Introduction to String Theory and D–Branes

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Abstract

These lecture notes cover a number of introductory topics in the area of string theory, and are intended as a foundation for many of the other lecture courses in TASI 2003. The standard topics in perturbative string theory are covered, including some basic conformal field theory, T–Duality and D–Branes. Some of the properties of the latter are uncovered using various techniques. Some standard properties of supersymmetric strings at strong coupling are reviewed, focusing on the role which branes play in uncovering those properties.

Contents

1 Introductory Remarks

These lectures are intended to serve as an introduction to some of the basic techniques, language and concepts in string theory and D–branes. by no stretch of the imagination will you learn everything about these topics from these notes. You should be able to use them to get a good sense for how things work, and this will allow you to understand the issues which are discussed in the other lecture courses in this school, which will assume a knowledge of basic string theory, and some previous experience with some of the “post–Second–Revolution” ideas. Once you get to a certain stage, the notes will need to be supplemented with more careful treatments of the technology, and you will be curious to know more about some of the topics and further applications that I did not have time or space to cover here. I recommend the excellent text of Polchinski[1], and also that of Green, Schwarz and Witten[2], which is still a brilliant text for many aspects of the subject. You might want to consult the recent book listed in ref.[4], in order to learn more about D–branes and further aspects of their applications. There are also many other sources, on the web (*e.g.,* www.arXiv.org) and elsewhere, of detailed reviews of various specialized topics, even whole string theory books[3].

These notes grew out of a series of notes, including those of ref.[5, 6, 7, 8], and I have also borrowed from parts of the book "D–Branes"[4]. I've included the basic material on stings and T–duality, and tried to include a little more of the conformal field theory language than I have used before in notes of this sort. It allows the language of the study of toroidal compactification to be developed quite efficiently and (I hope) clearly. The payoff is that is becomes easier to see how the heterotic strings work, and how they fit in with the other strings quite nicely. Furthermore, it allows one to explain a little about non–critical strings, a topic of considerable interest. It also sets up the important language used in string compactifications, and prepares one for concepts such as U–duality later on. There's a touch more geometry covered than I usually do, such as some basic tools for working on the tangent space any curved spacetime manifold. This comes into its own in later sections when I describe the anomalies which arise in the low energy effective actions of the various superstrings. The Green–Schwarz anomaly cancellation mechanism (the foundation of the First Revolution) is studied in gory detail. This is done for fun in section 6, but there is a bonus in section 8 when we look a bit more closely at gauge bundles and characteristic classes, allowing us to revisit the language of anomalies, now on the world–volumes of D–branes. The Green–Schwarz mechanism is then seen to be a basic paradigm for how extended objects with interesting massless fields on their world–volumes actually fit into the various field theories. I suspect that there will be some resonance with the lectures of Jeff Harvey in this area.

There is the obligatory section on strings at strong coupling. I've put in the standard classic D–brane probes of the duality, and outlined other aspects of how the D–branes get involved in the whole story, including how you can deduce the existence of other important branes, the NS5–branes and the M–branes. This is important basic lore, but I don't go too far into this treatment in order to save space. There are plenty of excellent reviews of this material in the literature I've already mentioned, and also in (for example) ref.[175].

I've included a table of contents for you which might help make this a useful reference. Enjoy, *—cvj.*

2 Relativistic Strings

This section is devoted to an introduction to bosonic strings and their quantization. There is no attempt made at performing a rigourous or exhaustive derivation of some of the various formulae we will encounter, since that would take us well away from the main goal. That goal is to understand some of how string theory incorporates some of the familiar spacetime physics that we know from low energy field theory, and then rapidly proceed to the point where many of the remarkable properties which make strings so different from field theory are manifest. That will be a good foundation for appreciating just what D–branes really are. The careful reader who needs to know more of the details behind some of what we will introduce is invited to consult texts devoted to the study of string theory.

2.1 Classical Bosonic Strings

Turning to strings, we parameterize the "world–sheet" which the string sweeps out with coordinates $(\sigma^1, \sigma^2) = (\tau, \sigma)$. The latter is a spatial coordinate, and for now, we take the string to be an open one, with $0 \leq \sigma \leq \pi$ running from one end to the other. The string's evolution in spacetime is described by the functions $X^\mu(\tau, \sigma)$, $\mu = 0, \cdots, D-1$, giving the shape of the string's world–sheet in target spacetime (see figure 1).

2.1.1 Two Actions

A natural object which arises from embedding the string into spacetime using the functions (or "map") $X^\mu(\tau, \sigma)$ is a two dimensional "induced" metric on the world–sheet:

$$h_{ab} = \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} \ . \tag{1}$$

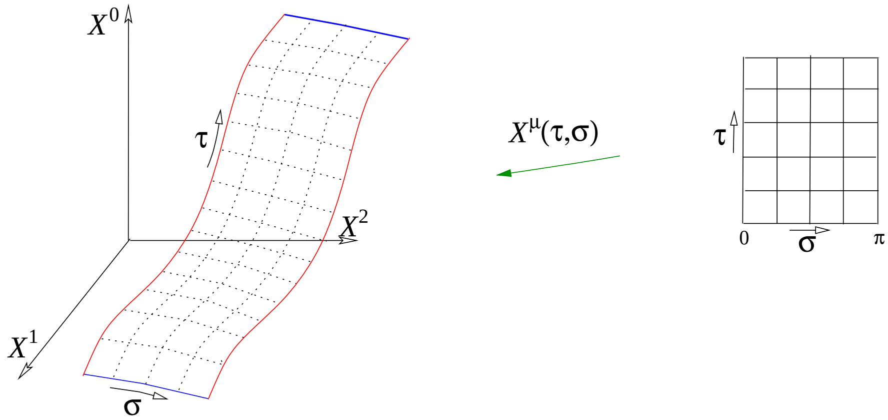


Figure 1: A string's world–sheet. The function $X^\mu(\tau,\sigma)$ embeds the world–sheet, parameterized by (τ,σ), into spacetime, with coordinates given by X^μ.

distances on the world–sheet as an object embedded in spacetime, and hence define an action analogous to one we would write for a particle: the total area swept out by the world–sheet:

$$S_o = -T\int dA = -T\int d\tau d\sigma\,(-\mathrm{det}h_{ab})^{1/2} \equiv \int d\tau d\sigma\;\mathcal{L}(\dot{X},X';\sigma,\tau)\ . \tag{2}$$

$$\begin{aligned} S_o &= -T\int d\tau d\sigma\left[\left(\frac{\partial X^\mu}{\partial\sigma}\frac{\partial X^\mu}{\partial\tau}\right)^2-\left(\frac{\partial X^\mu}{\partial\sigma}\right)^2\left(\frac{\partial X_\mu}{\partial\tau}\right)^2\right]^{1/2}\\ &= -T\int d\tau d\sigma\left[(X'\cdot\dot{X})^2-X'^2\dot{X}^2\right]^{1/2}\ , \end{aligned} \tag{3}$$

where X' means $\partial X/\partial\sigma$ and a dot means differentiation with respect to τ. This is the Nambu–Goto action. T is the tension of the string, which has dimensions of inverse squared length.

Varying the action, we have generally:

$$\begin{aligned} \delta S_o &= \int d\tau d\sigma\left\{\frac{\partial\mathcal{L}}{\partial\dot{X}^\mu}\delta\dot{X}^\mu+\frac{\partial\mathcal{L}}{\partial X'^\mu}\delta X'^\mu\right\}\\ &= \int d\tau d\sigma\left\{-\frac{\partial}{\partial\tau}\frac{\partial\mathcal{L}}{\partial\dot{X}^\mu}-\frac{\partial}{\partial\sigma}\frac{\partial\mathcal{L}}{\partial X'^\mu}\right\}\delta X^\mu+\int d\tau\left\{\frac{\partial\mathcal{L}}{\partial X'^\mu}\delta X'^\mu\right\}\Bigg|_{\sigma=0}^{\sigma=\pi}\ . \end{aligned} \tag{4}$$

Requiring this to be zero, we get:

$$\frac{\partial}{\partial\tau}\frac{\partial\mathcal{L}}{\partial\dot{X}^\mu}+\frac{\partial}{\partial\sigma}\frac{\partial\mathcal{L}}{\partial X'^\mu}=0\quad\text{and}\quad\frac{\partial\mathcal{L}}{\partial X'^\mu}=0\quad\text{at}\quad\sigma=0,\pi\ , \tag{5}$$

which are statements about the conjugate momenta:

$$\frac{\partial}{\partial\tau}P^\mu_\tau+\frac{\partial}{\partial\sigma}P^\mu_\sigma=0\quad\text{and}\quad P^\mu_\sigma=0\quad\text{at}\quad\sigma=0,\pi\ . \tag{6}$$

Here, P^μ_σ is the momentum running along the string (*i.e.,* in the σ direction) while P^μ_τ is the momentum running transverse to it. The total spacetime momentum is given by integrating up the infinitesimal (see figure 2):

$$dP^\mu = P^\mu_\tau d\sigma + P^\mu_\sigma d\tau\ . \tag{7}$$

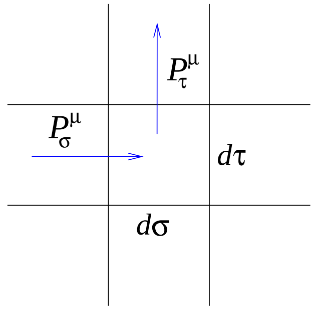


Figure 2: The infinitesimal momenta on the world sheet.

Actually, we can choose any slice of the world–sheet in order to compute this momentum. A most convenient one is a slice $\tau =$ constant, revealing the string in its original paramaterization: $P^\mu = \int P^\mu_\tau d\sigma$, but any other slice will do.

Similarly, one can define the angular momentum:

$$M^{\mu\nu} = \int (P^\mu_\tau X^\nu - P^\nu_\tau X^\mu) d\sigma \ . \tag{8}$$

It is a simple exercise to work out the momenta for our particular Lagrangian:

$$\begin{aligned} P^\mu_\tau &= T \frac{\dot{X}^\mu X'^2 - X'^\mu(\dot{X}\cdot X')}{\sqrt{(\dot{X}\cdot X')^2 - \dot{X}^2 X'^2}} \\ P^\mu_\sigma &= T \frac{X'^\mu \dot{X}^2 - \dot{X}^\mu(\dot{X}\cdot X')}{\sqrt{(\dot{X}\cdot X')^2 - \dot{X}^2 X'^2}} \ . \end{aligned} \tag{9}$$

It is interesting to compute the square of P^μ_σ from this expression, and one finds that

$$P^2_\sigma \equiv P^\mu_\sigma P_{\mu\sigma} = -2T^2 \dot{X}^2 \ . \tag{10}$$

This is our first (perhaps) non–intuitive classical result. We noticed that P_σ vanishes at the endpoints, in order to prevent momentum from flowing off the ends of the string. The equation we just derived implies that $\dot{X}^2 = 0$ at the endpoints, which is to say that they move at the speed of light.

We can introduce an equivalent action which does not have the square root form that the current one has. Once again, we do it by introducing a independent metric, $\gamma_{ab}(\sigma, \tau)$, on the world–sheet, and write the "Polyakov" action:

$$\begin{aligned} S &= -\frac{1}{4\pi\alpha'} \int d\tau d\sigma (-\gamma)^{1/2} \gamma^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} \\ &= -\frac{1}{4\pi\alpha'} \int d^2\sigma \, (-\gamma)^{1/2} \gamma^{ab} h_{ab} \ . \end{aligned} \tag{11}$$

If we vary γ, we get

$$\delta S = -\frac{1}{4\pi\alpha'} \int d^2\sigma \, \left\{ -\frac{1}{2}(-\gamma)^{1/2} \delta\gamma \gamma^{ab} h_{ab} + (-\gamma)^{1/2} \delta\gamma^{ab} h_{ab} \right\} \ . \tag{12}$$

Using the fact that $\delta\gamma = \gamma\gamma^{ab}\delta\gamma_{ab} = -\gamma\gamma_{ab}\delta\gamma^{ab}$, we get

$$\delta S = -\frac{1}{4\pi\alpha'} \int d^2\sigma \, (-\gamma)^{1/2} \delta\gamma^{ab} \left\{ h_{ab} - \frac{1}{2}\gamma_{ab}\gamma^{cd} h_{cd} \right\} \ . \tag{13}$$

Therefore we have

$$h_{ab} - \frac{1}{2}\gamma_{ab}\gamma^{cd}h_{cd} = 0 \ , \tag{14}$$

from which we can derive

$$\gamma^{ab}h_{ab} = 2(-h)^{1/2}(-\gamma)^{-1/2} \ , \tag{15}$$

and so substituting into S, we recover the Nambu–Goto action, S_o.

Let us note some of the symmetries of the action:

- Spacetime Lorentz/Poincaré:

$$X^\mu \to X'^\mu = \Lambda^\mu{}_\nu X^\nu + A^\mu \ ,$$

  where Λ is an $SO(1, D-1)$ Lorentz matrix and A^μ is an arbitrary constant D–vector. Just as before this is a trivial global symmetry of S (and also S_o), following from the fact that we wrote them in covariant form.

- Worldsheet Reparametrisations:

$$\begin{aligned} \delta X^\mu &= \zeta^a\partial_a X^\mu \\ \delta\gamma^{ab} &= \zeta^c\partial_c\gamma^{ab} - \partial_c\zeta^a\gamma^{cb} - \partial_c\zeta^b\gamma^{ac} \ , \end{aligned} \tag{16}$$

  for two parameters $\zeta^a(\tau, \sigma)$. This is a non–trivial local or "gauge" symmetry of S. This is a large extra symmetry on the world–sheet of which we will make great use.

- Weyl invariance:

$$\gamma_{ab} \to \gamma'_{ab} = \mathrm{e}^{2\omega}\gamma_{ab} \ , \tag{17}$$

  specified by a function $\omega(\tau, \sigma)$. This ability to do local rescalings of the metric results from the fact that we did not have to choose an overall scale when we chose γ^{ab} to rewrite S_o in terms of S. This can be seen especially if we rewrite the relation (15) as $(-h)^{-1/2}h_{ab} = (-\gamma)^{-1/2}\gamma_{ab}$.

We note here for future use that there are just as many parameters needed to specify the local symmetries (three) as there are independent components of the world-sheet metric. This is very useful, as we shall see.

2.2 String Equations of Motion

We can get equations of motion for the string by varying our action (11) with respect to the X^μ:

$$\begin{aligned} \delta S &= \frac{1}{2\pi\alpha'}\int d^2\sigma\, \partial_a\left\{(-\gamma)^{1/2}\gamma^{ab}\partial_b X_\mu\right\}\delta X^\mu \\ &\qquad -\frac{1}{2\pi\alpha'}\int d\tau\ (-\gamma)^{1/2}\partial_\sigma X_\mu \delta X^\mu\Big|_{\sigma=0}^{\sigma=\pi} \ , \end{aligned} \tag{18}$$

which results in the equations of motion:

$$\partial_a\left((-\gamma)^{1/2}\gamma^{ab}\partial_b X^\mu\right) \equiv (-\gamma)^{1/2}\nabla^2 X^\mu = 0 \ , \tag{19}$$

with *either*:

$$\left.\begin{aligned} X'^\mu(\tau,0) = 0 \\ X'^\mu(\tau,\pi) = 0 \end{aligned}\right\} \quad \begin{array}{c}\text{Open String} \\ \text{(Neumann b.c.'s)}\end{array} \tag{20}$$

or:

$$\left.\begin{aligned} X'^\mu(\tau,0) = X'^\mu(\tau,\pi) \\ X^\mu(\tau,0) = X^\mu(\tau,\pi) \\ \gamma_{ab}(\tau,0) = \gamma_{ab}(\tau,\pi) \end{aligned}\right\} \quad \begin{array}{c}\text{Closed String} \\ \text{(periodic b.c.'s)}\end{array} \tag{21}$$

We shall study the equation of motion (19) and the accompanying boundary conditions a lot later. We are going to look at the standard Neumann boundary conditions mostly, and then consider the case of Dirichlet conditions later, when we uncover D–branes, using T–duality. Notice that we have taken the liberty of introducing closed strings by imposing periodicity.

2.3 Further Aspects of the Two–Dimensional Perspective

The action (11) may be thought of as a two dimensional model of D bosonic fields $X^\mu(\tau,\sigma)$. This two dimensional theory has reparameterisation invariance, as it is constructed using the metric $\gamma_{ab}(\tau,\sigma)$ in a covariant way. It is natural to ask whether there are other terms which we might want to add to the theory which have similar properties. With some experience from General Relativity two other terms spring effortlessly to mind. One is the Einstein–Hilbert action (supplemented with a boundary term):

$$\chi = \frac{1}{4\pi}\int_{\mathcal{M}} d^2\sigma\,(-\gamma)^{1/2}R + \frac{1}{2\pi}\int_{\partial\mathcal{M}} ds K\ , \qquad (22)$$

where R is the two–dimensional Ricci scalar on the world–sheet $\mathcal{M}$ and K is the trace of the extrinsic curvature tensor on the boundary $\partial\mathcal{M}$. The other term is:

$$\Theta = \frac{1}{4\pi\alpha'}\int_{\mathcal{M}} d^2\sigma\,(-\gamma)^{1/2}\ , \qquad (23)$$

which is the cosmological term. What is their role here? Well, under a Weyl transformation (17), it can be seen that $(-\gamma)^{1/2}\to e^{2\omega}(-\gamma)^{1/2}$ and $R\to e^{-2\omega}(R-2\nabla^2\omega)$, and so χ is invariant, (because R changes by a total derivative which is canceled by the variation of K) but Θ is not.

So we will include χ, but not Θ in what follows. Let us anticipate something that we will do later, which is to work with Euclidean signature to help make sense of the topological statements to follow: γ_{ab} with signature $(-+)$ has been replaced by g_{ab} with signature $(++)$. Now, since as we said earlier, the full string action resembles two–dimensional gravity coupled to D bosonic "matter" fields X^μ, and the equations of motion are of course:

$$R_{ab} - \frac{1}{2}\gamma_{ab}R = T_{ab}\ . \qquad (24)$$

The left hand side vanishes identically in two dimensions, and so there are no dynamics associated to (22). The quantity χ depends only on the topology of the world–sheet (it is the Euler number) and so will only matter when comparing world sheets of different topology. This will arise when we compare results from different orders of string perturbation theory and when we consider interactions.

We can see this in the following: Let us add our new term to the action, and consider the string action to be:

$$\begin{aligned} S &= \frac{1}{4\pi\alpha'}\int_{\mathcal{M}} d^2\sigma\, g^{1/2}g^{ab}\partial_a X^\mu\partial_b X_\mu \\ &\qquad\qquad +\lambda\left\{\frac{1}{4\pi}\int_{\mathcal{M}} d^2\sigma\, g^{1/2}R + \frac{1}{2\pi}\int_{\partial\mathcal{M}} dsK\right\}\ , \end{aligned} \qquad (25)$$

where λ is —for now— and arbitrary parameter which we have not fixed to any particular value[1]. So what will λ do? Recall that it couples to Euler number, so in the full path integral defining the string theory:

$$\mathcal{Z} = \int \mathcal{D}X\mathcal{D}g\ e^{-S}\ , \qquad (26)$$

resulting amplitudes will be weighted by a factor $e^{-\lambda\chi}$, where $\chi = 2-2h-b-c$. Here, h, b, c are the numbers of handles, boundaries and crosscaps, respectively, on the world sheet. Consider figure 3. An emission and reabsorption of an open string results in a change $\delta\chi=-1$, while for a closed string it is $\delta\chi=-2$. Therefore, relative to the tree level open string diagram (disc topology), the amplitudes are weighted by e^{λ} and $e^{2\lambda}$, respectively. The quantity $g_s\equiv e^{\lambda}$ therefore will be called the closed string coupling. Note that it is the square of the open string coupling, which justifies the labeling we gave of the two three–string diagrams in figure 4. It is a striking fact that string theory dynamically determines its own coupling strength. (See figure 4.)

[1]Later, it will turn out that λ is not a free parameter. In the full string theory, it has dynamical meaning, and will be equivalent to the expectation value of one of the massless fields —the "dilaton"— described by the string.

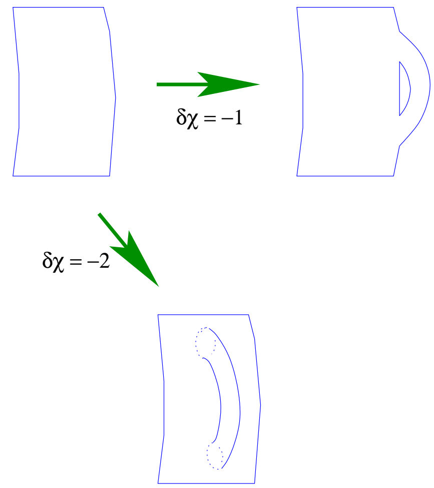


Figure 3: World–sheet topology change due to emission and reabsorption of open and closed strings

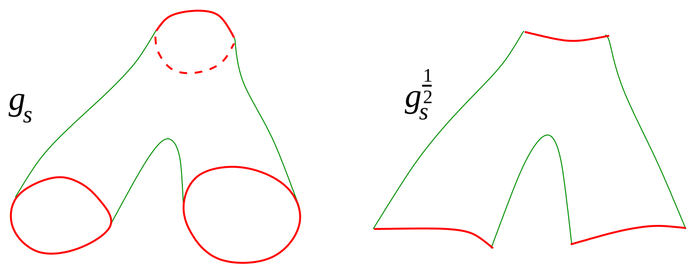


Figure 4: The basic three–string interaction for closed strings, and its analogue for open strings. Its strength, g_s, along with the string tension, determines Newton's gravitational constant G_N.

2.3.1 The Stress Tensor

Let us also note that we can define a two–dimensional energy–momentum tensor:

$$T^{ab}(\tau,\sigma) \equiv -\frac{2\pi}{\sqrt{-\gamma}}\frac{\delta S}{\delta\gamma_{ab}} = -\frac{1}{\alpha'}\left\{\partial^a X_\mu \partial^b X^\mu - \frac{1}{2}\gamma^{ab}\gamma_{cd}\partial^c X_\mu \partial^d X^\mu\right\} \ . \tag{27}$$

Notice that

$$T_a^a \equiv \gamma_{ab}T^{ab} = 0 \ . \tag{28}$$

This is a consequence of Weyl symmetry. Reparametrization invariance, $\delta_\gamma S' = 0$, translates here into (see discussion after equation (24))

$$T^{ab} = 0 \ . \tag{29}$$

These are the classical properties of the theory we have uncovered so far. Later on, we shall attempt to ensure that they are true in the quantum theory also, with interesting results.

2.4 Gauge Fixing

Now recall that we have three local or "gauge" symmetries of the action:

$$\begin{aligned} \text{2d reparametrizations}: \ \sigma,\tau &\rightarrow \tilde{\sigma}(\sigma,\tau), \tilde{\tau}(\sigma,\tau) \\ \text{Weyl}: \ \gamma_{ab} &\rightarrow \exp(2\omega(\sigma,\tau))\gamma_{ab} \ . \end{aligned} \tag{30}$$

The two dimensional metric γ_{ab} is also specified by three independent functions, as it is a symmetric 2×2 matrix. We may therefore use the gauge symmetries (see (16), (17)) to choose γ_{ab} to be a particular form:

$$\gamma_{ab} = \eta_{ab} e^{\gamma\varphi} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} e^{\gamma\varphi} \ , \tag{31}$$

i.e. the metric of two dimensional Minkowski, times a positive function known as a *conformal factor*. Here, γ is a constant and φ is a function of the world–sheet coordinates. In this "conformal" gauge, our X^μ equations of motion (19) become:

$$\left(\frac{\partial^2}{\partial\sigma^2} - \frac{\partial^2}{\partial\tau^2}\right) X^\mu(\tau,\sigma) = 0 \ , \tag{32}$$

the two dimensional wave equation. (In fact, the reader should check that the conformal factor cancels out entirely of the action in equation (11).) As the wave equation is $\partial_{\sigma^+}\partial_{\sigma^-}X^\mu = 0$, we see that the full solution to the equation of motion can be written in the form:

$$X^\mu(\sigma,\tau) = X_L^\mu(\sigma^+) + X_R^\mu(\sigma^-) \ , \tag{33}$$

where $\sigma^\pm \equiv \tau \pm \sigma$. Write $\sigma^\pm = \tau \pm \sigma$. This gives metric $ds^2 = -d\tau^2 + d\sigma^2 \to -d\sigma^+ d\sigma^-$. So we have $\eta_{-+} = \eta_{+-} = -1/2$, $\eta^{-+} = \eta^{+-} = -2$ and $\eta_{++} = \eta_{--} = \eta^{++} = \eta^{--} = 0$. Also, $\partial_\tau = \partial_+ + \partial_-$ and $\partial_\sigma = \partial_+ - \partial_-$.

Our constraints on the stress tensor become:

$$\begin{aligned} T_{\tau\sigma} &= T_{\sigma\tau} \equiv \frac{1}{\alpha'}\dot{X}^\mu X'_\mu = 0 \\ T_{\sigma\sigma} &= T_{\tau\tau} = \frac{1}{2\alpha'}\left(\dot{X}^\mu \dot{X}_\mu + X'^\mu X'_\mu\right) = 0 \ , \end{aligned} \tag{34}$$

or

$$\begin{aligned} T_{++} &= \frac{1}{2}(T_{\tau\tau} + T_{\tau\sigma}) = \frac{1}{\alpha'}\partial_+ X^\mu \partial_+ X_\mu \equiv \frac{1}{\alpha'}\dot{X}_L^2 = 0 \\ T_{--} &= \frac{1}{2}(T_{\tau\tau} - T_{\tau\sigma}) = \frac{1}{\alpha'}\partial_- X^\mu \partial_- X_\mu \equiv \frac{1}{\alpha'}\dot{X}_R^2 = 0 \ , \end{aligned} \tag{35}$$

and T_{-+} and T_{+-} are identically zero.

2.4.1 The Mode Decomposition

Our equations of motion (33), with our boundary conditions (20) and (21) have the simple solutions:

$$X^\mu(\tau,\sigma) = x^\mu + 2\alpha' p^\mu \tau + i(2\alpha')^{1/2} \sum_{n\neq 0} \frac{1}{n}\alpha_n^\mu e^{-in\tau}\cos n\sigma \ , \tag{36}$$

for the open string and

$$\begin{aligned} X^\mu(\tau,\sigma) &= X_R^\mu(\sigma^-) + X_L^\mu(\sigma^+) \\ X_R^\mu(\sigma^-) &= \frac{1}{2}x^\mu + \alpha' p^\mu \sigma^- + i\left(\frac{\alpha'}{2}\right)^{1/2} \sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-2in\sigma^-} \\ X_L^\mu(\sigma^+) &= \frac{1}{2}x^\mu + \alpha' p^\mu \sigma^+ + i\left(\frac{\alpha'}{2}\right)^{1/2} \sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}_n^\mu e^{-2in\sigma^+} \ , \end{aligned} \tag{37}$$

for the closed string, where, to ensure a real solution we impose $\alpha_{-n}^\mu = (\alpha_n^\mu)^*$ and $\tilde{\alpha}_{-n}^\mu = (\tilde{\alpha}_n^\mu)^*$.

Notice that the mode expansion for the closed string (37) is simply that of a pair of independent left and right moving traveling waves going around the string in opposite directions. The open string expansion (36) on the other hand, has a standing wave for its solution, representing the left and right moving sector reflected into one another by the Neumann boundary condition (20).

Note also that x^μ and p^μ are the centre of mass position and momentum, respectively. In each case, we can identify p^μ with the zero mode of the expansion:

$$\begin{aligned} \text{open string:} \quad \alpha_0^\mu &= (2\alpha')^{1/2} p^\mu ; \\ \text{closed string:} \quad \alpha_0^\mu &= \left(\frac{\alpha'}{2}\right)^{1/2} p^\mu \ . \end{aligned} \tag{38}$$

2.5 Conformal Invariance

Actually, we have not gauged away all of the local symmetry by choosing the gauge (31). We can do a left–right decoupled change of variables:

$$\sigma^+ \to f(\sigma^+) = \sigma'^+ ; \ \sigma^- \to g(\sigma^-) = \sigma'^- \ . \tag{39}$$

Then, as

$$\gamma'_{ab} = \frac{\partial \sigma^c}{\partial \sigma'^a} \frac{\partial \sigma^d}{\partial \sigma'^b} \gamma_{cd} \ , \tag{40}$$

we have

$$\gamma'_{+-} = \left(\frac{\partial f(\sigma^+)}{\partial \sigma^+} \frac{\partial g(\sigma^-)}{\partial \sigma^-} \right)^{-1} \gamma_{+-} \ . \tag{41}$$

However, we can undo this with a Weyl transformation of the form

$$\gamma'_{+-} = \exp(2\omega_L(\sigma^+) + 2\omega_R(\sigma^-))\gamma_{+-} \ , \tag{42}$$

if $\exp(-2\omega_L(\sigma^+)) = \partial_+ f(\sigma^+)$ and $\exp(-2\omega_R(\sigma^-)) = \partial_- g(\sigma^-)$. So we still have a residual "conformal" symmetry. As f and g are independent arbitrary functions on the left and right, we have an infinite number of conserved quantities on the left and right.

This is because the conservation equation $\nabla_a T^{ab} = 0$, together with the result $T_{+-} = T_{-+} = 0$, turns into:

$$\partial_- T_{++} = 0 \quad \text{and} \quad \partial_+ T_{--} = 0 \ , \tag{43}$$

but since $\partial_- f = 0 = \partial_+ g$, we have

$$\partial_-(f(\sigma^+)T_{++}) = 0 \quad \text{and} \quad \partial_+(g(\sigma^-)T_{--}) = 0 \ , \tag{44}$$

resulting in an infinite number of conserved quantities. The fact that we have this infinite dimensional conformal symmetry is the basis of some of the most powerful tools in the subject, for computing in perturbative string theory. We will return to it not too far ahead.

2.5.1 Some Hamiltonian Dynamics

Our Lagrangian density is

$$\mathcal{L} = -\frac{1}{4\pi\alpha'} \left(\partial_\sigma X^\mu \partial_\sigma X_\mu - \partial_\tau X^\mu \partial_\tau X_\mu \right) \ , \tag{45}$$

from which we can derive that the conjugate momentum to X^μ is

$$\Pi^\mu = \frac{\delta \mathcal{L}}{\delta(\partial_\tau X^\mu)} = \frac{1}{2\pi\alpha'} \dot{X}^\mu \ . \tag{46}$$

So we have the equal time Poisson brackets:

$$[X^\mu(\sigma), \Pi^\nu(\sigma')]_{\text{P.B.}} = \eta^{\mu\nu}\delta(\sigma - \sigma') \ , \tag{47}$$
$$[\Pi^\mu(\sigma), \Pi^\nu(\sigma')]_{\text{P.B.}} = 0 \ , \tag{48}$$

with the following results on the oscillator modes:

$$\begin{aligned} [\alpha_m^\mu, \alpha_n^\nu]_{\text{P.B.}} &= [\tilde{\alpha}_m^\mu, \tilde{\alpha}_n^\nu]_{\text{P.B.}} = im\delta_{m+n}\eta^{\mu\nu} \\ [p^\mu, x^\nu]_{\text{P.B.}} &= \eta^{\mu\nu}; \quad [\alpha_m^\mu, \tilde{\alpha}_n^\nu]_{\text{P.B.}} = 0 \ . \end{aligned} \tag{49}$$

We can form the Hamiltonian density

$$\mathcal{H} = \dot{X}^\mu \Pi_\mu - \mathcal{L} = \frac{1}{4\pi\alpha'}\left(\partial_\sigma X^\mu \partial_\sigma X_\mu + \partial_\tau X^\mu \partial_\tau X_\mu\right) \ , \tag{50}$$

from which we can construct the Hamiltonian H by integrating along the length of the string. This results in:

$$H = \int_0^\pi d\sigma\, \mathcal{H}(\sigma) = \frac{1}{2}\sum_{-\infty}^{\infty} \alpha_{-n}\cdot\alpha_n \qquad \text{(open)} \tag{51}$$
$$H = \int_0^{2\pi} d\sigma\, \mathcal{H}(\sigma) = \frac{1}{2}\sum_{-\infty}^{\infty} (\alpha_{-n}\cdot\alpha_n + \tilde{\alpha}_{-n}\cdot\tilde{\alpha}_n) \qquad \text{(closed)} \ .$$

(We have used the notation $\alpha_n \cdot \alpha_n \equiv \alpha_n^\mu \alpha_{n\mu}$)

The constraints $T_{++} = 0 = T_{--}$ on our energy–momentum tensor can be expressed usefully in this language. We impose them mode by mode in a Fourier expansion, defining:

$$L_m = \frac{T}{2}\int_0^\pi e^{-2im\sigma} T_{--} d\sigma = \frac{1}{2}\sum_{-\infty}^{\infty} \alpha_{m-n}\cdot\alpha_n \ , \tag{52}$$

and similarly for $\bar{L}_m$, using T_{++}. Using the Poisson brackets (49), these can be shown to satisfy the “Virasoro” algebra:

$$\begin{aligned} [L_m, L_n]_{\text{P.B.}} &= i(m-n)L_{m+n}; \quad \left[\bar{L}_m, \bar{L}_n\right]_{\text{P.B.}} = i(m-n)\bar{L}_{m+n}; \\ \left[\bar{L}_m, L_n\right]_{\text{P.B.}} &= 0 \ . \end{aligned} \tag{53}$$

Notice that there is a nice relation between the zero modes of our expansion and the Hamiltonian:

$$H = L_0 \qquad \text{(open)}; \quad H = L_0 + \bar{L}_0 \qquad \text{(closed)} \ . \tag{54}$$

So to impose our constraints, we can do it mode by mode and ask that $L_m = 0$ and $\bar{L}_m = 0$, for all m. Looking at the zeroth constraint results in something interesting. Note that

$$\begin{aligned} L_0 &= \frac{1}{2}\alpha_0^2 + 2\times\frac{1}{2}\sum_{n=1}^{\infty}\alpha_{-n}\cdot\alpha_n + \frac{D}{2}\sum_{n=1}^{\infty} n \\ &= \alpha' p^\mu p_\mu + \sum_{n=1}^{\infty}\alpha_{-n}\cdot\alpha_n \\ &= -\alpha' M^2 + \sum_{n=1}^{\infty}\alpha_{-n}\cdot\alpha_n \ . \end{aligned} \tag{55}$$

Requiring L_0 to be zero —diffeomorphism invariance— results in a (spacetime) mass relation:

$$M^2 = \frac{1}{\alpha'}\sum_{n=1}^{\infty}\alpha_{-n}\cdot\alpha_n \qquad \text{(open)} \ , \tag{56}$$

where we have used the zero mode relation (38) for the open string. A similar exercise produces the mass relation for the closed string:

$$M^2 = \frac{2}{\alpha'}\sum_{n=1}^{\infty}(\alpha_{-n}\cdot\alpha_n + \tilde{\alpha}_{-n}\cdot\tilde{\alpha}_n) \qquad \text{(closed)} \ . \tag{57}$$

These formulae (56) and (57) give us the result for the mass of a state in terms of how many oscillators are excited on the string. The masses are set by the string tension $T = (2\pi\alpha')^{-1}$, as they should be. Let us not dwell for too long on these formulae however, as they are significantly modified when we quantize the theory.

2.6 Quantised Bosonic Strings

For our purposes, the simplest route to quantisation will be to promote everything we met previously to operator statements, replacing Poisson Brackets by commutators in the usual fashion: $[\ \ ,\ \ ]_{\rm P.B.} \to -i[\ \ ,\ \ ]$. This gives:

$$\begin{aligned} [X^\mu(\tau,\sigma),\Pi^\nu(\tau,\sigma')] &= i\eta^{\mu\nu}\delta(\sigma-\sigma') \ ; \quad [\Pi^\mu(\tau,\sigma),\Pi^\nu(\tau,\sigma')] = 0 \\ [\alpha^\mu_m,\alpha^\nu_n] &= [\tilde{\alpha}^\mu_m,\tilde{\alpha}^\nu_n] \ = \ m\delta_{m+n}\eta^{\mu\nu} \\ [x^\nu,p^\mu] &= i\eta^{\mu\nu}; \quad [\alpha^\mu_m,\tilde{\alpha}^\nu_n] \ = \ 0 \ . \end{aligned} \tag{58}$$

One of the first things that we ought to notice here is that $\sqrt{m}\alpha^\mu_{\pm m}$ are like creation and annihilation operators for the harmonic oscillator. There are actually D independent families of them —one for each spacetime dimension— labelled by μ. In the usual fashion, we will define our Fock space such that $|0;k\!>$ is an eigenstate of p^μ with centre of mass momentum k^μ. This state is annihilated by α^ν_m.

What about our operators, the L_m? Well, with the usual "normal ordering" are to the right, the L_m are all fine when promoted to operators, except the Hamiltonian, L_0. It needs more careful definition, since α^μ_n and α^μ_{-n} *do not commute*. Indeed, as an operator, we have that

$$L_0 = \frac{1}{2}\alpha_0^2 + \sum_{n=1}^{\infty}\alpha_{-n}\cdot\alpha_n + \text{constant} \ , \tag{59}$$

where the apparently infinite constant is composed as copy of the infinite sum $(1/2)\sum_{n=1}^{\infty} n$ for each of the D families of oscillators. As is of course to be anticipated, this infinite constant can be regulated to give a finite answer, corresponding to the total zero point energy of all of the harmonic oscillators in the system.

2.7 The Constraints and Physical States

For now, let us not worry about the value of the constant, and simply impose our constraints on a state $|\phi\!>$ as[2]:

$$\begin{aligned} (L_0 - a)|\phi\!>= 0; &\qquad L_m|\phi\!>= 0 \quad \text{for } m>0 \ , \\ (\bar{L}_0 - a)|\phi\!>= 0; &\qquad \bar{L}_m|\phi\!>= 0 \quad \text{for } m>0 \ , \end{aligned} \tag{60}$$

[2]This assumes that the constant a on each side are equal. At this stage, we have no other choice. We have isomorphic copies of the same string modes on the left and the right, for which the values of a are by definition the same. When we have more than one consistent conformal field theory to choose from, then we have the freedom to consider having non–isomorphic sectors on the left and right. This is how the heterotic string is made, for example, as we shall see later.

where our infinite constant is set by a, which is to be computed. There is a reason why we have not also imposed this constraint for the L_{-m}'s. This is because the Virasoro algebra (53) in the quantum case is:

$$
\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{D}{12}(m^3-m)\delta_{m+n}; \quad [\bar{L}_m, L_n] = 0; \\
[\bar{L}_m, \bar{L}_n] &= (m-n)\bar{L}_{m+n} + \frac{D}{12}(m^3-m)\delta_{m+n} \ , \qquad (61)
\end{aligned}
$$

There is a central term in the algebra, which produces a non–zero constant when $m = n$. Therefore, imposing both L_m and L_{-m} would produce an inconsistency.

Note now that the first of our constraints (60) produces a modification to the mass formulae:

$$
\begin{aligned}
M^2 &= \frac{1}{\alpha'}\left(\sum_{n=1}^{\infty} \alpha_{-n}\cdot\alpha_n - a\right) \qquad \text{(open)} \qquad (62) \\
M^2 &= \frac{2}{\alpha'}\left(\sum_{n=1}^{\infty} (\alpha_{-n}\cdot\alpha_n + \tilde{\alpha}_{-n}\cdot\tilde{\alpha}_n) - 2a\right) \qquad \text{(closed)} \ .
\end{aligned}
$$

Notice that we can denote the (weighted) number of oscillators excited as $N = \sum \alpha_{-n}\cdot\alpha_n$ $(= \sum nN_n)$ on the left and $\bar{N} = \sum \tilde{\alpha}_{-n}\cdot\tilde{\alpha}_n$ $(= \sum n\bar{N}_n)$ on the right. N_n and $\bar{N}_n$ are the true count, on the left and right, of the number of copies of the oscillator labelled by n is present.

There is an extra condition in the closed string case. While $L_0 + \bar{L}_0$ generates time translations on the world sheet (being the Hamiltonian), the combination $L_0 - \bar{L}_0$ generates translations in σ. As there is no physical significance to where on the string we are, the physics should be invariant under translations in σ, and we should impose this as an operator condition on our physical states:

$$
(L_0 - \bar{L}_0)|\phi> = 0 \ , \qquad (63)
$$

which results in the "level–matching" condition $N = \bar{N}$, equating the number of oscillators excited on the left and the right. This is indeed the difference between the two equations in (60).

In summary then, we have two copies of the open string on the left and the right, in order to construct the closed string. The only extra subtlety is that we should use the correct zero mode relation (38) and match the number of oscillators on each side according to the level matching condition (63).

2.7.1 The Intercept and Critical Dimensions

Let us consider the spectrum of states level by level, and uncover some of the features, focusing on the open string sector. Our first and simplest state is at level 0, *i.e.,* no oscillators excited at all. There is just some centre of mass momentum that it can have, which we shall denote as k. Let us write this state as $|0;k>$. The first of our constraints (60) leads to an expression for the mass:

$$
(L_0 - a)|0;k> = 0 \qquad \Rightarrow \alpha' k^2 = a, \qquad \text{so} \quad M^2 = -\frac{a}{\alpha'} \ . \qquad (64)
$$

This state is a tachyonic state, having negative mass–squared (assuming $a > 0$.

The next simplest state is that with momentum k, and one oscillator excited. We are also free to specify a polarization vector ζ^μ. We denote this state as $|\zeta, k> \equiv (\zeta\cdot\alpha_{-1})|0;k>$; it starts out the discussion with D independent states. The first thing to observe is the norm of this state:

$$
\begin{aligned}
<\zeta;k||\zeta;k'> &= <0;k|\zeta^*\cdot\alpha_1\zeta\cdot\alpha_{-1}|0;k'> \\
&= \zeta^*_\mu\zeta_\nu <0;k|\alpha_1^\mu\alpha_{-1}^\nu|0;k'> \\
&= \zeta\cdot\zeta <0;k|0;k'> = \zeta\cdot\zeta(2\pi)^D\delta^D(k-k') \ , \qquad (65)
\end{aligned}
$$

where we have used the commutator (58) for the oscillators. From this we see that the time–like ζ's will produce a state with *negative norm*. Such states cannot be made sense of in a unitary theory, and are often called[3] "ghosts".

Let us study the first constraint:

$$(L_0 - a)|\zeta; k> = 0 \qquad \Rightarrow \quad \alpha' k^2 + 1 = a, \qquad M^2 = \frac{1-a}{\alpha'} \ . \tag{66}$$

The next constraint gives:

$$(L_1)|\zeta; k> = \sqrt{\frac{\alpha'}{2}} k \cdot \alpha_1 \zeta \cdot \alpha_{-1}|0; k> = 0 \qquad \Rightarrow, \qquad k \cdot \zeta = 0 \ . \tag{67}$$

Actually, at level 1, we can also make a special state of interest: $|\psi> \equiv L_{-1}|0; k>$. This state has the special property that it is orthogonal to any physical state, since $<\phi|\psi> = <\psi|\phi>^* = <0; k|L_1|\phi> = 0$. It also has $L_1|\psi> = 2L_0|0; k> = \alpha' k^2|0; k>$. This state is called a "spurious" state. So we note that there are three interesting cases for the level 1 physical state we have been considering:

1. $a < 1 \Rightarrow M^2 > 0$:
   - momentum k is timelike.
   - We can choose a frame where it is $(k, 0, 0, \ldots)$
   - Spurious state is not physical, since $k^2 \neq 0$.
   - $k \cdot \zeta = 0$ removes the timelike polarization. $D - 1$ states left
2. $a > 1 \Rightarrow M^2 < 0$:
   - momentum k is spacelike.
   - We can choose a frame where it is $(0, k_1, k_2, \ldots)$
   - Spurious state is not physical, since $k^2 \neq 0$
   - $k \cdot \zeta = 0$ removes a spacelike polarisation. $D - 1$ tachyonic states left, one which is including ghosts.
3. $a = 1 \Rightarrow M^2 = 0$:
   - momentum k is null.
   - We can choose a frame where it is $(k, k, 0, \ldots)$
   - Spurious state is physical *and* null, since $k^2 = 0$
   - $k \cdot \zeta = 0$ and $k^2 = 0$ remove two polarizations; $D - 2$ states left

So if we choose case (3), we end up with the special situation that we have a massless vector in the D dimensional target spacetime. It even has an associated gauge invariance: since the spurious state is physical and null, and therefore we can add it to our physical state with no physical consequences, defining an equivalence relation:

$$|\phi> \sim |\phi> + \lambda|\psi> \qquad \Rightarrow \qquad \zeta^\mu \sim \zeta^\mu + \lambda k^\mu \ . \tag{68}$$

Case (1), while interesting, corresponds to a massive vector, where the extra state plays the role of a longitudinal component. Case (2) seems bad. We shall choose case (3), where $a = 1$.

It is interesting to proceed to level two to construct physical and spurious states, although we shall not do it here. The physical states are massive string states. If we insert our level one choice $a = 1$ and see what

[3] These are not to be confused with the ghosts of the friendly variety —Faddeev–Popov ghosts. These negative norm states are problematic and need to be removed.

the condition is for the spurious states to be both physical and null, we find that there is a condition on the spacetime dimension[4]: $D = 26$.

In summary, we see that $a = 1$, $D = 26$ for the open bosonic string gives a family of extra null states, giving something analogous to a point of "enhanced gauge symmetry" in the space of possible string theories. This is called a "critical" string theory, for many reasons. We have the 24 states of a massless vector we shall loosely called the photon, A_μ, since it has a $U(1)$ gauge invariance (68). There is a tachyon of $M^2 = -1/\alpha'$ in the spectrum, which will not trouble us unduly. We will actually remove it in going to the superstring case. Tachyons will reappear from time to time, representing situations where we have an unstable configuration (as happens in field theory frequently). Generally, it seems that we should think of tachyons in the spectrum as pointing us towards an instability, and in many cases, the source of the instability is manifest.

Our analysis here extends to the closed string, since we can take two copies of our result, use the appropriate zero mode relation (38), and level matching. At level zero we get the closed string tachyon which has $M^2 = -4/\alpha'$. At level zero we get a tachyon with mass given by $M^2 = -4/\alpha'$, and at level 1 we get 24^2 massless states from $\alpha^\mu_{-1}\tilde{\alpha}^\nu_{-1}|0;k\!>$. The traceless symmetric part is the graviton, $G_{\mu\nu}$ and the antisymmetric part, $B_{\mu\nu}$, is sometimes called the Kalb–Ramond field, and the trace is the dilaton, Φ.

2.8 A Glance at More Sophisticated Techniques

Later we shall do a more careful treatment of our gauge fixing procedure (31) by introducing Faddeev–Popov ghosts (b, c) to ensure that we stay on our chosen gauge slice in the full theory. Our resulting two dimensional conformal field theory will have an extra sector coming from the (b, c) ghosts.

The central term in the Virasoro algebra (61) represents an anomaly in the transformation properties of the stress tensor, spoiling its properties as a tensor under general coordinate transformations. Generally:

$$\left(\frac{\partial\sigma'^+}{\partial\sigma^+}\right)^2 T'_{++}(\sigma'^+) = T_{++}(\sigma^+) - \frac{c}{12}\left\{\frac{2\partial^3_\sigma\sigma'\partial_\sigma\sigma' - 3\partial^2_\sigma\sigma'\partial^2_\sigma\sigma'}{2\partial_\sigma\sigma'\partial_\sigma\sigma'}\right\}\ , \tag{69}$$

where c is a number, the *central charge* which depends upon the content of the theory. In our case, we have D bosons, which each contribute 1 to c, for a total anomaly of D.

