Gravitational Perturbations in Plasmas and Cosmology

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Abstract

Gravitational perturbations can be in the form of scalars, vectors or tensors. This thesis focuses on the evolution of scalar perturbations in cosmology, and interactions between tensor perturbations, in the form of gravitational waves, and plasma waves.

The gravitational waves studied in this thesis are assumed to have small amplitudes and wavelengths much shorter than the background length scale, allowing for the assumption of a flat background metric. Interactions between gravitational waves and plasmas are described by the Einstein-Maxwell-Vlasov, or the Einstein-Maxwell-fluid equations, depending on the level of detail required. Using such models, linear wave excitation of various waves by gravitational waves in astrophysical plasmas are studied, with a focus on resonance effects. Furthermore, the influence of strong magnetic field quantum electrodynamics, leading to detuning of the gravitational wave-electromagnetic wave resonances, is considered. Various nonlinear phenomena, including parametric excitation and wave steepening are also studied in different astrophysical settings.

In cosmology the evolution of gravitational perturbations are of interest in processes such as structure formation and generation of large scale magnetic fields. Here, the growth of density perturbations in Kantowski-Sachs cosmologies with positive cosmological constant is studied.

Sammanfattning

Gravitationsstörningar finns i form av skalärer, vektorer och tensorer. Denna avhandling behandlar utvecklingen av skalära störningar inom kosmologi och växelverkan mellan tensorstörningar, i form av gravitationsvågor, och plasmavågor.

Gravitationsvågorna som studeras i avhandlingen antas ha små amplituder och våglängder mycket kortare än bakgrundens karakteristiska längdskala, vilket möjliggör antagandet av en plan bakgrundsmetrik. Växelverkan mellan gravitationsvågor och plasmor beskrivs av Einstein-Maxwell-Vlasov-, eller Einstein-Maxwell-vätskeekvationerna beroende på vilken grad av detaljinformation som krävs. Inom ramen för sådana modeller studeras linjär koppling av plasmavågor och gravitationsvågor i astrofysikaliska sammanhang, med fokus på resonanseffekter. Vidare undersöks modifieringen av resonansen mellan gravitationsvågor och elekromagnetiska vågor på grund av kvantelektrodynamiska effekter i starka magnetfält. Olika ickelinjära fenomen, bland annat parametrisk excitation och sk. *wave steepening* behandlas också i att antal astrofysikaliska sammanhang.

Studiet av tidsutvecklingen av gravitationsstörningar är av intresse inom kosmologi, då bland annat i processer såsom strukturformation och generering av storskaliga magnetiska fält. I denna avhandling studeras tillväxt av densitetsstörningar i Kantowski-Sachs kosmologier med positiv kosmologisk konstant.

Publications

The Thesis is based on the following papers:

I **"Nonlinear interactions between gravitational radiation and modified Alfvén modes in astrophysical dusty plasmas"**
M.Forsberg, G. Brodin, M. Marklund, P. K. Shukla, and J. Moortgat,
Phys. Rev. D, **74**, 064014 (2006)

II **"Harmonic generation of gravitational wave induced Alfvén waves"**
M. Forsberg and G. Brodin,
Phys. Rev. D, **77**, 024050 (2008)

III **"Interaction between gravitational waves and plasma waves in the Vlasov description"**
G. Brodin, M. Forsberg, M. Marklund and D. Eriksson,
J. Plasma Phys., **76**, 345, (2010)

IV **"Influence of strong field vacuum polarization on gravitational-electromagnetic wave interaction"**
M. Forsberg, D. Papadopoulos, and G. Brodin,
Phys. Rev. D, **82**, 024001 (2010)

V **"Linear theory of gravitational wave propagation in a magnetized, relativistic Vlasov plasma"**
M. Forsberg and G. Brodin,
Accepted for publication in *Phys. Rev. D.*

VI **“Density growth in Kantowski-Sachs cosmologies with cosmological constant”**
M. Bradley, P.K.S. Dunsby and M. Forsberg,
To be submitted.

Contents

Chapter 1

Introduction

This thesis details the propagation and interaction of gravitational perturbations on different backgrounds. Gravitation is assumed to be described by general relativity (GR), the currently most strongly supported theory of gravity. GR differs from Newtonian gravity in a number of ways, for example GR predicts the existence of gravitational waves (GWs), although the two theories coincide in the limit of low velocities and weak fields.

The gravitational processes discussed in this thesis will mostly involve GWs, whose interaction with various plasma waves and electromagnetic waves (EMWs) are studied and discussed in Papers I-V. Scalar gravitational perturbations in an anisotropic cosmology are also studied; this is done in Paper VI.

The fundamentals of GR and the various formalisms used in this thesis are detailed in Chapter 2. This includes an introduction to GWs, the 1+3 covariant formalism, the 1+1+2 covariant formalism and the construction of tetrads. Accordingly, GR is very much the theme of this thesis, since all papers on which this thesis is built contain various GR processes.

There are many textbooks and reviews directed towards GR and perturbations in GR, such as Refs. [1, 2, 3, 4] as well as books devoted to cosmology [5, 6] where the interested reader can find out more about the details of these subjects.

Chapter 3 contains basic plasma theory and the coupling of plasma physics, electrodynamics and GR. The coupling between GR and plasmas are of interest due to the presence of plasmas close to strong GW sources. As a consequence interactions between strong GWs and plasmas, including EMWs, become possible. Even if the GWs produced by such sources cannot be detected directly by observers on Earth, there is a possibility that waves and other phenomena resulting from such interactions are possible to observe by e.g. radio telescopes [7, 8]. Furthermore GW-plasma and GW-EMW interactions can be of interest in models of the early universe. Strong magnetic field quantum electrodynamical (QED) effects are also discussed in this chapter.

Further details on plasma physics can be found in textbooks such as Refs. [9, 10, 11]. Electromagnetic fields in cosmology are detailed in Ref. [12] and QED effects has been covered by e.g. Refs. [13, 14].

Chaper 4 is devoted to the study of some interesting nonlinear wave phenomena. Due to the nonlinear nature of plasmas such phenomena are commonly studied in the field of plasma physics. Phenomena of interest here include nonlinear wave steepening and three wave couplings. Textbooks detailing nonlinear waves and interactions include Refs. [15, 16, 17, 18, 19].

Chapter 2

General Relativity

General relativity (GR) describes gravitation not as forces acting instantaneously between masses, as is the case in Newtonian gravity, but instead as an effect of the curvature of *spacetime*. Effects of changes in the geometry propagate with the speed of light, c. A test mass, only affected by gravitation, will follow the straightest possible path on the curved spacetime; these paths are called *geodesics*. Locally geodesics will appear to be straight, since locally the curvature of spacetime can always be transformed away by the appropriate choice of inertial frame.

In GR the curvature of spacetime is determined by the distribution of matter and energy through Einstein's field equations (EFE)

$$G_{\mu\nu} = \kappa T_{\mu\nu} - \Lambda g_{\mu\nu} \ , \tag{2.1}$$

where $G_{\mu\nu}$ is the Einstein tensor, which contains term related to the curvature, Λ the cosmological constant, which can be neglected in most astrophysical applications, and $T_{\mu\nu}$ the energy-momentum tensor, containing the distribution of matter and energy. The constant κ, relating curvature with matter and energy density takes the value $\kappa = 8\pi G/c^4$, where G is the gravitational constant.

The mathematical description of GR is based on differential geometry, and spacetime in this case is a four dimensional Lorentzian manifold, $\mathcal{M}$, with a metric, $g_{\mu\nu}$, where each point on the manifold corresponds to an event. The metric, which describes the curvature of spacetime, is a tensor determining the distance between nearby points. There are numerous textbooks (see e.g. Refs. [20, 21]) where differential geometry is described in detail.

Notation and Conventions

Throughout this thesis it will be assumed that the metric has the signature $(-,+,+,+)$. Furthermore, tensor elements, such as $T^{\mu\cdots}{}_{\nu\ldots}$, will usually be referred to as tensors, since it is generally understood which basis is used. Strictly speaking the tensor is actually $\mathcal{T} = T^{\mu\cdots}{}_{\nu\ldots}\partial_\mu \otimes \ldots \otimes dx^\nu \otimes \ldots$, where $\{\partial_\mu\}$ is the vector basis, $\{dx^\mu\}$ the dual of the vector basis, and $\otimes$ denotes the tensor product. In this example the basis is a coordinate basis but, as will be discussed later, the basis can be chosen in different ways.

Greek indices, $\alpha, \beta, ... = 0, 1, 2, 3$, will be used in the coordinate formalism, and latin indices, $a, b, ..., h = 0, 1, 2, 3$, will be used in a tetrad basis. Latin indices, $i, j, ... = 1, 2, 3$, are reserved for the spatial parts of any basis.

The covariant derivative of a tensor $T_{\mu\nu}$ will be denoted by $\nabla_\sigma T_{\mu\nu}$ or $T_{\mu\nu;\sigma}$, while the partial derivative is written $\partial_\sigma T_{\mu\nu}$ or $T_{\mu\nu,\sigma}$.

2.1 Gravitational Waves

One interesting prediction of GR that has not yet been directly verified is the existence of gravitational waves (GWs). These are wave-like vacuum solutions to EFE, propagating with the speed of light. GWs manifest themselves as ripples in spacetime, which will alter the distance between test particles as they propagate past them. In a similar way as EMWs are generated from accelerated charges, GWs are generated from accelerated masses[1]. However, since the second time derivative of the mass dipole moment vanishes (due to conservation of momentum) there is no GW dipole radiation; in a multipole expansion of the radiation the lowest order non-vanishing part is the quadrupole.

