

# A novel approach to the computation of one-loop three- and four-point functions: I. The real mass case

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This article is the first of a series of three presenting an alternative method of computing the one-loop scalar integrals. This novel method enjoys a couple of interesting features as compared with the method closely following 't Hooft and Veltman adopted previously. It directly proceeds in terms of the quantities driving algebraic reduction methods. It applies to the three-point functions and, in a similar way, to the four-point functions. It also extends to complex masses without much complication. Lastly, it extends to kinematics more general than that of the physical, e.g., collider processes relevant at one loop. This last feature may be useful when considering the application of this method beyond one loop using generalized one-loop integrals as building blocks.  
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## 1. Introduction

Automated evaluations of loop multileg processes demand a fast and numerically stable evaluation of Feynman integrals. In particular, the calculation of two-loop three- and four-point functions in the general complex mass case remains challenging. Getting a reliable result using a multidimensional numerical integration and sector decomposition [1–5] has a high computing cost. The derivation of a fully analytic result remains beyond reach so far in the general mass case. In between, approaches based on Mellin–Barnes techniques [6–10] allow one to perform part of the integrals analytically, yet, as far as we understand, the number of integrals left over for numerical quadratures depends on the topologies considered and can remain rather costly. An alternative approach performing some/many of the Feynman parameter integrations analytically in a systematic way to reduce the number of integrations to be performed numerically would therefore be useful.

Such a working program was initiated in Ref. [11] for the calculation of massive two-loop  $N$ -point functions using analytically computed one-loop building blocks. This approach is based on the implementation of two-loop scalar  $N$ -point functions in  $n$  dimensions  ${}^{(2)}I_N^n$  as double integrals of the form:

$${}^{(2)}I_N^n \sim \sum \int_0^1 d\rho \int_0^1 d\xi W(\rho, \xi) {}^{(1)}\tilde{I}_{N'}^n(\rho, \xi)$$

where  $W(\rho, \xi)$  are some weighting functions whereas the  ${}^{(1)}\tilde{I}_{N'}^{n'}(\rho, \xi)$  are “generalized one-loop type”  $N'$ -point Feynman-type integrals<sup>1</sup>. The latter are “generalized” in the sense that the integration domain spanned by the Feynman parameters is no longer the usual simplex  $\{0 \leq z_j \leq 1, j = 1, \dots, N'; \sum_{j=1}^{N'} z_j = 1\}$  at work for the one-loop  $N'$ -point function but another domain, e.g., a hypercube or a cylinder with triangular basis, which depends on the topology of the two-loop  $N$ -point function considered. The generalization also concerns the underlying kinematics, which, besides external momenta, depends on two extra Feynman parameters  $\rho$  and  $\xi$ . The parameter space spanned by this kinematics is larger than the one spanned in one-loop  $N'$ -particle processes at colliders; e.g., Gram determinants may be all positive whereas this never happens in the physical region for one-loop  $N'$ -particle collider processes. Both these generalizations may be addressed by tuning case-by-case adaptations of the methods well established in the standard one-loop calculations [12] and using careful analytic continuations [13] to all cases considered, whose implementation can nevertheless be tricky. Alternatively, we hereby and in two related companion papers propose to develop a novel approach to address both these generalizations in a systematic way. More generally, for both the three- and four-point functions the approach presented will consider integrals of the form<sup>2</sup>

$$I \sim \int_{\Sigma} \frac{d^{\bar{n}}x}{D^{\alpha+1}} \tag{1.1}$$

where  $\bar{n} = 1, 2, \text{ or } 3$ ,  $D = X^T \cdot G \cdot X - 2V^T \cdot X - C$ , with an  $\bar{n} \times \bar{n}$  Gram matrix  $G$ , a column  $\bar{n}$ -vector  $V$ , and the  $x_j, j = 1, \dots, \bar{n}$  are the components of a column  $\bar{n}$ -vector  $X$  spanning the simplex  $\Sigma = \{0 \leq x_j \leq 1, j = 1, \dots, \bar{n} \mid \sum_{j=1}^{\bar{n}} x_j \leq 1\}$ . The method will make extensive use of the following Stokes-type identity<sup>3</sup> proven in Appendix A:

$$\frac{1}{D^{\alpha+1}} = \frac{1}{2\alpha \Delta_{\bar{n}}} \left[ \frac{\bar{n} - 2\alpha}{D^{\alpha}} - \nabla^T \cdot \left( \frac{X - G^{-1} \cdot V}{D^{\alpha}} \right) \right] \tag{1.2}$$

where  $\Delta_{\bar{n}} = V^T \cdot G^{-1} \cdot V + C$  and  $\nabla$  stands for the gradient w.r.t.  $X$ .

The present article and two companion papers [22,23] aim at presenting the method advocated to compute the building blocks  ${}^{(1)}\tilde{I}_{N'}^{n'}$  by applying it to the calculation of the standard one-loop three- and four-point functions as a “proof of concept”. It has been designed to be straightforwardly applied so as to, on one hand, trivialize the Feynman parameter integrations as boundary terms in the integrals defining the above building blocks  ${}^{(1)}\tilde{I}_{N'}^{n'}(\rho, \xi)$  in a systematic way, i.e., regardless of the shape of the integration domain of the Feynman parameters; and on the other hand to obtain all necessary analytic continuations in a systematic way as well. The primary aim of its comparison with the well established methods on this well studied case is not so much to readily provide an alternative competing first-line method to compute standard one-loop  $N$ -point functions, but rather use this comparison as a test-bench seeing the well established approach as a benchmark in the matter of efficiency, which provides guidance to improve and optimize the novel method before its application to compute the two-loop ingredients for which it has been designed. We aim at showing its ability to circumvent

<sup>1</sup> The “effective number of external legs”  $N'$  and “effective dimension”  $n'$  depend on the two-loop topology, in particular the number of internal lines  $I$ , and on the dimension  $n$ ; see Ref. [11].

<sup>2</sup> The contour prescription  $D := D - i\lambda$  is implicit in the sloppy integrals (1.1) and (2.8). Accordingly, the prescription  $\Delta_{\bar{n}} := \Delta_{\bar{n}} + i\lambda$  is induced in identity (1.2) when the latter is substituted in integrals (1.1) and (2.8); see Appendix A.

<sup>3</sup> The integration over the simplex defining the phase space of the Feynman parameters of this identity gives the relation between the  $N$ -point one-loop scalar functions in  $n$  and  $n + 2$  dimensions [14–21].

the subtleties of the various analytic continuations in the kinematical variables in a systematic way. We also aim at controlling the proliferation of dilogarithms in the closed-form expressions, which otherwise would hamper its use in further two-loop calculations. The computation of the building blocks  ${}^{(1)}\tilde{I}_{N'}^{n'}(\rho, \xi)$  by itself using this method will be presented in a future article.

Let us remind ourselves that the motivation behind this work is to study two-loop massive three- and four-point functions in a scalar theory. The case where some internal masses vanish may lead to soft/collinear divergent functions for which  $n$  and  $n'$  have to be taken away from 4. The computation of the “generalized one-loop function” in this case (restricted to the case where the phase space volume of the Feynman parameters is a simplex) is presented in a companion paper [22]. Nevertheless, even if no internal mass vanishes, in a general scalar theory with three- and four-leg vertices, some two-loop three- and four-point functions diverge in the UV region. In this case, the space-time dimension has to be taken slightly under 4 to regularize the Feynman integrals. It can be shown by power counting that, in this theory, the two-loop UV-divergent three- and four-point diagrams have four propagators, which implies that the associated “generalized one-loop functions” are two-point functions [11]. To compute analytically the latter, after doing the  $n' - 4$  expansion around 0, an integration over one Feynman parameter of a logarithm, at most to the power 2, whose argument is a second-order polynomial in this Feynman parameter has to be performed. This integration can be carried out without any difficulty. In more complicated field theories, e.g., gauge theories, the UV divergences can come from tensorial integrals. Although, in principle, the method presented can be used, this case is postponed to a future work. Since in this article, we focus on fully massive three- and four-point one-loop functions that are related to UV finite scalar two-loop functions, we set  $n$  and  $n'$  to 4 in the rest of this article.

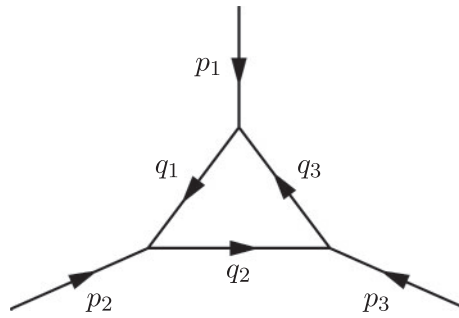
As already said, the work presented in the present article does not quite provide an alternative competing first-line method to compute standard one-loop  $N$ -point functions; nevertheless, it also provides a few interesting byproducts, we think, which are not manifest on the existing results. Most of the general results for three- and four-point scalar functions, valid for the complex mass case, expressed in terms of dilogarithms [12,24,25] are valid in the physical domain, except those given in Ref. [13] where the authors presented very compact results for the massive four-point scalar function whose validity has been extended by analytical continuation for kinematical ranges accessible beyond one loop. We also mention another general result for a scalar box in arbitrary dimensions [26] expressed in term of generalized hypergeometric functions but this result is hardly usable for practical computations. In a previous article [27] closely following 't Hooft and Veltman [12], 1D integral representations free of numerical instabilities caused by negative powers of Gram determinants were obtained for three-point functions in 4, 6, and 8 dimensions in the general complex mass case and they have been used in the `Go1em95` library [28,29]. The various analytic handlings used in Ref. [27] following Ref. [12], the successive cutting and pasting of integrals and the corresponding changes of variables involved, were rather intricate; they are even more so for the—as-yet unpublished—case of the four-point functions in four dimensions or more. Furthermore, in Refs. [12,24,25], the connection between the analytic results and the algebraic quantities  $\det(S)$  (the determinant of the kinematic matrix associated with a given Feynman diagram),  $\det(G)$  (the associated Gram determinant), and the algebraic reduction coefficients  $b_i$  involved in algebraic reduction methods such as in `Go1em` [30] (as well as similar ones corresponding to associated pinched diagrams) is badly blurred. Tracing back these algebraic ingredients in the coefficients and in the arguments of logarithms and dilogarithms in the final expressions obtained then requires cumbersome extra work. Our alternative approach circumvents at least some of the above difficulties. In this respect the method presented enjoys

a couple of interesting features as compared with the method primitively adopted in Ref. [27]. It directly proceeds in terms of the algebraic quantities  $\det(S)$ ,  $\det(G)$ ,  $b_i$  etc. The derivation of the one-loop integral representations happens to be more systematic and applies to the three- and four-point function cases in a similar way. It also extends to complex masses without much complication. It also applies to kinematical configurations beyond those relevant for collider processes at the one-loop order. The method exploits an integral identity that, as a byproduct, also allows one to compute angular integrals of eikonal terms describing real (massless) emission off massive emitters in closed form in a (relatively) simple way, while keeping the origin of the kinematical arguments involved in the final result more transparent.

At first glance a price to pay with the novel method seems to be an inherent increase of the number of dilogarithms involved compared to the 't Hooft–Veltman results—a doubling in the three-point case and worse in the four-point case—when the 1D integral representations are further computed in closed form. However, this increase can be counteracted in a simple way, at least in the three-point case. The four-point case still needs more work to resolve this issue.

The account of this approach amounts to an extensive technical matter. In particular, the treatment of the four-point function involves a multistep process, whereas the upgrade from the real mass case to the general complex mass case implies a patchwork of cases. Both deserve some elaboration. Gathering everything at once in a single article might seem indigestible; we therefore chose to split the presentation into a triptych for ease of reading. In part I, the present article, we revisit the one-loop scalar three-point function  $I_3^4$  with real internal masses to present the method, and apply it to the four-point function  $I_4^4$  with internal masses all real. The extension to general complex masses, especially for  $I_4^4$ , will constitute part II, presented in a companion article. The presence of some zero internal masses causes the appearance of infrared soft and/or collinear singularities and therefore requires some specific treatments, although the general line of thought sticks to the one of fully massive case. These specific cases with zero masses will thus be addressed separately in part III, forming yet another companion article.

In this article we start by considering the three-point function  $I_3^4$  with real internal masses considered as a warm-up in Sect. 2. We successively present two variants of the method. The simplest variant, which we call the “direct way”, is presented in Sect. 2.2.1. It is well suited for the three-point function; unfortunately it is not suitable for a handy extension to the case of the four-point function. In Sect. 2.2.2. we therefore tailor a more sophisticated alternative, called the “indirect way”. The latter comes across as rather artificial in the three-point case but it has the virtue of extending in a straightforward—albeit somewhat proliferating—way to the four-point case. We conclude this section by commenting on the apparent doubling of dilogarithms and the way to counteract this doubling. In Sect. 3, we consider the four-point function with internal masses all real; we follow the “indirect way”, which comprises one stage more than in the three-point case because of the larger number of Feynman parameters. Its implementation in four steps is presented in Sects. 3.1–3.4. We then discuss our results with respect to those of Ref. [12] and also discuss the proliferation of dilogarithms in Sect. 3.5. Lastly, we conclude. Various appendices gather a number of utilities: tools, proofs of steps, etc. We removed them from the main text to facilitate its reading but we consider them useful to the reader. Accordingly, in Appendix A, we present a Stokes-type identity, which is the master identity for the derivation of our result. Appendix B explains how to tune the power of the denominators to apply the master identity of Appendix A. Appendix C provides a detailed discussion on how to solve the equation  $\sum_{j=1}^N \mathcal{S}_{ij} b_j = 1, \forall i = 1 \dots N$  even in the case of peculiar kinematical configurations for which  $\det(G)$  or  $\det(S)$  and  $\det(G)$  vanish. Appendix D gives the



**Fig. 1.** A triangle showing the one-loop three-point function.

derivation of two integrals useful for the computation of three-point and four-point integrals, and Appendix E shows how to compute the last integration in a closed form in terms of dilogarithms. Lastly, Appendix F contains a discussion about the prescription of the imaginary part of  $\det(S)$ .

**2. Warm-up: The scalar three-point function  $I_3^4$**

Each internal line with momentum  $q_i$  stands for the propagator of a particle of mass  $m_i$ . We define the kinematic matrix  $S$ , which encodes all the information on the kinematics associated with this diagram by:

$$S_{ij} = (q_i - q_j)^2 - m_i^2 - m_j^2. \tag{2.1}$$

The squares of differences of two internal momenta can be written in terms of the internal masses  $m_i$  and the external invariants  $p_i^2$  so that  $S$  reads:

$$S = \begin{pmatrix} -2m_1^2 & p_2^2 - m_1^2 - m_2^2 & p_1^2 - m_1^2 - m_3^2 \\ p_2^2 - m_1^2 - m_2^2 & -2m_2^2 & p_3^2 - m_2^2 - m_3^2 \\ p_1^2 - m_1^2 - m_3^2 & p_3^2 - m_2^2 - m_3^2 & -2m_3^2 \end{pmatrix}. \tag{2.2}$$

The usual Feynman integral<sup>4</sup> representation of  $I_3^4$  is:

$$I_3^4 = - \int_0^1 \prod_{i=1}^3 dz_i \delta(1 - \sum_{i=1}^3 z_i) \left( -\frac{1}{2} Z^T \cdot S \cdot Z - i\lambda \right)^{-1}. \tag{2.3}$$

Here  $Z$  stands for a column 3-vector whose components are the  $z_i$ ,  $S$  is the  $3 \times 3$  kinematic matrix associated with the diagram of Fig. 1, and the superscript “ $T$ ” stands for the matrix transpose. Let us single out the subscript value  $a$  ( $a \in S_3 = \{1, 2, 3\}$ ) and write  $z_a$  as  $1 - \sum_{i \neq a} z_i$ . We find:

$$- Z^T \cdot S \cdot Z = \sum_{i,j \in S_3 \setminus \{a\}} G_{ij}^{(a)} z_i z_j - 2 \sum_{j \in S_3 \setminus \{a\}} V_j^{(a)} z_j - C^{(a)} \tag{2.4}$$

where the  $2 \times 2$  Gram matrix  $G^{(a)}$  and the column 2-vector  $V^{(a)}$  are defined by

<sup>4</sup> In this article and its companions, we use the notation  $-i\lambda$  for the Feynman prescription coming from propagators.

$$\begin{aligned}
 G_{ij}^{(a)} &= -(\mathcal{S}_{ij} - \mathcal{S}_{aj} - \mathcal{S}_{ia} + \mathcal{S}_{aa}), \quad i, j \neq a \\
 V_j^{(a)} &= \mathcal{S}_{aj} - \mathcal{S}_{aa}, \quad j \neq a \\
 C^{(a)} &= \mathcal{S}_{aa}.
 \end{aligned}
 \tag{2.5}$$

We label  $b$  and  $c$  the two elements of  $S_3 \setminus \{a\}$ , with  $b < c$ . We write the polynomial (2.4) with the  $2 \times 2$  matrix  $G^{(a)}$ , the column 2-vector  $V^{(a)}$ , the scalar  $C^{(a)}$  as:

$$D^{(a)}(z_b, z_c) = X^{(a)T} \cdot G^{(a)} \cdot X^{(a)} - 2 V^{(a)T} \cdot X^{(a)} - C^{(a)}, \quad X^{(a)} = \begin{bmatrix} z_b \\ z_c \end{bmatrix}. \tag{2.6}$$

We note the integration simplex  $\Sigma_{bc} = \{0 \leq z_b, z_c, z_b + z_c \leq 1\}$ .  $I_3^4$  can be written:

$$I_3^4 = -2 \int_{\Sigma_{bc}} \frac{dz_b dz_c}{D^{(a)}(z_b, z_c) - i\lambda}. \tag{2.7}$$

### 2.1. Step 1

Let us substitute identity (1.2) into Eqs. (2.7) or (1.1). Were the power  $\alpha + 1$  in the l.h.s. of Eq. (1.2) such that  $\bar{n} - 2\alpha = 0$ , only the boundary term, i.e., the second term in Eq. (1.2), would remain, thus making one integration in  $I$  trivial. In the case of the  $N = 3$ -point function; cf. Eq. (2.7),  $\bar{n} = N - 1$  is equal to 2. Imposing  $\bar{n} - 2\alpha = 0$  thus forces  $\alpha + 1 = 2$  in Eq. (1.2). However, in Eq. (2.7),  $1/D$  appears raised to the power 1, not 2. The idea is thus to adjust the power of the denominator by introducing an appropriate integral representation of the form

$$\frac{1}{D^{\alpha+1}} \sim \int_0^{+\infty} \frac{d\xi}{(D + \xi^\nu)^{\alpha'+1}} \tag{2.8}$$

( $\xi^\nu$  being some suitable power of  $\xi$ ) so that the power  $\alpha'$  of the effective denominator in this representation matches the request  $\bar{n} - 2\alpha' = 0$ . The auxiliary identity (2.8) is properly made explicit and derived in its general form in Appendix B. In the case of the three-point function in four dimensions  $I_3^4$  the relevant form for identity (2.8) is simply the familiar integral representation:

$$\frac{1}{D - i\lambda} = \int_0^{+\infty} \frac{d\xi}{(D + \xi - i\lambda)^2}. \tag{2.9}$$

The substitution of identity (2.9) into Eq. (2.7) provides a representation of  $I_3^4$  where  $(D^{(a)} + \xi)^2$  now replaces  $D^{(a)}$  in the integrand:

$$I_3^4 = -2 \int_0^{+\infty} d\xi \int_{\Sigma_{bc}} dx_b dx_c \frac{1}{(D^{(a)}(x_b, x_c) + \xi - i\lambda)^2}. \tag{2.10}$$

Identity (1.2) is then applied to the integrand of Eq. (2.10) considered as a function of the two variables  $x_b, x_c$  to be integrated first keeping  $\xi$  fixed, yielding:

$$\begin{aligned}
 I_3^4 &= \int_0^{+\infty} d\xi \int_{\Sigma_{bc}} dx_b dx_c \\
 &\times \left[ \frac{1}{\Delta_2 - \xi + i\lambda} \sum_{j \in S_3 \setminus \{a\}} \frac{\partial}{\partial x_j} \left( \frac{(X - (G^{(a)})^{-1} \cdot V^{(a)})_j}{D^{(a)}(x_b, x_c) + \xi - i\lambda} \right) \right].
 \end{aligned}
 \tag{2.11}$$

The use of the Stokes identity (1.2) in the integral representation (2.9) induces an apparent pole at  $\xi = \Delta_2$  in Eq. (2.11). Insofar as  $\Delta_2 \neq 0$  this apparent pole is no issue, as can be justified directly

as follows. It is manifestly no issue in the real mass case for which the  $-i\lambda$  contour prescription avoids this apparent pole anyway. In the general complex mass case the  $-i\lambda$  contour prescription is overruled by the finite imaginary part of  $\Delta_2$ , whose sign may, however, change in a continuous way with the kinematics: one might then worry about what happens whenever  $\text{Im}(\Delta_2)$  vanishes with a change of sign while  $\text{Re}(\Delta_2) > 0$ . In this respect we shall, however, notice that

$$D^{(a)}(x_b, x_c) + \Delta_2 = \left( X^{(a)} - (G^{(a)})^{-1} \cdot V^{(a)} \right)^T \cdot G^{(a)} \cdot \left( X^{(a)} - (G^{(a)})^{-1} \cdot V^{(a)} \right)$$

so that

$$\begin{aligned} & \sum_{j \in S_3 \setminus \{a\}} \frac{\partial}{\partial x_j} \left( \frac{(X - (G^{(a)})^{-1} \cdot V^{(a)})_j}{D^{(a)}(x_b, x_c) + \Delta_2 - i\lambda} \right) \\ &= \frac{1}{2} \sum_{j, k \in S_3 \setminus \{a\}} \frac{\partial}{\partial x_j} \left( (G^{(a)})_{jk}^{-1} \frac{\partial}{\partial x_k} \ln \left( D^{(a)}(x_b, x_c) + \Delta_2 - i\lambda \right) \right). \end{aligned} \quad (2.12)$$

This distribution (2.12), whether by direct calculation of the l.h.s. or by recognition on the r.h.s., proves to vanish identically outside its singular support contained in the region where  $D^{(a)}(x_b, x_c) + \Delta_2 = 0$ . On the other hand,  $\text{Im}(D^{(a)}(x_b, x_c)) < 0$  on the integration simplex  $\Sigma_{bc}$  where it is a convex linear combination of the imaginary parts of the masses squared. Thus, when  $\text{Im}(\Delta_2)$  vanishes,  $\text{Im}(D^{(a)}(x_b, x_c) + \Delta_2 - i\lambda) < 0$  on  $\Sigma_{bc}$ ; hence the residue of the apparent pole  $\xi = \Delta_2$  in Eq. (2.11) vanishes identically in  $x_b, x_c$  on  $\Sigma_{bc}$ .

The cases where  $\Delta_2$  vanishes require a separate examination. This implies that  $\det(\mathcal{S})$  vanishes. However, the condition  $\det(\mathcal{S}) = 0$  does not correspond to a kinematical singularity of the initial Feynman integral if the corresponding eigenvector of  $\mathcal{S}$  points outside the quadrant  $\{z_j \geq 0\}$ . In this case  $D^{(a)}(x_b, x_c)$  never vanishes on the simplex  $\Sigma_{bc}$  and the above argument based on Eq. (2.12) still applies. The case corresponding to the presence of infrared singularities in conjunction with some internal masses is addressed separately in the third article of the series. Lastly, if the condition  $\det(\mathcal{S}) = 0$  corresponds to a threshold singularity, the original Feynman integral is indeed singular and  $\xi = 0$  is an end-point singularity of the integral representation (2.11).