The ghosts do two crucial things: They contribute to the anomaly the amount -26, and therefore we can retain all our favourite symmetries for the dimension $D = 26$. They also cancel the contributions to the vacuum energy coming from the oscillators in the $\mu = 0, 1$ sector, leaving $D - 2$ transverse oscillators' contribution.

The regulated value of $-a$ is the vacuum or "zero point" energy (z.p.e.) of the transverse modes of the theory. This zero point energy is simply the Casimir energy arising from the fact that the two dimensional field theory is in a box. The box is the infinite strip, for the case of an open string, or the infinite cylinder, for the case of the closed string (see figure 5).

A periodic (integer moded) boson such as the types we have here, X^μ, each contribute $-1/24$ to the vacuum energy. So we see that in 26 dimensions, with only 24 contributions to count (see previous paragraph), we get that $-a = 24 \times (-1/24) = -1$. (Notice that from (59), this implies that $\sum_{n=1}^\infty n = -1/12$, which is in fact true (!) in ζ–function regularization.)

Later, we shall have world–sheet fermions ψ^μ as well, in the supersymmetric theory. They each contribute $1/2$ to the anomaly. World sheet superghosts will cancel the contributions from ψ^0, ψ^1. Each anti–periodic fermion will give a z.p.e. contribution of $-1/48$. Generally, taking into account the possibility of both periodicities for either bosons or fermions:

$$\begin{aligned} \text{z.p.e.} &= \frac{1}{2}\omega \qquad \text{for boson;} \qquad -\frac{1}{2}\omega \qquad \text{for fermion} \\ \omega &= \frac{1}{24} - \frac{1}{8}(2\theta - 1)^2 \qquad \begin{cases} \theta = 0 & \text{(integer modes)} \\ \theta = \frac{1}{2} & \text{(half–integer modes)} \end{cases} \end{aligned} \tag{70}$$

[4]We get a condition on the spacetime dimension here because level 2 is the first time it can enter our formulae for the norms of states, *via* the central term in the the Virasoro algebra (61).

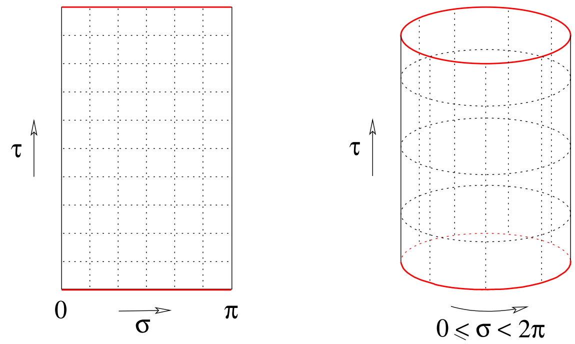


Figure 5: String world–sheets as boxes upon which live two dimensional conformal field theory.

This is a formula which we shall use many times in what is to come.

2.9 The Sphere, the Plane and the Vertex Operator

The ability to choose the conformal gauge, as first discussed in section 2.4, gives us a remarkable amount of freedom, which we can put to good use. The diagrams in figure 5 represent free strings coming in from $\tau = -\infty$ and going out to $\tau = +\infty$. Let us first focus on the closed string, the cylinder diagram. Working with Euclidean signature by taking $\tau \to -i\tau$, the metric on it is:

$$ds^2 = d\tau^2 + d\sigma^2 \ , \qquad -\infty < \tau < +\infty \qquad 0 < \sigma \leq 2\pi \ .$$

We can do the change of variables

$$z = e^{\tau - i\sigma} \ , \tag{71}$$

with the result that the metric changes to

$$ds^2 = d\tau^2 + d\sigma^2 \longrightarrow |z|^{-2} dz d\bar{z} \ .$$

This is conformal to the metric of the complex plane: $d\hat{s}^2 = dzd\bar{z}$, and so we can use this as our metric on the world–sheet, since a conformal factor $e^{\phi} = |z|^{-2}$ drops out of the action, as we already noticed.

The string from the infinite past $\tau = -\infty$ is mapped to the origin while the string in the infinite future $\tau = +\infty$ is mapped to the "point" at infinity. Intermediate strings are circles of constant radius $|z|$. See figure 6. The more forward–thinking reader who prefers to have the $\tau = +\infty$ string at the origin can use

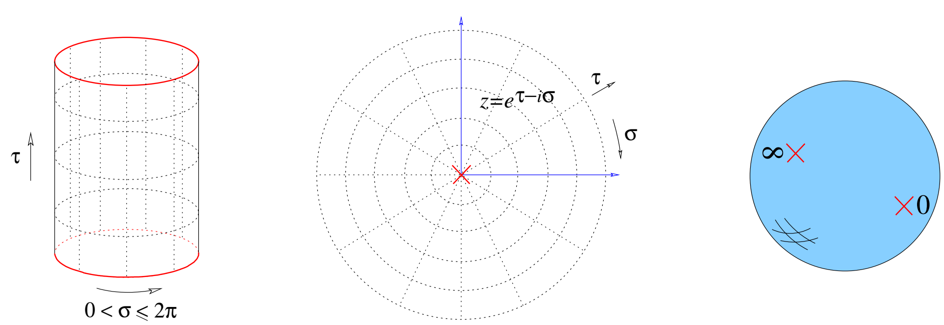


Figure 6: The cylinder diagram is conformal to the complex plane and the sphere.

the complex coordinate $\tilde{z} = 1/z$ instead.

One can even ask that *both* strings be placed at finite distance in z. Then we need a conformal factor which goes like $|z|^{-2}$ at $z = 0$ as before, but like $|z|^2$ at $z = \infty$. There is an infinite set of functions which do that, but one particularly nice choice leaves the metric:

$$ds^2 = \frac{4R^2 dz d\bar{z}}{(R^2 + |z|^2)^2} \ , \tag{72}$$

which is the familiar expression for the metric on a round S^2 with radius R, resulting from adding the point at infinity to the plane. See figure 6. The reader should check that the precise analogue of this process will relate the strip of the open string to the upper half plane, or to the disc. The open strings are mapped to points on the real axis, which is equivalent to the boundary of the disc. See figure 7.

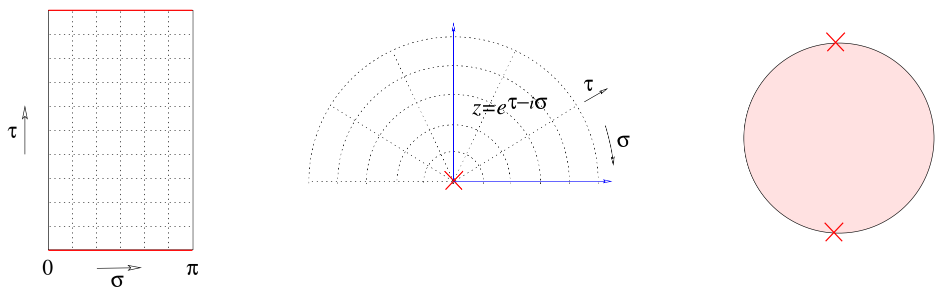


Figure 7: The strip diagram is conformal to the upper half of the complex plane and the disc.

We can go even further and consider the interaction with three or more strings. Again, a clever choice of function in the conformal factor can be made to map any tubes or strips corresponding to incoming strings to a point on the interior of the plane, or on the surface of a sphere (for the closed string) or the real axis of the upper half plane of the boundary of the disc (for the open string). See figure 8.

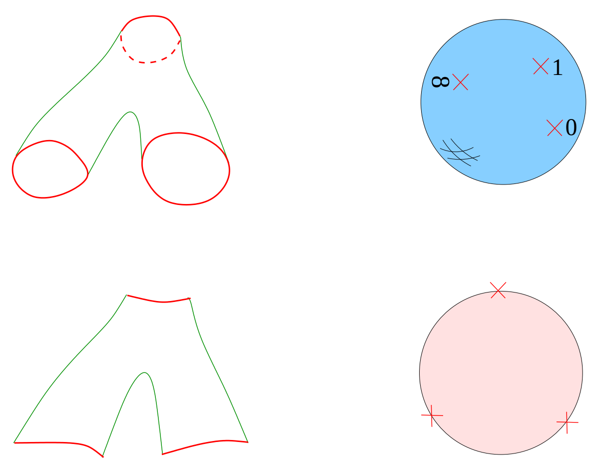


Figure 8: Mapping any number of external string states to the sphere or disc using conformal transformations.

2.9.1 Zero Point Energy From the Exponential Map

After doing the transformation to the z–plane, it is interesting to note that the Fourier expansions we have been working with to define the modes of the stress tensor become Laurent expansions on the complex plane, *e.g.*:

$$T_{zz}(z) = \sum_{m=-\infty}^{\infty} \frac{L_m}{z^{m+2}} \ .$$

One of the most straightforward exercises is to compute the zero point energy of the cylinder or strip (for a field of central charge c) by starting with the fact that the plane has no Casimir energy. One simply plugs the exponential change of coordinates $z = e^w$ into the anomalous transformation for the energy momentum tensor and compute the contribution to T_{ww} starting with T_{zz}:

$$T_{ww} = -z^2 T_{zz} - \frac{c}{24} \ ,$$

which results in the Fourier expansion on the cylinder, in terms of the modes:

$$T_{ww}(w) = -\sum_{m=-\infty}^{\infty} \left(L_m - \frac{c}{24}\delta_{m,0} \right) e^{i\sigma-\tau} \ .$$

2.9.2 States and Operators

There is one thing which we might worry about. Have we lost any information about the state that the string was in by performing this reduction of an entire string to a point? Should we not have some sort of marker with which we label each point with the properties of the string it came from? The answer is in the affirmative, and the object which should be inserted at these points is called a "vertex operator". Let us see where it comes from.

As we learned in the previous subsection, we can work on the complex plane with coordinate z. In these coordinates, our mode expansions (36) and (37) become:

$$X^\mu(z,\bar{z}) = x^\mu - i\left(\frac{\alpha'}{2}\right)^{1/2} \alpha_0^\mu \ln z\bar{z} + i\left(\frac{\alpha'}{2}\right)^{1/2} \sum_{n\neq 0} \frac{1}{n}\alpha_n^\mu \left(z^{-n} + \bar{z}^{-n}\right) \ , \qquad (73)$$

for the open string, and for the closed:

$$\begin{aligned} X^\mu(z,\bar{z}) &= X_L^\mu(z) + X_R^\mu(\bar{z}) \\ X_L^\mu(z) &= \frac{1}{2}x^\mu - i\left(\frac{\alpha'}{2}\right)^{1/2} \alpha_0^\mu \ln z + i\left(\frac{\alpha'}{2}\right)^{1/2} \sum_{n\neq 0} \frac{1}{n}\alpha_n^\mu z^{-n} \\ X_R^\mu(\bar{z}) &= \frac{1}{2}x^\mu - i\left(\frac{\alpha'}{2}\right)^{1/2} \tilde{\alpha}_0^\mu \ln \bar{z} + i\left(\frac{\alpha'}{2}\right)^{1/2} \sum_{n\neq 0} \frac{1}{n}\tilde{\alpha}_n^\mu \bar{z}^{-n} \ , \end{aligned} \qquad (74)$$

where we have used the zero mode relations (38). In fact, notice that:

$$\begin{aligned} \partial_z X^\mu(z) &= -i\left(\frac{\alpha'}{2}\right)^{1/2} \sum_n \alpha_n^\mu z^{-n-1} \\ \partial_{\bar{z}} X^\mu(\bar{z}) &= -i\left(\frac{\alpha'}{2}\right)^{1/2} \sum_n \tilde{\alpha}_n^\mu \bar{z}^{-n-1} \ , \end{aligned} \qquad (75)$$

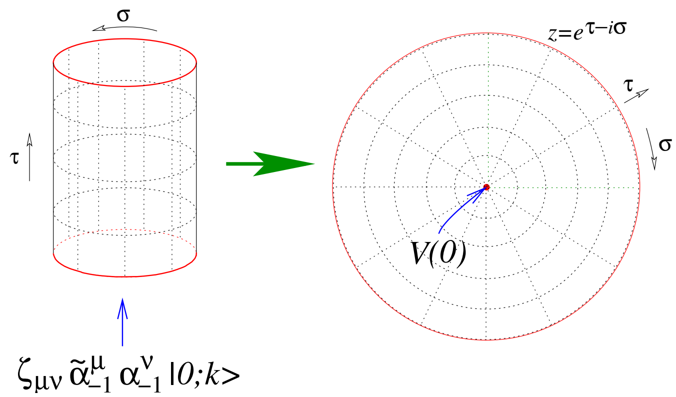


Figure 9: The correspondence between states and operator insertions. A closed string (graviton) state $\zeta_{\mu\nu}\alpha^{\mu}_{-1}\tilde{\alpha}^{\nu}_{-1}|0;k>$ is set up on the closed string at $\tau=-\infty$ and it propagates in. This is equivalent to inserting a graviton vertex operator $V^{\mu\nu}(z) =: \zeta_{\mu\nu}\partial_z X^{\mu}\partial_{\bar{z}}X^{\nu}e^{ik\cdot X}:$ at $z=0$.

and that we can invert these to get (for the closed string)

$$\alpha^{\mu}_{-n}=\left(\frac{2}{\alpha'}\right)^{1/2}\oint\frac{dz}{2\pi}z^{-n}\partial_z X^{\mu}(z)\qquad \tilde{\alpha}^{\mu}_{-n}=\left(\frac{2}{\alpha'}\right)^{1/2}\oint\frac{dz}{2\pi}\bar{z}^{-n}\partial_{\bar{z}}X^{\mu}(z)\ , \tag{76}$$

which are non–zero for $n\geq 0$. This is suggestive: Equations (75) define left–moving (holomorphic) and right–moving (anti–holomorphic) fields. We previously employed the objects on the left in (76) in making states by acting, *e.g.*, $\alpha^{\mu}_{-1}|0;k>$. The form of the right hand side suggests that this is equivalent to performing a contour integral around an insertion of a pointlike operator at the point z in the complex plane (see figure 9). For example, α^{μ}_{-1} is related to the residue $\partial_z X^{\mu}(0)$, while the α^{μ}_{-m} correspond to higher derivatives $\partial^m_z X^{\mu}(0)$. This is course makes sense, as higher levels correspond to more oscillators excited on the string, and hence higher frequency components, as measured by the higher derivatives.

The state with no oscillators excited (the tachyon), but with some momentum k, simply corresponds in this dictionary to the insertion of:

$$|0;k>\qquad\Longleftrightarrow\qquad\int d^2z :e^{ik\cdot X}: \tag{77}$$

We have integrated over the insertions' position on the sphere since the result should not depend upon our parameterization. This is reasonable, as it is the simplest form that allows the right behaviour under translations: A translation by a constant vector, $X^{\mu}\to X^{\mu}+A^{\mu}$, results in a multiplication of the operator (and hence the state) by a phase $e^{i\mathbf{k}\cdot\mathbf{A}}$. The normal ordering signs :: are there to remind that the expression means to expand and keep all creation operators to the left, when expanding in terms of the $\alpha_{\pm m}$'s.

The closed string level 1 vertex operator corresponds to the emission or absorption of $G_{\mu\nu}$, $B_{\mu\nu}$ and Φ:

$$\zeta_{\mu\nu}\alpha^{\mu}_{-1}\tilde{\alpha}^{\nu}_{-1}|0;k>\qquad\Longleftrightarrow\qquad\int d^2z\ :\zeta_{\mu\nu}\partial_z X^{\mu}\partial_{\bar{z}}X^{\nu}e^{ik\cdot X}: \tag{78}$$

where the symmetric part of $\zeta_{\mu\nu}$ is the graviton and the antisymmetric part is the antisymmetric tensor.

For the open string, the story is similar, but we get two copies of the relations (76) for the single set of modes α^{μ}_{-n} (recall that there are no $\tilde{\alpha}$'s). This results in, for example the relation for the photon:

$$\zeta_{\mu}\alpha^{\mu}_{-1}|0;k>\qquad\Longleftrightarrow\qquad\int dl\ :\zeta_{\mu}\partial_t X^{\mu}e^{ik\cdot X}:\ , \tag{79}$$

where the integration is over the position of the insertion along the real axis. Also, ∂_t means the derivative tangential to the boundary. The tachyon is simply the boundary insertion of the momentum $:e^{ik\cdot X}:$ alone.

2.10 Chan–Paton Factors

Let us endow the string endpoints with a slightly more interesting property. We can add non–dynamical degrees of freedom to the ends of the string without spoiling spacetime Poincaré invariance or world–sheet conformal invariance. These are called "Chan–Paton" degrees of freedom[25] and by declaring that their Hamiltonian is zero, we guarantee that they stay in the state that we put them in. In addition to the usual Fock space labels we have been using for the state of the string, we ask that each end be in a state i or j for i, j from 1 to N (see figure 10). We use a family of $N \times N$ matrices, λ^a_{ij}, as a basis into which to decompose

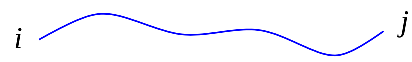


Figure 10: An open string with Chan–Paton degrees of freedom.

a string wavefunction

$$|k; a\rangle = \sum_{i,j=1}^{N} |k, ij\rangle \lambda^a_{ij}. \tag{80}$$

These wavefunctions are called "Chan–Paton factors". Similarly, all open string vertex operators carry such factors. For example, consider the tree–level (disc) diagram for the interaction of four oriented open strings in figure 11. As the Chan–Paton degrees of freedom are non–dynamical, the right end of string #1 must be

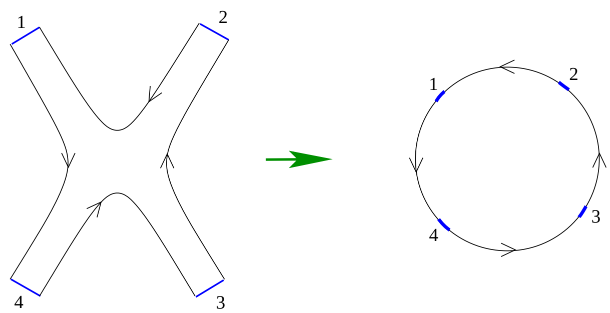


Figure 11: A four–point Scattering of open strings, and its conformally related disc amplitude.

in the same state as the left end of string #2, *etc.,* as we go around the edge of the disc. After summing over all the possible states involved in tying up the ends, we are left with a trace of the product of Chan–Paton factors,

$$\lambda^1_{ij}\lambda^2_{jk}\lambda^3_{kl}\lambda^4_{li} = \mathrm{Tr}(\lambda^1\lambda^2\lambda^3\lambda^4). \tag{81}$$

All open string amplitudes will have a trace like this and are invariant under a global (on the world–sheet) $U(N)$:

$$\lambda^i \rightarrow U\lambda^i U^{-1}, \tag{82}$$

under which the endpoints transform as $\mathbf{N}$ and $\bar{\mathbf{N}}$.

Notice that the massless vector vertex operator $V^{a\mu} = \lambda^a_{ij}\partial_t X^\mu \exp(ik \cdot X)$ transforms as the adjoint under the $U(N)$ symmetry. *This means that the global symmetry of the world–sheet theory is promoted to a gauge symmetry in spacetime.* It is a gauge symmetry because we can make a different $U(N)$ rotation at separate points $X^\mu(\sigma, \tau)$ in spacetime.

2.11 Unoriented Strings

2.11.1 Unoriented Open Strings

There is an operation of world sheet parity Ω which takes $\sigma \to \pi - \sigma$, on the open string, and acts on $z = e^{\tau - i\sigma}$ as $z \leftrightarrow -\bar{z}$. In terms of the mode expansion (73), $X^\mu(z,\bar{z}) \to X^\mu(-\bar{z}, -z)$ yields

$$\begin{aligned} x^\mu &\to x^\mu \\ p^\mu &\to p^\mu \\ \alpha_m^\mu &\to (-1)^m \alpha_m^\mu \ . \end{aligned} \tag{83}$$

This is a global symmetry of the open string theory and so, we can if we wish also consider the theory that results when it is gauged, by which we mean that only Ω–invariant states are left in the spectrum. We must also consider the case of taking a string around a closed loop. It is allowed to come back to itself only up to an over all action of Ω, which is to swap the ends. This means that we must include unoriented world–sheets in our analysis. For open strings, the case of the Möbius strip is a useful example to keep in mind. It is on the same footing as the cylinder when we consider gauging Ω. The string theories which result from gauging Ω are understandably called "unoriented string theories".

Let us see what becomes of the string spectrum when we perform this projection. The open string tachyon is even under Ω and so survives the projection. However, the photon, which has only one oscillator acting, does not:

$$\begin{aligned} \Omega|k\rangle &= +|k\rangle \\ \Omega\alpha_{-1}^\mu|k\rangle &= -\alpha_{-1}^\mu|k\rangle. \end{aligned} \tag{84}$$

We have implicitly made a choice about the sign of Ω as it acts on the vacuum. The choice we have made in writing equation (84) corresponds to the symmetry of the vertex operators (79): the resulting minus sign comes from the orientation reversal on the tangent derivative ∂_t (see figure 12).

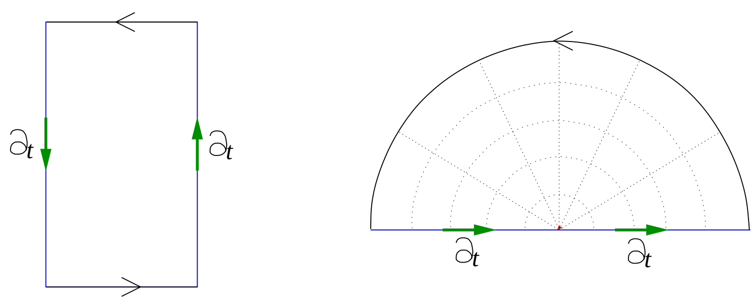


Figure 12: The action of Ω on the photon vertex operator can be deduced from seeing how exchanging the ends of the string changes the sign of the tangent derivative, ∂_t.

Fortunately, we have endowed the string's ends with Chan–Paton factors, and so there is some additional structure which can save the photon. While Ω reverses the Chan–Paton factors on the two ends of the string, it can have some additional action:

$$\Omega\lambda_{ij}|k, ij\rangle \ \to \ \lambda'_{ij}|k, ij\rangle, \quad \lambda' = M\lambda^T M^{-1}. \tag{85}$$

This form of the action on the Chan–Paton factor follows from the requirement that it be a symmetry of the amplitudes which have factors like those in equation (81).

If we act twice with Ω, this should square to the identity on the fields, and leave only the action on the Chan–Paton degrees of freedom. States should therefore be invariant under:

$$\lambda \to M M^{-T} \lambda M^T M^{-1}. \tag{86}$$

Now it should be clear that the λ must span a complete set of $N \times N$ matrices: If strings with ends labelled ik and jl are in the spectrum for *any* values of k and l, then so is the state ij. This is because jl implies lj by CPT, and a splitting–joining interaction in the middle gives $ik + lj \rightarrow ij + lk$.

Now equation (86) and Schur's lemma require MM^{-T} to be proportional to the identity, so M is either symmetric or antisymmetric. This gives two distinct cases, modulo a choice of basis[27]. Denoting the $n \times n$ unit matrix as I_n, we have the symmetric case:

$$M = M^T = I_N \tag{87}$$

In order for the photon $\lambda_{ij}\alpha^\mu_{-1}|k\rangle$ to be even under Ω and thus survive the projection, λ must be antisymmetric to cancel the minus sign from the transformation of the oscillator state. So $\lambda = -\lambda^T$, giving the gauge group $SO(N)$. For the antisymmetric case, we have:

$$M = -M^T = i \left[\begin{array}{cc} 0 & I_{N/2} \\ -I_{N/2} & 0 \end{array} \right] \tag{88}$$

For the photon to survive, $\lambda = -M\lambda^T M$, which is the definition of the gauge group $USp(N)$. Here, we use the notation that $USp(2) \equiv SU(2)$. Elsewhere in the literature this group is often denoted $Sp(N/2)$.

2.11.2 Unoriented Closed Strings

Turning to the closed string sector. For closed strings, we see that the mode expansion (74) for $X^\mu(z,\bar{z}) = X^\mu_L(z) + X^\mu_R(\bar{z})$ is invariant under a world–sheet parity symmetry $\sigma \rightarrow -\sigma$, which is $z \rightarrow -\bar{z}$. (We should note that this is a little different from the choice of Ω we took for the open strings, but more natural for this case. The two choices are related to other by a shift of π.) This natural action of Ω simply reverses the left– and right–moving oscillators:

$$\Omega\!: \qquad \alpha^\mu_n \leftrightarrow \tilde{\alpha}^\mu_n. \tag{89}$$

Let us again gauge this symmetry, projecting out the states which are odd under it. Once again, since the tachyon contains no oscillators, it is even and is in the projected spectrum. For the level 1 excitations:

$$\Omega\alpha^\mu_{-1}\tilde{\alpha}^\nu_{-1}|k\rangle = \tilde{\alpha}^\mu_{-1}\alpha^\nu_{-1}|k\rangle, \tag{90}$$

and therefore it is only those states which are symmetric under $\mu \leftrightarrow \nu$ —the graviton and dilaton— which survive the projection. The antisymmetric tensor is projected out of the theory.

2.12 World–sheet Diagrams

As stated before, once we have gauged Ω, we must allow for unoriented worldsheets, and this gives us rather more types of string worldsheet than we have studied so far. Figure 13 depicts the two types of one–loop diagram we must consider when computing amplitudes for the open string. The annulus (or cylinder) is on the left, and can be taken to represent an open string going around in a loop. The Möbius strip on the right is an open string going around a loop, but returning with the ends reversed. The two surfaces are constructed by identifying a pair of opposite edges on a rectangle, one with and the other without a twist. Figure 14 shows an example of two types of closed string one–loop diagram we must consider. On the left is a torus, while on the right is a Klein bottle, which is constructed in a similar way to a torus save for a twist introduced when identifying a pair of edges. In both the open and closed string cases, the two diagrams can be thought of as descending from the oriented case after the insertion of the normalised projection operator $\frac{1}{2}\mathrm{Tr}(1+\Omega)$ into one–loop amplitudes.

Similarly, the unoriented one–loop open string amplitude comes from the annulus and Möbius strip. We will discuss these amplitudes in more detail later. The lowest order unoriented amplitude is the projective plane $\mathbb{RP}^2$, which is a disk with opposite points identified. Shrinking the identified hole down, we recover the fact that $\mathbb{RP}^2$ may be thought of as a sphere with a crosscap inserted, where the crosscap is the result of

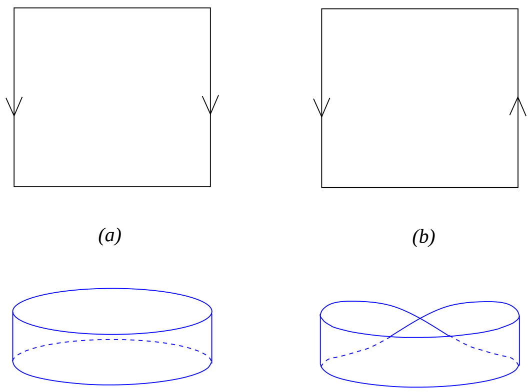


Figure 13: *(a)* Constructing a cylinder or annulus by identifying a pair of opposite edges of a rectangle. *(b)* Constructing a Möbius strip by identifying after a twist.

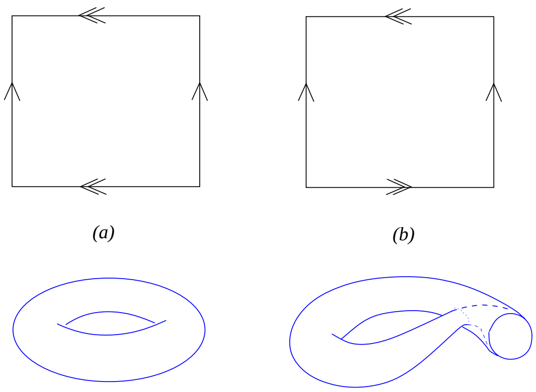


Figure 14: *(a)* Constructing a torus by identifying opposite edges of a rectangle. *(b)* Constructing a Klein bottle by identifying after a twist.

shrinking the identified hole. Actually, a Möbius strip can be thought of as a disc with a crosscap inserted, and a Klein Bottle is a sphere with two crosscaps. Since a sphere with a hole (one boundary) is the same as a disc, and a sphere with one handle is a torus, we can classify all world sheet diagrams in terms of the number of handles, boundaries and crosscaps that they have. In geeneral, each diagram is weighted by a factor $g_s^\chi = g_s^{2h-2+b+c}$ where h, b, c are the numbers of handles, boundaries and crosscaps, respectively.

2.13 Strings in Curved Backgrounds

So far, we have studied strings propagating in the (uncompactified) target spacetime with metric $\eta_{\mu\nu}$. While this alone is interesting, it is curved backgrounds of one sort or another which will occupy much of this book, and so we ought to see how they fit into the framework so far.

A natural generalisation of our action is simply to study the "σ–model" action:

$$S_\sigma = -\frac{1}{4\pi\alpha'}\int d^2\sigma\,(-g)^{1/2} g^{ab} G_{\mu\nu}(X)\partial_a X^\mu \partial_b X^\nu \ . \tag{91}$$

Comparing this to what we had before (11), we see that from the two dimensional point of view this still looks like a model of D bosonic fields X^μ, but with *field dependent* couplings given by the non–trivial spacetime metric $G_{\mu\nu}(X)$. This is an interesting action to study.

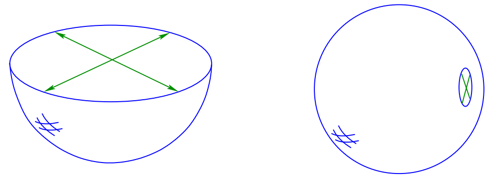


Figure 15: Constructing the projective plane $\mathbb{RP}^2$ by identifying opposite points on the disk. This is equivalent to a sphere with a crosscap insertion.

A first objection to this is that we seem to have cheated somewhat: Strings are supposed to generate the graviton (and ultimately any curved backgrounds) dynamically. Have we cheated by putting in such a background by hand? Or a more careful, less confrontational question might be: Is it consistent with the way strings generate the graviton to introduce curved backgrounds in this way? Well, let us see. Imagine, to start off, that the background metric is only locally a small deviation from flat space: $G_{\mu\nu}(X) = \eta_{\mu\nu} + h_{\mu\nu}(X)$, where h is small. Then, in conformal gauge, we can write in the Euclidean path integral (26):

$$e^{-S_\sigma} = e^{-S}\left(1 + \frac{1}{4\pi\alpha'}\int d^2z h_{\mu\nu}(X)\partial_z X^\mu \partial_{\bar{z}} X^\nu + \dots\right) , \tag{92}$$

and we see that if $h_{\mu\nu}(X) \propto g_s \zeta_{\mu\nu} \exp(ik \cdot X)$, where ζ is a symmetric polarization matrix, we are simply inserting a graviton emission vertex operator. So we are indeed consistent with that which we have already learned about how the graviton arises in string theory. Furthermore, the insertion of the full $G_{\mu\nu}(X)$ is equivalent in this language to inserting an exponential of the graviton vertex operator, which is another way of saying that a curved background is a "coherent state" of gravitons. It is clear that we should generalise our success, by including σ–model couplings which correspond to introducing background fields for the antisymmetric tensor and the dilaton:

$$S_\sigma = \frac{1}{4\pi\alpha'}\int d^2\sigma\, g^{1/2}\, \left\{(g^{ab}G_{\mu\nu}(X) + i\epsilon^{ab}B_{\mu\nu}(X))\partial_a X^\mu \partial_b X^\nu + \alpha'\Phi R\right\} , \tag{93}$$

where $B_{\mu\nu}$ is the background antisymmetric tensor field and Φ is the background value of the dilaton. The coupling for $B_{\mu\nu}$ is a rather straightforward generalisation of the case for the metric. The power of α' is there to counter the scaling of the dimension 1 fields X^μ, and the antisymmetric tensor accommodates the antisymmetry of B. For the dilaton, a coupling to the two dimensional Ricci scalar is the simplest way of writing a reparameterisation invariant coupling when there is no index structure. Correspondingly, there is no power of α' in this coupling, as it is already dimensionless.

It is worth noting at this point that α' is rather like $\hbar$ for this two dimensional theory, since the action is very large if $\alpha' \to 0$, and so this is a good limit to expand around. In this sense, the dilaton coupling is a one–loop term. Another thing to notice is that the $\alpha' \to 0$ limit is also like a "large spacetime radius" limit. This can be seen by scaling lengths by $G_{\mu\nu} \to r^2 G_{\mu\nu}$, which results in an expansion in α'/r^2. Large radius is equivalent to small α'.

The next step is to do a full analysis of this new action and ensure that in the quantum theory, one has Weyl invariance, which amounts to the tracelessness of the two dimensional stress tensor. Calculations (which we will not discuss here) reveal that[1, 2]:

$$T^a{}_a = -\frac{1}{2\alpha'}\beta^G_{\mu\nu} g^{ab}\partial_a X^\mu \partial_b X^\nu - \frac{i}{2\alpha'}\beta^B_{\mu\nu}\epsilon^{ab}\partial_a X^\mu \partial_b X^\nu - \frac{1}{2}\beta^\Phi R\, . \tag{94}$$

$$\beta^G_{\mu\nu} = \alpha'\left(R_{\mu\nu} + 2\nabla_\mu\nabla_\nu\Phi - \frac{1}{4}H_{\mu\kappa\sigma}H_\nu{}^{\kappa\sigma}\right) + O(\alpha'^2),$$

$$\beta^B_{\mu\nu} = \alpha'\left(-\frac{1}{2}\nabla^\kappa H_{\kappa\mu\nu} + \nabla^\kappa\Phi H_{\kappa\mu\nu}\right) + O(\alpha'^2), \tag{95}$$
$$\beta^\Phi = \alpha'\left(\frac{D-26}{6\alpha'} - \frac{1}{2}\nabla^2\Phi + \nabla_\kappa\Phi\nabla^\kappa\Phi - \frac{1}{24}H_{\kappa\mu\nu}H^{\kappa\mu\nu}\right) + O(\alpha'^2)\ ,$$

with $H_{\mu\nu\kappa} \equiv \partial_\mu B_{\nu\kappa} + \partial_\nu B_{\kappa\mu} + \partial_\kappa B_{\mu\nu}$. For Weyl invariance, we ask that each of these β–functions for the σ–model couplings actually vanish. (See section 3.4 for further explanation of this.) The remarkable thing is that these resemble *spacetime field equations for the background fields.* These field equations can be derived from the following spacetime action:

$$\begin{aligned} \mathrm{S} &= \frac{1}{2\kappa_0^2}\int d^DX(-G)^{1/2}e^{-2\Phi}\left[R + 4\nabla_\mu\Phi\nabla^\mu\Phi - \frac{1}{12}H_{\mu\nu\lambda}H^{\mu\nu\lambda}\right. \\ &\qquad \left. -\frac{2(D-26)}{3\alpha'} + O(\alpha')\right]\ . \end{aligned} \tag{96}$$

Note something marvellous, by the way: Φ is a background field which appears in the closed string theory σ–model multiplied by the Euler density. So comparing to (25) (and discussion following), we recover the remarkable fact that the string coupling g_s is not fixed, but is in fact given by the value of one of the background fields in the theory: $g_s = e^{<\Phi>}$. So the only free parameter in the theory is the string tension.

Turning to the open string sector, we may also write the effective action which summarises the leading order (in α') open string physics at tree level:

$$\mathrm{S} = -\frac{\mathrm{C}}{4}\int d^DX\, e^{-\Phi}\mathrm{Tr}F_{\mu\nu}F^{\mu\nu} + O(\alpha')\ , \tag{97}$$

with C a dimensionful constant which we will fix later. It is of course of the form of the Yang–Mills action, where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The field A_μ is coupled in σ–model fashion to the boundary of the world sheet by the boundary action:

$$\int_{\partial\mathcal{M}} d\tau\, A_\mu\partial_t X^\mu\ , \tag{98}$$

mimicking the form of the vertex operator (79).

One should note the powers of e^Φ in the above actions. Recall that the expectation value of e^Φ sets the value of g_s. We see that the appearance of Φ in the actions are consistent with this, as we have $e^{-2\Phi}$ in front of all of the closed string parts, representing the sphere (g_s^{-2}) and $e^{-\Phi}$ for the open string, representing the disc (g_s^{-1}).

Notice that if we make the following redefinition of the background fields:

$$\tilde{G}_{\mu\nu}(X) = e^{2\Omega(X)}G_{\mu\nu} = e^{4(\Phi_0-\Phi)/(D-2)}G_{\mu\nu}\ , \tag{99}$$

and use the fact that the new Ricci scalar can be derived using:

$$\tilde{R} = e^{-2\Omega}\left[R - 2(D-1)\nabla^2\Omega - (D-2)(D-1)\partial_\mu\Omega\partial^\mu\Omega\right]\ , \tag{100}$$

The action (96) becomes:

$$\begin{aligned} \mathrm{S} &= \frac{1}{2\kappa^2}\int d^DX(-\tilde{G})^{1/2}\left[\widetilde{R} - \frac{4}{D-2}\nabla_\mu\tilde{\Phi}\nabla^\mu\tilde{\Phi}\right. \\ &\qquad \left. -\frac{1}{12}e^{-8\tilde{\Phi}/(D-2)}H_{\mu\nu\lambda}H^{\mu\nu\lambda} - \frac{2(D-26)}{3\alpha'}e^{4\tilde{\Phi}/(D-2)} + O(\alpha')\right]\ , \end{aligned} \tag{101}$$

with $\tilde{\Phi} = \Phi - \Phi_0$, Looking at the part involving the Ricci scalar, we see that we have the form of the standard Einstein–Hilbert action (*i.e.,* we have removed the factor involving the dilaton Φ), with Newton's constant set by

$$\kappa \equiv \kappa_0 e^{\Phi_0} = (8\pi G_N)^{1/2} \ . \tag{102}$$

The standard terminology to note here is that the action (96) written in terms of the original fields is called the "string frame action", while the action (101) is referred to as the "Einstein frame action". It is in the latter frame that one gives meaning to measuring quantities like gravitational mass–energy. It is important to note the means, equation (99), to transform from the fields of one to another, depending upon dimension.

2.14 A Quick Look at Geometry

Now that we are firmly in curved spacetime, it is probably a good idea to gather some concepts, language and tools which will be useful to us in many places later on.

2.14.1 Working with the Local Tangent Frames

We can introduce "*vielbeins*" which locally diagonalize the metric[5]:

$$g_{\mu\nu}(x) = \eta_{ab} e^a_\mu(x) e^b_\nu(x).$$

The vielbeins form a basis for the tangent space at the point x, and orthonormality gives

$$e^a_\mu(x) e^{\mu b}(x) = \eta^{ab} \ .$$

These are interesting objects, connecting curved and tangent space, and transforming appropriately under the natural groups of each. It is a covariant vector under general coordinate transformations $x \to x'$:

$$e^a_\mu \to e'^a_\mu = \frac{\partial x^\nu}{\partial x'^\mu} e^a_\nu \ ,$$

and a contravariant vector under local Lorentz:

$$e^a_\mu(x) \to e'^a_\mu(x) = \Lambda^a{}_b(x) e^b_\mu(x) \ ,$$

where $\Lambda^a{}_b(x)\Lambda^c{}_d(x)\eta_{ac} = \eta_{bd}$ defines Λ as being in the Lorentz group $SO(1, D{-}1)$.

So we have the expected freedom to define our vielbein up to a local Lorentz transformation in the tangent frame. In fact the condition Λ is a Lorentz transformation guarantees that the metric is invariant under local Lorentz:

$$g_{\mu\nu} = \eta_{ab} e'^a{}_\mu e'^b{}_\nu \ . \tag{103}$$

Notice that we can naturally define a family of inverse vielbiens as well, by raising and lowering indices in the obvious way, $e^\mu_a = \eta_{ab} g^{\mu\nu} e^b_\nu$. (We use the same symbol for the vielbien, but the index structure will make it clear what we mean.) Clearly,

$$g^{\mu\nu} = \eta^{ab} e^\mu_a e^\nu_b \ , \quad e^\mu_b e^a_\mu = \delta^a_b \ . \tag{104}$$

In fact, the vielbien may be thought of as simply the matrix of coefficients of the transformation (that always exists, by the Equivalence Principle) which finds a locally inertial frame $\xi^a(x)$ from the general coordinates x^μ at the point $x = x_o$:

$$e^a_\mu(x) = \left.\frac{\partial \xi^a(x)}{\partial x^\mu}\right|_{x=x_o} \ .$$

[5] "Vielbein" means "many legs", adapted from the German. In $D = 4$ it is called a "vierbein". We shall offend the purists henceforth and not capitalise nouns taken from the German language into physics, such as "ansatz", "bremsstrahlung" and "gedankenexperiment".

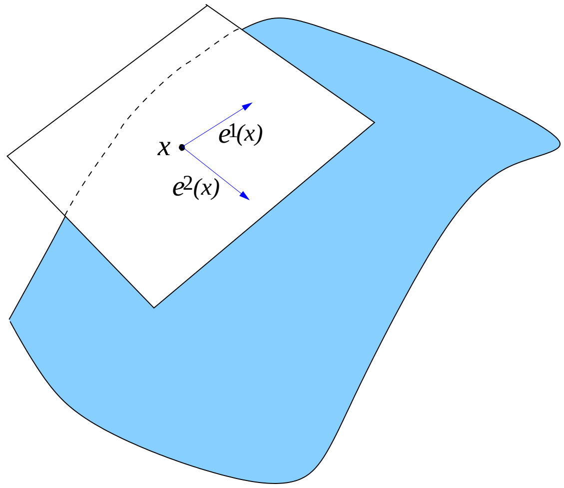


Figure 16: The local tangent frame to curved spacetime is a copy of Minkowski space, upon which the Lorentz group acts naturally.

2.14.2 Coordinate *vs.* Orthonormal Bases

The prototype contravariant vector in curved spacetime is in fact the object whose components are the infinitessimal coordinate displacements, dx^μ, since by the elementary chain rule, under $x \to x'$:

$$dx^\mu \to dx'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} dx^\nu \ . \tag{105}$$

They are often thought of as the coordinate basis elements, $\{dx^\mu\}$, for the "cotangent" space at the point x, and are a natural dual coordinate basis to that of the tangent space, the objects $\{\partial/\partial x^\mu\}$, *via* the perhaps obvious relation:

$$\frac{\partial}{\partial x^\mu} \cdot dx^\nu = \delta^\nu_\mu \ . \tag{106}$$

Of course, the $\{\partial/\partial x^\mu\}$ are the prototype covariant vectors:

$$\frac{\partial}{\partial x^\mu} \to \frac{\partial}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu} \frac{\partial}{\partial x^\nu} \ . \tag{107}$$

The things we usually think of as vectors in curved spacetime have a natural expansion in terms of these bases:

$$V = V^\mu \frac{\partial}{\partial x^\mu} \ , \qquad \text{or} \quad V = V_\mu dx^\mu \ ,$$

where the latter is sometimes called a "covector", and is also in fact a 1–form.

Yet another way of thinking of the vielbiens is as a means of converting that coordinate basis into a basis for the tangent space which is orthonormal, *via* $\{e^a = e^a_\mu(x) dx^\mu\}$. We see that we have defined a natural family of 1–forms. Similarly, using the inverse vielbiens, we can make an orthonormal basis for the dual tangent space *via* $e_a = e^\mu_a \partial/\partial x^\mu$.

As an example, for the two–sphere, S^2, of radius R, the metric in standard polar coordinates (θ, ϕ) is $ds^2 = R^2(d\theta^2 + \sin^2\theta d\phi^2)$ and so we have:

$$e^1_\theta = R \ , \ \ e^2_\phi = R \sin\theta \ , \ \ i.e., \ \ e^1 = R d\theta, \ \ e^2 = R \sin\theta \, d\phi \tag{108}$$

The things we think of as vectors, familiar from flat space, now have two natural settings. In the local frame, there is the usual vector property, under which the vector has Lorentz contravariant components

$V^a(x)$. But we can now relate this component to another object which has an index which is contravariant under general coordinate transformations, V^μ. These objects are related by our handy vielbiens: $V^a(x) = e^a_\mu(x)V^\mu$.

2.14.3 The Lorentz Group as a Gauge Group

The standard covariant derivative of *e.g.*, a contravariant vector V^μ, has a counterpart for $V^a = e^a_\mu V^\mu$:

$$D_\nu V^\mu = \partial_\nu V^\mu + \Gamma^\mu_{\nu\kappa}V^\kappa \quad \Rightarrow \quad D_\nu V^a = \partial_\nu V^a + \omega^a{}_{b\nu}V^b \ ,$$

where $\omega^a{}_{b\nu}$ is the *spin connection*, which we can write as a 1–form in either basis:

$$\omega^a{}_b = \omega^a{}_{b\mu}dx^\mu = \omega^a{}_{b\mu}e^\mu_c e^c_\nu dx^\nu = \omega^a{}_{bc}e^c \ .$$

We can think of the two Minkowski indices (a, b) from the space tangent structure as labeling components of ω as an $SO(1, D{-}1)$ matrix in the fundamental representation. So in an analogy with Yang–Mills theory, ω_μ is rather like a gauge potential and the gauge group is the Lorentz group. Actually, the most natural appearance of the spin connection is in the *structure equations* of Cartan. One defines the torsion T^a, and the curvature $R^a{}_b$, both 2–forms, as follows:

$$\begin{aligned} T^a &\equiv \frac{1}{2}T^a{}_{bc}e^a \wedge e^b = de^a + \omega^a{}_b \wedge e^b \\ R^a{}_b &\equiv \frac{1}{2}R^a{}_{bcd}e^c \wedge e^d = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b \ . \end{aligned} \tag{109}$$

Now consider a Lorentz transformation $e^a \to e'^a = \Lambda^a{}_b e^b$. It is amusing to work out how the torsion changes. Writing the result as $T'^a = \Lambda^a{}_b T^b$, the reader might like to check that this implies that the spin connection must transform as (treating everything as $SO(1, D-1)$ matrices):

$$\omega \to \Lambda\omega\Lambda^{-1} - d\Lambda \cdot \Lambda^{-1} \ , i.e., \quad \omega_\mu \to \Lambda\omega_\mu\Lambda^{-1} - \partial_\mu\Lambda \cdot \Lambda^{-1} \ , \tag{110}$$

or infinitesimally we can write $\Lambda = e^{-\Theta}$, and it is:

$$\delta\omega = d\Theta + [\omega, \Theta] \ . \tag{111}$$

A further check shows that the curvature two–form does

$$R \to R' = \Lambda R\Lambda^{-1} \ , \quad \text{or} \quad \delta R = [R, \Theta] \ , \tag{112}$$

which is awfully nice. This shows that the curvature 2–form is the analogue of the Yang–Mills field strength 2–form. The following rewriting makes it even more suggestive:

$$R^a{}_b = \frac{1}{2}R^a{}_{b\mu\nu}dx^\mu \wedge dx^\nu \ , \quad R^a{}_{b\mu\nu} = \partial_\mu\omega^a{}_{b\nu} - \partial_\nu\omega^a{}_{b\mu} + [\omega_\mu, \omega_\nu]^a{}_b \ .$$

2.14.4 Fermions in Curved Spacetime

Another great thing about this formalism is that it allows us to nicely discuss fermions in curved spacetime. Recall first of all that we can represent the Lorentz group with the Γ–matrices as follows. The group's algebra is:

$$[J_{ab}, J_{cd}] = -i(\eta_{ad}J_{bc} + \eta_{bc}J_{ad} - \eta_{ac}J_{bd} - \eta_{db}J_{ac}) \ , \tag{113}$$

with $J_{ab} = -J_{ba}$, and we can define *via* the Clifford algebra:

$$\{\Gamma^a, \Gamma^b\} = 2\eta^{ab} \ , \quad J^{ab} = -\frac{i}{4}\left[\Gamma^a, \Gamma^b\right] \ , \tag{114}$$

where the curved space Γ–matrices are related to the familiar flat (tangent) spacetime ones as $\Gamma^a = e^a_\mu(x)\Gamma^\mu(x)$, giving $\{\Gamma^\mu, \Gamma^\nu\} = 2g^{\mu\nu}$. With the Lorentz generators defined in this way, it is now natural to couple a fermion ψ to spacetime. We write a covariant derivative as:

$$D_\mu\psi(x) = \partial_\mu\psi(x) + \frac{i}{2}J_{ab}\omega^{ab}{}_\mu(x)\psi(x) \ , \tag{115}$$

and since the curved space Γ–matrices are now covariantly constant, we can write a sensible Dirac equation using this: $\Gamma^\mu D_\mu\psi = 0$.