GWs were originally predicted as early as 1916 [22], but their existence was highly debated until the late seventies. The discovery of the Hulse-Taylor binary pulsar in 1975 [23] provided the first indirect evidence of the existence of GWs. Measurements of the orbital period showed a slow decay of the orbits of the binary, indicating a loss of energy consistent with the loss due to emission of GWs predicted by

[1]This analogy gives the essence of the physics, but should not be taken too literally.

GR.

The main reason that GWs have proven to be so hard to detect directly is their weak coupling to matter and the vast distances to their sources. As an example it can be noted that the GW emission from a supernova collapse in a nearby galaxy will only perturb the distance between two test particles on Earth by roughly 10^{-20} times the original distance [24]. Nevertheless there are ambitious projects, both running and planned [25], and the hope is to be able to detect GWs in the near future.

Some of the sources considered to be interesting for the purpose of direct detection include massive binary stars [26], supernovae [27], neutron star quakes [28] and black holes during ringdown [29]. Furthermore, cosmological GWs, possibly generated in the early universe, see e.g. Ref. [30], might provide very interesting information if they could be detected.

2.1.1 Linear GWs in Vacuum

In what follows a short derivation of weak GWs in the high frequency limit in vacuum will be considered. Let $g^B_{\mu\nu}$ be a vacuum metric that satisfies the EFEs and assume that this metric is perturbed by the quantity $h_{\mu\nu}$, which is considered small, i.e. $h_{\mu\nu} \ll 1$ holds. The full metric can now be written

$$g_{\mu\nu} = g^B_{\mu\nu} + h_{\mu\nu} \ . \tag{2.2}$$

Since $h_{\mu\nu}$ is small, only terms to lowest order in $h_{\mu\nu}$ are considered, and thus the background metric can be used for raising and lowering indices of quantities proportional to $h_{\mu\nu}$ and its derivatives.

The Ricci tensor, $R_{\mu\nu}$, can now be split into a background part, $R^B_{\mu\nu}$ and a first order perturbation part, $R^P_{\mu\nu} \propto h_{\mu\nu}$, such that

$$R_{\mu\nu} = R^B_{\mu\nu} + R^P_{\mu\nu} + \mathcal{O}\left(h^2_{\mu\nu}\right) \ . \tag{2.3}$$

Since it is a vacuum spacetime, the Ricci tensor must be zero to all orders, i.e. $R^B_{\mu\nu} = R^P_{\mu\nu} = 0$. Expressing the first order Ricci tensor in terms of the metric perturbation yields

$$R^P_{\mu\nu} = \tfrac{1}{2}\left(\nabla_\nu\nabla_\mu h_\alpha{}^\alpha + \nabla^\alpha\nabla_\alpha h_{\mu\nu} - \nabla_\alpha\nabla_\nu h_\mu{}^\alpha - \nabla_\alpha\nabla_\mu h_\nu{}^\alpha\right) = 0 \ . \tag{2.4}$$

Assuming that the wavelength of the GW is much smaller than the background curvature, an approximation referred to as the *high frequency limit*, the order of the covariant derivatives in the last two terms of Eq. (2.4) can be exchanged. Furthermore, $h_{\mu\nu}$ can still be subjected to certain gauge transformations (see e.g. Refs. [1, 2] for more details on this). Choosing a gauge such that $\nabla_\alpha h^{\mu\alpha} = h^\alpha{}_\alpha = 0$, referred to as the *transverse traceless* (TT) gauge, reduces Eq. (2.4) to

$$\nabla^\alpha \nabla_\alpha h_{\mu\nu} = 0 \ . \tag{2.5}$$

As an example, the wave equation Eq. (2.5) for a GW propagating in the z-direction on a flat background reduces to

$$\left(-\frac{1}{c^2}\partial_t^2 + \partial_z^2\right) h_{+,\times} = 0 \ , \tag{2.6}$$

where $h_+ \equiv h_{11} = -h_{22}$ and $h_\times \equiv h_{12}$ and all other components of $h_{\mu\nu}$ are zero. As can be seen there are just two remaining independent degrees of freedom, i.e. two independent polarizations which differ only by a rotation of $\pi/4$ around the direction of propagation.

The energy density, $\mathcal{W}$, carried by a GW is obtained from the Landau-Lifshitz energy-momentum pseudo tensor [1]. In the case of linear GWs in the TT-gauge this reduces to

$$\mathcal{W} = \frac{1}{2\kappa c^2}\left(\dot{h}_+^2 + \dot{h}_\times^2\right) \ , \tag{2.7}$$

as detailed in [1].

2.2 Cosmology and the 1+3 Covariant Formalism

Cosmology is the study of the large scale structure of the universe and its evolution. It is often assumed that on sufficiently large scales, the universe is homogeneous and isotropic, which is referred to as the cosmological principle. Recent observations, see e.g. [31], indicates that this at least seems to be a good approximation. The most general models of the universe satisfying the cosmological principle are the Friedmann-Lemaître-Robertson-Walker (FLRW) cosmologies.

There are of course other cosmological models, where the cosmological principle is not invoked; in particular the class of models which are homogeneous, but not isotropic, consists of Bianchi types I-IX [5], and the Kantowski-Sachs solution [32]. These type of models, although maybe not describing the universe as a whole, might still be interesting to study. Also, many of these models may be close to FLRW models during large periods of time.

In order to provide a clearer description of the physics in a particular system it is often convenient to divide spacetime into a three dimensional space part and a one dimensional time part. This can be achieved by splitting with respect to a four-velocity field, u^μ, which can be seen as a field of fictitious observers (in cosmology this is usually taken to be the average four-velocity of the cosmic fluid). An observer will perceive space as the hypersurface perpendicular to its own four-velocity.

The chosen four-velocity can be used to construct the useful projection operators

$$h_{\mu\nu} \equiv g_{\mu\nu} + u_\mu u_\nu \ , \tag{2.8}$$

which projects on the local space perpendicular to u^μ, and

$$U^\mu{}_\nu \equiv -u^\mu u_\nu \ , \tag{2.9}$$

which projects parallel to u^μ.

With the help of these projection operators, two different derivative operators can be defined. The first of these is the *covariant time derivative*, denoted with a dot, which when acting on a second rank tensor $T^\mu{}_\nu$ produces

$$\dot{T}^\mu_\nu = u^\alpha \nabla_\alpha T^\mu_\nu \ . \tag{2.10}$$

The second kind of derivative operator, the *fully orthogonally projected covariant derivative*, $\tilde{\nabla}$, is defined as the covariant derivative with the projection $h_{\mu\nu}$ operator acting on all free indices, i.e.

$$\tilde{\nabla}_\tau T^\mu_\nu = h^\mu{}_\alpha h^\beta{}_\tau h^\gamma{}_\tau \nabla_\gamma T^\alpha_\beta \ . \tag{2.11}$$

Projections with $h_{\mu\nu}$ of vectors are denoted with angle brackets, so for a vector V^μ the corresponding projected vector is

$$V^{<\mu>} \equiv h^\mu{}_\alpha V^\alpha \ . \tag{2.12}$$

The projected symmetric trace-free (PSTF) part of a second rank tensor is also denoted by the use of angle brackets, such that the PSTF part of the tensor $T^{\mu\nu}$ is

$$T^{<\mu\nu>} \equiv \left[h_\alpha{}^{(\mu} h_\beta{}^{\nu)} - \tfrac{1}{3} h^{\mu\nu} h_{\alpha\beta}\right] T^{\alpha\beta} \ . \tag{2.13}$$

Given the four dimensional volume element $\epsilon_{\mu\nu\sigma\tau} = \epsilon_{[\mu\nu\sigma\tau]}$, where $\epsilon_{0123} = \sqrt{|\det g_{\mu\nu}|}$, it is useful to define the *rest-space volume element* as

$$\epsilon_{\mu\nu\sigma} \equiv u^\tau \epsilon_{\tau\mu\nu\sigma}. \tag{2.14}$$

With these ingredients the *curl* of the tensor $T^{\mu\nu}$ can be defined as

$$(\mathrm{curl}\ T)^{\mu\nu} \equiv \epsilon^{\alpha\beta<\mu} \tilde{\nabla}_\alpha T^{\nu>}{}_\beta. \tag{2.15}$$

Taking the covariant derivative of the chosen four-velocity field u^μ yields

$$\nabla_\mu u_\nu = -u_\mu \dot{u}_\nu + \tilde{\nabla}_\mu = -u_\mu \dot{u}_\nu + \tfrac{1}{3}\theta h_{\mu\nu} + \omega_{\mu\nu} + \sigma_{\mu\nu} \ , \tag{2.16}$$

where $\dot{u}_\mu$ is the acceleration, $\theta \equiv \tilde{\nabla}_\alpha u^\alpha$ the *expansion*, $\omega_{\mu\nu} \equiv \tilde{\nabla}_{[\mu} u_{\nu]}$ the *vorticity* and $\sigma_{\mu\nu} \equiv \tilde{\nabla}_{<\mu} u_{\nu>}$ the *shear*. For a more detailed description and interpretation of these quantities see Ref. [6]. Although models with nonzero vorticity has been studied by some authors (see e.g. Ref. [33]) they will not be considered here, so for the remainder of the thesis it will be assumed that $\omega_{\mu\nu} = 0$.

