion as in the proof of Lemma 7.2.3, the only difference being the value of λ_r considered there. Moreover and similarly to the $\chi = \gamma$ case we can use the remarkable property of the coefficients $B_\gamma^{(1)}$ and R_{s_1}

$$R_{s_1}(\alpha_1) = B_\gamma^{(1)}(\alpha_1, \frac{2}{\gamma}) R_{s_1}(\hat{s}_1(\alpha_1 - \frac{2}{\gamma}h_2))$$

to infer the following expansion for Φ^1 :

$$\Phi^1(z; \mathbf{c}) = |z|^{\frac{2}{\gamma} \langle Q - \alpha_1, e_1 \rangle} B_\gamma^{(1)}(\alpha_1, \frac{2}{\gamma}) R_{s_1}(\hat{s}_1(\alpha_1 - \frac{2}{\gamma}h_2)) \Phi_{\hat{s}_1(\alpha_1 - \gamma h_2)}(\mathbf{c}) + \mathfrak{R}_1(z).$$

Here $\mathfrak{R}_1(z)$ is a lower order term which can be processed like before using fusion asymptotics. This reasoning shows that it is a $o\left(|z|^{\frac{2}{\gamma} \langle Q - \alpha_1, e_1 \rangle + \varepsilon}\right)$ where ε is positive and is independent of $\langle \alpha_1, e_2 \rangle$. In particular for $\langle Q - \alpha_1, e_2 \rangle$ close enough to 0 we see that $\mathfrak{R}_1 = o\left(|z|^{\frac{2}{\gamma} \langle Q - \alpha_1, \rho \rangle}\right)$.

We now turn our attention to $\Phi^2(z; \mathbf{c})$. Then we can use the two-dimensional path decomposition for the planar, drifted Brownian motion $t \mapsto \mathbf{X}_{t+r}(0) - \mathbf{X}_r(0) + (\alpha_1 - Q)t$ to write that, using the notations from Subsection 3.3.4.1

$$\begin{aligned} & \mathbb{E} \left[e^{-\sum_{i=1}^2 e^{\gamma \langle \mathbf{c}, e_i \rangle} I_r^i(0)} \left(e^{-e^{\gamma \langle \mathbf{c}, e_1 \rangle} J_r^1(z)} - 1 \right) \left(e^{-e^{\gamma \langle \mathbf{c}, e_2 \rangle} J_r^2(z)} - 1 \right) \right] \\ &= \sum_{s \in W_{1,2}} \lambda_s \int_{\mathcal{C}} e^{\langle \alpha_1 - \hat{s}\alpha_1, \mathbf{M} \rangle} \mathbb{E} \left[\prod_{i=1}^2 e^{-e^{\gamma \langle \mathbf{c}, e_i \rangle} I_r^i(0)} \left(e^{-e^{\gamma \langle \Lambda_r + \mathbf{M}, e_1 \rangle} J_r^i(z; -\mathbf{M})} - 1 \right) \right] d\mathbf{M}, \end{aligned}$$

where $\Lambda_r := \mathbf{X}_r(0) + (\alpha_1 - Q)r + \frac{2}{\gamma}h_1 \ln |z| + \mathbf{c}$, and with the process from Subsection 3.3.4.1 that enters the definition of

$$J_r^i(z; -\mathbf{M}) := \int_0^{+\infty} e^{\gamma \langle \mathcal{B}_t^\nu, e_i \rangle} \int_0^{2\pi} \left| \frac{e^{-t-r+i\theta}}{z} - 1 \right|^{2 \langle h_1, e_i \rangle} F_i(e^{-t-r+i\theta}) M_Y^{\gamma e_i}(dt + r, d\theta)$$

having drift $\nu = \alpha_1 - Q$ and being started from $-\mathbf{M}$.

For fixed $s \in W_{1,2}$, we make the change of variable $\mathbf{M} \leftrightarrow \Lambda_r + \mathbf{M}$ to end up with

$$\mathbb{E} \left[e^{\langle \hat{s}\alpha_1 - \alpha_1, \Lambda_r \rangle} \int_{\mathcal{C} + \Lambda_r} e^{\langle \alpha_1 - \hat{s}\alpha_1, \mathbf{M} \rangle} \prod_{i=1}^2 e^{-e^{\gamma \langle \mathbf{c}, e_i \rangle} I_r^i(0)} \left(e^{-e^{\gamma \langle \mathbf{M}, e_i \rangle} J_r^i(z; \Lambda_r - \mathbf{M})} - 1 \right) \right] d\mathbf{M}.$$

As usual the first exponential term is a Girsanov transform whose effect is to shift the law of $(\mathbf{X}(x))_{|x| > e^{-r}}$ by $(\hat{s}\alpha_1 - \alpha_1)G(0, \cdot)$, and to shift the law of Λ_r to that of

$$\Lambda_r^s := \mathbf{c} + \mathbf{X}_r(0) + (\hat{s}\alpha_1 - Q)r + \frac{2}{\gamma}h_1 \ln |z|.$$

This allows to rewrite the latter as

$$|z|^{\frac{2}{\gamma}\langle\hat{s}\alpha_1-\alpha_1,h_1\rangle} e^{\langle\hat{s}\alpha_1-\alpha_1,\mathbf{c}\rangle} \mathbb{E} \left[\int_{\mathcal{C}+\Lambda_r^s} e^{\langle\alpha_1-\hat{s}\alpha_1,\mathbf{M}\rangle} \prod_{i=1}^2 e^{-e^{\gamma\langle\mathbf{c},e_i\rangle} \hat{I}_{s,r}^i(0)} \left(e^{-e^{\gamma\langle\mathbf{M},e_i\rangle} J_r^i(z;\Lambda_r-\mathbf{M})} - 1 \right) \right] d\mathbf{M}$$

where the notation $\hat{I}_{s,r}^i(0)$ means that α_1 is replaced by $\hat{s}\alpha_1$ in the definition of $I_r^i(0)$. Now one easily checks that $\hat{s}\alpha_1 - \frac{2}{\gamma}h_1 - Q$ belongs to the Weyl chamber $\mathcal{C}_-$ as soon as $\langle Q - \alpha_1, \rho \rangle < \frac{2}{\gamma}$, which we assumed to hold. Moreover if $s = s_1 s_2$ we see that

$$\langle \hat{s}_1 \hat{s}_2 \alpha_1 - \hat{s}_2 \hat{s}_1 \alpha_1, \hat{s}_1 \hat{s}_2 \alpha_1 - \frac{2}{\gamma} h_1 - Q \rangle = \langle \alpha_1 - Q, e_1 \rangle^2 + \langle \alpha_1 - Q, e_2 \rangle \langle \alpha_1 - Q, \rho \rangle + \frac{2}{\gamma} \langle Q - \alpha_1, e_2 \rangle$$

so that $\langle \hat{s}_1 \hat{s}_2 \alpha_1 - \hat{s}_2 \hat{s}_1 \alpha_1, \mathbf{c} + \Lambda_r^{s_1 s_2} \rangle \rightarrow +\infty$ almost surely. As a consequence the reasoning conducted in the proof of Theorem 6.3.1 shows that in this asymptotic the term corresponding to $s = s_1 s_2$ in the expectation term will be given by

$$|z|^{\frac{2}{\gamma}\langle Q-\alpha_1,\rho\rangle} R_{s_1 s_2}(\alpha_1) \Phi_{\hat{s}_1 \hat{s}_2 \alpha_1 - \chi h_1}(\mathbf{c}) + l.o.t.$$

Therefore it remains to consider the terms that correspond to $s = s_2 s_1$ and $s = s_1 s_2 s_1$. For $s = s_2 s_1$ the issue is that the limit $\lim_{r \rightarrow +\infty} \hat{I}_{s,r}^i(0)$ does not make sense, and therefore the asymptotic is governed by the fusion asymptotics (4.4.2). They yield the estimate

$$\mathbb{E} \left[\int_{\mathcal{C}+\Lambda_r^s} e^{\langle\alpha_1-\hat{s}\alpha_1,\mathbf{M}\rangle} \prod_{i=1}^2 e^{-e^{\gamma\langle\mathbf{c},e_i\rangle} \hat{I}_{s,r}^i(0)} \left(e^{-e^{\gamma\langle\mathbf{M},e_i\rangle} J_r^i(z;\Lambda_r-\mathbf{M})} - 1 \right) \right] d\mathbf{M} = o \left(e^{-r \left(\frac{\langle \alpha_1 - Q, e_2 \rangle^2}{4} - \eta \right)} \right)$$

for any positive η . As a consequence by choosing r large enough compared to $-\ln |z|$ we see that this expectation term is a $o \left(|z|^{\frac{2}{\gamma}\langle Q-\alpha_1, e_2 \rangle} \right)$. The corresponding term in the expansion of Φ^2 is thus seen to be a $o \left(|z|^{\frac{2}{\gamma}\langle Q-\alpha_1, \rho \rangle} \right)$ as desired.

As for the case where $s = s_1 s_2 s_1$, the leading term is $|z|^{\frac{2}{\gamma}\langle Q-\alpha_1, \rho \rangle}$ while the expectation term has the same feature in terms of fusion asymptotics. As a consequence it is easily seen to be a lower order term too.

Eventually this shows that for α_1 close to Q ,

$$\begin{aligned} \Phi_{\alpha_1, -\frac{2}{\gamma}h_1}(z; \mathbf{c}) &= \Phi_{\alpha_1, -\frac{2}{\gamma}h_1}(\mathbf{c}) + |z|^{\frac{2}{\gamma}\langle Q-\alpha_1, e_1 \rangle} R_{s_1}(\alpha_1) \Phi_{\hat{s}_1(\alpha_1 - \frac{2}{\gamma}h_2)}(\mathbf{c}) \\ &\quad + |z|^{\frac{2}{\gamma}\langle Q-\alpha_1, \rho \rangle} R_{s_1 s_2}(\alpha_1) \Phi_{\hat{s}_1 \hat{s}_2(\alpha_1 - \frac{2}{\gamma}h_3)}(\mathbf{c}) + \mathfrak{R}(z) \end{aligned} \tag{7.2.16}$$

where $\mathfrak{R}$ is a lower order term in z .

7.2.4.2 Extending the validity of Equation (7.2.16)

We have just provided an expansion of $\Phi_{\alpha_1, -\frac{2}{\gamma}h_1}(z; \mathbf{c})$ under the assumption that α_1 is close to Q . Using the *a priori* form of the expansion given by Theorem 7.1.1 we can actually extend the range of values for which this expansion is valid.

Indeed we can infer from Theorem 7.1.1 that the remainder term $\mathfrak{R}(z)$ can be put under the form

$$\mathfrak{R}(z) = \sum_{i=0}^2 |z|^{\frac{2}{\gamma}\langle Q-\alpha_1, \sum_{j=1}^i e_j \rangle} P_i(z, \bar{z}; \alpha_1) \tag{7.2.17}$$

with $P_i(z, \bar{z}; \alpha_1)$ a power series in $z, \bar{z}$, as soon as $\mathcal{H}$ makes sense. We will see below that $\mathcal{H}$ can be defined provided that $\gamma > 1$ by choosing α_1 close to $q\omega_1 + \frac{2}{\gamma}\omega_2$ with $\langle \alpha_1, e_2 \rangle < \frac{2}{\gamma}$. Therefore the expansion (7.2.17) is valid provided that α_1 satisfies such assumptions.

Now the left-hand side as well as the three first terms in the right-hand side in Equation (7.2.16) depend analytically on $\langle \alpha_1, e_2 \rangle$ for the whole range of values for which $\langle \alpha_1, e_2 \rangle < q$. As a consequence the power series $P_i(z, \bar{z}; \alpha_1)$ are actually analytic in $\langle \alpha_1, e_2 \rangle$ (in a complex neighbourhood of) for $\langle \alpha_1, e_2 \rangle < q$, which allows to extend the validity of Equation (7.2.17) for $\langle \alpha_1, e_2 \rangle$ close to q . But in that case we have proved that $\Re$ is such that

$$\Re(z) = o\left(|z|^{\frac{2}{\gamma}\langle Q-\alpha_1, \rho \rangle}\right),$$

so that $P_i(0, 0; \alpha_1) = 0$ for α_1 close to Q . By analyticity this equality extends for $\langle \alpha_1, e_2 \rangle < \frac{2}{\gamma}$ too, which is the framework where $\mathcal{H}$ makes sense. Therefore the remainder term $\Re$ does not contribute to the identification of the coefficients coming from the distinct expansions of $\mathcal{H}$ given by Theorem 7.1.1 and Lemma 7.2.6.

All in all, we can conclude that Lemma 7.2.6 does indeed hold. $\square$

7.2.5 . Conclusion of the proof of Theorem 7.2.2

7.2.5.1 Analyticity of the extension: proof of Proposition 7.2.1

To start with let us consider α_1 as in Lemma 7.2.3 in such a way that the set of weights $\alpha = (\alpha_1, -\gamma h_1, \alpha_2^*, \alpha_3) \in \mathcal{A}_4$ with $\alpha_2^* = \kappa\omega_2$ is non-empty. Note that this is possible for all values of γ : indeed if $\alpha_1 = \frac{2}{\gamma}\omega_1 + q\omega_2$, $\alpha_3 = \frac{2}{\gamma}\omega_2 + q\omega_1$ and $\alpha_2^* = q\omega_2$ then we have $\langle s, \omega_1 \rangle + \gamma = \frac{1}{3}(\frac{2}{\gamma} - \gamma)$, which is positive since $\gamma \in (0, \sqrt{2})$. Likewise $\langle s, \omega_2 \rangle = \frac{2}{3}(\frac{2}{\gamma} - \gamma)$ is positive too.

Then for such an α_1 , we have provided (under the assumptions of Lemma 7.2.3) an expansion of the four-point correlation function $\langle V_{-\gamma h_1}(z)V_{\alpha_1}(0)V_{\alpha_2^*}(1)V_{\alpha_3}(\infty) \rangle$ similar to that of Theorem 7.1.1. In particular by linear independence of the hypergeometric functions the coefficients that appear in this expansion can be identified, and therefore we see that we have the equalities

$$\begin{aligned} \frac{C_\gamma(\alpha_1 - \gamma h_2, \alpha_2^*, \alpha_3)}{A_\gamma^{(1)}(-\gamma h_1, \alpha_1, \alpha_2^*, \alpha_3)} &= \frac{C_\gamma(\alpha_1 - \gamma h_1, \alpha_2^*, \alpha_3)}{B_\gamma^{(1)}(\alpha_1, \gamma)} \quad \text{and} \\ \frac{R_{s_2}(\hat{s}_2(\alpha_1 - \gamma h_3))C_\gamma(\hat{s}_2(\alpha_1 - \gamma h_3), \alpha_2^*, \alpha_3)}{A_\gamma^{(2)}(-\gamma h_1, \alpha_1, \alpha_2^*, \alpha_3)} &= \frac{C_\gamma(\alpha_1 - \gamma h_1, \alpha_2^*, \alpha_3)}{B_\gamma^{(2)}(\alpha_1, \gamma)}. \end{aligned}$$

Using analyticity of three-point correlation functions as well as analyticity of A_γ and B we can extend the first equality to the whole range of values of α_1 such that $(\alpha_1 - \gamma h_i, \alpha_2^*, \alpha_3) \in \mathcal{A}_3$ for $i = 1, 2$ and the second equality for $(\hat{s}_2(\alpha_1 - \gamma h_3), \alpha_2^*, \alpha_3)$ and $(\alpha_1 - \gamma h_1, \alpha_2^*, \alpha_3)$ in $\mathcal{A}_3$.

We can proceed in a similar way by taking α_1 as considered in the statements of Lemma 7.2.5. Like above the set of $(\alpha_1, -\gamma h_1, \alpha_2^*, \alpha_3)$ satisfying the assumptions of Lemma 7.2.5 is non-empty for any value of $\gamma \in (0, \sqrt{2})$, by taking α_1 close to $q\omega_1 + \frac{2}{\gamma}\omega_2$,

while $\alpha_3 = \frac{2}{\gamma}\omega_1 + q\omega_2$ and $\alpha_2^* = q\omega_2$. This shows that

$$\frac{R_{s_1}(\hat{s}_1(\alpha_1 - \gamma h_2))C_\gamma(\hat{s}_1(\alpha_1 - \gamma h_2), \alpha_2^*, \alpha_3)}{A_\gamma^{(1)}(-\gamma h_1, \alpha_1, \alpha_2^*, \alpha_3)} = \frac{C_\gamma(\alpha_1 - \gamma h_1, \alpha_2^*, \alpha_3)}{B_\gamma^{(1)}(\alpha_1, \gamma)} \quad \text{and}$$

$$\frac{C_\gamma(\alpha_1 - \gamma h_3, \alpha_2^*, \alpha_3)}{A_\gamma^{(2)}(-\gamma h_1, \alpha_1, \alpha_2^*, \alpha_3)} = \frac{C_\gamma(\alpha_1 - \gamma h_1, \alpha_2^*, \alpha_3)}{B_\gamma^{(2)}(\alpha_1, \gamma)}$$

as soon as α_1 is such that $(\hat{s}(\alpha_1 - \gamma h_i), \alpha_2^*, \alpha_3)$ for $i = 2$ or $i = 3$ and $(\alpha_1 - \gamma h_1, \alpha_2^*, \alpha_3)$ belong to $\mathcal{A}_3$.

Now let us consider the map

$$F : \beta \mapsto C_\gamma(\beta - \gamma e_1, \alpha_2^*, \alpha_3) \frac{A_\gamma^{(1)}(-\gamma h_1, \beta + \gamma h_2, \alpha_2^*, \alpha_3)}{B^{(1)}(\beta + \gamma h_2)}.$$

Then this map is analytic in a complex neighbourhood of the set of β such that $(\beta - \gamma e_1, \alpha_2^*, \alpha_3)$. Moreover we have seen above that it coincides with $C_\gamma(\beta, \alpha_2^*, \alpha_3)$ provided that β is in addition such that $(\beta, \alpha_2^*, \alpha_3)$ belongs to $\mathcal{A}_3$. Likewise we have proved that it is equal to $R_{s_1}(\hat{s}_1(\alpha_1 - \gamma h_2))C_\gamma(\hat{s}_1(\alpha_1 - \gamma h_2), \alpha_2^*, \alpha_3)$ as soon as $(\hat{s}_1\beta, \alpha_2^*, \alpha_3) \in \mathcal{A}_3$. This shows that the map defined by setting $R_s(\hat{s}\beta)C_\gamma(\hat{s}\beta, \alpha_2^*, \alpha_3)$ where $s\beta \in Q + \mathcal{C}_-$ is actually equal to F , and is therefore analytic over a complex neighbourhood of the subset of β such that $(\beta, \alpha_2^*, \alpha_3)$ or $(\hat{s}_1\beta, \alpha_2^*, \alpha_3)$ belongs to $\mathcal{A}_3$.

The same reasoning applies when $s = s_2$. Namely we have obtained the equality

$$R_s(\hat{s}\beta)C_\gamma(\hat{s}\beta, \alpha_2^*, \alpha_3) = G(\beta), \quad G(\beta) := C_\gamma(\beta - \gamma \rho, \alpha_2^*, \alpha_3) \frac{A_\gamma^{(2)}(-\gamma h_1, \beta + \gamma h_3, \alpha_2^*, \alpha_3)}{B^{(2)}(\beta + \gamma h_3)}$$

for $s \in \{Id, s_2\}$ such that $(\hat{s}\beta, \alpha_2^*, \alpha_3) \in \mathcal{A}_3$, depending on the values of α_1 around $\frac{2}{\gamma}\omega_1 + q\omega_2$. Like above this shows that the map $R_s(\hat{s}\beta)C_\gamma(\hat{s}\beta, \alpha_2^*, \alpha_3)$ is analytic in a complex neighbourhood of the subset of β such that $(\beta, \alpha_2^*, \alpha_3)$ or $(\hat{s}_2\beta, \alpha_2^*, \alpha_3)$ belongs to $\mathcal{A}_3$.

Let us now look at what happens for other elements of the Weyl group W . It is readily seen that for any fixed $s \in W$, the map

$$\beta \mapsto R_s(\beta)C_\gamma(\hat{s}\beta, \alpha_2^*, \alpha_3)$$

is analytic in a complex neighbourhood of $\mathcal{U}_s(\alpha_2^*, \alpha_3) := \{\beta \in \mathbb{R}^2, (\hat{s}\beta, \alpha_2^*, \alpha_3) \in \mathcal{A}_3\}$. Moreover we have proved above that for $i = 1, 2$ the map defined by setting

$$\beta \mapsto R_{s_i}(\hat{s}\beta)C_\gamma(\hat{s}_i\hat{s}\beta, \alpha_2^*, \alpha_3)$$

was analytic in a complex neighbourhood of $\mathcal{U}_s(\alpha_2^*, \alpha_3) \cup \mathcal{U}_{s_i s}(\alpha_2^*, \alpha_3)$. Because the reflection coefficients enjoy the property that $R_{s_i}(\hat{s}\beta)R_s(\beta) = R_{s_i s}(\beta)$ we therefore see that the map from Equation (7.2.2) is actually analytic in a complex neighbourhood of $\mathcal{U}_s(\alpha_2^*, \alpha_3) \cup \mathcal{U}_{s_i s}(\alpha_2^*, \alpha_3)$. As a consequence this map is seen to be analytic in a complex neighbourhood of

$$\bigcup_{w \in \langle s_1, s_2 \rangle} \mathcal{U}_w(\alpha_2^*, \alpha_3)$$

where $\langle s_1, s_2 \rangle$ is the group generated by s_1 and s_2 . The latter being nothing but W , we infer that the map is analytic in a complex neighbourhood of $\mathcal{U}(\alpha_2^*, \alpha_3)$. This shows that as desired, the extension of the three-points correlation functions from Equation (7.2.2) is analytic in a complex neighbourhood of $\mathcal{U}(\alpha_2^*, \alpha_3)$, which was the statement of Proposition 7.2.1.