For each term of the sum in Eq. (2.11) the integration performed first is on the variable  $x_j$  on which the derivative acts, and we get:

$$\begin{aligned} I_3^4 &= \int_0^{+\infty} \frac{d\xi}{\Delta_2 - \xi + i\lambda} \\ & \left[ \int_0^1 dx_c \left( \frac{(1 - x_c) - ((G^{(a)})^{-1} \cdot V^{(a)})_b}{D^{(a)}(1 - x_c, x_c) + \xi - i\lambda} - \frac{-((G^{(a)})^{-1} \cdot V^{(a)})_b}{D^{(a)}(0, x_c) + \xi - i\lambda} \right) \right. \\ & \left. + \int_0^1 dx_b \left( \frac{(1 - x_b) - ((G^{(a)})^{-1} \cdot V^{(a)})_c}{D(x_b, 1 - x_b) + \xi - i\lambda} - \frac{-((G^{(a)})^{-1} \cdot V^{(a)})_c}{D(x_b, 0) + \xi - i\lambda} \right) \right]. \end{aligned} \quad (2.13)$$

After the change of variables  $x_b = x$  and  $x_c = 1 - x$ , the first and third terms in the integrand of the r.h.s. of Eq. (2.13) recombine and we get:

$$\begin{aligned} I_3^4 &= \int_0^{+\infty} \frac{d\xi}{\Delta_2 - \xi + i\lambda} \int_0^1 dx \left( \frac{1 - ((G^{(a)})^{-1} \cdot V^{(a)})_b - ((G^{(a)})^{-1} \cdot V^{(a)})_c}{D^{(a)}(x, 1 - x) + \xi - i\lambda} \right. \\ & \left. + \frac{((G^{(a)})^{-1} \cdot V^{(a)})_b}{D^{(a)}(0, x) + \xi - i\lambda} + \frac{((G^{(a)})^{-1} \cdot V^{(a)})_c}{D^{(a)}(x, 0) + \xi - i\lambda} \right). \end{aligned} \quad (2.14)$$

The quantities  $\Delta_2$  and  $(G^{(a)})^{-1} \cdot V^{(a)}$  in Eq. (2.14) are expressed simply<sup>5</sup> in terms of  $\det(\mathcal{S})$ ,  $\det(G)$ , and the coefficients  $\bar{b}_j$  such that  $\sum_{j=1}^3 \mathcal{S}_{ij} \bar{b}_j = \det(\mathcal{S}) e_i$  with  $e_i = 1$  for  $i = 1, 2, 3$  (cf. Eqs. (C.11) and (C.31) of Appendix C):

$$\begin{aligned}\Delta_2 &= \frac{\det(\mathcal{S})}{\det(G)} \\ ((G^{(a)})^{-1} \cdot V^{(a)})_b &= \frac{\bar{b}_b}{\det(G)} \\ ((G^{(a)})^{-1} \cdot V^{(a)})_c &= \frac{\bar{b}_c}{\det(G)} \\ 1 - ((G^{(a)})^{-1} \cdot V^{(a)})_b - ((G^{(a)})^{-1} \cdot V^{(a)})_c &= \frac{\bar{b}_a}{\det(G)}.\end{aligned}\quad (2.15)$$

The denominators  $D^{(a)}(0, x)$  and  $D^{(a)}(x, 0)$  straightforwardly read:

$$\begin{aligned}D^{(a)}(0, x) &= -(\mathcal{S}_{cc} - 2\mathcal{S}_{ca} + \mathcal{S}_{aa})x^2 - 2(\mathcal{S}_{ca} - \mathcal{S}_{aa})x - \mathcal{S}_{aa} \\ &= G^{\{b\}(a)}x^2 - 2V^{\{b\}(a)}x - C^{\{b\}(a)} \equiv D^{\{b\}(a)}(x)\end{aligned}\quad (2.16)$$

$$\begin{aligned}D^{(a)}(x, 0) &= -(\mathcal{S}_{bb} - 2\mathcal{S}_{ba} + \mathcal{S}_{aa})x^2 - 2(\mathcal{S}_{ba} - \mathcal{S}_{aa})x - \mathcal{S}_{aa} \\ &= G^{\{c\}(a)}x^2 - 2V^{\{c\}(a)}x - C^{\{c\}(a)} \equiv D^{\{c\}(a)}(x)\end{aligned}\quad (2.17)$$

and for  $D^{(a)}(x, 1-x)$  a simple calculation yields:

$$\begin{aligned}D^{(a)}(x, 1-x) &= -(\mathcal{S}_{bb} - 2\mathcal{S}_{bc} + \mathcal{S}_{cc})x^2 - 2(\mathcal{S}_{bc} - \mathcal{S}_{cc})x - \mathcal{S}_{cc} \\ &= G^{\{a\}(c)}x^2 - 2V^{\{a\}(c)}x - C^{\{a\}(c)} \equiv D^{\{a\}(c)}(x).\end{aligned}\quad (2.18)$$

The expressions in Eqs. (2.16)–(2.18) are one-variable counterparts of Eq. (2.6). For example, the equation making explicit  $D^{(a)}(x, 1-x)$  involves the pinched matrix  $\mathcal{S}^{\{a\}}$  in which line and column  $c$  are singled out so as to define the associated pinched Gram “matrix”  $G^{\{a\}(c)}$ , the “vector”  $V^{\{a\}(c)}$ , and the scalar  $C^{\{a\}(c)}$  as in Eqs. (2.4)–(2.5). The pinched matrix  $\mathcal{S}^{\{a\}}$  is itself built from the  $\mathcal{S}$  matrix by removing the line and column  $a$ ; cf. Ref. [30]. Similar explications hold for  $D^{(a)}(0, x)$  and  $D^{(a)}(x, 0)$ . The quantities in Eqs. (2.16)–(2.18) are precisely those involved in the integral representations of the one-loop two-point integrals corresponding to all possible pinchings of the triangle diagram in Fig. 1. In the three-point case at hand, the above pinched Gram “matrices” and “vectors” have actually one component only, thus the multiple superscript notation “ $\{a\}(c)$ ”, etc. may seem a clumsy sophistication. Yet it keeps track of the line and column that have been singled out and it encodes how quantities with the same pinching but distinct singled out lines and columns are related. For example,

$$D^{\{a\}(c)}(x) = D^{\{a\}(b)}(1-x),\quad (2.19)$$

i.e.,

<sup>5</sup> We remind ourselves that  $\det(G) \equiv \det G^{(a)}$  is independent of  $a$ .

$$\begin{aligned}
 G^{\{a\}(b)} &= G^{\{a\}(c)} \\
 V^{\{a\}(b)} &= -V^{\{a\}(c)} + G^{\{a\}(c)} \\
 C^{\{a\}(b)} &= C^{\{a\}(c)} + 2V^{\{a\}(c)} - G^{\{a\}(c)}.
 \end{aligned}
 \tag{2.20}$$

Likewise we have:

$$D^{\{b\}(a)}(x) = D^{\{b\}(c)}(1-x), \quad D^{\{c\}(a)}(x) = D^{\{c\}(b)}(1-x)
 \tag{2.21}$$

with corresponding relations among the  $G^{\{b\}(j)}, V^{\{b\}(j)}, C^{\{b\}(j)}$  for  $j \in S_3 \setminus \{b\}$ , and among the  $G^{\{c\}(k)}, V^{\{c\}(k)}, C^{\{c\}(k)}$  for  $k \in S_3 \setminus \{c\}$  respectively. The three-point integral (2.14) may then be written as a weighted sum over the  $\bar{b}_i$ :

$$I_3^4 = \sum_{i \in S_3} \frac{\bar{b}_i}{\det(G)} \int_0^{+\infty} \frac{d\xi}{\Delta_2 - \xi + i\lambda} \int_0^1 \frac{dx}{D^{\{i\}(i')}(x) + \xi - i\lambda}.
 \tag{2.22}$$

For the sake of definiteness, Eqs. (2.16)–(2.18) provide a value of  $i'$  for each  $i$  in Eq. (2.22). Let us, however, note that the assignment for  $i'$  in Eq. (2.22) is not unique. It can be changed relying on properties (2.19) and (2.21) in combination with changes of variables  $y = 1 - x$  in the terms in Eq. (2.22). For further convenience, we trade the assignment provided by Eqs. (2.16)–(2.18) for the alternative choice  $i' \equiv 1 + (i \text{ modulo } 3)$  in what follows. At this stage, one may argue that we have progressed by next to nothing as we performed one integration trivially yet at the price of introducing an extra integral so that two integrals still remain to be performed. However, representation (2.22) amounts to a handier form as we will see next.

## 2.2. Step 2

We can proceed further in two different ways. The first way henceforth called “direct” has the virtue of leading in a very few simple steps to the result that we obtained using the method *à la* ’t Hooft and Veltman; cf. Ref. [27]. Unfortunately, in the case of the four-point function we did not succeed in proceeding as simply. We have therefore formulated an alternative to the direct way, henceforth called “indirect”. It is somewhat academic to follow the latter for the three-point function, all the more so as it is uselessly more cumbersome than the direct way in this case. Yet we do the exercise as a warm-up before tackling the more involved case of the four-point function. As will be seen, the result that we obtain at first for the three-point function in the indirect way is *not quite* the same formula as the one obtained via the direct way. In particular, the indirect way would lead to a doubling of dilogarithmic terms compared to the result via the direct way if the last integral were explicitly performed in closed form. A comparison between the two may help understand how to recombine terms from the indirect way so as to counteract this doubling. The experience gained on this example may guide us to reverse a similar unwanted proliferation of dilogarithms in the case of the four-point function where only the indirect way is available. Let us now present the two ways in this subsection.

### 2.2.1. Direct way

We integrate directly the l.h.s. of Eq. (2.22) over the variable  $\xi$  first, keeping  $x$  fixed. This requires us to perform the following type of integral:

$$K = - \int_0^{+\infty} d\xi \frac{1}{(\xi - \Delta_2 - i\lambda)(\xi + D^{[i](i')}(x) - i\lambda)}, \quad (2.23)$$

which, using Eq. (D.4), gives<sup>6</sup>:

$$K = - \frac{1}{D^{[i](i')}(x) + \Delta_2} \left[ \ln \left( D^{[i](i')}(x) - i\lambda \right) - \ln \left( -\Delta_2 - i\lambda \right) \right]. \quad (2.24)$$

Last we substitute Eq. (2.24) into Eq. (2.22). To make the connection with the notations used in Ref. [27], we identify  $\Delta_2 = 1/B$ ,  $\bar{b}_i/\det(G) = b_i/B$  (cf. also Appendix C) and write

$$I_3^4 = - \sum_{i \in S_3} b_i \int_0^1 \frac{dx}{B D^{[i](i')}(x) + 1} \left[ \ln \left( D^{[i](i')}(x) - i\lambda \right) - \ln \left( -\frac{1}{B} - i\lambda \right) \right]. \quad (2.25)$$

$D^{[i](i')}(x)$  in Eq. (2.25) shall be identified with  $2 \times g_{(i)}(x)$  in Ref. [27]<sup>7</sup>. The above result thus coincides with the result obtained in Ref. [27] after symmetrization over the parameter  $\alpha$  using the method *à la* 't Hooft and Veltman. The present derivation is somewhat faster, though. The derivation of Eq. (2.25) holds true also in the complex mass case. In the real mass case, the difference of logarithms in Eq. (2.25) can be replaced by the logarithm of the ratio.

### 2.2.2. Indirect way

We perform the integration over  $x$  in Eq. (2.22) by writing the integrand as a derivative using identity (1.2) anew, this time for “ $\bar{n} = 1$ ”. Therefore identity (2.9) is not the relevant one to provide an integral representation of  $1/(D^{[i](i')} + \xi)$  with the appropriate power in the denominator so as to apply identity (1.2). Indeed, in order to keep only the boundary term in identity (1.2) we shall choose  $\alpha = 1/2$  whereas the power of the denominator in Eq. (2.22) is 1 not  $3/2$ . We thus have to customize an alternative representation of the type (2.8) providing shifts in powers of denominators by  $1/2$  instead of 1. A generalized representation including this type is derived in Appendix B. It also happens to be the one suited to the four-point function case as will be seen below. It takes the following form (see Eq. (B.1)):

$$\frac{1}{(D^{[i](i')})^{\mu-1/\nu}} = \frac{\nu}{B \left( \mu - \frac{1}{\nu}, \frac{1}{\nu} \right)} \int_0^{+\infty} \frac{d\rho}{(D^{[i](i')} + \rho^\nu)^\mu} \quad (2.26)$$

where  $B(y, z) = \Gamma(y) \Gamma(z) / \Gamma(y + z)$  and  $\Gamma(y)$  are the Euler Beta and Gamma functions [31]. The parameters  $\mu$  and  $\nu$  are chosen such that  $\mu - 1/\nu =$  the power of  $(D^{[i](i')} + \xi)$  in the integrand of Eq. (2.22), i.e., 1. The representation of  $1/(D^{[i](i')} + \xi)$  thus obtained is substituted into Eq. (2.22). This provides a representation of  $I_3^4$  with factors  $1/(D^{[i](i')} + \xi + \rho^\nu)^\mu$ . Identity (1.2) is then applied to this new integrand considered as a function of  $x$  seen as single integration variable (i.e.,  $\bar{n} = 1$ ), with  $\rho$  (and  $\xi$ ) seen as fixed. In order that the first term of identity (1.2) vanish,  $\mu - 1 = \alpha$  shall be chosen equal to  $1/2$ ; thus  $\mu = 3/2$  and  $\nu = 2$ . We substitute this integral representation into Eq. (2.22) and perform the integration over  $x$  explicitly. We get:

<sup>6</sup> We assume that the elements of the kinematic matrix  $S$  have been made dimensionless by an appropriate rescaling so that the arguments of the logarithms in Eq. (2.24) are dimensionless as well.

<sup>7</sup> In Ref. [27] we defined  $D(x_b, x_c) = \frac{1}{2} X^T \cdot G^{(a)} \cdot X - V^{(a)T} \cdot X - \frac{1}{2} S_{aa}$ , whereas in the present article the more convenient normalization  $D^{(a)}(x_b, x_c) = X^T \cdot G^{(a)} \cdot X - 2 V^{(a)T} \cdot X - S_{aa}$  is used, hence this factor 2 mismatch.

$$I_3^4 = - \sum_{i \in S_3} \frac{\bar{b}_i}{\det(G)} \frac{2}{B(1, 1/2)} \int_0^{+\infty} \frac{d\xi}{\Delta_2 - \xi + i\lambda} \int_0^{+\infty} \frac{d\rho}{\Delta_1^{[i]} - \xi - \rho^2 + i\lambda} \times \left[ \frac{(1 - (G^{[i](i')})^{-1} V^{[i](i')})}{(D^{[i](i')}(1) + \xi + \rho^2 - i\lambda)^{1/2}} + \frac{(G^{[i](i')})^{-1} V^{[i](i')}}{(D^{[i](i')}(0) + \xi + \rho^2 - i\lambda)^{1/2}} \right] \quad (2.27)$$

with  $\Delta_1^{[i]} = (G^{[i](i')})^{-1} (V^{[i](i')})^2 + C^{[i](i')}$ . In Eq. (2.27), one recognizes familiar algebraic quantities associated with the three possible pinchings of the triangle diagram (cf. Eqs. (C.11) and (C.31)):

$$\Delta_1^{[i]} = - \frac{\det(\mathcal{S}^{[i]})}{\det(G^{[i]})} \quad (2.28)$$

and

$$\begin{aligned} 1 - (G^{\{1\}(2)})^{-1} V^{\{1\}(2)} &= - \frac{\bar{b}_2^{\{1\}}}{\det(G^{\{1\}})}, & (G^{\{1\}(2)})^{-1} V^{\{1\}(2)} &= - \frac{\bar{b}_3^{\{1\}}}{\det(G^{\{1\}})} \\ 1 - (G^{\{2\}(3)})^{-1} V^{\{2\}(3)} &= - \frac{\bar{b}_3^{\{2\}}}{\det(G^{\{2\}})}, & (G^{\{2\}(3)})^{-1} V^{\{2\}(3)} &= - \frac{\bar{b}_1^{\{2\}}}{\det(G^{\{2\}})} \\ 1 - (G^{\{3\}(1)})^{-1} V^{\{3\}(1)} &= - \frac{\bar{b}_1^{\{3\}}}{\det(G^{\{3\}})}, & (G^{\{3\}(1)})^{-1} V^{\{3\}(1)} &= - \frac{\bar{b}_2^{\{3\}}}{\det(G^{\{3\}})}. \end{aligned} \quad (2.29)$$

Let us introduce<sup>8</sup>

$$\begin{aligned} \tilde{D}_{12} &\equiv D^{\{1\}(2)}(1) = D^{\{2\}(1)}(1) \equiv \tilde{D}_{21} \\ \tilde{D}_{23} &\equiv D^{\{2\}(3)}(1) = D^{\{3\}(2)}(1) \equiv \tilde{D}_{32} \\ \tilde{D}_{31} &\equiv D^{\{3\}(1)}(1) = D^{\{1\}(3)}(1) \equiv \tilde{D}_{13}. \end{aligned} \quad (2.30)$$

From Eq. (2.19) we also have:

$$\begin{aligned} D^{\{1\}(2)}(0) &= D^{\{1\}(3)}(1) = \tilde{D}_{13} \\ D^{\{2\}(3)}(0) &= D^{\{2\}(1)}(1) = \tilde{D}_{21} \\ D^{\{3\}(1)}(0) &= D^{\{3\}(2)}(1) = \tilde{D}_{32}. \end{aligned} \quad (2.31)$$

We can write Eq. (2.27) as the following weighted sum over the coefficients  $\bar{b}_i$  and  $\bar{b}_j^{[i]}$ :

$$I_3^4 = \sum_{i \in S_3} \sum_{j \in S_3 \setminus \{i\}} \frac{\bar{b}_i}{\det(G)} \frac{\bar{b}_j^{[i]}}{\det(G^{[i]})} L_3^4(\Delta_2, \Delta_1^{[i]}, \tilde{D}_{ij}) \quad (2.32)$$

with

$$\begin{aligned} L_3^4(\Delta_2, \Delta_1^{[i]}, \tilde{D}_{ij}) &= \int_0^{+\infty} \frac{d\xi}{(-\Delta_2 + \xi - i\lambda)} \\ &\times \int_0^{+\infty} \frac{d\rho}{(-\Delta_1^{[i]} + \xi + \rho^2 - i\lambda) (\tilde{D}_{ij} + \xi + \rho^2 - i\lambda)^{1/2}}. \end{aligned} \quad (2.33)$$

<sup>8</sup> This definition is chosen so that  $\tilde{D}_{ij} = 2 m_k^2$  where  $k \in S_3 \setminus \{i, j\}$ .

We first perform the  $\rho$  integration in Eq. (2.33) using Appendix D. In the real mass case at hand  $\text{Im}(-\Delta_1^{[i]} - i\lambda)$  and  $\text{Im}(\tilde{D}_{ij} - i\lambda)$  have the same (negative) sign; we can then use relation (D.8) to rewrite  $L_3^4(\Delta_2, \Delta_1^{[i]}, \tilde{D}_{ij})$  as:

$$L_3^4(\Delta_2, \Delta_1^{[i]}, \tilde{D}_{ij}) = \int_0^{+\infty} d\xi \frac{d\xi}{-\Delta_2 + \xi - i\lambda} \int_0^1 \frac{dz}{\xi - (1-z^2)\Delta_1^{[i]} + z^2\tilde{D}_{ij} - i\lambda}. \quad (2.34)$$

We perform the  $\xi$  integration first, using Eq. (D.4) to get:

$$L_3^4(\Delta_2, \Delta_1^{[i]}, \tilde{D}_{ij}) = \int_0^1 \frac{dz}{(\tilde{D}_{ij} + \Delta_1^{[i]})z^2 + \Delta_2 - \Delta_1^{[i]}} \left[ \ln\left((\tilde{D}_{ij} + \Delta_1^{[i]})z^2 - \Delta_1^{[i]} - i\lambda\right) - \ln(-\Delta_2 - i\lambda) \right]. \quad (2.35)$$

In the real mass case, the difference of logarithms in Eq. (2.35) can be rewritten as the logarithm of a ratio.

The quantities  $\Delta_2$  and  $\Delta_1^{[i]}$  and  $\tilde{D}_{ij}$  are expressed in terms of the various determinants and  $\bar{b}$  coefficients:

$$\tilde{D}_{ij} + \Delta_1^{[i]} = \frac{\bar{b}_j^{[i]2}}{\det(G^{[i]})} \quad (2.36)$$

$$\Delta_2 - \Delta_1^{[i]} = \frac{\bar{b}_i^2}{\det(G) \det(G^{[i]})} \quad (2.37)$$

$$-\Delta_1^{[i]} = \frac{\det(S^{[i]})}{\det(G^{[i]})} \quad (2.38)$$

$$\Delta_2 = \frac{\det(S)}{\det(G)}. \quad (2.39)$$

To derive Eqs. (2.36) and (2.37), the following identities are used:

$$\left(\bar{b}_k^{[i]}\right)^2 = -\det(S^{[i]}) + \tilde{D}_{ik} \det(G^{[i]}) \quad (2.40)$$

$$\bar{b}_i^2 = \det(S) \det(G^{[i]}) + \det(G) \det(S^{[i]}). \quad (2.41)$$

They are particular cases of the so-called Jacobi identity for determinants [32], of which various cases of interest for the issues discussed here were specified in Appendix A.2 of Ref. [27].

Using Eqs. (2.36)–(2.39) we obtain:

$$I_3^4 = \sum_{i \in S_3} \sum_{j \in S_3 \setminus \{i\}} \bar{b}_i \bar{b}_j^{[i]} \int_0^1 \frac{dz}{\det(G) \bar{b}_j^{[i]2} z^2 + \bar{b}_i^2} \times \left[ \ln\left(\frac{\bar{b}_j^{[i]2} z^2 + \det(S^{[i]})}{\det(G^{[i]})} - i\lambda\right) - \ln\left(-\frac{\det(S)}{\det(G)} - i\lambda\right) \right]. \quad (2.42)$$

Equation (2.42) can be recast in the form (2.25) obtained according to the direct way. To achieve this goal, we first make the change of variable  $s = z\bar{b}_j^{[i]}$ , so that  $I_3^4$  reads:

$$I_3^4 = \sum_{i \in S_3} \sum_{j \in S_3 \setminus \{i\}} \bar{b}_i \int_0^{\bar{b}_j^{(i)}} \frac{ds}{s^2 \det(G) + \bar{b}_i^2} \times \left[ \ln \left( \frac{s^2 + \det(\mathcal{S}^{(i)})}{\det(G^{(i)})} - i\lambda \right) - \ln \left( -\frac{\det(\mathcal{S})}{\det(G)} - i\lambda \right) \right]. \quad (2.43)$$

In the real mass case, the  $\bar{b}_j^{(i)}$  are real: the integration contours of the two integrals corresponding to the two values of  $j \in S_3 \setminus \{i\}$  in Eq. (2.43) both run along the real axis. The poles in the integrand of Eq. (2.43) are fake as their residues vanish by construction, and the straight contours of integration in Eq. (2.43) do not cross the cuts<sup>9</sup> of the logarithms. The two real-contour integrals corresponding to the two values of  $j \in S_3 \setminus \{i\}$  can thus be joined end-to-end into a single one in a straightforward way. To refer to the two elements of  $S_3 \setminus \{i\}$  in a definite way, let us introduce  $k \equiv 1 + ((i + 1) \text{ modulo } 3)$  and  $l \equiv 1 + (i \text{ modulo } 3)$ . Accordingly  $I_3^4$  can be recast into:

$$I_3^4 = \sum_{i \in S_3} \bar{b}_i \int_{-\bar{b}_k^{(i)}}^{\bar{b}_l^{(i)}} \frac{ds}{s^2 \det(G) + \bar{b}_i^2} \times \left[ \ln \left( \frac{s^2 + \det(\mathcal{S}^{(i)})}{\det(G^{(i)})} - i\lambda \right) - \ln \left( -\frac{\det(\mathcal{S})}{\det(G)} - i\lambda \right) \right]. \quad (2.44)$$

We then make the following change of variable:

$$s = -\bar{b}_k^{(i)} + (\bar{b}_k^{(i)} + \bar{b}_l^{(i)}) u = -\bar{b}_k^{(i)} - \det(G^{(i)}) u \quad (2.45)$$

such that  $s = -\bar{b}_k^{(i)} \leftrightarrow u = 0$  and  $s = \bar{b}_l^{(i)} \leftrightarrow u = 1$ . The numerator of the argument of the logarithm in Eq. (2.44) becomes:

$$\frac{s^2 + \det(\mathcal{S}^{(i)})}{\det(G^{(i)})} = \det(G^{(i)}) u^2 + 2\bar{b}_k^{(i)} u + \frac{\bar{b}_k^{(i)2} + \det(\mathcal{S}^{(i)})}{\det(G^{(i)})}. \quad (2.46)$$

Thanks to the identity (2.40), we recognize

$$\frac{\bar{b}_k^{(i)2} + \det(\mathcal{S}^{(i)})}{\det(G^{(i)})} = \tilde{D}_{ik} = -C^{(i)(l)}.$$

We can further identify

$$\det(G^{(i)}) = G^{(i)(l)} \\ \bar{b}_k^{(i)} = -V^{(i)(l)}$$

so that the complete identification reads:

$$\frac{s^2 + \det(\mathcal{S}^{(i)})}{\det(G^{(i)})} = D^{(i)(l)}(u). \quad (2.47)$$

<sup>9</sup> In anticipation, let us stress that the latter still holds in the complex mass case, although the justification of the second point requires a little more care.