The *Weyl conformal curvature tensor*, $C_{\mu\nu\sigma\tau}$, or Weyl tensor for short, is related to purely gravitational degrees of freedom, such as tidal forces, frame dragging and GWs. In a similar way as the Faraday tensor can be split into an electric and a magnetic part (this will be detailed in Chapter 3), the Weyl tensor can be split relative to u^μ into its "electric" and "magnetic" parts. The electric part of the Weyl tensor is defined as

$$E_{\mu\nu} \equiv C_{\mu\nu\alpha\beta} u^\alpha u^\beta \ , \tag{2.17}$$

and the magnetic part is

$$H_{\mu\nu} \equiv \tfrac{1}{2} \epsilon_{\mu\alpha\beta} C^{\alpha\beta}{}_{\nu\gamma} u^\gamma \ . \tag{2.18}$$

Note that both $E_{\mu\nu}$ and $H_{\mu\nu}$ are PSTF.

The energy and matter content can also be decomposed relative the observer four-velocity. This is done by splitting the energy-momentum tensor as

$$T_{\mu\nu} = \rho u_\mu u_\mu + p h_{\mu\nu} + 2 q_{(\mu} u_{\nu)} + \pi_{\mu\nu}, \tag{2.19}$$

where $\rho \equiv T_{\alpha\beta}u^{\alpha}u^{\beta}$ is the *relativistic energy density* relative to u^{μ}, $p \equiv (1/3)T^{\alpha\beta}h_{\alpha\beta}$ is the *isotropic pressure*, $q_{\mu} \equiv -T_{\alpha\beta}u^{\alpha}h^{\beta}{}_{\mu}$ is the *energy flux* and $\pi_{\mu\nu} \equiv T_{\alpha\beta}h^{\alpha}{}_{<\mu}h^{\beta}{}_{\nu>}$ the *anisotropic pressure.*

In order to fully describe the matter content, equations of state are needed. The equations of state relate the quantities obtained from Eq. (2.19) to each other, as governed by the physics of the situation. In most cosmological models, the equations of state are usually taken to be

$$p = (\gamma - 1)\rho \ , \ q_{\mu} = \pi_{\mu\nu} = 0, \tag{2.20}$$

where γ is referred to as the barytropic index. This is a special case of a *perfect fluid*, and the cases of $\gamma = 1$ and $\gamma = 4/3$ are generally referred to as "dust" and "radiation" respectively. The choice of a perfect fluid as matter content in the universe seems reasonable based on observations.

There are situations when non-perfect fluids can be considered in cosmological models, see e.g. Refs. [33, 34] , but in what follows it will be assumed that the equation of state is of the form (2.20).

2.2.1 Propagation equations and constraints

The evolution of spacetime, represented by the set $\{\dot{u}^{\mu}, \theta, \sigma_{\mu\nu}, E_{\mu\nu}, H_{\mu\nu}\}$ and its perfect fluid contents, represented by $\{\rho, \gamma\}$, is determined by the EFE (2.1) and its integrability conditions, resulting in three sets of equations.

The first set is obtained from the Ricci identities for the observer four-velocity, i.e.

$$2\nabla_{[\mu}\nabla_{\nu]}u^{\sigma} = R_{\mu\nu}{}^{\sigma}{}_{\tau}u^{\tau} \ , \tag{2.21}$$

where $R_{\mu\nu\sigma\tau}$ is the Riemann tensor. Using Eqs. (2.16) and (2.1) these identities split into two propagation equations and two constraints. The propagation equations are the *Raychaudhuri equation*

$$\dot{\theta} - \tilde{\nabla}_{\alpha}\dot{u}^{\alpha} = -\tfrac{1}{3}\theta^2 + \dot{u}^{\alpha}\dot{u}_{\alpha} - 2\sigma^2 + \tfrac{1}{2}\left(\rho + 3p\right) + \Lambda, \tag{2.22}$$

where $\sigma^2 \equiv \sigma^{\alpha\beta}\sigma_{\alpha\beta}/2$, and the *shear propagation equation*

$$\dot{\sigma}^{<\mu\nu>} - \tilde{\nabla}^{<\mu}\dot{u}^{\nu>} = -\tfrac{2}{3}\theta\sigma^{\mu\nu} + \dot{u}^{<\mu}\dot{u}^{\nu>} - \sigma^{<\mu}{}_{\alpha}\sigma^{\nu>\alpha} - E^{\mu\nu} \ , \tag{2.23}$$

and the constraint equations are the (0i)-equations

$$\tilde{\nabla}_{\alpha}\sigma^{\mu\alpha} - \frac{2}{3}\tilde{\nabla}^{\mu}\theta = 0 \ . \tag{2.24}$$

and the $H_{\mu\nu}$-equations

$$H^{\mu\nu} - (\mathrm{curl}\ \sigma)^{\mu\nu} = 0. \tag{2.25}$$

The second set of equations is obtained from the *twice-contracted Bianchi identities*, $G^{\mu\nu}{}_{;\nu} = 0$ which, together with the EFE (2.1), result in a propagation equation, known as the *energy conservation equation*

$$\dot{\rho} = -\theta\left(\rho + p\right) \ , \tag{2.26}$$

and a constraint, called the *momentum conservation equation*,

$$\tilde{\nabla}_{\mu} p = -\dot{u}_{\mu}\left(\rho + p\right) \ . \tag{2.27}$$

The third set of equations arises from the remaining Bianchi identities,

$$\nabla_{[\mu} R_{\nu\sigma]\tau\xi} = 0 \ , \tag{2.28}$$

together with the EFE (2.1). By rewriting the Riemann tensor in terms of the kinematical quantities and the electric and magnetic part of the Weyl tensor (see Ref. [6] for details), two propagation equations and two constraints are obtained. The first of the propagation equations is the $\dot{E}$*-equation*

$$\dot{E}^{<\mu\nu>} - (\mathrm{curl}\ H)^{\mu\nu} = -\tfrac{1}{2}\left(\rho + p\right)\sigma^{\mu\nu} - \theta E^{\mu\nu} + 3\sigma^{<\mu}{}_{\alpha} E^{\nu>\alpha} + 2\epsilon^{\alpha\beta<\mu}\dot{u}_{\alpha} H^{\nu>}_{\beta} \tag{2.29}$$

and the second one the $\dot{H}$*-equation*

$$\dot{H}^{<\mu\nu>} + (\mathrm{curl}\ E)^{\mu\nu} = -\theta H^{\mu\nu} + 3\sigma^{<\mu}{}_{\alpha} H^{\nu>\alpha} - 2\epsilon^{\alpha\beta<\mu}\dot{u}_{\alpha} E^{\nu>}_{\beta} \ . \tag{2.30}$$

The constraint equations are

$$\tilde{\nabla}_{\alpha} E^{\mu\alpha} - \tfrac{1}{3}\tilde{\nabla}^{\mu}\rho - \epsilon^{\mu\alpha\beta}\sigma_{\alpha\gamma} H^{\gamma}{}_{\beta} = 0 \ , \tag{2.31}$$

and

$$\tilde{\nabla}_{\alpha} H^{\mu\alpha} + \epsilon^{\mu\alpha\beta}\sigma_{\alpha\gamma} E^{\gamma}{}_{\beta} = 0 \ . \tag{2.32}$$

Note that from the two propagation equations (2.29) and (2.30) it is possible to see how GW solutions arises. Taking the time derivative of Eq. (2.29), and using the commutation relations to interchange the covariant time and fully orthogonally projected covariant derivatives, and then eliminating the $\dot{H}^{<\mu\nu>}$ term will result in a wave equation for the tensor $E^{<\mu\nu>}$. The commutation relations are useful tools since they can be used to construct propagation equations for other quantities, e.g. the density gradient.

The 1+3 covariant formalism, as well as the 1+1+2 covariant formalism presented below, are suitable for perturbative calculations. The perturbed quantities are in these methods represented by covariantly defined objects that vanish on the background, making the theory gauge invariant [35].

2.2.2 Harmonic Decomposition in the 1+3 Covariant Formalism

Similarly to a plane wave ansatz, the spatial and temporal dependence of a perturbed variable can be separated using an harmonic decomposition, provided that the background is homogeneous and isotropic. By introducing *scalar harmonics*, Q_k, satisfying

$$\tilde{\nabla}^2 Q_k = -\frac{k^2}{A^2} Q_k \ , \ \dot{Q}_k = 0 \ , \tag{2.33}$$

where A is the scale factor, any scalar Ψ can now be expressed as the sum

$$\Psi = \sum_k \Psi_k Q_k \ . \tag{2.34}$$

As Eq. (2.34) shows this decomposition is very similar to expanding a function into a Fourier series.

In a similar fashion as the expansion of scalar harmonics, vectors and tensors can be decomposed using *vector harmonics* and *tensor harmonics*. Details of this harmonic decomposition can be found in e.g. Refs. [6, 36].

2.3 1+1+2 Covariant Formalism

The 1+3 covariant formalism has many advantages, especially when the spacetime is an almost FLRW model. However, when examining models that at each point have a preferred spatial direction, such as locally rotationally symmetric (LRS) models, this formalism might not be so suitable.

By starting from the 1+3 covariant formalism and performing yet another split, this time with respect to a spatial vector n^μ, parallel to the direction of the anisotropy, such spacetimes can be conveniently modelled. This is referred to as the *1+1+2 covariant formalism* and is detailed in Ref. [37].

