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Approximation and perturbation methods

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Few problems in nature are amenable to an exact solution and hence when one proceeds from elegant problems of theory to messy complicated problems of practice one is forced to recourse to methods of approximation and perturbation. The development of such techniques has been natural in attempts to extract physically verifiable consequences from either exact solutions of general relativity or from specific astrophysical systems for which an exact solution is impossible to find. However, this should not be taken to imply giving up of mathematical rigour and an appeal to only physical intuition.

1. Approximation Methods

Though the topic of approximation methods have been with us since the inception of the theory of general relativity it received a fresh impetus with the observation of the secular acceleration in the mean orbital motion of the Binary Pulsar. Suddenly, the observations were getting to be accurate enough to make measurable higher order effects coming from general relativity and the theorist had also to update his tools and make more precise the conceptual foundations that formed the paradigm for matching the increasingly accurate observations to a theoretical model. The main class of approximation schemes that have been developed so far are the Post Newtonian Approximation(PNA) and the Post Minkowskian Approximation(PMA). Originally developed in the context of the solar system these old approximation schemes used a global coordinate system, a global weak field assumption and a single asymptotic expansion. The need to treat binary systems containing two neutron stars or black holes requires looking at regimes where strong field effects come into play. A more detailed description incorporating the clumpiness of the universe is also called for in cosmology. These new problems require new approximation methods characterised by the use of several coordinate systems and several asymptotic expansions[1].

What is the relation between the approximation methods and the exact theory? How do we go about investigating this? This was the theme of Alan Rendall's talk on 'Approximation methods in theory and practice'. It was concerned with the passage from practical use of approximations which are heuristically defined to rigorous theorems on how well and in what sense, the results of calculations of this kind, approximate solutions of the exact equations. A possible programme for such an undertaking could be : First, find a definition which on the one hand looks likely to provide a basis for rigorous theorems and on the other hand is relevant to practical calculations. Next, use this

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definition to prove that the approximations are qualitatively good in some sense. Finally, obtain quantitative estimates of the difference between exact and approximate solutions. For PMA a definition was given by Blanchet and Damour [2] which was tailored to their practical calculations and proved useful for questions of rigorous justification. Damour and Schmidt[3] showed on the basis of these definitions that in general circumstances PMA are asymptotic to solutions of Einstein's equation. That this is essentially optimal follows from the fact that the series obtained does not converge[4]. Quantitative estimates seem out of reach at present and is a challenge to the theory of partial differential equations which rarely yield quantitative information. On the other hand though a good definition of the newtonian limit of general relativity has existed for some time[5] its relation to PNA was unclear(however, see[6]). Recently, a definition of PNA has been given[7] which appears to have a good chance of linking theory and practice. Work is in progress towards justification of the PNA. The first step is the proof that the spherically symmetric Vlasov-Einstein system has a regular newtonian limit[8]. A comparison of the new definition of PNA with the traditional approach e.g. Chandrasekhar[9] shows that the most obvious difference is that in the former case all the variables including the matter variables are expanded in powers of $c\lambda{-1}$ whereas in the latter case only the variables describing the gravitational field are expanded. This leads to problems in regard to secular effects and quantitative information seems absolutely necessary to pin down good properties of PNA.

The new approximation methods are used mainly to investigate problems of motion and the generation problem in gravitational radiation theory. They are also used in problems of relativistic celestial mechanics as explained by Chongming Xu. He summarized the new formalism of Damour, Soffel and Xu[10] for studying general relativistic celestial mechanics of systems of N arbitrary, weakly self- gravitating, rotating bodies using a sophisticated version of the first PNA. It is characterised by use of a multi-reference system; a global one for describing overall dynamics of N bodies and N local systems to describe the internal structure of each body. The special features of the scheme are that the field equation and transformation laws are linear; the structure of the energy-momentum tensor is left open; each body is characterised by the Blanchet-Damour PN multipole moment[11] and external PN tidal moments. This allows one to obtain complete and explicit results for laws of translational motion at 1PN level for bodies with arbitrary composition and shapes. One can obtain an expression for the tidal moment in terms of PN multipole moments of other bodies. The discussion of PN spin motion requires in addition the Damour-Iyer[12] spin moments.

Moreschi talked about approximation methods around stationary systems. He argued that since the asymptotic symmetry group is the infinite dimensional BMS group and not the ten dimensional Poincaré group it led to an arbitrariness both in the definition of physical concepts associated with the system and also the best flat background to expand around. Consequently, approximation methods around a fixed stationary background metric can give a consistent description of the system at a fixed time at most[13].

Cutler commented briefly on their recent work[14] on calculation of inspiral waveforms using the Regge-Wheeler-Teukolsky perturbation formalism. The relevant equations were solved numerically to very high accuracy and a post newtonian expansion was fit to the numerical results. This lead to the surprising result that higher order terms were not getting smaller causing the template waveform to go out of phase with the general relativity waveform very quickly.