Furthermore, thanks to the identity (2.41), we recast the denominator  $[s^2 \det(G) + \bar{b}_i^2]$  in the integrand of Eq. (2.44) as:

$$\det(G) \left( -\bar{b}_k^{(i)} - \det(G^{(i)}) u \right)^2 + \bar{b}_i^2 = \det(G^{(i)}) \left[ \det(G) D^{(i)(l)}(u) + \det(S) \right].$$

Reminding ourselves that  $B = \det(G) / \det(S)$  and  $\bar{b}_i / \det(S) = b_i$ , we finally get:

$$I_3^4 = - \sum_{i \in S_3} b_i \int_0^1 \frac{du}{B D^{(i)(l)}(u) + 1} \left[ \ln \left( D^{(i)(l)}(u) - i\lambda \right) - \ln \left( -\frac{1}{B} - i\lambda \right) \right],$$

which is nothing but Eq. (2.25).

### 2.3. Taming the proliferation of dilogarithms

A comment is in order here regarding the dilogarithms generated by the computation of Eqs. (2.25) and (2.42) in closed form, performing the leftover integration after partial fraction decomposition. In the present case the number of dilogarithms generated by Eq. (2.25) seems quite non-minimal, and by Eq. (2.42) even less so. The latter contains 6 terms (cf. summation over  $i$  and  $j$ ) versus 3 (summation over  $i$  only) for both Eq. (2.25) and the original calculation of 't Hooft and Veltman. Furthermore each term of Eqs. (2.25) or (2.42) yields 8 dilogarithms: 4 generated by each of the *two* poles, i.e., the zeroes of the second-degree polynomial  $B D^{(i)(l)}(x) + 1$ , whereas the denominators in the original calculation of 't Hooft and Veltman were first degree in  $x$  only. This seemingly amounts to a twofold proliferation of dilogarithms (24 instead of 12) using Eq. (2.25) and, even a fourfold proliferation (48 instead of 12) using Eq. (2.42), w.r.t. the original calculation of 't Hooft and Veltman. The situation is actually not so bad.

We previously explained how pairs of terms in Eq. (2.42) can be recombined so as to recover Eq. (2.25). Thus we are left only with the relative twofold proliferation seemingly occurring in Eq. (2.25) w.r.t. to Ref. [12]. Let us show that this apparent doubling is actually fake. Let us rewrite Eq. (2.25) as<sup>10</sup>

$$I_3^4 = - \frac{1}{2 \sqrt{-\det(G)}} \sum_{i \in S_3} \int_0^1 dx \left[ \frac{1}{x - x_+} - \frac{1}{x - x_-} \right] \left[ \ln \left( D^{(i)(l)}(x) - i\lambda \right) - \ln \left( -\frac{1}{B} - i\lambda \right) \right] \quad (2.48)$$

where

$$x_{\pm} = - \frac{1}{\det(G^{(i)})} \left( \bar{b}_k^{(i)} \pm \frac{\bar{b}_i}{\sqrt{-\det(G)}} \right)$$

are the roots of the denominator in Eq. (2.44) and the index  $k$  is the element of  $S_3 \setminus \{i, l\}$ . Equation (2.48) involves 24 dilogarithms, i.e., twice as many as the result of Ref. [12]. To reduce their number let us first compute the following quantity:

$$K = \sum_{i \in S_3} \int_0^1 dx \left[ \frac{1}{x - x_+} + \frac{1}{x - x_-} \right] \left[ \ln \left( D^{(i)(l)}(x) - i\lambda \right) - \ln \left( -\frac{1}{B} - i\lambda \right) \right]. \quad (2.49)$$

<sup>10</sup> To be definite in case  $\det(G) > 0$ ,  $\sqrt{-\det(G)}$  is understood to be  $+i \sqrt{\det(G)}$ . This choice actually does not matter as the two roots are exchanged under  $x \leftrightarrow 1 - x$ .

Combining the two terms in the square bracket into a single denominator and using Eq. (2.47) yields:

$$K = \sum_{i \in S_3} \int_0^1 dx \frac{dD^{(i)(l)}(x)}{dx} \frac{1}{D^{(i)(l)}(x) + \frac{1}{B}} \left[ \ln \left( D^{(i)(l)}(x) - i\lambda \right) - \ln \left( -\frac{1}{B} - i\lambda \right) \right]. \quad (2.50)$$

We now make the change of variable  $t = D^{(i)(l)}(x)$ . The integrals over  $[0, 1]$  in Eq. (2.50) are traded for contour integrals in the complex plane<sup>11</sup>.  $K$  takes the form:

$$K = \left\{ \int_{D^{(1)(2)}(0)}^{D^{(1)(2)}(1)} dt + \int_{D^{(2)(3)}(0)}^{D^{(2)(3)}(1)} dt + \int_{D^{(3)(1)}(0)}^{D^{(3)(1)}(1)} dt \right\} F(t) \quad (2.51)$$

where

$$F(t) = \frac{1}{t + \frac{1}{B}} \left[ \ln(t - i\lambda) - \ln \left( -\frac{1}{B} - i\lambda \right) \right].$$

Note that  $D^{(i)(l)}(0)$  and  $D^{(i)(l)}(1)$  are internal masses, and from the correspondence (2.31) in Sect. 2.2.2 we see that the three integrals in Eq. (2.51) combine into an integral along a closed contour (triangle). Note also that the pole in  $F(t)$  is fake as the residue is zero by construction; thus we do not have to worry about the positions of the pole with respect to the triangle. In addition, in the case of complex masses, this triangle lies entirely below the real axis in the complex  $t$ -plane: in no case does it cross the real axis; consequently there is no worry with the discontinuity cuts of the various logarithms. Thus  $K = 0$ . To summarize, defining

$$I_i^\pm = -\frac{1}{2\sqrt{-\det(G)}} \int_0^1 \frac{dx}{x - x_\pm} \left[ \ln \left( D^{(i)(l)}(x) - i\lambda \right) - \ln \left( -\frac{1}{B} - i\lambda \right) \right] \quad (2.52)$$

“ $K = 0$ ” reads:

$$I_1^+ + I_2^+ + I_3^+ = - (I_1^- + I_2^- + I_3^-) \quad (2.53)$$

and these two mutually opposite sums of three terms happen to be subtracted one from another to yield  $I_3^4$  in Eq. (2.25), so that the three-point function is given by:

$$\begin{aligned} I_3^4 &= (I_1^+ + I_2^+ + I_3^+) - (I_1^- + I_2^- + I_3^-) \\ &= 2 (I_1^+ + I_2^+ + I_3^+), \end{aligned} \quad (2.54)$$

which involves 4 dilogarithms in each of the three terms  $I_1^+, I_2^+$ , and  $I_3^+$ , hence 12 distinct dilogarithms only, not 24 as hastily thought. In some intermediate step, the authors of Ref. [12] made use of some change of variable involving either of the two roots  $\alpha_\pm$  of some second-degree polynomial. Both changes of variables were usable and led to namely one combination of three  $I_i^\pm$  and its complementary with respect to Eq. (2.53), respectively<sup>12</sup>. The two combinations, albeit seemingly

<sup>11</sup> In the real mass case, the integration contours of the integrals in Eq. (2.51) still run along the real  $t$ -axis. In the complex mass case the contours more generally draw parabolic arcs in the complex  $t$ -plane.

<sup>12</sup> We do not specify which combination namely leads to the result of Ref. [12] here because it depends on the assignment adopted for the superscript  $l$ . The exercise is left to the masochistic reader!

distinct analytically, were guaranteed to take equal values in the approach of Ref. [12] since the three-point function does not depend on the choice of the change of variable made. Instead, our approach does not rely on any such choice triggering a breakdown of symmetry between two options. It thus yields a result that preserves this symmetry, being half the sum of the two expressions corresponding to  $\alpha_+$  and to  $\alpha_-$  in Ref. [12], yet at the expense of doubling the number of terms. The use of identity (2.53) to counteract the doubling is somehow a counterpart of the choice of  $\alpha_+$  vs.  $\alpha_-$  in Ref. [12].

Anticipating the absence of a direct way in the four-point function case, one may wonder whether the fourfold proliferation in Eq. (2.42) compared to Ref. [12] can be counteracted directly instead of recovering Eq. (2.25) first and then acting as explained just above. In this respect it shall be stressed that the actual number of dilogarithms generated by Eq. (2.42) is twice as small as naively counted because the integrand is even in  $x$ . Indeed, each of the 6 terms in Eq. (2.42), of the form

$$T = \int_0^1 dx \frac{\ln(P_1(x))}{P_2(x)}$$

where both  $P_j(x)$  are even second-degree trinomials in  $x$ , can be rewritten

$$T = \frac{1}{2} \int_{-1}^1 dx \frac{\ln(P_1(x))}{P_2(x)}.$$

Let us note  $\pm x_2$ , the two mutually opposite roots of  $P_2(x)$ . Partial fraction decomposition of  $1/P_2(x)$  leads to an integral of the form

$$\begin{aligned} T &\propto \int_{-1}^1 dx \frac{\ln(P_1(x))}{x - x_2} - \int_{-1}^1 dx \frac{\ln(P_1(x))}{x + x_2} \\ &= 2 \int_{-1}^1 dx \frac{\ln(P_1(x))}{x - x_2}, \end{aligned}$$

which yields only 4 dilogarithms instead of 8: Eq. (2.42) thus yields  $6 \times 4 = 24$  dilogarithms “only”. This fortunate feature of parity in the integration variable and its consequence will also be encountered in the case of the four-point function. One may wonder whether there is a straight way to reduce the number of dilogarithms, bypassing the step that reshuffles Eq. (2.42) into Eq. (2.25), which would instead exploit the parity in  $z$  of the integrand in Eq. (2.42) combined with some other trick. For the three-point function, an independent derivation of identity (2.53) alternative to the one above might be worked out directly from Eq. (2.42) computed in closed form, using a version of the five-dilogarithm Hill identity [33] supplemented by a couple of cancellations thereby made manifest (J. Ph. Guillet et al., manuscript in preparation).

### 3. Leg-up: The scalar four-point function $I_4^4$

The usual integral representation of  $I_4^4$  in terms of Feynman parameters is given by:

$$I_4^4 = \int_0^1 \prod_{i=1}^4 dz_i \delta(1 - \sum_{i=1}^4 z_i) \left( -\frac{1}{2} Z^T \cdot S \cdot Z - i\lambda \right)^{-2} \tag{3.1}$$

where  $Z$  is now a column 4-vector whose components are the  $z_i$ . In a way similar to Sect. 2 we arbitrarily single out the subscript value  $a$  ( $a \in S_4 = \{1, 2, 3, 4\}$ ), and write  $z_a$  as  $1 - \sum_{j \neq a} z_j$ . We find:

$$-Z^T \cdot S \cdot Z = \sum_{i,j \in S_4 \setminus \{a\}} G_{ij}^{(a)} z_i z_j - 2 \sum_{j \in S_4 \setminus \{a\}} V_j^{(a)} z_j - C^{(a)} \tag{3.2}$$

where the  $3 \times 3$  Gram matrix  $G^{(a)}$  and the column 3-vector  $V^{(a)}$  are defined by

$$\begin{aligned} G_{ij}^{(a)} &= -(\mathcal{S}_{ij} - \mathcal{S}_{aj} - \mathcal{S}_{ia} + \mathcal{S}_{aa}), \quad i, j \neq a \\ V_j^{(a)} &= \mathcal{S}_{aj} - \mathcal{S}_{aa}, \quad j \neq a \\ C^{(a)} &= \mathcal{S}_{aa}. \end{aligned} \tag{3.3}$$

We label  $b, c$ , and  $d$  the three elements of  $S_4 \setminus \{a\}$  with  $b < c < d$ . The polynomial (3.2) reads:

$$D^{(a)}(X) = X^T \cdot G^{(a)} \cdot X - 2 V^{(a)T} \cdot X - C^{(a)}, X = \begin{bmatrix} z_b \\ z_c \\ z_d \end{bmatrix}. \tag{3.4}$$

We note the integration simplex  $\Sigma_{bcd} = \{0 \leq z_b, z_c, z_d, z_b + z_c + z_d \leq 1\}$ .  $I_4^A$  can be written

$$I_4^A = 4 \int_{\Sigma_{bcd}} \frac{dz_b dz_c dz_d}{(D^{(a)}(X) - i\lambda)^2}. \tag{3.5}$$

Again, the dependence on  $G^{(a)}$ ,  $V^{(a)}$ , and  $C^{(a)}$  will arise through quantities independent of the actual choice of  $a$ . We will follow a similar strategy as for the three-point function in Sect. 2 *mutatis mutandis*.

### 3.1. Step 1

The idea is again to adjust the power of the denominator in the l.h.s. of Eq. (1.2) in such a way that only the boundary term in Eq. (1.2) remains. In the four-point function case at hand, cf. Eq. (3.5),  $\bar{n} = N - 1$  is equal to 3. Imposing  $\bar{n} - 2\alpha = 0$  implies  $\alpha = 3/2$ , and hence a power  $\alpha + 1 = 5/2$  in the l.h.s. of Eq. (1.2). In Eq. (3.5), however,  $D$  is raised to the power 2, not  $5/2$ . Identity (2.26) is used with  $\mu = 5/2$  and  $\nu = 2$  chosen such that  $\mu - 1/\nu = 2$  and  $2(\mu - 1) = 3$  to customize a denominator raised to a power shifted from 2 to  $5/2$ . The representation of  $1/D^2$  obtained is substituted into Eq. (3.5). Identity (1.2) is then applied to the new integrand seen as a function of the three variables  $z_b, z_c, z_d$ . Integration will be performed on the latter first, keeping  $\xi$  fixed. This yields:

$$I_4^A = \frac{8}{B(2, 1/2)} \int_0^{+\infty} d\xi \int_{\Sigma_{bcd}} \frac{dz_b dz_c dz_d}{(D^{(a)}(X) + \xi^2 - i\lambda)^{5/2}}. \tag{3.6}$$

Identity (1.2) recasts Eq. (3.6) into:

$$\begin{aligned} I_4^A &= -\frac{8}{3 B(2, 1/2)} \int_0^{+\infty} \frac{d\xi}{\Delta_3 - \xi^2 + i\lambda} \\ &\quad \times \int_{\Sigma_{bcd}} dz_b dz_c dz_d \left[ \sum_{j \in S_4 \setminus \{a\}} \frac{\partial}{\partial z_j} \left( \frac{z_j - ((G^{(a)})^{-1} \cdot V^{(a)})_j}{(D^{(a)}(X) + \xi^2 - i\lambda)^{3/2}} \right) \right] \end{aligned} \tag{3.7}$$

with  $\Delta_3 = V^{(a)T} \cdot (G^{(a)})^{-1} \cdot V^{(a)} + C^{(a)}$ . For each of the three terms  $j = b, c, d$  in Eq. (3.7) this makes the first integration over the corresponding  $z_j$  trivial, and we get:

$$I_4^A = -\frac{8}{3 B(2, 1/2)} \int_0^{+\infty} \frac{d\xi}{\Delta_3 - \xi^2 + i\lambda} \sum_{j \in S_4 \setminus \{a\}} \mathcal{T}_j \tag{3.8}$$

with

$$\begin{aligned}
 \mathcal{T}_b &= \int_{\Sigma_{cd}} dz_c dz_d \left[ \frac{(1 - z_c - z_d) - ((G^{(a)})^{-1} \cdot V^{(a)})_b}{(D^{(a)}(1 - z_c - z_d, z_c, z_d) + \xi^2 - i\lambda)^{3/2}} \right. \\
 &\quad \left. + \frac{((G^{(a)})^{-1} \cdot V^{(a)})_b}{(D^{(a)}(0, z_c, z_d) + \xi^2 - i\lambda)^{3/2}} \right] \\
 \mathcal{T}_c &= \int_{\Sigma_{bd}} dz_b dz_d \left[ \frac{(1 - z_b - z_d) - ((G^{(a)})^{-1} \cdot V^{(a)})_c}{(D^{(a)}(z_b, 1 - z_b - z_d, z_d) + \xi^2 - i\lambda)^{3/2}} \right. \\
 &\quad \left. + \frac{((G^{(a)})^{-1} \cdot V^{(a)})_c}{(D^{(a)}(z_b, 0, z_d) + \xi^2 - i\lambda)^{3/2}} \right] \\
 \mathcal{T}_d &= \int_{\Sigma_{bc}} dz_b dz_c \left[ \frac{(1 - z_b - z_c) - ((G^{(a)})^{-1} \cdot V^{(a)})_d}{(D^{(a)}(z_b, z_c, 1 - z_b - z_c) + \xi^2 - i\lambda)^{3/2}} \right. \\
 &\quad \left. + \frac{((G^{(a)})^{-1} \cdot V^{(a)})_d}{(D^{(a)}(z_b, z_c, 0) + \xi^2 - i\lambda)^{3/2}} \right].
 \end{aligned} \tag{3.9}$$

By an appropriate relabeling of the variables, all three terms  $\mathcal{T}_j$  in Eq. (3.8) can be gathered in one integral:

$$\begin{aligned}
 I_4^4 &= -\frac{8}{3B(2, 1/2)} \int_0^{+\infty} \frac{d\xi}{\Delta_3 - \xi^2 + i\lambda} \int_{\Sigma_{12}} dx_1 dx_2 \\
 &\times \left\{ \frac{(1 - x_1 - x_2) - ((G^{(a)})^{-1} \cdot V^{(a)})_b}{(D^{(a)}(1 - x_1 - x_2, x_1, x_2) + \xi^2 - i\lambda)^{3/2}} + \frac{((G^{(a)})^{-1} \cdot V^{(a)})_b}{(D^{(a)}(0, x_1, x_2) + \xi^2 - i\lambda)^{3/2}} \right. \\
 &+ \frac{(1 - x_1 - x_2) - ((G^{(a)})^{-1} \cdot V^{(a)})_c}{(D^{(a)}(x_1, 1 - x_1 - x_2, x_2) + \xi^2 - i\lambda)^{3/2}} + \frac{((G^{(a)})^{-1} \cdot V^{(a)})_c}{(D^{(a)}(x_1, 0, x_2) + \xi^2 - i\lambda)^{3/2}} \\
 &\left. + \frac{(1 - x_1 - x_2) - ((G^{(a)})^{-1} \cdot V^{(a)})_d}{(D^{(a)}(x_1, x_2, 1 - x_1 - x_2) + \xi^2 - i\lambda)^{3/2}} + \frac{((G^{(a)})^{-1} \cdot V^{(a)})_d}{(D^{(a)}(x_1, x_2, 0) + \xi^2 - i\lambda)^{3/2}} \right\}. \tag{3.10}
 \end{aligned}$$

The number of terms in Eq. (3.10) can be reduced. In the r.h.s. of Eq. (3.10) let us consider the first term of the second line of the curly bracket:

$$T_c \equiv \int_{\Sigma_{12}} dx_1 dx_2 \frac{(1 - x_1 - x_2) - ((G^{(a)})^{-1} \cdot V^{(a)})_c}{(D^{(a)}(x_1, 1 - x_1 - x_2, x_2) + \xi^2 - i\lambda)^{3/2}} \tag{3.11}$$

and perform the following change of variables:

$$\begin{cases} y_1 = 1 - x_1 - x_2 \\ y_2 = x_2 \end{cases} \Leftrightarrow \begin{cases} x_1 = 1 - y_1 - y_2 \\ x_2 = y_2 \end{cases}.$$

The new variables  $y_1$  and  $y_2$  still span the simplex  $\Sigma_{12}$ . Equation (3.11) becomes:

$$T_c = \int_{\Sigma_{12}} dy_1 dy_2 \frac{y_1 - ((G^{(a)})^{-1} \cdot V^{(a)})_c}{(D^{(a)}(1 - y_1 - y_2, y_1, y_2) + \xi^2 - i\lambda)^{3/2}}. \tag{3.12}$$

In a similar way, using  $t_1 = x_2$  and  $t_2 = 1 - x_1 - x_2$  we recast the first term of the third line of the curly bracket in the r.h.s. of Eq. (3.10) into:

$$\begin{aligned} T_d &\equiv \int_{\Sigma_{12}} dx_1 dx_2 \frac{(1 - x_1 - x_2) - ((G^{(a)})^{-1} \cdot V^{(a)})_d}{(D^{(a)}(x_1, x_2, 1 - x_1 - x_2) + \xi^2 - i\lambda)^{3/2}} \\ &= \int_{\Sigma_{12}} dt_1 dt_2 \frac{t_2 - ((G^{(a)})^{-1} \cdot V^{(a)})_d}{(D^{(a)}(1 - t_1 - t_2, t_1, t_2) + \xi^2 - i\lambda)^{3/2}}. \end{aligned} \quad (3.13)$$

We then recombine  $T_c$  and  $T_d$  with the first term of the first line of the curly bracket in the r.h.s. of Eq. (3.10), and  $I_4^4$  reads:

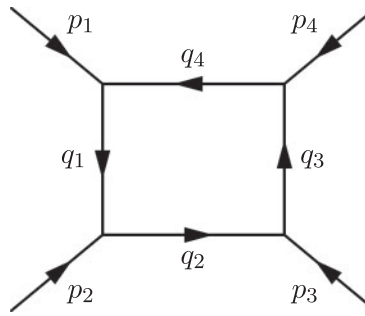
$$\begin{aligned} I_4^4 &= -\frac{8}{3B(2, 1/2)} \int_0^{+\infty} \frac{d\xi}{\Delta_3 - \xi^2 + i\lambda} \int_{\Sigma_{12}} dx_1 dx_2 \\ &\times \left\{ \frac{1 - \sum_{j \in S_4 \setminus \{a\}} ((G^{(a)})^{-1} \cdot V^{(a)})_j}{(D^{(a)}(1 - x_1 - x_1, x_1, x_2) + \xi^2 - i\lambda)^{3/2}} + \frac{((G^{(a)})^{-1} \cdot V^{(a)})_b}{(D^{(a)}(0, x_1, x_2) + \xi^2 - i\lambda)^{3/2}} \right. \\ &\left. + \frac{((G^{(a)})^{-1} \cdot V^{(a)})_c}{(D^{(a)}(x_1, 0, x_2) + \xi^2 - i\lambda)^{3/2}} + \frac{((G^{(a)})^{-1} \cdot V^{(a)})_d}{(D^{(a)}(x_1, x_2, 0) + \xi^2 - i\lambda)^{3/2}} \right\}. \end{aligned} \quad (3.14)$$

The quantities  $\Delta_3$  and  $((G^{(a)})^{-1} \cdot V^{(a)})_j$  involved in Eq. (3.14) are expressed in terms of  $\det(S)$ ,  $\det(G)$ , and the  $\bar{b}_j = b_j \det(S)$  (see Eqs. (C.11) and (C.31) of Appendix C):

$$\begin{aligned} \Delta_3 &= -\frac{\det(S)}{\det(G)} \\ ((G^{(a)})^{-1} \cdot V^{(a)})_j &= -\frac{\bar{b}_j}{\det(G)}, \quad j \in S_4 \setminus \{a\} \\ 1 - \sum_{j \in S_4 \setminus \{a\}} ((G^{(a)})^{-1} \cdot V^{(a)})_j &= -\frac{\bar{b}_a}{\det(G)}. \end{aligned} \quad (3.15)$$

The four new polynomials of  $x_1, x_2$  appearing in the denominators coincide with those involved in the integral representations of the triangle diagrams obtained by all possible pinchings of the box diagram of Fig. 2. Namely,  $D^{(a)}(0, x_1, x_2) = D^{\{b\}(a)}(x_1, x_2)$  comes out when pinching propagator  $b$  in the box, which corresponds to suppressing line and column  $b$  in the kinematic matrix  $S$  in which line and column  $a$  are singled out:

$$\begin{aligned} D^{(a)}(0, x_1, x_2) &= \tilde{X}^T \cdot G^{\{b\}(a)} \cdot \tilde{X} - 2 V^{\{b\}(a) T} \cdot \tilde{X} - C^{\{b\}(a)} \\ &= D^{\{b\}(a)}(x_1, x_2), \quad \tilde{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \end{aligned} \quad (3.16)$$



**Fig. 2.** A box picturing the one-loop four-point function.

Likewise for  $c, d, a$ :

$$\begin{aligned} D^{(a)}(x_1, 0, x_2) &= \tilde{X}^T \cdot G^{(c)(a)} \cdot \tilde{X} - 2 V^{(c)(a) T} \cdot \tilde{X} - C^{(c)(a)} \\ &= D^{(c)(a)}(x_1, x_2) \end{aligned}$$

$$\begin{aligned} D^{(a)}(x_1, x_2, 0) &= \tilde{X}^T \cdot G^{(d)(a)} \cdot \tilde{X} - 2 V^{(d)(a) T} \cdot \tilde{X} - C^{(d)(a)} \\ &= D^{(d)(a)}(x_1, x_2) \end{aligned}$$

whereas a simple calculation yields:

$$\begin{aligned} D^{(a)}(1 - x_1 - x_2, x_1, x_2) &= \tilde{X}^T \cdot G^{(a)(b)} \cdot \tilde{X} - 2 V^{(a)(b) T} \cdot \tilde{X} - C^{(a)(b)} \\ &= D^{(a)(b)}(x_1, x_2). \end{aligned}$$

$I_4^4$  thus reads:

$$I_4^4 = \frac{8}{3 B(2, 1/2)} \sum_{i=1}^4 \frac{\bar{b}_i}{\det(G)} \int_0^{+\infty} \frac{d\xi}{\Delta_3 - \xi^2 + i\lambda} \int_{\Sigma_{kl}} \frac{dx_k dx_l}{(D^{(i)(i')}(x_k, x_l) + \xi^2 - i\lambda)^{3/2}} \quad (3.17)$$

where, for definiteness, the  $i'$  assignment for each  $i$  is the one read on the four equations (3.16) above:  $i' = a$  for  $i = b, c, d$  and  $i' = b$  for  $i = a$ . In a similar way as with the three-point function case, this  $i'$  assignment can actually be changed relying on identities such as  $D^{(d)(a)}(x_1, x_2) = D^{(d)(b)}(1 - x_1 - x_2, x_2) = D^{(d)(c)}(x_1, 1 - x_1 - x_2)$  etc. combined with corresponding appropriate changes of variables that leave the integration simplex unchanged.