7.2.5.2 Operator Product Expansions and shift equations

Having proved that the extension $\alpha \mapsto R_s(\alpha)C_\gamma(\hat{s}\alpha, \alpha_2^*, \alpha_3)$ is analytic in a complex neighbourhood of $\mathcal{U}(\alpha_2^*, \alpha_3)$, we can denote by $C_\gamma(\alpha, \alpha_2^*, \alpha_3)$ this extension. Then as explained above, we know that the equalities

$$\frac{C_\gamma(\alpha_1 - \gamma h_2, \alpha_2^*, \alpha_3)}{A_\gamma^{(1)}(-\gamma h_1, \alpha_1, \alpha_2^*, \alpha_3)} = \frac{C_\gamma(\alpha_1 - \gamma h_1, \alpha_2^*, \alpha_3)}{B_\gamma^{(1)}(\alpha_1, \gamma)} \quad \text{and} \quad (7.2.18)$$

$$\frac{C_\gamma(\alpha_1 - \gamma h_3, \alpha_2^*, \alpha_3)}{A_\gamma^{(2)}(-\gamma h_1, \alpha_1, \alpha_2^*, \alpha_3)} = \frac{C_\gamma(\alpha_1 - \gamma h_1, \alpha_2^*, \alpha_3)}{B_\gamma^{(2)}(\alpha_1, \gamma)} \quad (7.2.19)$$

hold true in some open subset of $\mathcal{U}(\alpha_2^*, \alpha_3)$. By analyticity of the left and right-hand sides this equality extends to the whole range of values for which it makes sense. Put differently we recover Equation (7.2.3) in the case where $\chi = \gamma$.

When $\chi = \frac{2}{\gamma}$, the same reasoning remains valid. Namely thanks to Lemma 7.2.6 we know that as soon as the set of $(\alpha_1, -\frac{2}{\gamma}h_1, \alpha_2^*, \alpha_3) \in \mathcal{A}_4$ with α_1 as in the statement of Lemma 7.2.6 is non-empty we have the equalities

$$\frac{C_\gamma(\alpha_1 - \frac{2}{\gamma}h_2, \alpha_2^*, \alpha_3)}{A_\gamma^{(1)}(-\frac{2}{\gamma}h_1, \alpha_1, \alpha_2^*, \alpha_3)} = \frac{C_\gamma(\alpha_1 - \frac{2}{\gamma}h_1, \alpha_2^*, \alpha_3)}{B_\gamma^{(1)}(\alpha_1, \frac{2}{\gamma})} \quad \text{and} \quad (7.2.20)$$

$$\frac{C_\gamma(\alpha_1 - \frac{2}{\gamma}h_3, \alpha_2^*, \alpha_3)}{A_\gamma^{(2)}(-\frac{2}{\gamma}h_1, \alpha_1, \alpha_2^*, \alpha_3)} = \frac{C_\gamma(\alpha_1 - \frac{2}{\gamma}h_1, \alpha_2^*, \alpha_3)}{B_\gamma^{(2)}(\alpha_1, \frac{2}{\gamma})}. \quad (7.2.21)$$

Therefore to conclude for the proof of Theorem 7.2.2 it suffices to check that for any fixed value of $\gamma \in (0, \sqrt{2})$, if the set of $(\alpha_1, -\frac{2}{\gamma}h_1, \alpha_2^*, \alpha_3) \in \mathcal{A}_4$ is non-empty then we can find $(\alpha_1, -\frac{2}{\gamma}h_1, \alpha_2^*, \alpha_3)$ that meets the requirements of Lemma 7.2.6. Now one can check that as soon as $(\alpha_1, -\frac{2}{\gamma}h_1, \alpha_2^*, \alpha_3) \in \mathcal{A}_4$, we have $\langle s, \omega_1 \rangle < -\frac{1}{3} \left(\gamma + \frac{2}{\gamma} \right)$ so that for $\gamma \leq 1$ (which implies that $-\frac{1}{3} \left(\gamma + \frac{2}{\gamma} \right) \leq -\gamma$) this set is empty. Conversely if $\gamma > 1$ by choosing $\alpha_1 = \alpha_3 = q\omega_1 + \frac{2}{\gamma}\omega_2$ while $\alpha_2^* = q\omega_2$ then $\langle s, \omega_1 \rangle = -\frac{1}{3} \left(\gamma + \frac{2}{\gamma} \right) > -\gamma$ so that this set is non-empty, and by choosing the weights close to the above choice the assumptions of Lemma 7.2.6 are fulfilled. This wraps up the proof of Theorem 7.2.2.

7.3 . Shift equations and computation of the three-point correlation functions

This concluding section brings together the building blocks unveiled in the previous sections to provide a proof of one of the main statement of the present document, the computation of certain structure constants associated to the $\mathfrak{sl}_3$ Toda theory. The reasoning developed throughout the manuscript will allow us in this section to provide a rigorous derivation of the Fateev-Litvinov formula. But to do so we first give some background on the special function Υ and then bring the proof of this statement to its end.

7.3.1 . On the Fateev-Litvinov formula

The Fateev-Litvinov [52] formula proposed for the three-point correlation functions involves the special function Υ , which is ubiquitous in Liouville theory and more generally in Toda CFTs. One of the reasons why it is so is that it enjoys remarkable shift equations. These shift equations take the form²:

$$\Upsilon(z + \chi) = l\left(\frac{\chi}{2}\right) \left(\frac{\chi}{\sqrt{2}}\right)^{1-\chi z} \Upsilon(z), \quad (7.3.1)$$

valid for $z \in \mathbb{C}$ and $\chi \in \{\gamma, \frac{2}{\gamma}\}$. When $0 < \Re(z) < q = \gamma + \frac{2}{\gamma}$, this special function admits the integral representation

$$\ln \Upsilon(z) = \int_0^{+\infty} \left(\left(\frac{q}{2} - z\right)^2 \frac{e^{-t}}{2} - \frac{\left(\sinh\left(\left(\frac{q}{2} - z\right) \frac{t}{2\sqrt{2}}\right)\right)^2}{\sinh\left(\frac{t\gamma}{2\sqrt{2}}\right) \sinh\left(\frac{2\sqrt{2}t}{\gamma}\right)} \right) \frac{dt}{t}, \quad (7.3.2)$$

while the shift equations (7.3.1) allow to continue it to an analytic function over $\mathbb{C}$, with no poles and zeros given by the $(-\gamma\mathbb{N} - \frac{2}{\gamma}\mathbb{N}) \cup (q + \gamma\mathbb{N} + \frac{2}{\gamma}\mathbb{N})$, and which satisfies $\Upsilon(q - z) = \Upsilon(z)$.

Based on this special function, the Fateev-Litvinov formula [52] then takes the form

$$C_\gamma^{\text{FL}}(\alpha_1, \alpha_2^*, \alpha_3) = \left(\pi \mu l \left(\frac{\gamma^2}{2}\right) \left(\frac{\gamma}{\sqrt{2}}\right)^{2-\gamma^2} \right)^{\frac{\langle 2Q - \alpha_1 - \alpha_2^* - \alpha_3, \rho \rangle}{\gamma}} \frac{\Upsilon'(0)^2 \Upsilon(\kappa) \prod_{e \in \Phi^+} \Upsilon(\langle Q - \alpha_1, e \rangle) \Upsilon(\langle Q - \alpha_3, e \rangle)}{\prod_{1 \leq j, k \leq 3} \Upsilon\left(\frac{\kappa}{3} + \langle \alpha_1 - Q, h_j \rangle\right) + \langle \alpha_3 - Q, h_k \rangle} \quad (7.3.3)$$

where recall that $l(z) = \frac{\Gamma(z)}{\Gamma(1-z)}$.

The formula proposed in [52] for $\mathfrak{sl}_n$ Toda three-point correlation functions closely resembles that of the DOZZ formula [45, 141, 84] for Liouville theory. And actually the DOZZ formula can be recovered from the expression of the $\mathfrak{sl}_3$ Toda three-point correlation functions as the following statement discloses:

Lemma 7.3.1. *Assume that $\langle s, \omega_1 \rangle \rightarrow 0$ while $\langle s, \omega_2 \rangle$ remains positive. Then*

$$C_\gamma^{\text{FL}}(\alpha_1, \alpha_2^*, \alpha_3) \sim \frac{1}{\sqrt{2} \langle s, \omega_1 \rangle} C_{\sqrt{2}\gamma}^{\text{DOZZ}} \left(\frac{\langle \alpha_1, e_2 \rangle}{\sqrt{2}}, \frac{\langle \alpha_2^*, e_2 \rangle}{\sqrt{2}}, \frac{\langle \alpha_3, e_2 \rangle}{\sqrt{2}} \right). \quad (7.3.4)$$

Proof. As $\langle s, \omega_1 \rangle \rightarrow 0$, the prefactor converges to $\left(\pi \mu l \left(\frac{\gamma^2}{2}\right) \left(\frac{\gamma}{\sqrt{2}}\right)^{2-\gamma^2} \right)^{\frac{\langle 2Q - \bar{\alpha}, e_2 \rangle}{\gamma}}$ where we have used that $\langle s, \omega_1 \rangle = 0$ and denoted $\bar{\alpha} = \sum_{k=1}^3 \alpha_k$. Likewise by looking at

²Note that the special function considered here differs from the standard expression because of the convention on the length of the simple roots. One recovers the usual expression via the correspondence $\Upsilon(z) = \Upsilon_{\frac{\gamma}{\sqrt{2}}}(\frac{1}{\sqrt{2}} \cdot)$

each factor appearing in the product $\prod_{1 \leq j, k \leq 3} \Upsilon\left(\frac{\kappa}{3} + \langle \alpha_1 - Q, h_j \rangle + \langle \alpha_3 - Q, h_k \rangle\right)$ one can check that

$$C_\gamma^{\text{FL}}(\alpha_1, \alpha_2^*, \alpha_3) \sim \frac{1}{\langle \mathbf{s}, \omega_1 \rangle} \left(\pi \mu l \left(\frac{\gamma^2}{2} \right) \left(\frac{\gamma^2}{2} \right)^{2-\gamma^2} \right)^{\frac{\langle 2Q - \bar{\alpha}, e_2 \rangle}{\gamma}} \frac{\Upsilon'(0) \Upsilon(\kappa) \Upsilon(\langle Q - \alpha_1, e_2 \rangle) \Upsilon(\langle Q - \alpha_3, e_2 \rangle)}{\Upsilon\left(\frac{\langle \bar{\alpha} - 2Q, e_2 \rangle}{2}\right) \Upsilon\left(\frac{\langle \bar{\alpha}, e_2 \rangle}{2} - \langle \alpha_1, e_2 \rangle\right) \Upsilon\left(\frac{\langle \bar{\alpha}, e_2 \rangle}{2} - \langle \alpha_2^*, e_2 \rangle\right) \Upsilon\left(\frac{\langle \bar{\alpha}, e_2 \rangle}{2} - \langle \alpha_3, e_2 \rangle\right)}.$$

Using the fact that $\Upsilon(\langle Q - \alpha_3, e_2 \rangle) = \Upsilon(\langle \alpha_3, e_2 \rangle)$, via our convention on the Upsilon function the latter is nothing but the DOZZ formula, up to the normalization factor $\sqrt{2}$, which shows that $C_\gamma^{\text{FL}}(\alpha_1, \alpha_2^*, \alpha_3) \sim \frac{1}{\langle \mathbf{s}, \omega_1 \rangle} \frac{1}{\sqrt{2}} C_{\sqrt{2}\gamma}^{\text{DOZZ}}\left(\frac{\langle \alpha_1, e_2 \rangle}{\sqrt{2}}, \frac{\langle \alpha_2^*, e_2 \rangle}{\sqrt{2}}, \frac{\langle \alpha_3, e_2 \rangle}{\sqrt{2}}\right)$. $\square$

A counterpart statement also holds for the probabilistically defined correlation functions:

Lemma 7.3.2. *Assume that $\alpha_1, \alpha_3 \in Q + \mathcal{C}_-$ are close enough to Q so that $\kappa := \langle 2Q - \alpha_1 - \alpha_3, 3\omega_1 \rangle$ is such that $\kappa < q$. Then, as $\varepsilon \rightarrow 0$ with $\varepsilon > 0$,*

$$C_\gamma(\alpha_1, (\kappa + \varepsilon)\omega_2, \alpha_3) \sim \frac{3}{\sqrt{2}\varepsilon} C_{\sqrt{2}\gamma}^{\text{DOZZ}}\left(\frac{\langle \alpha_1, e_2 \rangle}{\sqrt{2}}, \frac{\langle \alpha_2^*, e_2 \rangle}{\sqrt{2}}, \frac{\langle \alpha_3, e_2 \rangle}{\sqrt{2}}\right). \quad (7.3.5)$$

Proof. Under the assumptions made on the weights, we see that the three-point correlation functions $C_\gamma(\alpha_1, (\kappa + \varepsilon)\omega_2, \alpha_3)$ admits the probabilistic representation

$$\left(\prod_{i=1}^2 \frac{\Gamma\left(\frac{\langle \mathbf{s}, \omega_i \rangle}{\gamma}\right) \mu_i^{-\frac{\langle \mathbf{s}, \omega_i \rangle}{\gamma}}}{\gamma} \right) \mathbb{E} \left[\prod_{i=1}^2 \left(\int_{\mathbb{C}} \frac{|1 - x_i|^{-\gamma \langle \alpha_2^*, e_i \rangle}}{|x_i|^{\gamma \langle \alpha_1, e_i \rangle}} |x_i|_+^{\gamma \langle \alpha_1 + \alpha_2^* + \alpha_3, e_i \rangle} M^{\gamma e_i}(d^2 x_i) \right)^{-\frac{\langle \mathbf{s}, \omega_i \rangle}{\gamma}} \right],$$

where $\langle \mathbf{s}, \omega_1 \rangle = \frac{\varepsilon}{3}$ while $\langle \mathbf{s}, \omega_2 \rangle$ converges towards $\langle 2Q - \alpha_1 - \alpha_3, e_1 \rangle$. Therefore as $\varepsilon \rightarrow 0$ we see that $C_\gamma(\alpha_1, (\kappa + \varepsilon)\omega_2, \alpha_3)$ is asymptotically equivalent to

$$\frac{3}{\varepsilon} \frac{\Gamma\left(\frac{\langle \mathbf{s}, \omega_2 \rangle}{\gamma}\right) \mu_i^{-\frac{\langle \mathbf{s}, \omega_2 \rangle}{\gamma}}}{\gamma} \mathbb{E} \left[\left(\int_{\mathbb{C}} \frac{|1 - x_2|^{-\gamma \kappa}}{|x_2|^{\gamma \langle \alpha_1, e_2 \rangle}} |x_2|_+^{\gamma \langle \alpha_1 + \alpha_2^* + \alpha_3, e_2 \rangle} M^{\gamma e_2}(d^2 x_2) \right)^{-\frac{\langle \mathbf{s}, \omega_2 \rangle}{\gamma}} \right].$$

To check that the expectation term does indeed coincide with the probabilistic representation of the DOZZ formula proved in [84], it remains to ensure that $2\langle \mathbf{s}, \omega_2 \rangle = \langle \alpha_1 + \alpha_2^* + \alpha_3 - 2Q, e_2 \rangle$ where the factor 2 comes from the fact that the simple root e_2 has norm $\sqrt{2}$. Using the fact that $2\omega_2 - e_2 = \omega_1$ the above equality boils down to $\langle \alpha_1 + \alpha_2^* + \alpha_3 - 2Q, \omega_1 \rangle = 0$, which was our assumption. $\square$

The second statement about the proposed expression for the three-point correlation function is concerned with shift equations, which take the form:

$$\frac{\mathfrak{C}_\gamma(\alpha_1 - \chi h_{i+1}, \alpha_2^*, \alpha_3)}{\mathfrak{C}_\gamma(\alpha_1 - \chi h_1, \alpha_2^*, \alpha_3)} = \frac{A_\gamma^{(i)}(-\chi h_1, \alpha_1, \alpha_2^*, \alpha_3)}{B^{(i)}(\alpha_1)} \quad \text{for } i = 1, 2 \text{ and } \chi \in \left\{ \gamma, \frac{2}{\gamma} \right\}. \quad (7.3.6)$$

Equation (7.3.6) follows from the shift equation (7.3.1) of the Υ function after some elementary but lengthy computations.

7.3.2 . From shift equations to three-point correlation functions

We are now in position to compute the desired formula for three-point correlation functions. Indeed under the assumption that $\gamma > 1$, we have already seen along the proof of Theorem 7.2.2 that the set of weights $\alpha = (\alpha, \alpha_1, \alpha_2^*, \alpha_3) \in \mathcal{A}_4$ such that $\alpha = -\chi h_1$ with $\chi \in \{\gamma, \frac{2}{\gamma}\}$ and $\alpha_2^* = \kappa \omega_2$ for $\kappa < q$ is non-empty and open in $\mathcal{A}_4$.

Now we have seen in Theorem 7.1.1 that for such weights, the associated four-point correlation functions can be expressed in terms of hypergeometric functions as follows:

$$\langle V_\alpha(z) V_{\alpha_1}(0) V_{\alpha_2^*}(1) V_{\alpha_3}(\infty) \rangle = |z|^{\chi \langle h_1, \alpha_1 \rangle} |z - 1|^{\frac{\chi \kappa}{3}} \mathcal{H}(z), \quad \text{where}$$

$$\mathcal{H}(z) = C_\gamma(\alpha + \alpha_1, \alpha_2^*, \alpha_3) \left(|\mathcal{H}_0(z)|^2 + \sum_{i=1}^2 A_\gamma^{(i)}(\alpha, \alpha_1, \alpha_2^*, \alpha_3) |\mathcal{H}_i(z)|^2 \right). \quad (7.3.7)$$

On the other hand we proved in Theorem 7.2.2 that under the same assumptions on the weights, this function $\mathcal{H}$ could be expressed using different coefficients that involve three-point correlation functions:

$$\langle V_\alpha(z) V_{\alpha_1}(0) V_{\alpha_2^*}(1) V_{\alpha_3}(\infty) \rangle = |z|^{\chi \langle h_1, \alpha_1 \rangle} |z - 1|^{\frac{\chi \kappa}{3}} \mathcal{H}(z), \quad \text{where}$$

$$\mathcal{H}(z) = \sum_{i=1}^3 B_\gamma^{(i)}(\alpha_1, \chi) C_\gamma(\alpha_1 - \chi h_i, \alpha_2^*, \alpha_3) |\mathcal{H}_{i-1}(z)|^2,$$

where $C_\gamma(\alpha_1 - \chi h_i, \alpha_2^*, \alpha_3)$ denotes the extension of C_γ defined by setting $C_\gamma(\alpha, \alpha_2^*, \alpha_3) := R_s(\alpha) C_\gamma(\hat{s}\alpha, \alpha_2^*, \alpha_3)$ where $s \in W$ is such that $\hat{s}\alpha \in Q + \mathcal{C}_-$. This extension is analytic in virtue of Proposition 7.2.1.

Combining these two equalities and because the hypergeometric functions are linearly independent, we get for $i = 1, 2$ and $\chi \in \{\gamma, \frac{2}{\gamma}\}$, the following shift equations:

$$\frac{C_\gamma(\alpha_1 - \chi h_{i+1}, \alpha_2^*, \alpha_3)}{C_\gamma(\alpha_1 - \chi h_i, \alpha_2^*, \alpha_3)} = \frac{A_\gamma^{(i)}(-\chi h_1, \alpha_1, \alpha_2^*, \alpha_3)}{B^{(i)}(\alpha_1)}. \quad (7.3.8)$$

In particular this shift equation allows to extend the map $\alpha \mapsto C_\gamma(\alpha, \alpha_2^*, \alpha_3)$ to an open complex neighbourhood of $\mathbb{R}^2$ of the form $\mathbb{R}^2 \times (-\delta, \delta)^2$ on which it is analytic.

Now we have seen before that the expression $C_\gamma^{\text{FL}}(\alpha_1, \alpha_2^*, \alpha_3)$ proposed for the three-point correlation functions satisfies the very same set of shift equations. As a consequence the map defined by setting

$$\alpha \mapsto \frac{C_\gamma^{\text{FL}}(\alpha_1, \alpha_2^*, \alpha_3)}{C_\gamma(\alpha, \alpha_2^*, \alpha_3)}$$

is analytic over $\mathbb{R}^2 \times (-\delta, \delta)^2$ and is periodic with periods χe_i for $\chi \in \{\gamma, \frac{2}{\gamma}\}$ and $i = 1, 2$.