2.14.5 Comparison to Differential Geometry

Let us make the connection to the usual curved spacetime formalism now, and fix what ω is in terms of the vielbiens (and hence the metric). Asking that the torsion vanishes is equivalent to saying that the vielbeins are covariantly constant, so that $D_\mu e^a_\nu = 0$. This gives $D_\mu V^a = e^{a\nu}D_\mu V_\nu$, allowing the two definitions of covariant derivatives to be simply related by using the vielbeins to convert the indices.

The fact that the metric is covariantly constant in terms of curved spacetime indices relates the affine connection to the metric connection, and in this language makes ω^{ab} antisymmetric in its indices. Finally, we get that

$$\omega^a{}_{b\mu} = e^a_\nu\nabla_\mu e^\nu_b = e^a_\nu(\partial_\mu e^\nu_b + \Gamma^\nu_{\mu\kappa}e^\kappa_b) \ .$$

We can now write covariant derivatives for objects with mixed indices (appropriately generalising the rule for terms to add depending upon the index structure), for example, on a vielbien:

$$D_\mu e^a_\nu = \partial_\mu e^a_\nu - \Gamma^\kappa_{\mu\nu}e^a_\kappa + \omega_\mu{}^a{}_b e^b_\nu \tag{116}$$

Revisiting our 2–sphere example, with bases given in equation (108), we can see that

$$\begin{aligned} 0 &= de^1 + \omega^1{}_2 \wedge e^2 = 0 + \omega^1{}_2 \wedge e^2 \ , \\ 0 &= de^2 + \omega^2{}_1 \wedge e^1 = R\cos\theta d\theta \wedge d\phi + \omega^2{}_1 \wedge e^1 \ , \end{aligned} \tag{117}$$

from which we see that $\omega^1{}_2 = -\cos\theta\, d\phi$. The curvature is:

$$R^1{}_2 = d\omega^1{}_2 = \sin\theta d\theta \wedge d\phi = \frac{1}{R^2}e^1 \wedge e^2 = R^1{}_{212}e^1 \wedge e^2 \ . \tag{118}$$

Notice that we can recover our friend the usual Riemann tensor if we pulled back the tangent space indices (a, b) on $R^a{}_{b\mu\nu}$ to curved space indices using the vielbiens e^μ_a.

3 A Closer Look at the World–Sheet

The careful reader has patiently suspended disbelief for a while now, allowing us to race through a somewhat rough presentation of some of the highlights of the construction of consistent relativistic strings. This enabled us, by essentially stringing lots of oscillators together, to go quite far in developing our intuition for how things work, and for key aspects of the language.

Without promising to suddenly become rigourous, it seems a good idea to revisit some of the things we went over quickly, in order to unpack some more details of the operation of the theory. This will allow us to develop more tools and language for later use, and to see a bit further into the structure of the theory.

3.1 Conformal Invariance

We saw in section 2.5 that the use of the symmetries of the action to fix a gauge left over an infinite dimensional group of transformations which we could still perform and remain in that gauge. These are conformal transformations, and the world–sheet theory is in fact conformally invariant. It is worth digressing a little and discussing conformal invariance in arbitrary dimensions first, before specializing to the case of two dimensions. You will find a reason to come back to conformal invariance in higher dimensions in the lectures of Juan Maldacena, so there is a point to this.

3.1.1 Diverse Dimensions

Imagine[9] that we do a change of variables $x \to x'$. Such a change, if invertible, is a "conformal transformation" if the metric is is invariant up to an overall scale $\Omega(x)$, which can depend on position:

$$g'_{\mu\nu}(x') = \Omega(x) g_{\mu\nu}(x) \ . \tag{119}$$

The name comes from the fact that angles between vectors are unchanged.

If we consider the infinitessimal change

$$x^\mu \to x'^\mu = x^\mu + \epsilon^\mu(x) \ , \tag{120}$$

then since the metric changes as:

$$g_{\mu\nu} \longrightarrow g'_{\mu\nu} = g_{\alpha\beta} \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} \ , \tag{121}$$

we get:

$$g'_{\mu\nu} = g_{\mu\nu} - (\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) \ , \tag{122}$$

and so we see that in order for this to be a conformal transformation,

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = F(x) g_{\mu\nu} \ , \tag{123}$$

where, by taking the trace of both sides, it is clear that:

$$F(x) = \frac{2}{D} g^{\mu\nu} \partial_\mu \epsilon_\nu \ .$$

It is enough to consider our metric to be Minkowski space, in Cartesian coordinates, *i.e.* $g_{\mu\nu} = \eta_{\mu\nu}$. We can take one more derivative ∂_κ of the expression (123), and then do the permutation of indices $\kappa \to \mu, \mu \to \nu, \nu \to \kappa$ twice, generating two more expressions. Adding together any two of those and subtracting the third gives:

$$2\partial_\mu \partial_\nu \epsilon_\kappa = \partial_\mu F \eta_{\nu\kappa} + \partial_\nu F \eta_{\kappa\mu} - \partial_\kappa F \eta_{\mu\nu} \ , \tag{124}$$

which yields

$$2\Box \epsilon_\kappa = (2-D)\partial_\kappa F \ . \tag{125}$$

We can take another derivative this expression to get $2\partial_\mu \Box \epsilon_\kappa = (2-D)\partial_\mu \partial_\kappa F$, which should be compared to the result of acting with $\Box$ on equation (123) to eliminate ϵ leaving:

$$\eta_{\mu\nu} \Box F = (2-D)\partial_\mu \partial_\nu F \quad \Longrightarrow \quad (D-1)\Box F = 0 \ , \tag{126}$$

where we have obtained the last result by contraction.

For general D we see that the last equations above ask that $\partial_\mu \partial_\nu F = 0$, and so F is linear in x. This means that ϵ is quadratic in the coordinates, and of the form:

$$\epsilon_\mu = A_\mu + B_{\mu\nu} x^\nu + C_{\mu\nu\kappa} x^\nu x^\kappa \ , \tag{127}$$

where C is symmetric in its last two indices. The parameter A_μ is obviously a translation. Placing the B term in equation (127) back into equation (123) yields that $B_{\mu\nu}$ is the sum of an antisymmetric part $\omega_{\mu\nu} = -\omega_{\nu\mu}$ and a trace part λ:

$$B_{\mu\nu} = \omega_{\mu\nu} + \lambda \eta_{\mu\nu} \ . \tag{128}$$

This represents a scale transformation by $1+\lambda$ and an infinitessimal rotation. Finally, direct substitution shows that

$$C_{\mu\nu\kappa} = \eta_{\mu\kappa} b_\nu + \eta_{\mu\nu} b_\kappa - \eta_{\nu\kappa} b_\mu \ , \tag{129}$$

Operation	Action	Generator
translations	$x'^{\mu} = x^{\mu} + A^{\mu}$	$P_{\mu} = -i\partial_{\mu}$
rotations	$x'^{\mu} = M^{\mu}{}_{\nu} x^{\nu}$	$L_{\mu\nu} = i(x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu})$
dilations	$x'^{\mu} = \lambda x^{\mu}$	$D = -ix^{\mu}\partial_{\mu}$
special conformal transf's	$x'^{\mu} = \dfrac{x^{\mu} - b^{\mu}x^2}{1 - 2(\mathbf{x}\cdot\mathbf{b}) - b^{\mu}x^2}$	$K_{\mu} = -i(2x_{\mu}x^{\nu}\partial_{\nu} - x^2\partial_{\mu})$

Table 1: The finite form of the Conformal Transformations and their infinitessimal generators.

and so the infinitessimal transformation which results is of the form

$$x'^{\mu} = x^{\mu} + 2(\mathbf{x}\cdot\mathbf{b})x^{\mu} - b^{\mu}x^2 \ , \tag{130}$$

which is called a "special conformal transformation". Its finite form can be written as:

$$\frac{x'^{\mu}}{x'^2} = \frac{x^{\mu}}{x^2} - b^{\mu} \ , \tag{131}$$

and so it looks like an inversion, then a translation, and then an inversion. We gather together all the transformations, in their finite form, in table 1.

Poincaré and dilatations together form a subgroup of the full conformal group, and it is indeed a special theory that has the full conformal invariance given by enlargement by the special conformal transformations.

It is interesting to examine the commutation relations of the generators, and to do so, we rewrite them as

$$\begin{aligned} J_{-1,\mu} &= \tfrac{1}{2}(P_{\mu} - K_{\mu}) \ , \qquad J_{0,\mu} = \tfrac{1}{2}(P_{\mu} + K_{\mu}) \ , \\ J_{-1,0} &= D \ , \quad J_{\mu\nu} = L_{\mu\nu} \ , \end{aligned} \tag{132}$$

with $J_{ab} = -J_{ba}$, $a, b = -1, 0, \ldots, D$, and the commutators are:

$$[J_{ab}, J_{cd}] = -i(\eta_{ad}J_{bc} + \eta_{bc}J_{ad} - \eta_{ac}J_{bd} - \eta_{db}J_{ac}) \ . \tag{133}$$

Note that we have defined an extra value for our indices, and η is now $\mathrm{diag}(-1, -1, +1, \ldots)$. This is the algebra of the group $SO(2, D)$ with $\frac{1}{2}(D+2)(D+1)$ parameters.

3.2 The Special Case of Two Dimensions

As we have already seen in section 2.5, the conformal transformations are equivalent to conformal mappings of the plane to itself, which is an infinite dimensional group. This might seem puzzling, since from what we saw just above, one might have expected $SO(2, 2)$, or in the case where we have Euclideanised the world–sheet, $SO(3, 1)$, a group with six parameters. Actually, this group is a very special subgroup of the infinite family, which is distinguished by the fact that the mappings are invertible. These are the *global* conformal transformations. Imagine that $w(z)$ takes the plane into itself. It can at worst have zeros and poles, (the map is not unique at a branch point, and is not invertible if there is an essential singularity) and so can be written as a ratio of polynomials in z. However, for the map to be invertible, it can only have a single zero, otherwise there would be an ambiguity determining the pre–image of zero in the inverse map. By working with the coordinate $\tilde{z} = 1/z$, in order to study the neighbourhood of infinity, we can conclude that it can only have a single simple pole also. Therefore, up to a trivial overall scaling, we have

$$z \to w(z) = \frac{az + b}{cz + d} \ , \tag{134}$$

where a, b, c, d are complex numbers, with for invertability, the determinant of the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

should be non–zero, and after a scaling we can choose $ad - bc = 1$. This is the group $SL(2,\mathbb{C})$ which is indeed isomorphic to $SO(3,1)$. In fact, since a, b, c, d is indistinguishable from $-a, -b, -c, -d$, the correct statement is that we have invariance under $SL(2,\mathbb{C})/\mathbb{Z}_2$.

For the open string we have the upper half plane, and so we are restricted to considering maps which preserve (say) the real axis of the complex plane. The result is that a, b, c, d must be real numbers, and the resulting group of invertible transformations is $SL(2,\mathbb{R})/\mathbb{Z}_2$. Correspondingly, the infinite part of the algebra is also reduced in size by half, as the holomorphic and antiholomorphic parts are no longer independent.

Notice that the dimension of the group $SL(2,\mathbb{C})$ is six, equivalent to three complex parameters. Often, in computations involving a number of operators located at points, z_i, a conventional gauge fixing of this invariance is to set three of the points to three values: $z_1 = 0, z_2 = 1, z_3 = \infty$. Similarly, the dimension of $SL(2,\mathbb{R})$ is three, and the convention used there is to set three (real) points on the boundary to $z_1 = 0, z_2 = 1, z_3 = \infty$.

3.2.1 States and Operators

A very important class of fields in the theory are those which transform under the $SO(2,D)$ conformal group as follows:

$$\phi(x^\mu) \longrightarrow \phi(x'^\mu) = \left|\frac{\partial x}{\partial x'}\right|^{\frac{\Delta}{D}} \phi(x^\mu) = \Omega^{\frac{\Delta}{2}} \phi(x^\mu) \ . \tag{135}$$

Here, $\left|\frac{\partial x}{\partial x'}\right|$ is the Jacobian of the change of variables. (Δ is the dimension of the field, as mentioned earlier.) Such fields are called "quasi–primary", and the correlation functions of some number of the fields will inherit such transformation properties:

$$<\phi_1(x_1)\cdots\phi_n(x_n)>= \left|\frac{\partial x}{\partial x'}\right|^{\frac{\Delta_1}{D}}_{x=x_1} \cdots \left|\frac{\partial x}{\partial x'}\right|^{\frac{\Delta_n}{D}}_{x=x_n} <\phi_1(x_1')\cdots\phi_n(x_n')> \tag{136}$$

In two dimensions, the relation is

$$\phi(z,\bar{z}) \longrightarrow \phi(z',\bar{z}') = \left(\frac{\partial z}{\partial z'}\right)^h \left(\frac{\partial \bar{z}}{\partial \bar{z}'}\right)^{\bar{h}} \phi(z,\bar{z}) \ , \tag{137}$$

where $\Delta = h + \bar{h}$, and we see the familiar holomorphic factorization. This mimics the transformation properties of the metric under $z \to z'(z)$:

$$g'_{z\bar{z}} = \left(\frac{\partial z}{\partial z'}\right)\left(\frac{\partial \bar{z}}{\partial \bar{z}'}\right) g_{z\bar{z}} \ ,$$

the conformal mappings of the plane. This is an infinite dimensional family, extending the expected six of $SO(2,2)$, which is the subset which is globally well–defined. The transformations (137) define what is called a "primary field", and the quasi–primaries defined earlier are those restricted to $SO(2,2)$. So a primary is automatically a quasi–primary, but not *vice–versa*.

In any dimension, we can use the definition (135) to construct a definition of a conformal field theory (CFT). First, we have a notion of a vacuum $|0>$ which is $SO(2,D)$ invariant, in which all the fields act. In such a theory, all of the fields can be divided into two categories: A field is either quasi–primary, or it is a linear combination of quasi–primaries and their derivatives. Conformal invariance imposes remarkably strong constraints on how the two– and three–point functions of the quasiprimary fields must behave. Obviously, for fields placed at positions x_i, translation invariance means that they can only depend on the differences $x_i - x_j$.

3.3 The Operator Product Expansion

In principle, we ought to be imagining the possibility of constructing a new field at the point x^μ by colliding together two fields at the same point. Let us label the fields as ϕ_k. Then we might expect something of the form:

$$\lim_{x\to y} \phi_i(x)\phi_j(y) = \sum_k c_{ij}{}^k(x-y)\phi_k(y) \ , \tag{138}$$

where the coefficients $c_{ij}{}^k(x-y)$ depend only on which operators (labelled by i, j) enter on the left. Given the scaling dimensions Δ_i for ϕ_i, we see that the coordinate behaviour of the coefficient should be:

$$c_{ij}{}^k(x-y) \sim \frac{1}{(x-y)^{\Delta_i+\Delta_j-\Delta_k}} \ .$$

This "Operator Product Expansion (OPE)" in conformal field theory is actually a convergent series, as opposed to the case of the OPE in ordinary field theory where it is merely an asymptotic series. An asymptotic series has a family of exponential contributions of the form $\exp(-L/|x-y|)$, where L is a length scale appropriate to the problem. Here, conformal invariance means that there is no length scale in the theory to play the role of L in an asymptotic expansion, and so the convergence properties of the OPE are stronger. In fact, the radius of convergence of the OPE is essentially the distance to the next operator insertion.

The OPE only really has sensible meaning if we define the operators as acting with a specific time ordering, and so we should specify that $x^0 > y^0$ in the above. In two dimensions, after we have continued to Euclidean time and work on the plane, the equivalent of time ordering is radial ordering (see figure 6). All OPE expressions written later will be taken to be appropriately time ordered.

Actually, the OPE is a useful way of giving us a definition of a normal ordering prescription in this operator language[6]. It follows from Wick's theorem, which says that the time ordered expression of a product of operators is equal to the normal ordered expression plus the sum of all contractions of pairs of operators in the expressions. The contraction is a number, which is computed by the correlator of the contracted operators.

$$\phi_i(x)\phi_j(y) = :\phi_i(x)\phi_j(y): + <\phi_i(x)\phi_j(y)> \ . \tag{139}$$

Actually, we can compute the OPE between objects made out of products of operators with this sort of way of thinking about it. We'll compute some examples later (for example in equations (156) and (158)) so that it will be clear that it is quite straightforward.

3.3.1 The Stress Tensor and the Virasoro Algebra

The stress–energy–momentum tensor's properties can be seen to be directly conformal invariance in many ways, because of its definition as a conjugate to the metric:

$$T^{\mu\nu} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g_{\mu\nu}} \ . \tag{140}$$

A change of variables of the form (120) gives, using equation (122):

$$S \longrightarrow S - \frac{1}{2}\int d^Dx\sqrt{-g}\,T^{\mu\nu}\delta g_{\mu\nu} = S + \frac{1}{2}\int d^Dx\sqrt{-g}\,T^{\mu\nu}\left(\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu\right) \ .$$

In view of equation (123), this is:

$$S \longrightarrow S + \frac{1}{D}\int d^Dx\sqrt{-g}\,T^\mu{}_\mu\partial_\nu\epsilon^\nu \ ,$$

[6] For free fields, this definition of normal ordering is equivalent to the definition in terms of modes, where the annihilators are placed to the right.

for a conformal transformation. So if the action is conformally invariant, then the stress tensor must be traceless, $T^\mu{}_\mu = 0$. We can formulate this more carefully using Noether's theorem, and also extract some useful information. Since the change in the action is

$$\delta S = \int d^D x\, \sqrt{-g}\, \partial_\mu \epsilon_\nu T^{\mu\nu} \ ,$$

given that the stress tensor is conserved, we can integrate by parts to write this as

$$\delta S = \int_\partial \epsilon_\nu T^{\mu\nu} dS_\mu \ .$$

We see that the current $j^\mu = T^{\mu\nu}\epsilon_\mu$, with ϵ_ν given by equation (123) is associated to the conformal transformations. The charge constructed by integrating over an equal time slice

$$Q = \int d^{D-1}x J^0 \ ,$$

is conserved, and it is responsible for infinitessimal conformal transformations of the fields in the theory, defined in the standard way:

$$\delta_\epsilon \phi(x) = \epsilon[Q, \phi] \ . \qquad (141)$$

In two dimensions, infinitesimally, a coordinate transformation can be written as

$$z \to z' = z + \epsilon(z) \ , \qquad \bar{z} \to \bar{z}' = \bar{z} + \bar{\epsilon}(\bar{z}) \ .$$

The tracelessness condition yields $T_{z\bar{z}} = T_{\bar{z}z} = 0$ and the conservation of the stress tensor is

$$\partial_z T_{zz}(z) = 0 = \partial_{\bar{z}} T_{\bar{z}\bar{z}}(\bar{z}) \ .$$

For simplicity, we shall often use the shorthand: $T(z) \equiv T_{zz}(z)$ and $\bar{T}(\bar{z}) \equiv T_{\bar{z}\bar{z}}(\bar{z})$. On the plane, an equal time slice is over a circle of constant radius, and so we can define

$$Q = \frac{1}{2\pi i} \oint \left(T(y)\epsilon(y) dy + \bar{T}(\bar{y})\bar{\epsilon}(\bar{y}) d\bar{y} \right) \ .$$

Infinitessimal transformations can then be constructed by an appropriate definition of the commutator $[Q, \phi(z)]$ of Q with a field ϕ:

Notice that this commutator requires a definition of two operators at a point, and so our previous discussion of the OPE comes into play here. We also have the added complication that we are performing a y–contour integration around one of the operators, inserted at z or $\bar{z}$. Under the integral sign, the OPE requires that $|z| < |y|$, when we have $Q\phi(y)$, or that $|z| > |y|$ if we have $\phi(y)Q$. The commutator requires the difference between these two, after consulting figure 17, can be seen in the limit $y \to z$ to simply result in the y contour integral around the point z of the OPE $T(z)\phi(y)$ (with a similar discussion for the anti–holomorphic case):

$$\delta_{\epsilon,\bar{\epsilon}} \phi(z, \bar{z}) = \frac{1}{2\pi i} \oint \left(\{T(y)\phi(z, \bar{z})\}\epsilon(y) dy + \{\bar{T}(\bar{y})\phi(z, \bar{z})\}\bar{\epsilon}(\bar{y}) d\bar{y} \right) \ . \qquad (142)$$

The result should simply be the infinitessimal version of the defining equation (137), which the reader should check is:

$$\delta_{\epsilon,\bar{\epsilon}} \phi(z, \bar{z}) = \left(h \frac{\partial \epsilon}{\partial z} \phi + \epsilon \frac{\partial \phi}{\partial z} \right) + \left(h \frac{\partial \bar{\epsilon}}{\partial \bar{z}} \phi + \bar{\epsilon} \frac{\partial \phi}{\partial \bar{z}} \right) \ . \qquad (143)$$

This defines the operator product expansions $T(z)\phi(z, \bar{z})$ and $\bar{T}(\bar{z})\phi(z, \bar{z})$ for us as:

$$\begin{aligned} T(y)\phi(z, \bar{z}) &= \frac{h}{(y - z)^2}\phi(z, \bar{z}) + \frac{1}{(y - z)}\partial_z \phi(z, \bar{z}) + \cdots \\ \bar{T}(\bar{y})\phi(z, \bar{z}) &= \frac{\bar{h}}{(\bar{y} - \bar{z})^2}\phi(z, \bar{z}) + \frac{1}{(\bar{y} - \bar{z})}\partial_z \phi(z, \bar{z}) + \cdots \end{aligned} \qquad (144)$$

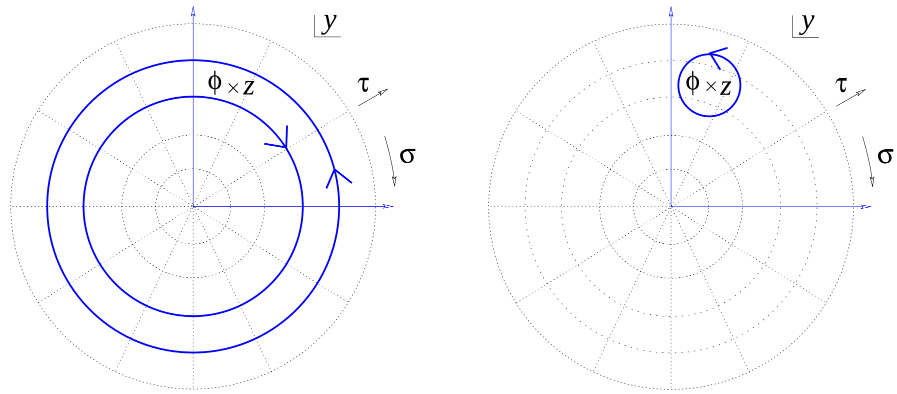


Figure 17: Computing the commutator between the generator Q, defined as a contour in the y–plane, and the operator ϕ, inserted at z. The result in the limit $y \to z$ is on the right.

where the ellipsis indicates that we have ignored parts which are regular (analytic). These OPEs constitute an alternative definition of a primary field with holomorphic and anti–holomorphic weights $h, \bar{h}$, often referred to simply as an $(h, \bar{h})$ primary.

We are at liberty to Laurent expand the infinitessimal transformation around $(z, \bar{z}) = 0$:

$$\epsilon(z) = -\sum_{n=-\infty}^{\infty} a_n z^{n+1} \ , \qquad \bar{\epsilon}(\bar{z}) = -\sum_{n=-\infty}^{\infty} \bar{a}_n \bar{z}^{n+1} \ ,$$

where the $a_n, \bar{a}_n$ are coefficients. The quantities which appear as generators, $\ell_n = z^{n+1}\partial_z, \bar{\ell}_n = \bar{z}^{n+1}\partial_{\bar{z}}$, satisfy the commutation relations

$$\begin{aligned} [\ell_n, \ell_m] &= (n-m)\ell_{n+m} \ , \\ [\ell_n, \bar{\ell}_m] &= 0 \ , \\ [\bar{\ell}_n, \bar{\ell}_m] &= (n-m)\bar{\ell}_{n+m} \ , \end{aligned} \qquad (145)$$

which is the classical version of the Virasoro algebra we saw previously in equation (53), or the quantum case in equation (61) with the central extension, $c = \bar{c} = 0$.

Now we can compare with what we learned here. It should be clear after some thought that $\ell_{-1}, \ell_0, \ell_1$ and their antiholomorphic counterparts form the six generators of the global conformal transformations generating $SL(2, \mathbb{C}) = SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$. In fact, $\ell_{-1} = \partial_z$ and $\bar{\ell}_{-1} = \partial_{\bar{z}}$ generate translations, $\ell_0 + \bar{\ell}_0$ generates dilations, $i(\ell_0 - \bar{\ell}_0)$ generates rotations, while $\ell_1 = z^2\partial_z$ and $\bar{\ell}_1 = \bar{z}^2\partial_{\bar{z}}$ generate the special conformal transformations.

Let us note some useful pieces of terminology and physics here. Recall that we had defined physical states to be those annihilated by the $\ell_n, \bar{\ell}_n$ with $n > 0$. Then ℓ_0 and $\bar{\ell}_0$ will measure properties of these physical states. Considering them as operators, we can find a basis of ℓ_0 and $\bar{\ell}_0$ eigenstates, with eigenvalues h and $\bar{h}$ (two independent numbers), which are the "conformal weights" of the state: $\ell_0|h> = h|h>$, $\bar{\ell}_0|\bar{h}> = \bar{h}|\bar{h}>$. Since the sum and difference of these operators are the dilations and the rotations, we can characterize the scaling dimension and the spin of a state or field as $\Delta = h + \bar{h}$, $s = h - \bar{h}$.

It is worth noting here that the stress–tensor itself is *not* in general a primary field of weight $(2, 2)$, despite the suggestive fact that it has two indices. There can be an anomalous term, allowed by the symmetries of the theory:

$$T(z)T(y) = \frac{c}{2}\frac{1}{(z-y)^4} + \frac{2}{(z-y)^2}T(y) + \frac{1}{z-y}\partial_y T(y) \ ,$$

$$\bar{T}(\bar{z})\bar{T}(\bar{y}) \quad = \quad \frac{\bar{c}}{2}\frac{1}{(\bar{z}-\bar{y})^4}+\frac{2}{(\bar{z}-\bar{y})^2}\bar{T}(\bar{y})+\frac{1}{\bar{z}-\bar{y}}\partial_{\bar{y}}\bar{T}(\bar{y}) \ . \qquad (146)$$

The holomorphic conformal anomaly c and its antiholomorphic counterpart $\bar{c}$, can in general be non–zero. We shall see this occur below.

It is worthwhile turning some of the above facts into statements about commutation relation between the modes of $T(z), \bar{T}(\bar{z})$, which we remind the reader are defined as:

$$\begin{aligned} T(z) \quad &= \quad \sum_{n=-\infty}^{\infty} L_n z^{-n-2} \ , \qquad L_n = \frac{1}{2\pi i}\oint dz\ z^{n+1}T(z) \ , \\ \bar{T}(\bar{z}) \quad &= \quad \sum_{n=-\infty}^{\infty} \bar{L}_n \bar{z}^{-n-2} \ , \qquad \bar{L}_n = \frac{1}{2\pi i}\oint d\bar{z}\ \bar{z}^{n+1}\bar{T}(\bar{z}) \ . \end{aligned} \qquad (147)$$

In these terms, the resulting commutator between the modes is that displayed in equation (61), with D replaced by $\bar{c}$ and c on the right and left.

The definition (143) of the primary fields ϕ translates into

$$[L_n,\phi(y)] = \frac{1}{2\pi i}\oint dz z^{n+1}T(z)\phi(y) = h(n+1)y^n\phi(y)+y^{n+1}\partial_y\phi(y) \ . \qquad (148)$$

It is useful to decompose the primary into its modes:

$$\phi(z) = \sum_{n=-\infty}^{\infty}\phi_n z^{-n-h} \ , \qquad \phi_n = \frac{1}{2\pi i}\oint dz\ z^{h+n-1}\phi(z) \ . \qquad (149)$$

In terms of these, the commutator between a mode of a primary and of the stress tensor is:

$$[L_n,\phi_m] = [n(h-1)-m]\phi_{n+m} \ , \qquad (150)$$

with a similar antiholomorphic expression. In particular this means that our correspondence between states and operators can be made precise with these expressions. $L_0|h\rangle = h|h\rangle$ matches with the fact that $\phi_{-h}|0\rangle = |h\rangle$ would be used to make a state, or more generally $|h,\bar{h}\rangle$, if we include both holomorphic and anti–holomorphic parts. The result $[L_0,\phi_{-h}] = h\phi_{-h}$ guarantees this.

In terms of the finite transformation of the stress tensor under $z\to z'$, the result (146) is

$$T(z) = \left(\frac{\partial z'}{\partial z}\right)^2 T(z') + \frac{c}{12}\left(\frac{\partial z'}{\partial z}\right)^{-2}\left[\frac{\partial z'}{\partial z}\frac{\partial^3 z'}{\partial z^3}-\frac{3}{2}\left(\frac{\partial^2 z'}{\partial z^2}\right)^2\right] \ , \qquad (151)$$

where the quantity multiplying $c/12$ is called the "Schwarzian derivative", $S(z,z')$. It is interesting to note (and the reader should check) that for the $SL(2,\mathbb{C})$ subgroup, the proper global transformations, $S(z,z')=0$. This means that the stress tensor is in fact a quasi–primary field, but not a primary field.

3.4 Revisiting the Relativistic String

Now we see the full role of the energy–momentum tensor which we first encountered in the previous section. Its Laurent coefficients there, L_n and $\bar{L}_n$, realized there in terms of oscillators, satisfied the Virasoro algebra, and so its role is to generate the conformal transformations. We can use it to study the properties of various operators in the theory of interest to us.

First, we translate our result of equation (34) into the appropriate coordinates here:

$$\begin{aligned} T(z) \quad &= \quad -\frac{1}{\alpha'} :\partial_z X^\mu(z)\partial_z X_\mu(z): \ , \\ \bar{T}(\bar{z}) \quad &= \quad -\frac{1}{\alpha'} :\partial_{\bar{z}} X^\mu(\bar{z})\partial_{\bar{z}} X_\mu(\bar{z}): \ . \end{aligned} \qquad (152)$$

We can use here our definition (139) of the normal ordering at the operator level here, which we construct with the OPE. To do this, we need to know the result for the OPE of ∂X^μ with itself. This we can get by observing that the propagator of the field $X^\mu(z,\bar{z}) = X(z) + \bar{X}(\bar{z})$ is

$$\begin{aligned} <X(z)^\mu X^\nu(y)> &= -\frac{\alpha'}{2}\eta^{\mu\nu}\log(z-y) \ , \\ <\bar{X}(\bar{z})^\mu \bar{X}^\nu(\bar{y})> &= -\frac{\alpha'}{2}\eta^{\mu\nu}\log(\bar{z}-\bar{y}) \ . \end{aligned} \tag{153}$$

By taking a couple of derivatives, we can deduce the OPE of $\partial_z X^\mu(z)$ or $\partial_{\bar{z}}\bar{X}^\mu(\bar{z})$:

$$\begin{aligned} \partial_z X^\mu(z)\partial_y X^\nu(y) &= -\frac{\alpha'}{2}\frac{\eta^{\mu\nu}}{(z-y)^2} + \cdots \\ \partial_{\bar{z}}\bar{X}^\nu(\bar{z})\partial_{\bar{y}}\bar{X}^\mu(\bar{y}) &= -\frac{\alpha'}{2}\frac{\eta^{\mu\nu}}{(\bar{z}-\bar{y})^2} + \cdots \end{aligned} \tag{154}$$

So in the above, we have, using our definition of the normal ordered expression using the OPE (see discussion below equation (139)):

$$T(z) = -\frac{1}{\alpha'} : \partial_z X^\mu(z)\partial_z X_\mu(z) := -\frac{1}{\alpha'}\lim_{y\to z}\left[\partial_z X^\mu(z)\partial_z X_\mu(y) - \frac{D}{(\bar{z}-\bar{y})^2}\right] \ , \tag{155}$$

with a similar expression for the anti–holomorphic part. It is now straightforward to evaluate the OPE of $T(z)$ and $\partial_z X^\nu(y)$. We simply extract the singular part of the following:

$$\begin{aligned} T(z)\partial_y X^\nu(y) &= \frac{1}{\alpha'} : \partial_z X^\mu(z)\partial_z X_\mu(z) : \partial_y X^\nu(y) \\ &= 2\cdot\frac{1}{\alpha'}\partial_z X^\mu(z) <\partial_z X_\mu(z)\partial_z X^\nu(y)> + \cdots \\ &= \partial_z X^\nu(z)\frac{1}{(z-y)^2} + \cdots \end{aligned} \tag{156}$$

In the above, we were instructed by Wick to perform the two possible contractions to make the correlator. The next step is to Taylor expand for small $(z-y)$: $X^\nu(z) = X^\nu(y) + (z-y)\partial_y X^\nu(y) + \cdots$, substitute into our result, to give:

$$T(z)\partial_y X^\nu(y) = \frac{\partial_y X^\nu(y)}{(z-y)^2} + \frac{\partial_y^2 X^\nu(y)}{z-y} + \cdots \tag{157}$$

and so we see from our definition in equation (144) that the field $\partial_z X^\nu(z)$ is a primary field of weight $h=1$, or a $(1,0)$ primary field, since from the OPE's (154), its OPE with $\bar{T}$ obviously vanishes. Similarly, the anti–holomorphic part is a $(0,1)$ primary. Notice that we should have suspected this to be true given the OPE we deduced in (154).

Another operator we used last section was the normal ordered exponentiation $V(z) =: \exp(i\mathbf{k}\cdot\mathbf{X}(z)) :$, which allowed us to represent the momentum of a string state. Here, the normal ordering means that we should not contract the various X's which appear in the expansion of the exponential with each other. We can extract the singular part to define the OPE with $T(z)$ by following our noses and applying the Wick procedure as before:

$$\begin{aligned} T(z)V(y) &= \frac{1}{\alpha'} : \partial_z X^\mu(z)\partial_z X_\mu(z) :: e^{i\mathbf{k}\cdot\mathbf{X}(y)} : \\ &= \frac{1}{\alpha'}(< \partial_z X^\mu(z) i\mathbf{k}\cdot\mathbf{X}(y) >)^2 : e^{i\mathbf{k}\cdot\mathbf{X}(y)} : \\ &\qquad +2\cdot\frac{1}{\alpha'}\partial_z X^\mu(z) < \partial_z X_\mu(z) i\mathbf{k}\cdot\mathbf{X}(y) >: e^{i\mathbf{k}\cdot\mathbf{X}(y)} : \end{aligned}$$

$$= \frac{\alpha' k^2}{4} \frac{1}{(z-y)^2} : e^{i\mathbf{k}\cdot\mathbf{X}(y)} : + \frac{i\mathbf{k}\cdot\partial_z \mathbf{X}(z)}{(z-y)} : e^{i\mathbf{k}\cdot\mathbf{X}(y)} :$$
$$= \frac{\alpha' k^2}{4} \frac{V(y)}{(z-y)^2} + \frac{\partial_y V(y)}{(z-y)} , \qquad (158)$$

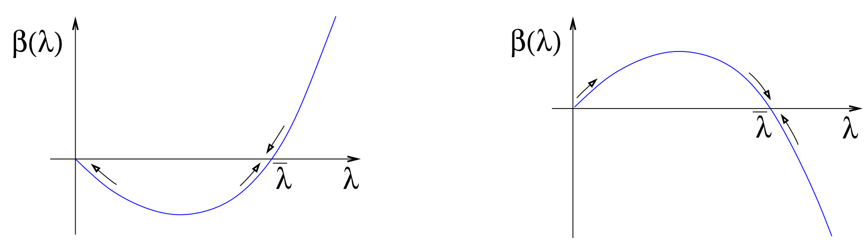
We have Taylor expanded in the last line, and throughout we only displayed explicitly the singular parts. The expressions tidy up themselves quite nicely if one realizes that the worst singularity comes from when there are two contractions with products of fields using up both pieces of $T(z)$. Everything else is either non–singular, or sums to reassemble the exponential after combinatorial factors have been taken into account. This gives the first term of the second line. The second term of that line comes from single contractions. The factor of two comes from making two choices to contract with one or other of the two identical pieces of $T(z)$, while there are other factors coming from the n ways of choosing a field from the term of order n from the expansion of the exponential. After dropping the non–singular term, the remaining terms (with the n) reassemble the exponential again. (The reader is advised to check this explicitly to see how it works.) The final result (when combined with the anti–holomorphic counterpart) shows that $V(y)$ is a primary field of weight $(\alpha' k^2/4, \alpha' k^2/4)$.

Now we can pause to see what this all means. Recall from section 2.9.2 that the insertion of states is equivalent to the insertion of operators into the theory, so that:

$$S \to S' = S + \lambda \int d^2z \mathcal{O}(z,\bar{z}) . \qquad (159)$$

In general, we may consider such an operator insertion for a general theory. For the theory to remain conformally invariant, the operator must be a *marginal* operator, which is to say that $\mathcal{O}(z,\bar{z})$ must at least have dimension $(1,1)$ so that the integrated operator is dimensionless. In principle, the dimension of the operator after the deformation (*i.e.* in the new theory defined by S') can change, and so the full condition for the operator is that it must remain $(1,1)$ after the insertion. It in fact defines a direction in the space of couplings, and λ can be thought of as an infinitessimal motion in that direction. The statement of the existence of a marginal operator is then referred to the existence of a "flat direction".

A useful picture to have in mind for later use is of a conformal field theory as a "fixed point" in the space of theories with coordinates given by the coefficients of possible operators such as in equation (159). (There is an infinite set of such perturbations and so the space is infinite dimensional.) In the usual reasoning using the renomalisation group (RG), once the operator is added with some value of the coupling, the theory (*i.e.* the value of the coupling) flows along an RG trajectory as the energy scale μ is changed. The "β–function", $\beta(\lambda) \equiv \mu\partial\lambda/\partial\mu$ characterizes the behaviour of the coupling. One can imagine the existence of *"fixed points"* of such flows, where $\beta(\lambda) = 0$ and the coupling tends to a specific value, as shown in the diagram.



On the left, $\bar{\lambda}$ is an "infrared (IR) fixed point", since the coupling is driven to it for decreasing μ, while on the right, $\bar{\lambda}$ is an "ultraviolet (UV) fixed point", since the coupling is driven to it for increasing μ. The origins of each diagram of course define a fixed point of the opposite type to that at $\bar{\lambda}$. A conformal field theory is then clearly such a fixed point theory, where the scale dependence of all couplings exactly vanishes. A *"marginal operator"* is an operator which when added to the theory, does not take it away from the fixed point. A *"relevant operator"* deforms a theory increasingly as μ goes to the IR, while an *"irrelevant operator"* is increasingly less important in the IR. This behaviour is reversed on going to the UV. When applied to a fixed point, such non–marginal operators can be used to deform fixed point theories away from the conformal point, often allowing us to find other interesting theories. $D=4$ Yang–Mills theories, for sufficiently few flavours of quark (like QCD), have negative β–function, and so behave roughly as the neighbourhood of the origin in the left diagram. *"Asymptotic freedom"* is the process of being driven to the origin (zero coupling) in the UV.

Returning to the specific case of the basic relativistic string, recall that the use of the tachyon vertex operator $V(z,\bar{z})$ corresponds to the addition of $\int d^2z\, V(z,\bar{z})$ to the action. We wish the theory to remain conformal (preserving the relativistic string's symmetries, as stressed in section 2), and so $V(z,\bar{z})$ must be (1,1). In fact, since our conformal field theory is actually free, we need do no more to check that the tachyon vertex is marginal. So we require that $(\alpha' k^2/4, \alpha' k^2/4) = (1,1)$. Therefore we get the result that $M^2 \equiv -k^2 = -4/\alpha'$, the result that we previously obtained for the tachyon.

Another example is the level 1 closed string vertex operator:

$$: \partial_z X^\mu \partial_{\bar{z}} X^\nu \exp(i\mathbf{k}\cdot\mathbf{X}) :$$

It turns out that there are no further singularities in contracting this with the stress tensor, and so the weight of this operator is $(1+\alpha' k^2/4, 1+\alpha' k^2/4)$. So, marginality requires that $M^2 \equiv -k^2 = 0$, which is the massless result that we encountered earlier.

Another computation that the reader should consider doing is to work out explicitly the $T(z)T(y)$ OPE, and show that it is of the form (146) with $c=D$, as each of the D bosons produces a conformal anomaly of unity. This same is true from the antiholomorphic sector, giving $\bar{c} = D$. Also, for open strings, we get the same amount for the anomaly. This result was alluded to in section 2. This is problematic, since this conformal anomaly prevents the full operation of the string theory. In particular, the anomaly means that the stress tensor's trace does not in fact vanish quantum mechanically.

This is all repaired in the next section, since there is another sector which we have not yet considered.

3.5 Fixing The Conformal Gauge

It must not be forgotten where all of the riches of the previous section —the conformal field theory— came from. We made a gauge choice in equation (31) from which many excellent results followed. However, despite everything, we saw that there is in fact a conformal anomaly equal to D (or a copy each on both the left and the right hand side, for the closed string). The problem is that we have not made sure that the gauge fixing was performed properly. This is because we are fixing a local symmetry, and it needs to be done dynamically in the path integral, just as in gauge theory. This is done with Faddeev–Popov ghosts in a very similar way to the methods used in field theory. Let us not go into the details of it here, but assume that the interested reader can look into the many presentations of the procedure in the literature. The key difference with field theory approach is that it introduces two ghosts, c^a and b_{ab} which are rank one and rank two tensors on the world sheet. The action for them is:

$$S^{\rm gh} = -\frac{1}{4\pi}\int d^2\sigma \sqrt{g} g^{ab} c^c \nabla_a b_{bc} \ , \tag{160}$$

and so b_{ab} and c^a, which are anti–commuting, are conjugates of each other.

3.5.1 Conformal Ghosts

Once the conformal gauge has been chosen, (see equation (31)) picking the diagonal metric, we have

$$S^{\rm gh} = -\frac{1}{2\pi}\int d^2z \left(c(z,\bar{z})\partial_{\bar{z}} b(z,\bar{z}) + \bar{c}(z,\bar{z})\partial_z \bar{b}(z,\bar{z})\right) \ . \tag{161}$$

From equation (160), the stress tensor for the ghost sector is:

$$T^{\rm gh}(z) =: c(z)\partial_{\bar{z}} b(z) : + : 2(\partial_{\bar{z}} c(z)) b(z) : \ , \tag{162}$$

with a similar expression for $\bar{T}_{\rm ghost}(\bar{z})$. Just as before, as the ghosts are free fields, with equations of motion $\partial_z c = 0 = \partial_z b$, we can Laurent expand them as follows:

$$b(z) = \sum_{n=-\infty}^{\infty} b_n\, z^{-n-2} \ , \qquad c(z) = \sum_{n=-\infty}^{\infty} c_n\, z^{-n+1} \ , \tag{163}$$

which follows from the property that b is of weight 2 and c is of weight -1, a fact which might be guessed from the structure of the action (160). The quantisation yields

$$\{b_m, c_n\} = \delta_{m+n} \ . \tag{164}$$

and the stress tensor is

$$L_n^{\rm gh} = \sum_{m=-\infty}^{\infty} (2n-m) : b_m c_{n-m} : -\delta_{n,0} \tag{165}$$

where we have a normal ordering constant -1, as in the previous sector.

$$[L_m^{\rm gh}, b_n] = (m-n) b_{m+n} \ , \qquad [L_m^{\rm gh}, c_n] = -(2m+n) c_{m+n} \ . \tag{166}$$

The OPE for the ghosts is given by

$$\begin{aligned} b(z)c(y) &= \frac{1}{(z-y)} + \cdots \ , \qquad c(z)b(y) = \frac{1}{(z-y)} + \cdots \ , \\ b(z)b(y) &= O(z-y) \qquad , \qquad c(z)c(y) = O(z-y) \ , \end{aligned} \tag{167}$$

where the second expression is obtained from the first by the anticommuting property of the ghosts. The second line also follows from the anticommuting property. There can be no non–zero result for the singular parts there.

As with everything for the closed string, we must supplement the above expressions with very similar ones referring to $\bar{z}$, $\bar{c}(\bar{z})$ and $\bar{b}(\bar{z})$. For the open string, we carry out the same procedures as before, defining everything on the upper half plane, reflecting the holomorphic into the anti–holomorpic parts, defining a single set of ghosts.

3.5.2 The Critical Dimension

Now comes the fun part. We can evaluate the conformal anomaly of the ghost system, by using the techniques for computation of the OPE that we refined in the previous section. We can do it for the ghosts in as simple a way as for the ordinary fields, using the expression (162) above. In the following, we will focus on the most singular part, to isolate the conformal anomaly term. This will come from when there are two contractions in each term. The next level of singularity comes from one contraction, and so on:

$$\begin{aligned} & T^{\rm gh}(z) T^{\rm gh}(y) \\ &= (: \partial_z b(z) c(z) : + : 2b(z)\partial_z c(z) :)(: \partial_y b(y) c(y) : + : 2b(y)\partial_y c(y) :) \\ &= \ : \partial_z b(z) c(z) :: \partial_y b(y) c(y) : + 2 : b(z)\partial_z c(z) :: \partial_y b(y) c(y) : \\ &+ \ 2 : \partial_z b(z) c(z) :: b(y)\partial_y c(y) : + 4 : b(z)\partial_z c(z) :: b(y)\partial_y c(y) : \\ &= \ <\partial_z b(z) c(y)><c(z)\partial_y b(y)> + 2 <b(z)c(y)><\partial_z c(z)\partial_y b(y)> \\ &+ \ 2<\partial_z b(z)\partial_y c(y)><c(z)b(y)> + 4<b(z)\partial_y c(y)><\partial_z c(z) b(y)> \\ &= \ -\frac{13}{(z-y)^4} \ , \end{aligned} \tag{168}$$

and so comparing with equation (146), we see that the ghost sector has conformal anomaly $c = -26$. A similar computation gives $\bar{c} = -26$. So recalling that the "matter" sector, consisting of the D bosons, has $c = \bar{c} = D$, we have achieved the result that the conformal anomaly vanishes in the case $D = 26$. This also applies to the open string in the obvious way.