Will presented the recent results of Iyer and Will[15] on gravitation radiation reaction in equation of motion of binary systems at PN order beyond the quadrupole approximation. The method is based on PN expression for energy and angular momentum flux to infinity and an assumption of energy and angular momentum balance. The arbitrariness in the formula is related to the coordinate system dependence of the radiation reaction formula. As mentioned earlier it is important to know this secular damping very accurately so that the theoretical template not lose phase with the observed signal.

2. Perturbations

The subject of perturbations in general relativity has developed into a specialized discipline on its own. Bulk of the work in this area has been in one of the following two topics: Cosmological Pertubations and Black Hole Perturbations.

2.1. *Cosmological Perturbations*

The basic questions that started these investigations was the attempt to understand the formation of structure in Cosmology and the growth of inhomogeneities in the expanding universe. Recently, a new covariant approach has been given to study the perturbations and this was summarized by George Ellis in his presentation.The gauge problem of perturbations in cosmology is the arbitrariness in the perturbed quantities arising from the arbitrariness in the choice of the map between the background spacetime and the real spacetime. The gauge problem in perturbed Robertson-Walker cosmologies has not been resolved in a satisfactory way. Bardeen's[16] introduction of gauge invariant variables was a major triumph. However, the formalism and method are not geometrically transparent, the split into scalar, vector and tensors is nonlocal/nonunique, it is not easily related to observations and cannot be easily extended beyond linear order because it is linearized ab initio. Moreover, the analysis is not invariant under general gauge transformations but only under a restricted set that respects the harmonic splitting. A more transparent gauge invariant formalism has been set up using fully covariant methods in terms of variables that are both gauge invariant and covariantly defined, leading to covariant evolution equations. The basic variables are spatial gradients of density, pressure and expansion of the cosmological fluid taken orthogonal to the fluid flow vector. Since they vanish in a Robertson-Walker universe they are gauge invariant and characterise inhomogeneities in the universe. The basic formalism is set up and applied to pressure free matter[17], perfect fluid [18], scalar field[19], multi-fluids and imperfect fluids[20]. Subtle effects due to rotation[21], relation to the Bardeen's approach[22], density waves in cosmology [23] and applications to newtonian cosmology[24] have also been investigated. The main advantages of this formalism is that the geometrical definition is clear, it is defined in an arbitrary spacetime(no background is needed), nonlinear equations can be obtained, the variables are observable in principle and finally there exist newtonian analogues of all equations. The effect of this programme has so far been to rederive standard results in a more transparent gauge invariant way as also more generally valid equations before linearizing about Robertson-Walker. The idea of density waves in cosmology is new. The formalism is also being used to study the Sachs-Wolfe effect[25] and clarify the gauge invariance of these calculations.

2.2. Black Hole Perturbations

The issues that led to the development of this subject were the following: Are black holes stable against small changes? Can one compute what happens when a test particle or radiation scatters off the black hole? What are the frequencies in which the black hole rings and how do the notes die off? The studies were made with all the analytically known black hole solutions: Schwarzschild, Reissner-Nordstrom, Kerr and Kerr-Newman e.g[26]. A variety of methods have been used in these investigations: scalar, vector and tensor harmonics, Newman-Penrose formalism, Debye-Hertz potentials, gauge invariant approaches e.g[27].

2.2.1. Separability of Wave Equations in Curved Backgrounds. One of the most remarkable results was the separability of the perturbation equations of Hamilton Jacobi, scalar, electromagnetic, gravitational, neutrino and electron fields in the background of the Kerr-Newman black hole. The miracle continues and some time back the separability of a Nambu-Goto string configuration in the Kerr background was also established[28]. All this led on to a more critical investigation of the question of separability, the operators commuting with the wave operator and operators whose eigenvalues the separation constants are. The status of issues related to the separability of wave equations on curved backgrounds was the subject of Ray McLenaghan's presentation. A symmetry operator of the equations satisfied by the variables of a physical system is a linear differential operator that maps the space of solutions into itself. The most familiar examples of symmetry operators are operators which commute with the differential operator appearing in the field equations. They are called constants of the motion and their eigenvalues are interpretable as quantum numbers of the system[29]. A remarkable example of such an operator is given by Carter and McLenaghan's[30] discovery of a first order commuting operator for the Dirac operator on Kerr spacetime by an analysis of the separation of variables procedure devised by Chandrasekhar[31]. They showed that these operators admit the separable solutions as eigenfunctions with the corresponding eigenvalues as separation constants and characterised one of them in terms of a valence two Killing spinor satisfying a skew-hermiticity condition. McLenaghan and Spindel[32] gave a tensorial expression for the most general first order commuting operator with the charged Dirac operator on a general curved background in terms of Killing-Yano tensors of valence one, two and three. A different situation arises when dealing with the conformally invariant Klein-Gordon, Dirac and Maxwell equations for zero rest mass particles where symmetry operators which are not necessarily commuting operators must be considered. In the case of the conformally invariant Klein-Gordon equation and the Dirac equation for the neutrino, the symmetry operators appear in the form of R-commuting operators that is operators whose commutators with the wave operator are proportional to it. All such operators up to the second order for the Klein-Gordon equation and up to first order for the Dirac equation have been characterised by Kamran and McLenaghan[33] in terms of conformal Killing vectors and conformal Killing tensors of valence two and conformal Killing-Yano tensors of valence one, two and three respectively. A corresponding analysis for Maxwell's equations has proved more elusive. However, Kalnins, Miller and Williams[34] recently found a second order symmetry operator for Maxwell's equations in the Kerr solution which characterises the separable solutions found previously by Teukolsky. The most general second order symmetry operator for Maxwell's equations on a general curved spacetime has been constructed[35]. This uses a conformal Killing vector, a conformal Killing tensor of valence two and a new valence four tensor with the same algebraic symmetries as the Weyl tensor which satisfies a first order differential equation which has similarities to both the conformal Killing equation and the conformal Killing-Yano equation. This tensor corresponds to a valence four Killing spinor; its properties and integrability conditions for its existence have been studied.