The spatial vector n^μ allows construction of the projection operator

$$N_\mu{}^\nu \equiv h_\mu{}^\nu - n_\mu n^\nu \ . \tag{2.35}$$

$N_\mu{}^\nu$ is used to project vectors and tensors perpendicular to n^μ (and u^μ). Any projected vector $V^{<\mu>}$ can now be decomposed with respect to n^μ as

$$V^{<\mu>} = V n^\mu + V^{\bar{\mu}} \ , \tag{2.36}$$

where $V \equiv n_\alpha V^\alpha$ and the bar over an index denotes projection with $N^{\mu\nu}$, i.e. $V^{\bar{\mu}} \equiv N^\mu{}_\alpha V^\alpha$. PSTF Tensors are decomposed as

$$T^{<\mu\nu>} = \left(n^\mu n^\nu - \tfrac{1}{2}N^{\mu\nu}\right) T + 2n^{(\mu}T^{\nu)} + T^{\{\mu\nu\}} \ , \tag{2.37}$$

where

$$T \equiv n^\alpha n^\beta T_{<\alpha\beta>} \ , \tag{2.38}$$

$$T^\mu \equiv N_\mu{}^\alpha n^\beta T_{<\alpha\beta>} \ , \tag{2.39}$$

$$T^{\{\mu\nu\}} \equiv \left(N_\alpha{}^{(\mu}N_\beta{}^{\nu)} - \tfrac{1}{2}N^{\mu\nu}N_{\alpha\beta}\right) T^{<\alpha\beta>} \ . \tag{2.40}$$

In the same fashion as the 1+3 split separates the time and spatial derivatives, the derivative $\tilde{\nabla}$ can be further divided into a derivative along n^μ, denoted with a “hat”, i.e.

$$\hat{T}_{\mu\nu} \equiv n^\alpha \tilde{\nabla}_\alpha T_{\mu\nu} \ , \tag{2.41}$$

and a derivative perpendicular to n^μ, denoted by δ, such that

$$\delta_\sigma T_{\mu\nu} \equiv N_\sigma{}^\gamma N_\mu{}^\alpha N_\nu{}^\beta T_{\alpha\beta} \ . \tag{2.42}$$

It is useful to define the surface element perpendicular to n^μ by

$$\epsilon_{\mu\nu} \equiv n^\alpha \epsilon_{\mu\nu\alpha} \ . \tag{2.43}$$

The fully orthogonally projected covariant derivative of the spatial vector n^μ can be decomposed, in much the same way as the decomposition of the derivative of the four-velocity in the 1+3 split, in order to obtain

$$\tilde{\nabla}_\mu n_\nu = n_\mu a_\nu + \tfrac{1}{2}N_{\mu\nu}\phi + \epsilon_{\mu\nu}\xi + \zeta_{\mu\nu} \ , \tag{2.44}$$

where $a_\mu \equiv \hat{n}_\mu$ is the "acceleration" , $\phi = \delta_\alpha n^\alpha$ the sheet expansion , $\xi \equiv \frac{1}{2}\epsilon_{\alpha\beta}\delta_\alpha n_\beta$ the "twisting", i.e. rotation of n^μ and $\zeta_{\mu\nu} \equiv \delta_{\{\mu}n_{\nu\}}$ the shear of n^μ, i.e. the distortion of the sheet.

Similarly, the covariant time derivative of n^μ can be decomposed as

$$\dot{n}^\mu = \mathcal{A}u_\mu + \dot{n}_{\bar{\mu}} \ , \tag{2.45}$$

where $\mathcal{A} \equiv n^\alpha \dot{u}_\alpha$.

2.3.1 Harmonic Decomposition in the 1+1+2 Covariant Formalism

Similar to what is done in the 1+3 formalism it is also possible to make a harmonic decomposition in the 1+1+2 split, but now there are two different harmonic functions to consider; one will be parallel to n^μ and the other one will be lying on the sheet perpendicular to n^μ.

The two different spatial derivatives, defined in Eqs. (2.41 - 2.42), can be used to construct two different Laplace operators

$$\delta^2 \equiv \delta^\alpha \delta_\alpha \ , \tag{2.46}$$

and

$$\hat{\Delta} \equiv n^\alpha \tilde{\nabla}_\alpha n^\beta \tilde{\nabla}_\beta \ , \tag{2.47}$$

The operator (2.46) can now be used to introduce the perpendicular scalar harmonic function, $Q_{k_\perp}$, satisfying

$$\hat{Q}_{k_\perp} = \dot{Q}_{k_\perp} = 0 \ \ , \ \ \delta^2 Q_{k_\perp} = -\frac{{k_\perp}^2}{{A_\perp}^2} Q_{k_\perp} \ , \tag{2.48}$$

where $A_\perp$ is the scale factor perpendicular to n^μ, which in a LRS model satisfies

$$\frac{\dot{A}_\perp}{A_\perp} = \frac{1}{3}\theta - \frac{1}{2}\Sigma \ , \tag{2.49}$$

where Σ is defined from the shear by $\sigma_{\mu\nu} = \Sigma\left(n_\mu n_\nu - \frac{1}{2}N_{\mu\nu}\right)$. In the same fashion the operator (2.47) can be used to define the parallel scalar harmonic function, $P_{k_\parallel}$, by demanding

$$\delta P_{k_\parallel} = \dot{P}_{k_\parallel} = 0 \ \ , \ \ \hat{\Delta} P_{k_\parallel} = -\frac{{k_\parallel}^2}{{A_\parallel}^2} P_{k_\parallel} \ , \tag{2.50}$$

holds, where $A_{\|}$ is the scale factor along n^μ, and

$$\frac{\dot{A}_{\|}}{A_{\|}} = \frac{1}{3}\theta + \Sigma \ , \tag{2.51}$$

holds if the model is LRS. Using these scalar harmonics any scalar quantity Ψ can be expressed as

$$\Psi = \sum_{k_{\|},k_{\perp}} \Psi_{k_{\|}k_{\perp}} P_{k_{\|}} Q_{k_{\perp}} . \tag{2.52}$$

Here only the decomposition of scalar perturbations are considered. Vectors and tensors can also be decomposed using vector and tensor harmonics, as detailed in [37, 38].

The 1+1+2 formalism is used in Paper VI, where scalar perturbations of a Kantowski-Sachs background with a nonzero cosmological constant are studied.

2.4 Tetrad Formalism

When working with spacetimes that lack any particular symmetries it can be useful to use a more general basis than the coordinate basis. A *tetrad* is a set of four linearly independent vectors, $\mathbf{e}_a$, related to the coordinate vector basis, ∂_μ, through $\mathbf{e}_a = e_a{}^\mu \partial_\mu$. To this vector basis there is a corresponding dual basis consisting of four one-forms $\omega^a = \omega^a{}_\mu dx^\mu$ such that

$$\omega^a \mathbf{e}_b = \delta^a_b \ . \tag{2.53}$$

Any tensor with elements $T^{\mu\cdots}{}_{\nu\ldots}$ in the coordinate description can now be expressed in the tetrad basis as

$$T^{a\ldots}{}_{b\ldots} = T^{\mu\ldots}{}_{\nu\ldots} \omega^a{}_\mu \ldots e_b{}^\nu \ldots \ . \tag{2.54}$$

In particular the metric in a coordinate basis, $g_{\mu\nu}$, is related to the metric in a tetrad basis, g_{ab}, by $g_{ab} = g_{\mu\nu} e_a{}^\mu e_b{}^\nu$.

The commutator of two tetrad basis vectors can be written

$$[\mathbf{e}_a, \mathbf{e}_b] = C^c{}_{ab} \mathbf{e}_c \ , \tag{2.55}$$

where $C^c{}_{ab}(x^d)$ are the *structure coefficients* of the basis. Note that in the case of a coordinate basis the structure coefficients are zero.

The covariant derivative of a vector V_a in the tetrad basis becomes

$$\nabla_a V_b = \mathbf{e}_a V_b - \Gamma^c_{ab} V_c \ , \tag{2.56}$$

where the *connection coefficients*, $\Gamma^a{}_{bc}$, analogous to the Christoffel symbols in a coordinate basis, are related to the structure coefficients and the metric by

$$\Gamma^a{}_{bc} = \tfrac{1}{2} g^{ad} \left(g_{db,c} + g_{dc,b} - g_{bc,d} \right) + \tfrac{1}{2} g^{ad} \left(g_{eb} C^e{}_{dc} + g_{ec} C^e{}_{db} \right) - \tfrac{1}{2} C^a{}_{bc} \ . \tag{2.57}$$

The Riemann tensor can be described in terms of Ricci rotation and structure coefficients as

$$R^a{}_{bcd} = \mathbf{e}_c \Gamma^a{}_{bd} - \mathbf{e}_d \Gamma^a{}_{bc} + \Gamma^e{}_{bd} \Gamma^a{}_{ec} - \Gamma^e{}_{bc} \Gamma^a{}_{ed} - C^e{}_{cd} \Gamma^a{}_{be} \ . \tag{2.58}$$

In a coordinate basis it can be seen, from Eqs. (2.57) and (2.58), that the Ricci rotation coefficients are equal to the Christoffel symbols and that the Riemann tensor takes its usual form (see e.g. [5]).