3.5.3 Further Aspects of Conformal Ghosts

Notice that the flat space expression (161) is also consistent with the stress tensor

$$T(z) =: \partial_z b(z)c(z) : -\kappa : \partial_z[b(z)c(z)] : , \tag{169}$$

for arbitrary κ, with a similar expression for the antiholomorphic sector. It is a useful exercise to use the OPE's of the ghosts given in equation (167) to verify that this gives b and c conformal weights $h = \kappa$ and $h = 1 - \kappa$, respectively. The case we studied above was $\kappa = 2$. Further computation (recommended) reveals that the conformal anomaly of this system is $c = 1 - 3(2\kappa - 1)^2$, with a similar expression for the antiholomorphic version of the above.

The case of fermionic ghosts will be of interest to us later. In that case, the action and stress tensor are just like before, but with $b \to \beta$ and $c \to \gamma$, where β and γ, are *fermionic*. Since they are fermionic, they have singular OPE's

$$\beta(z)\gamma(y) \quad = \quad -\frac{1}{(z-y)} + \cdots , \qquad \gamma(z)\beta(y) = \frac{1}{(z-y)} + \cdots \tag{170}$$

A computation gives conformal anomaly $3(2\kappa - 1)^2 - 1$, which in the case $\kappa = 3/2$, gives an anomaly of 11. In this case, they are the "superghosts", required by supersymmetry in the construction of superstrings later on.

3.6 Non–Critical Strings

Actually, decoupling of the function φ in section (2.4) is only exact for matter with $c = 26$. This does not mean that we must stay in the critical dimension, however. There is quite an industry in which string theory in very low dimensions are studied, specifically in two dimensions or fewer. How has can this be? Well, the answer is easy to state with the language we have developed here. In general φ does not decouple, but has an action:

$$S_{\rm L} = \frac{1}{4\pi\alpha'} \int d^2z \left\{ \partial\varphi\bar{\partial}\varphi + \frac{\alpha'}{2} Q\varphi R + \mu e^{\gamma\varphi} \right\} , \tag{171}$$

where R is the two dimensional Ricci scalar, and μ is a cosmological term. This is called the "Liouville model". The stress tensor for this model is readily computed as

$$T_{zz} = -\frac{1}{\alpha'}\partial\varphi\partial\varphi + \frac{Q}{2}\partial^2\varphi , \tag{172}$$

and from it, the central charge can be deduced to be $c_{\rm L} = 1 + 3Q^2$. This conclusion is unaffected by the case of $\mu \neq 0$. The case of $\mu \neq 0$ needs to be considered carefully. Think of it as small to begin with, controlling the insertion of the operator $e^{\gamma\varphi}$. The parameter γ is fixed by requiring that $e^{\gamma\varphi}$ is of unit weight, so that it acts as a marginal operator. The presence of Q in the stress tensor shifts the weight of $e^{\gamma\varphi}$ from $-\gamma^2/2$ to $-\gamma(\gamma - Q)/2$, and so the condition yields the relation $Q = 2/\gamma + \gamma$. Alternatively, this regime can be regarded as working in the limit of $\varphi \to -\infty$. This is weak coupling in the spacetime string theory as we see below. The main point to be made here is that one can adjust Q to get a total central charge of $c + c_{\rm L} = 26$, which sets $Q = \sqrt{(25 - c)/3}$. The reparametrisation ghosts then produce their $c = -26$ in order to cancel the conformal anomaly of the whole model.

Looking at the complete theory (where we have some number of bosonic fields with some central charge $c \leq 25$), one sees that φ is rather like an additional coordinate for the string with an interaction which soaks up the missing conformal anomaly to ensure a consistent string model. So the spacetime dimension is given by $D = c + 1$.

The case of 25 bosons from the matter sector, $c = 25$ recovers the usual situation, since in that case $Q = 0$ and so φ is coupled like a normal boson as well. Notice that γ is purely imaginary in that case, and so we can get a real metric from $g_{ab} = e^{\gamma\varphi}\delta_{ab}$ by Wick rotating φ to $i\varphi$. So φ plays the role of the time coordinate and we are back in 26 spacetime dimensions.

In general, the complete theory is often thought of as a conformal field theory of central charge c coupled to two dimensional gravity. The gravity here refers to the physics of the two dimensional metric. This nomenclature is consistent with the fact that the Riemann tensor in d dimensions has $d^2(d^2-1)/12$ independent components, so that in $d=2$ there is one degree of freedom, that represented by φ.

For example, with just one ordinary scalar X, (which has $c=1$), we have two dimensional string theory. The spacetime picture that is kept in mind for this model is that φ plays the role of a spatial coordinate, and time is recovered by Wick rotating X. Notice that the spacetime picture meshes very nicely with the σ–model approach we worked with in equation (93). Comparing our Liouville theory Lagrangian with that of the sigma model we see that the dilaton is set by φ as $\Phi = Q\varphi/(\sqrt{2\alpha'})$. Choose the spacetime metric as flat: $G_{\mu\nu}=\eta_{\mu\nu}$. This "linear dilaton vacuum" is a solution to the β–function equations (96) if $Q=\sqrt{(26-D)/3}=\sqrt{(25-c)/3}$, as we saw above (where we have $D=2$ for the case in hand). Since the string coupling is set by the dilaton *via* $g_s=e^{\Phi}$, we see that our earlier statement that $\varphi\to-\infty$ is weak coupling is indeed correct. Further to this, we see that the case of larger φ is at stronger coupling. Note that μ sets a natural scale where the cosmological term is of order one and the theory is firmly in the strong coupling regime, at $\varphi\sim\log(1/\mu)$. This is often referred to the "Liouville Wall", the demarcation between the strong and weak coupling regimes.

Turning to other cases, the most well–studied conformal field theories are the $c<1$ minimal models, an infinite family indexed by two integers (p,q) with

$$c = 1 - \frac{6(p-q)^2}{pq} \ . \tag{173}$$

These models are distinguished by (among other things) having a finite number of primary fields. They are unitary when $|p-q|=1$. The trivial model is the $(3,2)$, which has $c=0$. Two other famous unitary members of the series are $(4,3)$, which has $c=1/2$ and $(5,4)$, which has $c=7/10$ are the critical Ising model and the tricritical Ising model.

For the $(3,2)$ coupled to Liouville, we see that the model is just one with no extra embedding for the string at all, just the Liouville dimension. It is often referred to as "pure gravity", since the (non–Liouville) conformal field theory is trivial. In general, the study of these "non–critical string theories" is said to be the study of strings in $D\leq 2$.

Remarkably, the path integral for these models can be supplied with a definition using the techniques called "Matrix Models". Basically, the sum over world–sheet metrics is performed by studying the theory of an $N\times N$ matrix valued field, where N is large. The matrix integral can be expanded in terms of Feynmann diagrams, and for large N, the diagrams can be organized in terms of powers of $1/N$. This expansion in $1/N$ is the same as the g_s topological expansion of string perturbation theory, and the Feynmann diagrams act as a sort of regularized representation of the string world–sheets. We don't have time or space to study these models here, but it is a remarkable subject. In fact, it is in these simple string models that a lot of the important modern string ideas have their roots, such as non–perturbative string theory, the behaviour of string theory at high orders in perturbation theory, and non–perturbative relations between open and closed strings[7]. Recently, this area has been revisited, as it has become increasingly clear that there may be more lessons to be learned about the above topics and more, such as holography, tachyon condensation, and open–closed transitions[8].

There has yet to be presented a satisfactory interpretation for the physics for a vast range of dimensions, however. This is because it is only for the case $0\leq c\leq 1$ that Q and γ are both real. Outside this range, the above formulae of non–critical string theory await a physical interpretation.

[7] For a review of the subject, see for example refs.[10]

[8] There is no complete review of this area currently available (although see the last of refs.[10]), so it is recommended that refs.[11] be consulted, as their opening sections give a good guide to the literature, old and new.

3.7 The Closed String Partition Function

We have all of the ingredients we need to compute our first one–loop diagram[9]. It will be useful to do this as a warm up for more complicated examples later, and in fact we will see structures in this simple case which will persist throughout.

Consider the closed string diagram of figure 18*(a)*. This is a vacuum diagram, since there are no external strings. This torus is clearly a one loop diagram and in fact it is easily computed. It is distinguished topologically by having two completely independent one–cycles. To compute the path integral for this we are instructed, as we have seen, to sum over all possible metrics representing all possible surfaces, and hence all possible tori.

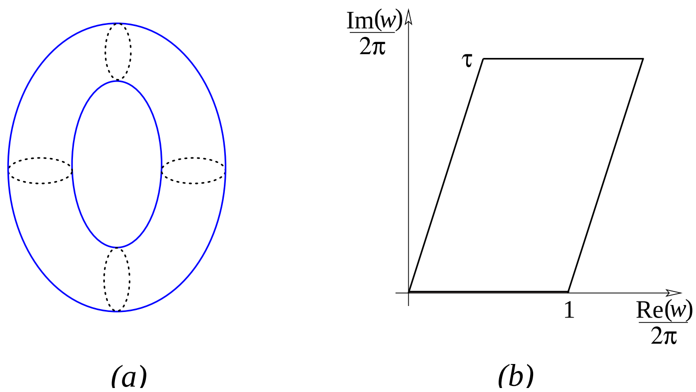


Figure 18: *(a)* A closed string vacuum diagram *(b)*. The flat torus and its complex structure.

Well, the torus is completely specified by giving it a flat metric, and a complex structure, τ, with $\mathrm{Im}\tau \geq 0$. It can be described by the lattice given by quotienting the complex w–plane by the equivalence relations

$$w \sim w + 2\pi n \ ; \quad w \sim w + 2\pi m\tau \ , \tag{174}$$

for any integers m and n, as shown in figure 18*(b)*. The two one–cycles can be chosen to be horizontal and vertical. The complex number τ specifies the *shape* of a torus, which cannot be changed by infinitesimal diffeomorphisms of the metric, and so we must sum over all of them. Actually, this naive reasoning will make us overcount by a lot, since in fact there are a lot of τ's which define the same torus. For example, clearly for a torus with given value of τ, the torus with $\tau+1$ is the same torus, by the equivalence relation (174). The full family of equivalent tori can be reached from any τ by the "modular transformations":

$$\begin{aligned} T &: \quad \tau \to \tau + 1 \\ S &: \quad \tau \to -\frac{1}{\tau} \ , \end{aligned} \tag{175}$$

which generate the group $SL(2,\mathbb{Z})$, which is represented here as the group of 2×2 unit determinant matrices with integer elements:

$$SL(2,\mathbb{Z}): \quad \tau \to \frac{a\tau + b}{c\tau + d} \ ; \quad \text{with} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \ , \quad ad - bc = 1 \ . \tag{176}$$

(It is worth noting that the map between tori defined by S exchanges the two one–cycles, therefore exchanging space and (Euclidean) time.) The full family of inequivalent tori is given not by the upper half plane $H_\perp$ (*i.e.,* τ such that $\mathrm{Im}\tau \geq 0$) but the quotient of it by the equivalence relation generated by the group of modular transformations. This is $\mathcal{F} = H_\perp / PSL(2,\mathbb{Z})$, where the P reminds us that we divide by the extra

[9] Actually, we've had them for some time now, essentially since section 2.

$\mathbb{Z}_2$ which swaps the sign on the defining $SL(2,\mathbb{Z})$ matrix, which clearly does not give a new torus. The commonly used fundamental domain in the upper half plane corresponding to the inequivalent tori is drawn in figure 19. Any point outside that can be mapped into it by a modular transformation. The fundamental region $\mathcal{F}$ is properly defined as follows: Start with the region of the upper half plane which is in the interval $(-\frac{1}{2},+\frac{1}{2})$ and above the circle of unit radius. we must then identify the two vertical edges, and also the two halves of the remaining segment of the circle. This produces a space which is smooth everywhere except for two points, $\tau=i$ and $\tau=e^{\frac{2\pi i}{3}}$ point about which there are conical singularities. They are fixed points of $SL(2,\mathbb{Z})$ fixed by the elements S, and ST respectively.

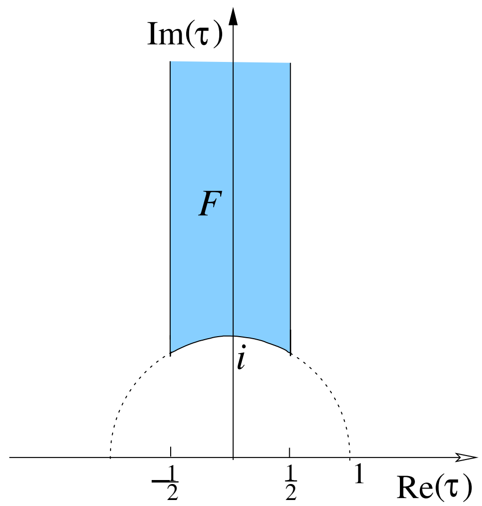


Figure 19: The space of inequivalent tori.

The string propagation on our torus can be described as follows. Imagine that the string is of length 1, and lies horizontally. Mark a point on the string. Running time upwards, we see that the string propagates for a time $t=2\pi\text{Im}\tau\equiv 2\pi\tau_2$. One it has got to the top of the diagram, we see that our marked point has shifted rightwards by an amount $x=2\pi\text{Re}\tau\equiv 2\pi\tau_1$. We actually already have studied the operators which perform these two operations. The operator for time translations is the Hamiltonian (54), $H=L_0+\bar{L}_0-(\text{c}+\bar{\text{c}})/24$ while the operator for translations along the string is the momentum $P=L_0-\bar{L}_0$ discussed above equation (63).

Recall that $\text{c}=\bar{\text{c}}=D-2=24$. So our vacuum path integral is

$$Z=\text{Tr}\left\{e^{-2\pi\tau_2 H}e^{2\pi \text{i}\tau_1 P}\right\}=\text{Tr}q^{L_0-\frac{c}{24}}\bar{q}^{\bar{L}_0-\frac{\bar{c}}{24}}\ . \tag{177}$$

Here, $q\equiv e^{2\pi \text{i}\tau}$, and the trace means a sum over everything which is discrete and an integral over everything which is continuous, which in this case, is simply τ.

This is easily evaluated, as the expressions for L_0 and $\bar{L}_0$ give a family of simple geometric sums. To see where the sums come from, let us look at one dimension, and so one family of oscillators α_n. We need to consider

$$\text{Tr}\,q^{L_0}=\text{Tr}\,q^{\sum_{n=0}^{\infty}\alpha_{-n}\alpha_n}\ .$$

We can see what the operator $q^{\sum_{n=0}^{\infty}\alpha_{-n}\alpha_n}$ means if we write it explicitly in a basis of all possible multiparticle states of the form $\alpha_{-n}|0\!>$, $(\alpha_{-n})^2|0\!>$, *etc.* :

$$q^{\alpha_{-n}\alpha_n} = \begin{pmatrix} 1 & & & & \\ & q^n & & & \\ & & q^{2n} & & \\ & & & q^{3n} & \\ & & & & \ddots \end{pmatrix} ,$$

and so clearly $\mathrm{Tr} q^{\alpha_{-n}\alpha_n} = \sum_{i=1}^{\infty}(q^n)^i = (1-q^n)^{-1}$, which is remarkably simple. The final sum over all modes is trivial, since

$$\mathrm{Tr}\, q^{\sum_{n=0}^{\infty}\alpha_{-n}\alpha_n} = \prod_{n=0}^{\infty} \mathrm{Tr}\, q^{\alpha_{-n}\alpha_n} = \prod_{n=0}^{\infty}(1-q^n)^{-1} \ .$$

We get a factor like this for all 24 dimensions, and we also get contributions from both the left and right to give the result.

Notice that if our modes were fermions, ψ_n, things would be even simpler. We would not be able to make multiparticle states $(\psi_{-n})^2|0\!>$, (Pauli), and so we only have a 2×2 matrix of states to trace in this case, and so we simply get

$$\mathrm{Tr}\, q^{\psi_{-n}\psi_n} = (1+q^n) \ .$$

Therefore the partition function is

$$\mathrm{Tr}\, q^{\sum_{n=0}^{\infty}\psi_{-n}\psi_n} = \prod_{n=0}^{\infty} \mathrm{Tr} q^{\psi_{-n}\psi_n} = \prod_{n=0}^{\infty}(1+q^n) \ .$$

Returning to the full problem, the result can be written as:

$$Z = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} Z(q) \ , \quad \text{where} \tag{178}$$

$$Z(q) = |\tau_2|^{-12}(q\bar{q})^{-1} \left| \prod_{n=1}^{\infty}(1-q^n)^{-24} \right|^2 = (\sqrt{\tau_2}\eta\bar{\eta})^{-24} \ , \tag{179}$$

is the "partition function", with Dedekind's function

$$\eta(q) \equiv q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n) \ ; \quad \eta\left(-\frac{1}{\tau}\right) = \sqrt{-\mathrm{i}\tau}\,\eta(\tau) \ . \tag{180}$$

This is a pleasingly simple result. One very interesting property it has is that it is actually "modular invariant". It is invariant under the T transformation in (174), since under $\tau \to \tau+1$, we get that $Z(q)$ picks up a factor $\exp(2\pi\mathrm{i}(L_0 - \bar{L}_0))$. This factor is precisely unity, as follows from the level matching formula (63).

Invariance of $Z(q)$ under the S transformation $\tau \to -1/\tau$ follows from the property mentioned in (180), after a few steps of algebra, and using the result $S:\ \tau_2 \to \tau_2/|\tau|^2$. Modular invariance of the partition function is a crucial property. It means that we are correctly integrating over all inequivalent tori, which is required of us by diffeomorphism invariance of the original construction. Furthermore, we are counting each torus only once, which is of course important.

Note that $Z(q)$ really deserves the name "partition function" since if it is expanded in powers of q and $\bar{q}$, the powers in the expansion —after multiplication by $4/\alpha'$— refer to the (mass)^2 level of excitations on the left and right, while the coefficient in the expansion gives the degeneracy at that level. The degeneracy is the number of partitions of the level number into positive integers. For example, at level 3 this is 3, since we have $\alpha_{-3}, \alpha_{-1}\alpha_{-2}$, and $\alpha_{-1}\alpha_{-1}\alpha_{-1}$.

The overall factor of $(q\bar{q})^{-1}$ sets the bottom of the tower of masses. Note for example that at level zero we have the tachyon, which appears only once, as it should, with $M^2 = -4/\alpha'$. At level one, we have the massless states, with multiplicity 24^2, which is appropriate, since there are 24^2 physical states in the graviton multiplet $(G_{\mu\nu}, B_{\mu\nu}, \Phi)$. Introducing a common piece of terminology, a term $q^{w_1}\bar{q}^{w_2}$, represents the appearance of a "weight" (w_1, w_2) field in the 1+1 dimensional conformal field theory, denoting its left–moving and right–moving weights or "conformal dimensions".

4 Strings on Circles and T–Duality

In this section we shall study the spectrum of strings propagating in a spacetime which has a compact direction. The theory has all of the properties we might expect from the knowledge that at low energy we are placing gravity and field theory on a compact space. Indeed, as the compact direction becomes small, the parts of the spectrum resulting from momentum in that direction become heavy, and hence less important, but there is much more. The spectrum has additional sectors coming from the fact that closed strings can wind around the compact direction, contributing states whose mass is proportional to the radius. Thus, they become light as the circle shrinks. This will lead us to T–duality, relating a string propagating on a large circle to a string propagating on a small circle.[18] This is just the first of the remarkable symmetries relating two string theories in different situations that we shall encounter here. It is a crucial consequence of the fact that strings are extended objects. Studying its consequences for open strings will lead us to D–branes, since T–duality will relate the Neumann boundary conditions we have already encountered to Dirichlet ones[13, 15], corresponding to open strings ending on special hypersurfaces in spacetime.

4.1 Closed Strings on a Circle

The mode expansion (74) for the closed string theory can be written as:

$$X^\mu(z,\bar{z}) = \frac{x^\mu}{2} + \frac{\tilde{x}^\mu}{2} - i\sqrt{\frac{\alpha'}{2}}(\alpha_0^\mu + \tilde{\alpha}_0^\mu)\tau + \sqrt{\frac{\alpha'}{2}}(\alpha_0^\mu - \tilde{\alpha}_0^\mu)\sigma + \text{oscillators} \ . \tag{181}$$

We have already identified the spacetime momentum of the string:

$$p^\mu = \frac{1}{\sqrt{2\alpha'}}(\alpha_0^\mu + \tilde{\alpha}_0^\mu) \ . \tag{182}$$

If we run around the string, *i.e.,* take $\sigma \to \sigma + 2\pi$, the oscillator terms are periodic and we have

$$X^\mu(z,\bar{z}) \to X^\mu(z,\bar{z}) + 2\pi\sqrt{\frac{\alpha'}{2}}(\alpha_0^\mu - \tilde{\alpha}_0^\mu) \ . \tag{183}$$

So far, we have studied the situation of non–compact spatial directions for which the embedding function $X^\mu(z,\bar{z})$ is single–valued, and therefore the above change must be zero, giving

$$\alpha_0^\mu = \tilde{\alpha}_0^\mu = \sqrt{\frac{\alpha'}{2}}p^\mu. \tag{184}$$

Indeed, momentum p^μ takes a continuum of values reflecting the fact that the direction X^μ is non–compact.

Let us consider the case that we have a compact direction, say X^{25}, of radius R. Our direction X^{25} therefore has period $2\pi R$. The momentum p^{25} now takes the discrete values n/R, for $n \in \mathbb{Z}$. Now, under $\sigma \sim \sigma + 2\pi$, $X^{25}(z,\bar{z})$ is not single valued, and can change by $2\pi wR$, for $w \in \mathbb{Z}$. Solving the two resulting equations gives:

$$\begin{aligned} \alpha_0^{25} + \tilde{\alpha}_0^{25} &= \frac{2n}{R}\sqrt{\frac{\alpha'}{2}} \\ \alpha_0^{25} - \tilde{\alpha}_0^{25} &= \sqrt{\frac{2}{\alpha'}}wR \end{aligned} \tag{185}$$

and so we have:

$$
\begin{aligned}
\alpha_0^{25} &= \left(\frac{n}{R}+\frac{wR}{\alpha'}\right)\sqrt{\frac{\alpha'}{2}} \equiv P_L\sqrt{\frac{\alpha'}{2}} \\
\tilde{\alpha}_0^{25} &= \left(\frac{n}{R}-\frac{wR}{\alpha'}\right)\sqrt{\frac{\alpha'}{2}} \equiv P_R\sqrt{\frac{\alpha'}{2}} \ .
\end{aligned} \tag{186}
$$

We can use this to compute the formula for the mass spectrum in the remaining uncompactified 24+1 dimensions, using the fact that $M^2 = -p_\mu p^\mu$, where now $\mu = 0, \ldots, 24$.

$$
\begin{aligned}
M^2 \ = \ -p^\mu p_\mu &= \frac{2}{\alpha'}(\alpha_0^{25})^2 + \frac{4}{\alpha'}(N-1) \\
&= \frac{2}{\alpha'}(\tilde{\alpha}_0^{25})^2 + \frac{4}{\alpha'}(\bar{N}-1) \ ,
\end{aligned} \tag{187}
$$

where $N, \bar{N}$ denote the total levels on the left– and right–moving sides, as before. These equations follow from the left and right $L_0, \bar{L}_0$ constraints. Recall that the sum and difference of these give the Hamiltonian and the level–matching formulae. Here, they are modified, and a quick computation gives:

$$
\begin{aligned}
&M^2 = \frac{n^2}{R^2} + \frac{w^2R^2}{\alpha'^2} + \frac{2}{\alpha'}\left(N+\tilde{N}-2\right) \\
&nw + N - \tilde{N} = 0 \ .
\end{aligned} \tag{188}
$$

The key features here are that there are terms in addition to the usual oscillator contributions. In the mass formula, there is a term giving the familiar contribution of the Kaluza–Klein tower of momentum states for the string (see the previous subsection), and a new term from the tower of winding states.

This latter term is a very stringy phenomenon. Notice that the level matching term now also allows a mismatch between the number of left and right oscillators excited, in the presence of discrete winding and momenta.

In fact, notice that we can get our usual massless Kaluza–Klein states[10] by taking

$$
n = w = 0 \ ; \qquad N = \bar{N} = 1 \ , \tag{189}
$$

exciting an oscillator in the compact direction. There are two ways of doing this, either on the left or the right, and so there are two $U(1)$'s following from the fact that there is an internal component of the metric and also of the antisymmetric tensor field. We can choose to identify the two gauge fields of this $U(1)\times U(1)$ as follows:

$$
A_{\mu(R)} \equiv \frac{1}{2}(G-B)_{\mu,25} \ ; \quad A_{\mu(L)} \equiv \frac{1}{2}(G+B)_{\mu,25} \ .
$$

We have written these states out explicitly, together with the corresponding spacetime fields, and the vertex operators (at zero momentum), below:

field	state	operator
$G_{\mu\nu}$	$(\alpha_{-1}^{\mu}\tilde{\alpha}_{-1}^{\nu}+\alpha_{-1}^{\nu}\tilde{\alpha}_{-1}^{\mu})\vert 0;k>$	$\partial X^\mu \partial X^\nu + \partial X^\mu \partial X^\nu$
$B_{\mu\nu}$	$(\alpha_{-1}^{\mu}\tilde{\alpha}_{-1}^{\nu}-\alpha_{-1}^{\nu}\tilde{\alpha}_{-1}^{\mu})\vert 0;k>$	$\partial X^\mu \bar{\partial} X^\nu - \partial X^\mu \bar{\partial} X^\nu$
$A_{\mu(R)}$	$\alpha_{-1}^{\mu}\tilde{\alpha}_{-1}^{25}\vert 0;k>$	$\partial X^\mu \bar{\partial} X^{25}$
$A_{\mu(L)}$	$\tilde{\alpha}_{-1}^{\mu}\alpha_{-1}^{25}\vert 0;k>$	$\partial X^{25} \bar{\partial} X^\mu$
$\phi \equiv \frac{1}{2}\log G_{25,25}$	$\alpha_{-1}^{25}\tilde{\alpha}_{-1}^{25}\vert 0;k>$	$\partial X^{25} \bar{\partial} X^{25}$

[10] We shall sometimes refer to Kaluza–Klein states as "momentum" states, to distinguish them from "winding" states, in what follows.

So we have these 25–dimensional massless states which are basically the components of the graviton and antisymmetric tensor fields in 26 dimensions, now relabeled. (There is also of course the dilaton Φ, which we have not listed.) There is a pair of gauge fields giving a $U(1)_L \times U(1)_R$ gauge symmetry, and in addition a massless scalar field ϕ. Actually, ϕ is a massless scalar which can have any background vacuum expectation value (vev), which in fact sets the radius of the circle. This is because the square root of the metric component $G_{25,25}$ is indeed the measure of the radius of the X^{25} direction.

4.2 T–Duality for Closed Strings

Let us now study the generic behaviour of the spectrum (188) for different values of R. For larger and larger R, momentum states become lighter, and therefore it is less costly to excite them in the spectrum. At the same time, winding states become heavier, and are more costly. For smaller and smaller R, the reverse is true, and it is gets cheaper to excite winding states while it is momentum states which become more costly.

We can take this further: As $R \to \infty$, all of the winding states *i.e.,* states with $w \neq 0$, become infinitely massive, while the $w = 0$ states with all values of n go over to a continuum. This fits with what we expect intuitively, and we recover the fully uncompactified result.

Consider instead the case $R \to 0$, where all of the momentum states *i.e.,* states with $n \neq 0$, become infinitely massive. If we were studying field theory we would stop here, as this would be all that would happen—the surviving fields would simply be independent of the compact coordinate, and so we have performed a dimension reduction. In closed string theory things are quite different: the pure winding states (*i.e.,* $n = 0$, $w \neq 0$, states) form a continuum as $R \to 0$, following from our observation that it is very cheap to wind around the small circle. *Therefore, in the $R \to 0$ limit, an effective uncompactified dimension actually reappears!*

Notice that the formula (188) for the spectrum is invariant under the exchange

$$n \leftrightarrow w \qquad \text{and} \qquad R \leftrightarrow R' \equiv \alpha'/R \ . \tag{190}$$

The string theory compactified on a circle of radius R' (with momenta and windings exchanged) is the "T–dual" theory[18], and the process of going from one theory to the other will be referred to as "T–dualising".

The exchange takes (see (186))

$$\alpha_0^{25} \to \alpha_0^{25}, \quad \tilde{\alpha}_0^{25} \to -\tilde{\alpha}_0^{25} \ . \tag{191}$$

The dual theories are identical in the fully interacting case as well (after a shift of the coupling to be discussed shortly):[19] Simply rewrite the radius R theory by performing the exchange

$$X^{25} = X^{25}(z) + X^{25}(\bar{z}) \longrightarrow X'^{25}(z,\bar{z}) = X^{25}(z) - X^{25}(\bar{z}) \ . \tag{192}$$

The energy–momentum tensor and other basic properties of the conformal field theory are invariant under this rewriting, and so are therefore all of the correlation functions representing scattering amplitudes, *etc.* The only change, as follows from equation (191), is that the zero mode spectrum in the new variable is that of the α'/R theory.

So these theories are physically identical. T–duality, relating the R and α'/R theories, is an exact symmetry of perturbative closed string theory. Shortly, we shall see that it is non–perturbatively exact as well.

It is important to note that the transformation (192) can be regarded as a spacetime parity transformation acting only on the right–moving (in the world sheet sense) degrees of freedom. We shall put this picture to good use in what is to come.

4.3 A Special Radius: Enhanced Gauge Symmetry

Given the relation we deduced between the spectra of strings on radii R and α'/R, it is clear that there ought to be something interesting about the theory at the radius $R = \sqrt{\alpha'}$. The theory should be self–dual, and this radius is the "self–dual radius". There is something else special about this theory besides just self–duality.

At this radius we have, using (186),

$$\alpha_0^{25} = \frac{(n+w)}{\sqrt{2}} \ ; \qquad \tilde{\alpha}_0^{25} = \frac{(n-w)}{\sqrt{2}} \ , \tag{193}$$

and so from the left and right we have:

$$\begin{aligned} M^2 \ = \ -p^\mu p_\mu \ &= \ \frac{1}{\alpha'}(n+w)^2 + \frac{4}{\alpha'}(N-1) \\ &= \ \frac{2}{\alpha'}(n-w)^2 + \frac{4}{\alpha'}(\bar{N}-1) \ . \end{aligned} \tag{194}$$

So if we look at the massless spectrum, we have the conditions:

$$(n+w)^2 + 4N = 4 \ ; \qquad (n-w)^2 + 4\bar{N} = 4 \ . \tag{195}$$

As solutions, we have the cases $n = w = 0$ with $N = 1$ and $\bar{N} = 1$ from before. These are include the vectors of the $U(1)\times U(1)$ gauge symmetry of the compactified theory.

Now however, we see that we have more solutions. In particular:

$$n = -w = \pm 1 \ , \quad N = 1 \ , \ \bar{N} = 0 \ ; \qquad n = w = \pm 1 \ , \quad N = 0 \ , \ \bar{N} = 1 \ . \tag{196}$$

The cases where the excited oscillators are in the non–compact direction yield two pairs of massless vector fields. In fact, the first pair go with the left $U(1)$ to make an $SU(2)$, while the second pair go with the right $U(1)$ to make another $SU(2)$. Indeed, they have the correct ± 1 charges under the Kaluza–Klein $U(1)$'s in order to be the components of the W–bosons for the $SU(2)_L\times SU(2)_R$ "enhanced gauge symmetries". The term is appropriate since there is an extra gauge symmetry at this special radius, given that new massless vectors appear there. When the oscillators are in the compact direction, we get two pairs of massless bosons. These go with the massless scalar ϕ to fill out the massless adjoint Higgs field for each $SU(2)$. These are the scalars whose vevs give the W–bosons their masses when we are away from the special radius.

The vertex operator for the change of radius, $\partial X^{25}\bar{\partial} X^{25}$, corresponding to the field ϕ, transforms as a $(\mathbf{3},\mathbf{3})$ under $SU(2)_L\times SU(2)_R$, and therefore a rotation by π in one of the $SU(2)$'s transforms it into minus itself. The transformation $R \to \alpha'/R$ is therefore the $\mathbb{Z}_2$ Weyl subgroup of the $SU(2)\times SU(2)$. Since T–duality is part of the spacetime gauge theory, this is a clue that it is an exact symmetry of the closed string theory, if we assume that non–perturbative effects preserve the spacetime gauge symmetry. We shall see that this assumption seems to fit with non–perturbative discoveries to be described later.

4.4 The Circle Partition Function

It is useful to consider the partition function of the theory on the circle. This is a computation as simple as the one we did for the uncompactified theory earlier, since we have done the hard work in working out L_0 and $\bar{L}_0$ for the circle compactification. Each non–compact direction will contribute a factor of $(\eta\bar{\eta})^{-1}$, as before, and the non–trivial part of the final τ–integrand, coming from the compact X^{25} direction is:

$$Z(q,R) = (\eta\bar{\eta})^{-1} \sum_{n,w} q^{\frac{\alpha'}{4}P_L^2} \bar{q}^{\frac{\alpha'}{4}P_R^2} \ , \tag{197}$$

where $P_{L,R}$ are given in (186). Our partition function is manifestly T–dual, and is in fact also modular invariant:

Under T, it picks us a phase $\exp(\pi i(P_L^2 - P_R^2))$, which is again unity, as follows from the second line in (188): $P_L^2 - P_R^2 = 2nw$. Under S, the role of the time and space translations as we move on the torus are exchanged, and this in fact exchanges the sums over momentum and winding. T–duality ensures that the

S–transformation properties of the exponential parts involving $P_{L,R}$ are correct, while the rest is S invariant as we have already discussed.

It is a useful exercise to expand this partition function out, after combining it with the factors from the other non–compact dimensions first, to see that at each level the mass (and level matching) formulae (188) which we derived explicitly is recovered.

In fact, the modular invariance of this circle partition function is part of a very important larger story. The left and right momenta $P_{L,R}$ are components of a special two dimensional lattice, $\Gamma_{1,1}$. There are two basis vectors $k = (1/R, 1/R)$ and $\hat{k} = (R, -R)$. We make the lattice with arbitrary integer combinations of these, $nk + w\hat{k}$, whose components are (P_L, P_R). (*c.f.* (186)) If we define the dot products between our basis vectors to be $k \cdot \hat{k} = 2$ and $k \cdot k = 0 = \hat{k} \cdot \hat{k}$, our lattice then has a Lorentzian signature, and since $P_L^2 - P_R^2 = 2nw \in 2\mathbb{Z}$, it is called "even". The "dual" lattice $\Gamma_{1,1}^*$ is the set of all vectors whose dot product with (P_L, P_R) gives an integer. In fact, our lattice is self–dual, which is to say that $\Gamma_{1,1} = \Gamma_{1,1}^*$. It is the "even" quality which guarantees invariance under T as we have seen, while it is the "self–dual" feature which ensures invariance under S. In fact, S is just a change of basis in the lattice, and the self–duality feature translates into the fact that the Jacobian for this is unity.

In fact, the special properties of the string theory at the self–dual radius is succinctly visible at all mass levels, by looking at the partition function (197). At the self–dual radius, it can be rewritten as a sum of squares of "characters" of the $SU(2)$ affine Lie algrebra:

$$Z(q, R = \sqrt{\alpha'}) = |\chi_1(q)|^2 + |\chi_2(q)|^2 \ , \tag{198}$$

where

$$\chi_1(q) \equiv \eta^{-1} \sum_n q^{n^2} \ , \quad \chi_2(q) \equiv \eta^{-1} \sum_n q^{(n+1/2)^2} \tag{199}$$

It is amusing to expand these out (after putting in the other factors of $(\eta\bar{\eta})^{-1}$ from the uncompactified directions) and find the massless states we discussed explicitly above.

It does not matter if an affine Lie algebra has not been encountered before by the reader. We can take this as an illustrative example, arising in a natural and instructive way. See the next section for further discussion.[16] In the language of two dimensional conformal field theory, there are additional left– and right–moving currents (*i.e.*, fields with weights (1,0) and (0,1)) present. We can construct them as vertex operators by exponentiating some of the existing fields. The full set of vertex operators of the $SU(2)_L \times SU(2)_R$ spacetime gauge symmetry:

$$\begin{aligned} SU(2)_L: &\quad \bar{\partial} X^\mu \partial X^{25}(z), \quad \bar{\partial} X^\mu \exp(\pm 2i X^{25}(z)/\sqrt{\alpha'}) \\ SU(2)_R: &\quad \partial X^\mu \bar{\partial} X^{25}(z), \quad \partial X^\mu \exp(\pm 2i X^{25}(\bar{z})/\sqrt{\alpha'}) \ , \end{aligned} \tag{200}$$

corresponding to the massless vectors we constructed by hand above.

4.4.1 Affine Lie Algebras

The key structure of an affine Lie algebra is just what we have seen arise naturally in this self–duality example. In addition to all of the nice structures that the conformal field theory has —most pertinently, the Virasoro algebra—, there is a family of unit weight operators, often constructed as vertex operators as we saw in equation (200), which form the Lie algebra of some group G. They are unit weight as measured either from the left or the right, and so we can have such structures on either side. Let us focus on the left. Then, as $(1,0)$ operators, $J^a(z)$, (a is a label) we have:

$$[L_n, J^a_m] = m J^a_{n+m} \ , \tag{201}$$

where

$$J^a_n = \frac{1}{2\pi i} \oint dz \ z^n J^a(z) \ , \tag{202}$$

and

$$[J^a_n, J^b_m] = if^{ab}{}_c J^c_{n+m} + mkd^{ab}\delta_{n+m} \ , \tag{203}$$

where it should be noticed that the zero modes of these currents form a Lie algebra, with structure constants $f^{ab}{}_c$. The constants d^{ab} define the inner product between the generators $(t^a, t^b) = d^{ab}$. Since in bosonic string theory a mode with index -1 creates a state which is massless in spacetime, J^a_{-1} can be placed either on the left with $\tilde{\alpha}^\mu_{-1}$ on the right (or *vice–versa*) to give a state $J^a_{-1}\tilde{\alpha}^\mu_{-1}|0>$ which is a massless vector $A^{\mu a}$ in the adjoint of G, for which the low energy physics must be Yang–Mills theory.

The full algebra is called an "affine Lie algebra", or a "current algebra", and sometimes a "Kac–Moody" algebra. In a standard normalization, k is an integer and is called the "level" of the affinisation. In the case that we first see this sort of structure, the string at a self–dual radius, the level is 1. The currents in this case are:

$$\begin{aligned} J^3(z) &= i\alpha'^{-1/2}\partial_z X^{25}(z) \ , \\ J^1(z) &= :\cos(2\alpha'^{-1/2}X^{25}(z)): \ , \quad J^2(z) =: \sin(2\alpha'^{-1/2}X^{25}(z)): \end{aligned}$$

which satisfy the algebra (203) with $f^{abc} = \epsilon^{abc}$, $k = 1$, and $d^{ab} = \frac{1}{2}\delta^{ab}$, as appropriate to the fundamental representation.

4.5 Toriodal Compactifications

It will be very useful later on for us to outline how things work more generally. The case of compactification on the circle encountered above can be easily generalized to compactification on the torus $T^d \simeq (S^1)^d$. Let us denote the compact dimensions by X^m, where $m, n = 1, \ldots, d$. Their periodicity is specified by

$$X^m \sim X^m + 2\pi R^{(m)}\mathrm{n}^m \ ,$$

where the n^m are integers and $R^{(m)}$ is the radius of the mth circle. The metric on the torus, G_{mn}, can be diagonalised into standard unit Euclidean form by the vielbeins e^a_m where $a, b = 1, \ldots, d$:

$$G_{mn} = \delta_{ab}e^a_m e^b_n \ ,$$

and it is convenient to use tangent space coordinates $X^a = X^m e^a_m$ so that the equivalence can be written:

$$X^a \sim X^a + 2\pi e^a_m \mathrm{n}^m \ .$$

We have defined for ourselves a lattice $\Lambda = \{e^a_m\mathrm{n}^m, \mathrm{n}^m \in \mathbb{Z}\}$. We now write our torus in terms of this as

$$T^d \equiv \frac{\mathbb{R}^d}{2\pi\Lambda} \ .$$

There are of course conjugate momenta to the X^a, which we denote as p^a. They are quantized, since moving from one lattice point to another, producing a change in the vector X by $\delta X \in 2\pi\Lambda$ are physically equivalent, and so single–valuedness of the wavefunction imposes $\exp(ip\cdot X) = \exp(ip\cdot[X+\delta X])$, *i.e.*:

$$p\cdot\delta X \in 2\pi\mathbb{Z} \ ,$$

from which we see that clearly

$$p^n = G^{mn}\mathrm{n}_m \ ,$$

where n_m are integers. In other words, the momenta live in the dual lattice, Λ^*, of Λ, defined by

$$\Lambda^* \equiv \{e^{*am}\mathrm{n}_m, \mathrm{n}_m \in \mathbb{Z}\} \ ,$$

where the inverse vielbiens $e^{*am}\mathrm{n}_m$ are defined in the usual way using the inverse metric:

$$e^{*am} \equiv e^a_m G^{mn} \ , \quad \text{or} \quad e^{*am}e^b{}_m = \delta^{ab} \ .$$

Of course we can have winding sectors as well, since as we go around the string *via* $\sigma \to \sigma + 2\pi$, we can change to a new point on the lattice characterized by a set of integers w^m, the winding number. Let us write out the string mode expansions. We have

$$X^a(\tau,\sigma) = X_L^a(\tau-\sigma) + X_R^a(\tau+\sigma) \ , \qquad \text{where}$$

$$\begin{aligned} X_L^a &= x_L^a - i\sqrt{\frac{\alpha'}{2}}p_L^a(\tau-\sigma) + \text{oscillators} \qquad x_L^a = \frac{x^a}{2} - \theta^a \\ p_L^a &= p^a + \frac{w^a R^{(a)}}{\alpha'} \equiv e^{*am}\text{n}_m + \frac{1}{\alpha'}e_m^a w^m \ , \end{aligned} \tag{204}$$

for the left, while on the right we have

$$\begin{aligned} X_R^a &= x_R^a - i\sqrt{\frac{\alpha'}{2}}p_R^a(\tau+\sigma) + \text{oscillators} \qquad x_R^a = \frac{x^a}{2} + \theta^a \\ p_R^a &= p^a - \frac{w^a R^{(a)}}{\alpha'} \equiv e^{*am}\text{n}_m - \frac{1}{\alpha'}e_m^a w^m \ . \end{aligned} \tag{205}$$

The action of the manifest T–duality symmetry is simply to act with a right–handed parity, as before, swapping $p_L \leftrightarrow p_L$ and $p_R \leftrightarrow -p_R$, and hence momenta and winding and $X_L \leftrightarrow X_L$ and $X_R \leftrightarrow -X_R$.

To see more, let us enlarge our bases for the two separate lattices Λ, Λ^* into a singe one, *via*:

$$\hat{e}_m = \frac{1}{\alpha'}\begin{pmatrix} e_m^a \\ -e_m^a \end{pmatrix} \ , \quad \hat{e}^{*m} = \begin{pmatrix} e^{*am} \\ e^{*am} \end{pmatrix} \ ,$$

and now we can write

$$\hat{p} = \begin{pmatrix} p_L^a \\ p_R^a \end{pmatrix} = \hat{e}_m w^m + \hat{e}^{*m}\text{n}_m \ ,$$

which lives in a $(d+d)$–dimensional lattice which we will call $\Gamma_{d,d}$. We can choose the metric on this space to be of Lorentzian signature (d,d), which is achieved by

$$G = \begin{pmatrix} \delta_{ab} & 0 \\ 0 & -\delta_{ab} \end{pmatrix} \ ,$$

and using this we see that

$$\begin{aligned} \hat{e}_m \cdot \hat{e}_n &= 0 = \hat{e}^{*m} \cdot \hat{e}^{*n} \\ \hat{e}_m \cdot \hat{e}^{*n} &= \frac{2}{\alpha'}\delta_n^m \ , \end{aligned} \tag{206}$$

which shows that the lattice is *self–dual*, since (up to a trivial overall scaling), the structure of the basis vectors of the dual is identical to that of the original: $\Gamma_{d,d}^* = \Gamma_{d,d}$. Furthermore, we see that the inner product between any two momenta is given by

$$(\hat{e}_m w^m + \hat{e}^{*m}\text{n}_m) \cdot (\hat{e}_n w'^n + \hat{e}^{*n}\text{n}\prime_n) = \frac{2}{\alpha'}(w^m \text{n}'_m + \text{n}_m w'^m) \ . \tag{207}$$

In other words, the lattice is *even*, because the inner product gives even integer multiples of $2/\alpha'$.

It is these properties that guarantee that the string theory is modular invariant[163]. The partition function for this compactification is the obvious generalisation of the expression given in (197):

$$Z_{T^d} = (\eta\bar{\eta})^{-d}\sum_{\Gamma_{d,d}} q^{\frac{\alpha'}{4}p_L^2}\bar{q}^{\frac{\alpha'}{4}p_R^2} \ , \tag{208}$$

where the $p_{L,R}$ are given in (205). Recall that the modular group is generated by $T:\tau\to\tau+1$, and $S:\tau\to -1/\tau$. So T–invariance follows from the fact that its action produces a factor $\exp(i\pi\alpha'(p_L^2-p_R^2)/2)=\exp(i\pi\alpha'(\hat{p}^2)/2)$ which is unity because the lattice is even, as shown in equation (207).

Invariance under S follows by rewriting the partition function $Z(-1/\tau)$ using the Poisson resummation formula[9] to get the result that

$$Z_\Gamma\left(-\frac{1}{\tau}\right)=\mathrm{vol}(\Gamma^*)Z_{\Gamma^*}(\tau)\ .$$

The volume of the lattice's unit cell is unity, for a self–dual lattice, since $\mathrm{vol}(\Lambda)\mathrm{vol}(\Lambda^*)=1$ for any lattice and its dual, and therefore S–invariance is demonstrated, and we can define a consistent string compactification.

We shall meet two very important examples of large even and self–dual lattices later in subsection 7.4. They are associated to the construction of the modular invariant partition functions of the ten dimensional $E_8\times E_8$ and $SO(32)$ heterotic strings.[23]

4.5.1 The Moduli Space of Compactifications

There is a large space of inequivalent lattices of the type under discussion, given by the shape of the torus (specified by background parameters in the metric G) and the fluxes of the B–field through it. We can work out this "moduli space" of compactifications. It would naively seem to be simply $O(d,d)$, since this is the space of rotations naturally acting, taking such lattices into each other, *i.e.*, starting with some reference lattice Γ_0, $\Gamma'=G\Gamma_0$ should be a different lattice. We must remember that the physics cares only about the values of p_L^2 and p_R^2, and so therefore we must count as equivalent any choices related by the $O(d)\times O(d)$ which acts independently on the left and right momenta: $G\sim G'G$, for $G'\in O(d)\times O(d)$. So at least *locally*, the space of lattices is isomorphic to

$$\mathcal{M}=\frac{O(d,d)}{O(d)\times O(d)}\ . \qquad (209)$$

A quick count of the dimension of this space gives $2d(2d-1)/2-2\times d(d-1)/2=d^2$, which fits nicely, since this is the number of independent components contained in the metric G_{mn}, $(d(d+1)/2)$ and the antisymmetric tensor field B_{mn}, $(d(d-1)/2)$, for which we can switch on constant values (sourced by winding).