Remark 7.3.3. *At this stage it is worth noticing that for the general case where $\gamma \in (0, \sqrt{2})$ the following shift equations hold true*

$$\frac{C_\gamma(\alpha_1, \alpha_2, \alpha_3)}{C_\gamma^{\text{FL}}(\alpha_1, \alpha_2^*, \alpha_3)} = \frac{C_\gamma(\alpha_1 + \gamma e, \alpha_2, \alpha_3)}{C_\gamma^{\text{FL}}(\alpha_1 + \gamma e, \alpha_2^*, \alpha_3)} \quad (7.3.9)$$

for every $e \in \Phi^+$. The main obstruction at the time being that prevents one from deriving the shift equation dual to Equation (7.3.9)

$$\frac{C_\gamma(\alpha_1, \alpha_2, \alpha_3)}{C_\gamma^{\text{FL}}(\alpha_1, \alpha_2^*, \alpha_3)} = \frac{C_\gamma(\alpha_1 + \frac{2}{\gamma}e, \alpha_2, \alpha_3)}{C_\gamma^{\text{FL}}(\alpha_1 + \frac{2}{\gamma}e, \alpha_2^*, \alpha_3)} \quad (7.3.10)$$

stems from the fact that for this we would need to define correlation functions with $\chi = \frac{2}{\gamma}$ and $\gamma \in (0, 1)$, which is technically demanding and out of range for the time being.

As soon as $\frac{\gamma^2}{2}$ is not a rational number, it is readily seen that the set $\gamma\mathbb{Z}e_1 + \gamma\mathbb{Z}e_2 + \frac{2}{\gamma}\mathbb{Z}e_1 + \frac{2}{\gamma}\mathbb{Z}e_2$ is dense in $\mathbb{R}^2$. Therefore under this assumption that $\frac{\gamma^2}{2} \notin \mathbb{Q}$, we get that $C_\gamma(\alpha_1, \alpha_2^*, \alpha_3) = a_\gamma(\alpha_2^*, \alpha_3)C_\gamma^{\text{FL}}(\alpha_1, \alpha_2^*, \alpha_3)$ for some constant $a_\gamma(\alpha_2^*, \alpha_3)$ independent of α_1 . By symmetry in the weights α we see that this constant is actually independent of these three variables. It can be evaluated using Lemmas 7.3.1 and 7.3.2, thanks to which it is found to be equal to 1. Therefore we have proved that $C_\gamma(\alpha_1, \alpha_2^*, \alpha_3) = C_\gamma^{\text{FL}}(\alpha_1, \alpha_2^*, \alpha_3)$ for $\frac{\gamma^2}{2} \in (1, \sqrt{2}) \setminus \mathbb{Q}$. Because both quantities are continuous in the variable γ this equality extends to all values of $\gamma \in [1, \sqrt{2})$. We can thus conclude that the following statement is true:

Theorem 7.3.4. *For $\gamma \in [1, \sqrt{2})$ assume that $V_{\alpha_2^*}$ is a semi-degenerate field and that $(\alpha_1, \alpha_2^*, \alpha_3)$ belongs to $\mathcal{A}_N$. Then*

$$C_\gamma(\alpha_1, \alpha_2^*, \alpha_3) = C_\gamma^{\text{FL}}(\alpha_1, \alpha_2^*, \alpha_3) \quad (7.3.11)$$

where the left-hand side is a probabilistic three-point correlation function.

8 - Fonctions de corrélations à quatre points : le bootstrap conforme

Le chapitre précédent nous a permis de calculer explicitement la valeur d'une famille de constantes de structure pour la théorie de Toda associée à $\mathfrak{g} = \mathfrak{sl}_3$. La dernière étape dans l'implémentation de la procédure du bootstrap conforme est la description d'une méthode récursive pour calculer les fonctions de corrélation avec un nombre plus élevé d'insertions à partir de celles à trois-points.

Dans cette perspective nous nous attacherons ici à expliciter comment calculer des fonctions de corrélation à quatre points en fonction des constantes de structure de la théorie – le même raisonnement restant valide lorsque sont considérées des fonctions de corrélation contenant plus d'insertions. Pour ce faire nous étudierons les propriétés spectrales d'un opérateur auto-adjoint : l'*Hamiltonien de Toda*.

Les éléments présentés ici sont issus d'un travail en cours mené conjointement avec Colin Guillarmou [32]. Nous n'esquisserons que les grandes lignes de ce travail afin que le manuscrit garde une taille raisonnable et pour ne pas faire appel à des notions trop diverses. Plus de détails sur les objets considérés ainsi que sur les preuves des résultats énoncés seront donnés dans un futur travail [32].

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8.1 . Conformal bootstrap in the $\mathfrak{sl}_3$ Toda theory

The derivation of the Fateev-Litvinov formula for a family of probabilistic three-point correlation functions represents the first step in the implementation of the conformal bootstrap method. We can then carry on the recursive procedure by showing that one can reduce the computation of a family of four-point correlation functions to that of the three-point computed in the previous chapter. In this perspective we aim to prove in a future work with Guillarmou [32] that:

Theorem 8.1.1 (In preparation). *Assume that the Vertex Operators $V_{\alpha_2^*}$ and $V_{\alpha_3^*}$ are semi-degenerate fields. Further assume that the weights α satisfy the assumption that for $i = 1, 2$:*

$$\begin{aligned} \langle \alpha_k - Q, e_i \rangle < 0 \quad \text{for all } 1 \leq k \leq 4, \quad \text{with} \\ \langle \alpha_1 + \alpha_2 - Q, \omega_i \rangle > 0 \quad \text{and} \quad \langle \alpha_3 + \alpha_4 - Q, \omega_i \rangle > 0. \end{aligned} \quad (8.1.1)$$

Then

$$\begin{aligned} & \langle V_{\alpha_1}(0) V_{\alpha_2^*}(z) V_{\alpha_3^*}(1) V_{\alpha_4}(\infty) \rangle \\ &= \frac{1}{24\pi^2} \int_{\mathbb{R}^2} C_\gamma(\alpha_1, \alpha_2^*, Q - iP) C_\gamma(Q + iP, \alpha_3^*, \alpha_4) |z|^{2(\Delta_Q + iP - \Delta_{\alpha_1} - \Delta_{\alpha_2})} |\mathcal{F}_P^T(z)|^2 dP \end{aligned} \quad (8.1.2)$$

where the conformal blocks $\mathcal{F}_P^T$ only depend on the central charge of the theory as well as the conformal and W -quantum weights Δ_{α_k} and $w(\alpha_k)$ for $1 \leq k \leq 4$.

In the general case where it is no longer assumed that two Vertex Operators are semi-degenerate fields we should be able to prove a similar statement—but in that case the three-point structure constants as well as the conformal blocks involved are not explicitly computed.

In the rest of this section we provide details on the proof of Theorem 8.1.1. In particular we will explain to what extent the formula (8.1.2) corresponds to the Plancherel formula associated to a self-adjoint operator which we will call the *Toda Hamiltonian*. We will provide in the subsequent sections more background on the notions presented, but will not enter into too much details in order to keep the length of the document reasonable. Some statements are not proved yet, we will make it clear when this is indeed the case.

8.1.1 . Hilbert space, W_3 algebra and the Hamiltonian for the free-field theory

In Section 8.2 we describe in which setting the Toda Hamiltonian can be defined. To do so we will carry a detailed study of the non-interacting or free-field theory, which corresponds to the case where the cosmological constants are taken equal to zero. This will provide the basic ingredients needed to prove the bootstrap formula (8.1.2). A particular attention will be paid to the description of the Hilbert space of the free-field theory as well as some properties of the Hamiltonian of the free-field theory, which will be at the core of the proof of the bootstrap formula (8.1.2) for the *interacting* $\mathfrak{sl}_3$ Toda CFT. Additional details on the framework presented here are provided in Section 8.2.

8.1.1.1 Hilbert space and the W_3 algebra

The Hilbert space of the $\mathfrak{sl}_3$ Toda theory can be identified to a space of maps $\mathbb{S}^1 = \mathbb{T} \rightarrow \mathfrak{a} \simeq \mathbb{R}^2$. This Hilbert space $\mathcal{H}_{\mathbb{T}} = L^2(\mathbb{R}^2 \times \Omega_{\mathbb{T}})$ is equipped with a measure $dc \otimes \mathbb{P}_{\mathbb{T}}$ such that a sample of a function with respect to it can be written under the form $c + \varphi$ where φ has the law of the GFF on the circle defined in Equation (3.2.17) and with c sampled according to the Lebesgue measure over $\mathbb{R}^2$. This Hilbert space thus comes naturally equipped with a L^2 scalar product $\langle \cdot | \cdot \rangle$.

Having defined this Hilbert space we can then introduce a family of operators $(\mathbf{A}_n)_{n \in \mathbb{Z}}$ acting on it. These operators satisfy the commutation relation of the Heisenberg algebra, so that they provide a Fock representation of the Heisenberg algebra within our probabilistic setting. Based on these operators we can define a *Segal-Sugawara representation* of the W_3 algebra, that is to say we construct a family of operators acting over $\mathcal{H}_{\mathbb{T}}$, $(\mathbf{L}_n^0, \mathbf{W}_m^0)_{n,m \in \mathbb{Z}}$, that satisfy the commutation relations of the W_3 algebra. We can likewise define a copy of these operators

$$(\tilde{\mathbf{L}}_n^0, \tilde{\mathbf{W}}_m^0)_{n,m \in \mathbb{Z}}$$

based on an another representation of the Heisenberg algebra, and that commutes with the first representation considered.

8.1.1.2 Hamiltonian of the free-field theory and its diagonalization

Based on this pair of family of operators we can introduce the Hamiltonian of the free-field theory by setting

$$\mathbf{H}^0 := \mathbf{L}_0^0 + \tilde{\mathbf{L}}_0^0.$$

This operator acting over $\mathcal{H}_{\mathbb{T}}$ is self-adjoint and can be explicitly diagonalized. Namely let us introduce for any $\alpha \in \mathbb{C}^2$ the map defined by setting

$$\psi_{\alpha}^0(c, \varphi) := e^{\langle \alpha - Q, c \rangle}.$$

Then these functions are generalized (since they are not elements of $L^2(\mathbb{R}^2 \times \Omega_{\mathbb{T}})$) eigenstates of $\mathbf{H}^0$ in that they satisfy

$$\mathbf{H}^0 \psi_{\alpha}^0 = 2\Delta_{\alpha} \psi_{\alpha}^0$$

with $\Delta_{\alpha} = \langle \frac{\alpha}{2}, Q - \frac{\alpha}{2} \rangle$.

Based on this set of eigenfunctions and thanks to the commutation relations of the W_3 algebra we can define additional eigenfunctions of $\mathbf{H}^0$ by considering their descendants. This is done by setting

$$\psi_{\alpha, \nu, \tilde{\nu}}^0 := \mathbf{W}_{-\nu^2}^0 \mathbf{L}_{-\nu^1}^0 \tilde{\mathbf{W}}_{-\tilde{\nu}^2}^0 \tilde{\mathbf{L}}_{-\tilde{\nu}^1}^0 \psi_{\alpha}^0$$

where we have set, for ν and $\tilde{\nu}$ two pairs of Young diagrams

$$\mathbf{L}_{-\nu}^0 := \mathbf{L}_{-\nu_{l(\nu)}}^0 \cdots \mathbf{L}_{-\nu_1}^0 \quad \text{where } l(\nu) \text{ is the last integer } k \text{ for which } \nu_k > 0$$

and likewise (for $\mathbf{W}$, $\tilde{\mathbf{W}}$ and $\tilde{\mathbf{L}}$). They satisfy

$$\mathbf{H}^0 \psi_{\alpha, \nu, \tilde{\nu}}^0 = 2\Delta_{\alpha} + |\nu| + |\tilde{\nu}|$$

with $|\nu| := \sum_{i=1}^2 \sum_{n \in \mathbb{N}} \nu_n^i$.

These generalized eigenfunctions allow to describe a Plancherel formula for the Hamiltonian derived from that associated to the Fourier transform on $L^2(\mathbb{R}^2 \times \Omega_{\mathbb{T}})$. Namely we show that the spectral decomposition of $\mathbf{H}^0$ takes the form:

$$\langle u|v \rangle_2 = \frac{1}{(2\pi)^2} \sum_{\substack{\nu, \tilde{\nu} \\ \nu', \tilde{\nu}'}} \int_{\mathbb{R}^2} \langle u|\psi_{Q+iP, \nu, \tilde{\nu}}^0 \rangle_2 \langle \psi_{Q+iP, \nu', \tilde{\nu}'}^0|v \rangle_2 F_{Q+iP}^{-1}(\nu, \nu') F_{Q+iP}^{-1}(\tilde{\nu}, \tilde{\nu}') dP$$

for any $u, v \in L^2(\mathbb{R}^2 \times \Omega_{\mathbb{T}})$ for some coefficients $F_{Q+iP}^{-1}(\nu, \nu')$.

8.1.2 . From the free-field to the Toda theory

In the interacting theory one needs to take into account the additional terms stemming from the fact that the cosmological constants are now chosen to be positive. This will in turn modify the Hamiltonian to be considered but our goal is to prove that a similar Plancherel formula holds true. We first explain what changes are to be made and then explain how the spectral analysis of this Hamiltonian can be carried. Using the general form of the Ward identities this will prove the formula (8.1.2).

8.1.2.1 Definition of the Hamiltonian

The Hamiltonian of the interacting theory can be defined from $\mathbf{H}^0$ by formally setting

$$\mathbf{H} = \mathbf{H}^0 + \sum_{i=1}^2 e^{\langle \gamma e_i, c \rangle} \int_{\mathbb{T}} e^{\langle \gamma e_i, \varphi(\theta) \rangle} d\theta$$

where the exponential that appears in the integral is defined as a GMC measure with respect to the GFF φ . However this writing may not make sense for the whole range of values of $\gamma \in (0, \sqrt{2})$ since the GMC measure is not well defined if $\gamma \geq 1$. Therefore in order to make sense of $\mathbf{H}$ one needs to provide an alternative definition of the Toda Hamiltonian. This can be done by defining the Hamiltonian from its quadratic form, but also by representing its propagator based on the dilation semigroup in connection with reflection positivity. We present these definitions in Section 8.3.

8.1.2.2 Spectral resolution of the Toda Hamiltonian

Having properly introduced the Toda Hamiltonian one then needs to compute its spectrum and understand its spectral properties. We will explain how the eigenfunctions $\psi_{\alpha, \nu, \tilde{\nu}}^0$ associated to the free Hamiltonian can be transformed into eigenvectors $\psi_{\alpha, \nu, \tilde{\nu}}$ of the Toda Hamiltonian satisfying the relation

$$\mathbf{H}\psi_{\alpha, \nu, \tilde{\nu}} = (2\Delta_{\alpha} + |\nu| + |\tilde{\nu}|) \psi_{\alpha, \nu, \tilde{\nu}}.$$

The Plancherel formula of the Toda Hamiltonian then takes the form of

$$\langle u|v \rangle_2 = \frac{1}{(2\pi)^2} \sum_{\substack{\nu, \tilde{\nu} \\ \nu', \tilde{\nu}'}} \int_{\mathbb{C}^2_-} \langle u|\psi_{Q+iP, \nu, \tilde{\nu}} \rangle_2 \langle \psi_{Q+iP, \nu', \tilde{\nu}'}|v \rangle_2 F_{Q+iP}^{-1}(\nu, \nu') F_{Q+iP}^{-1}(\tilde{\nu}, \tilde{\nu}') dP \quad (8.1.3)$$

for any suitable u, v .

Proving such a decomposition is actually very involved and relies on inputs coming from the theory of scattering in symmetric spaces, hence we will not enter into the details of the proof of this formula in this manuscript. An additional subtlety in the above writing is that for $\alpha = Q + iP$ the eigenfunctions $\psi_{\alpha, \nu, \tilde{\nu}}$ are not well-defined objects probabilistically speaking. In order to overcome this issue one needs to justify that such expressions can be analytically continued beyond the range of values prescribed by the probabilistic existence of these quantities, which is also particularly demanding. We will provide additional explanations on the way such a formula is derived in Section 8.3.

8.1.2.3 Ward identities and the bootstrap formula

Based on the Plancherel formula for the Toda Hamiltonian we are now in position to provide a justification of the conformal bootstrap formula (8.1.2). Indeed by choosing the two vectors u and v in Equation (8.1.3) in an appropriate way we can rewrite the L^2 scalar products that appear there as follows:

$$\begin{aligned}\langle u|v\rangle_2 &= \langle V_{\alpha_1}(0)V_{\alpha_2}(z)V_{\alpha_3}(1)V_{\alpha_4}(\infty)\rangle; \\ \langle u|\psi_{Q+iP, \nu, \tilde{\nu}}\rangle_2 &= \langle V_{\alpha_1}(0)V_{\alpha_2}(z)V_{Q+iP, \nu, \tilde{\nu}}(\infty)\rangle; \\ \langle \psi_{Q+iP, \nu', \tilde{\nu}'}|v\rangle_2 &= \langle V_{Q+iP, \nu', \tilde{\nu}'}(0)V_{\alpha_3}(1)V_{\alpha_4}(\infty)\rangle.\end{aligned}$$

The descendent fields $V_{Q+iP, \nu, \tilde{\nu}}$ are obtained by applying the operators of the W_3 algebra to the primary field V_{Q+iP} . Like before some care has to be taken when it comes to defining such quantities since they do not admit a probabilistic representation but rather correspond to their analytic continuation. We can thus write the following equality, valid under suitable assumptions on the weights:

$$\begin{aligned}\langle V_{\alpha_1}(0)V_{\alpha_2}(z)V_{\alpha_3}(1)V_{\alpha_4}(\infty)\rangle &= \frac{1}{(2\pi)^2} \sum_{\substack{\nu, \tilde{\nu} \\ \nu', \tilde{\nu}'}} \int_{\mathcal{C}_-^2} \\ \langle V_{\alpha_1}(0)V_{\alpha_2}(z)V_{Q+iP, \nu, \tilde{\nu}}(\infty)\rangle &\langle V_{Q+iP, \nu', \tilde{\nu}'}(0)V_{\alpha_3}(1)V_{\alpha_4}(\infty)\rangle F_{Q+iP}^{-1}(\nu, \nu') F_{Q+iP}^{-1}(\tilde{\nu}, \tilde{\nu}') dP.\end{aligned}$$

Now we may distinguish between two cases. First of all under the assumptions of Theorem 8.1.1 we can prove that the ratio between three-point correlation functions containing a descendent field and not is actually of the form

$$\frac{\langle V_{\alpha_1}(0)V_{\alpha_2}^*(1)V_{Q+iP, \nu, \tilde{\nu}}(\infty)\rangle}{\langle V_{\alpha_1}(0)V_{\alpha_2}^*(1)V_{Q+iP}(\infty)\rangle} = v(\alpha_1, \alpha_2^*, \nu)v(\alpha_1, \alpha_2^*, \tilde{\nu})$$

where the quantity d only depends on the conformal and W -weights Δ_α and $w(\alpha)$. This is due to the fact that the global Ward identities from Theorem 5.2.1 impose constraints on the descendent fields and in turn allow to reduce the knowledge of W -descendants to that of the weights Δ_α and $w(\alpha)$. The latter implies that we can rewrite the Plancherel formula under the form

$$\begin{aligned}&\langle V_{\alpha_1}(0)V_{\alpha_2}^*(z)V_{\alpha_3}(1)V_{\alpha_4}(\infty)\rangle \\ &= \frac{1}{4\pi^2} \int_{\mathcal{C}_-^2} C_\gamma(\alpha_1, \alpha_2^*, Q - iP)C_\gamma(Q + iP, \alpha_3^*, \alpha_4) |z|^{2(\Delta_{Q+iP} - \Delta_{\alpha_1} - \Delta_{\alpha_2})} |\mathcal{F}_P^T(z)|^2 dP\end{aligned}\tag{8.1.4}$$

where the conformal blocks are given by the (formal) power series

$$\mathcal{F}_P^T(z) = \sum_{\nu, \nu'} z^{|\nu|} v(\alpha_1, \alpha_2^*, \nu) F_{Q+iP}^{-1}(\nu, \nu') v(\alpha_3^*, \alpha_4, \nu').$$

A consequence of Theorem 8.1.1 is to ensure that this series has radius of convergence 1 for almost every $P \in \mathcal{C}_-$.

In the general case it is not clear at this stage whether the ratio

$$\frac{\langle V_{\alpha_1}(0) V_{\alpha_2}(1) V_{Q+iP, \nu, \tilde{\nu}}(\infty) \rangle}{\langle V_{\alpha_1}(0) V_{\alpha_2}(1) V_{Q+iP(\infty)} \rangle}$$

only depends on the conformal and quantum weights. To the best of our knowledge this question is not settled even in the physics literature. However there is some hope that our probabilistic representation can allow to address this issue.

8.2 . The free-field theory

In this preliminary section we want to explain where the formula (8.1.2) comes from by carrying a detailed study of the free-field theory. This will provide the basic ingredients needed to prove the bootstrap formula (8.1.2). In particular we will detail the framework in which the Toda Hamiltonian can be studied.

8.2.1 . Hilbert spaces and the W_3 algebra

We first describe in this subsection the *Hilbert space of the $\mathfrak{sl}_3$ Toda theory* and present a Segal-Sugawara representation of the W_3 algebra on this Hilbert space, based on the Heisenberg algebra which we represent in terms of operators acting on the Hilbert space of the free-field theory.