2.4.1 Orthonormal Frames and Three-Vector Notation

It is often convenient to choose the tetrad such that the metric g_{ab} is constant. In such a setting the connection coefficients (2.57) simplify considerably, resulting in

$$\Gamma_{abc} = \tfrac{1}{2} \left(C_{bac} + C_{cab} - C_{abc} \right) \ . \tag{2.59}$$

These are referred to as the *Ricci rotation coefficients* and, due to the definition of the structure coefficients (2.55), they are anti-symmetric in the first two indices, i.e. $\Gamma_{abc} = -\Gamma_{bac}$ holds.

The particular choice of $g_{ab} = \eta_{ab}$ is very useful since it implies that the metric is locally Minkowski everywhere. Such a tetrad is normally referred to as an *orthonormal frame* (ONF).

An ONF has a rather nice feature, especially when using perturbation theory, in that raising and lowering indices will not introduce additional terms coupled to the curvature, thus making the introduction and interpretation of perturbations clearer.

When using an ONF it often makes sense to introduce a three-vector notation, such that the three spatial components of the four-vector V^a are defined as $\mathbf{V} \equiv (V^1, V^2, V^3)$. Furthermore it is convenient to define the three dimensional del operator, similar to the usual del operator (sometimes referred to as the nabla operator) in vector calculus, by

$$\nabla \equiv (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) \ . \tag{2.60}$$

ONFs are used in Papers I-V when describing GWs as perturbations on a Minkowski spacetime.

2.4.2 GWs in an Orthonormal Frame

Here an example of the use of an ONF in the case of weak GWs in the TT gauge, propagating in the z-direction on a flat vacuum background will be considered. In this setting the line element is

$$ds^2 = -c^2dt^2 + (1+h_+)\,dx^2 + 2h_\times dxdy + (1-h_+)\,dy^2 + dz^2 \tag{2.61}$$

where $h_{\times,+} \ll 1$. Using an orthonormal frame the metric is expressed as

$$ds^2 = \eta_{ab}\omega^a\omega^b \ , \tag{2.62}$$

where the basis 1-forms $\{\omega^a\}$ are

$$\begin{aligned} \omega^0 &= cdt \ , \\ \omega^1 &= \left(1+\tfrac{1}{2}h_+\right)dx + \tfrac{1}{2}h_\times dy \ , \\ \omega^2 &= \left(1-\tfrac{1}{2}h_+\right)dy + \tfrac{1}{2}h_\times dx \ , \\ \omega^3 &= dz \ . \end{aligned} \tag{2.63}$$

The basis vectors, $\{\mathbf{e}_a\}$, corresponding to the basis one-forms are

$$\begin{aligned} \mathbf{e}_0 &= c^{-1}\partial_t \ , \\ \mathbf{e}_1 &= \left(1-\tfrac{1}{2}h_+\right)\partial_x - \tfrac{1}{2}h_\times\partial_y \ , \\ \mathbf{e}_2 &= \left(1+\tfrac{1}{2}h_+\right)\partial_y - \tfrac{1}{2}h_\times\partial_x \ , \\ \mathbf{e}_3 &= \partial_z \ . \end{aligned} \tag{2.64}$$

The nonzero Ricci rotation coefficients of this geometry are

$$\begin{aligned} &\Gamma^0_{\ 11} = -\Gamma^0_{\ 22} = \Gamma^1_{\ 01} = -\Gamma^2_{\ 02} = \frac{1}{2c}\partial_t h_+, \\ &\Gamma^0_{\ 12} = \Gamma^0_{\ 21} = \Gamma^1_{\ 02} = \Gamma^2_{\ 01} = \frac{1}{2c}\partial_t h_\times, \\ &\Gamma^1_{\ 31} = -\Gamma^2_{\ 32} = -\Gamma^3_{\ 11} = \Gamma^3_{\ 22} = \frac{1}{2}\partial_z h_+, \\ &\Gamma^1_{\ 32} = \Gamma^2_{\ 31} = -\Gamma^3_{\ 12} = -\Gamma^3_{\ 21} = \frac{1}{2}\partial_z h_\times. \end{aligned} \tag{2.65}$$

Chapter 3

Plasma Physics in General Relativity

Plasmas can be a wide variety of substances containing free charges (for example in the form of electrons and ions), which show a collective behaviour due to the long range of the Coulomb forces. The presence of free charges makes the plasma electrically conductive, and thus responsive to electromagnetic fields. Any charge imbalance will be neutralized, or screened, by attracting freely moving charges of opposite signs and repelling charges of the same sign. This leads to an exponential drop of the potential with distance to any free charge, a phenomena called *Debye shielding*. The characteristic length scale of the Debye shielding, λ_D, called the *Debye length*, can be seen as the distance from a charge imbalance at which the potential energy of the screening particle is roughly the same as its thermal energy. The Debye length of an electron-ion plasma is given by

$$\lambda_D = \left(\epsilon_0 KT/n_i e^2\right)^{1/2} \ , \tag{3.1}$$

where ϵ_0 is the dielectric constant in vacuum, K the Stefan-Boltzmann constant, T the temperature, n_i the ion number density, and e the elementary charge. Since charge imbalances are screened in this fashion, a plasma will be roughly neutral on length scales larger than λ_D, a property commonly referred to as *quasi-neutrality*, provided that other macroscopic parameters such as density and temperature are approximately constant on those length scales.

The collective behaviour of plasmas arise from the long range nature of the forces between particles. In a normal (i.e. non-charged) fluid or gas the particle-particle interaction is mainly due to collisions; the forces governing these interaction might be very strong, but the range is very short. In a plasma, particle interactions, in addition to collisions, consist of interactions where one charged particle influences several other charged particles at distances much greater than in the case of collisions. This is what gives plasmas their rich set of phenomena, allowing for a multitude of wave modes and instabilities. For a good and thorough introduction to plasma physics, see e.g. Refs [9, 10].

Plasmas in various forms can often be found around various astrophysical objects where GWs are generated, this includes accretion discs or electron-positron plasmas surrounding compact massive binaries, which is of relevance for the present thesis. As the GWs amplitudes are large close to their sources, it is of interest to study GW-plasma interactions in such regions, as has been done by many authors (see e.g. [39, 40, 41]).

Furthermore, plasma physics might play a role for mechanisms generating [42] or strengthening [43, 44] large scale magnetic fields, particularly in the early stage of the evolution of the universe.

Finally it is worth noting the existence of so called dusty plasmas, which might be of importance close to astrophysical GW sources, as investigated in Paper I. In addition to ions and electrons these plasmas contain dust particles, which might have a considerable mass compared to the ions, see e.g. Refs. [45, 46] for details.

3.1 Electrodynamics and General Relativity

Taking the spacetime curvature in account Maxwell's equations can be formulated as

$$\begin{aligned} F^{\mu\nu}{}_{;\nu} &= \mu_0 j^\mu, \\ F_{[\mu\nu;\sigma]} &= 0\ , \end{aligned} \tag{3.2}$$

where $F_{\mu\nu}$ is the Faraday tensor and j^μ is the four-current density. The Faraday tensor contains both the electric and magnetic fields. However, what is perceived as an electric or a magnetic field depends on the motion of the observer. If the Faraday tensor is specified in a system with four-velocity u^μ, the electric field perceived by observers in that system is

$$E_\mu = F_{\mu\alpha} u^\alpha \ , \tag{3.3}$$

and the magnetic field is

$$B_\mu = \tfrac{1}{2}\epsilon_{\mu\alpha\beta\gamma} F^{\alpha\beta} u^\gamma \ . \tag{3.4}$$

3.1.1 Electrodynamics in the 1+3 Covariant Formalism

This split of the Faraday tensor into a magnetic part and an electric part is to be expected in the 1+3 formalism, since all tensors are decomposed relative to the observer four-velocity. Expressing the Maxwell equations (3.2) in terms of electric and magnetic fields, and the observable quantities defined in Eq. (2.16), results in two propagation equations

$$\dot{E}^{<\mu>} - \epsilon^{\mu\alpha\beta}\tilde{\nabla}_\alpha B_\beta = -j^{<\mu>} - \tfrac{2}{3}\theta E^\mu + \sigma^{\mu\alpha}E_\alpha + \epsilon^{\mu\alpha\beta}\dot{u}_\alpha B_\beta , \tag{3.5}$$

$$\dot{B}^{<\mu>} + \epsilon^{\mu\alpha\beta}\tilde{\nabla}_\alpha E_\beta = -\tfrac{2}{3}\theta B^\mu + \sigma^{\mu\alpha}B_\alpha - \epsilon^{\mu\alpha\beta}\dot{u}_\alpha E_\beta \ , \tag{3.6}$$

and two constraints

$$\tilde{\nabla}_\alpha E^\alpha - \rho_e = 0 \ , \tag{3.7}$$

$$\tilde{\nabla}_\alpha B^\alpha = 0 \ , \tag{3.8}$$

where $\rho_e \equiv -j_\alpha u^\alpha$ is the charge density. Note that in Eqs. (3.5-3.8) it is assumed that the vorticity is zero (for a thorough derivation with a nonzero vorticity see Ref. [6]).