There are still a large number of discrete equivalences between the lattices, which follows from the fact that there is a discrete subgroup of $O(d,d)$, called $O(d,d,\mathbb{Z})$, which maps our reference lattice Γ_0 into itself: $\Gamma_0\sim G''\Gamma_0$. This is the set of discrete linear transformations generated by the subgroups of $SL(2d,\mathbb{Z})$ which preserves the inner product given in equations (206).

This group includes the T–dualities on all of the d circles, linear redefinitions of the axes, and discrete shifts of the B–field. The full space of torus compactifications is often denoted:

$$\mathcal{M}=O(d,d,\mathbb{Z})\backslash O(d,d)/[O(d)\times O(d)]\ , \qquad (210)$$

where we divide by one action under left multiplication, and the other under right.

Now we see that there is a possibility of much more than just the $SU(2)_L\times SU(2)_R$ enhanced gauge symmetry which we got in the case of a single circle. We can have this large symmetry from any of the d circles, of course but there is more, since there are extra massless states that can be made by choices of momenta from more than one circle, corresponding to weight one vertex operators. This will allow us to make very large enhanced gauge groups, up to rank d, as we shall see later in subsection 7.4.

4.6 Another Special Radius: Bosonisation

Before proceeding with the T–duality discussion, let us pause for a moment to remark upon something which will be useful later. In the case that $R=\sqrt{(\alpha'/2)}$, something remarkable happens. The partition function is:

$$Z\left(q,R=\sqrt{\frac{\alpha'}{2}}\right)=(\eta\bar\eta)^{-1}\sum_{n,w}q^{\frac{1}{2}\left(n+\frac{w}{2}\right)^2}\bar{q}^{\frac{1}{2}\left(n-\frac{w}{2}\right)^2}\ . \qquad (211)$$

Note that the allowed momenta at this radius are (*c.f.* (186)):

$$\begin{aligned} \alpha_0^{25} &= P_L\sqrt{\frac{\alpha'}{2}} = \left(n + \frac{w}{2}\right) \\ \tilde{\alpha}_0^{25} &= P_R\sqrt{\frac{\alpha'}{2}} = \left(n - \frac{w}{2}\right) \ , \end{aligned} \tag{212}$$

and so they span both integer and half–integer values. Now when P_L is an integer, then so is P_R and *vice–versa*, and so we have two distinct sectors, integer and half–integer. In fact, we can rewrite our partition function as a set of sums over these separate sectors:

$$Z_{R=\sqrt{\alpha'/2}} = \frac{1}{2}\left\{\left|\frac{1}{\eta}\sum_n q^{\frac{1}{2}n^2}\right|^2 + \left|\frac{1}{\eta}\sum_n (-1)^n q^{\frac{1}{2}n^2}\right|^2 + \left|\frac{1}{\eta}\sum_n q^{\frac{1}{2}\left(n+\frac{1}{2}\right)^2}\right|^2\right\} \ .$$

The middle sum is rather like the first, except that there is a -1 whenever n is odd. Taking the two sums together, it is just like we have performed the sum (trace) over all the integer momenta, but placed a projection onto even momenta, using the projector

$$P = \frac{1}{2}(1 + (-1)^n) \ . \tag{213}$$

In fact, an investigation will reveal that the third term can be written with a partner just like it save for an insertion of $(-1)^n$ also, but that latter sum vanishes identically. This all has a specific meaning which we will uncover shortly.

Notice that the partition function can be written in yet another nice way, this time as

$$Z_{R=\sqrt{\alpha'/2}} = \frac{1}{2}\left(|f_4^2(q)|^2 + |f_3^2(q)|^2 + |f_2^2(q)|^2\right) \ , \tag{214}$$

where, for here and for future use, let us define

$$\begin{aligned} f_1(q) \equiv &= q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1 - q^n) \equiv \eta(\tau) \\ f_2(q) \equiv &= \sqrt{2}q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1 + q^n) \\ f_3(q) \equiv &= q^{-\frac{1}{48}}\prod_{n=1}^{\infty}(1 + q^{n-\frac{1}{2}}) \\ f_4(q) \equiv &= q^{-\frac{1}{48}}\prod_{n=1}^{\infty}(1 - q^{n-\frac{1}{2}}) \ , \end{aligned} \tag{215}$$

and note that

$$f_2\left(-\frac{1}{\tau}\right) = f_4\left(\tau\right) \ ; \quad f_3\left(-\frac{1}{\tau}\right) = f_3\left(\tau\right) \ ; \tag{216}$$

$$f_3\left(\tau + 1\right) = f_4\left(\tau\right) \ ; \quad f_2\left(\tau + 1\right) = f_2\left(\tau\right) \ . \tag{217}$$

While the rewriting (214) might not look like much at first glance, this is in fact the partition function of a single Dirac fermion in two dimensions!: $Z(R = \sqrt{\alpha'/2}) = Z_{\rm Dirac}$. We have arrived at the result that a boson (at a special radius) is in fact equivalent to a fermion. This is called "bosonisation" or "fermionisation", depending upon one's perspective. How can this possibly be true?

The action for a Dirac fermion, $\Psi = (\Psi_L, \Psi_R)^T$ (which has two components in two dimensions) is, in conformal gauge:

$$S_{\rm Dirac} = \frac{\rm i}{2\pi}\int d^2\sigma\ \bar\Psi\gamma^a\partial_a\Psi = \frac{\rm i}{\pi}\int d^2\sigma\ \bar\Psi_L\bar\partial\Psi_L - \frac{\rm i}{\pi}\int d^2\sigma\ \bar\Psi_R\partial\Psi_R\ , \tag{218}$$

where we have used

$$\gamma^0 = {\rm i}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\ ,\quad \gamma^1 = {\rm i}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\ .$$

Now, as a fermion goes around the cylinder $\sigma\to\sigma+2\pi$, there are two types of boundary condition it can have: It can be periodic, and hence have integer moding, in which case it is said to be in the "Ramond" (R) sector. It can instead be antiperiodic, have half integer moding, and is said to be in the "Neveu–Schwarz" (NS) sector.

In fact, these two sectors in this theory map to the two sectors of allowed momenta in the bosonic theory: integer momenta to NS and half integer to R. The various parts of the partition function can be picked out and identified in fermionic language. For example, the contribution:

$$\left|f_3^2(q)\right|^2 \equiv \left|q^{-\frac{1}{24}}\right|^2\left|\prod_{n=1}^{\infty}(1+q^{n-\frac12})^2\right|^2\ ,$$

looks very fermionic, and is in fact the trace over the contributions from the NS sector fermions as they go around the torus. It is squared because there are two components to the fermion, Ψ and $\bar\Psi$. We have the squared modulus beyond that since we have the contribution from the left and the right.

The $f_4(q)$ contribution on the other hand, arises from the NS sector with a $(-)^F$ inserted, where F counts the number of fermions at each level. The $f_2(q)$ contribution comes from the R sector, and there is a vanishing contribution from the R sector with $(-1)^F$ inserted. We see that that the projector

$$P = \frac{1}{2}(1+(-1)^F) \tag{219}$$

is the fermionic version of the projector (213) we identified previously. Notice that there is an extra factor of two in front of the R sector contribution due to the definition of f_2. This is because the R ground state is in fact degenerate. The modes Ψ_0 and $\bar\Psi_0$ define two ground states which map into one another. Denote the vacuum by $|s\!>$, where s can take the values $\pm\frac12$. Then

$$\begin{aligned}\Psi_0|-\tfrac12\!>= 0\ ;\qquad &\bar\Psi_0|+\tfrac12\!>=0\ ;\\ \bar\Psi_0|-\tfrac12\!>=|+\tfrac12\!>\ ;\qquad &\Psi_0|+\tfrac12\!>=|-\tfrac12\!>\ ,\end{aligned} \tag{220}$$

and Ψ_0 and $\bar\Psi_0$ therefore form a representation of the two dimensional Clifford algebra. We will see this in more generality later on. In D dimensions there are $D/2$ components, and the degeneracy is $2^{D/2}$.

As a final check, we can see that the zero point energies work out nicely too. The mnemonic (70) gives us the zero point energy for a fermion in the NS sector as $-1/48$, we multiply this by two since there are two components and we see that that we recover the weight of the ground state in the partition function. For the Ramond sector, the zero point energy of a single fermion is $1/24$. After multiplying by two, we see that this is again correctly obtained in our partition function, since $-1/24+1/8=1/12$. It is awfully nice that the function $f_2^2(q)$ has the extra factor of $2q^{1/8}$, just for this purpose.

This partition function is again modular invariant, as can be checked using elementary properties of the f–functions (217): f_2 transforms into f_4 under the S transformation, while under T, f_4 transforms into f_3.

At the level of vertex operators, the correspondence between the bosons and the fermions is given by:

$$\begin{aligned}\Psi_L(z) = e^{i\beta X_L^{25}(z)}\ ;\quad &\bar\Psi_L(z) = e^{-i\beta X_L^{25}(z)}\ ;\\ \Psi_R(\bar z) = e^{i\beta X_R^{25}(\bar z)}\ ;\quad &\bar\Psi_R(\bar z) = e^{-i\beta X_R^{25}(\bar z)}\ ,\end{aligned} \tag{221}$$

where $\beta = \sqrt{2/\alpha'}$.

This makes sense, for the exponential factors define fields single–valued under $X^{25} \to X^{25} + 2\pi R$, at our special radius $R = \sqrt{\alpha'/2}$. We also have

$$\Psi_L(z)\bar{\Psi}_L(z) = \partial_z X^{25}\ ;\quad \Psi_R(\bar{z})\bar{\Psi}_R(\bar{z}) = \partial_{\bar{z}} X^{25}\ , \tag{222}$$

which shows how to combine two $(0, 1/2)$ fields to make a $(0, 1)$ field, with a similar structure on the left. Notice also that the symmetry $X^{25} \to -X^{25}$ swaps $\Psi_{L(R)}$ and $\bar{\Psi}_{L(R)}$, a symmetry of interest in the next subsection.

We shall briefly this bosonization/fermionization relation in later sections, where it will be useful to write vertex operators in various ways in the supersymmetric theories.

4.7 String Theory on an Orbifold

There is a rather large class of string vacua, called "orbifolds",[26] with many applications in string theory. We ought to study them, as many of the basic structures which will occur in their definition appear in more complicated examples later on.

The circle S^1, parametrised by X^{25} has the obvious $\mathbb{Z}_2$ symmetry $R_{25} : X^{25} \to -X^{25}$. This symmetry extends to the full spectrum of states and operators in the complete theory of the string propagating on the circle. Some states are even under R_{25}, while others are odd. Just as we saw before in the case of Ω, it makes sense to ask whether we can define another theory from this one by truncating the theory to the sector which is even. This would define string theory propagating on the "orbifold" space $S^1/\mathbb{Z}_2$. In defining this geometry, note that it is actually a line segment, where the endpoints of the line are actually "fixed points" of the $\mathbb{Z}_2$ action.

The point $X^{25} = 0$ is clearly such a point and the other is $X^{25} = \pi R \sim -\pi R$, where R is the radius of the original S^1. A picture of the orbifold space is given in figure 20. In order to check whether string theory

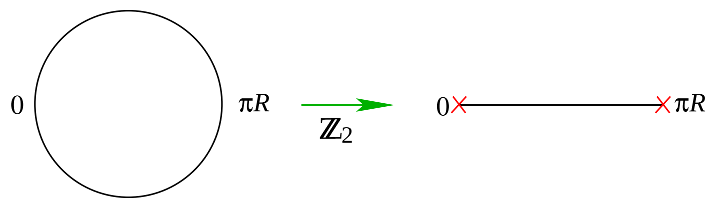


Figure 20: A $\mathbb{Z}_2$ orbifold of a circle, giving a line segment with two fixed points.

on this space is sensible, we ought to compute the partition function for it. We can work this out by simply inserting the projector

$$P = \frac{1}{2}(1 + R_{25})\ , \tag{223}$$

which will have the desired effect of projecting out the R_{25}–odd parts of the circle spectrum. So we expect to see two pieces to the partition function: a part that is $\frac{1}{2}$ times $Z_{\rm circle}$, and another part which is $Z_{\rm circle}$ with R_{25} inserted. Noting that the action of R_{25} is

$$R_{25} : \begin{cases} \alpha^{25}_n \to -\alpha^{25}_n \\ \tilde{\alpha}^{25}_n \to -\tilde{\alpha}^{25}_n \end{cases}\ , \tag{224}$$

the partition function is:

$$Z_{\rm orbifold} = \frac{1}{2}\left[Z(R,\tau) + 2\left(|f_2(q)|^{-2} + |f_3(q)|^{-2} + |f_4(q)|^{-2}\right)\right]\ , \tag{225}$$

The f_2 part is what one gets if one works out the projected piece, but there are two extra terms. From where do they come? One way to see that those extra pieces must be there is to realize that the first two parts on their own cannot be modular invariant. The first part is of course already modular invariant on its own, while the second part transforms (217) into f_4 under the S transformation, so it has to be there too. Meanwhile, f_4 transforms into f_3 under the T–transformation, and so that must be there also, and so on.

While modular invariance is a requirement, as we saw, what is the physical meaning of these two extra partition functions? What sectors of the theory do they correspond to and how did we forget them?

The sectors we forgot are very stringy in origin, and arise in a similar fashion to the way we saw windings appear in earlier sections. There, the circle may be considered as a quotient of the real line $\mathbb{R}$ by a translation $X^{25} \to X^{25} + 2\pi R$. There, we saw that as we go around the string, $\sigma \to \sigma + 2\pi$, the embedding map $X^{25}(\sigma)$ is allowed to change by any amount of the lattice, $2\pi Rw$. Here, the orbifold further imposes the equivalence $X^{25} \sim -X^{25}$, and therefore, as we go around the string, we ought to be allowed:

$$X^{25}(\sigma + 2\pi, \tau) = -X^{25}(\sigma, \tau) + 2\pi wR \ ,$$

for which the solution to the Laplace equation is:

$$X^{25}(z, \bar{z}) = x^{25} + \mathrm{i}\sqrt{\frac{\alpha'}{2}} \sum_{n=-\infty}^{\infty} \frac{1}{\left(n + \frac{1}{2}\right)} \left(\alpha^{25}_{n+\frac{1}{2}} z^{n+\frac{1}{2}} + \widetilde{\alpha}^{25}_{n+\frac{1}{2}} \bar{z}^{n+\frac{1}{2}} \right) \ , \qquad (226)$$

with $x^{25} = 0$ or πR, no zero mode α_0^{25} (hence no momentum), and no winding: $w = 0$.

This is a configuration of the string allowed by our equations of motion and boundary conditions and therefore has to be included in the spectrum. We have two identical copies of these "twisted sectors" corresponding to strings trapped at 0 and πR in spacetime. They are trapped, since x^{25} is fixed and there is no momentum.

Notice that in this sector, where the boson $X^{25}(w, \bar{w})$ is antiperiodic as one goes around the cylinder, there is a zero point energy of 1/16 from the twisted sector: it is a weight $(1/16, 1/16)$ field, in terms of where it appears in the partition function.

Schematically therefore, the complete partition function ought to be

$$Z_{\mathrm{orb.}} = \mathrm{Tr}_{\mathrm{untw'd}} \left(\frac{(1 + R_{25})}{2} q^{L_0 - \frac{1}{24}} \bar{q}^{\bar{L}_0 - \frac{1}{24}} \right) + \mathrm{Tr}_{\mathrm{tw'd}} \left(\frac{(1 + R_{25})}{2} q^{L_0 - \frac{1}{24}} \bar{q}^{\bar{L}_0 - \frac{1}{24}} \right) \qquad (227)$$

to ensure modular invariance, and indeed, this is precisely what we have in (225). The factor of two in front of the twisted sector contribution is because there are two identical twisted sectors, and we must sum over all sectors.

In fact, substituting in the expressions for the f–functions, one can discover the weight $(1/16, 1/16)$ twisted sector fields contributing to the vacuum of the twisted sector. This simply comes from the $q^{-1/48}$ factor in the definition of the $f_{3,4}$–functions. They appear inversely, and for example on the left, we have $1/48 = -\mathrm{c}/24 + 1/16$, where $\mathrm{c} = 1$.

Finally, notice that the contribution from the twisted sectors do not depend upon the radius R. This fits with the fact that the twisted sectors are trapped at the fixed points, and have no knowledge of the extent of the circle.

4.8 T–Duality for Open Strings: D–branes

Let us now consider the $R \to 0$ limit of the open string spectrum. Open strings do not have a conserved winding around the periodic dimension and so they have no quantum number comparable to w, so something different must happen, as compared to the closed string case. In fact, it is more like field theory: when $R \to 0$ the states with non–zero internal momentum go to infinite mass, but there is no new continuum of states coming from winding. So we are left with a theory in one dimension fewer. A puzzle arises when one remembers that theories with open strings have closed strings as well, so that in the $R \to 0$ limit the closed strings live in D spacetime dimensions but the open strings only in $D-1$.

This is perfectly fine, though, since the interior of the open string is indistinguishable from the closed string and so should still be vibrating in D dimensions. The distinguished part of the open string are the endpoints, and these are restricted to a $D-1$ dimensional hyperplane.

This is worth seeing in more detail. Write the open string mode expansion as

$$
\begin{aligned}
X^\mu(z,\bar z) &= X^\mu(z) + X^\mu(\bar z)\ , \\
X^\mu(z) &= \frac{x^\mu}{2} + \frac{x'^\mu}{2} - i\alpha' p_0^\mu \ln z + i\left(\frac{\alpha'}{2}\right)^{1/2} \sum_{n\neq 0} \frac{1}{n}\alpha_n^\mu z^{-n}\ , \\
X^\mu(\bar z) &= \frac{x^\mu}{2} - \frac{x'^\mu}{2} - i\alpha' p_0^\mu \ln \bar z + i\left(\frac{\alpha'}{2}\right)^{1/2} \sum_{n\neq 0} \frac{1}{n}\alpha_n^\mu \bar z^{-n}\ ,
\end{aligned}
\tag{228}
$$

where x'^μ is an arbitrary number which cancels out when we make the usual open string coordinate. Imagine that we place X^{25} on a circle of radius R. The T–dual coordinate is

$$
\begin{aligned}
X'^{25}(z,\bar z) &= X^{25}(z) - X^{25}(\bar z) \\
&= x'^{25} - i\alpha' p^{25} \ln\left(\frac{z}{\bar z}\right) + i(2\alpha')^{1/2} \sum_{n\neq 0} \frac{1}{n}\alpha_n^{25} e^{-in\tau}\sin n\sigma \\
&= x^{25} + 2\alpha' p^{25}\sigma + i(2\alpha')^{1/2} \sum_{n\neq 0} \frac{1}{n}\alpha_n^{25} e^{-in\tau}\sin n\sigma \\
&= x^{25} + 2\alpha' \frac{n}{R}\sigma + i(2\alpha')^{1/2} \sum_{n\neq 0} \frac{1}{n}\alpha_n^{25} e^{-in\tau}\sin n\sigma\ .
\end{aligned}
\tag{229}
$$

Notice that there is no dependence on τ in the zero mode sector. This is where momentum usually comes from in the mode expansion, and so we have no momentum. In fact, since the oscillator terms vanish at the endpoints $\sigma = 0, \pi$, we see that *the endpoints do not move in the X'^{25} direction!*. Instead of the usual Neumann boundary condition $\partial_n X \equiv \partial_\sigma X = 0$, we have $\partial_t X \equiv i\partial_\tau X = 0$. More precisely, we have the Dirichlet condition that the ends are at a fixed place:

$$
X'^{25}(\pi) - X'^{25}(0) = \frac{2\pi\alpha' n}{R} = 2\pi n R'.
\tag{230}
$$

In other words, the values of the coordinate X'^{25} at the two ends are equal up to an integral multiple of the periodicity of the dual dimension, corresponding to a string that winds as in figure 21.

This picture is consistent with the fact that under T–duality, the definition of the normal and tangential derivatives get exchanged:

$$
\begin{aligned}
\partial_n X^{25}(z,\bar z) &= \frac{\partial X^{25}(z)}{\partial z} + \frac{\partial X^{25}(\bar z)}{\partial \bar z} = \partial_t X'^{25}(z,\bar z) \\
\partial_t X^{25}(z,\bar z) &= \frac{\partial X^{25}(z)}{\partial z} - \frac{\partial X^{25}(\bar z)}{\partial \bar z} = \partial_n X'^{25}(z,\bar z)\ .
\end{aligned}
\tag{231}
$$

Notice that this all pertains to just the direction which we T–dualised, X^{25}. So the ends are still free to move in the other 24 spatial dimensions, which constitutes a hyperplane called a "D–brane". There are 24 spatial directions, so we shall denote it a D24–brane.

4.9 Chan–Paton Factors and Wilson Lines

The picture becomes even more rich when we include Chan–Paton factors and Wilson lines[28]. Let us pause to recall the latter.

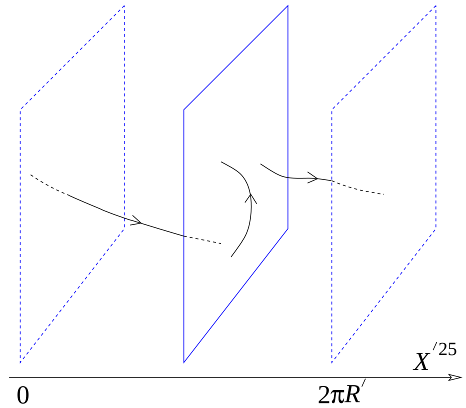


Figure 21: Open strings with endpoints attached to a hyperplane. The dashed planes are periodically identified. The strings shown have winding numbers zero and one.

Wilson line configurations in a gauge theory are of relevance when spacetime has non–trivial topology. Let us consider the case that the coordinate X^{25} is compactified on a circle of radius R. Consider the case that we have gauge group $U(1)$. We can make the following choice of constant background gauge potential:

$$A_{25}(X^\mu) = -\frac{\theta}{2\pi R} = -i\Lambda^{-1}\frac{\partial\Lambda}{\partial X^{25}} \ , \tag{232}$$

where $\Lambda(X^{25}) = e^{-\frac{i\theta X^{25}}{2\pi R}}$. This is clearly pure gauge, but only locally. There still exists non–trivial physics. Form the gauge invariant quantity ("Wilson Line"):

$$W_q = \exp\left(iq\oint dX^{25}A_{25}\right) = e^{-iq\theta} \ . \tag{233}$$

Where does this observable show up? Imagine a point particle of charge q under the $U(1)$. Its action can be written as:

$$S = \int d\tau\left\{\frac{1}{2}\dot{X}^\mu\dot{X}_\mu - iqA_\mu\dot{X}^\mu\right\} = \int d\tau\mathcal{L} \ . \tag{234}$$

The last term is just $-iq\int A = -iq\int A_\mu dx^\mu$, in the language of forms. This is the natural coupling of a world volume to an antisymmetric tensor, as we shall see.)

Recall that in the path integral we are computing e^{-S}. So if the particle does a loop around X^{25} circle, it will pick up a phase factor of W_q. Notice that the conjugate momentum to X^μ is

$$\Pi^\mu = i\frac{\partial\mathcal{L}}{\partial\dot{X}^\mu} = i\dot{X}^\mu \ , \qquad \text{except for} \qquad \Pi^{25} = i\dot{X}^{25} - \frac{q\theta}{2\pi R} = \frac{n}{R} \ ,$$

where the last equality results from the fact that we are on a circle. Now we can of course gauge away A with the choice Λ^{-1}, but it will be the case that as we move around the circle, *i.e.,* $X^{25} \to X^{25} + 2\pi R$, the particle (and all fields) of charge q will pick up a phase $e^{iq\theta}$. So the canonical momentum is shifted to:

$$p^{25} = \frac{n}{R} + \frac{q\theta}{2\pi R} \ . \tag{235}$$

We shall use this result in what follows.

Returning to the oriented open string and consider the case of $U(N)$. When we compactify the X^{25} direction, we can include a Wilson line

$$A_{25} = \mathrm{diag}\{\theta_1, \theta_2, \ldots, \theta_N\}/2\pi R \ ,$$

which generically breaks $U(N) \to U(1)^N$. Locally this is pure gauge,

$$A_{25} = -i\Lambda^{-1}\partial_{25}\Lambda, \qquad \Lambda = \mathrm{diag}\{e^{iX^{25}\theta_1/2\pi R}, e^{iX^{25}\theta_2/2\pi R}, \ldots, e^{iX^{25}\theta_1/2\pi R}\} \ . \tag{236}$$

We can gauge A_{25} away, but since the gauge transformation is not periodic, the fields pick up a phase

$$\mathrm{diag}\left\{e^{-i\theta_1}, e^{-i\theta_2}, \ldots, e^{-i\theta_N}\right\} \tag{237}$$

under $X^{25} \to X^{25} + 2\pi R$. What is the effect in the dual theory? Due to the phase (237) the open string momenta are now fractional.

As the momentum is dual to winding number, we conclude that the fields in the dual description have fractional winding number, *i.e.,* their endpoints are no longer on the same hyperplane. Indeed, a string whose endpoints are in the state $|ij\rangle$ picks up a phase $e^{i(\theta_j-\theta_i)}$, so their momentum is $(2\pi n + \theta_j - \theta_i)/2\pi R$. Modifying the endpoint calculation (230) then gives

$$X'^{25}(\pi) - X'^{25}(0) = (2\pi n + \theta_j - \theta_i)R'. \tag{238}$$

In other words, up to an arbitrary additive constant, the endpoint in state i is at position

$$X'^{25} \ = \ \theta_i R' \ = \ 2\pi\alpha' A_{25,ii}. \tag{239}$$

We have in general N hyperplanes at different positions as depicted in figure 22.

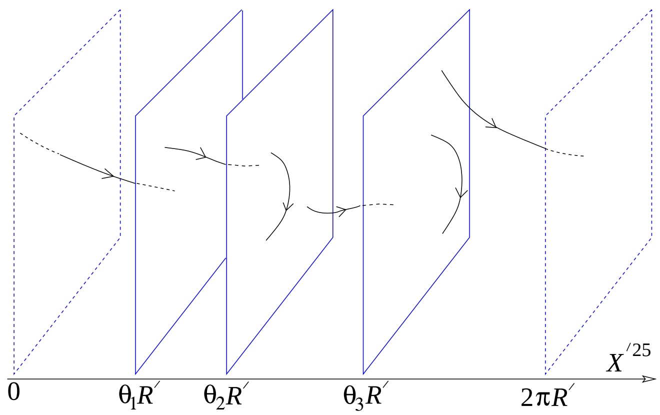


Figure 22: Three D–branes at different positions, with various strings attached.

4.10 D–Brane Collective Coördinates

Clearly, the whole picture goes through if several coordinates

$$X^m = \{X^{25}, X^{24}, \ldots, X^{p+1}\} \tag{240}$$

are periodic, and we rewrite the periodic dimensions in terms of the dual coordinates. The open string endpoints are then confined to N $(p+1)$–dimensional hyperplanes, the D$(p+1)$–branes. The Neumann conditions on the world sheet, $\partial_n X^m(\sigma^1, \sigma^2) = 0$, have become Dirichlet conditions $\partial_t X'^m(\sigma^1, \sigma^2) = 0$ for the dual coordinates. In this terminology, the original 26 dimensional open string theory theory contains N D25–branes. A 25–brane fills space, so the string endpoint can be anywhere: it just corresponds to an ordinary Chan–Paton factor.

It is natural to expect that the hyperplane is dynamical rather than rigid.[12] For one thing, this theory still has gravity, and it is difficult to see how a perfectly rigid object could exist. Rather, we would expect that the hyperplanes can fluctuate in shape and position as dynamical objects. We can see this by looking at the massless spectrum of the theory, interpreted in the dual coordinates.

Taking for illustration the case where a single coordinate is dualised, consider the mass spectrum. The $D-1$ dimensional mass is

$$\begin{aligned} M^2 &= (p^{25})^2 + \frac{1}{\alpha'}(N-1) \\ &= \left(\frac{[2\pi n + (\theta_i - \theta_j)]R'}{2\pi\alpha'}\right)^2 + \frac{1}{\alpha'}(N-1). \end{aligned} \tag{241}$$

Note that $[2\pi n + (\theta_i - \theta_j)]R'$ is the minimum length of a string winding between hyperplanes i and j. Massless states arise generically only for non–winding (*i.e.,* $n = 0$) open strings whose end points are on the same hyperplane, since the string tension contributes an energy to a stretched string. We have therefore the massless states (with their vertex operators):

$$\begin{aligned} &\alpha_{-1}^{\mu}|k;ii\rangle, \qquad V = \partial_t X^{\mu}, \\ &\alpha_{-1}^{m}|k;ii\rangle, \qquad V = \partial_t X^{m} = \partial_n X'^{m}. \end{aligned} \tag{242}$$

The first of these is a gauge field living on the D–brane, with $p+1$ components tangent to the hyperplane, $A^{\mu}(\xi^a)$, $\mu, a = 0,\ldots,p$. Here, $\xi^{\mu} = x^{\mu}$ are coordinates on the D–branes' world–volume. The second was the gauge field in the compact direction in the original theory. In the dual theory it becomes the transverse position of the D–brane (see (239)). From the point of view of the world–volume, it is a family of scalar fields, $\Phi^m(\xi^a)$, $(m = p+1,\ldots,D-1)$ living there.

We saw this in equation (239) for a Wilson line, which was a constant gauge potential. Now imagine that, as genuine scalar fields, the Φ^m vary as we move around on the world–volume of the D–brane. This therefore embeds the brane into a variable place in the transverse coordinates. This is simply describing a specific *shape* to the brane as it is embedded in spacetime. The $\Phi^m(\xi^a)$ are exactly analogous to the embedding coordinate map $X^{\mu}(\sigma,\tau)$ with which we described strings in the earlier sections.

The values of the gauge field backgrounds describe the shape of the branes as a soliton background, then. Meanwhile their quanta describe fluctuations of that background. This is the same phenomenon which we found for our description of spacetime in string theory. We started with strings in a flat background and discover that a massless closed string state corresponds to fluctuations of the geometry. Here we found first a flat hyperplane, and then discovered that a certain open string state corresponds to fluctuations of its shape. Remarkably, these open string states are simply gauge fields, and this is one of the reasons for the great success of D–branes. There are other branes in string theory (as we shall see in section 11) and they have other types of field theory describing their collective dynamics. D–branes are special, in that they have a beautiful description using gauge theory. Ultimately, we can use the long experience of working with gauge theories to teach us much about D–branes, the geometry of D–branes and the string theories in which they live can teach us a lot about gauge theories. This is the basis of the dialogue between gauge theory and geometry which is common in the field at present.

It is interesting to look at the $U(N)$ symmetry breaking in the dual picture where the brane can move transverse to their world–volumes. When no D–branes coincide, there is just one massless vector each, or $U(1)^N$ in all, the generic unbroken group. If k D–branes coincide, there are new massless states because strings which are stretched between these branes can achieve vanishing length. Thus, there are k^2 vectors, forming the adjoint of a $U(k)$ gauge group.[28, 29] This coincident position corresponds to $\theta_1 = \theta_2 = \cdots = \theta_k$ for some subset of the original $\{\theta\}$, so in the original theory the Wilson line left a $U(k)$ subgroup unbroken. At the same time, there appears a set of k^2 massless scalars: the k positions are promoted to a matrix. This is curious and hard to visualize, but plays an important role in the dynamics of D–branes.[29] Note that if all N branes are coincident, we recover the $U(N)$ gauge symmetry.

While this picture seems a bit exotic, and will become more so in the unoriented theory, the reader should note that all we have done is to rewrite the original open string theory in terms of variables which are more natural in the limit $R \ll \sqrt{\alpha'}$. Various puzzling features of the small–radius limit become clear in the T–dual picture.

Observe that, since T–duality interchanges Neumann and Dirichlet boundary conditions, a further T–duality in a direction tangent to a Dp–brane reduces it to a D$(p-1)$–brane, while a T–duality in a direction orthogonal turns it into a D$(p+1)$–brane.

4.11 T–Duality for Unoriented Strings

4.11.1 Orientifolds.

The $R \to 0$ limit of an unoriented theory also leads to a new extended object. Recall that the effect of T–duality can also be understood as a one–sided parity transformation.

For closed strings, the original coordinate is $X^m(z,\bar{z}) = X^m(z) + X^m(\bar{z})$. We have already discussed how to project string theory with these coordinates by Ω. The dual coordinate is $X'^m(z,\bar{z}) = X^m(z) - X^m(\bar{z})$. The action of world sheet parity reversal is to exchange $X^\mu(z)$ and $X^\mu(\bar{z})$. This gives for the dual coordinate:

$$X'^m(z,\bar{z}) \leftrightarrow -X'^m(\bar{z},z) \ . \tag{243}$$

This is the product of a world–sheet and a spacetime parity operation.

In the unoriented theory, strings are invariant under the action of Ω, while in the dual coordinate the theory is invariant under the product of world–sheet parity and a spacetime parity. This generalisation of the usual unoriented theory is known as an "orientifold", a term which mixes the term "orbifold" with orientation reversal.

Imagine that we have separated the string wavefunction into its internal part and its dependence on the centre of mass, x^m. Furthermore, take the internal wavefunction to be an eigenstate of Ω. The projection then determines the string wavefunction at $-x^m$ to be the same as at x^m, up to a sign. In practice, the various components of the metric and antisymmetric tensor satisfy *e.g.,*

$$\begin{aligned} G_{\mu\nu}(x^\mu,-x^m) &= G_{\mu\nu}(x^\mu,x^m), \qquad & B_{\mu\nu}(x^\mu,-x^m) &= -B_{\mu\nu}(x^\mu,x^m), \\ G_{\mu n}(x^\mu,-x^m) &= -G_{\mu n}(x^\mu,x^m), \qquad & B_{\mu n}(x^\mu,-x^m) &= B_{\mu n}(x^\mu,x^m), \\ G_{mn}(x^\mu,-x^m) &= G_{mn}(x^\mu,x^m), \qquad & B_{mn}(x^\mu,-x^m) &= -B_{mn}(x^\mu,x^m) \ . \end{aligned} \tag{244}$$

In other words, when we have k compact directions, the T–dual spacetime is the torus T^{25-k} modded by a $\mathbb{Z}_2$ reflection in the compact directions. So we are instructed to perform an orbifold construction, modified by the extra sign. In the case of a single periodic dimension, for example, the dual spacetime is the line segment $0 \le x^{25} \le \pi R'$. The reader should remind themselves of the orbifold construction in section 4.7. At the ends of the interval, there are fixed "points" , which are in fact spatially 24–dimensional planes.

Looking at the projections (244) in this case, we see that on these fixed planes, the projection is just like we did for the Ω–projection of the 25+1 dimensional theory in section 2.11: The theory is unoriented there, and half the states are removed. These orientifold fixed planes are called "O–planes" for short. For this case, we have two O24–planes. (For k directions we have 2^k O$(25-k)$–planes arranged on the vertices of a hypercube.) In particular, we can usefully think of the original case of $k=0$ as being on an O25–plane.

While the theory is unoriented on the O–plane, away from the orientifold fixed planes, the local physics is that of the *oriented* string theory. The projection relates the physics of a string at some point x^m to the string at the image point $-x^m$.

In string perturbation theory, orientifold planes are not dynamical. Unlike the case of D–branes, there are no string modes tied to the orientifold plane to represent fluctuations in its shape. Our heuristic argument in the previous subsection that gravitational fluctuations force a D–brane to move dynamically does not apply to the orientifold fixed plane. This is because the identifications (244) become *boundary conditions* at the fixed plane, such that the incident and reflected gravitational waves cancel. For the D–brane, the reflected wave is higher order in the string coupling.

The orientifold construction was discovered *via* T–duality[12] and independently from other points of view.[30, 14] One can of course consider more general orientifolds which are not simply T–duals of toroidal compactifications. The idea is simply to combine a group of discrete symmetries with Ω such that the resulting group of operations (the "orientifold group", G_Ω) is itself a symmetry of some string theory. One then has the right to ask what the nature of the projected theory obtained by dividing by G_Ω might be. This is a fruitful way of construction interesting and useful string vacua.[31] We shall have more to say about this later, since in superstring theory we shall find that O–planes, like D–branes , are sources of various closed string sector fields. Therefore there will be additional consistency conditions to be satisfied in constructing an orientifold, amounting to making sure that the field equations are satisfied.

4.11.2 Orientifolds and D–Branes

So far our discussion of orientifolds was just for the closed string sector. Let us see how things are changed in the presence of open strings. In fact, the situation is similar. Again, let us focus for simplicity on a single compact dimension. Again there is one orientifold fixed plane at 0 and another at $\pi R'$. Introducing $SO(N)$ Chan–Paton factors, a Wilson line can be brought to the form

$$\mathrm{diag}\{\theta_1, -\theta_1, \theta_2, -\theta_2, \cdots, \theta_{N/2}, -\theta_{N/2}\}. \tag{245}$$

Thus in the dual picture there are $\frac{1}{2}N$ D–branes on the line segment $0 \le X'^{25} < \pi R'$, and $\frac{1}{2}N$ at their image points under the orientifold identification.

Strings can stretch between D–branes and their images as shown in figure 23. The generic gauge group

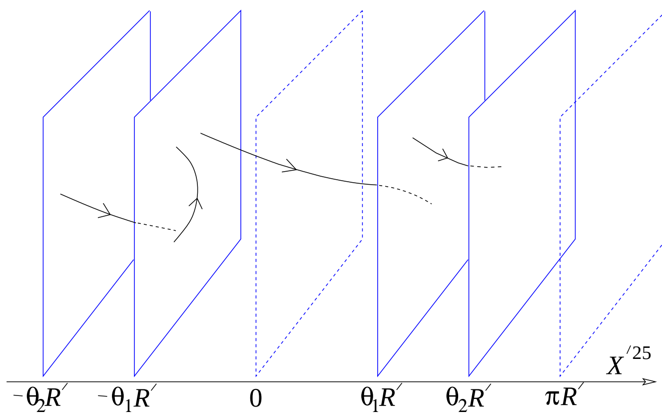


Figure 23: Orientifold planes at 0 and $\pi R'$. There are D–branes at $\theta_1 R'$ and $\theta_2 R'$, and their images at $-\theta_1 R'$ and $-\theta_2 R'$. Ω acts on any string by a combination of a spacetime reflection through the planes and reversing the orientation arrow.

is $U(1)^{N/2}$, where all branes are separated. As in the oriented case, if m D–branes are coincident there is a $U(m)$ gauge group. However, now if the m D–branes in addition lie at one of the fixed planes, then strings stretching between one of these branes and one of the image branes also become massless and we have the right spectrum of additional states to fill out $SO(2m)$. The maximal $SO(N)$ is restored if all of the branes are coincident at a single orientifold plane. Note that this maximally symmetric case is asymmetric between the two fixed planes. Similar considerations apply to $USp(N)$. As we saw before, the difference between the appearance of the two groups is in a sign on the matrix M as it acts on the string wavefunction. Later, we shall see that this sign is correlated with the sign of the charge and tension of the orientifold plane.

We should emphasize that there are $\frac{1}{2}N$ dynamical D–branes but an N–valued Chan–Paton index. An interesting case is when $k+\frac{1}{2}$ D–branes lie on a fixed plane, which makes sense because the number $2k+1$ of indices is integer. A brane plus image can move away from the fixed plane, but the number of branes remaining is always half–integer.

5 Background Fields and World–Volume Actions

T–duality is clearly a remarkable phenomenon which is highly indicative of the different view string theory has of spacetime from that of field theories. This heralds a rather rich landscape of possibilities for new physics, and indeed T–duality will govern much of what we will study in the rest of this book, either directly or indirectly. So far, we have uncovered it at the level of the string spectrum, and have used it to discover D–branes and orientifolds. However, we have so far restricted ourselves to flat spacetime backgrounds, with none of the other fields in the string spectrum switched on. In this section, we shall study the action of T–duality when the massless fields of the string theory take on non–trivial values, giving us curved backgrounds and/or gauge fields on the world–volume of the D–branes. It is also important to uncover further aspects of the dynamics of D–branes in non–trivial backgrounds, and we shall also uncover an action to describe this here.

5.1 T–duality in Background Fields

The first thing to notice is that T–duality acts non–trivially on the dilaton, and therefore modifies the string coupling: [20, 21] After dimensional reduction on a circle of radius R, the effective 25–dimensional string coupling read off from the reduced string frame supergravity action is now $g_s = e^{\Phi}(2\pi R)^{-1/2}$. Since the resulting 25–dimensional theory is supposed to have the same physics, by T–duality, as a theory with a dilaton $\tilde{\Phi}$, compactified on a circle of radius R', it is required that this coupling is equal to $\tilde{g}_s = e^{\tilde{\Phi}}(2\pi R')^{-1/2}$, the string coupling of the dual 25–dimensional theory:

$$e^{\tilde{\Phi}} = e^{\Phi}\frac{\alpha'^{1/2}}{R} \ . \tag{246}$$

This is just part of a larger statement about the T–duality transformation properties of background fields in general. Starting with background fields $G_{\mu\nu}$, $B_{\mu\nu}$ and Φ, let us first T–dualise in one direction, which we shall label X^{25}, as before. In other words, X^{25} is a direction which is a circle of radius R, and the dual circle X'^{25} is a circle of radius $R' = \alpha'/R$.

We may start with the two–dimensional sigma model (93) with background fields $G_{\mu\nu}$, $B_{\mu\nu}$, Φ, and assume that locally, all of the fields are independent of the direction X^{25}. In this case, we may write an equivalent action by introducing a Lagrange multiplier, which we shall call X'^{25}:

$$\begin{aligned} S_\sigma &= \frac{1}{4\pi\alpha'}\int d^2\sigma g^{1/2}\Big\{ g^{ab}\left[G_{25,25}v_av_b + 2G_{25,\mu}v_a\partial_bX^\mu + G_{\mu\nu}\partial_aX^\mu\partial_bX^\nu\right] \\ &+ i\epsilon^{ab}\left[2B_{25,\mu}v_a\partial_bX^\mu + B_{\mu\nu}\partial_aX^\mu\partial_bX^\nu + 2X'^{25}\partial_av_b\right] + \alpha' R\Phi\Big\} \ . \end{aligned} \tag{247}$$

Since the equation of motion for the Lagrange multiplier is

$$\frac{\partial\mathcal{L}}{\partial X'^{25}} = i\epsilon^{ab}\partial_av_b = 0 \ ,$$

we can write a solution as $v_b = \partial_b\phi$ for any scalar ϕ, which we might as well call X^{25}, since upon substitution of this solution back into the action, we get our original action (93).

Instead, we can find the equation of motion for the quantity v_a:

$$\begin{aligned} &\frac{\partial\mathcal{L}}{\partial v_a} - \frac{\partial}{\partial\sigma_b}\left(\frac{\partial\mathcal{L}}{\partial(\partial_bv_a)}\right) = 0 \\ &= g^{ab}\left[G_{25,25}v_b + G_{25,\mu}\partial_bX^\mu\right] + i\epsilon^{ab}\left[B_{25,\mu}\partial_bX^\mu + \partial_bX'^{25}\right] \ , \end{aligned} \tag{248}$$

which, upon solving it for v_a and substituting back into the equations gives an action of the form (93), but with fields $\tilde{G}_{\mu\nu}$ and $\tilde{B}_{\mu\nu}$ given by:

$$\begin{aligned}\widetilde{G}_{25,25} &= \frac{1}{G_{25,25}}\ ; \qquad e^{2\widetilde{\Phi}} = \frac{e^{2\Phi}}{G_{25,25}}\ ,\\ \widetilde{G}_{\mu 25} &= \frac{B_{\mu 25}}{G_{25,25}}\ ; \qquad \widetilde{B}_{\mu 25} = \frac{G_{\mu 25}}{G_{25,25}}\ ,\\ \widetilde{G}_{\mu\nu} &= G_{\mu\nu} - \frac{G_{\mu 25}G_{\nu 25} - B_{\mu 25}B_{\nu 25}}{G_{25,25}}\ ,\\ \widetilde{B}_{\mu\nu} &= B_{\mu\nu} - \frac{B_{\mu 25}G_{\nu 25} - G_{\mu 25}B_{\nu 25}}{G_{25,25}}\ ,\end{aligned} \tag{249}$$

where a *one loop* (not tree level) world–sheet computation (*e.g.*, by checking the β–function equations again, or by considering the new path integral measure induced by integrating out v_a), gives the new dilaton. This fits with the fact that it couples at the next order in α' (which plays the role of $\hbar$ on the world–sheet) as discussed previously.

Of course, we can T–dualise on many (say d) independent circles, forming a torus T^d. It is not hard to deduce that one can succinctly write the resulting T–dual background as follows. If we define the $D{\times}D$ metric

$$E_{\mu\nu} = G_{\mu\nu} + B_{\mu\nu}\ , \tag{250}$$

and if the circles are in the directions X^i, $i = 1, \dots, d$, with the remaining directions labelled by X^a, then the dual fields are given by

$$\begin{aligned}\widetilde{E}_{ij} &= E^{ij}\ , \qquad \widetilde{E}_{aj} = E_{ak}E^{kj}\ , \qquad e^{2\widetilde{\Phi}} = e^{2\Phi}\det(E^{ij})\ ,\\ \widetilde{E}_{ab} &= E_{ab} - E_{ai}E^{ij}E_{jb}\ ,\end{aligned} \tag{251}$$

where $E_{ik}E^{kj} = \delta_i^{\ j}$ defines E^{ij} as the inverse of E_{ij}. We will find this succinct form of the $O(d,d)$ T–duality transformation very useful later on.

5.2 A First Look at the D–brane World–Volume Action

The D–brane is a dynamical object, and as such, feels the force of gravity. In fact, it must be able to respond to the values of the various background fields in the theory. This is especially obvious if one recalls that the D–branes' location and shaped is controlled (in at least one way of describing them) by the open strings which end on them. These strings respond to the background fields in ways we have already studied (we have written world–sheet actions for them), and so should the D–branes. We must find a world–volume action describing their dynamics.

If we introduce coordinates ξ^a, $a = 0, \dots p$ on the brane, we can begin to write an action for the dynamics of the brane in terms of fields living on the world–volume in much the same way that we did for the string, in terms of fields living on the world–sheet. The background fields will act as generalized field–dependent couplings. As we discussed before, the fields on the brane are the embedding $X^\mu(\xi)$ and the gauge field $A_a(\xi)$. We shall ignore the latter for now and concentrate just on the embedding part. By direct analogy to the particle and string case studied in section 2, the action is

$$S_p = -T_p \int d^{p+1}\xi\, e^{-\Phi} \det{}^{1/2} G_{ab}\ , \tag{252}$$

where G_{ab} is the induced metric on the brane, otherwise known as the "pull–back" of the spacetime metric $G_{\mu\nu}$ to the brane:

$$G_{ab} \equiv \frac{\partial X^\mu}{\partial \xi^a}\frac{\partial X^\nu}{\partial \xi^b} G_{\mu\nu}\ . \tag{253}$$

T_p is the tension of the Dp–brane, which we shall discuss at length later. The dilaton dependence $e^{-\Phi} = g_s^{-1}$ arises because this is an open string tree level action, and so this is the appropriate function of the dilaton to introduce.

The world–volume reparametrisation invariant action we have just written is in terms of the determinant of the metric. It is a common convention to leave the a, b indices dangling in writing this action and its generalizations, and we shall adopt that somewhat loose notation here. More careful authors sometimes use other symbols, like $\det^{1/2} P[G]$, where the P denotes the pullback, and G means the metric, now properly thought of as a matrix whose determinant is to be taken. Here, the meaning of what we write using the looser notation should always be clear from the context.