### 3.2. Step 2

We proceed further as we did for the three-point function following the “indirect way”, iterating the procedure to integrate over one more  $x_i$  explicitly. Let us define the quantities  $J_i$  by:

$$J_i = \int_{\Sigma_{kl}} \frac{dx_k dx_l}{(D^{(i)(i')}(x_k, x_l) + \xi^2 - i\lambda)^{3/2}}. \quad (3.18)$$

Here again we do not apply identity (1.2) directly because the power of the numerator is not the appropriate one. We now deal with  $\bar{n} = 2$  variables of integration;  $\alpha$  shall thus be equal to 1 in order to keep the boundary term only, whereas the exponent of the denominator in Eq. (3.18) happens to be 3/2, not  $\alpha + 1 = 2$ . To get the appropriate power, we again make use of Eq. (2.26) with  $\mu = 2$ ,  $\nu = 2$  and  $J_i$  is represented by:

$$J_i = \frac{2}{B(3/2, 1/2)} \int_0^{+\infty} d\rho \int_{\Sigma_{kl}} \frac{dx_k dx_l}{(D^{(i)(i')}(x_k, x_l) + \xi^2 + \rho^2 - i\lambda)^2}. \quad (3.19)$$

We use identity (1.2) in which “ $D$ ” is interpreted as  $(D^{\{i\}(i')} + \xi^2 + \rho^2)$ . We note  $k, l \in S_4 \setminus \{i, i'\}$  with  $k < l$ . We get:

$$J_i = -\frac{1}{B(3/2, 1/2)} \int_0^{+\infty} \frac{d\rho}{\Delta_2^{\{i\}} - \xi^2 - \rho^2 + i\lambda} \int_{\Sigma_{kl}} dx_k dx_l \times \left[ \sum_{j \in S_4 \setminus \{i, i'\}} \frac{\partial}{\partial x_j} \left( \frac{x_j - ((G^{\{i\}(i')})^{-1} \cdot V^{\{i\}(i')})_j}{(D^{\{i\}(i')}(x_k, x_l) + \xi^2 + \rho^2 - i\lambda)} \right) \right] \tag{3.20}$$

with  $\Delta_2^{\{i\}} = V^{\{i\}(i')} T \cdot (G^{\{i\}(i')})^{-1} \cdot V^{\{i\}(i')} + C^{\{i\}(i')}$ . In each term “ $j$ ” the integration performed over  $x_j$  first is trivialized as a boundary term and we get for  $J_i$ :

$$J_i = -\frac{1}{B(3/2, 1/2)} \int_0^{+\infty} \frac{d\rho}{\Delta_2^{\{i\}} - \xi^2 - \rho^2 + i\lambda} \times \left[ \int_0^1 dx_l \left( \frac{(1 - x_l) - ((G^{\{i\}(i')})^{-1} \cdot V^{\{i\}(i')})_k}{(D^{\{i\}(i')}(1 - x_l, x_l) + \xi^2 + \rho^2 - i\lambda)} + \frac{((G^{\{i\}(i')})^{-1} \cdot V^{\{i\}(i')})_k}{(D^{\{i\}(i')}(0, x_l) + \xi^2 + \rho^2 - i\lambda)} \right) + \int_0^1 dx_k \left( \frac{(1 - x_k) - ((G^{\{i\}(i')})^{-1} \cdot V^{\{i\}(i')})_l}{(D^{\{i\}(i')}(x_k, 1 - x_k) + \xi^2 + \rho^2 - i\lambda)} + \frac{((G^{\{i\}(i')})^{-1} \cdot V^{\{i\}(i')})_l}{(D^{\{i\}(i')}(x_k, 0) + \xi^2 + \rho^2 - i\lambda)} \right) \right]. \tag{3.21}$$

Making the change of variable  $x_k = x$  and  $x_l = 1 - x$  in the first and third terms inside the square bracket of Eq. (3.21) and gathering the terms in the same integral, we get:

$$J_i = -\frac{1}{B(3/2, 1/2)} \int_0^{+\infty} \frac{d\rho}{\Delta_2^{\{i\}} - \xi^2 - \rho^2 + i\lambda} \times \int_0^1 dx \left[ \frac{((G^{\{i\}(i')})^{-1} \cdot V^{\{i\}(i')})_k}{(D^{\{i\}(i')}(0, x) + \xi^2 + \rho^2)} + \frac{((G^{\{i\}(i')})^{-1} \cdot V^{\{i\}(i')})_l}{(D^{\{i\}(i')}(x, 0) + \xi^2 + \rho^2)} + \frac{1 - \sum_{j \in S_4 \setminus \{i, i'\}} ((G^{\{i\}(i')})^{-1} \cdot V^{\{i\}(i')})_j}{(D^{\{i\}(i')}(x, 1 - x) + \xi^2 + \rho^2 - i\lambda)} \right]. \tag{3.22}$$

We recognize

$$\Delta_2^{\{i\}} = \frac{\det(S^{\{i\}})}{\det(G^{\{i\}})}$$

$$((G^{\{i\}(i')})^{-1} \cdot V^{\{i\}(i')})_j = \frac{\bar{b}_j^{\{i\}}}{\det(G^{\{i\}})}, \quad j \in S_4 \setminus \{i, i'\} \tag{3.23}$$

$$1 - \sum_{j \in S_4 \setminus \{i, i'\}} ((G^{\{i\}(i')})^{-1} \cdot V^{\{i\}(i')})_j = \frac{\bar{b}_{i'}^{\{i\}}}{\det(G^{\{i\}})}.$$

We recall that the coefficients  $\bar{b}_j^{\{i\}}$  are equal to  $b_j^{\{i\}} / \det(S^{\{i\}})$  with the  $b_j^{\{i\}}$  such that  $\sum_{l \in S_4 \setminus \{i\}} S_{kl}^{\{i\}} b_l^{\{i\}} = 1$ . In denominators, the polynomials  $D^{\{i\}(i')}(0, 1 - x)$ ,  $D^{\{i\}(i')}(x, 0)$ , and  $D^{\{i\}(i')}(x, 1 - x)$  also have a

simple interpretation. We collect the dictionary:

$$\begin{aligned}
 D^{(a)}(0, 0, x) &= D^{(b)(a)}(0, x) &= D^{(c)(a)}(0, x) &= D^{(b,c)(a)}(x) \\
 D^{(a)}(0, x, 0) &= D^{(b)(a)}(x, 0) &= D^{(d)(a)}(0, x) &= D^{(b,d)(a)}(x) \\
 D^{(a)}(x, 0, 0) &= D^{(c)(a)}(x, 0) &= D^{(d)(a)}(x, 0) &= D^{(c,d)(a)}(x) \\
 D^{(a)}(0, x, 1-x) &= D^{(b)(a)}(x, 1-x) &= D^{(a)(b)}(x, 1-x) &= D^{(a,b)(d)}(x) \\
 D^{(a)}(x, 0, 1-x) &= D^{(c)(a)}(x, 1-x) &= D^{(a)(b)}(0, 1-x) &= D^{(a,c)(d)}(x) \\
 D^{(a)}(x, 1-x, 0) &= D^{(d)(a)}(x, 1-x) &= D^{(a)(b)}(1-x, 0) &= D^{(a,d)(c)}(x).
 \end{aligned}
 \tag{3.24}$$

$D^{(i,j)(k)}(x)$  is involved in the integral representation of the two-point diagram obtained by pinching the two propagators  $i$  and  $j \neq i$  in the box diagram. The  $k$  assigned to each pair  $\{i, j\}$  in the fourth column of Table (3.24) corresponds to singling out line and column  $k \in S_4 \setminus \{i, j\}$  in the twice-pinched matrix  $\mathcal{S}^{\{i,j\}}$ . Alternatively,  $D^{(i,j)(l)}(x)$  with  $l \in S_4 \setminus \{i, j, k\}$  corresponds to singling out line and column  $l$  in  $\mathcal{S}^{\{i,j\}}$  and is related to  $D^{(i,j)(k)}(x)$  by:

$$\begin{aligned}
 D^{(i,j)(k)}(x) &= G^{(i,j)(k)} x^2 - 2 V^{(i,j)(k)} x - C^{(i,j)(k)} \\
 &= D^{(i,j)(l)}(1-x).
 \end{aligned}
 \tag{3.25}$$

For any pair  $\{i, j\}$ ,  $k$  can thus be traded for  $l$  in Table (3.24) modulo the change of variable  $x' = 1-x$ , which leaves unchanged the corresponding term in Eq. (3.22).  $I_4^4$  is rewritten as a sum of 12 terms:

$$\begin{aligned}
 I_4^4 &= -\frac{8}{3 B(2, 1/2) B(3/2, 1/2)} \sum_{i \in S} \sum_{j \in S \setminus \{i\}} \frac{\bar{b}_i}{\det(G)} \frac{\bar{b}_j^{(i)}}{\det(G^{(i)})} \\
 &\times \int_0^{+\infty} \frac{d\xi}{\Delta_3 - \xi^2 + i\lambda} \int_0^{+\infty} \frac{d\rho}{\Delta_2^{(i)} - \xi^2 - \rho^2 + i\lambda} \int_0^1 \frac{dx}{D^{(i,j)(k)}(x) + \xi^2 + \rho^2 - i\lambda}.
 \end{aligned}
 \tag{3.26}$$

### 3.3. Step 3

We iterate once more the procedure, adjusting the power of the denominator of Eq. (3.26) using identity (2.26) with  $\mu = 3/2$ ,  $\nu = 2$ , so as to transform the integrand into an  $x$  derivative using identity (1.2) for  $\bar{n} = 1$ ,  $\alpha = 1/2$ . This trivializes the integration over  $x$ —albeit at the price of an extra integration. Let us introduce

$$K_{ij} = \int_0^1 \frac{dx}{D^{(i,j)(k)}(x) + \xi^2 + \rho^2 - i\lambda}.
 \tag{3.27}$$

With the help of Eq. (2.26),  $K_{ij}$  reads:

$$K_{ij} = \frac{2}{B(1, 1/2)} \int_0^{+\infty} d\sigma \int_0^1 dx \frac{1}{(D^{(i,j)(k)}(x) + \xi^2 + \rho^2 + \sigma^2 - i\lambda)^{3/2}}.
 \tag{3.28}$$

Using identity (1.2), we get:

$$\begin{aligned}
 K_{ij} &= -\frac{2}{B(1, 1/2)} \int_0^{+\infty} \frac{d\sigma}{\Delta_1^{(i,j)} - \xi^2 - \rho^2 - \sigma^2 + i\lambda} \\
 &\times \int_0^1 dx \frac{\partial}{\partial x} \left[ \frac{x - (G^{(i,j)(k)})^{-1} V^{(i,j)(k)}}{(D^{(i,j)(k)}(x) + \xi^2 + \rho^2 + \sigma^2 - i\lambda)^{1/2}} \right]
 \end{aligned}
 \tag{3.29}$$

with  $\Delta_1^{\{i,j\}} = (G^{\{i,j\}(k)})^{-1} (V^{\{i,j\}(k)})^2 + C^{\{i,j\}(k)}$ . The trivial integration over  $x$  reads:

$$K_{ij} = -\frac{2}{B(1, 1/2)} \int_0^{+\infty} \frac{d\sigma}{\Delta_1^{\{i,j\}} - \xi^2 - \rho^2 - \sigma^2 + i\lambda} \times \left[ \frac{1 - (G^{\{i,j\}(k)})^{-1} V^{\{i,j\}(k)}}{(D^{\{i,j\}(k)}(1) + \xi^2 + \rho^2 + \sigma^2 - i\lambda)^{1/2}} + \frac{(G^{\{i,j\}(k)})^{-1} V^{\{i,j\}(k)}}{(D^{\{i,j\}(k)}(0) + \xi^2 + \rho^2 + \sigma^2 - i\lambda)^{1/2}} \right]. \quad (3.30)$$

We recognize<sup>13</sup>:

$$\begin{aligned} \Delta_1^{\{i,j\}} &= -\frac{\det(\mathcal{S}^{\{i,j\}})}{\det(G^{\{i,j\}})} \\ (G^{\{i,j\}(k)})^{-1} \cdot V^{\{i,j\}(k)} &= -\frac{\bar{b}_l^{\{i,j\}}}{\det(G^{\{i,j\}})}, \quad l \in S_4 \setminus \{i, j, k\} \\ 1 - (G^{\{i,j\}(k)})^{-1} \cdot V^{\{i,j\}(k)} &= -\frac{\bar{b}_k^{\{i,j\}}}{\det(G^{\{i,j\}})}. \end{aligned} \quad (3.31)$$

Furthermore,  $D^{\{i,j\}(k)}(0)$  and  $D^{\{i,j\}(k)}(1)$  are proportional to internal masses squared:

$$\begin{aligned} D^{\{i,j\}(k)}(1) &= 2m_l^2 \equiv \tilde{D}_{ijk} \\ D^{\{i,j\}(k)}(0) &= 2m_k^2 \equiv \tilde{D}_{ijl} \end{aligned} \quad (3.32)$$

for  $k \in S_4 \setminus \{i, j\}$  and  $l \in S_4 \setminus \{i, j, k\}$ . Since  $D^{\{i,j\}(k)}(0) = D^{\{i,j\}(l)}(1)$ ,  $\tilde{D}_{ijk}$  is completely symmetric in all three mutually distinct indices  $i, j, k$ . We thus rewrite  $K_{ij}$  as a sum over the  $\bar{b}_k^{\{i,j\}}$ :

$$K_{ij} = \frac{2}{B(1, 1/2)} \sum_{k \in S_4 \setminus \{i,j\}} \frac{\bar{b}_k^{\{i,j\}}}{\det(G^{\{i,j\}})} \times \int_0^{+\infty} \frac{d\sigma}{(\Delta_1^{\{i,j\}} - \xi^2 - \rho^2 - \sigma^2 + i\lambda) (\tilde{D}_{ijk} + \xi^2 + \rho^2 + \sigma^2 - i\lambda)^{1/2}}. \quad (3.33)$$

By substitution of Eq. (3.33) into Eq. (3.26)  $I_4^4$  reads:

$$I_4^4 = \sum_{i \in S_4} \sum_{j \in S_4 \setminus \{i\}} \sum_{k \in S_4 \setminus \{i,j\}} \frac{\bar{b}_i}{\det(G)} \frac{\bar{b}_j^{\{i\}}}{\det(G^{\{i\}})} \frac{\bar{b}_k^{\{i,j\}}}{\det(G^{\{i,j\}})} L_4^4(\Delta_3, \Delta_2^{\{i\}}, \Delta_1^{\{i,j\}}, \tilde{D}_{ijk}) \quad (3.34)$$

with

$$\begin{aligned} L_4^4(\Delta_3, \Delta_2^{\{i\}}, \Delta_1^{\{i,j\}}, \tilde{D}_{ijk}) &= \kappa \int_0^{+\infty} \frac{d\xi}{(\xi^2 - \Delta_3 - i\lambda)} \int_0^{+\infty} \frac{d\rho}{(\rho^2 + \xi^2 - \Delta_2^{\{i\}} - i\lambda)} \\ &\times \int_0^{+\infty} \frac{d\sigma}{(\sigma^2 + \rho^2 + \xi^2 - \Delta_1^{\{i,j\}} - i\lambda) (\sigma^2 + \rho^2 + \xi^2 + \tilde{D}_{ijk} - i\lambda)^{1/2}} \\ \kappa &= \frac{16}{3 B(2, 1/2) B(3/2, 1/2) B(1, 1/2)}. \end{aligned} \quad (3.35)$$

<sup>13</sup> “ $\det(G^{\{i,j\}})$ ” is merely a fancy notation to keep some unity in formulas, as  $G^{\{i,j\}}$  reduces to one single scalar.

In summary, steps 1 to 3 traded three nested integrals over the initial Feynman parameters spanning the 3D simplex for three nested integrals over the real positive half line. The procedure generated the natural appearance of familiar ingredients of the algebraic reduction: this has been the main benefit so far. We also note that the r.h.s. of Eq. (3.34) is manifestly independent of the successive choices made to single out lines and columns in the kinematic matrix  $\mathcal{S}$  and its subsequent pinchings. We now have to compute the triple integral on the first octant.

### 3.4. Step 4

Let us consider  $L_4^4(\Delta_3, \Delta_2^{\{i\}}, \Delta_1^{\{i,j\}}, \tilde{D}_{ijk})$  and focus on the real mass case.  $\Delta_3, \Delta_2^{\{i\}}, \Delta_1^{\{i,j\}}$ , and  $\tilde{D}_{ijk}$  being all real, and thus all imaginary parts come from the  $-i\lambda$  prescription, the situation is the simplest possible. In contrast, the extension to the general complex mass case leads to a branching of cases, which, albeit systematic, is more profuse than what happened for the three-point function. We therefore postpone the extension to the general complex mass case to a separate article. We split step 4 itself into a succession of substeps that will make the further complex mass extension easier.

**a.** Let us first consider the nested  $\sigma$  integral in  $L_4^4(\Delta_3, \Delta_2^{\{i\}}, \Delta_1^{\{i,j\}}, \tilde{D}_{ijk})$  given by:

$$M_1(\xi^2 + \rho^2) = \int_0^{+\infty} \frac{d\sigma}{(\sigma^2 + \xi^2 + \rho^2 - \Delta_1^{\{i,j\}} - i\lambda)(\sigma^2 + \xi^2 + \rho^2 + \tilde{D}_{ijk} - i\lambda)^{1/2}}. \quad (3.36)$$

We use identity (D.8) with  $A = \xi^2 + \rho^2 - \Delta_1^{\{i,j\}} - i\lambda$  and  $B = \xi^2 + \rho^2 + \tilde{D}_{ijk} - i\lambda$  to transform the r.h.s. of Eq. (3.36) into an integral of only one factor:

$$M_1(\xi^2 + \rho^2) = \int_0^1 \frac{dz}{z^2 (\tilde{D}_{ijk} + \Delta_1^{\{i,j\}}) + \xi^2 + \rho^2 - \Delta_1^{\{i,j\}} - i\lambda}. \quad (3.37)$$

**b.** Next we nest  $M_1(\xi^2 + \rho^2)$  in the  $\rho$  integration, which reads:

$$\begin{aligned} M_2(\xi^2) &= \int_0^{+\infty} \frac{d\rho}{\rho^2 + \xi^2 - \Delta_2^{\{i\}} - i\lambda} M_1(\xi^2 + \rho^2) \\ &= \int_0^1 dz \int_0^{+\infty} \frac{d\rho}{(\rho^2 + \xi^2 - \Delta_2^{\{i\}} - i\lambda) ( \rho^2 + z^2 (\tilde{D}_{ijk} + \Delta_1^{\{i,j\}}) + \xi^2 - \Delta_1^{\{i,j\}} - i\lambda )}. \end{aligned} \quad (3.38)$$

The  $\rho$  integration is performed first, using partial fraction decomposition and Eq. (2.26). We get:

$$\begin{aligned} M_2(\xi^2) &= \frac{1}{2} B(1/2, 1/2) \\ &\times \int_0^1 \frac{dz}{z^2 (\tilde{D}_{ijk} + \Delta_1^{\{i,j\}}) + (\Delta_2^{\{i\}} - \Delta_1^{\{i,j\}})} \left[ \frac{1}{(\xi^2 - \Delta_2^{\{i\}} - i\lambda)^{1/2}} \right. \\ &\quad \left. - \frac{1}{(z^2 (\tilde{D}_{ijk} + \Delta_1^{\{i,j\}}) + (\xi^2 - \Delta_1^{\{i,j\}} - i\lambda))^{1/2}} \right]. \end{aligned} \quad (3.39)$$

**c.** We then substitute  $M_2(\xi^2)$  given by Eq. (3.39) into the remaining  $\xi$  integration providing  $L_4^4(\Delta_3, \Delta_2^{\{i\}}, \Delta_1^{\{i,j\}}, \tilde{D}_{ijk})$ :

$$L_4^4(\Delta_3, \Delta_2^{\{i\}}, \Delta_1^{\{i,j\}}, \tilde{D}_{ijk}) = \kappa \int_0^{+\infty} \frac{d\xi}{\xi^2 - \Delta_3 - i\lambda} M_2(\xi^2). \quad (3.40)$$

Normalization factors greatly simplify in  $L_4^4(\Delta_3, \Delta_2^{\{i\}}, \Delta_1^{\{i,j\}}, \tilde{D}_{ijk})$  as

$$\frac{1}{2} B(1/2, 1/2) \kappa = 2$$

and we exchange the integrations over  $\xi$  and  $z$  in Eq. (3.40):

$$L_4^4(\Delta_3, \Delta_2^{\{i\}}, \Delta_1^{\{i,j\}}, \tilde{D}_{ijk}) = 2 \int_0^1 \frac{dz}{z^2 (\tilde{D}_{ijk} + \Delta_1^{\{i,j\}}) + (\Delta_2^{\{i\}} - \Delta_1^{\{i,j\}})} \tag{3.41}$$

$$\times \int_0^{+\infty} \frac{d\xi}{\xi^2 - \Delta_3 - i\lambda} \left[ \frac{1}{(\xi^2 - \Delta_2^{\{i\}} - i\lambda)^{1/2}} - \frac{1}{(z^2 (\tilde{D}_{ijk} + \Delta_1^{\{i,j\}}) + (\xi^2 - \Delta_1^{\{i,j\}} - i\lambda))^{1/2}} \right].$$

d. Keeping  $z$  fixed, the nested  $\xi$  integral in the last two lines of Eq. (3.41) can be split into two integrals, each of which being of the same type as the one in Eq. (3.36). Each integral can thus be recast in a form similar to Eq. (3.37). For convenience we introduce:

$$\left. \begin{aligned} P_{ijk} &= \tilde{D}_{ijk} + \Delta_1^{\{i,j\}} \\ R_{ij} &= \Delta_2^{\{i\}} - \Delta_1^{\{i,j\}} \\ Q_i &= \Delta_3 - \Delta_2^{\{i\}} \\ T &= -\Delta_3 \end{aligned} \right\} \Leftrightarrow \left\{ \begin{aligned} P_{ijk} + R_{ij} + Q_i + T &= \tilde{D}_{ijk} \\ R_{ij} + Q_i + T &= -\Delta_1^{\{i,j\}} \\ Q_i + T &= -\Delta_2^{\{i\}} \\ T &= -\Delta_3 \end{aligned} \right. \tag{3.42}$$