3.1.2 Electrodynamics in an Orthonormal Frame

In an ONF it might be advantageous to express the electric and magnetic fields using a three-vector notation (see Section 2.4.1), e.g. for easier comparison with physically relevant processes in flat spacetime previously analysed in such a formalism. The electric and magnetic fields are then described by the three-vectors $\mathbf{B} = (B_1, B_2, B_3)$ and $\mathbf{E} = (E_1, E_2, E_3)$ respectively. The Maxwell equations (3.2) now split into the four equations

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}(\rho + \rho_E) \ , \quad (3.9)$$

$$\nabla \cdot \mathbf{B} = \frac{\rho_B}{c\epsilon_0} \ , \quad (3.10)$$

$$\frac{1}{c}\mathbf{e}_0\mathbf{E} - \nabla \times \mathbf{B} = -\mu_0 (\mathbf{j} + \mathbf{j}_E) \ , \quad (3.11)$$

$$\mathbf{e}_0\mathbf{B} + \frac{\nabla \times \mathbf{E}}{c} = -\mu_0 \mathbf{j}_B \ , \quad (3.12)$$

where the *effective charge densities*, ρ_E, and ρ_B are

$$\rho_E \equiv -\epsilon_0 \left(\Gamma^i{}_{ji} E^j + \epsilon^{ijk} \Gamma^0{}_{ij} c B_k \right) \ ,$$
$$\rho_B \equiv -\epsilon_0 \left(\Gamma^i{}_{ji} c B^j - \epsilon^{ijk} \Gamma^0{}_{ij} E_k \right) \ , \quad (3.13)$$

and the *effective current densities*, $\mathbf{j}_E$, and $\mathbf{j}_B$ are

$$j^i_E \equiv \frac{1}{\mu_0} \left[\frac{1}{c} \left(\Gamma^i_{j0} - \Gamma^i{}_{0j} \right) E^j + \frac{1}{c} \Gamma^j{}_{0j} E^i - \epsilon^{ijk} (\Gamma^0{}_{0j} B_k + \Gamma^m{}_{jk} B_m) \right] ,$$
$$j^i_B \equiv \frac{1}{\mu_0} \left[\left(\Gamma^i{}_{j0} - \Gamma^i{}_{0j} \right) B^j + \Gamma^j{}_{0j} B^i + \frac{1}{c} \epsilon^{ijk} \left(\Gamma^0{}_{0j} E_k + \Gamma^m{}_{jk} E_m \right) \right] , \quad (3.14)$$

Eqs. (3.9-3.12) strongly resembles regular, non-GR electrodynamics but with effects from the curvature included in the effective charge and current densities and the del operator, $\nabla = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$. The difference of the standard del operator and that in curved space time is seen from the expression for the basis vectors, as for example in Eq. (2.64).

3.1.3 Strong Magnetic Field QED Effects

In the presence of strong electromagnetic fields, that is when the field strength approaches or surpasses the Schwinger critical field strength, i.e. $E \gtrsim E_{cr} \equiv m_e^2 c^3/\hbar e$ in case of electric fields and $B \gtrsim B_{cr} \equiv E_{cr}/c$ in case of magnetic fields, QED effects will become important. These QED effects arise due to interactions between virtual particles and the background field, and cause an effective polarization and magnetization of the vacuum which will affect photon propagation. Provided the soft photon approximation is valid, i.e. the photon energy is smaller than the electron rest mass, an effective field theory including all orders of one-loop photon-photon interaction diagrams (see [13, 14] for details) can be constructed. By imposing the additional condition that the background fields are slowly varying [1] , the effect of all these processes are included in the Heisenberg-Euler Lagrangian density

$$\begin{aligned} \mathcal{L} &= -\frac{1}{\mu_0}\mathcal{F} - A_\alpha j^\alpha - \frac{\alpha}{2\pi\mu_0 e^2}\int_0^{i\infty}\frac{ds}{s^3}e^{-eE_{cr}s/c} \\ &\times \left[(es)^2 ab\coth(eas)\cot(ebs) - \frac{(es)^2}{3}(a^2-b^2) - 1\right] \end{aligned} \quad (3.15)$$

where A_a is the four-potential, j^a the four-current and α the fine structure constant. The auxiliary quantities a and b are defined by

$$a = \left(\sqrt{\mathcal{F}^2+\mathcal{G}^2}+\mathcal{F}\right)^{1/2} , \quad (3.16)$$

$$b = \left(\sqrt{\mathcal{F}^2+\mathcal{G}^2}-\mathcal{F}\right)^{1/2} , \quad (3.17)$$

where the electromagnetic invariants $\mathcal{F}$ and $\mathcal{G}$ are

$$\mathcal{F} = (1/4)F_{ab}F^{ab} , \quad (3.18)$$

$$\mathcal{G} = (1/4)F_{ab}\hat{F}^{ab} , \quad (3.19)$$

and the dual of the Faraday tensor is $\hat{F}^{ab} = \frac{1}{2}\epsilon^{abcd}F_{cd}$.

When the background field is a pure magnetic field the situation simplifies considerably. Using the Lagrangian density (3.15) to derive the Maxwell equations on a curved background in an ONF results in the modification of Eqs. (3.9) and (3.11) to

$$\nabla\cdot\mathbf{E} = \frac{1}{\epsilon_0}\left(\frac{\rho}{\gamma_F}+\rho_E\right), \quad (3.20)$$

and

$$\frac{1}{c}e_0\mathbf{E} - \nabla\times\mathbf{B} = -\mu_0\left(\mathbf{j}_Q + \frac{\mathbf{j}}{\gamma_F} + \mathbf{j}_E\right), \quad (3.21)$$

respectively, while leaving Eqs. (3.10) and (3.12) unchanged. The expressions for the factor γ_F and the current density due to the QED

[1] "Slowly varying" is in comparison with the QED scales, i.e. the Compton frequency and the Compton wavelength, hence the processes may still be fast compared to the GW scales.

polarization and magnetization, j_Q can be found in Paper IV, where a study of how GWs interact with an ultra strong magnetic field is made, showing that the QED effects lead to a detuning of the GW-EMW wave resonances. Furthermore it might be worth noting that the integral in the Lagrangian (3.15) can be calculated explicitly in this case; this is also done in Paper IV.

3.2 Kinetic Plasma Description

In an ONF the equation of motion due to gravitation and Lorentz force of a charged particle with mass m and charge q, subjected to the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$, is given by

$$p^a \mathbf{e}_a \mathbf{p} = q \left[\gamma m \mathbf{E} + \mathbf{p} \times \mathbf{B} \right] - \gamma m \mathbf{G} \tag{3.22}$$

where $G^i = \Gamma^i{}_{ab} p^a p^b / \gamma m$, $\gamma = \sqrt{1 + p^a p_a / m^2}$ and p^a is the four-momentum of the particle. In a plasma consisting of a large number of particles it is not possible to keep track of all individual particles and their effect on the fields. Therefore it is common to use less detailed descriptions in order to capture the essence of the behaviour of the plasma.

In the kinetic plasma description it is assumed that each plasma species, s, can be described by a *distribution function*, $f_s(x^a, \mathbf{p})$, defined as the ensemble average number of point particles per unit unit phase space. The initial distribution function is usually taken to be a reasonable, smooth function, and can often be a thermodynamic equilibrium distribution with some imposed perturbation.

The evolution of the distribution function is governed by the *Bolzmann equation*

$$\mathcal{L}\left[f_s \right] = C \ , \tag{3.23}$$

where C describes collisions, and $\mathcal{L}$ is the *Liouville operator*, which can be written

$$c\mathbf{e}_0 + \frac{\mathbf{p} \cdot \nabla}{\gamma m} + \left[q \left(\mathbf{E} + \frac{\mathbf{p} \times \mathbf{B}}{\gamma m} \right) - \mathbf{G} \right] \cdot \nabla_{\mathbf{p}} = 0 \ , \tag{3.24}$$

where $\nabla_{\mathbf{p}} = (\partial_{p_1}, \partial_{p_2}, \partial_{p_3})$. When the collision term is neglected Eq. (3.23) reduces to the *Vlasov equation*

$$c\mathbf{e}_0 f_s + \frac{\mathbf{p} \cdot \nabla f_s}{\gamma m} + \left[q \left(\mathbf{E} + \frac{\mathbf{p} \times \mathbf{B}}{\gamma m} \right) - \mathbf{G} \right] \cdot \nabla_{\mathbf{p}} f_s = 0 \ . \tag{3.25}$$

The Vlasov equation can be thought of as an equation describing the conservation of phase space volume occupied by a set of neighbouring (in phase space) particles (cf. Liouvilles theorem [9]).

Macroscopic plasma quantities are obtained by integrating products of the distribution function over momentum space. For example, the number density of particles of species s, denoted n_s, is obtained from the distribution function by integrating over the three-momenta, i.e.

$$n_s(x^a) = \int d^3\mathbf{p} f_s(x^a, \mathbf{p}) \ , \tag{3.26}$$

and the bulk three-momentum of particles of species s is obtained from

$$\mathbf{P}_s = \frac{1}{n_s} \int d^3\mathbf{p} \ \mathbf{p} f_s \ . \tag{3.27}$$

The four-current j^a is obtained by summation over the four-current contributions from the different species

$$j^a = \sum_s q_s \int d^3\mathbf{p} \ \frac{p^a}{\gamma m} f_s \ , \tag{3.28}$$

and finally the contribution to the energy-momentum tensor from all particle species in a plasma is obtained by

$$T_{ab}^{(PL)} = \sum_s \int d^3\mathbf{p} \ \frac{p_a p_b}{\gamma m} f_s \ . \tag{3.29}$$

A general relativistic plasma can be described completely by the EFE (2.1), the Vlasov equation (3.25), and the Maxwell equations (3.9-3.12). The kinetic plasma description is used in papers III and V.