Of course, this cannot be the whole story, and indeed it is clear that we shall need a richer action, since the rules of T–duality action on the background fields mean that T–dualising to a D$(p+1)$– or D$(p-1)$–brane's action will introduce a dependence on $B_{\mu\nu}$, since it mixes with components of the metric. Furthermore, there will be mixing with components of a world–volume gauge field, since some of kinetic terms for the transverse fields, $\partial_a X^m$, $m = p+1, \ldots, D-1$, implicit in the action (253), will become derivatives of gauge fields, $2\pi\alpha' \partial_a A_m$ according to the rules of T–duality for open strings deduced in the previous section. We shall construct the full T–duality respecting action in the next subsection. Before we do that, let us consider what we can learn about the tension of the D–brane from this simple action, and what we learned about the transformation of the dilaton.

The tension of the brane controls its response to outside influences which try to make it change its shape, absorb energy, *etc.*, just as we saw for the tension of a string. We shall compute the actual value of the tension in section 6. Here, we are going to uncover a useful recursion relation relating the tensions of different D–branes, which follows from T–duality[72, 32]. The mass of a Dp–brane wrapped around a p–torus T^p is

$$T_p e^{-\Phi} \prod_{i=1}^{p} (2\pi R_i) \ . \tag{254}$$

T–dualising on the single direction X^p and recalling the transformation (246) of the dilaton, we can rewrite the mass (254) in the dual variables:

$$T_p (2\pi\sqrt{\alpha'}) e^{-\Phi'} \prod_{i=1}^{p-1} (2\pi R_i) \ = \ T_{p-1} e^{-\Phi'} \prod_{i=1}^{p-1} (2\pi R_i) \ . \tag{255}$$

Hence,

$$T_p = T_{p-1}/2\pi\sqrt{\alpha'} \qquad \Rightarrow \qquad T_p = T_{p'} (2\pi\sqrt{\alpha'})^{p'-p} \ . \tag{256}$$

Where we performed the duality recursively to deduce the general relation.

The next step is to take into account new couplings for the embedding coordinates/fields which result of other background spacetime fields like the antisymmetric tensor $B_{\mu\nu}$. This again appears as an induced tensor B_{ab} on the worldvolume, *via* a formula like (253).

It is important to notice that that there is a restriction due to spacetime gauge symmetry on the precise combination of B_{ab} and A^a which can appear in the action. The combination $B_{ab} + 2\pi\alpha' F_{ab}$ can be understood as follows. In the world–sheet sigma model action of the string, we have the usual closed string term (93) for B and the boundary action (98) for A. So the fields appear in the combination:

$$\frac{1}{2\pi\alpha'} \int_{\mathcal{M}} B + \int_{\partial\mathcal{M}} A \ . \tag{257}$$

We have written everything in terms of differential forms, since B and A are antisymmetric. For example $\int A \equiv \int A_a d\xi^a$.

This action is invariant under the spacetime gauge transformation $\delta A = d\lambda$. However, the spacetime gauge transformation $\delta B = d\zeta$ will give a surface term which must be canceled with the following gauge transformation of A: $\delta A = -\zeta/2\pi\alpha'$. So the combination $B + 2\pi\alpha' F$, where $F = dA$ is invariant under both symmetries; This is the combination of A and B which must appear in the action in order for spacetime gauge invariance to be preserved.

5.2.1 World–Volume Actions from Tilted D–Branes

There are many ways to deduce pieces of the world–volume action. One way is to redo the computation for Weyl invariance of the complete sigma model, including the boundary terms, which will result in the $(p+1)$–dimensional equations of motion for the world–volume fields G_{ab}, B_{ab} and A_a. One can then deduce the $p+1$–dimensional world–volume action from which those equations of motion may be derived. We will comment on this below.

Another way, hinted at in the previous subsection, is to use T–duality to build the action piece by piece. For the purposes of learning more about how the branes work, and in view of the various applications we will put the branes to, this second way is perhaps more instructive.

Consider[41] a D2–brane extended in the X^1 and X^2 directions, and let there be a constant gauge field F_{12}. (We leave the other dimensions unspecified, so the brane could be larger by having extent in other directions. This will not affect our discussion.) We can choose a gauge in which $A_2 = X^1 F_{12}$. Now consider T–dualising along the x^2–direction. The relation (239) between the potential and coordinate gives

$$X'^2 = 2\pi\alpha' X^1 F_{12} \ , \tag{258}$$

This says that the resulting D1–brane is tilted at an angle[11]

$$\theta = \tan^{-1}(2\pi\alpha' F_{12}) \tag{259}$$

to the X^2–axis! This gives a geometric factor in the D1–brane world–volume action,

$$S \sim \int_{\rm D1} ds = \int dX^1 \sqrt{1+(\partial_1 X'^2)^2} = \int dX^1 \sqrt{1+(2\pi\alpha' F_{12})^2} \ . \tag{260}$$

We can always boost the D–brane to be aligned with the coordinate axes and then rotate to bring $F_{\mu\nu}$ to block-diagonal form, and in this way we can reduce the problem to a product of factors like (260) giving a determinant:

$$S \sim \int d^D X \ \det{}^{1/2}(\eta_{\mu\nu} + 2\pi\alpha' F_{\mu\nu}) \ . \tag{261}$$

This is the Born–Infeld action.[45]

In fact, this is the complete action (in a particular "static" gauge which we will discuss later) for a space–filling D25–brane in flat space, and with the dilaton and antisymmetric tensor field set to zero. In the language of section 2.13, Weyl invariance of the open string sigma model (98) amounts to the following analogue of (96) for the open string sector:

$$\beta^A_\mu = \alpha' \left(\frac{1}{1-(2\pi\alpha' F)^2} \right)^{\nu\lambda} \partial_{(\nu} F_{\lambda)\mu} = 0 \ , \tag{262}$$

these equations of motion follow from the action. In fact, in contrast to the Maxwell action written previously (97), and the closed string action (96), this action is true to all orders in α', although only for slowly varying field strengths; there are corrections from derivatives of $F_{\mu\nu}$.[35]

5.3 The Dirac–Born–Infeld Action

We can uncover a lot of the rest of the action by simply dimensionally reducing. Starting with (261), where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ as usual (we will treat the non–Abelian case later) let us assume that $D-p-1$ spatial coordinates are very small circles, small enough that we can neglect all derivatives with respect to

[11] The reader concerned about achieving irrational angles and hence densely filling the (x^1, x^2) torus should suspend disbelief until section 8. There, when we work in the fully consistent quantum theory of superstrings, it will be seen that the fluxes are quantized in just the right units to make this sensible.

those directions, labelled X^m, $m = p+1, \ldots, D-1$. (The uncompactified coordinates will be labelled X^a, $a = 0, \ldots, p$.) In this case, the matrix whose determinant appears in (261) is:

$$\begin{pmatrix} N & -A^T \\ A & M \end{pmatrix} , \tag{263}$$

where

$$N = \eta_{ab} + 2\pi\alpha' F_{ab} \ ; \qquad M = \delta_{mn} \ ; \qquad A = 2\pi\alpha'\partial_a A_m \ . \tag{264}$$

Using the fact that its determinant can be written as $|M||N + A^T M^{-1} A|$, our action becomes[59]

$$S \sim -\int d^{p+1}X \ \det{}^{1/2}(\eta_{ab} + \partial_a X^m \partial_b X_m + 2\pi\alpha' F_{ab}) \ , \tag{265}$$

up to a numerical factor (coming from the volume of the torus we reduced on. Once again, we used the T–duality rules (239) to replace the gauge fields in the T–dual directions by coordinates: $2\pi\alpha' A_m = X^m$.

This is (nearly) the action for a Dp–brane and we have uncovered how to write the action for the collective coordinates X^m representing the fluctuations of the brane transverse to the world–volume. There now remains only the issue of putting in the case of non–trivial metric, $B_{\mu\nu}$ and dilaton. This is easy to guess given that which we have encountered already:

$$S_p = -T_p \int d^{p+1}\xi \, e^{-\Phi} \det{}^{1/2}(G_{ab} + B_{ab} + 2\pi\alpha' F_{ab}) \ . \tag{266}$$

This is the Dirac–Born–Infeld Lagrangian, for arbitrary background fields. The factor of the dilaton is again a result of the fact that all of this physics arises at open string tree level, hence the factor g_s^{-1}, and the B_{ab} is in the right place because of spacetime gauge invariance. T_p and G_{ab} are in the right place to match onto the discussion we had when we computed the tension. Instead of using T–duality, we could have also deduced this action by a generalisation of the sigma model methods described earlier, and in fact this is how it was first derived in this context.[37]

We have re–introduced independent coordinates ξ^a on the world–volume. Note that the actions given in equations (260) and (265) were written using a choice where we aligned the world–volume with the first $p+1$ spacetime coordinates as $\xi^a = X^a$, leaving the $D-p-1$ transverse coordinates called X^m. We can always do this using world–volume and spacetime diffeomorphism invariance. This choice is called the "static gauge", and we shall use it quite a bit in these notes. Writing this out (for vanishing dilaton) using the formula (253) for the induced metric, for the case of $G_{\mu\nu} = \eta_{\mu\nu}$ we see that we get the action (265).

5.4 The Action of T–Duality

It is amusing[44, 54] to note that our full action obeys (as it should) the rules of T–duality which we already wrote down for our background fields. The action for the Dp–brane is built out of the determinant $|E_{ab} + 2\pi\alpha' F_{ab}|$, where the $(a, b = 0, \ldots p)$ indices on E_{ab} mean that we have performed the pullback of $E_{\mu\nu}$ (defined in (250)) to the world–volume. This matrix becomes, if we T–dualise on n directions labelled by X^i and use the rules we wrote in (251):

$$\left| \begin{matrix} E_{ab} - E_{ai}E^{ij}E_{jb} + 2\pi\alpha' F_{ab} & -E_{ak}E^{kj} - \partial_a X^i \\ E^{ik}E_{kb} + \partial_b X^i & E^{ij} \end{matrix} \right| , \tag{267}$$

which has determinant $|E^{ij}||E_{ab} + 2\pi\alpha' F_{ab}|$. In forming the square root, we get again the determinant needed for the definition of a T–dual DBI action, as the extra determinant $|E^{ij}|$ precisely cancels the determinant factor picked up by the dilaton under T–duality. (Recall, E^{ij} is the inverse of E_{ij}.)

Furthermore, the tension $T_{p'}$ comes out correctly, because there is a factor of $\Pi_i^n(2\pi R_i)$ from integrating over the torus directions, and a factor $\Pi_i^n(R_i/\sqrt{\alpha'})$ from converting the factor $e^{-\Phi}$, (see (246)), which fits nicely with the recursion formula (256) relating T_p and $T_{p'}$.

The above was done as though the directions on which we dualised were all Neumann or all Dirichlet. Clearly, we can also extrapolate to the more general case.

5.5 Non–Abelian Extensions

For N D–branes the story is more complicated. The various fields on the brane representing the collective motions, A_a and X^m, become matrices valued in the adjoint. In the Abelian case, the various spacetime background fields (here denoted F_μ for the sake of argument) which can appear on the world–volume typically depend on the transverse coordinates X^m in some (possibly) non–trivial way. In the non–Abelian case, with N D–branes, the transverse coordinates are really $N \times N$ matrices, $2\pi\alpha'\Phi^m$, since they are T–dual to non–Abelian gauge fields as we learned in previous sections, and so inherit the behaviour of gauge fields (see equation (239)). We write them as $\Phi^m = X^m/(2\pi\alpha')$. So not only should the background fields F_μ depend on the Abelian part, but they ought to possibly depend (implicitly or explicitly) on the full non–Abelian part as $F(\Phi)_\mu$ in the action.

Furthermore, in (266) we have used the partial derivatives $\partial_a X^\mu$ to pull back spacetime indices μ to the world–volume indices, a, *e.g.,* $F_a = F_\mu \partial_a X^\mu$, and so on. To make this gauge covariant in the non–Abelian case, we should pull back with the covariant derivative: $F_a = F_\mu \mathcal{D}_a X^\mu = F_\mu(\partial_a X^\mu + [A_a, X^\mu])$.

With the introduction of non–Abelian quantities in all of these places, we need to consider just how to perform a trace, in order to get a gauge invariant quantity to use for the action. Starting with the fully Neumann case (261), a first guess is that things generalise by performing a trace (in the fundamental of $U(N)$) of the square rooted expression. The meaning of the Tr needs to be stated, It is proposed that is means the "symmetric" trace, denoted "STr" which is to say that one symmetrises over gauge indices, consequently ignoring all commutators of the field strengths encountered while expanding the action. [46] (This suggestion is consistent with various studies of scattering amplitudes and also the BPS nature of various non–Abelian soliton solutions. There is still apparently some ambiguity in the definition which results in problems beyond fifth order in the field strength. [47])

Once we have this action, we can then again use T–duality to deduce the form for the lower dimensional, Dp–brane actions. The point is that we can reproduce the steps of the previous analysis, but keeping commutator terms such as $[A_a, \Phi^m]$ and $[\Phi^m, \Phi^n]$. We will not reproduce those steps here, as they are similar in spirit to that which we have already done (for a complete discussion, the reader is invited to consult some of the literature[48].) The resulting action is:

$$\begin{aligned} S_p &= -T_p \int d^{p+1}\xi \, e^{-\Phi}\mathcal{L} \ , \quad \text{where} \\ \mathcal{L} &= \mathrm{STr}\left\{ \det{}^{1/2}[E_{ab} + E_{ai}(Q^{-1}-\delta)^{ij}E_{jb} + 2\pi\alpha' F_{ab}] \det{}^{1/2}[Q^i{}_j] \right\} \ , \end{aligned} \tag{268}$$

where $Q^i{}_j = \delta^i{}_j + i2\pi\alpha'[\Phi^i, \Phi^k]E_{kj}$, and we have raised indices with E^{ij}.

5.6 D–Branes and Gauge Theory

In fact, we are now in a position to compute the constant C in equation (97), by considering N D25–branes, which is the same as an ordinary (fully Neumann) N–valued Chan–Paton factor. Expanding the D25–brane Lagrangian (261) to second order in the gauge field, we get

$$-\frac{T_{25}}{4}(2\pi\alpha')^2 e^{-\Phi}\mathrm{Tr}F_{\mu\nu}F^{\mu\nu}, \tag{269}$$

with the trace in the fundamental representation of $U(N)$. This gives the precise numerical relation between the open and closed string couplings.

Actually, with Dirichlet and Neumann directions, performing the same expansion, and in addition noting that

$$\det[Q^i{}_j] = 1 - \frac{(2\pi\alpha')^2}{4}[\Phi^i, \Phi^j][\Phi^i, \Phi^j] + \cdots \ , \tag{270}$$

one can write the leading order action (268) as

$$S_p = -\frac{T_p(2\pi\alpha')^2}{4}\int d^{p+1}\xi \, e^{-\Phi}\mathrm{Tr}\left[F_{ab}F^{ab} + 2\mathcal{D}_a\Phi^i\mathcal{D}_a\Phi^i + [\Phi^i, \Phi^j]^2\right]\Big\} \ . \tag{271}$$

This is the dimensional reduction of the D–dimensional Yang–Mills term, displaying the non–trivial commutator for the adjoint scalars. This is an important term in many modern applications. Note that the $(p+1)$–dimensional Yang–Mills coupling for the theory on the branes is

$$g^2_{\rm YM,\it p} = g_s T_p^{-1}(2\pi\alpha')^{-2} \ . \tag{272}$$

This is worth noting[66]. With the superstring value of T_p which we will compute later, it is used in many applications to give the correct relation between gauge theory couplings and string quantities.

5.7 BPS Lumps on the World–volume

We can of course treat the Dirac–Born–Infeld action as an interesting theory in its own right, and seek for interesting solutions of it. These solutions will have both a $(p+1)$–dimensional interpretation and a D–dimensional one.

We shall not dwell on this in great detail, but include a brief discussion here to illustrate an important point, and refer to the literature for more complete discussions.[58] More details will appear when we get to the supersymmetric case. One can derive an expression for the energy density contained in the fields on the world–volume:

$$\mathcal{E}^2 = E^aE^bF_{ca}F_{cb} + E^aE^bG_{ab} + \det(G+2\pi\alpha' F) \ , \tag{273}$$

where here the matrix F_{ab} contains only the magnetic components (*i.e.* no time derivatives) and E^a are the electric components, subject to the Gauss Law constraint $\vec{\nabla}\cdot\vec{E}=0$. Also, as before

$$G_{ab} = \eta_{ab} + \partial_a X^m \partial_b X^m \ , \qquad m = p+1,\ldots,D-1 \ . \tag{274}$$

Let us consider the case where we have no magnetic components and only one of the transverse fields, say X^{25}, switched on. In this case, we have

$$\mathcal{E}^2 = (1\pm\vec{E}\cdot\vec{\nabla}X^{25})^2 + (\vec{E}\mp\vec{\nabla}X^{25})^2 \ , \tag{275}$$

and so we see that we have the Bogomol'nyi condition

$$\mathcal{E} \geq |\vec{E}\cdot\vec{\nabla}X^{25}| + 1 \ . \tag{276}$$

This condition is saturated if $\vec{E}=\pm\vec{\nabla}X^{25}$. In such a case, we have

$$\nabla^2 X^{25} = 0 \qquad \Rightarrow \qquad X^{25} = \frac{c_p}{r^{p-2}} \ , \tag{277}$$

an harmonic solution, where c_p is a constant to be determined.

The total energy (beyond that of the brane itself) is, integrating over the world–volume:

$$\begin{aligned} E_{\rm tot} &= \lim_{\epsilon\to\infty} T_p \int_\epsilon^\infty r^{p-1}dr d\Omega_{p-1}(\vec{\nabla}X^{25})^2 = \lim_{\epsilon\to\infty} T_p \frac{c_p^2(p-2)\Omega_{p-1}}{\epsilon^{p-2}} \\ &= \lim_{\epsilon\to\infty} T_p c_p (p-2)\Omega_{p-1} X^{25}(\epsilon) \ , \end{aligned} \tag{278}$$

where Ω_{p-1} is the volume of the sphere S^{p-1} surrounding our point charge source, and we have cut off the divergent integral by integrating down to $r=\epsilon$. (We will save the case of $p=1$ for later [131, 63].) Now we can choose[1] a value of the electric flux such that we get $(p-2)c_p\Omega_{p-1}T_p = (2\pi\alpha')^{-1}$. Putting this into our equation for the total energy, we see that the (divergent) energy of our configuration is:

$$E_{\rm tot} = \frac{1}{2\pi\alpha'} X^{25}(\epsilon) \ . \tag{279}$$

[1]In the supersymmetric case, this has a physical meaning, since overall consistency of the D–brane charges set a minimum electric flux. Here, it is a little more arbitrary, and so we choose a value by hand to make the point we wish to illustrate.

What does this mean? Well, recall that $X^{25}(\xi)$ gives the transverse position of the brane in the X^{25} direction. So we see that the brane has grown a semi–infinite spike at $r = 0$, and the base of this spike is our point charge. The interpretation of the divergent energy is simply the (infinite) length of the spike multiplied by a mass per unit length. But this mass per unit length is precisely the fundamental string tension $T = (2\pi\alpha')^{-1}$! In other words, the spike solution is the fundamental string stretched perpendicular to the brane and ending on it, forming a point electric charge, known as a "BIon". See figure 24(a). In fact, a general BIon includes the non–linear corrections to this spike solution, which we have neglected here, having only written the linearized solution.

It is a worthwhile computation to show that if test source with the same charges is placed on the brane, there is no force of attraction or repulsion between it and the source just constructed, as would happen with pure Maxwell charges. This is because our sources have in addition to electric charge, some scalar (X^{25}) charge, which can also be attractive or repulsive. In fact, the scalar charges are such that the force due to electromagnetic charges is canceled by the force of the scalar charge, another characteristic property of these solutions, which are said to be "Bogomol'nyi–Prasad–Sommerfield" (BPS)–saturated.[64, 65] We shall encounter solutions with this sort of behaviour a number of times in what is to follow.

Because of this property, the solution is easily generalized to include any number of BIons, at arbitrary positions, with positive and negative charges. The two choices of charge simply represents strings either leaving from, or arriving on the brane. See figure 24(b).

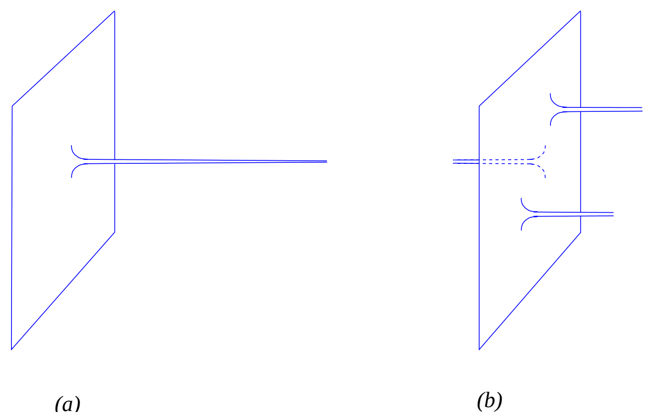


Figure 24: The D dimensional interpretation of the BIon solution. (a) It is an infinitely long spike representing a fundamental string ending on the D–brane. (b) BIons are BPS and therefore can be added together at no cost to make a multi–BIon solution.

6 D–Brane Tension and Boundary States

We have already stated that since the D–brane is a dynamical object, and couples to gravity, it should have a mass per unit volume. This tension will govern the strength of its response to outside influences which try to make it change its shape, absorb energy, *etc.* We have already computed a recursion relation (256) for the tension, which follows from the underlying T–duality which we used to discover D–branes in the first place.

In this section we shall see in detail just how to compute the value of the tension for the D–brane, and also for the orientifold plane. While the numbers that we will get will not (at face value) be as useful as the analogous quantities for the supersymmetric case, the structure of the computation is extremely important. The computation puts together many of the things that we have learned so far in a very elegant manner which lies at the heart of much of what will follow in more advanced sections.

Along the way, we will see that D–branes can be constructed and studied in an alternative formalism known as the "Boundary State" formalism, which is essentially conformal field theory with certain sorts of boundaries included[36]. For much of what we will do, it will be a clearly equivalent way of formulating things which we also say (or have already said) based on the spacetime picture of D–branes. However, it should be noted that it is much more than just a rephrasing since it can be used to consistently formulate D–branes in many more complicated situations, even when a clear spacetime picture is not available. The method becomes even more useful in the supersymmetric situation, since it provides a natural way of constructing stable D–brane vacua of the superstring theories which do not preserve any supersymmetries, a useful starting point for exploring dualities and other non–perturbative physics in dynamical regimes which ultimately may have relevance to observable physics.

6.1 The D–brane Tension

6.1.1 An Open String Partition Function

Let us now compute the D–brane tension T_p. As noted previously, it is proportional to g_s^{-1}. We can in principle calculate it from the gravitational coupling to the D–brane, given by the disk with a graviton vertex operator in the interior. However, it is much easier to obtain the absolute normalization in the following manner. Consider two parallel Dp–branes at positions $X'^{\mu} = 0$ and $X'^{\mu} = Y^{\mu}$. These two objects can feel each other's presence by exchanging closed strings as shown in figure 25. This string graph is an annulus,

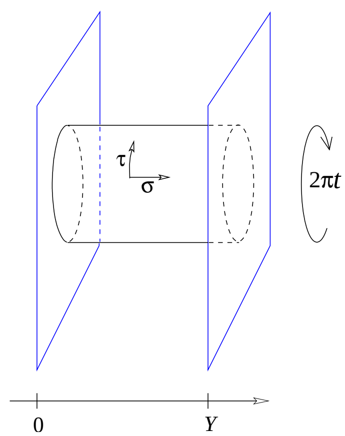


Figure 25: Exchange of a closed string between two D–branes. This is equivalent to a vacuum loop of an open string with one end on each D–brane.

with no vertex operators. It is therefore as easily calculated as our closed string one loop amplitudes done earlier in section 3.

In fact, this is rather like an open string partition function, since the amplitude can be thought of as an open string going in a loop. We should sum over everything that goes around in the loop. Once we have computed this, we will then change our picture of it as an open string one–loop amplitude, and look at it as a closed string amplitude for propagation between one D–brane and another. We can take a low energy limit of the result to focus on the massless closed string states which are being exchanged. Extracting the poles from graviton and dilaton exchange (we shall see that the anti–symmetric tensor does not couple in this limit) then give the coupling T_p of closed string states to the D–brane.

Let us parametrised the string world–sheet as $(\sigma^2 = \tau, \sigma^1 = \sigma)$ where now τ is periodic and runs from 0 to $2\pi t$, and σ runs (as usual) from 0 to π. This vacuum graph (a cylinder) has the single modulus t, running from 0 to ∞. If we slice horizontally, so that $\sigma^2 = \tau$ is world–sheet time, we get an open string going in a loop. If we instead slice vertically, so that σ is time, we see a single closed string propagating in the tree channel.

Notice that the world–line of the open string boundary can be regarded as a vertex connecting the vacuum to the single closed string, *i.e.,* a one–point closed string vertex, which is a useful picture in a "boundary state" formalism, which we will develop a bit further shortly. This diagram will occur explicitly again in many places in our treatment of this subject. String theory produces many examples where one–loop gauge/field theory results (open strings) are related to tree level geometrical/gravity results. This is all organized by diagrams of this form, and is the basis of much of the gauge theory/geometry correspondences to be discussed.

Let us consider the limit $t \to 0$ of the loop amplitude. This is the ultra–violet limit for the open string channel, since the circle of the loop is small. However, this limit is correctly interpreted as an *infrared* limit of the closed string. (This is one of the earliest "dualities" of string theory, discussed even before it was known to be a theory of strings.) Time–slicing vertically shows that the $t \to 0$ limit is dominated by the lowest lying modes in the closed string spectrum. This all fits with the idea that there are no "ultraviolet limits" of the moduli space which could give rise to high energy divergences. They can always be related to amplitudes which have a handle pinching off. This physics is controlled by the lightest states, or the long distance physics. (This relationship is responsible for the various "UV/IR" relations which are a popular feature of current research.)

One–loop vacuum amplitudes are given by the Coleman–Weinberg [38, 39] formula, which can be thought of as the sum of the zero point energies of all the modes:

$$\mathcal{A} = V_{p+1} \int \frac{d^{p+1}k}{(2\pi)^{p+1}} \int_0^\infty \frac{dt}{2t} \sum_I e^{-2\pi\alpha' t(k^2+M_I^2)}. \tag{280}$$

Here the sum I is over the physical spectrum of the string, *i.e.,* the transverse spectrum, and the momentum k is in the $p+1$ extended directions of the D–brane world–sheet.

The mass spectrum is given by a familiar formula

$$M^2 = \frac{1}{\alpha'}\left(\sum_{n=1}^{\infty} \alpha^i_{-n}\alpha^i_n - 1\right) + \frac{Y\cdot Y}{4\pi^2\alpha'^2} \tag{281}$$

where Y^m is the separation of the D–branes. The sums over the oscillator modes work just like the computations we did before, giving

$$\mathcal{A} = 2V_{p+1}\int_0^\infty \frac{dt}{2t}(8\pi^2\alpha' t)^{-\frac{(p+1)}{2}} e^{-Y\cdot Y t/2\pi\alpha'} f_1(q)^{-24} \ . \tag{282}$$

Here $q = e^{-2\pi t}$, and the overall factor of 2 is from exchanging the two ends of the string.

Compare our open string appearance of $f_1(q)$, for $q = e^{-2\pi t}$ with the expressions for $f_1(q)$, ($q = e^{2\pi\tau}$) defined in our closed string discussion in (215). Here the argument is real. The translation between definitions is done by setting $t = -\mathrm{Im}\,\tau$. From the modular transformations (217), we can deduce some useful asymptotics. While the asymptotics as $t \to \infty$ are obvious, we can get the $t \to 0$ asymptotics using (217):

$$\begin{aligned} f_1(e^{-2\pi/s}) &= \sqrt{s}\, f_1(e^{-2\pi s}) \ , \ f_3(e^{-2\pi/s}) = f_3(e^{-2\pi s}) \ , \\ f_2(e^{-2\pi/s}) &= f_4(e^{-2\pi s}) \ . \end{aligned}$$

In the present case, (using the asymptotics of the previous paragraph)

$$\mathcal{A} = 2V_{p+1}\int_0^\infty \frac{dt}{2t}(8\pi^2\alpha' t)^{-\frac{(p+1)}{2}} e^{-Y\cdot Y t/2\pi\alpha'} t^{12}\left(e^{2\pi/t} + 24 + \ldots\right). \tag{283}$$

The leading divergence is from the tachyon and is the usual bosonic string artifact not relevant to this discussion. The massless pole, from the second term, is

$$\begin{aligned} \mathcal{A}_{\rm massless} &\sim V_{p+1}\frac{24}{2^{12}}(4\pi^2\alpha')^{11-p}\pi^{(p-23)/2}\Gamma((23-p)/2)|Y|^{p-23} \\ &= V_{p+1}\frac{24\pi}{2^{10}}(4\pi^2\alpha')^{11-p}G_{25-p}(Y) \end{aligned} \tag{284}$$

where $G_d(Y)$ is the massless scalar Green's function in d dimensions:

$$G_d(Y) = \frac{\pi^{d/2}}{4}\Gamma\left(\frac{d}{2}-1\right)\frac{1}{Y^{d-2}} \ . \tag{285}$$

Here, $d = 25 - p$, the dimension of the space transverse to the brane.

6.2 A Background Field Computation

We must do a field theory calculation to work out the amplitude for the exchange of the graviton and dilaton between a pair of D–branes. Our result can the be compared to the low energy string result above to extract the value of the tension. We need propagators and couplings as per the usual field theory computation. The propagator is from the bulk action (96) and the couplings are from the D–brane action (266), but we must massage them a bit in order to find them.

In fact, we should work in the Einstein frame, since that is the appropriate frame in which to discuss mass and energy, because the dilaton and graviton don't mix there. We do this (recall equation (99)) by sending the metric $G_{\mu\nu}$ to $\tilde{G}_{\mu\nu} = \exp(4(\Phi_0 - \Phi)/(D-2))G_{\mu\nu}$, which gives the metric in equation (101). Let us also do this in the Dirac–Born–Infeld action (266), with the result:

$$S_p^E = -\tau_p \int d^{p+1}\xi\, e^{-\tilde{\Phi}} \det{}^{1/2}(e^{\frac{4\tilde{\Phi}}{D-2}}\tilde{G}_{ab} + B_{ab} + 2\pi\alpha' F_{ab}) \ , \tag{286}$$

where $\tilde{\Phi} = \Phi - \Phi_0$ and $\tau_p = T_p e^{-\Phi_0}$ is the physical tension of the brane; it is set by the background value, Φ_0, of the dilaton.

The next step is to linearize about a flat background, in order to extract the propagator and the vertices for our field theory. We simply write the metric as $G_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}(X)$, and expand up to second order in $h_{\mu\nu}$. Also, if we do this with the action (286) as well, we see that the antisymmetric field $B_{ab} + 2\pi\alpha' F_{ab}$ do not contribute at this order, and so we will drop them in what follows[2].

Pick the gauge:

$$F_\mu \equiv \eta^{\rho\sigma}(\partial_\rho h_{\sigma\mu} - \frac{1}{2}\partial_\mu h_{\rho\sigma}) = 0 \ , \tag{287}$$

and introduce the gauge choice into the Lagrangian *via* the addition of a gauge fixing term:

$$\mathcal{L}_{\rm fix} = -\frac{\eta^{\mu\nu}}{4\kappa^2}F_\mu F_\nu \ . \tag{288}$$

The result for the bulk action is:

$$S_{\rm bulk} = -\frac{1}{2\kappa^2}\int d^DX \left\{ \frac{1}{2}\left[\eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho} - \frac{2}{D-2}\eta^{\mu\nu}\eta^{\rho\sigma}\right] h_{\mu\nu}\partial^2 h_{\rho\sigma} + \frac{4}{D-2}\tilde{\Phi}\partial^2\tilde{\Phi} \right\} \ , \tag{289}$$

and the interaction terms from the Dirac–Born–Infeld action are:

$$S_{\rm brane} = -\tau_p \int d^{p+1}\xi \left(\left(\frac{2p - D + 4}{D-2}\right)\tilde{\Phi} - \frac{1}{2}h_{aa} \right) \ , \tag{290}$$

where the trace on the metric was in the $(p+1)$–dimensional world–volume of the Dp–brane.

Now it is easy to work out the momentum space propagators for the graviton and the dilaton:

$$\begin{aligned} < h_{\mu\nu}h_{\rho\sigma} > \ &= \ -\frac{2i\kappa^2}{k^2}\left[\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho} - \frac{2}{D-2}\eta_{\mu\nu}\eta_{\rho\sigma}\right] \ ; \\ < \tilde{\Phi}\tilde{\Phi} > \ &= \ -\frac{i\kappa^2(D-2)}{4k^2} \ , \end{aligned} \tag{291}$$

[2]This fits with the intuition that the D–brane should not be a source for the antisymmetric tensor field. The source for it is the fundamental closed string itself. We shall come back to this point later.

for momentum k.

The reader might recognize the graviton propagator as the generalisation of the four dimensional case. If the reader has not encountered it before, the resulting form should be thought of as entirely consistent with gauge invariance for a massless spin two particle.

All we need to do is compute two tree level Feynman diagrams, one for exchange of the dilaton and one for the exchange of the graviton, and add the result. The vertices are given in action (290). The result is (returning to position space):

$$\begin{aligned}\mathcal{A}_{\rm massless} &= V_{p+1}T_p^2\kappa_0^2G_{25-p}(Y)\left\{\frac{D-2}{4}\left(\frac{2p-D+4}{D-2}\right)^2+\frac{1}{2}\left[2(p+1)-\frac{2}{D-2}(p+1)^2\right]\right\}\\ &= \frac{D-2}{4}V_{p+1}T_p^2\kappa_0^2G_{25-p}(Y)\end{aligned} \qquad (292)$$

and so after comparing to our result from the string theory computation (284) we have:

$$T_p=\frac{\sqrt{\pi}}{16\kappa_0}(4\pi^2\alpha')^{(11-p)/2}\ . \qquad (293)$$

This agrees rather nicely with the recursion relation (256). We can also write it in terms of the physical value of the D–brane tension, which includes a factor of the string coupling $g_s=e^{\Phi_0}$,

$$\tau_p=\frac{\sqrt{\pi}}{16\kappa}(4\pi^2\alpha')^{(11-p)/2} \qquad (294)$$

where $\kappa=\kappa_0 g_s$, and we shall use τ_p this to denote the tension when we include the string coupling henceforth, and reserve T for situations where the string coupling is included in the background field $e^{-\Phi}$. (This will be less confusing than it sounds, since it will always be clear from the context which we mean.)

As promised, the tension τ_p of a Dp–brane is of order g_s^{-1}, following from the fact that the diagram connecting the brane to the closed string sector is a disc diagram. An immediate consequence of this is that they will produce non–perturbative effects of order $\exp(-1/g_s)$ in string theory, since their action is of the same order as their mass. This is consistent with anticipated behaviour from earlier studies of toy non–perturbative string theories[95], the very $D\leq 1$ string theories already mentioned in section 3.6. The precise numbers in formula (293) will not concern us much beyond these sections, since we will derive a new one for the superstring case later.

6.3 The Orientifold Tension

The O–plane, like the D–brane, couples to the dilaton and metric. The most direct amplitude to use to compute the tension is the same as in the previous section, but with $\mathbb{RP}^2$ in place of the disk; *i.e.,* a crosscap replaces the boundary loop. The orientifold identifies X^m with $-X^m$ at the opposite point on the crosscap, so the crosscap is localized near one of the orientifold fixed planes. However, once again, it is easier to organize the computation in terms of a one–loop diagram, and then extract the parts we need.

6.3.1 Another Open String Partition Function

To calculate this *via* vacuum graphs, the cylinder has one or both of its boundary loops replaced by crosscaps. This gives the Möbius strip and Klein bottle, respectively. To understand this, consider figure 26, which shows two copies of the fundamental region for the Möbius strip.

The lower half is identified with the reflection of the upper, and the edges $\sigma^1=0,\pi$ are boundaries. Taking the lower half as the fundamental region gives the familiar representation of the Möbius strip as a strip of length $2\pi t$, with ends twisted and glued. Taking instead the left half of the figure, the line $\sigma^1=0$ is a boundary loop while the line $\sigma^1=\pi/2$ is identified with itself under a shift $\sigma^2\to\sigma^2+2\pi t$ plus reflection of σ^1: it is a crosscap. The same construction applies to the Klein bottle, with the right and left edges

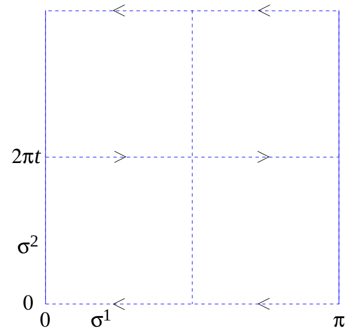


Figure 26: Two copies of the fundamental region for the Möbius strip.

now identified. Another way to think of the Möbius strip amplitude we are going to compute here is as representing the exchange of a closed string between a D—brane and its mirror image, as shown in figure 27. The identification with a twist is performed on the two D–branes, turning the cylinder into a Möbius strip. The Möbius strip is given by the vacuum amplitude

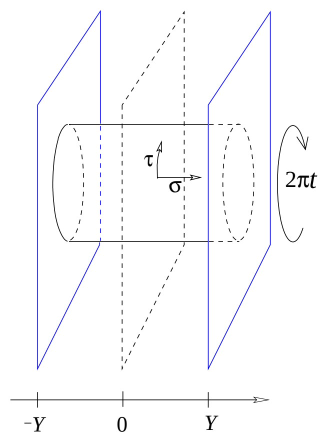


Figure 27: The Möbius strip as the exchange of closed strings between a brane and its mirror image. The dotted plane is the orientifold plane.

$$\mathcal{A}_{\rm M} = V_{p+1}\int \frac{d^{p+1}k}{(2\pi)^{p+1}}\int_0^\infty \frac{dt}{2t}\sum_i \frac{\Omega_i}{2} e^{-2\pi\alpha' t(p^2+M_I^2)}, \tag{295}$$

where Ω_I is the Ω eigenvalue of state i. The oscillator contribution to Ω_I is $(-1)^n$ from equation (84). Actually, in the directions orthogonal to the brane and orientifold there are two additional signs in Ω_I which cancel. One is from the fact that world–sheet parity contributes an extra minus sign in the directions with Dirichlet boundary conditions (this is evident from the mode expansions we shall list later, in equations (425)). The other is from the fact that spacetime reflection produces an additional sign.

For the $SO(N)$ open string the Chan–Paton factors have $\frac{1}{2}N(N+1)$ even states and $\frac{1}{2}N(N-1)$ odd for a net of $+N$. For $USp(N)$ these numbers are reversed for a net of $-N$. Focus on a D–brane and its image, which correspondingly contribute ± 2. The diagonal elements, which contribute to the trace, are those where one end is on the D–brane and one on its image. The total separation is then $Y^m = 2X^m$. Then,

$$\mathcal{A}_{\rm M} \;=\; \pm V_{p+1}\int_0^\infty \frac{dt}{2t}(8\pi^2\alpha' t)^{-\frac{(p+1)}{2}} e^{-2\vec{Y}\cdot\vec{Y}t/\pi\alpha'}\left[q^{-2}\prod_{k=1}^\infty (1+q^{4k-2})^{-24}(1-q^{4k})^{-24}\right]$$

The factor in braces is

$$f_3(q^2)^{-24} f_1(q^2)^{-24} = (2t)^{12} f_3(e^{-\pi/2t})^{-24} f_1(e^{-\pi/2t})^{-24} = (2t)^{12}\left(e^{\pi/2t} - 24 + \dots\right) . \quad (296)$$

One thus finds a pole

$$\mp 2^{p-12} V_{p+1} \frac{3\pi}{2^6} (4\pi^2\alpha')^{11-p} G_{25-p}(Y) . \quad (297)$$

This is to be compared with the field theory result

$$\frac{D-2}{2} V_{p+1} T_p T'_p \kappa_0^2 G_{25-p}(Y) , \quad (298)$$

where T'_p is the O–plane tension. A factor of 2 as compared to the earlier field theory result of equation (292) comes because the spacetime boundary forces all the flux in one direction. Thus the O–plane and D–brane tensions are related

$$\tau'_p = \mp 2^{p-13} \tau_p . \quad (299)$$

A similar calculation with the Klein bottle gives a result proportional to $\tau_p'^2$.

Noting that there are 2^{25-p} O–planes (recall that one doubles the number every time another new direction is T–dualised, starting with a single D25–brane), the total charge of an O–plane source must be $\mp 2^{12}\tau_p$. Now, by Gauss' law, the total source must vanish because the volume of the torus T^p on which we are working is finite and the flux must end on sinks and sources.

So we conclude that there are $2^{(D-2)/2} = 2^{12}$ D–branes (times two for the images) and that the gauge group is $SO(2^{13}) = SO(2^{D/2})$. [40] For this group the "tadpoles" associated with the dilaton and graviton, representing violations of the field equations, cancel at order g_s^{-1}. This has no special significance in the bosonic string, as the one loop g_s^0 tadpoles are nonzero and imaginary due to the tachyon instability, but similar considerations will give a restriction on anomaly free Chan–Paton gauge groups in the superstring.

6.4 The Boundary State Formalism

The asymptotics (283) can be interpreted in terms of a sum over closed string states exchanged between the two D–branes. One can write the cylinder path integral in a Hilbert space formalism treating σ_1 rather than σ_2 as time. It then has the form

$$\langle B| e^{-(L_0+\tilde{L}_0)\pi/t} |B\rangle \quad (300)$$

where the *"boundary state"* $|B\rangle$ is the closed string state created by the boundary loop.

Let us unpack this formalism a little, seeing where this all comes from. Recall that a Dp–brane is specified by the following open string boundary conditions:

$$\partial_\sigma X^\mu|_{\sigma=0,\pi} = 0 , \qquad \mu = 0, \dots, p ;$$
$$X^m|_{\sigma=0,\pi} = Y^m , \qquad m = p+1, \dots, D-1 . \quad (301)$$

Now we have to reinterpret this as a closed string statement. This involves exchanging τ and σ. So we write, focusing on the initial time:

$$\partial_\tau X^\mu|_{\tau=0} = 0 , \qquad \mu = 0, \dots, p ;$$
$$X^m|_{\tau=0} = Y^m , \qquad m = p+1, \dots, D-1 . \quad (302)$$

Recall that in the quantum theory we pass to an operator formalism, and so the conditions above should be written as an operator statement, where we are operating on some state in the Hilbert space. This defines for us then the boundary state $|B>$:

$$\partial_\tau X^\mu|_{\tau=0} |B> = 0 , \qquad \mu = 0, \dots, p ;$$
$$(X^m|_{\tau=0} - Y^m)|B> = 0 , \qquad m = p+1, \dots, D-1 . \quad (303)$$

As with everything we did in section 2, we can convert our equations above into a statement about the modes:

$$\begin{aligned}
(\alpha_n^\mu + \tilde{\alpha}_{-n}^\mu)|B> &= 0\ , \qquad \mu = 0, \ldots, p\ ; \\
(\alpha_n^m - \tilde{\alpha}_{-n}^m)|B> &= 0\ , \qquad m = p+1, \ldots, D-1\ ; \\
p^\mu|B> &= 0\ , \qquad \mu = 0, \ldots, p\ ; \\
(x^m - Y^m)|B> &= 0\ , \qquad m = p+1, \ldots, D-1\ .
\end{aligned} \tag{304}$$

As before, we either use only $D-2$ of the oscillator modes here (ignoring $\mu = 0, 1$) or we do everything covariantly and make sure that we include the ghost sector and impose BRST invariance. We shall do the former here.

The solution to the condition above can we found by analogy with the (perhaps) familiar technology of coherent states in harmonic oscillator physics.

$$|B>= \mathcal{N}_p \delta(x^m - Y^m) \left(\prod_{n=1}^{\infty} e^{-\frac{1}{n}\alpha_{-n}\cdot\mathcal{S}\cdot\tilde{\alpha}_{-n}} \right) |0>\ . \tag{305}$$

The object $\mathcal{S} = (\eta^{\mu\nu}, -\delta^{mn})$ is just shorthand for the fact that the dot product must be the usual Lorentz one in the directions parallel to the brane, but there is a minus sign for the transverse directions.

The normalization constant is determined by simply computing the closed string amplitude directly in this formalism. The closed string is prepared in a boundary state that corresponds to a D–brane, and it propagates for a while, ending in a similar boundary state at position $\vec{Y}$:

$$\mathcal{A} = <B|\Delta|B>\ , \tag{306}$$

where Δ is the closed string propagator.

How is this object constructed? Well, we might expect that it is essentially the inverse of $H_{\rm cl} = 2(L_0 + \bar{L}_0 - 2)/\alpha'$, the closed string Hamiltonian, which we can easily represent as:

$$\Delta = \frac{\alpha'}{2}\int_0^1 d\rho \rho^{L_0+\bar{L}_0-3}\ ,$$

and we must integrate the modulus $\ell = -\log\rho$ of the cylinder from 0 to ∞. We must remember, however, that a physical state $|\phi>$ is annihilated by $L_0 - \bar{L}_0$, and so we can modify our propagator so that it only propagates such states:

$$\Delta = \frac{\alpha'}{2}\int_0^1 d\rho \frac{d\phi}{2\pi}\int_0^{2\pi} \frac{d\phi}{2\pi} \rho^{L_0+\bar{L}_0-3} e^{i\phi(L_0-\bar{L}_0)}\ ,$$

which, after the change of variable to $z = \rho e^{i\phi}$, gives

$$\Delta = \frac{\alpha'}{4\pi}\int_{|z|\leq 1} \frac{dz d\bar{z}}{|z|^2} z^{L_0-1}\bar{z}^{\bar{L}_0-1}\ .$$

Computing the amplitude (306) using this definition of the propagator is a straightforward exercise, similar in spirit to what we did in the open string sector. We get geometric sums over the oscillator modes resulting from traces, and integrals over the continuous quantities. If we make the choices $|z| = e^{-\pi s}$ and $dz d\bar{z} = -\pi e^{-2\pi s} ds d\phi$ for our closed string cylinder, the result is:

$$\mathcal{A} = \mathcal{N}_p^2 V_{p+1} \frac{\alpha'\pi}{2}(2\pi\alpha')^{-\frac{25-p}{2}} \int_0^\infty \frac{ds}{s} s^{-\frac{25-p}{2}} e^{-Y\cdot Y/s 2\pi\alpha'} f_1(q)^{-24}\ . \tag{307}$$

Here $q = e^{-2\pi/s}$.

Now we can compare to the open string computation, which is the result in equation (282). We must do a modular transformation $s = -1/t$, and using the modular transformation properties given in equations (217), we find exactly the open string result if we have

$$\mathcal{N}_p = \frac{T_p}{2} \ ,$$

where T_p is the brane tension (293) computed earlier.

This is a very useful way of formulating the whole D–brane construction. In fact, the boundary state constructed above is just a special case of a sensible conformal field theory object. It is a state which can arise in the conformal field theory with boundary. Not all boundary states have such a simple spacetime interpretation as the one we made here. We see therefore that D–branes, if interpreted simply as resulting from the introduction of open string sectors into closed string theory, have a worldsheet formulation which does not necessarily always have a spacetime interpretation as its counterpart. Similar things happen in closed string conformal field theory: There are very many conformal field theories which are perfectly good string vacua, which have no spacetime interpretation in terms of an unambiguous target space geometry. It is natural that this also be true for the open string sector.