8.2.1.1 The Hilbert space of the $\mathfrak{sl}_3$ Toda theory

The Hilbert space of the $\mathfrak{sl}_3$ Toda CFT can be realized starting from the space of maps $\mathbb{S}^1 = \mathbb{T} \rightarrow \mathfrak{a} \simeq \mathbb{R}^2$. Namely, let us introduce for $n \in \mathbb{Z}$ and $1 \leq j \leq 2$ the map $e_{n,j}(e^{i\theta}) := e^{in\theta} v_j$, where recall that $(v_j)_{j=1,2}$ is any orthonormal basis of $\mathbb{R}^2$. Then the mode expansion of an element φ of $L^2(\mathbb{S}^1 \rightarrow \mathbb{R}^2)$

$$\varphi := c + \sum_{n>0} \sum_{i=1}^2 (\varphi_{n,i} e_{n,i} + \bar{\varphi}_{n,i} e_{-n,i}) \quad (8.2.1)$$

establishes a correspondence between this L^2 space and the space $\mathbb{R}^2 \times \Omega_{\mathbb{T}}$, where $\Omega_{\mathbb{T}} := (\mathbb{R}^4)^{\mathbb{N}^*}$. The latter comes naturally equipped with a cylinder sigma-algebra $\Sigma_{\mathbb{T}} := \mathcal{B}^{\otimes \mathbb{N}^*}$ and a probability measure $\mathbb{P}_{\mathbb{T}}$ under which the modes $\varphi_{n,i}$, for $n > 0$ and $i = 1, 2$, are centered, complex independent Gaussian variables with variance $\frac{1}{2n}$. By considering such a measure over $\Omega_{\mathbb{T}}$ we see that a GFF on the circle $\mathbb{S}^1$ defined in Equation (3.2.17) corresponds to a sample with respect to this probability measure. In what follows we will consider the Hilbert space $\mathcal{H}_{\mathbb{T}}$ corresponding to the L^2 space of $\mathbb{R}^2 \times \Omega_{\mathbb{T}}$ when equipped

with the measure $d\mathbf{c} \otimes \mathbb{P}_{\mathbb{T}}$, where $d\mathbf{c}$ is the Lebesgue measure on $\mathbb{R}^2$ (which is *not* a probability measure). The associated scalar product is denoted by $\langle \cdot | \cdot \rangle_2$:

$$\langle f | g \rangle_2 = \int_{\mathbb{R}^2} \mathbb{E} \left[f(\mathbf{c}, \varphi) \overline{g(\mathbf{c}, \varphi)} \right] d\mathbf{c} \quad (8.2.2)$$

where $\mathbb{E}$ is associated to the probability measure $\mathbb{P}_{\mathbb{T}}$.

8.2.1.2 Heisenberg algebra

We now introduce a family of operators acting on our Hilbert space $\mathcal{H}_{\mathbb{T}}$. For this purpose we start by introducing the subset $\mathcal{S}$ of $L^2(\Omega_{\mathbb{T}})$ defined as the linear span of smooth maps that depend only on a finite number of coordinates $\varphi_{n,i}$. On such maps we can consider operators $(\partial_{n,i})_{n \in \mathbb{Z}, 1 \leq i \leq 2}$, that correspond to the holomorphic derivatives with respect to the modes $\varphi_{n,i}$:

$$\partial_{n,i} F \left((\varphi_{m,j})_{\substack{|m| \leq N \\ j=1,2}} \right) := \frac{\partial F}{\partial \varphi_{n,i}} \left((\varphi_{m,j})_{\substack{|m| \leq N \\ j=1,2}} \right).$$

Using these operators we can form

$$\partial_n F := \sum_{i=1}^2 \frac{\partial F}{\partial \varphi_{n,i}} \mathbf{v}_i \text{ and likewise set } \varphi_n := \varphi_{n,1} \mathbf{v}_1 + \varphi_{n,2} \mathbf{v}_2 \quad (8.2.3)$$

so that ∂_n is the gradient with respect to the variable $\varphi_n \in \mathbb{R}^2$. The operators $\partial_{n,k}$ can be recovered by using that $\partial_{n,k} F = \langle \partial_n F, \mathbf{v}_k \rangle$.

We now present the creation and annihilation operators that will allow to define a Fock representation of the Heisenberg algebra. These operators act on the Hilbert space $\mathcal{C}_{\infty}$ defined by

$$\mathcal{C}_{\infty} := \text{Span} \{ \psi(c) F, \quad \psi \in C^{\infty}(\mathbb{R}^2) \text{ and } F \in \mathcal{S} \}. \quad (8.2.4)$$

To start with we introduce 2 copies of the Heisenberg algebra by setting

$$\begin{aligned} \mathbf{A}_{n,j} &:= \frac{i}{2} \partial_{n,j} \quad \text{for positive } n \\ \mathbf{A}_{0,j} &:= \frac{i}{2} (\partial_{0,j} + \langle Q, \mathbf{v}_j \rangle) \quad \text{for } n = 0 \\ \mathbf{A}_{n,j} &:= \frac{i}{2} (\partial_{n,j} + 2n \varphi_{-n,j}) \quad \text{for negative } n, \end{aligned} \quad (8.2.5)$$

where $j = 1, 2$. These operators are unitary and form 2 independent copies of the Heisenberg algebra:

$$[\mathbf{A}_{n,i}, \mathbf{A}_{m,j}] = \frac{n}{2} \delta_{n,-m} \delta_{i,j}. \quad (8.2.6)$$

The adjoint with respect to the $L^2(\mathbb{R}^2 \times \Omega_{\mathbb{T}})$ Hermitian scalar product is seen to be given by

$$\mathbf{A}_{n,i}^* = -\mathbf{A}_{-n,i} \text{ for } n \geq 0 \quad \text{and} \quad \mathbf{A}_{0,i}^* = \mathbf{A}_{0,i} - i \langle Q, \mathbf{v}_i \rangle. \quad (8.2.7)$$

Like before, of particular interest is the vectorial version of these operators, defined via

$$\mathbf{A}_n = \sum_{i=1}^2 \mathbf{A}_{n,i} \mathbf{v}_i. \quad (8.2.8)$$

It is such that for any u and v in $\mathbb{R}^2$, $[\langle u, \mathbf{A}_n \rangle, \langle v, \mathbf{A}_m \rangle] = \langle u, v \rangle \frac{n}{2} \delta_{n,-m}$.

We can likewise define other Fock representations of the Heisenberg algebra by taking anti-holomorphic derivatives rather than holomorphic ones:

$$\begin{aligned}\tilde{\mathbf{A}}_{n,j} &:= \frac{i}{2} \partial_{-n,j} \quad \text{for positive } n \\ \tilde{\mathbf{A}}_{0,j} &:= \frac{i}{2} (\partial_{0,j} + Q) \quad \text{for } n = 0 \\ \tilde{\mathbf{A}}_{n,j} &:= \frac{i}{2} (\partial_{-n,j} + 2n\varphi_{n,j}) \quad \text{for negative } n.\end{aligned}\tag{8.2.9}$$

Based on these representations of the Heisenberg algebra we are almost in position to introduce the Hamiltonian operator associated to the $\mathfrak{sl}_3$ Toda CFT. Before that, we define the self-adjoint and nonnegative operator $\mathbf{P}$ acting on $\mathcal{S} \subset \mathcal{C}_\infty$:

$$\mathbf{P} := 2 \sum_{n \geq 1} \langle \mathbf{A}_{-n}, \mathbf{A}_n \rangle + \langle \tilde{\mathbf{A}}_{-n}, \tilde{\mathbf{A}}_n \rangle \tag{8.2.10}$$

where with the above notation is meant that

$$\langle \mathbf{A}_{-n}, \mathbf{A}_n \rangle = \sum_{i,j=1}^2 \langle \mathbf{v}_i, \mathbf{v}_j \rangle \mathbf{A}_{-n,i} \mathbf{A}_{n,j} = (\mathbf{A}_{-n,1} \mathbf{A}_{n,1} + \mathbf{A}_{-n,2} \mathbf{A}_{n,2}).$$

The eigenfunctions of $\mathbf{P}$ are indexed by 4-tuples (k^1, k^2, l^1, l^2) of elements of

$$\mathcal{N} := \{ \mathbf{k} \in \mathbb{N}^{\mathbb{N}^*}, \quad k_n = 0 \text{ for } n \text{ large enough} \}$$

with eigenvalues of the form $|\mathbf{k}| + |\mathbf{l}|$ where $|\mathbf{k}| = \sum_{i=1}^2 \sum_{n>0} n k_n^i$. This allows to decompose $L^2(\Omega_{\mathbb{T}})$ as follows:

$$L^2(\Omega_T) = \oplus_{\mathbf{k}, \mathbf{l} \in \mathcal{N}^2} \ker(\mathbf{P} - |\mathbf{k}| - |\mathbf{l}|). \tag{8.2.11}$$

The corresponding eigenspaces can be explicitly described. Namely for $\mathbf{k}^1, \mathbf{k}^2$ and $\mathbf{l}^1, \mathbf{l}^2$ in $\mathcal{N}$ let us set

$$\pi_{\mathbf{k}, \mathbf{l}} := \prod_{n \geq 0} \prod_{i=1}^2 \mathbf{A}_{-n,i}^{k_n^i} \tilde{\mathbf{A}}_{-n,i}^{l_n^i} 1, \tag{8.2.12}$$

where 1 is simply the constant function. Then the commutation relations (8.2.6) allow to claim that $\pi_{\mathbf{k}, \mathbf{l}}$ is an element of $\mathcal{S}$ that satisfies $\mathbf{P} \pi_{\mathbf{k}, \mathbf{l}} = (|\mathbf{k}| + |\mathbf{l}|) \pi_{\mathbf{k}, \mathbf{l}}$. The eigenspace of $\mathbf{P}$ in $L^2(\Omega_{\mathbb{T}})$ corresponding to the eigenvalue $N \in \mathbb{N}$ is spanned by the $(\pi_{\mathbf{k}, \mathbf{l}})_{|\mathbf{k}|+|\mathbf{l}|=N}$, so that the family $(\pi_{\mathbf{k}, \mathbf{l}})_{\mathbf{k}, \mathbf{l} \in \mathcal{N}^2}$ form a basis of $L^2(\Omega_{\mathbb{T}})$.

8.2.1.3 Segal-Sugawara representation of the W_3 algebra

Using the above operators and based on its Segal-Sugawara construction, we can introduce a representation of the W_3 algebra by means of operators acting on $\mathcal{S}$, which in particular contains a representation of the Virasoro algebra. For this purpose let us recall that the normal ordered product for the Heisenberg algebra is defined by the convention that annihilation operators are on the right, that is we denote for any u, v in $\mathbb{R}^2$

$$: \langle u, \mathbf{A}_n \rangle \langle v, \mathbf{A}_m \rangle := \begin{cases} \langle u, \mathbf{A}_n \rangle \langle v, \mathbf{A}_m \rangle & \text{if } m > 0 \\ \langle v, \mathbf{A}_m \rangle \langle u, \mathbf{A}_n \rangle & \text{if } n > 0 \end{cases}$$

When both n and m are non-positive the order becomes irrelevant since the operators then commute.

With this convention at hand we define for any integer $n \in \mathbb{Z}$

$$\mathbf{L}_n^0 := -i(n+1)\langle Q, \mathbf{A}_n \rangle + \sum_{m \in \mathbb{Z}} : \langle \mathbf{A}_{n-m}, \mathbf{A}_m \rangle : . \quad (8.2.13)$$

Over $\mathcal{C}_\infty$, this family of operators satisfies the commutation relations of the Virasoro algebra with central charge $c = 2 + 6|Q|^2$:

$$[\mathbf{L}_n^0, \mathbf{L}_m^0] = (n-m)\mathbf{L}_{n+m}^0 + \frac{c}{12}(n^3-n)\delta_{n,-m}. \quad (8.2.14)$$

In a similar fashion as above we can define another representation of the Virasoro algebra that would correspond to the anti-holomorphic derivatives. It takes the form

$$\tilde{\mathbf{L}}_n^0 := -i(n+1)\langle Q, \tilde{\mathbf{A}}_n \rangle + \sum_{m \in \mathbb{Z}} : \langle \tilde{\mathbf{A}}_{n-m}, \tilde{\mathbf{A}}_m \rangle : .$$

Thanks to Equation (8.2.7) these two representations enjoy the property that for all $n \in \mathbb{N}$:

$$(\mathbf{L}_n^0)^* = \mathbf{L}_{-n}^0 \quad \text{and} \quad (\tilde{\mathbf{L}}_n^0)^* = \tilde{\mathbf{L}}_{-n}^0. \quad (8.2.15)$$

In the W_3 algebra there is yet a second family of operators that we need to introduce. It is defined by the expression

$$\begin{aligned} \mathbf{W}_n^0 &= iq^2(n+1)(n+2)\langle h_2, \mathbf{A}_n \rangle \\ &- 2q \sum_{m \in \mathbb{Z}} (n-m+1) \left(: \langle h_2 - h_1, \mathbf{A}_{n-m} \rangle \langle h_1, \mathbf{A}_m \rangle : + : \langle h_3 - h_2, \mathbf{A}_{n-m} \rangle \langle h_3, \mathbf{A}_m \rangle : \right) \\ &+ 8i \sum_{n_1+n_2+n_3=n} : \langle h_1, \mathbf{A}_{n_1} \rangle \langle h_2, \mathbf{A}_{n_2} \rangle \langle h_3, \mathbf{A}_{n_3} \rangle : . \end{aligned} \quad (8.2.16)$$

It is common to rescale this family of operators by an extra factor in order for the commutation relations to admit an elegant form. Namely let us introduce the notation

$$\mathbf{W}_n^0 := i \sqrt{\frac{48}{22+5c}} \mathbf{W}_n^0. \quad (8.2.17)$$

Thanks to Equation (8.2.6) this family of operators enjoys the following commutation relations with the Virasoro algebra

$$[\mathbf{L}_m^0, \mathbf{W}_n^0] = (2m-n)\mathbf{W}_{m+n}^0 \quad (8.2.18)$$

while the commutation rule for the $(\mathbf{W}_n^0)_{n \in \mathbb{Z}}$ is rather complicated (see for instance [25]) but irrelevant here. Altogether the algebra of operators on $\mathcal{S}$ generated by $(\mathbf{L}_n^0, \mathbf{W}_m^0)_{n,m \in \mathbb{Z}}$ forms a representation of the W_3 -algebra. Likewise we can define another copy of the W_3 algebra associated to anti-holomorphic derivatives $(\tilde{\mathbf{L}}_n^0, \tilde{\mathbf{W}}_m^0)_{n,m \in \mathbb{Z}}$. These two representations of the W_3 algebra enjoy the additional property that, thanks to Equation (8.2.7),

$$(\mathbf{W}_n^0)^* = \mathbf{W}_{-n}^0 \quad \text{and} \quad (\tilde{\mathbf{W}}_n^0)^* = \tilde{\mathbf{W}}_{-n}^0. \quad (8.2.19)$$

For future reference we stress that both $\mathbf{L}_0^0$ and $\mathbf{W}_0^0$ are self-adjoint. However and to simplify the computations we will often work with $\mathbf{W}_n$ in what follows.

8.2.2 . The Hamiltonian of the free-field theory

Having defined above a representation of the W_3 algebra, we now consider a special operator acting over $L^2(\mathbb{R}^2 \times \Omega_{\mathbb{T}})$ that we will refer to as the Hamiltonian of the non-interacting $\mathfrak{sl}_3$ Toda CFT. It is defined by setting

$$\mathbf{H}^0 := \mathbf{L}_0^0 + \widetilde{\mathbf{L}}_0^0 = -\frac{1}{2}\Delta_c + \frac{1}{2}|Q|^2 + \mathbf{P}. \quad (8.2.20)$$

Thanks to Equation (8.2.15) this Hamiltonian is seen to be self-adjoint.

8.2.2.1 Diagonalization of the free Hamiltonian

We start by considering (generalized) eigenfunctions for the Laplace operator. They are defined for $\alpha \in \mathbb{C}^2$ by

$$\psi_\alpha^0(\mathbf{c}, \varphi) := e^{\langle \alpha - Q, \mathbf{c} \rangle}, \quad \text{which are such that} \quad \frac{1}{4}(-\Delta_c + |Q|^2) \psi_\alpha^0 = \Delta_\alpha \psi_\alpha^0 \quad (8.2.21)$$

with $\Delta_\alpha = \langle \frac{\alpha}{2}, Q - \frac{\alpha}{2} \rangle$ like before. In particular ψ_α^0 is such that $\mathbf{H}^0 \psi_\alpha^0 = \Delta_\alpha \psi_\alpha^0$. We have seen before that $\mathbf{P}$ admits a set of eigenfunctions $\pi_{\mathbf{k}, \mathbf{l}}$ where $\mathbf{k}$ and $\mathbf{l} \in \mathcal{N}$, that are constructed by application of creation operators starting from the constant function 1. We can actually reproduce the same scheme but starting from any ψ_α^0 and construct in such a way eigenfunctions of $\mathbf{H}^0$. However to preserve the fact that the maps thus defined are indeed eigenfunctions of $\mathbf{H}^0$ we shall not use the creation operators $\mathbf{A}_{-n}$ but rather the operators $\mathbf{L}_{-n}^0$ and $\mathbf{W}_{-n}^0$. This leads us to introducing the so-called descendant states of ψ_α^0 as follows.

First given a Young diagram¹ ν , we introduce the shorthand

$$\mathbf{L}_{-\nu}^0 := \mathbf{L}_{-\nu_{l(\nu)}}^0 \cdots \mathbf{L}_{-\nu_1}^0 \quad \text{where } l(\nu) \text{ is the last integer } k \text{ for which } \nu_k > 0.$$

The same notation applies in the same way to $\mathbf{W}$, $\widetilde{\mathbf{W}}$ and $\widetilde{\mathbf{L}}$. The descendant states of ψ_α^0 are then defined by setting

$$\psi_{\alpha, \nu, \widetilde{\nu}}^0 := \mathbf{W}_{-\nu^2}^0 \mathbf{L}_{-\nu^1}^0 \widetilde{\mathbf{W}}_{-\widetilde{\nu}^2}^0 \widetilde{\mathbf{L}}_{-\widetilde{\nu}^1}^0 \psi_\alpha^0 \quad (8.2.22)$$

for any four-tuple of Young diagrams $(\nu, \widetilde{\nu}) = (\nu^i, \widetilde{\nu}^i)_{i=1,2}$.

Using the commutation relations of the W_3 algebra (8.2.14) and (8.2.18), straightforward computations show that by doing so we indeed end up with a set of generalized eigenfunctions for $\mathbf{H}_0$, for which the associated eigenvalues are given by

$$2\Delta_\alpha + |\nu| + |\widetilde{\nu}| \quad (8.2.23)$$

with $|\nu| := \sum_{i=1}^2 \sum_{n \in \mathbb{N}} \nu_n^i$. The reason why these are only *generalized* eigenfunctions is that the $\psi_{\alpha, \nu, \widetilde{\nu}}^0$ do not belong to $L^2(\mathbb{R}^2 \times \Omega_{\mathbb{T}})$ due to the lack of integrability in the $\mathbf{c}$ variable but rather to a weighted space $e^{\beta|\mathbf{c}|} L^2(\mathbb{R}^2 \times \Omega_{\mathbb{T}})$ for $\beta > |\Re(\alpha - Q)|$.

¹Young diagrams are non-increasing and eventually-zero sequences of integers. We denote their set by $\mathcal{T}$.

8.2.2.2 On the generalized eigenfunctions

From their explicit expression we can infer that these eigenfunctions enjoy the following properties:

Proposition 8.2.1. *There exist polynomials $\mathcal{Q}_{\alpha,\nu,\tilde{\nu}}$ depending on finitely many elements of $\Omega_{\mathbb{T}}$ such that*

$$\psi_{\alpha,\nu,\tilde{\nu}}^0 = \mathcal{Q}_{\alpha,\nu,\tilde{\nu}} \psi_{\alpha}^0. \quad (8.2.24)$$

These polynomials are such that for $\alpha \in \mathbb{C}$,

$$\langle \mathcal{Q}_{2Q-\bar{\alpha},\nu,\tilde{\nu}} | \mathcal{Q}_{\alpha,\nu',\tilde{\nu}'} \rangle_{L^2(\Omega_{\mathbb{T}})} = \delta_{|\nu|,|\nu'|} \delta_{|\tilde{\nu}|,|\tilde{\nu}'|} F_{\alpha}(\nu, \nu') F_{\alpha}(\tilde{\nu}, \tilde{\nu}'). \quad (8.2.25)$$

Moreover for any $P \in \mathbb{R}^2$ and $N \geq 0$, the matrix $(F_{Q+iP}(\nu, \nu'))_{|\nu|=|\nu'|=N}$ is invertible and the polynomials $(\mathcal{Q}_{Q+iP,\nu,\tilde{\nu}})_{|\nu|+|\tilde{\nu}|=N}$ span the eigenspace $\ker(\mathbf{P} - N)$.