3.3 Multifluid Description

When the level of detail provided by kinetic theory is not required, it may be useful to adopt a simpler approach and treat the plasma species as a set of separate, interpenetrating fluids. Electrons and one or more ion species, as well as positrons and dust particles, may all be included in this model; each species having its own fluid equation.

The approach may be to view each species as parts of a fluid energy-momentum tensor, supplemented by appropriate equations of state. The fluid equation for each species can be derived by taking the divergence of the energy-momentum tensor, using the Maxwell equations, and projecting along the fluid four-momentum. Details of the multifluid description can be found in e.g. Ref [47].

Alternatively, the multifluid description can be derived from kinetic theory. Multiplying the Vlasov equation (3.25) by appropriate functions of the three momentum, and integrating over momentum space provides equations governing the evolution of the macroscopic quantities, such as the fluid momentum and number density. Furthermore, this will also give an idea of the applicability of the equations of state.

The multifluid description is used in paper II.

3.4 Magnetohydrodynamic Description

In many cases the different species in a plasma are ions and electrons and, since electrons are much lighter than ions, the electrons move on a completely different, much faster time scale than the ions. In the case of low frequency plasma phenomena the motion of electrons can be regarded as instantaneous, thus any deviation from neutrality caused by the ion motion is immediately nullified by the electron response. This can be seen as the ion fluid dragging the lighter electron fluid along. Furthermore, when the plasma is magnetized, the ions are bound to the magnetic field. In this setting the plasma can be described as a single, electrically conducting, magnetized fluid. This is usually referred to as the magnetohydrodynamic (MHD) description.

The MHD model can be derived from multifluid theory, and depends on a number of rather restrictive assumptions. It is hard to find systems where all of these assumptions are valid simultaneously, but the MHD description can nevertheless be useful in many systems due to its simplicity. Furthermore, experience has shown that the MHD model is more accurate than would be expected from the formal validity conditions [10]. For a more detailed description of MHD models, see Ref. [10].

Presence of even heavier dust particles in an otherwise pure electron-ion MHD plasma leads to a modification of MHD theory, which is described in Refs. [45, 46] . Such a modified MHD model is used in paper I.

Chapter 4

Wave Interactions in Plasmas

Plasma systems are inherently nonlinear, even in flat spacetime, which in most cases makes exact modelling of plasma waves very difficult. However, if the wave amplitudes are sufficiently small, nonlinear effects can be neglected and the plasma waves can be described by linear wave theory (see Ref. [9]). The linearised system of wave equations can be written in the form

$$\hat{W}_{op}\Psi = \mathbf{S} \ , \tag{4.1}$$

where $\hat{W}_{op}$ is the wave operator, Ψ the vector representing the wave variables and $\mathbf{S}$ a vector describing source terms.

As long as the wavelength, λ, is much smaller than the characteristic length-scale of the background, L, and there is no source term, the solution of Eq. (4.1) is a superposition of plane waves, i.e.

$$\Psi = \sum_{\mathbf{k},\omega} \Psi_{\mathbf{k},\omega} e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} \ , \tag{4.2}$$

in three-vector notation (which will be used throughout this chapter) where $\mathbf{k}$ is the wave vector and ω the frequency. A matrix, $\mathbf{W}_{\mathbf{k},\omega}$, can be constructed from the operator $\hat{W}_{op}$ by using (4.2) as an ansatz and letting $\partial_t \to -i\omega$ and $\nabla \to i\mathbf{k}$. Now, since the different vectors $\Psi_{\mathbf{k},\omega}$ are independent, Eq. (4.1) leads to

$$\mathbf{W}_{\mathbf{k},\omega}\Psi_{\mathbf{k},\omega} = 0 \ . \tag{4.3}$$

For nontrivial solutions, i.e. $\Psi_{\mathbf{k},\omega} \neq 0$, to exist, the determinant of the matrix $\mathbf{W}_{\mathbf{k},\omega}$ must be zero. Factorising the determinant yields

$$\det\left(\mathbf{W}_{\mathbf{k},\omega}\right) = \prod_{n=1}^{N} D_n(\omega, \mathbf{k}) \ , \tag{4.4}$$

i.e., a product of N polynomials $D_n(\omega, \mathbf{k})$. The determinant is zero if any of the polynomials are zero, and each solution $D_n(\omega, \mathbf{k}) = 0$, called a *dispersion relation*, represents a different wave mode. Each eigenvector of a corresponding eigenvalue provides the polarisation vector for all wave variables (EM-fields, velocity fields, etc.).

The dispersion relation of a wave mode describes how the wave vector and the frequency are related. This relationship can used to determine the *phase velocity*, $\mathbf{v}_{ph} \equiv \omega\mathbf{k}/k^2$, and the *group velocity*, $v_g \equiv d\omega/d\mathbf{k}$ of a wave mode.

Nonlinear theory can be divided into coherent (phase dependent) and non-coherent processes. The latter usually includes a turbulent (broadband) spectra, and will not be considered here.

4.1 Three Wave Coupling and Parametric Excitation

By relaxing the linear approximation and including second order nonlinear terms, three different wave modes can be nonlinearly coupled. Assuming that for each wave mode, $n = 0, 1, 2$, the wave can be described as an approximately linear wave

$$\psi_n = \tfrac{1}{2}\left(\tilde{\psi}_n e^{i[\mathbf{k}_n\cdot\mathbf{r} - \omega_n t]} + \tilde{\psi}_n^* e^{-i[\mathbf{k}_n\cdot\mathbf{r} - \omega_n t]}\right) \tag{4.5}$$

where * denotes complex conjugate and ψ is a scalar[1]. Here $\tilde{\psi}_n$ is allowed to vary slowly in time, i.e. $\partial_t\tilde{\psi}_n \ll \omega_n\tilde{\psi}_n$. The wave vectors and frequencies satisfy the dispersion relations $D_n(\mathbf{k}_n, \omega_n) = 0$.

For each wave mode, the wave equation (4.1), can now be written

$$\hat{W}_{op}^{(n)}\psi_n = S^{(n)}(\psi_1, \psi_2, \psi_3) \ , \tag{4.6}$$

[1] By assuming that the waves can be described by scalars the discussion is somewhat simplified, but it should be stressed that the same principles can be applied to the eigenvectors found from Eq. (4.4).

where the source term is a function of the different waves. Using the dispersion relation, the wave equation of the wave-mode $n = 1$ can now be written

$$i\omega_0 \frac{d\tilde{\psi}_0}{dt} e^{i[\mathbf{k}_0\cdot\mathbf{r}-\omega_0 t]} - i\omega_0 \frac{d\tilde{\psi}_0^*}{dt} e^{-i[\mathbf{k}_0\cdot\mathbf{r}-\omega_0 t]} = S_0(\psi_0, \psi_1, \psi_2) \ . \tag{4.7}$$

Second order terms in S_0 (higher order terms are neglected) contain products of the different wave amplitudes, such as

$$\begin{aligned} \psi_1\psi_2 &= \tilde{\psi}_1\tilde{\psi}_2 e^{i[(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{r}-(\omega_1+\omega_2)t]} + \tilde{\psi}_1^*\tilde{\psi}_2 e^{i[(-\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{r}-(-\omega_1+\omega_2)t]} \\ &+ \tilde{\psi}_1\tilde{\psi}_2^* e^{i[(\mathbf{k}_1-\mathbf{k}_2)\cdot\mathbf{r}-(\omega_1-\omega_2)t]} + \tilde{\psi}_1^*\tilde{\psi}_2^* e^{i[(-\mathbf{k}_1-\mathbf{k}_2)\cdot\mathbf{r}-(-\omega_1-\omega_2)t]} . \end{aligned} \tag{4.8}$$

For $n = 0$ terms in S_0 proportional to $\psi_1\psi_2$ in the right hand side of Eq. (4.7) will be resonant if the matching conditions

$$\omega_0 = \omega_1 + \omega_2 \ , \tag{4.9}$$

and

$$\mathbf{k}_0 = \mathbf{k}_1 + \mathbf{k}_2 \ , \tag{4.10}$$

are fulfilled. The non-resonant terms also cause nonlinear fluctuations, but these will be insignificant. Ignoring the non-resonant terms, performing the same calculations for all three wave modes, and using these matching conditions, leads to the *coupled mode equations*

$$\begin{aligned} \frac{d\tilde{\psi}_0}{dt} &= c_0\tilde{\psi}_1\tilde{\psi}_2 \ , \\ \frac{d\tilde{\psi}_1}{dt} &= c_1\tilde{\psi}_0\tilde{\psi}_2^* \ , \\ \frac{d\tilde{\psi}_2}{dt} &= c_2\tilde{\psi}_0^*\tilde{\psi}_1 \ , \end{aligned} \tag{4.11}$$

where c_0, c_1 and c_2 are the *coupling coefficients*. Analytical solutions of this system can be found in terms of Jacobian elliptic functions, see Ref. [15].