2.2.2. Quasi-Normal Modes. At late times, all perturbations of the black hole are radiated away like the last pure dying tones of a ringing bell. To describe this 'ringing' the notion of quasinormal modes (QNM) was introduced around 1970. QNM's are the sourceless perturbations of spacetime and in the case of the time-evolution of small perturbations of a Schwarzschild black hole are governed by a one-dimensional wave equation. The QNM frequencies are characteristic of the black hole, and (in the Schwarzschild case) depend only on its mass. QNM's excited during for example a gravitational collapse may be eventually detected. Thus the determination of QNM's of black holes is an important problem on which considerable effort and progress has been made in the last three years. Nils Andersson summarized the current status of these calculations as follows. Recently, Nollert and Schmidt [36] proved that the QNM's should be properly defined as poles of the Green's function to the Laplace transformed wave equation. In a simplified picture, the desired solutions correspond to boundary conditions of purely outgoing waves arriving at spatial infinity, and purely ingoing waves crossing the event horizon. The desired solutions to the radial problem increase exponentially towards spatial infinity and the event horizon. To identify a QNM solution an exponentially decreasing solution must be singled out from the exponentially increasing one in the asymptotic region. Hence, the determination of QNM frequencies is a delicate problem. During the last ten years several attempts to determine the QNM frequencies have been made. Leaver[37] determined accurate values for them using a continued fraction approach. Recently Leaver's results have been confirmed as reliable using numerical integration in the complex coordinate plane [38]. For gravitational perturbations, recent double-precision calculations by Leaver (unpublished) agree to nine decimal places with the numerical integration results. Nollert and Schmidt have also verified Leaver's results. These three independent investigations of the problem yield results that agree perfectly for the first ten modes. A semi-analytical phase-integral method has also been applied to the problem [39]. A powerful phase-integral formula determining QNM frequencies for a Schwarzschild black hole has been derived by Andersson and Linnæus[40] . The results obtained from this formula are in good agreement with the numerical results mentioned above. This method may also prove to be powerful in situations more general than the Schwarzschild case, where the methods mentioned above are difficult to apply.

Omar Ortiz talked about recent work[41] on hyberbolizing the heat equation. Systems described by parabolic or hyperbolic-parabolic systems have arbitrarily high propagation velocities incompatible with relativity. One therefore looks for a one parameter family of hyberbolic systems that goes over to the parabolic system in an appropriate limit. For the heat diffusion equation, such an analysis can be done, using as parameter the reciprocal of the maximum speed of propagation. The solution to this family equals the solution to the heat equation plus terms that vanish when the maximum speed of propagation becomes infinite.

3. Open Questions

We conclude with a list of important open questions that remain in these areas[42, 43]. (i) Characterisation of the Teukolsky parameter for gravitational perturbations in Kerr background by a symmetry operator. (ii) Solve the perturbation problem for Kerr-Newman *i.e* exhibit the relevant symmetry operator and its solutions. (iii) Extend the results of separability to other backgrounds like Cosmological and String backgrounds. Do generalized Hertz potentials exist? And if they do, how do we find them? (iv) What can we say about the completeness of QNM's in the neighbourhood of the black hole? (v) Investigate the relation between Einstein's Theory and PNA. (vi) Prove existence theorems for various sources since validity of approximation methods cannot be judged otherwise. (vii) Put procedures used to relate near zone approximations with far fields on a sound basis since they are always used in most applications to astrophysical systems. Normally one relates a higher order PN description of sources to PMA of higher order. (viii) Are different PNA schemes like Chandrasekhar's, Damour- Soffel- Xu and Rendall's equivalent? (ix) Do PNA methods have a fundamental limitation? Can their convergence be improved using better numerical techniques? Are some variables better than others? Or do we need a very different approximation scheme?

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