Proof. Using the explicit expression of the generators of the Heisenberg algebra $\mathbf{A}_n$ we see that we can write

$$\psi_{\alpha,\nu,\tilde{\nu}}^0 = \psi_{\alpha}^0 \left(\mathbf{W}_{-\nu^2}^{0,\alpha} \mathbf{L}_{-\nu^1}^{0,\alpha} \widetilde{\mathbf{W}}_{-\tilde{\nu}^2}^{0,\alpha} \widetilde{\mathbf{L}}_{-\tilde{\nu}^1}^{0,\alpha} 1 \right)$$

where in the above expression operators with an exponent α are defined by replacing $\mathbf{A}_0$ by $\frac{i}{2}\alpha$. For instance

$$\mathbf{L}_{-n}^{0,\alpha} = \begin{cases} i\langle \alpha - (n+1)Q, \mathbf{A}_n \rangle + \sum_{m \neq 0,n} \mathbf{A}_{n-m} \mathbf{A}_m & \text{for } n \neq 0. \\ \Delta_{\alpha} + 2 \sum_{m \geq 0} \mathbf{A}_{-m} \mathbf{A}_m & \text{for } n = 0. \end{cases}$$

These operators viewed as acting on (a proper subset of) $L^2(\Omega_{\mathbb{T}})$ now satisfy

$$(\mathbf{L}_n^{0,\alpha})^* = \mathbf{L}_{-n}^{0,2Q-\bar{\alpha}} \text{ and } (\mathbf{W}_n^{0,\alpha})^* = \mathbf{W}_{-n}^{0,2Q-\bar{\alpha}}.$$

Note that the commutation relations of the W_3 algebra are still satisfied.

Now using the fact that the set of polynomials of $L^2(\Omega_{\mathbb{T}})$ that depend on finitely many variables is stable by the action of the Heisenberg algebra we infer Equation (8.2.24). Moreover we also see that

$$\langle \mathcal{Q}_{2Q-\bar{\alpha},\nu,\tilde{\nu}} | \mathcal{Q}_{\alpha,\nu',\tilde{\nu}'} \rangle_{L^2(\Omega_{\mathbb{T}})} = \langle 1 | \mathbf{L}_{\nu^1}^{0,\alpha} \mathbf{W}_{\nu^2}^{0,\alpha} \mathbf{W}_{-(\nu')^2}^{0,\alpha} \mathbf{L}_{-(\nu')^1}^{0,\alpha} \widetilde{\mathbf{L}}_{\tilde{\nu}^1}^{0,\alpha} \widetilde{\mathbf{W}}_{\tilde{\nu}^2}^{0,\alpha} \widetilde{\mathbf{L}}_{(\tilde{\nu}')^1}^{0,\alpha} \widetilde{\mathbf{W}}_{(\tilde{\nu}')^2}^{0,\alpha} 1 \rangle_{L^2(\Omega_{\mathbb{T}})}.$$

To start with and based on the commutation rules together with the fact that for $n > 1$ $\mathbf{L}_n 1 = \mathbf{W}_n 1 = 0$, we can reproduce the arguments developed along the proof of [66, Lemma A.1] to deduce that $\mathbf{L}_{\nu^1}^{0,\alpha} \mathbf{W}_{\nu^2}^{0,\alpha} \mathbf{W}_{-(\nu')^2}^{0,\alpha} \mathbf{L}_{-(\nu')^1}^{0,\alpha} 1 = 0$ as soon as $|\nu| > |\nu'|$. By symmetry in ν and ν' this implies that $\langle \mathcal{Q}_{2Q-\bar{\alpha},\nu,\tilde{\nu}} | \mathcal{Q}_{\alpha,\nu',\tilde{\nu}'} \rangle_{L^2(\Omega_{\mathbb{T}})}$ has an expression of the form $F_{\alpha}(\nu, \nu')$. Since the two copies of the W_3 algebra are independent the formula (8.2.25) follows via the very same reasoning.

Eventually we see that the matrix $(F_{Q+iP}(\nu, \nu'))_{|\nu|=|\nu'|=N}$ is invertible thanks to the determinant formula from [102] or [136, Equation 5.16]:

$$\det(F_{\alpha}(\nu, \nu'))_{|\nu|=|\nu'|=N} = \kappa_N \prod_{\substack{1 \leq r,s \leq N \\ rs \leq N}} \prod_{e \in \Phi} \left(\langle \alpha, e \rangle + r\gamma + s\frac{2}{\gamma} \right)^{p^2(N-rs)} \quad (8.2.26)$$

for κ_N independent of α and c , and with $p^2(k)$ denoting the number of pairs of Young diagrams ν with $|\nu| = k$. Since this determinant vanishes only for weights α for which

$\langle \alpha, e \rangle = (1-r)\gamma + (1-s)\frac{2}{\gamma}$ for some $e \in \Phi$ and $r, s \in \mathbb{N}^*$ the claim follows. This implies that the polynomials $(\mathcal{Q}_{Q+iP, \nu, \emptyset})_{|\nu|=N}$ are linearly independent, while the form of Equation (8.2.25) allows to show that more generally the $(\mathcal{Q}_{Q+iP, \nu, \emptyset})_{\nu \in \mathcal{T}^2}$ are independent, and likewise that $(\mathcal{Q}_{Q+iP, \nu, \tilde{\nu}})_{\nu, \tilde{\nu} \in \mathcal{T}^2}$. A specialization of this statement is that the $(\mathcal{Q}_{Q+iP, \nu, \tilde{\nu}})_{|\nu|+|\tilde{\nu}|=N}$ are linearly independent. Because they belong to the eigenspace $\ker(\mathbf{P} - N)$, by a dimensional argument (since this space is spanned by the $(\pi_{k,l})_{|k|+|l|=N}$) we see that the family of the $(\mathcal{Q}_{Q+iP, \nu, \tilde{\nu}})_{|\nu|+|\tilde{\nu}|=N}$ is actually a basis of $\ker(\mathbf{P} - N)$. $\square$

8.2.2.3 A spectral decomposition for the free Hamiltonian

We are now in position to describe a Plancherel formula for the Hamiltonian derived from that associated to the Fourier transform on $L^2(\mathbb{R}^2 \times \Omega_{\mathbb{T}})$. This spectral decomposition takes the form:

Proposition 8.2.2. *For any $u, v \in L^2(\mathbb{R}^2 \times \Omega_{\mathbb{T}})$ we have the following decomposition:*

$$\langle u|v \rangle_2 = \frac{1}{(2\pi)^2} \sum_{\substack{\nu, \tilde{\nu} \\ \nu', \tilde{\nu}'}} \int_{\mathbb{R}^2} \langle u | \psi_{Q+iP, \nu, \tilde{\nu}}^0 \rangle_2 \langle \psi_{Q+iP, \nu', \tilde{\nu}'}^0 | v \rangle_2 F_{Q+iP}^{-1}(\nu, \nu') F_{Q+iP}^{-1}(\tilde{\nu}, \tilde{\nu}') dP \quad (8.2.27)$$

with coefficients $F_{Q+iP}^{-1}(\nu, \nu')$ being that of the inverse matrix of $(F_{Q+iP}(\nu, \nu'))_{|\nu|=|\nu'|=j}$.

Proof. To start with assume that u and v both belong to an eigenspace $\ker(\mathbf{P} - N)$. Then over such an eigenspace $\mathbf{H}^0$ is simply given by $-\frac{1}{2}\Delta_c + \frac{1}{2}|Q|^2 + N$, for which the spectral resolution is given using the Fourier transform on $\mathbb{R}^2$ (which implies that $\mathbf{H}^0$ has absolutely continuous spectrum). In particular over this eigenspace we can write down the Plancherel formula which takes the form of Equation (8.2.27) since this eigenspace is spanned, in virtue of Proposition 8.2.1, by the polynomials $(\mathcal{Q}_{Q+iP, \nu, \tilde{\nu}})_{|\nu|+|\tilde{\nu}|=N}$.

In general, Equation (8.2.27) follows by recalling that we have the decomposition from Equation (8.2.11)

$$L^2(\Omega_T) = \oplus_{\nu, \tilde{\nu} \in \mathcal{T}^2} \ker(\mathbf{P} - |\nu| - |\tilde{\nu}|),$$

so that $\mathbf{H}^0$ is unitary equivalent to $\oplus_{\nu, \tilde{\nu} \in \mathcal{T}^2} \left(-\frac{1}{2}\Delta_c + \frac{1}{2}|Q|^2 + |\nu| + |\tilde{\nu}|\right)$. $\square$

In passing we stress that the above proof shows that $\mathbf{H}^0$ is diagonalized by the family of generalized (but not L^2) eigenstates $(e^{i(P, c)} \mathcal{Q}_{Q+iP, \nu, \tilde{\nu}})_{c \in \mathbb{R}^2, \nu, \tilde{\nu} \in \mathcal{T}^2}$.

8.2.3 . Reflection positivity

Using the GFFs introduced above allows to identify the Hilbert space $\mathcal{H}_{\mathbb{T}}$ with a space of functionals of fields $\phi : \mathbb{D} \rightarrow \mathbb{R}^2$. Namely let us introduce $\mathcal{A}_{\mathbb{D}}$ to be the sigma-algebra on $\mathbb{R}^2 \times \Omega$ generated by maps of the form $\phi \mapsto \langle \phi, f \rangle_{\mathbb{D}}$ for $f \in C_0^\infty(\mathbb{D} \rightarrow \mathbb{R}^2)$.

We then consider $\mathcal{F}_{\mathbb{D}}$ the set of $\mathbb{C}$ -valued, $\mathcal{A}_{\mathbb{D}}$ -measurable functions, equipped with the sesquilinear form

$$(F, G)_{\mathbb{D}} := \langle \theta F \bar{G} \rangle_{\gamma, 0}, \text{ where } \langle F \rangle_{\gamma, 0} := \int_{\mathbb{R}^2} e^{-2\langle Q, c \rangle} \mathbb{E} \left[F \left(\mathbf{X} + c - \frac{Q}{2} \ln g \right) \right] dc. \quad (8.2.28)$$

In the above, the action of θ on $\mathcal{F}_{\mathbb{D}}$ has been defined by $\theta F(\phi) := F(\phi \circ \theta - 2Q \ln |\cdot|)$. For future reference, we stress that using the decomposition (3.2.19) of the GFF the above sesquilinear form can actually be rewritten as

$$(F, G)_{\mathbb{D}} = \int_{\mathbb{R}^2} e^{-2\langle Q, c \rangle} \mathbb{E} \left[\mathbb{E}_{\varphi} [F(\mathbf{X}_1 + P\varphi + c)] \mathbb{E}_{\varphi} [\overline{G(\mathbf{X}_2 + P\varphi + c)}] \right] dc \quad (8.2.29)$$

where $\mathbf{X}_1$ and $\mathbf{X}_2$ are two independent GFFs with same law as $\mathbf{X}_{\mathbb{D}}$. The associated semi-norm will be denoted $\|\cdot\|_{\mathbb{D}}$ hereafter. For future reference we note that, using Cauchy-Schwartz inequality, this semi-norm satisfies

$$\|F\|_{\mathbb{D}} \leq \langle F^2 \rangle_{\gamma, 0}^{\frac{1}{2}}. \quad (8.2.30)$$

Now we would like to identify $\mathcal{F}_{\mathbb{D}}$ (or at least a subspace of it) with $\mathcal{H}_{\mathbb{T}}$. For this purpose let us introduce the map U_0 defined by

$$U_0 F(c, \varphi) := e^{-\langle Q, c \rangle} \mathbb{E}_{\varphi} [F(\mathbf{X}_{\mathbb{D}} + P\varphi + c)], \quad (8.2.31)$$

where $F \in \mathcal{F}_{\mathbb{D}}$ is such that $U_0(|F|) < \infty$ $dc \otimes d\mathbb{P}_{\mathbb{T}}$ -almost everywhere, together with the subspace $\mathcal{F}_{\mathbb{D}}^{0,2}$ of $\mathcal{F}_{\mathbb{D}}$ of such maps for which in addition $\|U_0 F\|_2 < \infty$.

One issue is that the sesquilinear form defined via Equation (8.2.28) is only non-negative over $\mathcal{F}_{\mathbb{D}}^{0,2}$. However if we denote by $\mathcal{N}_0 := \{F \in \mathcal{F}_{\mathbb{D}}^{0,2}, (F, F)_{\mathbb{D}} = 0\}$ the null set, we can consider the Hilbert space $\mathcal{H}_{\mathbb{D}}^0$ which is the completion with respect to $(\cdot, \cdot)_{\mathbb{D}}$ of the quotient space $\mathcal{F}_{\mathbb{D}}^{0,2}/\mathcal{N}_0$. The connection between the Hilbert space $\mathcal{H}_{\mathbb{D}}^0$ thus defined and the one, $\mathcal{H}_{\mathbb{T}}$, considered before, can now be properly stated:

Proposition 8.2.3. *The sesquilinear form (8.2.28) is non-negative over $\mathcal{F}_{\mathbb{D}}^{0,2}$ and descends to a unitary map from $\mathcal{H}_{\mathbb{D}}^0$ to $\mathcal{H}_{\mathbb{T}}$.*

Non-negativity of the bilinear form is usually referred to as *reflection positivity*.

Proof. To start with note that thanks to Equation (8.2.29) we can write that over $\mathcal{F}_{\mathbb{D}}^{0,2}$ we have $(F, G)_{\mathbb{D}} = \langle U_0 F | U_0 G \rangle_2$, and in particular the bilinear form $(\cdot, \cdot)_{\mathbb{D}}$ is non-negative and descends to an isometry on $\mathcal{H}_{\mathbb{D}}^0$. As a consequence proving that it is unitary boils down to the statement that it is onto, and for this it is enough to show that we can find a subset of $L^2(\mathbb{R}^2 \times \Omega_{\mathbb{T}})$ whose linear span is dense in $L^2(\mathbb{R}^2 \times \Omega_{\mathbb{T}})$ and that lies in the image of U_0 . For this purpose we consider elements of $L^2(\mathbb{R}^2 \times \Omega_{\mathbb{T}})$ of the form $e^{-\langle Q, c \rangle} \rho(c) e^{\langle \varphi, h \rangle_{\mathbb{T}}}$ where $\rho \in C_c^\infty(\mathbb{R}^2)$ and $h \in C^\infty(\mathbb{T} \rightarrow \mathbb{R}^2)$ has zero mean over $\mathbb{T}$.

Indeed the linear span of this set is dense in $L^2(\mathbb{R}^2 \times \Omega_{\mathbb{T}})$, and besides if

$$F_\varepsilon(\mathbf{X} + c) := \rho(\langle \mathbf{X}, g_\varepsilon \rangle_{\mathbb{D}} + c) e^{\langle \mathbf{X} + c, f_\varepsilon \rangle_{\mathbb{D}} - \frac{1}{2} \langle f_\varepsilon, G_{\mathbb{D}} f_\varepsilon \rangle} \text{ with} \\ g_\varepsilon(z) := \varepsilon^{-1} \eta \left(\frac{1 - |z|}{\varepsilon} \right) \text{ and } f_\varepsilon = (Ph)g_\varepsilon$$

with η a smooth mollifier with support in $[1, 2]$, then $\lim_{\varepsilon \rightarrow 0} U_0(F_\varepsilon)(c, \varphi) = e^{-\langle Q, c \rangle} \rho(c) e^{\langle \varphi, h \rangle_{\mathbb{T}}}$ where the limit is taken in $L^2(\mathbb{R}^2 \times \Omega_{\mathbb{T}})$. Indeed

$$\begin{aligned} U_0(F_\varepsilon)(c, \varphi) &= e^{-\langle Q, c \rangle} e^{\langle P\varphi + c, f_\varepsilon \rangle_{\mathbb{D}}} \mathbb{E}_\varphi \left[\rho(\langle \mathbf{X}_{\mathbb{D}} + P\varphi, g_\varepsilon \rangle_{\mathbb{D}} + c) e^{\langle \mathbf{X}_{\mathbb{D}}, f_\varepsilon \rangle_{\mathbb{D}} - \frac{1}{2} \langle f_\varepsilon, G_{\mathbb{D}} f_\varepsilon \rangle} \right] \\ &= e^{-\langle Q, c \rangle} e^{\langle P\varphi, f_\varepsilon \rangle_{\mathbb{D}}} \mathbb{E}_\varphi [\rho(\langle \mathbf{X}_{\mathbb{D}}, g_\varepsilon \rangle_{\mathbb{D}} + c)] \end{aligned}$$

where we have used the fact that $\langle c, f_\varepsilon \rangle_{\mathbb{D}} = \langle P\varphi, g_\varepsilon \rangle_{\mathbb{D}} = 0$ (since both f_ε and $P\varphi$ have zero mean on circles), together with independence of the Gaussian random variables $\langle \mathbf{X}_{\mathbb{D}}, g_\varepsilon \rangle_{\mathbb{D}}$ and $\langle \mathbf{X}_{\mathbb{D}}, f_\varepsilon \rangle_{\mathbb{D}}$. Along the same lines as in [66, Proposition 3.1] the latter converges in $L^2(\mathbb{R}^2 \times \Omega_{\mathbb{T}})$ to the desired limit $e^{-\langle Q, c \rangle} \rho(c) e^{\langle \varphi, h \rangle_{\mathbb{T}}}$. $\square$

8.2.4 . Hamiltonian: dilation semigroup and quadratic form

To conclude for this first section we present a few properties of the free Hamiltonian $\mathbf{H}^0$ that will be key in the derivation of the bootstrap formula (8.1.2). Namely we provide two alternative representations of this operator, in terms of the dilation semigroup and as a quadratic form, and describe a few consequences of it.

8.2.4.1 Dilation semigroup

Given $q \in \mathbb{D}$, the dilation map s_q allows to define an operator S_q over $\mathcal{F}_{\mathbb{D}}^0$ by setting

$$S_q F(\phi) := F(\phi \circ s_q + Q \ln |q|) \quad (8.2.32)$$

where the extra term $Q \ln |q|$ stems from the conformal covariance enjoyed by Toda CFTs, see Equation (4.2.8).

Proposition 8.2.4. *Viewed as a family of operators over $\mathcal{H}_{\mathbb{D}}^0$, $(S_q)_{q \in \mathbb{D}}$ is a strongly continuous contraction semigroup. In particular there exists a positive self-adjoint operator $\mathbf{H}_*^0$ with domain $\mathcal{D}(\mathbf{H}_*^0) \subset L^2(\mathbb{R}^2 \times \Omega_{\mathbb{T}})$ such that*

$$U_0 S_{e^{-t}} U_0^{-1} = e^{-t \mathbf{H}_*^0}. \quad (8.2.33)$$

Proof. This family being easily seen to be a semigroup, we only need to prove that it is contracting, and for this we aim to show that for $F \in \mathcal{F}_{\mathbb{D}}$ such that $\langle F^2 \rangle_{\gamma, 0} < \infty$ we have $\|S_q F\|_{\mathbb{D}} \leq \|F\|_{\mathbb{D}}^{1-2^{-k}} \langle F^2 \rangle_{\gamma, 0}^{2^{-k-1}}$ for any positive integer k . This is indeed enough since the set of such F is dense in $\mathcal{F}_{\mathbb{D}}$ (e.g. by considering $F_n := F \mathbb{1}_{|F| < n}$, which is such that $\|F - F_n\| \rightarrow 0$ as $n \rightarrow \infty$) so that by letting $k \rightarrow \infty$ we see that $\|S_q F\|_{\mathbb{D}} \leq \|F\|_{\mathbb{D}}$ as desired.

Now using the fact that the adjoint of S_q is nothing but $S_{\bar{q}}$ (which follows from the Möbius covariance of the correlation functions (4.2.8)) we can use Cauchy-Schwartz inequality to write that

$$\|S_q F\|_{\mathbb{D}}^2 = (F, S_{|q|^2} F)_{\mathbb{D}} \leq \|F\|_{\mathbb{D}} \|S_{|q|^2} F\|$$

which iterated yields $\|S_q F\|_{\mathbb{D}} \leq \dots \leq \|F\|_{\mathbb{D}}^{1-2^{-k}} \|S_{|q|^2} F\|_{\mathbb{D}}^{2^{-k}}$. Now thanks to Equation (8.2.30) the latter can be rewritten as $\|S_q F\|_{\mathbb{D}} \leq \|F\|_{\mathbb{D}}^{1-2^{-k}} \langle F^2 \rangle_{\gamma, 0}^{2^{-k-1}}$, which was our claim. $\square$

8.2.4.2 Connection with Subsection 8.2.1

So far we have defined two Hamiltonians acting over $L^2(\mathbb{R}^2 \times \Omega_{\mathbb{T}})$, either using the dilation semigroup as in Equation (8.2.33) or by direct construction based on the Heisenberg algebra as done in Subsection 8.2.1. The following statement allows to relate both operators using a Feynman-Kac formula:

Proposition 8.2.5. *For any $f \in L^2(\mathbb{R}^2 \times \Omega_{\mathbb{T}})$ and positive t ,*

$$U_0 S_{e^{-t}} U_0^{-1} f = e^{-t\mathbf{H}^0} f = e^{-\frac{|Q|^2}{2}t} \mathbb{E}_{\varphi} [f(\mathbf{c} + B_t, \varphi_t)], \quad (8.2.34)$$

where $\varphi_t(e^{i\theta}) := P\varphi(e^{-t+i\theta}) + Y_t(\theta)$ from Equation (3.3.6). In particular $\mathbf{H}^0_* = \mathbf{H}^0$.

Proof. To start with recall that we have the decomposition of the GFF inside $\mathbb{D}$ given by

$$\mathbf{X}(e^{-t+i\theta}) + \mathbf{c} = B_t + Y_t(\theta) + P\varphi(e^{-t+i\theta}) + \mathbf{c} = \varphi_t(\theta) + \mathbf{c} + B_t.$$

Therefore with f as above we have

$$U_0 S_{e^{-t}} U_0^{-1} f = e^{-\langle Q, \mathbf{c} \rangle} \mathbb{E}_{\varphi} [f(\mathbf{c} + B_t - Qt, \varphi_t) e^{\langle Q, \mathbf{c} + B_t - Qt \rangle}]$$

where the exponential term can be interpreted as a Girsanov transform whose effect is to shift the law of the Brownian motion by Qt . The variance of this term is $|Q|^2 t$, so that the above can be rewritten as $e^{-\frac{|Q|^2}{2}t} \mathbb{E}_{\varphi} [f(\mathbf{c} + B_t, \varphi_t)]$ as expected.