The wave energy W_n of the wave ψ_n is proportional to $|\psi_n|^2$, such that it is possible to express Eqs. (4.11) in terms of the wave energies as

$$\frac{dW_n}{dt} = -\omega_n V_n \ . \tag{4.12}$$

Whenever a system has the property $V_0 = V_1 = V_2$, the total wave energy of the system is conserved, i.e. $W \equiv \sum_n W_n$ = constant. Furthermore, this also implies the *Manley-Rowe relations* [15],

$$\frac{d}{dt}\left(\frac{W_1}{\omega_1} - \frac{W_2}{\omega_2}\right) = \frac{d}{dt}\left(\frac{W_0}{\omega_0} + \frac{W_1}{\omega_1}\right) = \frac{d}{dt}\left(\frac{W_0}{\omega_0} + \frac{W_2}{\omega_2}\right) = 0 \ . \quad (4.13)$$

The Manley-Rowe relations describe the proportion of wave energy exchanged between waves. One interpretation of this is to view the interaction quantum-mechanically: $N = V/\hbar\omega_0$ quanta with energy $\hbar\omega_0$ and momenta $\hbar\mathbf{k}_0$ decays into N pairs of quanta with energies $\hbar\omega_1$ and $\hbar\omega_2$ and momenta $\hbar\mathbf{k}_1$ and $\hbar\mathbf{k}_2$. While the Manley-Rowe relations have a quantum mechanical interpretation, it should be stressed that these relations apply equally well to systems described by purely classical equations. The reason is that non-dissipative physical systems generally have an underlaying Hamiltonian structure, which is the principal cause of Eqs. (4.13) [48].

If one of the waves in the system (4.11) is externally imposed and can be considered to act as a continuous input of energy, a *pump wave*, the system can become unstable. This can be explained as follows: Assuming that ψ_0 acts as a pump wave implies that $d\tilde{\psi}_0/dt = 0$. This in turn will lead to solutions to the system (4.11) of the form $\tilde{\psi}_{1,2} \propto e^{\Gamma t}$ where the growth rate is found to be $\Gamma = \sqrt{c_1 c_2}\left|\tilde{\psi}_0\right|$. Thus, if $\sqrt{c_1 c_2}$ is positive, (which will be the case if ω_0 has the highest frequency, which is a further consequence of the Manley-Rowe relations), the solutions will grow exponentially. This process is called *parametric excitation* and is considered in Papers I and III.

4.2 Nonlinear Wave Propagation

In this section the propagation of a single wave mode, whose amplitude is sufficiently large for nonlinear profile modifications to be important, will be considered. As a rule of thumb, this requires wave perturbations that are not too much smaller than the corresponding background variables.

When confronted with waves having large amplitudes, linear theory might not be enough to accurately model the physics. As an example, consider a system where the propagation of a wave described by $\psi(z,t)$ is governed by the equation

$$[\partial_t + \mathcal{V}(\psi)\,\partial_z]\,\psi = 0 \ . \tag{4.14}$$

This equation may be relevant for e.g. surface waves on water [16], ion-acoustic waves [10] or Compressional Alfvén waves, where the latter case is studied in Paper II. Since the wave-velocity depends on Ψ, a simple (e.g. sinusoidal) wave profile will be nonlinearly deformed, as will be outlined below.

Making a change of variable such that

$$t = \tau \ , \ z = \zeta + \int_0^\tau d\tau' \ \mathcal{V}\left(\psi(\zeta',\tau')\right) \ , \tag{4.15}$$

implies

$$\partial_t = \partial_\tau - \frac{\mathcal{V}(\psi)}{\mathcal{R}}\partial_\zeta \ , \tag{4.16}$$

and

$$\partial_z = \frac{1}{\mathcal{R}}\partial_\zeta \ , \tag{4.17}$$

where

$$\mathcal{R} \equiv 1 + \int_0^\tau d\tau' \ \frac{\partial\mathcal{V}\left(\psi(\zeta',\tau')\right)}{\partial\zeta} \ . \tag{4.18}$$

The solution to Eq. (4.14) can now be written $\psi(z,t) = \psi(\zeta)$. Now Eq. (4.17) shows that in a comoving system $\partial_z \to \infty$ if $\mathcal{R} \to 0$. Typically, when $\mathcal{R}$ changes with time, the wave undergoes *wave steepening* and, if $\mathcal{R}$ equals zero at some finite time, *wave breaking* will occur. In general terms wave breaking is the phenomena when the wave profile becomes multi-valued at some point(s).

Wave steepening might be thought of as the wave having a speed dependent on the deviation from equilibrium, causing an initial wave profile to deform. If the different speeds cause parts of the wave to overtake other parts, wave breaking occurs. This is what happens with water waves close to the beach; the wave-top overtakes the rest of the wave causing the water to come crashing down. Another example of wave steepening can be found in ion acoustic waves, which is explained in [10].

The convective derivative nonlinearities that always occur in fluid dynamics can induce a similar behaviour as is discussed here, even for wave modes where Eq. (4.14) does not apply [49].

The nonlinear wave phenomena discussed in Chaper IV applies to a broad range of plasma systems. Other examples of important nonlinear processes include e.g. soliton formation, self-focusing and wave collapse [50, 51], which is outside of the scope of the present thesis, however.

Summary of Papers

Paper I

Nonlinear interactions between gravitational radiation and modified Alfvén modes in astrophysical dusty plasmas

In this paper a multi-fluid plasma model including dust-particles is detailed. The main significance of the dust is to enable new wave modes. Furthermore, we present an investigation of nonlinear interactions between Gravitational Radiation and modified Alfvén modes - so called Alfvén-Rao modes - in astrophysical dusty plasmas. Assuming that stationary charged dust grains form a neutralising background in an electron-ion-dust plasma, we obtain the three wave coupling coefficients, which are shown to fulfill the Manley-Rowe symmetries. From the coupling coefficients the growth rates of the Alfvén-Rao modes is calculated. The threshold value of the gravitational wave amplitude associated with convective stabilisation is particularly small if the gravitational frequency is close to twice the modified Alfvén wave-frequency.

In this work I did all of the calculations, some of which was checked by the co-authors. I also participated in the discussions of the interpretation of our results.

Paper II

Harmonic generation of gravitational wave induced Alfvén waves

Using multifluid theory adapted to the MHD-regime we examine the nonlinear evolution of Alfvén waves, excited by gravitational waves from merging binary pulsars. We derive a wave equation for strongly nonlinear and Alfvén waves. Due to the low frequency the Alfvén waves are initially weakly dispersive. During the evolution, the nonlinear wave steepening leads to strong dispersion, associated with strong harmonic generation. We find that the harmonic generation is eventually saturated due to dispersive effects, and use this to estimate the resulting spectrum. Finally we discuss the possibility of observing the above process.

In this work I did the vast majority of the analytical calculation, and all of the numerical calculations. I contributed to the presentation through numerous discussions of the astrophysical applications.

Paper III

Interaction between gravitational waves and plasma waves in the Vlasov description

The nonlinear interaction between electromagnetic, electrostatic Langmuir waves and GWs in a Vlasov plasma is considered. By using an orthonormal tetrad description the three-wave coupling coefficients are computed, which e.g. determines the growth rate for parametric instabilities. Comparing with previous results, it is found that the present theory leads to algebraic expression that are much reduced, as compared to those computed using a coordinate frame formalism. Furthermore, the Manley-Rowe symmetries are discussed. Moreover, we calculate the back-reaction on the gravitational waves, and a simple energy conservation law is deduced in the limit of a cold plasma.

In this work I did a majority of the calculations, and took part in the discussions regarding the interpretation of the results.

Paper IV

Influence of strong field vacuum polarization on gravitational-electromagnetic wave interaction

The interaction between gravitational and electromagnetic waves in the presence of a static magnetic field is studied using an effective field theory based on the Heisenberg-Euler Lagrangian. The field strength of the static field is allowed to surpass the Schwinger critical field, such that the QED effects of vacuum polarization and magnetization are significant. This is of relevance for certain astrophysical objects, e.g. pulsars and magnetars. Equations governing the interaction are derived and analyzed. It turns out that the energy conversion from gravitational to electromagnetic waves can be significantly altered due to the QED effects. The consequences of our results are discussed.

In this work I did the vast majority of the analytical calculation, some of which was confirmed by the co-authors. I also did all of the numerical calculations and the figures. Furthermore, I wrote parts of the article, and contributed to the rest of the presentation through several discussions.

Paper V

Linear theory of gravitational wave propagation in a magnetized, relativistic Vlasov plasma.

We consider propagation of gravitational waves in a magnetized plasma, using the linearized Maxwell-Vlasov equations coupled to Einstein's equations. A set of coupled electromagnetic-gravitational wave equations are derived, that can be straightforwardly reduced to a single dispersion relation. We demonstrate that there is a number of different resonance effects that can enhance the influence of the plasma on the gravitational waves. In particular cyclotron modes can be excited when the GW frequency is twice the cyclotron frequency, extraordinary modes when the GW frequency matches the plasma frequency, and compressional Alfvén waves when the relativistic Alfvén velocity is close to the speed of light.

In this work I did all calculations leading up to the general dispersion relation. I also wrote most of the technical parts of the article. I *contributed to the rest of the presentation, for example by discussing the astrophysical implications.*

Paper VI

Density growth in Kantowski-Sachs cosmologies with cosmological constant

Here we consider the evolution of density perturbations in Kantowski-Sachs cosmologies with a positive cosmological constant. These perturbations are represented by gradients of density, expansion, shear and one more auxiliary variable needed to close the system. All of these variables are zero on the background, and hence gauge invariant. Their time evolution is obtained by taking the spatial gradients of the evolution equations and using commutation relations between the covariant time derivatives and the fully orthogonally projected covariant derivatives. The solutions to this system are analyzed both analytically and numerically. In particular the effects of anisotropy and the behaviour close to bounces is considered.

In this work I did analytical calculations as well as all of the numerics. Furthermore, I contributed to the presentation through several discussions.

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