The recasting reads:

$$L_4^4(\Delta_3, \Delta_2^{\{i\}}, \Delta_1^{\{i,j\}}, \tilde{D}_{ijk}) = 2 \int_0^1 \frac{dz}{z^2 P_{ijk} + R_{ij}} \tag{3.43}$$

$$\times \left[ \int_0^1 \frac{dy}{y^2 (Q_i + T) + (1 - y^2) T - i\lambda} - \int_0^1 \frac{dy}{y^2 (z^2 P_{ijk} + R_{ij} + Q_i + T) + (1 - y^2) T - i\lambda} \right].$$

e. We trade  $z$  for  $u = yz$  so that  $L_4^4(\Delta_3, \Delta_2^{\{i\}}, \Delta_1^{\{i,j\}}, \tilde{D}_{ijk})$  takes the form:

$$L_4^4(\Delta_3, \Delta_2^{\{i\}}, \Delta_1^{\{i,j\}}, \tilde{D}_{ijk}) = -2 \int_0^1 du \int_u^1 \frac{dy y}{u^2 P_{ijk} + y^2 R_{ij}} \tag{3.44}$$

$$\times \left[ \frac{1}{u^2 P_{ijk} + y^2 (R_{ij} + Q_i) + T - i\lambda} - \frac{1}{y^2 Q_i + T - i\lambda} \right].$$

f. We trade  $y$  for  $x = y^2$ :

$$L_4^4(\Delta_3, \Delta_2^{\{i\}}, \Delta_1^{\{i,j\}}, \tilde{D}_{ijk}) = - \int_0^1 du \int_{u^2}^1 \frac{dx}{u^2 P_{ijk} + x R_{ij}} \tag{3.45}$$

$$\times \left[ \frac{1}{u^2 P_{ijk} + x (R_{ij} + Q_i) + T - i\lambda} - \frac{1}{x Q_i + T - i\lambda} \right].$$

g. We perform a partial fraction decomposition w.r.t.  $x$  in the integrand of Eq. (3.45):

$$\begin{aligned} & \frac{1}{u^2 P_{ijk} + x R_{ij}} \left[ \frac{1}{u^2 P_{ijk} + x (R_{ij} + Q_i) + T - i\lambda} - \frac{1}{x Q_i + T - i\lambda} \right] \\ &= \frac{1}{u^2 P_{ijk} Q_i - R_{ij} (T - i\lambda)} \left[ \frac{R_{ij} + Q_i}{u^2 P_{ijk} + x (R_{ij} + Q_i) + T - i\lambda} - \frac{Q_i}{x Q_i + T - i\lambda} \right]. \end{aligned} \quad (3.46)$$

h. Lastly the  $x$  integration is readily performed and we finally get:

$$\begin{aligned} L_4^4(\Delta_3, \Delta_2^{\{i\}}, \Delta_1^{\{i,j\}}, \tilde{D}_{ijk}) &= - \int_0^1 \frac{du}{u^2 P_{ijk} Q_i - R_{ij} (T - i\lambda)} \\ &\quad \times \left[ \ln \left( \frac{u^2 P_{ijk} + (R_{ij} + Q_i + T) - i\lambda}{u^2 (P_{ijk} + R_{ij} + Q_i) + T - i\lambda} \right) \right. \\ &\quad \left. - \ln \left( \frac{Q_i + T - i\lambda}{u^2 Q_i + T - i\lambda} \right) \right]. \end{aligned} \quad (3.47)$$

Using the following identities:

$$\begin{aligned} \bar{b}_i^2 &= - \det(\mathcal{S}^{\{i\}}) \det(G) - \det(\mathcal{S}) \det(G^{\{i\}}) \\ \bar{b}_j^{\{i\}2} &= \det(\mathcal{S}^{\{i,j\}}) \det(G^{\{i\}}) + \det(\mathcal{S}^{\{i\}}) \det(G^{\{i,j\}}) \\ \bar{b}_k^{\{i,j\}2} &= \tilde{D}_{ijk} \det(G^{\{i,j\}}) - \det(\mathcal{S}^{\{i,j\}}) \end{aligned} \quad (3.48)$$

(see Appendix A.2 on Jacobi identities for determinants in Ref. [27]), the coefficients  $P_{ijk}$ ,  $R_{ij}$ ,  $Q_i$ , and  $T$  defined in Eqs. (3.42) can be recast in terms of the  $\bar{b}_i$ ,  $\bar{b}_j^{\{i\}}$  and  $\bar{b}_k^{\{i,j\}}$  using the group of equations (3.48):

$$\begin{aligned} P_{ijk} &= \frac{\bar{b}_k^{\{i,j\}2}}{\det(G^{\{i,j\}})} \\ R_{ij} &= \frac{\bar{b}_j^{\{i\}2}}{\det(G^{\{i,j\}}) \det(G^{\{i\}})} \\ Q_i &= \frac{\bar{b}_i^2}{\det(G^{\{i\}}) \det(G)}. \end{aligned}$$

With  $\Delta_3$ ,  $\Delta_2^{\{i\}}$ ,  $\Delta_1^{\{i,j\}}$ , and  $\tilde{D}_{ijk}$ , given by the groups of Eqs. (3.15), (3.23), (3.31), and (3.32) respectively, Eq. (3.47) then translates into:

$$\begin{aligned} L_4^4(\Delta_3, \Delta_2^{\{i\}}, \Delta_1^{\{i,j\}}, \tilde{D}_{ijk}) &= - \det(G) \det(G^{\{i\}}) \det(G^{\{i,j\}}) \\ &\quad \times \int_0^1 \frac{du}{u^2 \bar{b}_i^2 \bar{b}_k^{\{i,j\}2} - \bar{b}_j^{\{i\}2} (\det(\mathcal{S}) - i\lambda \text{sign}(\det(G)))} \\ &\quad \times \left[ \ln \left( \frac{u^2 \tilde{D}_{ijk} - (1 - u^2) \Delta_1^{\{i,j\}} - i\lambda}{u^2 \tilde{D}_{ijk} - (1 - u^2) \Delta_3 - i\lambda} \right) \right. \\ &\quad \left. - \ln \left( \frac{-\Delta_2^{\{i\}} - i\lambda}{-u^2 \Delta_2^{\{i\}} - (1 - u^2) \Delta_3 - i\lambda} \right) \right], \end{aligned} \quad (3.49)$$

which is useful in view of making practical numerical implementation.

Let us note that the apparent poles in the integrand of the r.h.s. of Eq. (3.47) are fake as their residues vanish<sup>14</sup>. Furthermore, in the real mass case at hand,  $(u^2 P_{ijk} + R_{ij} + Q_i + T)$ ,  $(u^2 (P_{ijk} + R_{ij} + Q_i) + T)$ , and  $(u^2 Q_i + T)$  are all real when  $u$  spans  $[0, 1]$ : each logarithm of ratios in Eq. (3.47) can be safely split into a difference of two logarithms, the arguments of which keeping an (infinitesimal) imaginary part of constant (negative) sign as  $u$  spans  $[0, 1]$ . The remaining integration can be performed in closed form in terms of dilogarithms. This computation is provided in Appendix E.

### 3.5. Comparison with Ref. [12] and proliferation of dilogarithms

We compared our result (3.47) with Ref. [12]. Beyond mere numerical comparisons, the respective analytical expressions do not compare easily, and formula (3.47) involves a proliferation of dilogarithms, namely 288 vs. 108 for Ref. [12].

Let us first remind ourselves of a few features of the latter. The method of Ref. [12] linearizes the dependence of the integrand in some integration variables to facilitate two integrations. For this purpose two parameters  $\alpha$  and  $\beta$  are introduced and successively adjusted as roots of some second-degree equations, which generates a fourfold arbitrariness in the procedure. All choices of  $\alpha$  and  $\beta$  are equivalent; they lead to results that look formally the same, and provide the same numerical result. Yet the analytic forms when  $\alpha$  and  $\beta$  are made explicit in terms of the primary parameters depend on the choice made. In contrast our iterative decomposition closely related to algebraic reduction and pinching operations preserves a kind of symmetry—a “democracy” among all possible single pinchings and double pinchings. This symmetry is explicitly broken in Ref. [12] whenever any choice for  $\alpha$  and  $\beta$  is made. Therefore, the comparison of our result shall be carried out with the double symmetrization of Ref. [12] w.r.t. both  $\alpha$  and  $\beta$ , not with Ref. [12] *per se*. This double symmetrization increases the number of terms w.r.t. Ref. [12]. The price to pay for our method compared to Ref. [12] is thus that some extra work shall be carried out to counteract the corresponding proliferation. This already occurred for the three-point function, and we saw that the issue was easily overcome in that case. The situation for the four-point function is more complex and requires some elaboration and we need to carry out the comparison in some more detail.

Consecutively to the choice of  $\alpha$  the calculation of Ref. [12] splits into three contributions depending on the choice of  $\alpha$ —let us call them “ $\alpha$ -sectors”. In each  $\alpha$ -sector, the subsequent choice of  $\beta$  (which depends both on the sector considered and explicitly on  $\alpha$ ) leads again to three contributions—or “ $\beta$ -subsectors”. This provides  $I_4^4$  as a sum of 1D integrals. In each of the  $\beta$ -subsectors the integrand is given by a logarithm of some rational fraction, divided by some second-degree polynomial. The rational fractions in the logarithms are of the form  $(\mathcal{L} - i\lambda)/(\mathcal{Q} - i\lambda)$  in which  $\mathcal{L}$  and  $\mathcal{Q}$  are first- and second-degree polynomials respectively. After partial fraction decomposition of these denominators the values of the logarithms at each pole are subtracted so that the poles are made manifestly fake in each term separately. Whereas the  $\mathcal{Q}$  are independent of the choices of  $\alpha$  and  $\beta$ , the  $\mathcal{L}$  are independent of the choice of  $\beta$  but do depend explicitly on  $\alpha$ . The locations of the fake poles in each  $\alpha$ -sector depend on  $\beta$  explicitly but not on  $\alpha$ , yet there are relations between different poles from different  $\alpha$ -sectors. Each one of the three  $\alpha$ -sectors in Ref. [12] leads to three  $\mathcal{L}$  and three  $\mathcal{Q}$ ; however, this does not lead to nine distinct  $\mathcal{L}$  and  $\mathcal{Q}$  overall. Among the nine  $\mathcal{Q}$ , three of them appear

<sup>14</sup> This can be checked explicitly in Eq. (3.47) in which the residue contributions of the two logarithms cancel against each other. These fake poles are an artifact of the partial fraction decomposition (3.46).

twice (among the last two lines in Eq. (6.26) of Ref. [12]) whereas the three others (in the second line in Eq. (6.26)) appear only once. Each of the  $\mathcal{Q}$  happens to correspond to a double-pinching pattern of the box diagram, and indeed there are only six such distinct double pinchings. There is no connection between splitting in sectors and double pinchings, though, and the correspondence between the polynomials  $\mathcal{Q}$  and the double pinchings is identified a posteriori. The  $\mathcal{L}$  happen to coincide whenever the  $\mathcal{Q}$  correspondingly involved in the rational fractions respectively coincide, yet the  $\mathcal{L}$  look “atypical” and have no simple interpretation, e.g., in terms of pinching or otherwise, at least to us.

The terms  $\ln(u^2 P_{ijk} + (R_{ij} + Q_i + T - i\lambda))$  in Eq. (3.47) in our approach coincide with the  $\ln(\mathcal{Q})$  in Ref. [12]. This can be seen by first absorbing  $\bar{b}_k^{i,j}$  by the rescaling of the integration variable  $u$  as  $t = \bar{b}_k^{i,j} u$ , then performing the sum  $\sum_k$  explicitly introducing  $y \in [0, 1]$  such that  $t = -\bar{b}_l^{i,j} - \det(G^{i,j})y$ . However, the pole terms weighting these identical logarithms do not match in any clear way between Eq. (3.47) in our approach and Ref. [12]. Furthermore, the two other logarithms  $\ln(u^2 Q_i + T - i\lambda)$  and  $\ln(u^2 (P_{ijk} + (R_{ij} + Q_i) + T - i\lambda))$  in our approach are atypical and their contributions do not match with those of the atypical  $\ln(\mathcal{L})$  of Ref. [12] in any clear way either.

As for taming the proliferation of dilogarithms in Eq. (3.47) in our approach, the use of the five-dilogarithm Hill identity combined with the property of parity in  $u$  of the involved integrals allows us to reduce the number of dilogarithms by a factor of two, from 288 to 144, still somewhat more than in Ref. [12]: some extra work would still be required to further reduce this number. Note also that the advantage of the method used here is that the roots of the denominators and of the arguments of the logarithms are expressed in terms of the determinants of the Gram matrix and the  $\mathcal{S}$  matrix as well as those of the corresponding pinched matrices. From that, some relations between these different roots can be deduced. These relations could be used to reduce in a systematic way using modern tools like  $G$  symbols the number of dilogarithms. This future work is postponed to a future publication (J. Ph. Guillet et al., manuscript in preparation).

#### 4. Summary and outlook

In this article we have presented a novel approach to the computation of one-loop three- and four-point functions. The method naturally proceeds in terms of algebraic kinematical invariants involved in reduction algorithms and applies to general kinematics beyond the one relevant for one-loop collider processes; it thereby offers a potential application to the calculation of processes beyond one loop using one-loop (generalized)  $N$ -point functions as building blocks. In addition, limiting behaviors in a number of specific regimes with  $\det(\mathcal{S}) \rightarrow 0$  or  $\det(G) \rightarrow 0$  or both simultaneously is presented. They are restricted to cases that do not lead to Landau singularities and that are relevant for NLO kinematics. The treatment of the leftover cases will be postponed to subsequent work if relevant. For the sake of pedagogy, the method was hereby exposed on “ordinary” three- and four-point functions in four dimensions in the real mass case. It can be extended in several respects, and each of these respects will be tackled in two companion papers briefly advertised in what follows.

In the first companion paper we will extend the presented framework to the general complex mass case. Whereas the extension will branch into a profusion of cases, the general philosophy of the approach extends straightforwardly, and the profusion of cases can be tamed and reunified by means of 1D contour integrals in the complex plane encoding analyticity properties. This companion paper will also provide a formula to express  $\text{Im}(\det(\mathcal{S}))$  in terms of widths.

In the second companion paper we will extend the presented framework to the case where some vanishing internal masses cause infrared soft and/or collinear divergences. The method extends in a straightforward way, once a few intermediate steps and tools are appropriately adapted.

The proliferation of dilogarithms in the expression of the four-point function computed in closed form with the present method requires some extra work to be better apprehended, in order to counteract it. This issue will be addressed in a future article.

This method can of course also be applied to compute  $N$ -point one-loop functions with any  $N \geq 5$  but this is pointless because in these cases the first step boils down to recovering well known formulae that reduce the  $N$ -point one-loop function to one-loop ones with fewer points [15–17,34].

The last goal is to provide the generalized one-loop building blocks entering as integrands in the computation of two-loop three- and four-point functions by means of an extra numerical double integration.

### In memoriam

Various ideas and techniques used in this work were initiated by Prof. Shimizu after a visit to LAPTh. He explained to us his ideas about the numerical computation of scalar two-loop three- and four-point functions, he shared his notes partly in English, partly in Japanese with us and he encouraged us to push this project forward. J.Ph.G. would like to thank Shimizu-sensei for giving him a taste of Japanese culture and for his kindness.

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### Appendix A. A Stokes-type identity

In this appendix, we derive the master identity (1.2) that enables us to perform one integral trivially. Let us consider the column  $\bar{n}$ -vector

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{\bar{n}} \end{bmatrix}$$

and the polynomial  $D$  of  $\bar{n}$  variables defined by:

$$D = X^T \cdot G \cdot X - 2V^T \cdot X - C \quad (\text{A.1})$$

where  $G$  is a symmetric  $\bar{n} \times \bar{n}$  matrix and  $V$  a column  $\bar{n}$ -vector. The subscript  $T$  stands for the transpose. Throughout this appendix we assume the matrix  $G$  to be invertible. Let  $\nabla$  stand for:

$$\nabla = \begin{bmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \vdots \\ \frac{\partial}{\partial x_{\bar{n}}} \end{bmatrix}$$

so that  $\nabla f$  is the gradient of the scalar field  $f$  whereas  $\nabla^T \cdot F$  is the divergence of the  $\bar{n}$ -vector field  $F$ . The gradient of  $D$  is given by:

$$\nabla D = 2 (G \cdot X - V)$$

so that

$$\nabla^T \cdot \left( \frac{X - G^{-1} \cdot V}{D^\alpha} \right) = \frac{\bar{n}}{D^\alpha} - \frac{2\alpha}{D^{\alpha+1}} [(G \cdot X - V)^T \cdot (X - G^{-1} \cdot V)]. \quad (\text{A.2})$$

The square bracket in the r.h.s. of Eq. (A.2) may be written:

$$(G \cdot X - V)^T \cdot (X - G^{-1} \cdot V) = D + \Delta_{\bar{n}} \quad (\text{A.3})$$

$$\Delta_{\bar{n}} = V^T \cdot G^{-1} \cdot V + C. \quad (\text{A.4})$$

Substituting Eqs. (A.3), (A.4) into Eq. (A.2) thus leads to

$$\nabla^T \cdot \left( \frac{X - G^{-1} \cdot V}{D^\alpha} \right) = \frac{\bar{n} - 2\alpha}{D^\alpha} - \frac{2\alpha \Delta_{\bar{n}}}{D^{\alpha+1}} \quad (\text{A.5})$$

from which identity (1.2) immediately follows whenever  $\Delta_{\bar{n}} \neq 0$ . Let us remark in passing that

$$\Delta_{\bar{n}} = (-1)^{\bar{n}} \frac{\det(S)}{\det(G)}$$

as shown in Appendix C.

## Appendix B. How to tune powers of denominators

The following identity is used with real  $\mu, \nu$  such that  $\mu > 1/\nu > 0$  to tune powers in denominators:

$$\int_0^\infty \frac{d\xi}{(D + \xi^\nu)^\mu} = \frac{1}{\nu} B\left(\mu - \frac{1}{\nu}, \frac{1}{\nu}\right) \frac{1}{D^{\mu-1/\nu}} \quad (\text{B.1})$$

where  $B(x, y)$  is the Euler Beta function defined by

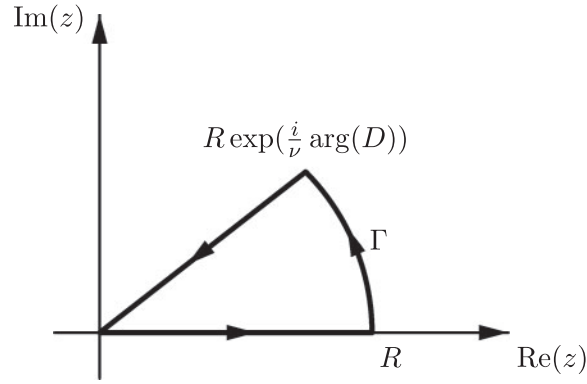
$$B(x, y) = \frac{\Gamma(x) \Gamma(y)}{\Gamma(x + y)}.$$

The integral in the l.h.s. of Eq. (B.1) is convergent for  $\mu > 1/\nu > 0$ . Identity (B.1) is easily established for  $D$  real writing

$$\int_0^\infty \frac{d\xi}{(D + \xi^\nu)^\mu} = \frac{1}{\Gamma(\mu)} \int_0^\infty d\xi \int_0^{+\infty} du u^{\mu-1} e^{-u(D+\xi^\nu)} \quad (\text{B.2})$$

then making the change of variable  $\xi = v^{1/\nu}$  and performing the integration over  $v$  first. For  $D$  complex, identity (B.1) still holds provided  $\nu \mu > 1$ . Indeed, we may equivalently

- either analytically continue the r.h.s. of identity (B.1) to  $|\arg(D)| < \pi$
- or consider the integral of  $F(\xi) = (D + \xi^\nu)^{-\mu}$  along the closed contour of integration in the complex  $\xi$ -plane defined by Fig. B.1. According to the Cauchy theorem,



**Fig. B.1.** Integration contour  $\mathcal{C}$  in the  $z$  complex plane.

$$\begin{aligned}
 0 &= \oint_{\mathcal{C}} d\xi F(\xi) \\
 &= \int_0^R d\xi F(\xi) + \int_{\Gamma} d\xi F(\xi) + \int_{R e^{i \arg(D)/v}}^0 d\xi F(\xi). \tag{B.3}
 \end{aligned}$$

When  $R \rightarrow +\infty$ , the first integral of Eq. (B.3) leads to the sought-after one, the second integral is  $\mathcal{O}(R^{1-\nu\mu}) \rightarrow 0$  provided  $\nu > 1/\mu > 0$ , and the path of the third integral may be parametrized by  $\xi = \rho D^{1/\nu}$ ,  $0 \leq \rho \leq R|D|^{-1/\nu}$  so as to get:

$$\begin{aligned}
 \int_{R e^{i/v \arg(D)}}^0 d\xi F(\xi) &= - \int_0^{R|D|^{-1/\nu}} d\rho \frac{d\rho D^{1/\nu}}{[(\rho^\nu + 1)D]^\mu} \\
 &\rightarrow - \frac{1}{D^{\mu-1/\nu}} \int_0^\infty \frac{d\rho}{(\rho^\nu + 1)^\mu}. \tag{B.4}
 \end{aligned}$$

The integral in the r.h.s. of Eq. (B.4) is namely the one just computed above for  $D = 1$ , hence:

$$\int_0^\infty \frac{d\xi}{(D + \xi^\nu)^\mu} = \frac{1}{\nu} B\left(\mu - \frac{1}{\nu}, \frac{1}{\nu}\right) \frac{1}{D^{\mu-1/\nu}}. \tag{B.5}$$

### Appendix C. $b$ and $B$ in terms of $G^{(a)}$ and $V^{(a)}$

In this appendix, we show how to express the  $b$  coefficients in term of  $G^{(a)}$  and  $V^{(a)}$ . We assume that the dimension of the  $\mathcal{S}$  matrix is  $N$  and we single out its last line and column to build the associated Gram matrix  $G^{(N)}$ . We introduce the  $b_i$  [30], which are solutions of the equation:

$$\mathcal{S}.b = e \tag{C.1}$$

with

$$b = \begin{bmatrix} b_1 \\ \vdots \\ b_N \end{bmatrix} \tag{C.2}$$

$$e = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}. \tag{C.3}$$

The  $\mathcal{S}$  matrix can be related to the block matrix  $\widehat{\mathcal{S}}^{(N)}$  whose blocks are proportional to the Gram matrix  $G^{(N)}$ , to the vector  $V^{(N)}$ , and to the scalar  $\mathcal{S}_{NN}$  defined in Sects. 2 and 3:

$$\widehat{\mathcal{S}}^{(N)} = \left[ \begin{array}{c|c} -G^{(N)} & V^{(N)} \\ \hline V^{(N)T} & \mathcal{S}_{NN} \end{array} \right]. \quad (\text{C.4})$$

The subtraction of column and line  $N$  leading from  $\mathcal{S}$  to  $\widehat{\mathcal{S}}^{(N)}$  corresponds to the following matricial product:

$$\widehat{\mathcal{S}}^{(N)} = L_{-N} \cdot \mathcal{S} \cdot C_{-N} \quad (\text{C.5})$$

with

$$L_{-N} = \left[ \begin{array}{ccc|c} \vdots & & & -1 \\ \cdots & \mathbf{1}_{N-1} & \cdots & \vdots \\ \vdots & & & -1 \\ \hline & & & \\ 0 & \cdots & 0 & 1 \end{array} \right], \quad C_{-N} = L_{-N}^T. \quad (\text{C.6})$$

Left multiplication of  $\mathcal{S}$  by  $L_{-N}$  subtracts line  $N$  of  $\mathcal{S}$  from each of the lines 1 to  $N - 1$  of  $\mathcal{S}$ . Likewise right multiplication of  $\mathcal{S}$  by  $C_{-N}$  subtracts column  $N$  of  $\mathcal{S}$  from each of the columns 1 to  $N - 1$ . The matrix  $L_{-N}$  is invertible ( $\det(L_{-N}) = 1$ ) and  $L_{-N}^{-1}$  is the matrix  $L_{+N}$  obtained from  $L_{-N}$  by replacing all “ $-1$ ” by “ $+1$ ” in the last column, so that left multiplication by  $L_{+N}$  adds line  $N$  to each of the lines 1 to  $N - 1$ , and likewise with  $C_{+N} = C_{-N}^{-1}$  vs.  $C_{-N}$ .