On the other hand, a standard property of the Brownian motion is that $e^{t\frac{\partial^2}{2}} f(\mathbf{c}, \varphi) = \mathbb{E} [f(\mathbf{c} + B_t, \varphi)]$ while the same reasoning as the one conducted in the proof of [66, Proposition 4.1] shows that $e^{-t\mathbf{P}} f(\mathbf{c}, \varphi) = \mathbb{E}_{\varphi} [f(\mathbf{c}, \varphi_t)]$. This implies that

$$e^{-t\mathbf{H}^0} = e^{-t\left(\frac{-\partial_{\mathbf{c}}^2 + |Q|^2}{2} + \mathbf{P}\right)} f = e^{-\frac{|Q|^2}{2}t} \mathbb{E}_{\varphi} [f(\mathbf{c} + B_t, \varphi_t)].$$

□

8.2.4.3 Quadratic form

There is yet a third way to make sense of $\mathbf{H}^0$, that closely resembles that from Subsection 8.2.1, based on the quadratic form associated to it. Defining $\mathbf{H}^0$ in such a way will appear to be necessary for the model associated to the Toda theory as soon as $\gamma > 1$. To this end let us introduce for two elements $u, v \in \mathcal{C}$, the subset of $\mathcal{C}^{\infty}$ whose elements are compactly supported in the $\mathbf{c}$ variable, the quantity:

$$\mathcal{Q}^0(u, v) := \frac{1}{2} \int_{\mathbb{R}^2} (\langle \nabla_{\mathbf{c}} u, \nabla_{\mathbf{c}} \bar{v} \rangle + |Q|^2 \langle u, \bar{v} \rangle + 2\langle \mathbf{P}u, \bar{v} \rangle) d\mathbf{c}. \quad (8.2.35)$$

By doing so we define a sesquilinear form but also a quadratic form $\mathcal{Q}^0(u) := \mathcal{Q}^0(u, u)$. This quadratic form provides an alternative way of making sense of the Hamiltonian $\mathbf{H}^0$ as the following statement discloses:

Proposition 8.2.6. *There exists a unique self-adjoint operator $\mathbf{H}^0$ (its Friedrichs extension) such that for any $u \in \mathcal{D}(\mathbf{H}^0)$, $\mathbf{H}^0 u$ is the unique element of $L^2(\mathbb{R}^2 \times \Omega_{\mathbb{T}})$ for which*

$$\mathcal{Q}^0(u, v) = \langle \mathbf{H}^0 u | v \rangle_2$$

for all $v \in \mathcal{D}(\mathcal{Q}^0)$. Here $\mathcal{D}(\mathcal{Q}^0)$ is the domain of (the closure of) the quadratic form $\mathcal{Q}_0$, while $\mathbf{H}^0$ is defined by

$$\mathcal{D}(\mathbf{H}^0) := \{u \in \mathcal{D}(\mathcal{Q}^0), \quad \exists C > 0, \forall v \in \mathcal{D}(\mathcal{Q}^0) \quad |\mathcal{Q}^0(u, v)| \leq C \|v\|_2\}.$$

It is readily checked that $\mathbf{H}^0$ coincides with the one defined in Subsection 8.2.1.

Proof. Let us assume for now that the quadratic form $\mathcal{Q}^0$ is closable with domain $\mathcal{D}(\mathcal{Q}^0)$. Then its closure (still denoted) $\mathcal{Q}^0$ is easily seen to be lower semi-bounded in the sense that for any $u \in \mathcal{D}(\mathcal{Q}^0)$, $\mathcal{Q}^0(u) \geq \frac{|\mathcal{Q}|^2}{2} \|u\|_2^2$ (which follows from self-adjointness of $\mathbf{P}^{1/2}$). Under these assumptions the reasoning developed in the proof of [91, Theorem 8.1] provides a construction of the Friedrichs extension of this closed, lower-semibounded quadratic form, thus existence of $\mathbf{H}^0$. Therefore proving Proposition 8.2.6 boils down to showing that $\mathcal{Q}^0$ is closable.

Saying that $\mathcal{Q}^0$ is closable amounts to saying that the map j that associates to any equivalence class $[u]$ of Cauchy sequences of $\mathcal{C}$ (under the relation $\lim_n u_n \sim \lim_n v_n$ when $\mathcal{Q}^0(u_n - v_n) \rightarrow 0$) its limit $\lim_n u_n$ in $L^2(\mathbb{R}^2 \times \Omega_{\mathbb{T}})$ is injective and continuous. Indeed based on lower semi-boundedness of $\mathcal{Q}^0$ a Cauchy sequence for $\mathcal{Q}^0$ is also Cauchy in $L^2(\mathbb{R}^2 \times \Omega_{\mathbb{T}})$ so that the above is well-defined. This property also implies continuity of the map j . For injectivity, let us consider u a Cauchy sequence for $\mathcal{Q}^0$ such that $u_n \rightarrow 0$ in $L^2(\mathbb{R}^2 \times \Omega_{\mathbb{T}})$. Then from the expression of $\mathcal{Q}^0$ we infer that both ∇u and $\mathbf{P}^{\frac{1}{2}}u$ are Cauchy in $L^2(\mathbb{R}^2 \times \Omega_{\mathbb{T}})$ and thus converge. These limits are seen to be equal to zero by testing them against functions in $\mathcal{C}$ (that forms a dense subset of $L^2(\mathbb{R}^2 \times \Omega_{\mathbb{T}})$). As a consequence $\mathcal{Q}^0(u) = 0$ too, showing injectivity of the map j . $\square$

8.2.5 . Eigenstates of the Hamiltonian and holomorphic currents

Building on the probabilistic framework introduced in the previous subsection, we provide here an alternative definition of the generalized eigenstates of the Hamiltonian $\mathbf{H}^0$. This definition relies on the *holomorphic currents* that we considered in Chapter 5. Indeed we will relate these currents to the representation of the W_3 algebra (and thus of the generalized eigenfunctions of $\mathbf{H}^0$) introduced above.

8.2.5.1 Mode of the currents and the W_3 algebra

Indeed the family of operators $(\mathbf{L}_n^0, \mathbf{W}_m^0)_{n,m \in \mathbb{Z}}$ defining the representation of the W_3 algebra studied in this document can be recovered thanks to these holomorphic currents. Indeed recall that they admit (formal) mode expansions of the form

$$\mathbf{T}(z) = \sum_{n \in \mathbb{Z}} \frac{\mathbf{L}_n^0}{z^{n+2}} \quad \text{and} \quad \mathbf{W}(z) = \sum_{n \in \mathbb{Z}} \frac{\mathbf{W}_n^0}{z^{n+3}}.$$

Having such an expansion would imply that the modes $(\mathbf{L}_n^0, \mathbf{W}_m^0)_{n,m \in \mathbb{Z}}$ can be recovered using contour integrals. Namely let us introduce for positive δ the contour integrals

$$\oint_{\mathcal{C}_\delta} f(w) dw := \int_0^{2\pi} f(\delta e^{i\theta}) i \delta e^{i\theta} d\theta \quad \text{and} \quad \oint_{\mathcal{C}_\delta} f(w) d\bar{w} := - \int_0^{2\pi} f(\delta e^{i\theta}) i \delta e^{-i\theta} d\theta$$

for $f : \mathbb{D} \rightarrow \mathbb{C}$, and more generally for $f = f(\mathbf{u}, \mathbf{v}) : \mathbb{D}^k \times \mathbb{D}^j \rightarrow \mathbb{C}$ the nested contour integrals

$$\oint_{\mathcal{C}_{\delta, \tilde{\delta}}} f(\mathbf{u}, \mathbf{v}) d\mathbf{u} d\mathbf{v} := \oint_{\mathcal{C}_{\delta_k}} \cdots \oint_{\mathcal{C}_{\delta_1}} \oint_{\mathcal{C}_{\tilde{\delta}_j}} \cdots \oint_{\mathcal{C}_{\tilde{\delta}_1}} f(\mathbf{u}, \mathbf{v}) du_1 \cdots du_k d\bar{v}_1 \cdots d\bar{v}_j \quad (8.2.36)$$

where δ is such that $0 < \delta_1 < \cdots < \delta_k < 1$ (and likewise for $\tilde{\delta}$). In order to make rigorous the definition of the modes of the currents, we first to introduce the proper framework in which this should apply.

For $0 < \delta < 1$ let us define a subset $\mathcal{F}_\delta$ of $\mathcal{F}_{\mathbb{D}}$ by setting

$$\mathcal{F}_\delta := \left\{ \prod_{i=1}^l \langle \mathbf{X} + c, g_i \rangle_{\mathbb{D}} e^{\langle \mathbf{X} + c, f \rangle}; l \geq 0 \text{ and } f, g_i \in \mathcal{E}_\delta \right\} \quad \text{where} \quad (8.2.37)$$

$$\mathcal{E}_\delta := \left\{ f(e^{-t+i\theta}) = \sum_{|n| \leq N} f_n(t) e^{in\theta} \text{ with } f_n \in C_0^\infty((-\ln \delta, +\infty) \rightarrow \mathbb{R}^2), N \geq 0 \right\}.$$

Using this setting the following statement allows to provide a rigorous meaning to the mode expansion described above:

Proposition 8.2.7. *Assume that $F \in \mathcal{F}_\delta$ with $\delta < \delta_1 \vee \tilde{\delta}_1$. Then*

$$\begin{aligned} & \mathbf{W}_{-\nu^2}^0 \mathbf{L}_{-\nu^1}^0 \widetilde{\mathbf{W}}_{-\tilde{\nu}^2}^0 \widetilde{\mathbf{L}}_{-\tilde{\nu}^1}^0 U_0 F \\ &= \frac{1}{(2i\pi)^{l(\nu)+l(\tilde{\nu})}} \oint_{\mathcal{C}_{\delta, \tilde{\delta}}} \mathbf{w}^{2-\nu^2} \mathbf{t}^{1-\nu^1} \bar{\mathbf{v}}^{2-\tilde{\nu}^2} \bar{\mathbf{s}}^{1-\tilde{\nu}^1} U_0 \left(\mathbf{W}(\mathbf{w}) \mathbf{T}(\mathbf{t}) \widetilde{\mathbf{W}}(\mathbf{v}) \widetilde{\mathbf{T}}(\mathbf{s}) F \right) d\mathbf{w} d\mathbf{t} d\mathbf{v} d\mathbf{s} \end{aligned} \quad (8.2.38)$$

where the last expression is defined by the limit in $e^{\beta|c|} L^2(\mathbb{R}^2 \times \Omega_{\mathbb{T}})$ for $\beta > |f_0 - Q|$: $\lim_{\varepsilon \rightarrow 0} U_0 \left(\mathbf{W}_\varepsilon(\mathbf{w}) \mathbf{T}_\varepsilon(\mathbf{t}) \widetilde{\mathbf{W}}_\varepsilon(\mathbf{v}) \widetilde{\mathbf{T}}_\varepsilon(\mathbf{s}) F \right)$. The contour integral corresponds to that in Equation (8.2.36) via the identification $w_{l(\nu^2)}^{2-\nu^2} \cdots w_1^{2-\nu_1^2} t_{l(\nu^1)}^{1-\nu_1^1} \cdots t_1^{1-\nu_1^1} = u_k^{2-\nu_{l(\nu^2)}^2} \cdots u_1^{1-\nu_1^1}$ and likewise for v and s .

This statement is a consequence of the following lemma, which allows to relate more generally mode expansion of the free-field and the representation of the Heisenberg algebra considered:

Lemma 8.2.8. *Assume that $w_1, \dots, w_r$ are distinct points in the unit disk $\mathbb{D}$. Then*

for $F \in \mathcal{F}_\delta$

$$\begin{aligned}
& \lim_{\varepsilon \rightarrow 0} \frac{1}{(2i\pi)^r} \oint_{\mathcal{C}_\delta} w_1^{l_1-1} \cdots w_r^{l_r-1} U_0 \left(\prod_{s=1}^r \left(: \prod_{k=1}^{p_s} \langle u_{k,s}, \partial^{m_{k,s}} \mathbf{X}_\varepsilon(w_s) \rangle : \right) F \right) d\mathbf{w} \\
&= \left[\sum_{\substack{n_1, \dots, n_{p_r} \in \mathbb{Z} \\ \sum_k n_k + m_{k,r} = l_r}} : \prod_{k=1}^{p_r} (n_k + 1)_{m_{k,r}-1} \langle u_{k,r}, i\mathbf{A}_{n_k} \rangle : \right] \cdots \\
& \quad \left[\sum_{\substack{n_1, \dots, n_{p_1} \in \mathbb{Z} \\ \sum_k n_k + m_{k,1} = l_1}} : \prod_{k=1}^{p_1} (n_k + 1)_{m_{k,1}-1} \langle u_{k,1}, i\mathbf{A}_{n_k} \rangle : \right] U_0 F
\end{aligned} \tag{8.2.39}$$

where the contour of integration is chosen with $0 < \delta < \delta_1 < \cdots < \delta_r$, and where $(x)_m = x(x+1)\cdots(x+m-1)$ is a Pochhammer symbol.

Specializing the above to $F(\mathbf{X} + \mathbf{c}) = S_{e^{-s}} \psi_\alpha^0$ allows to recover the (generalized) eigenfunctions of $\mathbf{H}^0$ using these contour integrals: for any $t > 0$ such that $e^{-t} < \delta$,

$$\begin{aligned}
\psi_{\alpha, \nu, \tilde{\nu}}^0 &= \frac{e^{2t\Delta_\alpha}}{(2i\pi)^{l(\nu)+l(\tilde{\nu})}} \oint_{\mathcal{C}_{\delta, \tilde{\delta}}} \mathbf{w}^{2-\nu^2} \mathbf{t}^{1-\nu^1} \tilde{\mathbf{v}}^{2-\tilde{\nu}^2} \tilde{\mathbf{s}}^{1-\tilde{\nu}^1} \\
& \quad U_0 \left(\mathbf{W}(\mathbf{w}) \mathbf{T}(\mathbf{t}) \widetilde{\mathbf{W}}(\mathbf{v}) \widetilde{\mathbf{T}}(\mathbf{s}) S_{e^{-s}} U_0(\psi_\alpha^0) \right) d\mathbf{w} d\mathbf{t} d\mathbf{v} d\mathbf{s}.
\end{aligned} \tag{8.2.40}$$

This representation will be crucial in the definition of the quantities that we denoted $v(\alpha_1, \alpha_2, \nu)$ before, a key step in the proof of the bootstrap formula (8.1.2). The analog of these Ward identities in the setting of the free-field theory corresponds to the study of expressions of the form

$$\frac{1}{(2i\pi)^{l(\nu)}} \oint_{\mathcal{C}_{\delta, \tilde{\delta}}} \mathbf{w}^{2-\nu^2} \mathbf{t}^{1-\nu^1} \mathbb{E} \left[\mathbf{W}(\mathbf{w}) \mathbf{T}(\mathbf{t}) V_{\alpha_0}(0) \prod_{k=1}^N V_{\alpha_k}(z_k) \right] d\mathbf{w} d\mathbf{t},$$

which amounts to considering, instead of a Vertex Operator $V_{\alpha_0}(0)$, one if its descendants via the representation (8.2.40). Following the reasoning developed in Chapter 5 the Ward identities allow to simplify the above expression as the following statement discloses:

Proposition 8.2.9. Assume that $z_1, \dots, z_N$ belong to $\mathbb{D} \setminus \delta\mathbb{D}$, while $0 < |t_j| < |w_i| < \delta$ for $1 \leq i \leq r$ and $1 \leq j \leq s$. Then for $n \geq 0$:

$$\begin{aligned}
& \frac{1}{2i\pi} \oint_{\delta} (w')^{2-n} \mathbb{E} \left[\mathbf{W}(w') \mathbf{W}(\mathbf{w}) \mathbf{T}(\mathbf{t}) V_{\alpha_0}(0) \prod_{k=1}^N V_{\alpha_k}(z_k) \right] dw' \\
&= - \sum_{l=1}^N \left(\frac{(n-1)(n-2)w(\alpha_l)}{2z_l^n} - \frac{(n-2)\mathcal{W}_{-1}^{(l)}}{z_l^{n-1}} + \frac{\mathcal{W}_{-2}^{(l)}}{z_l^{n-2}} \right) \mathbb{E} \left[\mathbf{W}(\mathbf{w}) \mathbf{T}(\mathbf{t}) V_{\alpha_0}(0) \prod_{k=1}^N V_{\alpha_k}(z_k) \right].
\end{aligned} \tag{8.2.41}$$

Likewise for the stress-energy tensor we have

$$\begin{aligned} & \frac{1}{2i\pi} \oint_{\delta} (t')^{1-n} \mathbb{E} \left[\mathbf{T}(t') \mathbf{T}(t) V_{\alpha_0}(0) \prod_{k=1}^N V_{\alpha_k}(z_k) \right] dt' \\ &= - \sum_{l=1}^N \left(-\frac{(n-1)\Delta_{\alpha_l}}{z_l^n} + \frac{\mathcal{L}_{-1}^{(l)}}{z_l^{n-1}} \right) \mathbb{E} \left[\mathbf{T}(t) \prod_{k=1}^N V_{\alpha_k}(z_k) \right]. \end{aligned} \quad (8.2.42)$$

This statement shows that one can reduce the computation of “correlation functions” containing one arbitrary descendent field can be reduced to that of correlation functions that contain only Virasoro descendants at the order 1 and W -descendants at the order 2.

We will not prove these statements in this manuscript since they are rather long and not very informative. They rely on algebraic manipulations that generalize the ones considered along the proof of Theorem 5.1.3. More details are to be found in [32].

8.3 . Towards the conformal bootstrap formula

The previous section was dedicated to the introduction of the background necessary to make sense of the free-field theory associated to the $\mathfrak{sl}_3$ Toda CFT. Along this section and based on the objects defined above we will define a similar framework for the $\mathfrak{sl}_3$ Toda CFT. We will shed light on the main steps leading to the proof of Theorem 8.1.1 but will not enter the details of the computations to keep the document readable.

8.3.1 . From the free-field theory to Toda

8.3.1.1 The framework

Let us briefly explain how to adapt the setting presented in the previous section to that of the $\mathfrak{sl}_3$ Toda CFT. Like before we work with the sigma-algebra $\mathcal{A}_{\mathbb{D}}$ as well as $\mathcal{F}_{\mathbb{D}}$, but this time equipped with the bilinear form

$$\begin{aligned} (F, G)_{\mathbb{D}} &:= \langle \theta F \bar{G} \rangle_{\gamma, \mu}, \quad \text{where} \\ \langle F \rangle_{\gamma, \mu} &:= \int_{\mathbb{R}^2} e^{-2\langle Q, c \rangle} \mathbb{E} \left[F \left(\mathbf{X} + c - \frac{Q}{2} \ln g \right) e^{-\mu \sum_{i=1}^2 e^{\gamma \langle c, e_i \rangle} M^{\gamma e_i}(\mathbb{C})} \right] dc \end{aligned} \quad (8.3.1)$$

with $M^{\gamma e_i}$ the GMC measure defined from the GFF $\mathbf{X}$. Like before, we can use the decomposition (3.2.19) of the GFF $\mathbf{X}$ to put the latter under the form

$$\begin{aligned} (F, G)_{\mathbb{D}} &= \int_{\mathbb{R}^2} e^{-2\langle Q, c \rangle} \mathbb{E} \left[\mathbb{E}_{\varphi} \left[F(\mathbf{X}_1 + P\varphi + c) e^{-\mu \sum_{i=1}^2 e^{\gamma \langle c, e_i \rangle} M_1^{\gamma e_i}(\mathbb{D})} \right] \right. \\ &\quad \left. \mathbb{E}_{\varphi} \left[G(\mathbf{X}_2 + P\varphi + c) e^{-\mu \sum_{i=1}^2 e^{\gamma \langle c, e_i \rangle} M_2^{\gamma e_i}(\mathbb{D})} \right] \right] dc \end{aligned} \quad (8.3.2)$$

with $M_k^{\gamma e_i}$ is the GMC measure defined from the GFF $\mathbf{X}_k + P\varphi$, for $k = 1, 2$. We also have the inequality derived from Equation (8.2.30):

$$\|F\|_{\mathbb{D}} \leq \langle F^2 \rangle_{\gamma, \mu}^{\frac{1}{2}}. \quad (8.3.3)$$

We likewise introduce the analog of the map U_0 by setting

$$UF(\mathbf{c}, \varphi) := e^{-\langle Q, \mathbf{c} \rangle} \mathbb{E}_\varphi \left[F(\mathbf{c} + \mathbf{X}_\mathbb{D} + P\varphi) e^{-\mu \sum_{i=1}^2 e^{\gamma(\mathbf{c}, e_i)} M^{\gamma e_i}(\mathbb{D})} \right], \quad (8.3.4)$$

where $M^{\gamma e_i}$ is the GMC measure associated to the GFF $\mathbf{X}_\mathbb{D} + P\varphi$, and for $F \in \mathcal{F}_\mathbb{D}$ such that $U(|F|) < \infty$ $d\mathbf{c} \otimes d\mathbb{P}_\mathbb{T}$ -almost everywhere. The natural set over which U acts is the subspace $\mathcal{F}_\mathbb{D}^2$ of $\mathcal{F}_\mathbb{D}$ of such maps that satisfy $\|UF\|_2 < \infty$.