7 Supersymmetric Strings

The previous five sections' discussion of bosonic strings allowed us to uncover a great deal of the structure essential to understanding D–branes and other background solutions, in addition to the basic concepts used in discussing and working with critical string theory.

At the back of our mind was always the expectation that we would move on to include supersymmetry. Two of the main reasons are that we can remove the tachyon from the spectrum and that we will be able to use supersymmetry to endow many of our results with extra potency, since stability and non–renormalisation arguments will allow us to extrapolate beyond perturbation theory.

Let us set aside D–branes and T–duality for a while and use the ideas we discussed earlier to construct the supersymmetric string theories which we need to carry the discussion further. There are five such theories. Three of these are the *"superstrings"*, while two are the *"heterotic strings"*[3].

7.1 The Three Basic Superstring Theories

7.1.1 Open Superstrings: Type I

Let us go back to the beginning, almost. We can generalise the bosonic string action we had earlier to include fermions. In conformal gauge it is:

$$S = \frac{1}{4\pi}\int_{\mathcal{M}} d^2\sigma \left\{ \frac{1}{\alpha'}\partial X^\mu \bar{\partial} X_\mu + \psi^\mu \bar{\partial}\psi_\mu + \tilde{\psi}^\mu \partial \tilde{\psi}_\mu \right\} \tag{308}$$

where the open string world–sheet is the strip $0 < \sigma < \pi$, $-\infty < \tau < \infty$.

Recall that α' is the loop expansion parameter analogous to $\hbar$ on worldsheet. It is therefore natural for the fermions' kinetic terms to be normalised in this way. We get a modification to the energy–momentum tensor from before (which we now denote as T_B, since it is the bosonic part):

$$T_B(z) = -\frac{1}{\alpha'}\partial X^\mu \partial X_\mu - \frac{1}{2}\psi^\mu \partial \psi_\mu \ , \tag{309}$$

which is now accompanied by a fermionic energy–momentum tensor:

$$T_F(z) = i\frac{2}{\alpha'}\psi^\mu \partial X_\mu \ . \tag{310}$$

[3]A looser and probably more sensible nomenclature is to call them all "superstrings", but we'll choose the catch–all term to be the one we used for the title of this section.

This enlarges our theory somewhat, while much of the logic of what we did in the purely bosonic story survives intact here. Now, one extremely important feature which we encountered in section 4.6 is the fact that the equations of motion admit two possible boundary conditions on the world–sheet fermions consistent with Lorentz invariance. These are denoted the "Ramond" (R) and the "Neveu–Schwarz" (NS) sectors:

$$\begin{array}{lll} \text{R:} & \psi^\mu(0,\tau)=\tilde{\psi}^\mu(0,\tau) & \psi^\mu(\pi,\tau)=\tilde{\psi}^\mu(\pi,\tau) \\ \text{NS:} & \psi^\mu(0,\tau)=-\tilde{\psi}^\mu(0,\tau) & \psi^\mu(\pi,\tau)=\tilde{\psi}^\mu(\pi,\tau) \end{array} \tag{311}$$

We are free to choose the boundary condition at, for example the $\sigma=\pi$ end, in order to have a + sign, by redefinition of $\tilde{\psi}$. The boundary conditions and equations of motion are summarised by the "doubling trick": Take just left–moving (analytic) fields ψ^μ on the range 0 to 2π and define $\tilde{\psi}^\mu(\sigma,\tau)$ to be $\psi^\mu(2\pi-\sigma,\tau)$. These left–moving fields are periodic in the Ramond (R) sector and antiperiodic in the Neveu–Schwarz (NS).

On the complex z–plane, the NS sector fermions are half–integer moded while the R sector ones are integer, and we have:

$$\psi^\mu(z)=\sum_r \frac{\psi^\mu_r}{z^{r+1/2}}\ ,\ \text{where } r\in\mathbb{Z}\,\text{or } r\in\mathbb{Z}+\tfrac{1}{2} \tag{312}$$

and canonical quantisation gives

$$\{\psi^\mu_r,\psi^\nu_s\}=\{\tilde{\psi}^\mu_r,\tilde{\psi}^\nu_s\}=\eta^{\mu\nu}\delta_{r+s}\ . \tag{313}$$

Similarly we have

$$\begin{aligned} T_B(z) &= \sum_{m=-\infty}^{\infty}\frac{L_m}{z^{m+2}} \quad \text{as before, and} \\ T_F(z) &= \sum_r \frac{G_r}{z^{r+3/2}}\ ,\ \text{where } r\in\mathbb{Z}\ (\text{R})\ \text{ or }\ \mathbb{Z}+\tfrac{1}{2}\ (\text{NS}) \end{aligned} \tag{314}$$

Correspondingly, the Virasoro algebra is enlarged, with the non–zero (anti) commutators being

$$\begin{aligned} [L_m,L_n] &= (m-n)L_{m+n}+\frac{c}{12}(m^3-m)\delta_{m+n} \\ \{G_r,G_s\} &= 2L_{r+s}+\frac{c}{12}(4r^2-b)\delta_{r+s} \\ [L_m,G_r] &= \frac{1}{2}(m-2r)G_{m+r}\ , \end{aligned} \tag{315}$$

where b is 1 or 0, respectively, for the NS or R sector, respectively, and

$$\begin{aligned} L_m &= \frac{1}{2}\sum_n :\alpha_{n-m}\cdot\alpha_m: +\frac{1}{4}\sum_r(2r-m):\psi_{m-r}\cdot\psi_r: +a\delta_{m,0} \\ G_r &= \sum_n \alpha_n\cdot\psi_{r-n}\ . \end{aligned} \tag{316}$$

In the above, c is the total contribution to the conformal anomaly, which is $D+D/2$, where D is from the D bosons while $D/2$ is from the D fermions.

The values of D and a are again determined by any of the methods mentioned in the discussion of the bosonic string. For the superstring, it turns out that $D=10$ and $a=0$ for the R sector and $a=-1/2$ for the NS sector. This comes about because the contributions from the X^0 and X^1 directions are canceled by the Faddeev–Popov ghosts as before, and the contributions from the ψ^0 and ψ^1 oscillators are canceled by the superghosts. Then, the computation uses the mnemonic/formula given in equation (70).

$$\begin{aligned} \text{NS sector:}\quad \text{zpe} &= 8\left(-\frac{1}{24}\right)+8\left(-\frac{1}{48}\right)=-\frac{1}{2}\ , \\ \text{R sector:}\quad \text{zpe} &= 8\left(-\frac{1}{24}\right)+8\left(\frac{1}{24}\right)=0\ . \end{aligned} \tag{317}$$

As before, there is a physical state condition imposed by annihilating with the positive modes of the (super) Virasoro generators:

$$G_r|\phi\rangle = 0\ ,\ \ r > 0\ ;\quad L_n|\phi\rangle = 0\ ,\ n > 0\ ;\quad (L_0 - a)|\phi\rangle = 0\ . \tag{318}$$

The L_0 constraint leads to a mass formula:

$$M^2 = \frac{1}{\alpha'}\left(\sum_{n,r}\alpha_{-n}\cdot\alpha_n + r\psi_{-r}\cdot\psi_r - a\right)\ . \tag{319}$$

In the NS sector the ground state is a Lorentz singlet and is assigned odd fermion number, *i.e.,* under the operator $(-1)^F$, it has eigenvalue -1.

In order to achieve spacetime supersymmetry, the spectrum is projected on to states with even fermion number. This is called the "GSO projection",[67] and for our purposes, it is enough to simply state that this obtains spacetime supersymmetry, as we will show at the massless level. A more complete treatment —which gets it right for all mass levels— is contained in the full superconformal field theory. The GSO projection there is a statement about locality with the gravitino vertex operator. Yet another way to think of its origin is as a requirement of modular invariance.

Since the open string tachyon clearly has $(-1)^F = -1$, it is removed from the spectrum by GSO. This is our first achievement, and justifies our earlier practice of ignoring the tachyons appearance in the bosonic spectrum in what has gone before. From what we will do for the rest of the this book, the tachyon will largely remain in the wings, but it (and other tachyons) do have a role to play, since they are often a signal that the vacuum wants to move to a (perhaps) more interesting place.

Massless particle states in ten dimensions are classified by their $SO(8)$ representation under Lorentz rotations, that leave the momentum invariant: $SO(8)$ is the "little group" of $SO(1,9)$. The lowest lying surviving states in the NS sector are the eight transverse polarizations of the massless open string photon, A^μ, made by exciting the ψ oscillators:

$$\psi^\mu_{-1/2}|k\rangle,\qquad M^2 = 0\ . \tag{320}$$

These states clearly form the vector of $SO(8)$. They have $(-)^F = 1$ and so survive GSO.

In the R sector the ground state energy always vanishes because the world–sheet bosons and their superconformal partners have the same moding. The Ramond vacuum has a 32–fold degeneracy, since the ψ^μ_0 take ground states into ground states. The ground states form a representation of the ten dimensional Dirac matrix algebra

$$\{\psi^\mu_0, \psi^\nu_0\} = \eta^{\mu\nu}\ . \tag{321}$$

(Note the similarity with the standard Γ–matrix algebra, $\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu}$. We see that $\psi^\mu_0 \equiv \Gamma^\mu/\sqrt{2}$.)

For this representation, it is useful to choose this basis:

$$\begin{aligned} d^\pm_i &= \frac{1}{\sqrt{2}}\left(\psi^{2i}_0 \pm i\psi^{2i+1}_0\right)\qquad i = 1,\cdots,4 \\ d^\pm_0 &= \frac{1}{\sqrt{2}}\left(\psi^1_0 \mp \psi^0_0\right)\ . \end{aligned} \tag{322}$$

In this basis, the Clifford algebra takes the form

$$\{d^+_i, d^-_j\} = \delta_{ij}\ . \tag{323}$$

The $d^\pm_i$, $i = 0,\cdots,4$ act as creation and annihilation operators, generating the $2^{10/2} = 32$ Ramond ground states. Denote these states

$$|s_0, s_1, s_2, s_3, s_4\rangle = |\mathbf{s}\rangle \tag{324}$$

where each of the s_i takes the values $\pm\frac{1}{2}$, and where

$$d_i^-|-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2}\rangle = 0 \tag{325}$$

while d_i^+ raises s_i from $-\frac{1}{2}$ to $\frac{1}{2}$. This notation has physical meaning: The fermionic part of the ten dimensional Lorentz generators is

$$S^{\mu\nu} = -\frac{i}{2}\sum_{r\in\mathbf{Z}+\kappa}[\psi^\mu_{-r},\psi^\nu_r] \ , \tag{326}$$

(recall equation (114)). The states (324) above are eigenstates of $S_0 = iS^{01}$, $S_i = S^{2i,2i+1}$, with s_i the corresponding eigenvalues. Since by construction the Lorentz generators (326) always flip an even number of s_i, the Dirac representation **32** decomposes into a **16** with an even number of $-\frac{1}{2}$'s and **16′** with an odd number.

The physical state conditions (318), on these ground states, reduce to $G_0 = (2\alpha')^{1/2}p_\mu\psi^\mu_0$. (Note that $G_0^2 \sim L_0$.) Let us pick the (massless) frame $p^0 = p^1$. This becomes

$$G_0 = \alpha'^{1/2}p_1\Gamma_0\left(1-\Gamma_0\Gamma_1\right) = 2\alpha'^{1/2}p_1\Gamma_0\left(\tfrac{1}{2}-S_0\right) \ , \tag{327}$$

which means that $s_0 = \frac{1}{2}$, giving a sixteen–fold degeneracy for the *physical* Ramond vacuum. This is a representation of $SO(8)$ which decomposes into $\mathbf{8_s}$ with an even number of $-\frac{1}{2}$'s and $\mathbf{8_c}$ with an odd number. One is in the **16** and the **16′**, but the two choices, **16** or **16′**, are physically equivalent, differing only by a spacetime parity redefinition, which would therefore swap the $\mathbf{8_s}$ and the $\mathbf{8_c}$.

In the R sector the GSO projection amounts to requiring

$$\sum_{i=1}^4 s_i = 0 \quad (\text{mod } 2), \tag{328}$$

picking out the $\mathbf{8_s}$. Of course, it is just a convention that we associated an even number of $\frac{1}{2}$'s with the $\mathbf{8_s}$; a physically equivalent discussion with things the other way around would have resulted in $\mathbf{8_c}$. The difference between these two is only meaningful when they are both present, and at this stage we only have one copy, so either is as good as the other.

The ground state spectrum is then $\mathbf{8}_v\oplus\mathbf{8}_s$, a vector multiplet of $D=10, \mathcal{N}=1$ spacetime supersymmetry. Including Chan–Paton factors gives again a $U(N)$ gauge theory in the oriented theory and $SO(N)$ or $USp(N)$ in the unoriented. This completes our tree–level construction of the open superstring theory.

Of course, we are not finished, since this theory is (on its own) inconsistent for many reasons. One such reason (there are many others) is that it is anomalous. Both gauge invariance and coordinate invariance have anomalies arising because it is a chiral theory: *e.g.,* the fermion $\mathbf{8_s}$ has a specific chirality in spacetime. The gauge and gravitational anomalies are very useful probes of the consistency of any theory. These show up quantum inconsistencies of the theory resulting in the failure of gauge invariance and general coordinate invariance, and hence must be absent.

Another reason we will see that the theory is inconsistent is that, as we learned in section 4, the theory is equivalent to some number of space–filling D9–branes in spacetime, and it will turn out later that these are positive electric sources of a particular 10–form field in the theory. The field equation for this field asks that all of its sources must simply vanish, and so we must have a negative source of this same field in order to cancel the D9–branes' contribution. This will lead us to the closed string sector *i.e.* one–loop, the same level at which we see the anomaly.

Let us study some closed strings. We will find three of interest here. Two of them will stand in their own right, with two ten dimensional supersymmetries, while the third will have half of that, and will be anomalous. This latter will be the closed string sector we need to supplement the open string we made here, curing its one–loop anomalies.

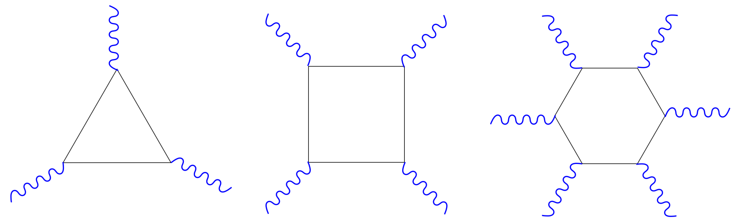


Figure 28: One–loop diagrams displaying anomalies in dimensions four, six, and ten, respectively.

7.1.2 Gauge and Gravitational Anomalies

The beauty of the anomaly is that it is both a UV and an IR tool. UV since it represents the failure to be able to find a consistent regulator at the quantum level and IR since it cares only about the massless sector of the theory: Any potentially anomalous variations for the *effective* action $\Gamma = \ln Z$ should be written as the variation of a local term which allows it to be canceled by adding a local counterterm. Massive fields always give effectively local terms at long distance.

An anomaly in D dimensions arises from complex representations of the Lorentz group which include chiral fermions in general but also bosonic representations if $D = 4k+2$, *e.g.*, the rank $2k+1$ (anti) self–dual tensor. The anomalies are controlled by the so–called "hexagon" diagram which generalizes the (perhaps more familiar) triangle of four dimensional field theory or a square in six dimensions. See figure 28.

The external legs are either gauge bosons, gravitons, or a mixture. We shall not spend any time on the details[56], but simply state that consistency demands that the structure of the anomaly,

$$\delta \ln Z = \frac{i}{(2\pi)^{D/2}} \int \hat{I}_D \ ,$$

is in terms of a D–form $\hat{I}_D$, polynomial in traces of even powers of the field strength 2–forms $F = dA + A^2$ and $R = d\omega + \omega^2$. (Recall section 2.14.) It is naturally related to a $(D+2)$–form polynomial $\hat{I}_{D+2}$ which is gauge invariant and written as an exact form $\hat{I}_{D+2} = d\hat{I}_{D+1}$. The latter is not gauge invariant, but its variation is another exact form:

$$\delta \hat{I}_{D+1} = d\hat{I}_D \ . \tag{329}$$

A key example of this is the Chern–Simons 3–form, which is discussed in section 7.1.3. See also section 7.1.4 for explicit expressions in dimensions $D = 4k + 2$. We shall see that the anomalies are a useful check of the consistency of string spectra that we construct in various dimensions.

7.1.3 The Chern–Simons Three–Form

The Chern–Simons 3–form is a very important structure which will appear in a number of places, and it is worth pausing a while to consider its properties. Recall that we can write the gauge potential, and the field strength as Lie Algebra–valued forms: $A = t^a A^a_\mu dx^\mu$, where the t^a are generators of the Lie algebra. We can write the Yang–Mills field strength as a matrix–valued 2–form, $F = t^a F^a_{\mu\nu} dx^\mu \wedge dx^\nu$. We can define the Chern–Simons 3–form as

$$\omega_{3\mathrm{Y}} = \mathrm{Tr}\left(A \wedge F - \frac{1}{3} A \wedge A \wedge A\right) = \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right) \ .$$

One interesting thing about this object is that we can write:

$$d\omega_{3\mathrm{Y}} = \mathrm{Tr}\left(F \wedge F\right) \ .$$

Furthermore, under a gauge transformation $\delta A = d\Lambda + [A, \Lambda]$:

$$\delta\omega_{3\mathrm{Y}} = \mathrm{Tr}(d\Lambda dA) = d\omega_2 \ , \quad \omega_2 = \mathrm{Tr}(\Lambda dA) \ .$$

So its gauge variation, while not vanishing, is an exact 3–form. Note that there is a similar structure in the pure geometry sector. From section 2.14, we recall that the potential analogous to A is the spin connection 1–form $\omega^a{}_b = \omega^a{}_{b\mu} dx^\mu$, with a and b being Minkowski indices in the space tangent to the point x^μ in spacetime and so ω is an $SO(D{-}1,1)$ matrix in the fundamental representation. The curvature is a 2–form $R^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b = R^a{}_{b\mu\nu} dx^\mu \wedge dx^\nu$, and the gauge transformation is now $\delta\omega = d\Theta + [\omega, \Theta]$. We can define:

$$\omega_{3\mathrm{L}} = \mathrm{tr}\left(\omega \wedge d\omega + \frac{2}{3}\omega \wedge \omega \wedge \omega\right) \ ,$$

with similar properties to $\omega_{3\mathrm{Y}}$, above. Here tr means trace on the indices a, b.

7.1.4 A list of Anomaly Polynomials

It is useful to list here some anomaly polynomials for later use. In ten dimensions, the contributions to the polynomial come from three sorts of field, the spinors $\mathbf{8}_{s,c}$, the gravitinos $\mathbf{56}_{c,s}$, and the 5th rank antisymmetric tensor field strength with its self–dual and anti–self–dual parts. The anomalies for each pair within each sort are equal and opposite in sign, *i.e.*, $\hat{I}_{12}^{\mathbf{8}_s} = -\hat{I}_{12}^{\mathbf{8}_c}$, *etc.* and we have:

$$\begin{aligned}
\hat{I}_{12}^{\mathbf{8}_s} = & - \frac{\mathrm{Tr}(F^6)}{1440} \\
& + \frac{\mathrm{Tr}(F^4)\mathrm{tr}(R^2)}{2304} - \frac{\mathrm{Tr}(F^2)\mathrm{tr}(R^4)}{23040} - \frac{\mathrm{Tr}(F^2)[\mathrm{tr}(R^2)]^2}{18432} \\
& + \frac{n\mathrm{tr}(R^6)}{725760} + \frac{n\mathrm{tr}(R^4)\mathrm{tr}(R^2)}{552960} + \frac{n[\mathrm{tr}(R^2)]^3}{1327104} \ . \\
\hat{I}_{12}^{\mathbf{56}_c} = & - 495\frac{\mathrm{tr}(R^6)}{725760} + 225\frac{\mathrm{tr}(R^4)\mathrm{tr}(R^2)}{552960} - 63\frac{[\mathrm{tr}(R^2)]^3}{1327104} \ . \\
\hat{I}_{12}^{\mathbf{35}_+} = & + 992\frac{\mathrm{tr}(R^6)}{725760} - 448\frac{\mathrm{tr}(R^4)\mathrm{tr}(R^2)}{552960} + 128\frac{[\mathrm{tr}(R^2)]^3}{1327104} \ ,
\end{aligned}$$

and n is the dimension of the gauge representation under which the spinor transforms, for which we use the trace denoted Tr. We also have suppressed the use of $\wedge$, for brevity. For $D = 6$, there are anomaly 8–forms. We denote the various fields by their transformation properties of the $D = 6$ little group $SO(4) \sim SU(2) \times SU(2)$:

$$\begin{aligned}
\hat{I}_8^{(\mathbf{1},\mathbf{2})} = & + \frac{\mathrm{Tr}(F^4)}{24} - \frac{\mathrm{Tr}(F^2)\mathrm{tr}(R^2)}{96} + \frac{n\mathrm{tr}(R^4)}{5760} + \frac{n[\mathrm{tr}(R^2)]^2}{4608} \ . \\
\hat{I}_8^{(\mathbf{3},\mathbf{2})} = & + 245\frac{\mathrm{tr}(R^4)}{5760} - 43\frac{[\mathrm{tr}(R^2)]^2}{4608} \ . \\
\hat{I}_8^{(\mathbf{3},\mathbf{1})} = & + 28\frac{\mathrm{tr}(R^4)}{5760} - 8\frac{[\mathrm{tr}(R^2)]^2}{4608} \ .
\end{aligned}$$

Note that the first two are for complex fermions. For real fermions, one must divide by two. For completeness, for $D = 2$ we list the three analogous anomaly 4–forms:

$$\hat{I}_4^{1/2} = \frac{n\mathrm{tr}(R^2)}{48} - \frac{\mathrm{Tr}(F^2)}{2} \ , \qquad \hat{I}_4^{3/2} = -23\frac{\mathrm{tr}(R^2)}{48} \ , \qquad \hat{I}_4^{0} = \frac{\mathrm{tr}(R^2)}{48} \ .$$

It is amusing to note that the anomaly polynomials can be written in terms of geometrical characteristic classes. This should be kept at the back of the mind for a bit later, in section 9.5.

7.2 Closed Superstrings: Type II

Just as we saw before, the closed string spectrum is the product of two copies of the open string spectrum, with right– and left–moving levels matched. In the open string the two choices for the GSO projection were equivalent, but in the closed string there are two inequivalent choices, since we have to pick two copies to make a closed string.

Taking the same projection on both sides gives the "type IIB" case, while taking them opposite gives "type IIA". These lead to the massless sectors

$$\begin{aligned}\text{Type IIA:} \quad & (\mathbf{8_v} \oplus \mathbf{8_s}) \otimes (\mathbf{8_v} \oplus \mathbf{8_c}) \\ \text{Type IIB:} \quad & (\mathbf{8_v} \oplus \mathbf{8_s}) \otimes (\mathbf{8_v} \oplus \mathbf{8_s}) \ .\end{aligned} \tag{330}$$

Let us expand out these products to see the resulting Lorentz ($SO(8)$) content. In the NS–NS sector, this is

$$\mathbf{8_v} \otimes \mathbf{8_v} = \Phi \oplus B_{\mu\nu} \oplus G_{\mu\nu} = \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35}. \tag{331}$$

In the R–R sector, the IIA and IIB spectra are respectively

$$\begin{aligned}\mathbf{8_s} \otimes \mathbf{8_c} &= [1] \oplus [3] = \mathbf{8_v} \oplus \mathbf{56_t} \\ \mathbf{8_s} \otimes \mathbf{8_s} &= [0] \oplus [2] \oplus [4]_+ = \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35}_+.\end{aligned} \tag{332}$$

Here $[n]$ denotes the n–times antisymmetrised representation of $SO(8)$, and $[4]_+$ is self–dual. Note that the representations $[n]$ and $[8-n]$ are the same, as they are related by contraction with the 8–dimensional ϵ–tensor. The NS–NS and R–R spectra together form the bosonic components of $D = 10$ IIA (nonchiral) and IIB (chiral) supergravity respectively; We will write their effective actions shortly.

In the NS–R and R–NS sectors are the products

$$\begin{aligned}\mathbf{8_v} \otimes \mathbf{8_c} &= \mathbf{8_s} \oplus \mathbf{56_c} \\ \mathbf{8_v} \otimes \mathbf{8_s} &= \mathbf{8_c} \oplus \mathbf{56_s}.\end{aligned} \tag{333}$$

The $\mathbf{56_{s,c}}$ are gravitinos. Their vertex operators are made roughly by tensoring a NS field ψ^μ with a vertex operator $\mathcal{V}_\alpha = e^{-\varphi/2}\mathbf{S}_\alpha$, where the latter is a "spin field", made by bosonising the d_i's of equation (322) and building:

$$\mathbf{S} = \exp\left[i\sum_{i=0}^{4} s_i H^i\right] \ ; \quad d_i = e^{\pm \mathrm{i} H^i} \ . \tag{334}$$

(The factor $e^{-\varphi/2}$ is the bosonisation (see section 4.6) of the Faddeev–Popov ghosts, about which we will have nothing more to say here.) The resulting full gravitino vertex operators, which correctly have one vector and one spinor index, are two fields of weight $(0,1)$ and $(1,0)$, respectively, depending upon whether ψ^μ comes from the left or right. These are therefore holomorphic and anti–holomorphic world–sheet currents, and the symmetry associated to them in spacetime is the supersymmetry. In the IIA theory the two gravitinos (and supercharges) have opposite chirality, and in the IIB the same.

Consider the vertex operators for the R–R states.[1] This will involve a product of spin fields, [70] one from the left and one from the right. These again decompose into antisymmetric tensors, now of $SO(9,1)$:

$$V = \mathcal{V}_\alpha \mathcal{V}_\beta (\Gamma^{[\mu_1} \cdots \Gamma^{\mu_n]} C)_{\alpha\beta} G_{[\mu_1 \cdots \mu_n]}(X) \tag{335}$$

with C the charge conjugation matrix. In the IIA theory the product is $\mathbf{16} \otimes \mathbf{16'}$ giving even n (with $n \cong 10 - n$) and in the IIB theory it is $\mathbf{16} \otimes \mathbf{16}$ giving odd n. As in the bosonic case, the classical equations of motion follow from the physical state conditions, which at the massless level reduce to $G_0 \cdot V = \tilde{G}_0 \cdot V = 0$. The relevant part of G_0 is just $p_\mu \psi_0^\mu$ and similarly for $\tilde{G}_0$. The p_μ act by differentiation on G, while ψ_0^μ acts on the spin fields as it does on the corresponding ground states: as multiplication by Γ^μ. Noting the identity

$$\Gamma^\nu \Gamma^{[\mu_1} \cdots \Gamma^{\mu_n]} = \Gamma^{[\nu} \cdots \Gamma^{\mu_n]} + \left(\delta^{\nu\mu_1}\Gamma^{[\mu_2} \cdots \Gamma^{\mu_n]} + \text{perms}\right) \tag{336}$$

and similarly for right multiplication, the physical state conditions become

$$dG = 0 \qquad\qquad d^*G = 0. \tag{337}$$

These are the Bianchi identity and field equation for an antisymmetric tensor field strength. This is in accord with the representations found: in the IIA theory we have odd–rank tensors of $SO(8)$ but even–rank tensors of $SO(9,1)$ (and reversed in the IIB), the extra index being contracted with the momentum to form the field strength. It also follows that R–R amplitudes involving elementary strings vanish at zero momentum, so strings do not carry R–R charges[4].

As an aside, when the dilaton background is nontrivial, the Ramond generators have a term $\Phi_{,\mu}\partial\psi^\mu$, and the Bianchi identity and field strength pick up terms proportional to $d\Phi\wedge G$ and $d\Phi\wedge {}^*G$. The Bianchi identity is nonstandard, so G is not of the form dC. Defining $G' = e^{-\Phi}G$ removes the extra term from both the Bianchi identity and field strength. The field G' is thus decoupled from the dilaton. In terms of the action, the fields G in the vertex operators appear with the usual closed string $e^{-2\Phi}$ but with non-standard dilaton gradient terms. The fields we are calling G' (which in fact are the usual fields used in the literature, and so we will drop the prime symbol in the sequel) have a dilaton–independent action.

The type IIB theory is chiral since it has different numbers of left moving fermions from right–moving. Furthermore, there is a self–dual R–R tensor. These structures in principle produce gravitational anomalies, and it is one of the miracles (from the point of view of the low energy theory) of string theory that the massless spectrum is in fact anomaly free. There is a delicate cancellation between the anomalies for the $\mathbf{8}_c$'s and for the $\mathbf{56}_s$'s and the $\mathbf{35}_+$. The reader should check this by using the anomaly polynomials in section 7.1.4, (of course, put $n = 1$ and $F = 0$) to see that

$$-2\hat{I}_{12}^{\mathbf{8}_s} + 2\hat{I}_{12}^{\mathbf{56}_c} + \hat{I}_{12}^{\mathbf{35}_+} = 0\ , \tag{338}$$

which is in fact miraculous, as previously stated[101].

7.2.1 Type I from Type IIB, The Prototype Orientifold

As we saw in the bosonic case, we can construct an unoriented theory by projecting onto states invariant under world sheet parity, Ω. In order to get a consistent theory, we must of course project a theory which is invariant under Ω to start with. Since the left and right moving sectors have the same GSO projection for type IIB, it is invariant under Ω, so we can again form an unoriented theory by gauging. We cannot gauge Ω in type IIA to get a consistent theory, but see later.

Projecting onto $\Omega = +1$ interchanges left–moving and right–moving oscillators and so one linear combination of the R–NS and NS–R gravitinos survives, so there can be only one supersymmetry remaining. In the NS–NS sector, the dilaton and graviton are symmetric under Ω and survive, while the antisymmetric tensor is odd and is projected out. In the R–R sector, by counting we can see that the $\mathbf{1}$ and $\mathbf{35}_+$ are in the symmetric product of $\mathbf{8_s}\otimes\mathbf{8_s}$ while the $\mathbf{28}$ is in the antisymmetric. The R–R state is the product of right– and left–moving fermions, so there is an extra minus in the exchange. Therefore it is the $\mathbf{28}$ that survives. The bosonic massless sector is thus $\mathbf{1}\oplus\mathbf{28}\oplus\mathbf{35}$, and together with the surviving gravitino, this give us the $D = 10$ $N = 1$ supergravity multiplet.

Sadly, this supergravity is in fact anomalous. The delicate balance (338) between the anomalies from the various chiral sectors, which we noted previously, vanishes since one each of the $\mathbf{8}_c$ and $\mathbf{56}_s$, and the $\mathbf{35}_+$, have been projected out. Nothing can save the theory unless there is an additional sector to cancel the anomaly.[101]

This sector turns out to be $N = 1$ supersymmetric Yang–Mills theory, with gauge group $SO(32)$ or $E_8\times E_8$. Happily, we already know at least one place to find the first choice: We can use the low–energy (massless) sector of $SO(32)$ unoriented open superstring theory. This fits nicely, since as we have seen before, at one loop open strings couple to closed strings. We will not be able to get gauge group $E_8\times E_8$ from

[4]The reader might wish to think of this as analogous to the discovery that a moving electric point source generates a magnetic field, but of course is not a basic magnetic monopole source.

perturbative open string theory (Chan–Paton factors can't make this sort of group), but we will see shortly that there is another way of getting this group, but from a closed string theory.

The total anomaly is that of the gravitino, dilatino and the gaugino, the latter being charged in the adjoint of the gauge group:

$$I_{12} = -\hat{I}_{12}^{\mathbf{8}_s}(R) + \hat{I}_{12}^{\mathbf{56}_c}(R) + \hat{I}_{12}^{\mathbf{8}_s}(F,R) \tag{339}$$

Using the polynomials given in section 7.1.4, it should be easily seen that there is an irreducible term

$$(n-496)\frac{\mathrm{tr}(R^6)}{725760}\ , \tag{340}$$

which must simply vanish, and so n, the dimension of the group, must be 496. Since $SO(32)$ and $E_8\times E_8$ both have this dimension, this is encouraging. That the rest of the anomaly cancels is a very delicate and important story which deserves some attention. We will do that in the next section.

Finishing the present discussion, in the language we learned in section 4.11.1, we put a single (space–filling) O9–plane into type IIB theory, making the type IIB theory into the unoriented $N=1$ closed string theory. This is anomalous, but we can cancel the resulting anomalies by adding 16 D9–branes. Another way of putting it is that (as we shall see) the O9–plane has 16 units of C_{10} charge, which cancels that of 16 D9–branes, satisfying the equations of motion for that field.

We have just constructed our first (and in fact, the simplest) example of an "orientifolding" of a superstring theory to get another. More complicated orientifolds may be constructed by gauging combinations of Ω with other discrete symmetries of a given string theory which form an "orientifold group" G_Ω under which the theory is invariant.[31] Generically, there will be the requirement to cancel anomalies by the addition of open string sectors (*i.e.* D–branes), which results in consistent new string theory with some spacetime gauge group carried by the D–branes. In fact, these projections give rise to gauge groups containing any of $U(n)$, $USp(n)$ factors, and not just $SO(n)$ sectors.

7.3 The Green–Schwarz Mechanism

Let us finish showing that the anomalies of $\mathcal{N}=1$, $D=10$ supergravity coupled to Yang–Mills do vanish for the groups $SO(32)$ and $E_8\times E_8$. We have already shown above that the dimension of the group must be $n=496$. Some algebra shows that the rest of the anomaly (339), for *this* value of n can be written suggestively as:

$$\begin{aligned} I_{12}^{(n=496)} &= \frac{1}{3\times 2^8}Y_4X_8 \\ &\quad -\frac{1}{1440}\left(\mathrm{Tr}_{\rm adj}(F^6)-\frac{\mathrm{Tr}_{\rm adj}(F^2)\mathrm{Tr}_{\rm adj}(F^4)}{48}+\frac{[\mathrm{Tr}_{\rm adj}(F^2)]^3}{14400}\right)\ , \end{aligned} \tag{341}$$

where

$$\begin{aligned} Y_4 &= \mathrm{tr}(R^2)-\frac{1}{30}\mathrm{Tr}_{\rm adj}(F^2)\ , \\ X_8 &= \frac{\mathrm{Tr}_{\rm adj}(F^4)}{3}-\frac{[\mathrm{Tr}_{\rm adj}(F^2)]^2}{900}-\frac{\mathrm{Tr}_{\rm adj}(F^2)\mathrm{tr}(R^2)}{30}+\mathrm{tr}(R^4)+\frac{[\mathrm{tr}(R^2)]^2}{4}\ . \end{aligned} \tag{342}$$

On the face of it, it does not really seem possible that this can be canceled, since the gaugino carries gauge charge and nothing else does, and so there are a lot of gauge quantities which simply stand on their own. This seems hopeless because we have so far restricted ourselves to quantum anomalies arising from the gauge and gravitational sector. If we include the rank two R–R potential $C_{(2)}$ in a cunning way, we can generate a mechanism for canceling the anomaly. Consider the interaction

$$S_{\rm GS} = \frac{1}{3\times 2^6(2\pi)^5\alpha'}\int C_{(2)}\wedge X_8\ . \tag{343}$$

It is invariant under the usual gauge transformations

$$\delta A = d\Lambda + [A, \Lambda] \; ; \quad \delta\omega = d\Theta + [\omega, \Theta] \; , \tag{344}$$

since it is constructed out of the field strengths F and R. It is also invariant under the 2–form potential's standard transformation $\delta C_{(2)} = d\lambda$. Let us however give $C_{(2)}$ another gauge transformation rule. While A and ω transform under (344), let it transform as:

$$\delta C_{(2)} = \frac{\alpha'}{4}\left(\frac{1}{30}\mathrm{Tr}(\Lambda F) - \mathrm{tr}(\Theta R)\right) \; . \tag{345}$$

Then the variation of the action does not vanish, and is:

$$\delta S_{\rm GS} = \frac{1}{3\times 2^8 (2\pi)^5}\int\left[\frac{1}{30}\mathrm{Tr}(\Lambda F) - \mathrm{tr}(\Theta R)\right]\wedge X_8 \; .$$

However, using the properties of the Chern–Simons 3–form discussed in section 7.1.3, this classical variation can be written as descending *via* the consistency chain of equation (329) from precisely the 12–form polynomial given in the first line of equation (342), but with a minus sign. Therefore we cancel that offending term with this classical modification of the transformation of $C_{(2)}$. Later on, when we write the supergravity action for this field in the type I model, we will use the modified field strength:

$$\widetilde{G}^{(3)} = dC^{(2)} - \frac{\alpha'}{4}\left[\frac{1}{30}\omega_{3\rm Y}(A) - \omega_{3\rm L}(\Omega)\right] \; , \tag{346}$$

where because of the transformation properties of the Chern–Simons 3–form (see section 7.1.3), $\widetilde{G}^{(3)}$ is gauge invariant under the new transformation rule (345).

It is worth noting here that this is a quite subtle mechanism. We are canceling the anomaly generated by a one loop diagram with a tree–level graph. It is easy to see what the tree level diagram is. The kinetic term for the modified field strength will have its square appearing, and so looking at its definition (346), we see that there is a vertex coupling $C_{(2)}$ to two gauge bosons or to two gravitons. There is another vertex which comes from the interaction (343) which couples $C_{(2)}$ to four particles, pairs of gravitons and pairs of gauge bosons, or a mixture. So the tree level diagram in figure 29 can mix with the hexagon anomaly of figure 28.

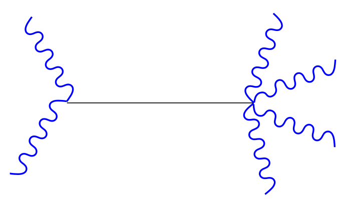


Figure 29: The tree which cures the $\mathcal{N}=1$ $D=10$ anomalies. A 2–form field is exchanged.

Somehow, the terms in the second line must cancel amongst themselves. Miraculously, they do for a number of groups, $SO(32)$ and $E_8\times E_8$ included. For the first group, it follows from the fact that for the group $SO(n)$, we can write:

$$\begin{aligned}
\mathrm{Tr}_{\rm adj}(t^6) &= (n-32)\mathrm{Tr}_{\rm f}(t^6) + 15\mathrm{Tr}_{\rm f}(t^2)\mathrm{Tr}_{\rm f}(t^4) \; ; \\
\mathrm{Tr}_{\rm adj}(t^4) &= (n-8)\mathrm{Tr}_{\rm f}(t^4) + 3\mathrm{Tr}_{\rm f}(t^2)\mathrm{Tr}_{\rm f}(t^2) \; ; \\
\mathrm{Tr}_{\rm adj}(t^2) &= (n-2)\mathrm{Tr}_{\rm f}(t^2) \; ,
\end{aligned} \tag{347}$$

where "f" denotes the fundamental representation. For E_8, we have that

$$\begin{aligned} \mathrm{Tr}_{\mathrm{adj}}(t^6) &= \frac{1}{7200}[\mathrm{Tr}_{\mathrm{f}}(t^2)]^3 \ , \\ \mathrm{Tr}_{\mathrm{adj}}(t^4) &= \frac{1}{100}[\mathrm{Tr}_{\mathrm{f}}(t^2)]^2 \ . \end{aligned} \tag{348}$$

In checking these (which of course the reader will do) one should combine the traces as $\mathrm{Tr}_{G_1\times G_2} = \mathrm{Tr}_{G_1} + \mathrm{Tr}_{G_2}$, *etc.*

Overall, the results[101] of this subsection are quite remarkable, and generated a lot of excitement which we now call the First Superstring Revolution. This excitement was of course justified, since the discovery of the mechanism revealed that there were consistent superstring theories with considerably intricate structures with promise for making contact with the physics that we see in Nature.

7.4 The Two Basic Heterotic String Theories

In addition to the three superstring theories briefly constructed above, there are actually two more supersymmetric string theories which live in ten dimensions. In addition, they have non–Abelian spacetime gauge symmetry, and they are also free of tachyons. These are the "Heterotic Strings".[23] The fact that they are chiral, have fermions and non–Abelian gauge symmetry meant that they were considered extremely attractive as starting points for constructing "realistic" phenomenology based on string theory. It is in fact remarkable that one can come tantalizingly close to naturally realizing many of the features of the Standard Model of particle physics by starting with, say, the $E_8\times E_8$ Heterotic String, while remaining entirely in the perturbative regime. This was the focus of much of the First Superstring Revolution. Getting many of the harder questions right led to the search for non–perturbative physics, which ultimately led us to the Second Superstring Revolution, and the realization that all of the other string theories were just as important too, because of duality.

One of the more striking things about the heterotic strings, from the point of view of what we have done so far, is the fact that they have non–Abelian gauge symmetry and are still closed strings. The $SO(32)$ of the type I string theory comes from Chan–Paton factors at the ends of the open string, or in the language we now use, from 16 coincident D9–branes.

We saw a big hint of what is needed to get spacetime gauge symmetry in the heterotic string in section 4. Upon compactifying bosonic string theory on a circle, at a special radius of the circle, an enhanced $SU(2)_L\times SU(2)_R$ gauge symmetry arose. From the two dimensional world–sheet point of view, this was a special case of a current algebra, which we uncovered further in section 4.3. We can take two key things away from that section for use here. The first is that we can generalise this to a larger non–Abelian gauge group if we use more bosons, although this would seem to force us to have many compact directions. The second is that there were identical and *independent* structures coming from the left and the right to give this result. So we can take, say, the left hand side of the construction and work with it, to produce a single copy of the non–Abelian gauge group in spacetime.

This latter observation is the origin of the word "heterotic" which comes from "heterosis". The theory is a hybrid of two very different constructions on the left and the right. Let us take the right hand side to be a copy of the right hand side of the superstrings we constructed previously, and so we use only the right hand side of the action given in equation (308) (with closed string boundary conditions).

Then the usual consistency checks give that the critical dimension is of course ten, as before: The central charge (conformal anomaly) is $-26+11=15$ from the conformal and superconformal ghosts. This is canceled by ten bosons and their superpartners since they contribute to the anomaly an amount $10\times1+10\times\frac{1}{2}=15$. The left hand side is in fact a purely bosonic string, and so the anomaly is canceled to zero by the -26 from the conformal ghosts and there must be the equivalent of 26 bosonic degrees of freedom, producing 26×1 to the anomaly.

How can the theory make sense as a ten dimensional theory? The answer to this question is just what gives the non–Abelian gauge symmetry. Sixteen of the bosons are periodic, and so may be thought of as compactified on a torus $T^{16}\simeq(S^1)^{16}$ with very specific properties.

Those properties are such that the generic $U(1)^{16}$ one might have expected from such a toroidal compactification is enhanced to one of two special rank 16 gauge groups: $SO(32)$, or $E_8 \times E_8$, *via* the very mechanism we saw in section 4: The torus is "self–dual". The remaining 10 non–compact bosons on the left combine with the 10 on the right to make the usual ten spacetime coordinates, on which the usual ten dimensional Lorentz group $SO(1,9)$ acts.

7.4.1 $SO(32)$ and $E_8 \times E_8$ From Self–Dual Lattices

The requirements are simple to state. We are required to have a sixteen dimensional lattice, according to the above discussion, and so we can apply the results of section 4, but there is a crucial difference. Recalling what we learned there, we see that since we only have a left–moving component to this lattice, we do not have the Lorentzian signature which arose there, but only a *Euclidean* signature. But all of the other conditions apply: It must be even, in order to build gauge bosons as vertex operators, and it must be self–dual, to ensure modular invariance.

The answer turns out the be quite simple. There are only two choices, since even self–dual Euclidean lattices are very rare (They only exist when the dimension is a multiple of eight). For sixteen dimensions, there is either $\Gamma_8 \times \Gamma_8$ or Γ_{16}. The lattice Γ_8 is the collection of points:

$$(n_1, n_2, \ldots, n_8) \quad \text{or} \quad (n_1 + \tfrac{1}{2}, n_2 + \tfrac{1}{2}, \ldots, n_8 + \tfrac{1}{2}) \ , \quad \sum_i n_i \in 2\mathbb{Z} \ ,$$

with $\sum_i n_i^2 = 2$. The integer lattice points are actually the root lattice of $SO(16)$, with which the 120 dimensional adjoint representation is made. The half integer points construct the spinor representation of $SO(16)$. A bit of thought shows that it is just like the construction we made of the spinor representations of $SO(8)$ previously; the entries are only $\pm\frac{1}{2}$ in 8 different slots, with only an even number of minus signs appearing, which again gives a squared length of two. There are $2^7 = 128$ possibilities, which is the dimension of the spinor representation. The total dimension of the representation we can make is $120 + 128 = 248$ which is the dimension of E_8. The sixteen dimensional lattice is made as the obvious tensor product of two copies of this, giving gauge group $E_8 \times E_8$, which is 496 dimensional.

The lattice Γ_{16} is extremely similar, in that it is:

$$(n_1, n_2, \ldots, n_{16}) \quad \text{or} \quad (n_1 + \tfrac{1}{2}, n_2 + \tfrac{1}{2}, \ldots, n_{16} + \tfrac{1}{2}) \ , \quad \sum_i n_i \in 2\mathbb{Z} \ ,$$

with $\sum_i n_i^2 = 2$. Again, we see that the integer points make the root lattice of $SO(32)$, but there is more. There is a spinor representation of $SO(32)$, but it is clear that since $16 \times 1/4 = 4$, the squared length is twice as large as it need to be to make a massless vector, and so the gauge bosons remain from the adjoint of $SO(32)$, which is 496 dimensional. In fact, the full structure is more than $SO(32)$, because of this spinor representation. It is not quite the cover, which is $Spin(32)$ because the conjugate spinor and the vector representations are missing. It is instead written as $Spin(32)/\mathbb{Z}_2$. In fact, $SO(32)$ in the quotient of $Spin(32)$ by another $\mathbb{Z}_2$.

Actually, before concluding, we should note that there is an alternative construction to this one using left–moving fermions instead of bosons. This is easily arrived at from here using what we learned about fermionisation in section 4.6. From there, we learn that we can trade in each of the left–moving bosons here for *two* left–moving Majorana–Weyl fermions, giving a fermionic construction with 32 fermions $\mathbf{\Psi}^i$. The construction divides the fermions into the NS and R sectors as before, which correspond to the integer and half–integer lattice sites in the above discussion. The difference between the two heterotic strings is whether the fermions are split into two sets with independent boundary conditions (giving $E_8 \times E_8$) or if they have all the same boundary conditions ($SO(32)$).

In this approach, there is a GSO projection, which in fact throws out a tachyon, *etc.*. Notice that in the R sector, the zero modes of the 32 Ψ will generate a spinor and conjugate spinor $\mathbf{2^{31}} + \mathbf{2^{\bar{31}}}$ of $SO(32)$ for much the same reasons as we saw a $16 + \bar{1}6$ in the construction of the superstring. Just as there, a GSO projection arises in the construction, which throws out the conjugate spinor, leaving the sole massive spinor we saw arise in the direct lattice approach.

7.5 The Massless Spectrum

In the case we must consider here, we can borrow a lot of what we learned in section 4.5 with hardly any adornment. We have sixteen compact left–moving bosons, X^i, which, together with the allowed momenta P^i, define a lattice Γ. The difference between this lattice and the ones we considered in section 4.5 is that there is no second part coming from a family of right–moving momenta, and hence it is only half the expected dimension, and with a purely Euclidean signature. This sixteen dimensional lattice must again be self–dual and even. This amounts to the requirement of modular invariance, just as before. More directly, we can see what effect this has on the low–lying parts of the spectrum.