From Eq. (C.5), it is clear that the two matrices  $\mathcal{S}$  and  $\widehat{\mathcal{S}}^{(N)}$  have the same determinant. So  $\det(\mathcal{S})$  can be computed by expanding with respect to the last line of Eq. (C.4); this yields:

$$\det(\mathcal{S}) = \mathcal{S}_{NN} \det(-G^{(N)}) + \sum_{j=1}^{N-1} (-1)^{N+j} V_j^{(N)} \overline{G}_j^{(N)}. \quad (\text{C.7})$$

The quantity

$$\overline{G}_j^{(N)} = \left| \begin{array}{c} -G_{\text{column } j \text{ suppressed}}^{(N)} \\ \vdots \\ V^{(N)} \end{array} \right|$$

is in its turn expanded w.r.t. to its last column  $V^{(N)}$ :

$$\overline{G}_j^{(N)} = (-1)^{i-1} \sum_{i=1}^{N-1} \text{Cof}[G^{(N)}]_{ij} V_i^{(N)} \quad (\text{C.8})$$

where  $\text{Cof}[G^{(N)}]$  is the matrix of cofactors of  $G^{(N)}$ . Substituting Eq. (C.8) into Eq. (C.7) we get:

$$\det(\mathcal{S}) = (-1)^{N-1} \left\{ \mathcal{S}_{NN} \det(G) + V^{(N)T} \cdot \text{Cof}[G^{(N)}] \cdot V^{(N)} \right\} \quad (\text{C.9})$$

or, in other words,

$$\mathcal{S}_{NN} + V^{(N)T} \cdot (G^{(N)})^{-1} \cdot V^{(N)} = (-1)^{N-1} \frac{\det(\mathcal{S})}{\det(G)}. \quad (\text{C.10})$$

This is to compare with Eq. (A.4), reminding ourselves that the constant term  $C$  of the polynomial  $D$  is precisely  $S_{NN}$  (cf. Eq. (2.5) or Eq. (3.3)), so we conclude that:

$$\Delta_{\bar{n}} = (-1)^{N-1} \frac{\det(\mathcal{S})}{\det(G)}. \tag{C.11}$$

Equation (C.1) can be reformulated in terms of  $\widehat{\mathcal{S}}^{(N)}$ . Inverting Eq. (C.5), we get:

$$\widehat{\mathcal{S}}^{(N)} \cdot \widehat{b} = \widehat{e} \tag{C.12}$$

with

$$\widehat{b} \equiv C_{+N} \cdot b = \begin{bmatrix} \beta^{(N)} \\ \text{---} \\ B \end{bmatrix} \quad \text{and} \quad \widehat{e} \equiv L_{-N} \cdot e = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \text{---} \\ 1 \end{bmatrix} \tag{C.13}$$

where  $\beta^{(N)}$  is the  $N - 1$ -vector:

$$\beta^{(N)} = \begin{bmatrix} b_1 \\ \vdots \\ b_{N-1} \end{bmatrix} \tag{C.14}$$

and

$$B = \sum_{i=1}^N b_i. \tag{C.15}$$

We will discuss the solutions of Eq. (C.12) in the following cases: 1) the determinants of the matrices  $\mathcal{S}$  and  $G^{(N)}$  do not vanish, 2) the determinant of the matrix  $\mathcal{S}$  is different from 0 but the determinant of the Gram matrix  $G^{(N)}$  vanishes, and 3) both determinants of the  $\mathcal{S}$  and  $G^{(N)}$  matrices vanish.

*Appendix C.1. Case  $\det(\mathcal{S}) \neq 0$  and  $\det(G) \neq 0$*

Let us parametrize  $(\widehat{\mathcal{S}}^{(N)})^{-1}$  as

$$(\widehat{\mathcal{S}}^{(N)})^{-1} = \left[ \begin{array}{c|c} Q & W \\ \text{---} & \text{---} \\ W^T & U \end{array} \right]. \tag{C.16}$$

The solution of Eq. (C.12) then reads:

$$\beta^{(N)} = W \tag{C.17}$$

$$B = U. \tag{C.18}$$

In order to compute the different parameters, Eq. (C.16) is substituted in the equation  $\widehat{\mathcal{S}}^{(N)} \cdot (\widehat{\mathcal{S}}^{(N)})^{-1} = \mathbb{1}_N$ . We get the following linear system:

$$- G^{(N)} \cdot Q + V^{(N)} \cdot W^T = \mathbb{1}_{N-1} \tag{C.19}$$

$$-G^{(N)} \cdot W + V^{(N)} U = 0 \quad (\text{C.20})$$

$$Q \cdot V^{(N)} + S_{NN} W = 0 \quad (\text{C.21})$$

$$V^{(N)T} \cdot W + S_{NN} U = 1. \quad (\text{C.22})$$

Assuming that  $G^{(N)}$  is invertible, Eq. (C.20) is solved into

$$W = U (G^{(N)})^{-1} \cdot V^{(N)}. \quad (\text{C.23})$$

Substituting Eq. (C.23) into Eq. (C.22) we get:

$$1 = U \left[ V^{(N)T} \cdot (G^{(N)})^{-1} \cdot V^{(N)} + S_{NN} \right], \quad (\text{C.24})$$

i.e.,

$$U = \frac{1}{\Delta_{\bar{n}}} = (-1)^{N-1} \frac{\det(G)}{\det(S)}. \quad (\text{C.25})$$

At this level, the solution of Eq. (C.12) is fully determined. Nevertheless, it is interesting to continue and compute the remaining parameter  $Q$ . For that, let us substitute Eqs. (C.23) and (C.25) into Eq. (C.19), which becomes:

$$-G^{(N)} \cdot Q = \mathbb{1}_{N-1} - U V^{(N)} \cdot V^{(N)T} \cdot (G^{(N)})^{-1}, \quad (\text{C.26})$$

i.e.,

$$Q = -(G^{(N)})^{-1} + U \left[ (G^{(N)})^{-1} \cdot V^{(N)} \right] \cdot \left[ (G^{(N)})^{-1} \cdot V^{(N)} \right]^T. \quad (\text{C.27})$$

As a consistency check, we substitute Eq. (C.27) into the l.h.s. of Eq. (C.21), which becomes:

$$\begin{aligned} \text{l.h.s. of Eq. (C.21)} &= -(G^{(N)})^{-1} \cdot V^{(N)} \left\{ 1 - U \left[ V^{(N)T} \cdot (G^{(N)})^{-1} \cdot V^{(N)} + S_{NN} \right] \right\} \\ &= -(G^{(N)})^{-1} \cdot V^{(N)} \underbrace{\left[ 1 - U (-1)^{N-1} \frac{\det(S)}{\det(G)} \right]}_0 = 0, \end{aligned} \quad (\text{C.28})$$

i.e., Eq. (C.21) is indeed fulfilled.

In a nutshell, to solve Eq. (C.12), we use results (C.17), (C.18), (C.23), (C.25) and read:

$$\beta^{(N)} = B (G^{(N)})^{-1} \cdot V^{(N)}, \quad (\text{C.29})$$

$$B = (-1)^{N-1} \frac{\det(G)}{\det(S)}. \quad (\text{C.30})$$

Note that the identity (C.30) has already been given in Refs. [15,30]. In term of  $\bar{b}_i = b_i \det \mathcal{S}$ , the solutions given by Eqs. (C.29) and (C.30) become<sup>15</sup>:

$$\bar{b}_{j \neq N} = (-1)^{N-1} \det(G) \left( \left[ G^{(N)} \right]^{-1} \cdot V^{(N)} \right)_j, \quad \bar{b}_N = (-1)^{N-1} \det(G) - \sum_{j \neq N} \bar{b}_j, \quad (\text{C.31})$$

which is indeed the results used in Sects. 2 and 3.

Although Eqs. (C.11) and (C.31) provide the main results of this appendix, we will discuss now the peculiar cases where some determinants vanish limiting ourselves to cases that do not lead to Landau singularities and whose associated kinematics is relevant to colliders.

### Appendix C.2. Case $\det(\mathcal{S}) \neq 0$ and $\det(G) = 0$

We now assume  $\det(G) = 0$ ; therefore we can no longer proceed as in Sect. C.1 to solve Eqs. (C.19) and (C.20). The system {(C.20),(C.22)} is identical to the resolution of  $\mathcal{S} \cdot b = e$  in terms of  $G^{(N)}$  and  $V^{(N)}$  in the case at hand. When  $\det(\mathcal{S}) \neq 0$  and  $\det(G) = 0$ ,  $B$  vanishes; cf. Eq. (C.30), from which we infer

$$U = 0. \quad (\text{C.32})$$

To solve Eq. (C.20)  $W$  shall thus belong to  $\text{Ker } G^{(N)}$ . In the case at hand  $\text{Ker } G^{(N)}$  is  $1D^{16}$  and it is the eigenspace of  $\text{Cof}[G^{(N)}]$  associated with the only non-vanishing eigenvalue  $\gamma_1$  of the latter. Let  $u_1$  be a normalized vector spanning  $\text{Ker } G^{(N)}$ . We parametrize

$$W = v u_1 \quad (\text{C.33})$$

where  $v$  is determined by solving Eq. (C.22), which, using Eq. (C.32), reads:

$$v = \left( V^{(N)T} \cdot u_1 \right)^{-1}. \quad (\text{C.34})$$

We notice that  $V^{(N)T} \cdot u_1 \neq 0$  since otherwise Eq. (C.22) cannot be fulfilled<sup>17</sup>. At this level, we hold all the information to solve Eq. (C.12). Nevertheless, let us go on and also determine the parameter  $Q$ .

For that, we substitute the expression of  $W$  into Eq. (C.19), which becomes:

$$- G^{(N)} \cdot Q = \mathbb{1}_{N-1} - \frac{V^{(N)} \cdot u_1^T}{V^{(N)T} \cdot u_1}. \quad (\text{C.35})$$

<sup>15</sup> Instead of subtracting column and line  $N$  while solving Eq. (C.12) we could have subtracted any other column and line  $a \neq N$ . Equation (C.31) would then read

$$\bar{b}_{j \neq a} = (-1)^{N-1} \det(G) \left( \left[ G^{(a)} \right]^{-1} \cdot V^{(a)} \right)_j, \quad \bar{b}_a = (-1)^{N-1} \det(G) - \sum_{j \neq a} \bar{b}_j.$$

The column vector  $\bar{b}$  defined by Eq. (C.31) seems to depend on the choice of the line and column singled out but actually it does *not*. When  $\det \mathcal{S} \neq 0$ ,  $\bar{b}$  fulfills  $\mathcal{S} \cdot \bar{b} = (\det \mathcal{S}) e$  whose solution is manifestly  $a$ -independent.

<sup>16</sup> For  $N = 3, 4$ ,  $\dim(\text{Ker } G^{(N)}) > 1$  leads to degenerate kinematics; cf. the discussion at the end of Sect. C.3.

<sup>17</sup> Notice also that since  $\text{Cof}[G^{(N)}]$  is a real symmetric matrix,  $\text{Cof}[G^{(N)}] = \gamma_1 u_1 \cdot u_1^T$  and then  $\det(\mathcal{S}) = (-1)^{N-1} \gamma_1 (V^{(N)T} \cdot u_1)^2$ , which is by assumption different from zero, so  $V^{(N)T} \cdot u_1 \neq 0$ .

We look for the pseudo inverse, more precisely the Moore–Penrose pseudo inverse (MPPI)<sup>18</sup> [35] of  $G^{(N)}$ , i.e.,  $H$  such that:

$$G^{(N)} \cdot H \cdot G^{(N)} = G^{(N)} \tag{C.36}$$

$$H \cdot G^{(N)} \cdot H = H \tag{C.37}$$

$$\left(G^{(N)} \cdot H\right)^\dagger = G^{(N)} \cdot H \tag{C.38}$$

$$\left(H \cdot G^{(N)}\right)^\dagger = H \cdot G^{(N)}. \tag{C.39}$$

In order for Eq. (C.35) to admit solutions the following compatibility condition has to hold:

$$\left[\mathbb{1}_{N-1} - G^{(N)} \cdot H\right] \cdot \left[\mathbb{1}_{N-1} - \frac{V^{(N)} \cdot u_1^T}{V^{(N)T} \cdot u_1}\right] = 0.$$

$[\mathbb{1}_{N-1} - G^{(N)} \cdot H]$  is the (orthogonal) projector onto  $\text{Ker } G^{(N)}$ , i.e.,  $[\mathbb{1}_{N-1} - G^{(N)} \cdot H] = u_1 \cdot u_1^T$  thus we have indeed

$$u_1 \cdot u_1^T \cdot \left[\mathbb{1}_{N-1} - \frac{V^{(N)} \cdot u_1^T}{V^{(N)T} \cdot u_1}\right] = 0$$

and  $Q$  is of the form

$$Q = -H \cdot \left[\mathbb{1}_{N-1} - \frac{V^{(N)} \cdot u_1^T}{V^{(N)T} \cdot u_1}\right] + u_1 \cdot n^T \tag{C.40}$$

where  $n$  is a so-far arbitrary (complex)  $(N - 1)$ -vector. Equation (C.40) is a matricial counterpart of the usual unknown  $(N - 1)$ -vector spanning  $\text{Ker } G^{(N)}$  in the singular vectorial case (it is “the general solution of the homogeneous equation  $G^{(N)} \cdot X = 0$ ”). We shall further request  $Q$  to be symmetric (as  $(\widehat{\mathcal{S}}^{(N)})^{-1}$  has to be so!). We have:

$$\begin{aligned} Q - Q^T &= \frac{1}{u_1^T \cdot V^{(N)}} \left[ H \cdot V^{(N)} \cdot u_1^T - u_1 \cdot V^{(N)T} \cdot H \right] + \left[ u_1 \cdot n^T - n \cdot u_1^T \right] \\ &= \left[ \frac{1}{u_1^T \cdot V^{(N)}} H \cdot V^{(N)} - n \right] \cdot u_1^T - u_1 \cdot \left[ \frac{1}{u_1^T \cdot V^{(N)}} H \cdot V^{(N)} - n \right]^T. \end{aligned} \tag{C.41}$$

This vanishes if and only if

$$\frac{1}{u_1^T \cdot V^{(N)}} H \cdot V^{(N)} - n = \xi u_1,$$

i.e.,

$$n = \frac{1}{u_1^T \cdot V^{(N)}} H \cdot V^{(N)} - \xi u_1 \tag{C.42}$$

---

<sup>18</sup> Any other pseudo inverse could also be used yet the MPPI provides extra properties—namely the MPPI  $H$  is symmetric and commutes with  $G^{(N)}$ , and  $G^{(N)} \cdot H$  is symmetric and nilpotent, i.e., a projector—convenient for further use.

where the scalar  $\xi$  is fixed by substituting into Eq. (C.21). We get:

$$-H \cdot V^{(N)} \left[ 1 - \frac{(u_1^T \cdot V^{(N)})}{(u_1^T \cdot V^{(N)})} \right] + \left[ \frac{V^{(N)T} \cdot H \cdot V^{(N)}}{(u_1^T \cdot V^{(N)})} + \frac{\mathcal{S}_{NN}}{(u_1^T \cdot V^{(N)})} - \xi (u_1^T \cdot V^{(N)}) \right] = 0,$$

i.e.,

$$\xi = \frac{1}{(u_1^T \cdot V^{(N)})^2} \left[ \mathcal{S}_{NN} + V^{(N)T} \cdot H \cdot V^{(N)} \right]. \tag{C.43}$$

Collecting the results of Eqs. (C.40), (C.42), and (C.43), the parameter  $Q$  is given by:

$$Q = -H \cdot \left[ \mathbb{1}_{N-1} - \frac{V^{(N)} \cdot u_1^T}{V^{(N)T} \cdot u_1} \right] + \frac{1}{u_1^T \cdot V^{(N)}} u_1 \cdot \left\{ V^{(N)} \cdot H - \frac{[\mathcal{S}_{NN} + V^{(N)T} \cdot H \cdot V^{(N)}]}{u_1^T \cdot V^{(N)}} \right\}, \tag{C.44}$$

which completes the extraction of  $(\widehat{\mathcal{S}}^{(N)})^{-1}$ . Using Eq. (C.44), we can verify that Eq. (C.20) is indeed verified.

To sum up, putting together the results of Eqs. (C.32)–(C.34) and (C.44), the solution of Eq. (C.12) is given by:

$$B = 0 \tag{C.45}$$

$$\beta^{(N)} = \frac{1}{(V^{(N)T} \cdot u_1)} u_1 \tag{C.46}$$

with  $u_1$  a normalized vector spanning  $\text{Ker } G^{(N)}$ . Note that the solution given by Eqs. (C.29) and (C.30) smoothly matches the solution given by Eqs. (C.45), (C.46) in the limit  $\det(G) \rightarrow 0$  as shown in Sect. C.4; cf. Eqs. (C.79), (C.81).

*Appendix C.3. Case  $\det(S) = 0$  and  $\det(G) = 0$*

We seek to express the MPPI  $\widehat{\Sigma}$  of  $\widehat{\mathcal{S}}^{(N)}$  in terms of  $V^{(N)}$  and the MPPI  $H$  of  $G^{(N)}$ . We parametrize  $\widehat{\Sigma}$  as

$$\widehat{\Sigma} = \left[ \begin{array}{c|c} Q & W \\ \hline W^T & U \end{array} \right]. \tag{C.47}$$

We thus have:

$$\widehat{\mathcal{S}}^{(N)} \cdot \widehat{\Sigma} = \left[ \begin{array}{c|c} -G^{(N)} \cdot Q + V^{(N)} \cdot W^T & -G^{(N)} \cdot W + U V^{(N)} \\ \hline (Q \cdot V^{(N)} + \mathcal{S}_{NN} W)^T & V^{(N)T} \cdot W + \mathcal{S}_{NN} U \end{array} \right]. \tag{C.48}$$

Let us warn the reader that we will not determine the MPPI  $\widehat{\Sigma}$  in general. Since we are interested in solving the equation  $S \cdot b = e$ , we restrict ourselves here to the case when this equation admits solutions, i.e., when the compatibility condition is fulfilled. The compatibility condition for Eq. (C.12) in terms of the MPPI is  $[\mathbb{1}_N - \widehat{\mathcal{S}}^{(N)} \cdot \widehat{\Sigma}] \cdot \widehat{e} = 0$ . This condition explicitly reads:

$$-G^{(N)} \cdot W + U V^{(N)} = 0 \tag{C.49}$$

$$V^{(N)T} \cdot W + \mathcal{S}_{NN} U = 1. \quad (\text{C.50})$$

Furthermore, the Moore–Penrose extra conditions require  $\widehat{\mathcal{S}}^{(N)} \cdot \widehat{\Sigma}$  to be a projector: it shall be symmetric, implying

$$Q \cdot V^{(N)} + \mathcal{S}_{NN} W = 0 \quad (\text{C.51})$$

$$\left[ -G^{(N)} \cdot Q + V^{(N)} \cdot W^T \right]^T = -G^{(N)} \cdot Q + V^{(N)} \cdot W^T \quad (\text{C.52})$$

together with the nilpotent request:

$$\left[ -G^{(N)} \cdot Q + V^{(N)} \cdot W^T \right]^2 = -G^{(N)} \cdot Q + V^{(N)} \cdot W^T. \quad (\text{C.53})$$

Together with MP conditions (C.51)–(C.53), compatibility conditions (C.49)–(C.50) make the extraction of  $\widehat{\Sigma}$  easier.

A solution such that  $U = 0$  would require  $W$  to belong to  $\text{Ker } G^{(N)}$  in order to fulfill Eq. (C.49) and not be orthogonal to  $V^{(N)}$  in order to fulfill Eq. (C.50). Let us assume  $\text{Ker } G^{(N)}$  to be 1D and spanned by a normalized vector  $u_1$ : one would have  $\text{Cof}[G^{(N)}] = \gamma_1 u_1 \cdot u_1^T$  with  $\gamma_1 \neq 0$ , thus  $V^{(N)T} \cdot \text{Cof}[G^{(N)}] \cdot V^{(N)} = \gamma_1 (u_1^T \cdot V^{(N)})^2 \neq 0$ , which would be inconsistent with  $\det(\mathcal{S}) = 0$  and  $\det(G) = 0$  simultaneously. Consequently:

- either  $U = 0$  and  $\text{Rank } G^{(N)}$  has to be  $\leq N - 3$  and thus  $\text{Cof}[G^{(N)}] = 0$  identically, which corresponds to degenerate kinematics;
- or  $\text{Rank } G^{(N)} = (N - 2)$ , in which case  $U$  cannot be 0.

Let us ignore the first alternative and focus on the second one. Indeed, the former leads to degenerate kinematics for the case  $N = 3, 4$  not relevant for colliders. Nevertheless, as explained in the introduction, since a two-loop scalar  $N$ -point function can be written as a sum of double integrals whose integrands are some “generalized” one-loop Feynman integrals, the kinematics of the latter depends on two parameters. So, one may wonder if such degenerate kinematics is met when these parameters run over the unit square. If relevant, this issue will be tackled later on.

In order for Eq. (C.49) to admit solutions, the following compatibility condition is fulfilled:

$$\left[ \mathbb{1}_{N-1} - G^{(N)} \cdot H \right] \cdot V^{(N)} = 0 \Leftrightarrow V^{(N)} \perp \text{Ker } G^{(N)}, \quad (\text{C.54})$$

$H$  being the MPPI of  $G^{(N)}$ . Equation (C.49) is then solved in the form:

$$W = UH \cdot V^{(N)} + k, \quad k \in \text{Ker } G^{(N)}. \quad (\text{C.55})$$

So far  $k$  remains arbitrary. Notwithstanding,  $(V^{(N)T} \cdot k) = 0$ ; substituting Eq. (C.55) into Eq. (C.50) thus yields:

$$U \left[ \mathcal{S}_{NN} + \left( V^{(N)T} \cdot H \cdot V^{(N)} \right) \right] = 1. \quad (\text{C.56})$$

Let us define  $\Omega$  as the factor of  $U$  in Eq. (C.78), namely:

$$\Omega = \left[ \mathcal{S}_{NN} + \left( V^{(N)T} \cdot H \cdot V^{(N)} \right) \right]. \quad (\text{C.57})$$

Since  $\widehat{\mathcal{S}}^{(N)}$  does admit one unique MPPI  $\widehat{\Sigma}$ , whose  $U$  has to obey Eq. (C.56), the combination  $\Omega$  involved in Eq. (C.56) is necessarily non-zero, and Eq. (C.56) fixes  $U = \Omega^{-1}$ .

Let us now determine  $Q$  fulfilling Eqs. (C.51)–(C.53). Using Eqs. (C.49)–(C.51), the l.h.s. of Eq. (C.53) may be expanded as

$$\begin{aligned} & \left[ -G^{(N)} \cdot Q + V^{(N)} \cdot W^T \right]^2 \\ &= \left( G^{(N)} \cdot Q \right)^2 - G^{(N)} \cdot \left( Q \cdot V^{(N)} \right) \cdot W^T \\ &\quad - V^{(N)} \cdot \left( W^T \cdot G^{(N)} \right) \cdot Q + V^{(N)} \cdot \left( W^T \cdot V^{(N)} \right) \cdot W^T \\ &= \left( G^{(N)} \cdot Q \right)^2 + \mathcal{S}_{NN} U V^{(N)} \cdot W^T - U V^{(N)} \cdot V^{(N)T} \cdot Q + (1 - \mathcal{S}_{NN} U) V^{(N)} \cdot W^T. \end{aligned}$$

Thus condition (C.53) may be replaced by:

$$\left( G^{(N)} \cdot Q \right)^2 + \left( G^{(N)} \cdot Q \right) - U V^{(N)} \cdot V^{(N)T} \cdot Q = 0. \quad (\text{C.58})$$

Inspired by the cases where  $\widehat{\mathcal{S}}^{(N)}$  is invertible, let us seek  $Q$  of the form

$$Q = -H + y \cdot y^T$$

with the  $(N - 1)$ -column vector  $y$  to be determined. We get:

$$\left( G^{(N)} \cdot Q \right) = -\left( G^{(N)} \cdot H \right) + G^{(N)} \cdot y \cdot y^T \quad (\text{C.59})$$

$$\begin{aligned} \left( G^{(N)} \cdot Q \right)^2 &= \left( G^{(N)} \cdot H \right)^2 - \left( G^{(N)} \cdot H \cdot G^{(N)} \right) \cdot y \cdot y^T \\ &\quad + \left( G^{(N)} \cdot y \right) \cdot \left[ -\left( y^T \cdot G^{(N)} \right) \cdot H + \left( y^T \cdot G^{(N)} \cdot y \right) y^T \right], \end{aligned} \quad (\text{C.60})$$

where  $H$  is the pseudo inverse of  $G^{(N)}$ ,  $(G^{(N)} \cdot H)$  is nilpotent (cf. Eqs. (C.36) and (C.37)) and “(C.60)+(C.59)” simplifies to:

$$\left( G^{(N)} \cdot Q \right)^2 + \left( G^{(N)} \cdot Q \right) = \left( G^{(N)} \cdot y \right) \cdot \left[ -\left( y^T \cdot G^{(N)} \right) \cdot H + \left( y^T \cdot G^{(N)} \cdot y \right) y^T \right] \quad (\text{C.61})$$

and Eq. (C.58) reads:

$$\begin{aligned} & \left( G^{(N)} \cdot y \right) \cdot \left[ -\left( y^T \cdot G^{(N)} \right) \cdot H + \left( y^T \cdot G^{(N)} \cdot y \right) y^T \right] \\ & - U V^{(N)} \cdot \left[ -V^{(N)T} \cdot H + \left( V^{(N)T} \cdot y \right) \cdot y^T \right] = 0. \end{aligned} \quad (\text{C.62})$$

At this point we recall that  $V^{(N)}$  fulfills the compatibility condition (C.54) and we make the following ansatz:

$$y = \zeta H \cdot V^{(N)} \quad (\text{C.63})$$

with  $\zeta$  to be determined. Let us note that actually, since  $Q$  quadratically depends on  $y$ , only  $\zeta^2$  not  $\zeta$  *per se* is involved in the expression of  $Q$ . The l.h.s. of Eq. (C.62), with the help of Eq. (C.54), becomes:

l.h.s. of Eq. (C.62)

$$\begin{aligned}
&= \zeta^2 \left( G^{(N)} \cdot H \cdot V^{(N)} \right) \\
&\quad \cdot \left[ - \left( V^{(N)T} \cdot H \cdot G^{(N)} \right) \cdot H + \zeta^2 \left( V^{(N)T} \cdot H \cdot G^{(N)} \cdot H \cdot V^{(N)} \right) \cdot V^{(N)T} \cdot H \right] \\
&\quad - U V^{(N)} \cdot \left[ - V^{(N)T} \cdot H + \zeta^2 \left( V^{(N)T} \cdot H \cdot V^{(N)} \right) V^{(N)T} \cdot H \right] \\
&= (\zeta^2 - U) \left[ -1 + \zeta^2 \left( V^{(N)T} \cdot H \cdot V^{(N)} \right) \right] V^{(N)} \cdot V^{(N)T} \cdot H, \tag{C.64}
\end{aligned}$$

which vanishes for

$$\zeta^2 = U. \tag{C.65}$$

We shall now enforce the symmetry property (C.52):

$$\begin{aligned}
&- G^{(N)} \cdot Q + V^{(N)} \cdot W^T \\
&= - G^{(N)} \cdot \left[ -H + U \left( H \cdot V^{(N)} \right) \cdot \left( H \cdot V^{(N)} \right)^T \right] + U V^{(N)} \cdot V^{(N)T} \cdot H + V^{(N)} \cdot k^T \\
&= - G^{(N)} \cdot H + V^{(N)} \cdot k^T \tag{C.66}
\end{aligned}$$

where the last equality is obtained using the compatibility condition (C.54). Due to the fact that the MPPI  $H$  of  $G^{(N)}$  is symmetric and does commute with  $G^{(N)}$ , the r.h.s. of Eq. (C.66) is thus symmetric if and only if  $k = 0$ , which now completely fixes  $W$ .