8.3.1.2 Correlation functions of Vertex Operators

Using this framework we can also define the correlation functions of Vertex Operators using the L^2 structure of $\mathcal{F}_\mathbb{D}$ as follows. To start with let us introduce for $z_1, \dots, z_N \in \mathbb{D}$ and $\alpha_1, \dots, \alpha_N \in \mathbb{R}^2$ the notation

$$U_\alpha(\mathbf{z}) := \lim_{\varepsilon \rightarrow 0} U \left(\prod_{k=1}^N V_{\alpha_k, \varepsilon}(z_k) \right). \quad (8.3.5)$$

Then thanks to Girsanov's theorem the above limit makes sense as soon as $\langle \alpha_k - Q, e_i \rangle < 0$ for all $1 \leq k \leq N$ and $i = 1, 2$, and defines an element in the weighted space $e^{\beta_\alpha(\mathbf{c})} L^2(\mathbb{R}^2 \times \Omega_\mathbb{T})$ where $\beta_\alpha = \langle \sum_k \alpha_k - Q, \mathbf{c} \rangle$ for $\mathbf{c} \in \mathcal{C}_-$ and has exponential decay in the other directions. This allows to write the correlation functions under the form

$$\left\langle \prod_{k=1}^N V_{\alpha_k}(z_k) \prod_{k=1}^{N'} V_{\beta_k}(w_k) \right\rangle = \left(\prod_{k=1}^{N'} |w_k|^{-4\Delta_{\beta_k}} \right) \langle U_\alpha(\mathbf{z}) | U_\beta(\theta(\mathbf{w})) \rangle_2 \quad (8.3.6)$$

for $z_1, \dots, z_N \in \mathbb{D}$ and $\theta(w_1), \dots, \theta(w_{N'}) \in \mathbb{D}$.

8.3.2 . The Hamiltonian from the dilation semigroup

In a similar fashion as for the free-field theory, we will provide two alternative definitions (which will later be seen to be equivalent) to the Toda Hamiltonian. The first way to make sense of it is to proceed like in Subsection 8.2.4.1 by defining it using the dilation semigroup.

8.3.2.1 Hilbert space and Hamiltonian

Like before, a natural associated Hilbert space can be defined as the Hilbert space completion $\mathcal{H}_\mathbb{D}$ (with respect to $(\cdot, \cdot)_\mathbb{D}$) of the quotient space $\mathcal{F}_\mathbb{D}^2 / \mathcal{N}$ where the null set is $\mathcal{N} := \{F \in \mathcal{F}_\mathbb{D}^2, (F, F)_\mathbb{D} = 0\}$. The correspondence between this Hilbert space and the one introduced before is made thanks to the following statement, analogous to Proposition 8.2.3:

Proposition 8.3.1. *The bilinear form (8.3.1) can be extended over $\mathcal{F}_\mathbb{D}^2$ where it is non-negative. Moreover the map U descends to a unitary map from $\mathcal{H}_\mathbb{D}$ to $\mathcal{H}_\mathbb{T}$.*

Proof. This follows from Proposition 8.2.3 since $UF = U_0 \left(F e^{-\mu \sum_{i=1}^2 e^{\gamma(\mathbf{c}, e_i)} M^{\gamma e_i}(\mathbb{D})} \right)$. $\square$

The above statement combined with the action of the dilation semigroup on $\mathcal{H}_\mathbb{D}$ allows to define an Hamiltonian as the following discloses:

Proposition 8.3.2. Viewed as a family of operators over $\mathcal{H}_{\mathbb{D}}$, $(S_q)_{q \in \mathbb{D}}$ is a strongly continuous contraction semigroup. In particular there exists a positive self-adjoint operator $\mathbf{H}_*$ with domain $\mathcal{D}(\mathbf{H}_*) \subset L^2(\mathbb{R}^2 \times \Omega_{\mathbb{T}})$ such that

$$US_{e^{-t}}U^{-1} = e^{-t\mathbf{H}_*}. \quad (8.3.7)$$

Proof. The arguments developed in the Proof of Proposition 8.2.3 still apply in this context thanks to Equation (8.3.3). $\square$

8.3.2.2 A Feynman-Kac formula

Thanks to this definition of the Toda Hamiltonian we can adapt the reasoning developed above and provide the following Feynman-Kac formula, analogous to Equation (8.2.34):

Proposition 8.3.3. For any $f \in L^2(\mathbb{R}^2 \times \Omega_{\mathbb{T}})$ and $t \geq 0$:

$$e^{-t\mathbf{H}_*}f = e^{-\frac{|Q|^2}{2}t} \mathbb{E}_{\varphi} \left[f(\mathbf{c} + B_t, \varphi_t) e^{-\mu \sum_{i=1}^2 e^{\gamma \langle \mathbf{c}, e_i \rangle} \int_{\mathbb{A}_{0,t}} |z|^{-\gamma q} M^{\gamma e_i}(dz)} \right] \quad (8.3.8)$$

where $\mathbb{A}_{0,t} := \{z \in \mathbb{C}, \quad e^{-t} < |z| < 1\}$.

Proof. To start with let us note that $S_{e^{-t}}(U^{-1}f)$ only depends on the sigma-algebra generated by the $(\mathbf{X}(z))_{1 \geq |z| \geq e^{-t}}$. As a consequence we can write that

$$e^{-t\mathbf{H}_*}f = e^{-\langle Q, \mathbf{c} \rangle} \mathbb{E}_{\varphi} \left[S_{e^{-t}}(U^{-1}f) \mathbb{E} \left[e^{-\mu \sum_{i=1}^2 e^{\gamma \langle \mathbf{c}, e_i \rangle} \int_{\mathbb{D}} M^{\gamma e_i}(dz)} \middle| \mathbf{X}(z), 1 \geq |z| \geq e^{-t} \right] \right].$$

Now the conditional expectation can be dealt with using the Markov property of the GFF, namely the fact that for $|z| < e^{-t}$ we can write $\mathbf{X}$ under the form $\mathbf{X}(e^{-t} \cdot) = \tilde{\mathbf{X}}_{\mathbb{D}} + P(s_{e^{-t}}\mathbf{X})$ where $\tilde{\mathbf{X}}_{\mathbb{D}}$ has the law of $\mathbf{X}_{\mathbb{D}}$ and is independent of $(\mathbf{X}(z))_{1 \geq |z| \geq e^{-t}}$. Therefore the conditional expectation is seen to be given by

$$e^{-\mu \sum_{i=1}^2 e^{\gamma \langle \mathbf{c}, e_i \rangle} \int_{\mathbb{A}_{0,t}} M^{\gamma e_i}(dz)} \mathbb{E} \left[e^{-\mu \sum_{i=1}^2 e^{\gamma \langle \mathbf{c}, e_i \rangle} \int_{e^{-t}\mathbb{D}} M^{\gamma e_i}(dz)} \middle| \mathbf{X}(z), 1 \geq |z| \geq e^{-t} \right],$$

where in the integral that appears in the last expectation term we can make the change of variable $z \leftrightarrow e^{-t}z$ to rewrite it as

$$\mathbb{E}_{S_{e^{-t}}\mathbf{X}|_{\mathbb{S}^1}} \left[e^{-\mu \sum_{i=1}^2 e^{\gamma \langle \mathbf{c}, e_i \rangle} \int_{\mathbb{D}} M_{\tilde{\mathbf{X}}_{\mathbb{D}} + P(s_{e^{-t}}\mathbf{X})}^{\gamma e_i}(dz)} \right].$$

The latter is therefore equal to $S_{e^{-t}}(e^{\langle Q, \mathbf{c} \rangle} U1)$ where 1 is the constant function. Recollecting terms we end up with

$$\begin{aligned} e^{-t\mathbf{H}_*}f &= e^{-\langle Q, \mathbf{c} \rangle} \mathbb{E}_{\varphi} \left[S_{e^{-t}} \left(e^{\langle Q, \mathbf{c} \rangle} f \right) e^{-\mu \sum_{i=1}^2 e^{\gamma \langle \mathbf{c}, e_i \rangle} \int_{\mathbb{A}_{0,t}} M^{\gamma e_i}(dz)} \right] \\ &= e^{-\langle Q, \mathbf{c} \rangle} \mathbb{E}_{\varphi} \left[e^{\langle Q, \mathbf{c} + B_t - Qt \rangle} f(\mathbf{c} + B_t - Qt, \varphi_t) e^{-\mu \sum_{i=1}^2 e^{\gamma \langle \mathbf{c}, e_i \rangle} \int_{\mathbb{A}_{0,t}} M^{\gamma e_i}(dz)} \right]. \end{aligned}$$

The Girsanov transform has the effect of shifting the law of B_s by Qs for $s \leq t$. This accounts for the additional singularity $|z|^{-\gamma \langle Q, e_i \rangle}$ that shows up in the integral, and yields

$$e^{-t\mathbf{H}_*}f = e^{-\frac{|Q|^2}{2}t} \mathbb{E}_{\varphi} \left[f(\mathbf{c} + B_t, \varphi_t) e^{-\mu \sum_{i=1}^2 e^{\gamma \langle \mathbf{c}, e_i \rangle} \int_{\mathbb{A}_{0,t}} |z|^{-\gamma q} M^{\gamma e_i}(dz)} \right].$$

$\square$

8.3.3 . The Hamiltonian from the quadratic form

There is a second way to define the Toda Hamiltonian, more explicit and based on the reasoning conducted in Subsection 8.3.1. However in order to make sense of it we will need to define it through its quadratic form, in a similar fashion as in Subsection 8.2.4.3. The reason for such a definition is that GMC measures that should appear in its expression may not make sense probabilistically speaking for the whole range of values of γ .

8.3.3.1 Quadratic forms

To overcome this issue we proceed in the same way as in the construction of GMC measures by considering a sequence of regularized potentials

$$V_i^{(k)} : \varphi \mapsto \int_{\mathbb{T}} e^{\gamma \langle \varphi^{(k)}(\theta), e_i \rangle - \frac{\gamma^2}{2} \mathbb{E}[\langle \varphi^{(k)}(\theta), e_i \rangle^2]} d\theta \quad \text{for } k \geq 1 \text{ and } i = 1, 2.$$

In the above we have defined

$$\varphi^{(k)} := \sum_{1 \leq l \leq k} \sum_{i=1}^2 (\varphi_{n,i} \mathbf{e}_{n,i} + \bar{\varphi}_{n,i} \mathbf{e}_{-n,i})$$

based on the mode expansion (8.2.1) of φ . We then introduce a sequence of quadratic forms by setting for $u, v \in \mathcal{C}$

$$\mathcal{Q}^{(k)}(u, v) := \frac{1}{2} \int_{\mathbb{R}^2} \left(\langle \nabla_{\mathbf{c}} u, \nabla_{\mathbf{c}} \bar{v} \rangle + |Q|^2 \langle u, \bar{v} \rangle + 2 \langle \mathbf{P} u, \bar{v} \rangle + 2\mu \sum_{i=1}^2 e^{\gamma \langle \mathbf{c}, e_i \rangle} V_i^{(k)} \langle u, \bar{v} \rangle \right) d\mathbf{c} \quad (8.3.9)$$

The arguments developed in the proof of Proposition 8.2.6 show that to these quadratic forms one can associate a self-adjoint operator $\mathbf{H}^{(k)}$ such that for $u \in \mathcal{D}(\mathbf{H}^{(k)})$ and $v \in \mathcal{D}(\mathcal{Q}^{(k)})$

$$\mathcal{Q}^{(k)}(u, v) = \langle \mathbf{H}^{(k)} u | v \rangle_2.$$

Our next task is to investigate the limiting behaviour of the above sequence of quadratic forms. For this purpose we define

$$\mathcal{Q}(u, v) := \lim_{k \rightarrow \infty} \mathcal{Q}^{(k)}(u, v). \quad (8.3.10)$$

Our goal is to show the following:

Proposition 8.3.4. *The quadratic form $\mathcal{Q}$ is closable. Its Friedrichs extension gives rise to an operator $\mathbf{H}$ that satisfies $\mathbf{H} = \mathbf{H}_*$.*

8.3.3.2 Proof of Proposition 8.3.4

In order to prove such as statement, we will rely on the following Feynman-Kac formula, reminiscent of Equation (8.3.8):

Proposition 8.3.5. *For any $f \in L^2(\mathbb{R}^2 \times \Omega_{\mathbb{T}})$ and $k \in \mathbb{N}$,*

$$e^{-t\mathbf{H}^{(k)}} f = e^{-\frac{|Q|^2}{2}t} \mathbb{E}_{\varphi} \left[f(\mathbf{c} + B_t, \varphi_t) e^{-\mu \sum_{i=1}^2 \int_0^t e^{\gamma \langle \mathbf{c} + B_s, e_i \rangle} V_i^{(k)}(\varphi_s) ds} \right] \quad (8.3.11)$$

Proof. To prove such a statement we will first provide an intermediate Feynman-Kac formula for the quantity

$$\left(e^{-\frac{t}{n} \mathbf{H}^0} e^{-\frac{t}{n} \mu \sum_{i=1}^2 e^{\gamma \langle \mathbf{c}, e_i \rangle} V_i^{(k)}} \right)^n f, \quad \text{with } n \text{ positive,}$$

and then deduce the general case using Trotter-Kato's formula.

Along the same lines as in the proof of Equation (8.2.34) we see that for $f \in L^2(\mathbb{R}^2 \times \Omega_{\mathbb{T}})$

$$\left(e^{-\frac{t}{n} \mathbf{H}^0} e^{-\frac{t}{n} \mu \sum_{i=1}^2 e^{\gamma \langle \mathbf{c}, e_i \rangle} V_i^{(k)}} \right)^n f = e^{-\frac{|Q|^2}{2} t} \mathbb{E}_{\varphi} \left[f(\mathbf{c} + B_t, \varphi_t) e^{-\mu \sum_{i=1}^2 \sum_{j=1}^n e^{\gamma \langle \mathbf{c} + B_{jt/n}, e_i \rangle} V_i^{(k)}(\varphi_{jt/n})} \right].$$

The sum that appears in the above exponential is nothing but a Riemann sum, that converges almost surely to $\int_0^t e^{\gamma \langle \mathbf{c} + B_s, e_i \rangle} V_i^{(k)}(\varphi_s) ds$ since the process $s \mapsto e^{\gamma \langle \mathbf{c} + B_s, e_i \rangle} V_i^{(k)}(\varphi_s)$ is almost surely continuous. This shows that

$$\lim_{n \rightarrow +\infty} \left(e^{-\frac{t}{n} \mathbf{H}^0} e^{-\frac{t}{n} \mu \sum_{i=1}^2 e^{\gamma \langle \mathbf{c}, e_i \rangle} V_i^{(k)}} \right)^n f = e^{-\frac{|Q|^2}{2} t} \mathbb{E}_{\varphi} \left[f(\mathbf{c} + B_t, \varphi_t) e^{-\mu \sum_{i=1}^2 \int_0^t e^{\gamma \langle \mathbf{c} + B_s, e_i \rangle} V_i^{(k)}(\varphi_s) ds} \right]$$

where, using Jensen's inequality, the above limit is seen to hold in $L^2(\mathbb{R}^2 \times \Omega_{\mathbb{T}})$.

Now we can rely on [91, Theorem S.21] to see that the above limits is actually given by

$$\lim_{n \rightarrow +\infty} \left(e^{-\frac{t}{n} \mathbf{H}^0} e^{-\frac{t}{n} \mu \sum_{i=1}^2 e^{\gamma \langle \mathbf{c}, e_i \rangle} V_i^{(k)}} \right)^n f = e^{-t \mathbf{H}^{(k)}} f$$

where the limit is understood in $L^2(\mathbb{R}^2 \times \Omega_{\mathbb{T}})$. □

Thanks to this Feynman-Kac formula we are able to relate $\mathcal{Q}$ to the quadratic form associated to $\mathbf{H}_*$, which we denote by $\mathcal{Q}_*$ and which is defined by $\mathcal{Q}_*(u, v) := \langle \mathbf{H}_* u | v \rangle_2$.

Lemma 8.3.6. *For any u and v in $\mathcal{C}$ $\mathcal{Q}(u, v) = \mathcal{Q}_*(u, v)$.*

Proof. To start with we note that over $\mathcal{C} \subset \mathcal{D}(\mathcal{Q}_*)$ we have

$$\mathcal{Q}_*(u, v) = \lim_{t \rightarrow 0} \left\langle \frac{u - e^{-t \mathbf{H}_*} u}{t} | v \right\rangle_2$$

where the right-hand side can be rewritten using the Feynman-Kac formula (8.3.8).

As a consequence we need to check that we can write an expansion of the form

$D_t = \langle u | v \rangle_2 - t \mathcal{Q}(u, v) + o(t)$ as $t \rightarrow 0$ where

$$D_t := e^{-\frac{|Q|^2}{2} t} \int_{\mathbb{R}^2} \mathbb{E} \left[u(\mathbf{c} + B_t, \varphi_t) v(\mathbf{c}, \varphi) e^{-\mu \sum_{i=1}^2 e^{\gamma \langle \mathbf{c}, e_i \rangle} V_i(t)} \right] d\mathbf{c}, \quad V_i(t) := \int_{\mathbb{A}_{0,t}} |z|^{-\gamma q} M^{\gamma e_i}(dz).$$

For this purpose we split D_t as follows:

$$\begin{aligned} D_t &= D_t^0 + D_t^1 + D_t^2, \quad \text{where} \quad D_t^0 = e^{-\frac{|Q|^2}{2} t} \int_{\mathbb{R}^2} \mathbb{E} [u(\mathbf{c} + B_t, \varphi_t) v(\mathbf{c}, \varphi)] d\mathbf{c}, \\ D_t^1 &= e^{-\frac{|Q|^2}{2} t} \int_{\mathbb{R}^2} \mathbb{E} \left[u(\mathbf{c}, \varphi) v(\mathbf{c}, \varphi) \left(e^{-\mu \sum_{i=1}^2 e^{\gamma \langle \mathbf{c}, e_i \rangle} V_i(t)} - 1 \right) \right] d\mathbf{c}, \\ D_t^2 &= e^{-\frac{|Q|^2}{2} t} \int_{\mathbb{R}^2} \mathbb{E} \left[(u(\mathbf{c} + B_t, \varphi_t) - u(\mathbf{c}, \varphi)) v(\mathbf{c}, \varphi) \left(e^{-\mu \sum_{i=1}^2 e^{\gamma \langle \mathbf{c}, e_i \rangle} V_i(t)} - 1 \right) \right] d\mathbf{c} \end{aligned}$$

and treat each term separately. We wish to prove that:

$$D_t^0 = \langle u, |v \rangle_2 - t \mathcal{Q}_0(u, v) + o(t), \quad D_t^1 = t (\mathcal{Q}_0(u, v) - \mathcal{Q}(u, v)) + o(t), \quad D_t^2 = o(t). \quad (8.3.12)$$

This would prove Lemma 8.3.6.