Last, we shall check whether Eq. (C.51) is fulfilled:

$$\begin{aligned}
&Q \cdot V^{(N)} + S_{NN} W \\
&= \left[ -H + U \left( H \cdot V^{(N)} \right) \cdot \left( H \cdot V^{(N)} \right)^T \right] \cdot V^{(N)} + S_{NN} U H \cdot V^{(N)} \\
&= \left[ -1 + U \left( S_{NN} + V^{(N)T} \cdot H \cdot V^{(N)} \right) \right] H \cdot V^{(N)}, \tag{C.67}
\end{aligned}$$

which indeed vanishes by virtue of Eq. (C.56).

We have to distinguish between the two cases.

*Appendix C.3.1.  $\det(S) = 0$  and  $[\mathbb{1}_N - \widehat{S}^{(N)} \cdot \widehat{\Sigma}] \cdot \widehat{e} = 0$*

If  $\det(S) = 0$ , yet with  $[\mathbb{1}_N - \widehat{S}^{(N)} \cdot \widehat{\Sigma}] \cdot \widehat{e} = 0$  then  $\det(G) = 0$ . Indeed, were  $\det(G)$  non-zero, the solution of Eq. (C.49) would be given as in Eq. (C.29) by  $\beta^{(N)} = U (G^{(N)})^{-1} \cdot V^{(N)}$ , whose substitution into Eq. (C.50) would again lead to  $U [V^{(N)T} \cdot (G^{(N)})^{-1} \cdot V^{(N)} + S_{NN}] = U (-1)^{N-1} \det(S) / \det(G) = 1$ . However, as the coefficient of  $U$  would vanish, no  $U$  could satisfy the latter equation, and thus Eqs. (C.49), (C.50) would have no solution; this is likewise for Eq. (C.12), which would contradict the compatibility condition. Let us restrict ourselves to the case “ $\dim \text{Ker } G^{(N)} = 1$ ” (cf. the discussion in this subsection). We can then use the results of this section to express  $\widehat{\Sigma} \cdot \widehat{e}$ :

$$U = \Omega^{-1} = \left[ S_{NN} + \left( V^{(N)T} \cdot H \cdot V^{(N)} \right) \right]^{-1} \tag{C.68}$$

$$W = \Omega^{-1} H \cdot V^{(N)}. \tag{C.69}$$

The solutions of Eq. (C.12) are given by:

$$\widehat{b} = \widehat{\Sigma} \cdot \widehat{e} + \left[ \mathbb{1}_N - \widehat{S}^{(N)} \cdot \widehat{\Sigma} \right] \cdot n \quad (\text{C.70})$$

where  $n$  is an arbitrary  $N$ -vector. In terms of the block components of the  $\widehat{\Sigma}$  matrix, these solutions become:

$$\beta^{(N)} = \Omega^{-1} H \cdot V^{(N)} + \left( \mathbb{1}_{N-1} - G^{(N)} \cdot H \right) \cdot \tilde{n} \quad (\text{C.71})$$

$$B = \Omega^{-1} \quad (\text{C.72})$$

where  $\tilde{n}$  is a  $N - 1$ -vector formed by the  $N - 1$  first components of  $n$ . The fact that  $\det(G)$  has to vanish can also be seen as a direct consequence of the compatibility condition required for Eq. (C.1) equivalent to Eq. (C.12):

$$[\mathbb{1}_N - S \cdot \Sigma] \cdot e = 0$$

with  $\Sigma$  the MPPI of  $S$ . If  $S$  is *real*, as can be shown, e.g., by diagonalization of  $S$  (and  $\Sigma$ ) this compatibility condition is equivalent to  $e \perp \text{Ker } S$  (where “ $\perp$ ” is understood as the Euclidean scalar product  $u \cdot v = \sum_{i=1}^N u_i v_i$ ): this means that *all*  $u \in \text{Ker } S$  fulfill the condition  $e^T \cdot u = 0$ . This condition has been shown in Appendix E of Ref. [27] to imply the existence of an eigenvector<sup>19</sup> for  $G^{(N)}$  with a corresponding vanishing eigenvalue; hence  $\det(G) = 0$ .

This can be extended to a *complex*  $S$  as shown in what follows. Although not real  $S$  is symmetric and thus admits a Takagi decomposition [36–42] of the form  $S = F \cdot S_D \cdot F^T$ , where  $S_D = \text{diag}(s_j, j = 1, \dots, N)$  is a real non-negative diagonal matrix and  $F$  is a  $N \times N$  unitary matrix. Let  $f_k$  be the column vectors of  $F$  so that the Takagi decomposition of  $S$  reads:

$$S = \sum_{j \in J} s_j f_j \cdot f_j^T$$

where  $J$  is the set of index values corresponding to the non-zero elements of  $S_D$ . Let us also define the set of index values  $K$  corresponding to vanishing elements of  $S_D$ . The  $f_j, j \in K$  are mutually orthogonal, normalized eigenvectors of  $S$  associated with eigenvalue zero, i.e., form an orthonormal basis of  $\text{Ker } S$ . A Takagi decomposition of the MPPI  $\Sigma$  of  $S$  is provided by:

$$\Sigma = \sum_{l \in J} s_l^{-1} f_l^* \cdot f_l^\dagger$$

so that

$$[\mathbb{1}_N - S \cdot \Sigma] = \sum_{l \in K} f_l \cdot f_l^\dagger,$$

i.e.,  $[\mathbb{1}_N - S \cdot \Sigma]$  is the (orthogonal) projector onto  $\text{Ker } S$  and the consistency condition  $[\mathbb{1}_N - S \cdot \Sigma] \cdot e = 0$  again means, i.e.,  $e \perp \text{Ker } S$  (where “ $\perp$ ” now refers to the *Hermitian* product  $u^\dagger \cdot v = \sum_{i=1}^N u_i^* v_i$ ).

<sup>19</sup> Such a vector can be formed by taking the  $N - 1$  first components of any non-vanishing vector in  $\text{Ker } S$ .

Whether  $\mathcal{S}$  is real or complex, “ $e \perp \text{Ker } \mathcal{S}$ ” implies that “ $\det \mathcal{S} = 0$ ” does *not* correspond to any kinematical singularity. Indeed such a singularity is characterized by a point satisfying the Landau conditions [43,44], which may occur either on the boundary of the domain of integration (end-point singularity as for an infrared or collinear singularity), or inside the domain of integration and where a pinching of the integration hypercontour occurs (pinching singularity as, e.g., for a threshold singularity). To such a point corresponds an eigenvector  $x \in \text{Ker } \mathcal{S}$  belonging to the first hyperquadrant defined by  $\{\sum_{j=1}^N x_j > 0, x_i \geq 0 \text{ for all } i = 1, \dots, N\}$ . In contrast, in the present case, “ $e \perp \text{Ker } \mathcal{S}$ ” prevents any element of  $\text{Ker } \mathcal{S}$  from belonging to this hyperquadrant. Let us notice that the converse is not true: there are situations where  $e$  may not be “ $\perp \text{Ker } \mathcal{S}$ ” yet no vector of  $\text{Ker } \mathcal{S}$  may belong to the first hyperquadrant either. For instance, considering the fake threshold at  $s = (m_1 - m_2)^2$  in the one-loop two-point function with distinct internal masses  $m_1^2 \neq m_2^2$ , all non-vanishing two component-vectors of  $\text{Ker } \mathcal{S}$  are found to have components with opposite signs, which means that the point in the integrand of the one-loop function fulfilling the Landau conditions lies outside the integration domain.

*Appendix C.3.2.  $\det(\mathcal{S}) = 0$  and  $[\mathbb{1}_N - \widehat{\mathcal{S}}^{(N)} \cdot \widehat{\Sigma}] \cdot \widehat{e} \neq 0$*

If  $\det(\mathcal{S}) = 0$  but the condition  $[\mathbb{1}_N - \widehat{\mathcal{S}} \cdot \widehat{\Sigma}] \cdot \widehat{e} = 0$  is not fulfilled, Eq. (C.12) has no solution. This situation occurs in cases of practical relevance such the one leading to Landau singularities. In deriving the representation of  $I_N^4$  using the Stokes-type identity the coefficients  $\bar{b}$ , defined when  $\det(G) \neq 0$  by

$$\bar{b}_{j \neq a} = (-1)^{N-1} \det(G) \left( [G^{(a)}]^{-1} \cdot V^{(a)} \right)_j, \bar{b}_a = (-1)^{N-1} \det(G) - \sum_{j \neq a} \bar{b}_j, \quad (\text{C.73})$$

naturally emerge. When  $\det(\mathcal{S}) = 0$  and  $[\mathbb{1}_N - \widehat{\mathcal{S}} \cdot \widehat{\Sigma}] \cdot \widehat{e} \neq 0$ , there is no  $b$  fulfilling  $\mathcal{S} \cdot b = e$  yet  $\bar{b}$  defined by Eq. (C.73) still exists and each of its components can be seen as the limits of the previous case (C.31) when  $\det(\mathcal{S}) \rightarrow 0$ , and  $\bar{b}$  fulfills  $\mathcal{S} \cdot \bar{b} = 0$ .

*Appendix C.4. Continuity between solutions given by Eqs. (C.29), (C.30) and Eqs. (C.45), (C.46)*

In spite of the *discontinuity in the algebraic treatment* to compute the expression of  $(\widehat{\mathcal{S}}^{(N)})^{-1}$  when  $\det(G) = 0$  vs.  $\neq 0$ , we expect the *expression* of  $(\widehat{\mathcal{S}}^{(N)})^{-1}$  itself to enjoy continuity in  $\det(G)$ . Let us investigate this issue.

The expression  $B = (-1)^{N-1} \det(G) / \det(\mathcal{S})$  vanishes continuously in  $\det(G)$ . Whatever  $\det(G)$  is, we recall that the real symmetric matrices  $G^{(N)}$  and  $\text{Cof}[G^{(N)}]$ , which fulfill  $G^{(N)} \cdot \text{Cof}[G^{(N)}] = \text{Cof}[G^{(N)}] \cdot G^{(N)} = \det(G) \mathbb{1}_{N-1}$ , commute and can be diagonalized using a common Euclidean orthonormal eigenbasis  $\{u'_j\}_{j=1, \dots, N-1}$ :

$$\begin{aligned} G^{(N)} &= \sum_{j=1}^{N-1} g'_j u'_j \cdot u'_j{}^T \\ \text{Cof}[G^{(N)}] &= \sum_{j=1}^{N-1} \gamma'_j u'_j \cdot u'_j{}^T \\ \mathbb{1}_{N-1} &= \sum_{j=1}^{N-1} u'_j \cdot u'_j{}^T \end{aligned}$$

where  $g'_j \gamma'_j = \det(G)$  for all  $j = 1, \dots, N - 1$ . The label “ $j=1$ ” labels the (unique) eigendirection corresponding to the eigenvalue that makes  $\det(G)$  vanish. The “prime notation” stands for the generic case  $\det(G) \neq 0$ , with corresponding quantities computed in Sect. C.2, whereas unprimed quantities are the respective values when  $\det(G)$  vanishes. We introduce

$$H' = \sum_{j=2}^{N-1} (g'_j)^{-1} u'_j \cdot u'^T_j \tag{C.74}$$

and rewrite

$$\begin{aligned} \sum_{k=2}^{N-1} \gamma'_k u'_k \cdot u'^T_k &= g'_1 \gamma'_1 \sum_{k=2}^{N-1} (g'_k)^{-1} u'_k \cdot u'^T_k \\ &= g'_1 \gamma'_1 H' \end{aligned} \tag{C.75}$$

so that

$$\text{Cof}[G^{(N)}] = \gamma'_1 [u'_1 \cdot u'^T_1 + g'_1 H']. \tag{C.76}$$

We can rewrite  $\det(S)$

$$\det(S) = (-1)^{N-1} \gamma'_1 \left\{ g'_1 \left[ \mathcal{S}_{NN} + (V^{(N)T} \cdot H' \cdot V^{(N)}) \right] + (u'^T_1 \cdot V^{(N)})^2 \right\}. \tag{C.77}$$

It is convenient to introduce

$$\Omega' = \left[ \mathcal{S}_{NN} + (V^{(N)T} \cdot H' \cdot V^{(N)}) \right]. \tag{C.78}$$

The expression  $B = (-1)^{N-1} \det(G) / \det(S)$  also takes the form:

$$(-1)^{N-1} \frac{\det(G)}{\det(S)} = \frac{g'_1}{g'_1 \Omega' + (u'^T_1 \cdot V^{(N)})^2}. \tag{C.79}$$

The r.h.s. of Eq. (C.79) vanishes continuously with  $g'_1$  while keeping  $(u'^T_1 \cdot V^{(N)}) \neq 0$ .

When  $\det(G) \neq 0$  the expression of  $W' = B (G^{(N)})^{-1} \cdot V^{(N)}$ , rewritten as

$$W' = \frac{1}{g'_1 \Omega' + (u'^T_1 \cdot V^{(N)})^2} \left\{ u'_1 (u'^T_1 \cdot V^{(N)}) + g'_1 H' \cdot V^{(N)} \right\}, \tag{C.80}$$

has a smooth limit when  $g'_1 \rightarrow 0$  while  $(u'^T_1 \cdot V^{(N)})$  is kept  $\neq 0$ , i.e.,  $\det(G) \rightarrow 0$  while  $\det(S) \neq 0$ :

$$W' \rightarrow \frac{1}{(u'^T_1 \cdot V^{(N)})} u_1, \tag{C.81}$$

namely the expression for  $W$  found in Eqs. (C.33), (C.34).

Let us now focus on  $Q$ . With  $G^{(N)}$  invertible we found (cf. Eq. (C.27))

$$Q' = -(G^{(N)})^{-1} + B (G^{(N)})^{-1} \cdot V^{(N)} \cdot V^{(N)T} \cdot (G^{(N)})^{-1}.$$

As just seen,  $B V^{(N)T} \cdot (G^{(N)})^{-1} \rightarrow (u_1^T \cdot V^{(N)})^{-1} u_1^T$  but in each term giving  $Q'$  there is an extra  $(G^{(N)})^{-1}$ , which becomes wild onto the  $u_1$  direction, thus  $Q'$  written in this form is an indeterminate  $\infty - \infty$ . We rewrite  $Q'$  splitting the various contributions on  $u_1'$  vs. the  $u'_{j \geq 2}$ :

$$\begin{aligned}
- Q' &= \left\{ (g_1')^{-1} u_1' \cdot u_1'^T + \sum_{j=2}^{N-1} (g_j')^{-1} u_j' \cdot u_j'^T \right\} \\
&\quad - \frac{(-1)^{N-1}}{\det(\mathcal{S})} \left\{ (g_1')^{-1} u_1' \cdot u_1'^T \cdot V^{(N)} + \sum_{j=2}^{N-1} (g_j')^{-1} u_j' \cdot u_j'^T \cdot V^{(N)} \right\} \\
&\quad \cdot \left\{ \gamma_1' V^{(N)T} \cdot u_1' \cdot u_1'^T + \sum_{k=2}^{N-1} \gamma_k' V^{(N)T} \cdot u_k' \cdot u_k'^T \right\}. \tag{C.82}
\end{aligned}$$

Using Eqs. (C.74) and (C.75), Eq. (C.82) becomes:

$$\begin{aligned}
- Q' &= \left\{ (g_1')^{-1} u_1' \cdot u_1'^T + H' \right\} \\
&\quad - \frac{(-1)^{N-1} \gamma_1'}{\det(\mathcal{S})} \left\{ (g_1')^{-1} u_1' \left( u_1'^T \cdot V^{(N)} \right) + H' \cdot V^{(N)} \right\} \\
&\quad \cdot \left\{ \left( V^{(N)T} \cdot u_1' \right) u_1'^T + g_1' V^{(N)T} \cdot H' \right\}, \tag{C.83}
\end{aligned}$$

which can be recast into

$$\begin{aligned}
- Q' &= H' + (g_1')^{-1} u_1' \cdot u_1'^T \left\{ 1 - \frac{(-1)^{N-1}}{\det(\mathcal{S})} \gamma_1' \left( V^{(N)T} \cdot u_1' \right)^2 \right\} \\
&\quad - \frac{(-1)^{N-1}}{\det(\mathcal{S})} \gamma_1' \left( u_1'^T \cdot V^{(N)} \right) \left\{ u_1' \cdot V^{(N)T} \cdot H' + H' \cdot V^{(N)} \cdot u_1'^T \right\} \\
&\quad - \frac{(-1)^{N-1}}{\det(\mathcal{S})} \gamma_1' g_1' H' \cdot V^{(N)} \cdot V^{(N)T} \cdot H'. \tag{C.84}
\end{aligned}$$

Using Eq. (C.77) we rewrite the first bracket in Eq. (C.84) as

$$\begin{aligned}
&g_1' \left\{ 1 - \frac{(-1)^{N-1}}{\det(\mathcal{S})} \gamma_1' \left( V^{(N)T} \cdot u_1' \right)^2 \right\} \\
&= \frac{(-1)^{N-1}}{\det(\mathcal{S})} \gamma_1' \Omega' \tag{C.85}
\end{aligned}$$

in which the indeterminate  $\infty - \infty$  has been canceled, leaving a net contribution with a finite limit when  $g_1'$  will be sent to 0. Finally using Eq. (C.77)  $Q'$  takes the following form:

$$\begin{aligned}
- Q' &= H' + \frac{\Omega'}{g_1' \Omega' + (u_1'^T \cdot V^{(N)})^2} u_1' \cdot u_1'^T \\
&\quad - \frac{(u_1'^T \cdot V^{(N)})}{g_1' \Omega' + (u_1'^T \cdot V^{(N)})^2} \left[ u_1' \cdot V^{(N)T} \cdot H' + H' \cdot V^{(N)} \cdot u_1'^T \right] \\
&\quad - \frac{g_1'}{g_1' \Omega' + (u_1'^T \cdot V^{(N)})^2} H' \cdot V^{(N)} \cdot V^{(N)T} \cdot H'. \tag{C.86}
\end{aligned}$$

In the limit  $g'_1 \rightarrow 0$  while  $(u_1'^T \cdot V^{(N)})$  is kept  $\neq 0$ , the last term of Eq. (C.86)  $\rightarrow 0$  and

$$Q' \rightarrow H + \frac{\Omega}{(u_1'^T \cdot V^{(N)})^2} u_1 \cdot u_1^T - \frac{1}{(u_1'^T \cdot V^{(N)})} \left[ u_1 \cdot (V^{(N)T} \cdot H) + (H \cdot V^{(N)}) \cdot u_1^T \right], \quad (\text{C.87})$$

i.e., the form found in Eq. (C.44)!

This study has yielded interesting byproducts: Eq. (C.76) relates in some sense the matrix  $\text{Cof}[G^{(N)}]$  (which provides the inverse of  $G^{(N)}$  as long as  $\det(G) \neq 0$ ) and the MPPI  $H$  of  $G^{(N)}$  when  $\det(G) = 0$ , leading to Eqs. (C.77) and (C.79). Equation (C.86) will also help understand what make the case “ $\det(G) = 0, \det(S) \neq 0$ ” vs. the combined limit “ $\det(G) = 0, \det(S) = 0$  simultaneously” depart from each other, as it interpolates between these two limit cases in some sense.

Let us compare  $\widehat{\Sigma}$  obtained in Sect. C.3 with  $(\widehat{S}^{(N)})^{-1}$  extracted in Sect. C.1. If one formally imposes the compatibility condition  $(u_1'^T \cdot V^{(N)}) = 0$  first while keeping  $g'_1 \neq 0$ , then sends  $g'_1 \rightarrow 0$ , the ingredients  $W$  and  $Q$  building  $\widehat{\Sigma}$  are simply obtained from those building  $(\widehat{S}^{(N)})^{-1}$  by replacing  $(G^{(N)})^{-1}$  by  $H'$ ;  $U$  is obtained likewise by reconsidering Eq. (C.79): the dependence in  $g'_1$  that causes the indeterminate  $0/0$  in  $B$  cancels and the r.h.s. of Eq. (C.79) is just  $\Omega'^{-1} \rightarrow \Omega^{-1}$ . With the *prior* formal imposition of compatibility condition (C.54), the limit  $g'_1 \rightarrow 0$  is the *combined* limit  $\det(S) \rightarrow 0, \det(G) \rightarrow 0$  keeping their ratio  $\Omega$  fixed; cf. Eq. (C.79). This remark helps us understand how the case addressed in this subsection and the one covered in the previous subsection of this appendix depart from each other. The latter instead was the limit  $g'_1 \rightarrow 0$  while  $(u_1'^T \cdot V^{(N)})$  was kept  $\neq 0$  so that  $\det(S)$  was kept non-zero. Keeping  $(u_1'^T \cdot V^{(N)}) \neq 0$  and sending  $g'_1 \rightarrow 0$  (as done in the previous subsection) then sending  $(u_1'^T \cdot V^{(N)}) \rightarrow 0$  vs. imposing  $(u_1'^T \cdot V^{(N)}) = 0$  first then sending  $g'_1 \rightarrow 0$  (as done in the present subsection) faces a mismatch between the orderings of limits. This is best seen considering the various scalar factors that weight the tensor structures decomposing  $Q'$  in Eq. (C.86). Let us consider, for instance, the last term in Eq. (C.86):

$$\text{last term of Eq. (C.86)} = \frac{g'_1}{g'_1 \Omega' + (u_1'^T \cdot V^{(N)})^2} H' \cdot V^{(N)} \cdot V^{(N)T} \cdot H';$$

taking  $g'_1 \rightarrow 0$  while keeping  $(u_1'^T \cdot V^{(N)}) \neq 0$  makes this term drop from the expression of  $Q'$ . In contrast, taking first  $(u_1'^T \cdot V^{(N)}) = 0$  makes  $g'_1$  cancel from the scalar factor, which then has the finite limit  $\Omega^{-1}$  when  $g'_1 \rightarrow 0$ ; hence the contribution  $\propto H \cdot V^{(N)} \cdot V^{(N)T} \cdot H / \Omega$  to  $Q$  in the “combined limit” case. Conversely, let us consider the third term in Eq. (C.86) given by:

$$\text{third term of Eq. (C.86)} = \frac{(u_1'^T \cdot V^{(N)})}{g'_1 \Omega' + (u_1'^T \cdot V^{(N)})^2} \left[ u_1' \cdot V^{(N)T} \cdot H' + H' \cdot V^{(N)} \cdot u_1'^T \right].$$

When taking  $g'_1 \rightarrow 0$  while keeping  $(u_1'^T \cdot V^{(N)}) \neq 0$  this term yields the finite contribution  $\propto [u_1' \cdot V^{(N)T} \cdot H + H \cdot V^{(N)} \cdot u_1'^T] / (u_1'^T \cdot V^{(N)})$  to  $Q'$ . In contrast, taking first  $(u_1'^T \cdot V^{(N)}) = 0$  makes this term drop from the expression of  $Q'$  in the “combined limit” case.

#### Appendix D. Integrals occurring in the computation of $L_3^4$ and $L_4^4$

In what follows  $A$  and  $B$  are assumed dimensionless and complex valued, the signs of their real parts are unknown, and the signs of their imaginary parts may or may not be the same either.

### Appendix D.1. First kind

The computation of  $I_3^4$  involves the computation in closed form of the following kind of integral:

$$K = \int_0^\infty \frac{d\xi}{(\xi + A)(\xi + B)}. \quad (\text{D.1})$$

After partial fraction decomposition the r.h.s. of Eq. (D.1) becomes:

$$K = \frac{1}{B-A} \int_0^\infty d\xi \left[ \frac{1}{\xi + A} - \frac{1}{\xi + B} \right]. \quad (\text{D.2})$$

The r.h.s. of Eq. (D.1) converges at infinity although each term separately grows logarithmically. Let us introduce a cut-off  $\Lambda$  and write Eq. (D.2) as:

$$K = \lim_{\Lambda \rightarrow +\infty} \tilde{K}(\Lambda), \quad \tilde{K}(\Lambda) = \frac{1}{B-A} \int_0^\Lambda d\xi \left[ \frac{1}{\xi + A} - \frac{1}{\xi + B} \right]. \quad (\text{D.3})$$

The  $\xi$  integration yields:

$$\tilde{K}(\Lambda) = \frac{1}{B-A} \left[ \ln \left( \frac{1 + A/\Lambda}{A} \right) - \ln \left( \frac{1 + B/\Lambda}{B} \right) \right]$$

so that

$$K = \frac{1}{B-A} [\ln(B) - \ln(A)] \quad (\text{D.4})$$

regardless of the signs of  $\text{Im}(A)$  vs.  $\text{Im}(B)$ .

### Appendix D.2. Second kind

The integral

$$J(\nu) = \int_0^{+\infty} \frac{d\xi}{(\xi^\nu + A) \sqrt{\xi^\nu + B}} \quad (\text{D.5})$$

arises in some intermediate step in the computation of three- and four-point functions for real masses. When no internal masses are vanishing it arises for  $\nu = 2$ . The integral need not be computed in closed form and shall instead be recast in an alternative, handier form cleared from any radical.

In the case of real masses,  $\text{Im}(A)$  and  $\text{Im}(B)$  are always of the same sign. Whenever this is the case, the use of the celebrated Feynman “trick” is justified and leads to:

$$\frac{1}{(\xi^2 + A) \sqrt{\xi^2 + B}} = \frac{1}{2} \int_0^1 \frac{dx x^{-\frac{1}{2}}}{(\xi^2 + (1-x)A + xB)^{3/2}}. \quad (\text{D.6})$$

$J(2)$  is readily rewritten as:

$$J(2) = \frac{1}{2} \int_0^{+\infty} d\xi \int_0^1 \frac{dx x^{-\frac{1}{2}}}{(\xi^2 + (1-x)A + xB)^{3/2}}. \quad (\text{D.7})$$

Then the  $\xi$  integration is performed first, using Eq. (B.1) of Appendix B. Performing the change of variable  $z = \sqrt{x}$  in the result obtained yields:

$$J(2) = \int_0^1 \frac{dz}{(1-z^2)A + z^2B}. \quad (\text{D.8})$$

### Appendix E. Basic integrals in terms of dilogarithms and logarithms: $K$ -type integrals

The computations of the various  $N$ -point functions in closed form can be reduced to the calculation of integrals of simple types. The  $K$ -type is of the form

$$K = \int_b^a du \frac{\ln(Au^2 + B) - \text{“subtracted term”}}{u^2 - u_0^2}$$

where  $u_0^2 \neq -B/A$ . In the real mass case,  $A$  is real and the non-vanishing imaginary part of the complex quantity  $B$  is infinitesimal. The integration contour runs along the positive real axis; more precisely, this case involves the segment between  $[0, 1]$ .

In the real mass case,  $u_0^2$  may primitively appear in calculations either as a real parameter, or slightly shifted away from the real axis by an infinitesimal contour prescription  $\text{Im}(u_0^2) = i s \lambda$  with  $s = \pm$ . In the calculation of three-point functions presented in this article  $K$ -type integrals naturally arise with real parameters  $u_0^2$  always in conjunction with a “subtracted term” equal to  $\ln(Au_0^2 + B)$ , so that the poles at  $u = \pm u_0$  are actually fake. Alternatively, the calculation of four-point functions involves  $K$ -type integrals with parameters  $u_0^2$  slightly shifted away from the real axis, which makes the integrals well defined even with a vanishing “subtracted term”. In the latter case, however, framing the calculation with a vanishing “subtracted term” may result in an annoying mismatch between the infinitesimal imaginary part of  $B$  shifting the integration contour from the cut of  $\ln(Au^2 + B)$  in the numerator, and the contour prescription for the pole  $1/(u^2 - u_0^2)$ . This mismatch may lead to possible apparent ambiguities on the signs of infinitesimal imaginary parts of some arguments in  $\text{Li}_2$  functions, such as  $\text{Li}_2[(u_0 \pm 1)/(u_0 - \bar{u})]$  with  $\bar{u} = \sqrt{-B/A}$  when the signs of the infinitesimal  $\text{Im}(u_0)$  and  $\text{Im}(\bar{u})$  happen to be the same while  $\text{Re}[(u_0 \pm 1)/(u_0 - \bar{u})] > 1$ . Sordid fiddling would then be required to solve these ambiguities. This annoyance can be avoided by framing the calculation so as to involve  $K$ -type integrals with a “subtracted term” equal to  $\ln(A \text{Re}(u_0^2) + B)$  instead of 0, which effectively allows us to ignore the infinitesimal imaginary part in  $u_0^2$ , as will be shown below.

We hereby compute the above type of integral. This appendix often makes use of the identity

$$\ln(z) = \ln(-z) + i\pi S(z) \quad , \quad S(z) \equiv \text{sign}(\text{Im}(z)). \tag{E.1}$$

Let us parametrize  $u_0^2$  by  $\text{Re}(u_0^2) = u_0^{2R}$  and  $\text{Im}(u_0^2) = s\lambda$  with  $s = \pm$  and note  $u_0 \equiv \sqrt{u_0^2}$ . We define:

$$K_{0,1}^R(A, B, u_0^2) = \int_0^1 du \frac{\ln(Au^2 + B) - \ln(Au_0^{2R} + B)}{u^2 - u_0^2}. \tag{E.2}$$

Using partial fraction decomposition, Eq. (E.2) can be written as:

$$K_{0,1}^R(A, B, u_0^2) = \frac{1}{2u_0} \int_0^1 du \left( \frac{1}{u - u_0} - \frac{1}{u + u_0} \right) (\ln(Au^2 + B) - \ln(Au_0^{2R} + B)). \tag{E.3}$$

Introducing  $\bar{u}^2 \equiv -B/A$  and  $S_u \equiv S(-\bar{u}^2)$ , the logarithmic term  $\ln(Au^2 + B)$  can be split as<sup>20</sup>

$$\ln(Au^2 + B) = \ln(A - i\lambda S_u) + \ln(u - \bar{u}) + \ln(u + \bar{u}). \tag{E.4}$$

<sup>20</sup> This comes from the splitting of  $\ln(ab)$  with  $a$  real but not necessarily  $> 0$  and  $b$  is complex non-real, for which [12]

$$\ln(ab) = \ln(a - i\lambda S_b) + \ln(b), \quad S_b = \text{sign}(\text{Im}(b)).$$

It is convenient to introduce  $\tilde{u}_0$  defined by:

$$\tilde{u}_0 \equiv \begin{cases} i s \sqrt{-u_0^{2R}} & \text{if } u_0^{2R} < 0 \\ \sqrt{u_0^{2R}} & \text{if } u_0^{2R} > 0 \end{cases}.$$

The subtracted term can in its turn be split as

$$\ln(A u_0^{2R} + B) = \ln(A - i \lambda S_u) + \ln(\tilde{u}_0 - \bar{u}) + \ln(\tilde{u}_0 + \bar{u}) + \eta(\tilde{u}_0 - \bar{u}, \tilde{u}_0 + \bar{u}) \quad (\text{E.5})$$

where the function  $\eta(z_1, z_2)$  is given by:

$$\begin{aligned} \eta(z_1, z_2) &= \ln(z_1 z_2) - \ln(z_1) - \ln(z_2) \\ &= \begin{cases} +2 i \pi & \text{if } \text{Im}(z_1) \geq 0, \text{Im}(z_2) \geq 0, \text{ and } \text{Im}(z_1 z_2) < 0 \\ 2 i \pi & \text{if } \text{Im}(z_1) < 0, \text{Im}(z_2) < 0, \text{ and } \text{Im}(z_1 z_2) \geq 0 \\ -2 i \pi & \text{if } \text{Im}(z_1) = 0, \text{Im}(z_2) = 0, \text{Re}(z_1) < 0, \text{ and } \text{Re}(z_2) < 0 \\ 0 & \text{otherwise} \end{cases}. \end{aligned} \quad (\text{E.6})$$

Incorporating identities (E.4) and (E.5) into Eq. (E.3), we get:

$$\begin{aligned} &K_{0,1}^R(A, B, u_0^2) \\ &= \frac{1}{2 u_0} \left[ \bar{R}(u_0, \bar{u}, \tilde{u}_0) + \bar{R}(u_0, -\bar{u}, \tilde{u}_0) - \bar{R}(-u_0, \bar{u}, -\tilde{u}_0) - \bar{R}(-u_0, -\bar{u}, -\tilde{u}_0) \right. \\ &\quad \left. - \eta(\tilde{u}_0 - \bar{u}, \tilde{u}_0 + \bar{u}) \ln\left(\frac{u_0 - 1}{u_0}\right) + \eta(-\tilde{u}_0 - \bar{u}, -\tilde{u}_0 + \bar{u}) \ln\left(\frac{u_0 + 1}{u_0}\right) \right] \end{aligned} \quad (\text{E.7})$$

where we have defined:

$$\bar{R}(u_0, \bar{u}, \tilde{u}_0) = \int_0^1 du \frac{\ln(u - \bar{u}) - \ln(\tilde{u}_0 - \bar{u})}{u - u_0}. \quad (\text{E.8})$$

Using Eq. (E.1),  $[\bar{R}(u_0, \bar{u}, \tilde{u}_0) - \bar{R}(-u_0, -\bar{u}, -\tilde{u}_0)]$  can be further recast into

$$\begin{aligned} &\bar{R}(u_0, \bar{u}, \tilde{u}_0) - \bar{R}(-u_0, -\bar{u}, -\tilde{u}_0) \\ &= \int_{-1}^1 du \frac{\ln(u - \bar{u}) - \ln(\tilde{u}_0 - \bar{u})}{u - u_0} + i \pi [S(\bar{u}) + S(\tilde{u}_0 - \bar{u})] \ln\left(\frac{u_0}{1 + u_0}\right). \end{aligned} \quad (\text{E.9})$$

For convenience, let us define

$$\bar{R}'(u_0, \bar{u}, \tilde{u}_0) = \int_{-1}^1 du \frac{\ln(u - \bar{u}) - \ln(\tilde{u}_0 - \bar{u})}{u - u_0}. \quad (\text{E.10})$$

A comment is in order here. One shall be cautious that in Eqs. (E.8) and (E.10), the denominators of the integrands (pole terms) involve  $u_0$  whereas  $\tilde{u}_0$  is involved in the numerators (logarithmic subtracted terms). Notwithstanding,

- i) when  $\text{Re}(u_0^2) < 0$ , the pole in Eqs. (E.8) and (E.10) is well away from the integration contour, the contour prescription of the pole is irrelevant, and  $u_0$  can be harmlessly traded for  $\tilde{u}_0$ ;

- ii) when  $\text{Re}(u_0^2) > 0$ ,  $\tilde{u}_0$  is real and  $u_0 = \tilde{u}_0 + i s \lambda$ , in which case the contour prescription of the pole also proves to be ineffective. Indeed, let us rewrite the r.h.s. of Eq. (E.10) explicitly as:

$$\int_{-1}^1 du \frac{\ln(u - \bar{u}) - \ln(\tilde{u}_0 - \bar{u})}{u - \tilde{u}_0 - i s \lambda};$$

the numerator vanishes  $\propto (u - \tilde{u}_0)$  at  $u = \tilde{u}_0$  so that the pole at  $u = \tilde{u}_0$ , whether  $\tilde{u}_0$  is on the integration contour or not, is actually fake. Only the prescription coming from the infinitesimal  $\text{Im}(\bar{u})$  in  $\ln(u - \bar{u})$  matters, which determines on which side of the cut the integration contour runs. Thus  $\bar{R}'(u_0, \bar{u}, \tilde{u}_0)$  is equivalently<sup>21</sup> given by:

$$\bar{R}'(u_0, \bar{u}, \tilde{u}_0) = \int_{-1}^1 du \frac{\ln(u - \bar{u}) - \ln(\tilde{u}_0 - \bar{u})}{u - \tilde{u}_0} \equiv R'(\bar{u}, \tilde{u}_0). \tag{E.11}$$

This is so because the subtracted term involves  $\tilde{u}_0$ , not  $u_0$ . If instead the subtracted term were  $\ln(u_0 - \bar{u})$  a potential conflict might occur between the signs of  $\text{Im}(u_0)$  vs.  $\text{Im}(\bar{u})$ . This might result into a non-vanishing residue  $[\ln(\tilde{u}_0 - \bar{u}) - \ln(u_0 - \bar{u})]$  whenever  $\tilde{u}_0 - \text{Re}(\bar{u}) < 0$  and  $\text{sign}(\text{Im}(u_0 - \bar{u})) = \text{sign}(\text{Im}(\bar{u}))$ . No such subtlety is faced with complex masses as will be met in the second article, for which poles and cuts are generically well away from the integration contour.

In conclusion, Eq. (E.10) can be safely traded for Eq. (E.11) in all cases. The integral  $R'(y, z)$  can be computed following the main line of the computation in Appendix B of Ref. [12], more generally for two complex numbers  $y$  and  $z$  ( $z$  can be real or complex, whereas  $\text{Im}(y)$  shall be kept non-vanishing) as this will also be relevant in the complex mass case. We first make the change of variable  $x = u - y$ :

$$R'(y, z) = \int_{-1-y}^{1-y} dx \frac{1}{x + y - z} (\ln(x) - \ln(z - y)). \tag{E.12}$$

Next we deform the straight contour  $[-1 - y, 1 - y]$  of integration in Eq. (E.12) into the dihedron  $[-1 - y, 0] \cup [0, 1 - y]$  in the complex  $x$ -plane. This procedure is legitimate since 1) the pole at  $x = z - y$  is fake and thus we do not have to care about the location of  $(z - y)$  w.r.t. the triangle  $\{-1 - y, 0, 1 - y\}$ , and 2) none of the paths  $[0, \pm 1 - y]$  crosses the cut of  $\ln(x)$ . We split the integral as

$$\int_{-1-y}^{1-y} = \int_0^{1-y} - \int_0^{-1-y}$$

and  $R'(y, z)$  becomes:

$$R'(y, z) = \int_0^{1-y} dx \frac{1}{x + y - z} (\ln(x) - \ln(z - y)) - \int_0^{-1-y} dx \frac{1}{x + y - z} (\ln(x) - \ln(z - y)). \tag{E.13}$$

We then make the changes of variable  $x = (1 - y) t$  in the first integral and  $x = (-1 - y) t$  in the second one, so that  $R'(y, z)$  now reads:

$$R'(y, z) = \int_0^1 dt \left\{ \frac{d}{dt} \ln \left( 1 + t \frac{1 - y}{y - z} \right) \right\} (\ln((1 - y) t) - \ln(z - y))$$

<sup>21</sup> The limit  $\lambda \rightarrow 0$  of the integrand pointwise in  $u$  provides an “integrable hat in the variable  $u$ ”, which controls the limit  $\lambda \rightarrow 0$  of the integral defining  $\bar{R}'(u_0, \bar{u}, \tilde{u}_0)$ , towards the value  $R'(\bar{u}, \tilde{u}_0)$ ; cf. Lebesgue’s theorem of dominated convergence.

$$- \int_0^1 dt \left\{ \frac{d}{dt} \ln \left( 1 - t \frac{1+y}{y-z} \right) \right\} (\ln(-(1+y)t) - \ln(z-y)). \quad (\text{E.14})$$

The two integrals of Eq. (E.14) are performed by integration by parts. Using the identity  $\text{Li}_2(w) = -\text{Li}_2(1-w) + \pi^2/6 - \ln(w) \ln(1-w)$  we finally get:

$$\begin{aligned} R'(y, z) = & \text{Li}_2 \left( \frac{z+1}{z-y} \right) - \text{Li}_2 \left( \frac{z-1}{z-y} \right) \\ & + \eta \left( -1-y, \frac{1}{z-y} \right) \ln \left( \frac{z+1}{z-y} \right) - \eta \left( 1-y, \frac{1}{z-y} \right) \ln \left( \frac{z-1}{z-y} \right). \end{aligned} \quad (\text{E.15})$$

The quantity  $[\bar{R}(u_0, -\bar{u}, \tilde{u}_0) - \bar{R}(-u_0, \bar{u}, -\tilde{u}_0)]$  can be computed along the same lines, and we end up with:

$$\begin{aligned} \bar{R}(u_0, \bar{u}, \tilde{u}_0) + \bar{R}(u_0, -\bar{u}, \tilde{u}_0) = & \bar{R}(-u_0, \bar{u}, -\tilde{u}_0) + \bar{R}(-u_0, -\bar{u}, -\tilde{u}_0) \\ & + \mathcal{F}(\tilde{u}_0, \bar{u}) + i\pi [S(\tilde{u}_0 - \bar{u}) + S(\tilde{u}_0 + \bar{u})] \ln \left( \frac{u_0}{1+u_0} \right) \end{aligned} \quad (\text{E.16})$$

where

$$\begin{aligned} \mathcal{F}(x, y) = & \text{Li}_2 \left( \frac{x+1}{x-y} \right) - \text{Li}_2 \left( \frac{x-1}{x-y} \right) + \text{Li}_2 \left( \frac{x+1}{x+y} \right) - \text{Li}_2 \left( \frac{x-1}{x+y} \right) \\ & - \eta \left( 1-y, \frac{1}{x-y} \right) \ln \left( \frac{x-1}{x-y} \right) + \eta \left( -1-y, \frac{1}{x-y} \right) \ln \left( \frac{x+1}{x-y} \right) \\ & - \eta \left( 1+y, \frac{1}{x+y} \right) \ln \left( \frac{x-1}{x+y} \right) + \eta \left( -1+y, \frac{1}{x+y} \right) \ln \left( \frac{x+1}{x+y} \right). \end{aligned} \quad (\text{E.17})$$

Putting everything together in Eq. (E.7), things can be further rearranged using Eq. (E.1) by noting that, for any two complex numbers  $a$  and  $b$ ,

$$\eta(a, b) - \eta(-a, -b) = -i\pi [S(a) + S(b)]. \quad (\text{E.18})$$

We finally get<sup>22</sup> for  $K_{0,1}^R(A, B, u_0^2)$ :

$$K_{0,1}^R(A, B, u_0^2) = \frac{1}{2\tilde{u}_0} \left[ \mathcal{F}(\tilde{u}_0, \bar{u}) - \eta(\tilde{u}_0 - \bar{u}, \tilde{u}_0 + \bar{u}) \ln \left( \frac{u_0 - 1}{u_0 + 1} \right) \right]. \quad (\text{E.19})$$

In particular, when  $\tilde{u}_0$  is real all the  $\eta$  functions appearing explicitly or through  $\mathcal{F}(\tilde{u}_0, \bar{u})$  in Eq. (E.19) vanish.

## Appendix F. Prescription for the imaginary part of $\det(\mathcal{S})$ : Real masses, i.e., the infinitesimal Im part

In this appendix, we discuss the sign of the imaginary part of  $\det(\mathcal{S})$  for the real mass case.  $I_N^4$  involves the quadratic form

$$\mathcal{Q}(z) = -z^T \cdot \mathcal{S} \cdot z - i\lambda, \quad z^T = [z_1, \dots, z_N] \quad (\text{F.1})$$

<sup>22</sup>  $u_0$  is harmlessly replaced by  $\tilde{u}_0$  in the prefactor  $1/(2u_0)$  in the expression of  $K_{0,1}^R(A, B, u_0^2)$ .

with  $\lambda$  real positive<sup>23</sup>. The explicit calculation of  $I_N^4$  involves the square root of  $\det(\mathcal{S})$ , which carefully keeps track of the  $i\lambda$  prescription in  $\mathcal{Q}(z)$ .

Since the  $z_j$  sum to 1,  $\mathcal{Q}(z)$  can be rewritten as:

$$\mathcal{Q}(z) = -z^T \cdot \mathcal{S} \cdot z - i\lambda \left( \sum_{i=1}^N z_i \right) \left( \sum_{j=1}^N z_j \right) = -z^T \cdot [\mathcal{S} + i\lambda E] \cdot z \quad (\text{F.2})$$

$$E_{ij} = 1 \text{ for all } i, j = 1, \dots, N. \quad (\text{F.3})$$

Thus, the sought-after prescription is provided by:

$$\det(\mathcal{S}) := \det[\mathcal{S} + i\lambda E]. \quad (\text{F.4})$$

$\mathcal{S}$  is implicitly assumed to be invertible in what follows.

Let us write the r.h.s. of Eq. (F.4) as:

$$\det[\mathcal{S} + i\lambda E] = (i\lambda)^N \det(\mathcal{S}) \det \left[ \mathcal{S}^{-1} \cdot E + \left( \frac{1}{i\lambda} \right) \mathbb{1}_N \right]. \quad (\text{F.5})$$

The second determinant in the r.h.s. of Eq. (F.5) is the value of the characteristic polynomial of the matrix  $-K$  with  $K = \mathcal{S}^{-1} \cdot E$  evaluated at  $x = 1/(i\lambda)$ . The matrix  $K$  is given by

$$K = \begin{bmatrix} b_1 & \cdots & b_1 & \cdots & b_1 \\ \vdots & & \vdots & & \vdots \\ b_j & \cdots & b_j & \cdots & b_j \\ \vdots & & \vdots & & \vdots \\ b_N & \cdots & b_N & \cdots & b_N \end{bmatrix} \quad (\text{F.6})$$

$$b_j = \sum_{k=1}^N (\mathcal{S}^{-1})_{jk} \quad (\text{F.7})$$

$$\text{tr}[K] = \sum_{j=1}^N b_j = B = (-1)^{N-1} \frac{\det(G)}{\det(\mathcal{S})} \quad (\text{F.8})$$

(for Eq. (F.8) cf. Appendix C); i.e., all lines of  $\mathcal{S}^{-1} \cdot E$  are proportional to each other, thus  $\mathcal{S}^{-1} \cdot E$  has rank 1: all its eigenvalues but one are zero. Consequently, the coefficients of all but the first two monomials of highest and next-to-highest degrees in  $x$  of  $\det[K + x \mathbb{1}_N]$  vanish and

$$\begin{aligned} \det[K + x \mathbb{1}_N] &= x^N + x^{N-1} \text{tr}[K] \\ &= x^N + x^{N-1} B. \end{aligned} \quad (\text{F.9})$$

One thus has:

$$\begin{aligned} \det[\mathcal{S} + i\lambda E] &= \det(\mathcal{S}) + i\lambda B \det(\mathcal{S}) \\ &= \det(\mathcal{S}) + i\lambda (-1)^{N-1} \det(G), \end{aligned} \quad (\text{F.10})$$

<sup>23</sup>  $\lambda$  being infinitesimal or finite is irrelevant here.

i.e., the prescription for the sign of “the Im part of  $\det(S)$ ” is “ $+\lambda \operatorname{sign}((-1)^{N-1} \det(G))$ ” (as long as  $\det(G) \neq 0$ ). Let us also notice that the proof holds whether  $\lambda$  is infinitesimal or finite. We can remark that the prescription coming naturally for all  $\Delta_{\bar{n}}$  in Sects. 2 and 3, namely a  $+i\lambda$ , after applying the Stokes-type identity, is the one obtained by taking into account the prescription given by Eq. (F.4) (supplemented by Eq. (F.10)) in Eq. (C.11).

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