First of all note that $D_t^0 = \langle e^{-t\mathbf{H}_0} | v \rangle_2$ so that since $u, v \in \mathcal{D}(\mathcal{Q}_0)$ we already know that $D_t^0 = \langle u, | v \rangle_2 - t\mathcal{Q}_0(u, v) + o(t)$. Therefore we can turn to D_t^1 , which we rewrite as

$$D_t^1 = -\mu \sum_{i=1}^2 e^{\gamma \langle \mathbf{c}, e_i \rangle} e^{-\frac{|Q|^2}{2}t} \int_{\mathbb{R}^2} \mathbb{E} [u(\mathbf{c}, \varphi) v(\mathbf{c}, \varphi) V_i(t)] d\mathbf{c} + l.o.t.$$

The leading term can be dealt with using Girsanov's theorem A since, by setting for $z \in \mathbb{D}$ $\varphi_n^z = \varphi_n + \frac{z^n}{\sqrt{n}} \gamma e_i$,

$$\begin{aligned} \mathbb{E} [u(\mathbf{c}, \varphi) v(\mathbf{c}, \varphi) V_i(t)] &= \int_{\mathbb{A}_{0,t}} |z|^{-\gamma q} \mathbb{E} [u(\mathbf{c}, \varphi^z) v(\mathbf{c}, \varphi^z)] dz \\ &= t \int_0^{2\pi} \mathbb{E} [u(\mathbf{c}, \varphi^{e^{i\theta}}) v(\mathbf{c}, \varphi^{e^{i\theta}})] d\theta \\ &\quad + \int_0^t \int_0^{2\pi} \left(e^{\gamma q s} \mathbb{E} [u(\mathbf{c}, \varphi^{e^{-s+i\theta}}) v(\mathbf{c}, \varphi^{e^{-s+i\theta}})] - \mathbb{E} [u(\mathbf{c}, \varphi^{e^{i\theta}}) v(\mathbf{c}, \varphi^{e^{i\theta}})] \right) d\theta ds. \end{aligned}$$

This shows that $D_t^1 = t(\mathcal{Q}_0(u, v) - \mathcal{Q}(u, v)) + \mathfrak{R}_t^1$ where the remainder term is defined by

$$\begin{aligned} \mathfrak{R}_t^1 &= e^{-\frac{|Q|^2}{2}t} \int_{\mathbb{R}^2} \mathbb{E} \left[u(\mathbf{c}, \varphi) v(\mathbf{c}, \varphi) \left(e^{-\mu \sum_{i=1}^2 e^{\gamma \langle \mathbf{c}, e_i \rangle} V_i(t)} - 1 + \mu \sum_{i=1}^2 e^{\gamma \langle \mathbf{c}, e_i \rangle} V_i(t) \right) \right] d\mathbf{c} \\ &\quad + e^{-\frac{|Q|^2}{2}t} \int_{\mathbb{R}^2} \int_0^t \int_0^{2\pi} \left(e^{\gamma q s} \mathbb{E} [u(\mathbf{c}, \varphi^{e^{-s+i\theta}}) v(\mathbf{c}, \varphi^{e^{-s+i\theta}})] - \mathbb{E} [u(\mathbf{c}, \varphi^{e^{i\theta}}) v(\mathbf{c}, \varphi^{e^{i\theta}})] \right) d\theta ds d\mathbf{c}. \end{aligned}$$

This remainder term has been controlled along the proof of [66, Lemma 5.4] and was shown there to be a $o(t)$. Therefore D_t^1 has the desired expansion. Likewise it was proved in [66, Lemma 5.4] that $D_t^2 = o(t)$ too, which allows to wrap up the proof. $\square$

An immediate consequence of this statement is the fact that the quadratic form $\mathcal{Q}$ is closable. We denote by $\mathcal{D}(\mathcal{Q})$ the completion of $\mathcal{C}$ with respect to the norm induced by $\mathcal{Q}$. We are now in position to finish up with the proof of Proposition 8.3.4, which boils down to the following statement:

Lemma 8.3.7. *Denote by $\mathbf{R}_\lambda^{(k)}$ (resp. $\mathbf{R}_{*,\lambda}$) the resolvent family associated to the semigroup $e^{-t\mathbf{H}^{(k)}}$ (resp. $e^{-t\mathbf{H}_*}$). Then for $\lambda > 0$ $\mathbf{R}_{*,\lambda}$ maps $\mathcal{C}$ into $\mathcal{D}(\mathcal{Q})$ where it coincides with $\mathbf{R}_\lambda = \lim_{k \rightarrow +\infty} \mathbf{R}_\lambda^{(k)}$.*

By density of $\mathcal{C}$ in $L^2(\mathbb{R}^2 \times \Omega_{\mathbb{T}})$ this statement shows that both resolvent families $\mathbf{R}_\lambda$ and $\mathbf{R}_{*,\lambda}$ coincide, hence so do the generators of their semigroup, that is $\mathbf{H} = \mathbf{H}_*$.

Proof. Consider $F \in \mathcal{C}$ and define the sequence in $L^2(\mathbb{R}^2 \times \Omega_{\mathbb{T}})$ $u_k := \mathbf{R}_\lambda^{(k)}$ as well as $u := \mathbf{R}_{*,\lambda} \in L^2(\mathbb{R}^2 \times \Omega_{\mathbb{T}})$. Our goal is to prove that $\lim_{k \rightarrow +\infty} u_k = u$, where the limit is meant in the semi-norm induced by $\mathcal{Q}$. Indeed this would show that $u \in \mathcal{D}(\mathcal{Q})$ and that both resolvent agree on over $\mathcal{C}$.

To start with recall that the resolvent admits the representation for $\lambda > 0$:

$$\mathbf{R}_\lambda^{(k)} = \int_0^{+\infty} e^{-\lambda t} e^{-t\mathbf{H}^{(k)}} dt.$$

Thus the Feynman-Kac formula 8.3.8 together with convergence of the GMC measure term appearing there shows that $u_k \rightarrow u$ in $L^2(\mathbb{R}^2 \times \Omega_{\mathbb{T}})$. Moreover we see that $\sup_{k \geq 0} \mathcal{Q}^{(k)}(u_k) < \infty$ since by definition $\mathcal{Q}^{(k)}(u_k) = \langle F|u_k \rangle_2 - \lambda \|u_k\|_2^2 \leq \|F\|_2 \|u_k\|_2 - \lambda \|u_k\|_2^2$. These two assertions imply that we have weak convergence in $L^2(\mathbb{R}^2 \times \Omega_{\mathbb{T}})$ of the sequences $(\nabla_{\mathbf{c}} u_k)_{k \geq 0}$ and $(\mathbf{P}^{\frac{1}{2}} u_k)_{k \geq 0}$, with weak limit given by $\nabla_{\mathbf{c}} u$ and $\mathbf{P}^{\frac{1}{2}} u$. The convergence is seen to hold in the strong sense by reproducing the computations in the proof of [66, Proposition 5.5]. Likewise the sequence of $(u_{k,i}^z)_{k \geq 0}$ strongly converges in $e^{-\frac{\gamma}{2}\langle \mathbf{c}, e_i \rangle} L^2(\mathbb{T} \times \mathbb{R}^2 \times \Omega_{\mathbb{T}})$ for $i = 1, 2$, where $u_{k,i}^z(\varphi_n)_n = u_k(\varphi_n + \frac{z^n}{\sqrt{n}} \gamma e_i)_n$. In brief we have seen that

$$(u_k, \nabla_{\mathbf{c}} u_k, \mathbf{P}^{\frac{1}{2}} u_k, u_{k,i}^z) \rightarrow (u, \nabla_{\mathbf{c}} u, \mathbf{P}^{\frac{1}{2}} u, u_i^z)$$

in $(L^2(\mathbb{R}^2 \times \Omega_{\mathbb{T}}))^3 \times_{i=1,2} e^{-\frac{\gamma}{2}\langle \mathbf{c}, e_i \rangle} L^2(\mathbb{T} \times \mathbb{R}^2 \times \Omega_{\mathbb{T}})$. Now the Hilbert space $\mathcal{D}(\mathcal{Q})$, which consists of equivalence class $[u]$ of Cauchy sequences of $\mathcal{C}$ (under the relation $\lim_n u_n \sim \lim_n v_n$ when $\mathcal{Q}(u_n - v_n) \rightarrow 0$), can be identified with the closure in $(L^2(\mathbb{R}^2 \times \Omega_{\mathbb{T}}))^3 \times_{i=1,2} e^{-\frac{\gamma}{2}\langle \mathbf{c}, e_i \rangle} L^2(\mathbb{T} \times \mathbb{R}^2 \times \Omega_{\mathbb{T}})$ of the set of such quadruples $(u, \nabla_{\mathbf{c}} u, \mathbf{P}^{\frac{1}{2}} u, u_i^z)$. This shows that $u \in \mathcal{D}(\mathcal{Q})$ as desired. $\square$

8.4 . Proof of the conformal bootstrap formula

In this concluding section we sketch the main steps leading to the proof of Theorem 8.1.1. The statements that appear here are not proved yet but we expect to establish their validity in our future work [32] to which we redirect for additional details.

8.4.1 . Spectral resolution of the $\mathfrak{sl}_3$ Toda Hamiltonian

Having properly introduced and studied some basic properties of the Hamiltonian associated to the $\mathfrak{sl}_3$ Toda CFT, the next step is to understand the spectral resolution of this positive, self-adjoint operator over $L^2(\mathbb{R}^2 \times \Omega_{\mathbb{T}})$. Namely our main goal is to show that the following Plancherel formula holds true:

Theorem 8.4.1 (In preparation). *There exists a family $(\psi_{Q+iP, \nu, \tilde{\nu}})_{P \in \mathbb{R}^2; \nu, \tilde{\nu} \in \mathcal{T}^2}$ of elements of the weighted space $e^{-\beta(\mathbf{c})} L^2(\mathbb{R}^2 \times \Omega_{\mathbb{T}})$ for some $\beta : \mathbb{R}^2 \rightarrow \mathbb{R}^+$ such that for any $u, v \in e^{\beta(\mathbf{c})} L^2(\mathbb{R}^2 \times \Omega_{\mathbb{T}})$:*

$$\langle u|v \rangle_2 = \frac{1}{(2\pi)^2} \sum_{\substack{\nu, \tilde{\nu} \in \mathcal{T}^2 \\ \nu', \tilde{\nu}' \in \mathcal{T}^2}} \int_{\mathcal{C}_-} \langle u|\psi_{Q+iP, \nu, \tilde{\nu}} \rangle_2 \langle \psi_{Q+iP, \nu', \tilde{\nu}'}|v \rangle_2 F_{Q+iP}^{-1}(\nu, \nu') F_{Q+iP}^{-1}(\tilde{\nu}, \tilde{\nu}') dP. \quad (8.4.1)$$

This Plancherel formula is very close to the bootstrap formula (8.1.2) by choosing appropriate u and v . However some extra care is necessary to understand the matrix elements $\langle u|\psi_{Q+iP, \nu, \tilde{\nu}} \rangle_2$ that appear in Equation (8.4.1). For this purpose we also provide in this section a definition of these quantities based on the probabilistic representation of the generalized eigenfunctions $\psi_{\alpha, \nu, \tilde{\nu}}$ for $\alpha \in \mathbb{R}^2$:

Proposition 8.4.2 (In preparation). *For $\langle \alpha - Q, e_i \rangle$ small enough for $i = 1, 2$ set*

$$\psi_{\alpha, \nu, \tilde{\nu}} := \lim_{t \rightarrow +\infty} e^{t(2\Delta_\alpha + |\nu| + |\tilde{\nu}|)} e^{-t\mathbf{H}} \psi_{\alpha, \nu, \tilde{\nu}}^0. \quad (8.4.2)$$

Then the limit holds in a weighted space of the form $e^{\beta_\alpha(c)} L^2(\mathbb{R}^2 \times \Omega_{\mathbb{T}})$. Moreover for u as above the quantity $\langle u | \psi_{\alpha, \nu, \tilde{\nu}} \rangle_2$ is holomorphic in α over a connected subset

$$W_{\nu, \tilde{\nu}} \quad \text{of} \quad \{ \Re(\langle \alpha - Q, e_i \rangle) \leq 0 \quad \text{for all} \quad i = 1, 2 \}.$$

The proof of these two statements is based on the understanding of the spectral properties of the Toda Hamiltonian thanks to *scattering theory*. We will conduct the study of this operator in [32]. The need for such a study of this self-adjoint operator stems from the specific shape of the potential term that appears in this definition. Indeed recall that the c dependence of the Hamiltonian is governed by a term of the form

$$-\frac{1}{2}\Delta_c + \sum_{i=1}^2 \mu_i e^{\langle \gamma_{e_i}, c \rangle} V^i$$

where V^i is positive and defined using a GMC measure on $\mathbb{S}^1$. This operator has a wall of potential in the directions where $c, e_i \rightarrow +\infty$ for some $i = 1, 2$ while it vanishes when $\langle c, e_i \rangle \rightarrow -\infty$ for all $i = 1, 2$. This is reminiscent of the operators studied in the *scattering theory in symmetric spaces*, as considered for instance in [95]. The understanding of the Toda Hamiltonian thus involves conjugation of ideas developed there together with techniques developed in [13, 94] for tensor products of self-adjoint operators. This in turn allows to construct the resolvent of the Toda Hamiltonian thanks to the Liouville Hamiltonian whose spectral analysis has been carried in [66]. The fact that the potential depends on this two-dimensional c variable is one of the main difficulties of this model compared to the one studied in Liouville theory. We stress that proving these two statements is really involved.

8.4.2 . Ward identities and descendent fields

As explained in the previous section, the spectral resolution of the Hamiltonian $\mathbf{H}$ involves the consideration of the descendent $(\psi_{\alpha, \nu, \tilde{\nu}})_{\nu, \tilde{\nu} \in \mathcal{T}^2}$ of the primary field ψ_α where $\alpha \in Q + i\mathbb{R}^2$. The conformal bootstrap corresponding to a Plancherel formula with respect to $\mathbf{H}$, correlation functions that contain such descendent fields naturally arise in this decomposition.

The following statement allows to relate such correlation functions and the structure constants of the $\mathfrak{sl}_3$ Toda CFT computed in the previous sections:

Proposition 8.4.3 (In preparation). *Assume that α_2^* is colinear to ω_1 , and that α_1, α_2^* satisfy for $i = 1, 2$:*

$$\begin{aligned} \langle \alpha_k - Q, e_i \rangle < 0 \text{ for } k = 1, 2; \quad \langle \alpha_1 + \alpha_2^* - Q, \omega_i \rangle > 0. \quad \text{Then} \\ \langle U(V_{\alpha_1}(0)V_{\alpha_2^*}(z)) | \psi_{Q+iP, \nu, \tilde{\nu}} \rangle_2 = C_\gamma(\alpha_1, \alpha_2^*, Q - iP) |z|^{2(\Delta_{Q+iP} - \Delta_{\alpha_1} - \Delta_{\alpha_2^*})} \\ \times z^{|\nu|} \bar{z}^{|\tilde{\nu}|} v(\Delta_{\alpha_i}, \mathbf{w}(\alpha_i) | \nu) v(\Delta_{\alpha_i}, \mathbf{w}(\alpha_i) | \tilde{\nu}) \end{aligned} \quad (8.4.3)$$

with $\alpha_3 = Q + iP$, and where the rational functions v only depend on the $\mathfrak{sl}_3$ Toda CFT via the variables indicated.

In order to prove such a statement we will rely on the probabilistic representation of such three-point correlation functions when the weight $Q + iP$ is actually real (that is $P \in i\mathbb{R}^2$) and then use analyticity of these objects to extend the range of values for which it is valid. Namely in order to prove Proposition 8.4.3 we first provide an expression of general correlation functions $\langle U_\alpha(z) | \psi_{\alpha, \nu, \tilde{\nu}} \rangle_2$ for α and α such that the probabilistic representation makes sense. It takes the form

$$\begin{aligned} \langle \psi_{\alpha, \nu, \tilde{\nu}} | U_\alpha(\theta(z)) \rangle_2 &= \left(\prod_{k=1}^N |z_k|^{4\Delta_{\alpha_k}} \right) \frac{1}{(2i\pi)^{l(\nu)+l(\tilde{\nu})}} \times \lim_{t \rightarrow +\infty} \\ &\oint_{\mathcal{C}_{e^{-t}\delta, e^{-t}\tilde{\delta}}} w^{2-\nu^2} t^{1-\nu^1} \bar{v}^{2-\tilde{\nu}^2} \bar{s}^{1-\tilde{\nu}^1} \langle \mathbf{W}(w) \mathbf{T}(t) \widetilde{\mathbf{W}}(v) \widetilde{\mathbf{T}}(s) V_\alpha(0) \prod_{k=1}^N V_{\alpha_k}(z_k) \rangle_t dw dt dv ds \end{aligned} \quad (8.4.4)$$

where we have introduced the notation

$$\langle F \rangle_t := \int_{\mathbb{R}^2} e^{-2\langle Q, c \rangle} \mathbb{E} \left[F(\mathbf{X} + c) e^{-\mu \sum_{i=1}^2 e^{\gamma \langle c, e_i \rangle} M^{\gamma e_i}(\mathbb{C} \setminus e^{-t}\mathbb{D})} \right] dc,$$

with like before, $\mathbf{W}(w) = i\sqrt{\frac{48}{22+5c}} \mathbf{W}(w)$ and where the contour of integration is around 0.

Based on this representation we will then derive Ward identities for such correlation functions. They are given by the equality

$$\begin{aligned} \frac{1}{2i\pi} \oint_{\delta} (w')^{2-n} \langle \mathbf{W}(w') \mathbf{W}(w) \mathbf{T}(t) V_{\alpha_0}(0) \prod_{k=1}^N V_{\alpha_k}(z_k) \rangle_{t, \varepsilon} dw' &= \mathfrak{R}_t + o_\varepsilon(1) \\ - \sum_{l=1}^N \left(\frac{(n-1)(n-2)w(\alpha_l)}{2z_l^n} - \frac{(n-2)\mathbf{W}_{-1}^{(l)}}{z_l^{n-1}} + \frac{\mathbf{W}_{-2}^{(l)}}{z_l^{n-2}} \right) &\langle \mathbf{W}(w) \mathbf{T}(t) V_{\alpha_0}(0) \prod_{k=1}^N V_{\alpha_k}(z_k) \rangle_{t, \varepsilon} \end{aligned} \quad (8.4.5)$$

for $n \geq 0$, $e^{-t} > \delta$ and $\varepsilon > 0$, and where the remainder term $\mathfrak{R}_t$ is such that, uniformly on w and t in $\delta\mathbb{D}$ and for some constant C ,

$$|\mathfrak{R}_t| \leq C \max_{i=1,2} e^{-t(2-|\nu|-\langle \alpha_0, \gamma e_i \rangle)}.$$

However these identities being valid only when the weight $\alpha \in \mathbb{R}^2$ does not belong to the spectrum, a concluding analyticity argument is necessary to extend these identities to the setting of Proposition 8.4.3. Assuming that such a continuation is possible we see that by combining the two above equations the quantity $\langle U(V_{\alpha_1}(0)V_{\alpha_2}(z)) | \psi_{Q+iP, \nu, \tilde{\nu}} \rangle_2$ can be expressed using only the conformal and W -quantum weights $(\Delta_{\alpha_i}, w(\alpha_i))_{1 \leq i \leq 3}$ as well as the W -descendants at the order 1 and 2 of the primary fields V_{α_1} and V_{α_2} . Now we can use the global Ward identities of Theorem 5.2.1 to see that we can actually express all these descendants in terms of the weights $w(\alpha_i)$, $1 \leq i \leq 3$ and the descendent $\mathbf{W}_{-1}V_{\alpha_2}$. We now distinguish between two cases. First of all of $V_{\alpha_2} = V_{\alpha_2}^*$ is semi-degenerate then the latter can be expressed using Virasoro descendants and therefore depends only on the weights. This shows that Proposition 8.4.3 holds true. If we do not

make this assumption then this is no longer true and it is far from clear at this stage that the ratio considered there does indeed only depend on the weights. We hope that our probabilistic framework would allow to address this issue. Additional details and proofs of these statements will be provided in the work in progress [32].

8.4.3 . Conclusion: proof of Theorem 8.1.1

To conclude for this section we gather all the elements of proof disclosed in the previous paragraphs to derive the bootstrap formula (8.1.2). We will only present the main arguments and some justifications are of course necessary to be fully rigorous, and additional details on the reasonings developed here will be described in the future article [32].

We start from the fact that four-point correlation functions can be represented using the sesquilinear form $\langle \cdot | \cdot \rangle_2$. More precisely we can write that

$$\langle V_{\alpha_1}(0)V_{\alpha_2}(z)V_{\alpha_3}(1)V_{\alpha_4}(\infty) \rangle_{\gamma,\mu} = \langle U(V_{\alpha_1}(0)V_{\alpha_2}(z)) | U(V_{\alpha_4}(0)V_{\alpha_3}(1)) \rangle_2$$

where the elements $U(V_{\alpha_1}(0)V_{\alpha_2}(z))$ appearing in the L^2 product belong to a weighted space $e^{\beta_{\alpha_1,2}(c)} L^2(\mathbb{R}^2 \times \Omega_{\mathbb{T}})$. Under the assumptions of Theorem 8.1.1 we are in position to apply Theorem 8.4.1, which states that the spectral resolution of $\mathbf{H}$ takes the form

$$\begin{aligned} & \left\langle U(V_{\alpha_1}(0)V_{\alpha_2}(z)) | U(V_{\alpha_4}(0)V_{\alpha_3}(1)) \right\rangle_2 \\ &= \frac{1}{(2\pi)^2} \sum_{\substack{\nu, \tilde{\nu} \in \mathcal{T}^2 \\ \nu', \tilde{\nu}' \in \mathcal{T}^2}} \int_{\mathcal{C}_-} \langle U(V_{\alpha_1}(0)V_{\alpha_2}(z)) | \psi_{Q+iP, \nu, \tilde{\nu}} \rangle_2 \\ & \quad \langle \psi_{Q+iP, \nu', \tilde{\nu}'} | U(V_{\alpha_4}(0)V_{\alpha_3}(1)) \rangle_2 F_{Q+iP}^{-1}(\nu, \nu') F_{Q+iP}^{-1}(\tilde{\nu}, \tilde{\nu}') dP. \end{aligned}$$

Then we can apply Proposition 8.4.3 which explains how to express the scalar products that appear in this expression. We can therefore deduce that

$$\begin{aligned} & \langle V_{\alpha_1}(0)V_{\alpha_2}(z)V_{\alpha_3}(1)V_{\alpha_4}(\infty) \rangle_{\gamma,\mu} \\ &= \frac{1}{(2\pi)^2} \int_{\mathcal{C}_-} C_{\gamma,\mu}(\alpha_1, \alpha_2, Q - iP) C_{\gamma,\mu}(Q + iP, \alpha_3, \alpha_4) |z|^{2(\Delta_{Q+iP} - \Delta_{\alpha_1} - \Delta_{\alpha_2})} |\mathcal{F}_P(z)|^2 dP \end{aligned}$$

where the conformal blocks are defined by the expression

$$\mathcal{F}_P(z) = \sum_{\nu, \nu' \in \mathcal{T}^2} z^{|\nu|} v(\Delta_{\alpha_i}, w(\alpha_i) | \nu) F_{Q+iP}^{-1}(\nu, \nu') v(\Delta_{\alpha_i}, w(\alpha_i) | \nu'). \quad (8.4.6)$$

The above is nothing but the desired formula. This wraps up the proof of our main statement on the conformal bootstrap procedure for the $\mathfrak{sl}_3$ Toda CFT, Theorem 8.1.1.

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