Recall that the NS and R sector of the right hand side has zero point energy equal to $-1/2$ and 0, respectively. Recall that we then make, after the GSO projection, the vector $\mathbf{8}_v$, and its superpartner the spinor $\mathbf{8}_s$ from these two sectors. On the left hand side, we have the structure of the bosonic string, with zero point energy -1. There is no GSO projection on this side, and so potentially we have the tachyon, $|0\rangle$, the familiar massless states $\alpha^\mu_{-1}|0\rangle$, and the current algebra elements $J^a_{-1}|0\rangle$. These must be tensored together with the right hand side's states, but we must be aware that the level–matching condition is modified. To work out what it is we must take the difference between the correctly normalised *ten dimensional* M^2 operators on each side. We must also recall that in making the ten dimensional M^2 operator, we are left with a remainder, the contribution to the internal momentum $\alpha' p_L^2/4$. The result is:

$$\frac{\alpha' p_L^2}{4} + N - 1 = \tilde{N} - \begin{cases} -\frac{1}{2} \\ 0 \end{cases} ,$$

where the choice corresponds to the NS or R sectors.

Now we can see how the tachyon is projected out of the theory, even without a GSO projection on the left. The GSO on the right has thrown out the tachyon there, and so we start with $\tilde{N} = \frac{1}{2}$ there. The left tachyon is $N = 0$, but this is not allowed, and we must have the even condition $\alpha' p_L^2/2 = 2$ which corresponds to switching on a current J^a_{-1}, making a massless state. If we do not have this state excited, then we can also make a massless state with $N = 1$, corresponding to $\alpha^\mu_{-1}|0\rangle$.

The massless states we can make by tensoring left and right, respecting level–matching are actually familiar: In the NS–NS sector, we have $\alpha^\mu_{-1}\psi^\nu_{-\frac{1}{2}}|0\rangle$, which is the graviton, $G_{\mu\nu}$ antisymmetric tensor $B_{\mu\nu}$ and dilaton Φ in the usual way. We also have $J^a_{-1}\psi^\mu_{-\frac{1}{2}}|0\rangle$, which gives an $E_8 \times E_8$ or $SO(32)$ gauge boson, $A^{\mu a}$. In the NS–R sector, we have $\alpha^\mu_{-1}|0\rangle_\alpha$ which is the gravitino, ψ^μ_α. Finally, we have $J^a_{-1}|0\rangle_\alpha$, which is the superpartner of the gauge boson, λ^a_α. In the language we used earlier, we can write the left hand representations under $SO(8) \times G$ (where G is $SO(32)$ or $E_8 \times E_8$) as $(\mathbf{8}_v, \mathbf{1})$ or $(\mathbf{1}, \mathbf{496})$. Then the tensoring is

$$\begin{aligned} (\mathbf{8}_v, \mathbf{1}) \otimes (\mathbf{8}_v + \mathbf{8}_s) &= (\mathbf{1}, \mathbf{1}) + (\mathbf{35}, \mathbf{1}) + (\mathbf{28}, \mathbf{1}) + (\mathbf{56}, \mathbf{1}) + (\mathbf{8}', \mathbf{1}) \ , \\ (\mathbf{1}, \mathbf{496}) \otimes (\mathbf{8}_v + \mathbf{8}_s) &= (\mathbf{8}_v, \mathbf{496}) + (\mathbf{8}_s, \mathbf{496}) \ . \end{aligned}$$

So we see that we have again obtained the $\mathcal{N} = 1$ supergravity multiplet, coupled to a massless vector. The effective theory which must result at low energy must have the same gravity sector, but since the gauge fields arise at closed string tree level, their Lagrangian must have a dilaton coupling $e^{2\Phi}$, instead of e^Φ for the open string where the gauge fields arise at open string tree level.

7.6 The Ten Dimensional Supergravities

Just as we saw in the case of the bosonic string, we can truncate consistently to focus on the massless sector of the string theories, by focusing on low energy limit $\alpha' \to 0$. Also as before, the dynamics can be summarised in terms of a low energy effective (field theory) action for these fields, commonly referred to as "supergravity".

The bosonic part of the low energy action for the type IIA string theory in ten dimensions may be written (*c.f.* (96)) as (the wedge product is understood):[1, 2, 71]

$$S_{\rm IIA} = \frac{1}{2\kappa_0^2}\int d^{10}x(-G)^{1/2}\left\{e^{-2\Phi}\left[R+4(\nabla\Phi)^2-\frac{1}{12}(H^{(3)})^2\right]\right.$$
$$\left.-\frac{1}{4}(G^{(2)})^2-\frac{1}{48}(G^{(4)})^2\right\}-\frac{1}{4\kappa_0^2}\int B^{(2)}dC^{(3)}dC^{(3)}\ . \qquad (349)$$

As before $G_{\mu\nu}$ is the metric in string frame, Φ is the dilaton, $H^{(3)} = dB^{(2)}$ is the field strength of the NS–NS two form, while the Ramond-Ramond field strengths are $G^{(2)} = dC^{(1)}$ and $G^{(4)} = dC^{(3)} + H^{(3)} \wedge C^{(1)}$.[5]

For the bosonic part in the case of type IIB, we have:

$$S_{\rm IIB} = \frac{1}{2\kappa_0^2}\int d^{10}x(-G)^{1/2}\left\{e^{-2\phi}\left[R+4(\nabla\Phi)^2-\frac{1}{12}(H^{(3)})^2\right]\right.$$
$$\left.-\frac{1}{12}(G^{(3)}+C^{(0)}H^{(3)})^2-\frac{1}{2}(dC^{(0)})^2-\frac{1}{480}(G^{(5)})^2\right\}$$
$$+\frac{1}{4\kappa_0^2}\int\left(C^{(4)}+\frac{1}{2}B^{(2)}\,C^{(2)}\right)G^{(3)}\,H^{(3)}\ . \qquad (350)$$

Now, $G^{(3)} = dC^{(2)}$ and $G^{(5)} = dC^{(4)} + H^{(3)}C^{(2)}$ are R–R field strengths, and $C^{(0)}$ is the RR scalar. (Note that we have canonical normalizations for the kinetic terms of forms: there is a prefactor of the inverse of $-2\times p!$ for a p–form field strength.) There is a small complication due to the fact that we require the R–R four form $C^{(4)}$ to be self dual, or we will have too many degrees of freedom. We write the action here and remind ourselves to always impose the self duality constraint on its field strength $F^{(5)} = dC^{(4)}$ by hand in the equations of motion: $F^{(5)} = {}^*F^{(5)}$.

Equation (99) tells us that in ten dimensions, we must use:

$$\widetilde{G}_{\mu\nu} = e^{(\Phi_0-\Phi)/2}G_{\mu\nu}\ . \qquad (351)$$

to convert these actions to the Einstein frame. As before, (see discussion below (101)) Newton's constant will be set by

$$2\kappa^2 \equiv 2\kappa_0^2 g_s^2 = 16\pi G_N = (2\pi)^7\alpha'^4 g_s^2\ , \qquad (352)$$

where the latter equality can be established by (for example) direct examination of the results of a graviton scattering computation. We will see that it gives a very natural normalization for the masses and charges of the various branes in the theory. Also g_s is set by the asymptotic value of the dilaton at infinity: $g_s \equiv e^{\Phi_0}$.

Those were the actions for the ten dimensional supergravities with thirty–two supercharges. Let us consider those with sixteen supercharges. For the bosonic part of type I, we can construct it by dropping the fields which are odd under Ω and then adding the gauge sector, plus a number of cross terms which result from canceling anomalies, as we discussed in subsection 7.2.1:

$$S_{\rm I} = \frac{1}{2\kappa_0^2}\int d^{10}x(-G)^{1/2}\left\{e^{-2\Phi}\left[R+4(\nabla\phi)^2\right]-\frac{1}{12}(\widetilde{G}^{(3)})^2-\frac{\alpha'}{8}e^{-\Phi}{\rm Tr}(F^{(2)})^2\right\}\ . \qquad (353)$$

Here, $\widetilde{G}^{(3)}$ is a modified field strength for the 2–form potential, defined in equation (346). Recall that this modification followed from the requirement of cancellation of the anomaly *via* the Green–Schwarz mechanism.

We can generate the heterotic low–energy action using a curiosity which will be meaningful later. Notice that a simple redefinition of fields:

$$G_{\mu\nu}(\text{type I}) \quad = \quad e^{-\Phi}G_{\mu\nu}(\text{heterotic})$$

[5] This can be derived by dimensional reduction from the structurally simpler eleven dimensional supergravity action, presented in section 11, but at this stage, this relation is a merely formal one. We shall see a dynamical connection later.

$$\begin{aligned}
\Phi(\text{type I}) &= -\Phi(\text{heterotic}) \\
\widetilde{G}^{(3)}(\text{type I}) &= \widetilde{H}^{(3)}(\text{heterotic}) \\
A_\mu(\text{type I}) &= A_\mu(\text{heterotic}) \ ,
\end{aligned} \tag{354}$$

takes one from the type I Lagrangian to:

$$S_{\rm H} = \frac{1}{2\kappa_0^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left\{R + 4(\nabla\phi)^2 - \frac{1}{12}(\widetilde{H}^{(3)})^2 - \frac{\alpha'}{8}{\rm Tr}(F^{(2)})^2\right\} , \tag{355}$$

where (renaming $C^{(2)} \to B^{(2)}$)

$$\widetilde{H}^{(3)} = dB^{(2)} - \frac{\alpha'}{4}\left[\frac{1}{30}\omega_{3\rm Y}(A) - \omega_{3\rm L}(\Omega)\right] \ . \tag{356}$$

This is the low energy effective Lagrangian for the heterotic string theories. Note that in (355), α' is measured in heterotic units of length.

We can immediately see two key features about these theories. The first was anticipated earlier: Their Lagrangian for the gauge fields have a dilaton coupling $e^{-2\Phi}$, since they arise at closed string tree level, instead of $e^{-\Phi}$ for the open string where where the gauge fields arise at open string tree level. The second observation is that since from equation.(354) the dilaton relations tell us that $g_s(\text{type I}) = g_s^{-1}(\text{heterotic})$, there is a non–perturbative connection between these two theories, although they are radically different in perturbation theory. We are indeed *forced* to consider these theories when we study the type I string in the limit of strong coupling.

7.7 Heterotic Toroidal Compactifications

Much later, it will be of interest to study simple compactifications of the heterotic strings, and the simplest result from placing them on tori[164, 165]. Our interest here is not in low energy particle physics phenomenology, as this would require us to compactify on more complicated spaces to break the large amount of supersymmetry and gauge symmetry. Instead, we shall see that it is quite instructive, on the one hand, and on the other hand, studying various superstring compactifications with D–brane sectors taken into account will produce vacua which are in fact strong/weak coupling dual to heterotic strings on tori. This is another remarkable consequence of duality which forces us to consider the heterotic strings even though they cannot have D–brane sectors.

Actually, there is not much to do. From our work in section 7.4 and from that in section 4.5, it is easy to see what the conditions for the consistency of a heterotic toroidal compactification must be. Placing some of the ten dimensions on a torus T^d will give us the possibility of having windings, and right–moving momenta. In addition, the gauge group can be broken by introducing Wilson lines (see section 4.9) on the torus for the gauge fields A^μ. This latter choice breaks the gauge group to the maximal Abelian subgroup, which is $U(1)^{16}$.

The compactification simply enlarges our basic sixteen dimensional Euclidean lattice from $\Gamma_8\oplus\Gamma_8$ or Γ_{16} by two dimensions of Lorentzian signature $(1,1)$ for each additional compact direction, for the reasons we already discussed in section 4.5. So we end up with a lattice with signature $(16+d,d)$, on which there must be an action of $O(d,16+d)$ generating the lattices.

Again, we will have that there is a physical equivalence between some of these lattices, because physics only depends on p_L^2 and p_R^2, and further, there will be the discrete equivalences corresponding to the action of a T–duality group, which is $O(d,16+d,\mathbb{Z})$.

The required lattices are completely classified, as a mathematical exercise. In summary, the space of inequivalent toroidal compactifications turns out to be:

$$\mathcal{M}_{T^d} = [O(d)\times O(d+16)]\backslash O(d,d+16)/O(d,d+16,\mathbb{Z}) \ . \tag{357}$$

Notice, after a quick computation, that the dimension of this space is $d^2 + 16d$. So in addition to the fields $G_{\mu\nu}, B_{\mu\nu}$ and Φ, we have that number of extra massless scalars in the $\mathcal{N} = 2$, $D = 6$ low energy theory. The first part of the result comes, as before from the available constant components, G_{mn} and B_{mn}, of the internal metric and antisymmetric tensor on T^d. The remaining part comes from the sixteen generic constant internal gauge bosons (the Wilson lines), A_m for each circle.

Let us compute what the generic gauge group of this compactified model is. There is of course the $U(1)^{16}$ from the original current algebra sector. In addition, there is a $U(1) \times U(1)$ coming from each compact dimension, since we have Kaluza–Klein reduction of the metric and antisymmetric tensor. Therefore, the generic gauge group is $U(1)^{16+2d}$.

To get something less generic, we must tune some moduli to special points. Of course, we can choose to switch off some of the Wilson lines, getting non–Abelian gauge groups from the current algebra sector, restoring an $E_8 \times E_8 \times U(1)^{2d}$ or $SO(32) \times U(1)^{2d}$ gauge symmetry. We also have the possibility of enhancing the Kaluza–Klein factor by tuning the torus to special points. We simply need to make states of the form $\exp(ik_L \cdot X_L)\psi^\mu_{-1/2}|0>$, where we can have left–moving momenta of $\alpha' p_L^2/2 = 2$, (we are referring to the components of p_L which are in the torus T^d). This will give any of the A–D–E series of gauge groups up to a rank $2d$ in this sector.

The reader will have noticed that we only gave one family of lattices for each dimension d of the torus. We did not have one choice for the $E_8 \times E_8$ string and another for the $SO(32)$ string. In other words, as soon as we compactify one heterotic string on a circle, we find that we could have arrived at the same spectrum by compactifying the other heterotic string on a circle. This is of course T–duality. It is worth examining further, and we do this in section 8.1.3.

7.8 Superstring Toroidal Compactification

The placement of the superstrings on tori is at face value rather less interesting than the heterotic case, and so we will not spend much time on it here, although will return to it later when we revisit T–duality, and again when we study U–duality.

Imagine that we compactify one of our superstring theories on the torus T^d. We simply ask that d of the directions are periodic with some chosen radius, as we did in section 4.5 for the bosonic string. This does not not affect any of our discussion of supercharges, *etc*, and we simply have a $(10 - d)$–dimensional theory with the same amount of supersymmetry as the ten dimensional theory which we started with. As discussed in section 4.4, there is a large $O(d, d, \mathbb{Z})$ pattern of T–duality groups available to us. There are also Kaluza–Klein gauge groups $U(1)^{2d}$ coming from the internal components of the graviton and the anti–symmetric tensor. In addition, there are Kaluza–Klein gauge groups coming from the possibility of some of the R–R sector antisymmetric tensors having internal indices. Note that there aren't the associated enhanced gauge symmetries present at special radii, since the appropriate objects which would have arisen in a current algebra, J^a_{-1}, do not give masses states in spacetime, and in any case level matching would have forbidden them from being properly paired with $\psi^\mu_{-1/2}$ to give a spacetime vector.

To examine the possibilities, it is probably best to study a specific example, and we do the case of placing the type IIA string theory on T^5. Let us first count the gauge fields. This can be worked out simply by counting the number of ways of wrapping the metric and the various p–form potentials (with p odd) in the theory on the five circles of the T^5 to give a one–form in the remaining five non–compact directions. From the NS–NS sector there are 5 Kaluza–Klein gauge bosons and 5 gauge bosons from the antisymmetric tensor. There are 16 gauge bosons from the dimensional reduction of the various R–R forms: The breakdown is 10+5+1 from the forms $C^{(3)}$, $C^{(5)}$ and $C^{(1)}$, respectively, since *e.g.* there are 10 independent ways of making two out of the three indices of $C^{(3)}$ be any two out of the five internal directions, and so on. Finally, in five dimensions, one can form a two form field strength from the Hodge dual $^*H$ of the 3–form field strength of the NS–NS $B_{\mu\nu}$, thus defining another gauge field.

So the gauge group is generically $U(1)^{27}$. There are in fact a number of massless fields corresponding to moduli representing inequivalent sizes and shapes for the T^5. We can count them easily. We have the $5^2 = 25$ components coming from the graviton and anti–symmetric tensor field. From the R–R sector there is only one way of getting a scalar from $C^{(5)}$, and 5 and 10 ways from $C^{(1)}$ and $C^{(3)}$, respectively. This gives 41 moduli. Along with the dilaton, this gives a total of 42 scalars for this compactification.

By now, the reader should be able to construct the very same five dimensional spectrum but starting with the type IIB string and placing it on T^5. This is a useful exercise in preparation for later. The same phenomenon will happen with any torus, T^d. Thus we begin to uncover the fact that the type IIA and type IIB string theories are (T–dual) equivalent to each other when placed on circles. We shall examine this in more detail in section 8.1, showing that the equivalence is exact.

The full T–duality group is actually $O(5,5;\mathbb{Z})$. It acts on the different sectors independently, as it ought to. For example, for the gauge fields, it mixes the first 10 NS–NS gauge fields among themselves, and the 16 R–R gauge fields among themselves, and leaves the final NS–NS field invariant. Notice that the fields fill out sensible representations of $O(5,5;\mathbb{Z})$. Thinking of the group as roughly $SO(10)$, those familiar with numerology from grand unification might recognize that the sectors are transforming as the **10**, **16**, and **1**.

A little further knowledge will lead to questions about the fact that $\mathbf{10}\oplus\mathbf{16}\oplus\mathbf{1}$ is the decomposition of the **27** (the fundamental representation) of the group E_6, but we should leave this for a later time, when we come to discuss U–duality in section 11.

7.9 A Superstring Orbifold: The K3 Manifold

Before we go further, let us briefly revisit the idea of strings propagating on an orbifold, and take it a bit further. Imagine that we compactify one of our closed string theories on the four torus, T^4. Let us take the simple case where there the torus is simply the product of four circles, S^1, each with radius R. Let us choose that the four directions (say) x^6, x^7, x^8 and x^9 are periodic with period $2\pi R$. The resulting six dimensional theory has $\mathcal{N}=4$ supersymmetry.

Let us orbifold the theory by the $\mathbb{Z}_2$ group which has the action

$$\mathbf{R}:\quad x^6,x^7,x^8,x^9\to -x^6,-x^7,-x^8,-x^9\ , \tag{358}$$

which is clearly a good symmetry to divide by. The $\mathbb{R}^4$ is naturally acted on by rotations, $SO(4)\sim SU(2)_L\times SU(2)_R$. We can choose to let $\mathbf{R}$ be embedded in the $SU(2)_L$ which acts on the $\mathbb{R}^4$. This will leave an $SU(2)_R$ which descends to the six dimensions as a global symmetry. It is in fact the R–symmetry of the remaining $D=6$, $\mathcal{N}=2$ model. We shall use this convention a number of times in what is to come.

7.9.1 The Orbifold Spectrum

We can construct the resulting six dimensional spectrum by first working out (say) the left—moving spectrum, seeing how it transforms under $\mathbf{R}$ and then tensoring with another copy from the right in order to construct the closed string spectrum.

Let us now introduce a bit of notation which will be useful in the future. Use the label x^m, $m=6,7,8,9$ for the orbifolded directions, and use x^μ, $\mu=0,\ldots,5$, for the remaining. Let us also note that the ten dimensional Lorentz group is decomposed as

$$SO(1,9)\supset SO(1,5)\times SO(4)\ .$$

We shall label the transformation properties of our massless states in the theory under the $SU(2)\times SU(2)=SO(4)$ Little group. Just as we did before, it will be useful in the Ramond sector to choose a labeling of the states which refers to the rotations in the planes (x^0,x^1), (x^2,x^3), *etc.,* as eigenstates $s_0, s_1 ... s_4$ of the operator S^{01}, S^{23}, *etc.*, (see equations (324) and (326) and surrounding discussion). With this in mind, we can list the states on the left which survive the GSO projection:

sector	state	$\mathbf{R}$ charge	$SO(4)$ charge
NS	$\psi^{\mu}_{-\frac{1}{2}}\|0;k>$	$+$	$(\mathbf{2},\mathbf{2})$
	$\psi^{m}_{-\frac{1}{2}}\|0;k>$	$-$	$\mathbf{4}(\mathbf{1},\mathbf{1})$
R	$\|s_1s_2s_3s_4>;\ s_1=+s_2,\ s_3=-s_4$	$+$	$\mathbf{2}(\mathbf{2},\mathbf{1})$
	$\|s_1s_2s_3s_4>;\ s_1=-s_2,\ s_3=+s_4$	$-$	$\mathbf{2}(\mathbf{1},\mathbf{2})$

Crucially, we should also examine the "twisted sectors" which will arise, in order to make sure that we get a modular invariant theory. The big difference here is that in the twisted sector, the moding of the fields in the x^m directions is shifted. For example, the bosons are now half–integer moded. We have to recompute the zero point energies in each sector in order to see how to get massless states (see (70)):

$$
\begin{aligned}
\text{NS sector:}\quad & 4\left(-\frac{1}{24}\right)+4\left(-\frac{1}{48}\right)+4\left(\frac{1}{48}\right)+4\left(\frac{1}{24}\right)=0\ ,\\
\text{R sector:}\quad & 4\left(-\frac{1}{24}\right)+4\left(\frac{1}{24}\right)+4\left(\frac{1}{48}\right)+4\left(-\frac{1}{48}\right)=0.
\end{aligned}
\tag{359}
$$

This is amusing, both the Ramond and NS sectors have zero vacuum energy, and so the integer moded sectors will give us degenerate vacua. We see that it is only states $|s_1s_2>$ which contribute from the R–sector (since they are half integer moded in the x^m directions) and the NS sector, since it is integer moded in the x^m directions, has states $|s_3s_4>$.

It is worth seeing in (359) how we achieved this ability to make a massless field in this case. The single twisted sector ground state in the bosonic orbifold theory with energy 1/48, was multiplied by 4 since there are four such orbifolded directions. Combining this with the contribution from the four unorbifolded directions produced just the energy needed to cancel the contribution from the fermions.

The states and their charges are (after imposing GSO):

sector	state	$\mathbf{R}$ charge	$SO(4)$ charge
NS	$\|s_3s_4>;\ s_3=-s_4$	$+$	$\mathbf{2}(\mathbf{1},\mathbf{1})$
R	$\|s_1s_2>;\ s_1=-s_2$	$-$	$(\mathbf{1},\mathbf{2})$

Now we are ready to tensor. Recall that we could have taken the opposite GSO choice here to get a left moving with the identical spectrum, but with the swap $(\mathbf{1},\mathbf{2})\leftrightarrow(\mathbf{2},\mathbf{1})$. Again we have two choices: Tensor together two identical GSO choices, or two opposite. In fact, since six dimensional supersymmetries are chiral, and the orbifold will keep only two of the four we started with, we can write these choices as $(0,2)$ or $(1,1)$ supersymmetry, resulting from type IIB or IIA on K3. It is useful to tabulate the result for the bosonic spectra for the untwisted sector:

sector	$SO(4)$ charge
NS–NS	$(\mathbf{3},\mathbf{3})+(\mathbf{1},\mathbf{3})+(\mathbf{3},\mathbf{1})+(\mathbf{1},\mathbf{1})$
	$10(\mathbf{1},\mathbf{1})+6(\mathbf{1},\mathbf{1})$
R–R (IIB)	$2(\mathbf{3},\mathbf{1})+4(\mathbf{1},\mathbf{1})$
	$2(\mathbf{1},\mathbf{3})+4(\mathbf{1},\mathbf{1})$
R–R (IIA)	$4(\mathbf{2},\mathbf{2})$
	$4(\mathbf{2},\mathbf{2})$

and for the twisted sector:

sector	$SO(4)$ charge
NS–NS	$3(\mathbf{1},\mathbf{1})+(\mathbf{1},\mathbf{1})$
R–R (IIB)	$(\mathbf{1},\mathbf{3})+(\mathbf{1},\mathbf{1})$
R–R (IIA)	$(\mathbf{2},\mathbf{2})$

Recall now that we have two twisted sectors for each orbifolded circle, and hence there are 16 twisted sectors in all, for $T^4/\mathbb{Z}_2$. Therefore, to make the complete model, we must take sixteen copies of the content of the twisted sector table above.

Now let identify the various pieces of the spectrum. The gravity multiplet $G_{\mu\nu} + B_{\mu\nu} + \Phi$ is in fact the first line of our untwisted sector table, coming from the NS–NS sector, as expected. The field B can be seen to be broken into its self–dual and anti–self–dual parts $B^+_{\mu\nu}$ and $B^-_{\mu\nu}$, transforming as $(\mathbf{1},\mathbf{3})$ and $(\mathbf{3},\mathbf{1})$. There are sixteen other scalar fields, $((\mathbf{1},\mathbf{1}))$, from the untwisted NS–NS sector. The twisted sector NS–NS sector has 4×16 scalars. Not including the dilaton, there are 80 scalars in total from the NS–NS sector.

Turning to the R–R sectors, we must consider the cases of IIA and IIB separately. For type IIA, there are 8 one–forms (vectors, $(\mathbf{2},\mathbf{2})$) from the untwisted sector and 16 from the twisted, giving a total of 24 vectors, and have a generic gauge group $U(1)^{24}$.

For type IIB, the untwisted R–R sector contains three self–dual and three anti–self–dual tensors, while there are an additional 16 self–dual tensors $(\mathbf{1},\mathbf{3})$. We therefore have 19 self–dual $C^+_{\mu\nu}$ and 3 anti–self–dual $C^-_{\mu\nu}$. There are also eight scalars from the untwisted R–R sector and 16 scalars from the twisted R–R sector. In fact, including the dilaton, there are 105 scalars in total for the type IIB case.

7.9.2 Another Miraculous Anomaly Cancellation

This type IIB spectrum is chiral, as already mentioned, and in view of what we studied in earlier sections, the reader must be wondering whether or not it is anomaly–free. It actually is, and it is a worthwhile exercise to check this, using the polynomials in section 7.1.4.

The cancellation is so splendid that we cannot resist explaining it in detail here. To do so we should be careful to understand the $\mathcal{N}=2$ multiplet structure properly. A sensible non–gravitational multiplet has the same number of bosonic degrees of freedom as fermionic, and so it is possible to readily write out the available ones given what we have already seen. (Or we could simply finish the tensoring done in the last section, doing the NS–R and R–NS parts to get the fermions.) Either way, table 2 has the multiplets listed. The 16 components of the supergravity bosonic multiplet is accompanied by two copies of the 16 components

Multiplet	Bosons	Fermions
vector	$(\mathbf{2},\mathbf{2})+4(\mathbf{1},\mathbf{1})$	$2(\mathbf{1},\mathbf{2})+2(\mathbf{2},\mathbf{1})$
SD tensor	$(\mathbf{1},\mathbf{3})+5(\mathbf{1},\mathbf{1})$	$4(\mathbf{2},\mathbf{1})$
ASD tensor	$(\mathbf{3},\mathbf{1})+5(\mathbf{1},\mathbf{1})$	$4(\mathbf{2},\mathbf{1})$
supergravity	$(\mathbf{3},\mathbf{3})+(\mathbf{3},\mathbf{1})+(\mathbf{1},\mathbf{3})+(\mathbf{1},\mathbf{1})$	$2(\mathbf{3},\mathbf{2})+2(\mathbf{2},\mathbf{1})$ or $2(\mathbf{2},\mathbf{3})+2(\mathbf{1},\mathbf{2})$

Table 2: The structure of the $\mathcal{N}=2$ multiplets in $D=6$.

making up a gravitino and a dilatino. These two copies are the same chirality for type IIB and opposite for type IIA.

The next thing to do is to repackage the spectrum we identified earlier in terms of these multiplets. First, notice that the supergravity multiplet has one $(\mathbf{1},\mathbf{1})$, four $(\mathbf{2},\mathbf{1})$'s and one $(\mathbf{1},\mathbf{3})$. With four other scalars, we can make a full tensor multiplet. (The other $(\mathbf{3},\mathbf{1})$, which is an anti–self dual piece makes up the rest of $B_{\mu\nu}$.) That gives us 19 complete self dual tensor multiplets in total and 2 complete anti–self dual ones since the last one is not complete. Since there are five scalars in a tensor multiplet this accounts for the 105 scalars that we have.

So we can study the anomaly now, knowing what (anti–) self dual tensors, and fermions we have. Consulting section 7.1.4, we note that the polynomials listed for the fermions are for complex fermions, and so we must divide them by 2 to get the ones appropriate for the real components we have counted in the orbifolding. Putting it together according to what we have said above for the content of the spectrum, we have:

$$19\hat{I}_8^{(\mathbf{1},\mathbf{3})} + 19\times 4\hat{I}_8^{(\mathbf{2},\mathbf{1})} + 2\hat{I}_8^{(\mathbf{3},\mathbf{1})} + 2\times 4\hat{I}_8^{(\mathbf{2},\mathbf{1})} + 2\hat{I}_8^{(\mathbf{3},\mathbf{2})} + \hat{I}_8^{(\mathbf{3},\mathbf{1})} = 0\ , \qquad (360)$$

where we have listed, respectively, the contribution of the 19 self dual tensors, the two anti self dual tensors, the two gravitinos, and the remaining piece of the supergravity multiplet. That this combination of polynomials vanishes is amazing[103].

7.9.3 The K3 Manifold

Quite remarkably, there is a geometrical interpretation of all of those data presented in the previous subsections in terms of compactifying type II string theory on a smooth manifold. The manifold is K3. It is a four dimensional manifold containing 22 independent two–cycles, which are topologically two–spheres more properly described as the complex surface $\mathbb{CP}^1$ in this context. Correspondingly the space of two forms which can be integrated over these two cycles is 22 dimensional. So we can choose a basis for this space. Nineteen of them are self–dual and three of them are anti–self–dual, in fact. The space of metrics on K3 is in fact parametrised by 58 numbers.

In compactifying the type II superstrings on K3, the ten dimensional gravity multiplet and the other R–R fields gives rise to six dimensional fields by direct dimensional reduction, while the components of the fields in the K3 give other fields. The six dimensional gravity multiplet arises by direct reduction from the NS–NS sector, while 58 scalars arise, parameterizing the 58 dimensional space of K3 metrics which the internal parts of the metric, G_{mn}, can choose. Correspondingly, there are 22 scalars arising from the 19+3 ways of placing the internal components of the antisymmetric tensor, B_{mn} on the manifold. A commonly used terminology is that the form has been "wrapped" on the 22 two–cycles to give 22 scalars.

In the R–R sector of type IIB, there is one scalar in ten dimensions, which directly reduces to a scalar in six. There is a two–form, which produces 22 scalars, in the same way as the NS–NS two form did. The self–dual four form can be integrated over the 22 two cycles to give 22 two forms in six dimensions, 19 of them self–dual and 3 anti–self–dual. Finally, there is an extra scalar from wrapping the four form entirely on K3. This is precisely the spectrum of fields which we computed directly in the type IIB orbifold.

Alternatively, while the NS–NS sector of type IIA gives rise to the same fields as before, there is in the R–R sector a one form, three form and five form. The one form directly reduces to a one form in six dimensions. The three form gives rise to 22 one forms in six dimensions while the five form gives rise to a single one form. We therefore have 24 one forms (generically carrying a $U(1)$ gauge symmetry) in six dimensions. This also completes the smooth description of the type IIA on K3 spectrum, which we computed directly in the orbifold limit. See section 7.9.5 for a significant comment on this spectrum.

7.9.4 Blowing Up the Orbifold

The connection between the orbifold and the smooth K3 manifold is as follows:[74] K3 does indeed have a geometrical limit which is $T^4/\mathbb{Z}_2$, and it can be arrived at by tuning enough parameters, which corresponds here to choosing the vev's of the various scalar fields. Starting with the $T^4/\mathbb{Z}_2$, there are 16 fixed points which look locally like $\mathbb{R}^4/\mathbb{Z}^2$, a singular point of infinite curvature. It is easy to see where the 58 geometric parameters of the K3 metric come from in this case. Ten of them are just the symmetric G_{mn} constant components, on the internal directions. This is enough to specify a torus T^4, since the hypercube of the lattice in $\mathbb{R}^4$ is specified by the ten angles between its unit vectors, $\mathbf{e}^m \cdot \mathbf{e}^n$. Meanwhile each of the 16 fixed points has 3 scalars associated to its metric geometry. (The remaining fixed point NS–NS scalar in the table is from the field B, about which we will have more to say later.)

The three metric scalars can be tuned to resolve or "blow up" the fixed point, and smooth it out into the $\mathbb{CP}^1$ which we mentioned earlier. (This accounts for 16 of the two–cycles. The other six correspond to the six $\mathbb{Z}_2$ invariant forms $dX^m \wedge dX^n$ on the four–torus.) The smooth space has a known metric, the "Eguchi–Hanson" metric,[80] which is *locally* asymptotic to $\mathbb{R}^4$ (like the singular space) but with a global $\mathbb{Z}_2$ identification. Its metric is:

$$ds^2 = \left(1 - \left(\frac{a}{r}\right)^4\right)^{-1} dr^2 + r^2 \left(1 - \left(\frac{a}{r}\right)^4\right) (d\psi + \cos\theta d\phi)^2 + r^2 d\Omega_2^2 \ . \tag{361}$$

Here, $d\Omega_2$ is the metric on a round S^2, written in polar coordinates as $d\theta^2 + \sin^2\theta d\phi^2$. The angles θ, ϕ, ψ are the S^3 Euler angles, where $0 < \theta \leq \pi$, $0 < \phi \leq 2\pi$, $0 < \psi \leq 4\pi$.

The point $r = a$ is an example of a "bolt" singularity. Near there, the space is topologically $\mathbb{R}^2_{r\psi} \times S^2_{\theta\phi}$, with the S^2 of radius a, and the singularity is a coordinate one provided ψ has period 2π.

Let us see how this works. To examine the potential singularity at $r = a$, look *near* $r = a$. Choose, if you will, $r = a + \varepsilon$ for small ε, and:

$$ds^2 = \frac{a}{4\varepsilon}\left[d\varepsilon^2 + 16\varepsilon^2(d\psi + \cos\theta d\phi)^2\right] + (a^2 + 2a\varepsilon)d\Omega_2^2 \ ,$$

which as $\varepsilon \to 0$ is obviously topologically looking locally like $\mathbb{R}^2_{\varepsilon,\psi} \times S^2_{\theta,\phi}$, where the S^2 is of radius a. (Globally, there is a fibred structure due to the $d\psi d\phi$ cross term.) Incidentally, this is perhaps the quickest way to see that the Euler number or "Euler characteristic" of the space has to be equal to that of an S^2, which is 2. There is a potential "bolt" singularity at $r = 0$.

It is a true singularity for arbitrary choices of periodicity $\Delta\psi$ of ψ, since there is a conical deficit angle in the plane. In other words, we have to ensure that as we get to the origin of the plane, $\varepsilon = 0$, the ψ–circles have circumference 2π, no more or less. Infinitesimally, we make those measures with the metric, and so the condition is:

$$2\pi = \lim_{\varepsilon\to 0}\left(\frac{d(2\sqrt{a}\varepsilon^{1/2})\Delta\psi}{d\varepsilon\sqrt{(a/4)}\varepsilon^{-1/2}}\right) \ ,$$

which gives $\Delta\psi = 2\pi$. So in fact, we must spoil our S^3 which was a nice orbit of the $SU(2)$ isometry, by performing an $\mathbb{Z}_2$ identification on ψ, giving it half its usual period. In this way, the "bolt" singularity $r = a$ is just a harmless artifact of coordinates.[79, 78] Also, we are left with an $SO(3) = SU(2)/\mathbb{Z}_2$ isometry of the metric. The space at infinity is $S^3/\mathbb{Z}_2$, just like an $\mathbb{R}^4/\mathbb{Z}_2$ fixed point. For small enough a, the Eguchi–Hanson space can b neatly slotted into the space left after cutting out the neighbourhood of the fixed point. The bolt is in fact the $\mathbb{CP}^1$ of the blowup mentioned earlier. The parameter a controls the size of the $\mathbb{CP}^1$, while the other two parameters correspond to how the $\mathbb{R}^2$ (say) is oriented in $\mathbb{R}^4$.

The Eguchi–Hanson space is the simplest example of an "Asymptotically Locally Euclidean" (ALE) space, which K3 can always be tuned to resemble locally. These spaces are classified [81] according to their identification at infinity, which can be any discrete subgroup, Γ, [82] of the $SU(2)$ which acts on the S^3 at infinity, to give S^3/Γ. These subgroups have been characterized by McKay, [83] and have an A–D–E classification. The metrics on the A–series are known explicitly as the Gibbons–Hawking metrics, [87] and Eguchi–Hanson is in fact the simplest of this series, corresponding to A_1. [88] We shall later use a D–brane as a probe of string theory on a $\mathbb{R}^4/\mathbb{Z}_2$ orbifold, an example which will show that the string theory correctly recovers all of the metric data (361) of these fixed points, and not just the algebraic data we have seen here.

For completeness, let us compute one more thing about K3 using this description. The Euler characteristic, in this situation, can be written in two ways[78]

$$\begin{aligned} \chi(K3) &= \frac{1}{32\pi^2}\int_{K3}\sqrt{g}\left(R_{abcd}R^{abcd} - 4R_{ab}R^{ab} + R^2\right) \\ &= \frac{1}{32\pi^2}\int_{K3}\sqrt{g}\epsilon_{abcd}R^{ab}R^{cd} \\ &= -\frac{1}{16\pi^2}\int_{K3}\mathrm{Tr}R\wedge R = 24 \ . \end{aligned} \qquad (362)$$

Even though no explicit metric for K3 has been written, we can compute χ as follows. [76, 78] If we take a manifold M, divide by some group G, remove some fixed point set F, and add in some set of new manifolds N, one at each point of F, the Euler characteristic of the new manifold is

$$\chi = \frac{\chi(M) - \chi(F)}{|G|} + \chi(N) \ . \qquad (363)$$

Here, $G = \mathbf{R} \equiv \mathbb{Z}_2$, and the Euler characteristic of the Eguchi–Hanson space is equal to 2, as we saw earlier. That of a point is 1, and of the torus is zero. We therefore get

$$\chi(K3) = -\frac{16}{2} + 16 \times 2 = 24 \ , \tag{364}$$

which will be of considerable use later on. So we have constructed the consistent, supersymmetric string propagation on the K3 manifold, using orbifold techniques.

7.9.5 Anticipating a String/String Duality in $D = 6$

We have seen that for type IIA we have an $\mathcal{N} = 2$, $D = 6$ supergravity with 80 additional scalars and 24 gauge bosons with a generic gauge group $U(1)^{24}$. The attentive reader will have noticed an apparent coincidence between the result for the spectrum of type IIA on K3 and another six dimensional spectrum which we obtained earlier. That was the spectrum of the heterotic string compactified on T^4, obtained in section 7.7 (put $d = 4$ in the results there). The moduli space of compactifications is in fact

$$O(20, 4, \mathbb{Z})\backslash O(20, 4)/[O(20) \times O(4)]$$

on both sides. We have seen where this comes from on the heterotic side. On the type IIA side it arises too. Start with the known

$$O(19, 3, \mathbb{Z})\backslash O(19, 3)/[O(19) \times O(3)]$$

for the standard moduli space of K3's (you should check that this has 57 parameters; there is an additional one for the volume). It acts on the 19 self–dual and 3 anti–self–dual 2–cycles. This classical geometry is supplemented by stringy geometry arising from $B_{\mu\nu}$, which can have fluxes on the 22 2–cycles, giving the missing 22 parameters. We will not prove here that the moduli space is precisely as above, and hence the same as globally and locally as the heterotic one, as it will take us beyond the scope of these notes.

Perturbatively, the coincidence of the spectra must be an accident. The two string theories in $D = 10$ are extremely dissimilar. One has twice the supersymmetry of the other and is simpler, having no large gauge group, while the other is chiral. We place the simpler theory on a complicated space (K3) and the more complex theory on a simple space T^4 and result in the same spectrum. The theories cannot be T–dual since the map would have to mix things which are unrelated by properties of circles. The only duality possible would have to go beyond perturbation theory. We will study such situations briefly in section duality. Note also that there is something missing. At special points in the heterotic moduli space we have seen that it is possible to get large enhanced non–Abelian gauge groups. There is no sign of that here in how we have described the type IIA string theory using conformal field theory. In fact it is possible to go beyond conformal field theory and describe these special points using D–branes although time will not permit us to do that here. You should be able to anticipate it though. There is an A–D–E classification of the singularities of K3, as already mentioned, where a two dimensional locus (a family of intersecting $\mathbb{CP}^1$s) shrinks to zero size. In type IIA, we can wrap D2–branes on this locus. They appear as particles in the six–dimensional theory. When the cycles shrink, these particles become massless, and are precisely the new massless states required to act give the enhanced gauge symmetry needed to match the dual heterotic spectrum. You can learn more about this from reviews in the literature[86, 152].

8 Supersymmetric Strings and T–Duality

8.1 T–Duality of Supersymmetric Strings

We noticed in section 7.8, when considering the low energy spectrum of the type II superstrings compactified on tori, that there is an equivalence between them. We saw much the same things happen for the heterotic strings in section 7.7 too. This is of course T–duality, as we should examine it further here and check that it is the familiar exact equivalence. Just as in the case of bosonic strings, doing this when there are open string sectors present will uncover D–branes of various dimensions.

8.1.1 T–Duality of Type II Superstrings

T–duality on the closed oriented Type II theories has a somewhat more interesting effect than in the bosonic case.[16, 12] Consider compactifying a single coordinate X^9, of radius R. In the $R \to \infty$ limit the momenta are $p_R^9 = p_L^9$, while in the $R \to 0$ limit $p_R^9 = -p_L^9$. Both theories are $SO(9,1)$ invariant but under *different* $SO(9,1)$'s. T–duality, as a right–handed parity transformation (see (191)), reverses the sign of the right–moving $X^9(\bar{z})$; therefore by superconformal invariance it does so on $\tilde{\psi}^9(\bar{z})$. Separate the Lorentz generators into their left-and right–moving parts $M^{\mu\nu} + \widetilde{M}^{\mu\nu}$. Duality reverses all terms in $\widetilde{M}^{\mu 9}$, so the $\mu 9$ Lorentz generators of the T–dual theory are $M^{\mu 9} - \widetilde{M}^{\mu 9}$. In particular this reverses the sign of the helicity $\tilde{s}_4$ and so switches the chirality on the right–moving side. If one starts in the IIA theory, with opposite chiralities, the $R \to 0$ theory has the same chirality on both sides and is the $R \to \infty$ limit of the IIB theory, and *vice–versa*. In short, T–duality, as a one–sided spacetime parity operation, reverses the relative chiralities of the right– and left–moving ground states. The same is true if one dualises on any odd number of dimensions, whilst dualising on an even number returns the original Type II theory.

Since the IIA and IIB theories have different R–R fields, T_9 duality must transform one set into the other. The action of duality on the spin fields is of the form

$$S_\alpha(z) \to S_\alpha(z), \qquad \tilde{S}_\alpha(\bar{z}) \to P_9 \tilde{S}_\alpha(\bar{z}) \tag{365}$$

for some matrix P_9, the parity transformation (9-reflection) on the spinors. In order for this to be consistent with the action $\tilde{\psi}^9 \to -\tilde{\psi}^9$, P_9 must anticommute with Γ^9 and commute with the remaining Γ^μ. Thus $P_9 = \Gamma^9\Gamma^{11}$ (the phase of P_9 is determined, up to sign, by hermiticity of the spin field). Now consider the effect on the R–R vertex operators (335). The Γ^{11} just contributes a sign, because the spin fields have definite chirality. Then by the Γ–matrix identity (336), the effect is to add a 9-index to G if none is present, or to remove one if it is. The effect on the potential C ($G = dC$) is the same. Take as an example the Type IIA vector C_μ. The component C_9 maps to the IIB scalar C, while the $\mu \neq 9$ components map to $C_{\mu 9}$. The remaining components of $C_{\mu\nu}$ come from $C_{\mu\nu 9}$, and so on.

Of course, these relations should be translated into rules for T–dualising the spacetime fields in the supergravity actions (349) and (350). The NS–NS sector fields' transformations are the same as those shown in equations (249),(251), while for the R–R potentials:[73]

$$\begin{aligned}
\tilde{C}^{(n)}_{\mu\cdots\nu\alpha 9} &= C^{(n-1)}_{\mu\cdots\nu\alpha} - (n-1)\frac{C^{(n-1)}_{[\mu\cdots\nu|9}G_{|\alpha]9}}{G_{99}} \\
\tilde{C}^{(n)}_{\mu\cdots\nu\alpha\beta} &= C^{(n+1)}_{\mu\cdots\nu\alpha\beta 9} + nC^{(n-1)}_{[\mu\cdots\nu\alpha}B_{\beta]9} + n(n-1)\frac{C^{(n-1)}_{[\mu\cdots\nu|9}B_{|\alpha|9}G_{|\beta]9}}{G_{99}}
\end{aligned} \tag{366}$$

8.1.2 T–Duality of Type I Superstrings

Just as in the case of the bosonic string, the action of T–duality in the open and unoriented open superstring theory produces D–branes and orientifold planes. Having done it once, (say on X^9 with radius R), we get a T_9–dual theory on the line interval $S^1/\mathbb{Z}_2$, where $\mathbb{Z}_2$ acts as the reflection $X^9 \to -X^9$. The S^1 has radius $R' = \alpha'/R$). There are 16 D8–branes and their mirror images (coming from the 16 D9–branes), together with two orientifold O8–planes located at $X^9 = 0, \pi R'$. This is called the "Type I′" theory (and sometimes the "Type IA" theory, and then the usual open string it "Type IB"), about which we will have more to say later as well.

Starting with the type IB theory, *i.e.*, 16 D9–branes and one O9–plane, we can carry this out n times on n directions, giving us 16 D$(9-n)$ and their mirror images through 2^n O$(9-n)$–planes arranged on the hypercube of fixed points of $T^n/\mathbb{Z}_2$, where the $\mathbb{Z}_2$ acts as a reflection in the n directions. If n is odd, the bulk theory away from the planes and branes is type IIA string theory, while we are back in type IIB otherwise.

Let us focus here on a single D–brane, taking a limit in which the other D–branes and the O–planes are distant and can be ignored. Away from the D–brane, only closed strings propagate. The local physics is that of the Type II theory, with two gravitinos. This is true even though we began with the unoriented Type I theory which has only a single gravitino. The point is that the closed string begins with two gravitinos, one with the spacetime supersymmetry on the right–moving side of the world–sheet and one on the left. The orientation projection of the Type I theory leaves one linear combination of these. But in the T–dual theory, the orientation projection does not constrain the local state of the string, but relates it to the state of the (distant) image gravitino. Locally there are two independent gravitinos, with equal chiralities if n, (the number of dimensions on which we dualised) is even and opposite if n is odd.

This is all summarised nicely by saying that while the type I string theory comes from projecting the type IIB theory by Ω, the T–dual string theories come from projecting type II string theory compactified on the torus T^n by $\Omega\prod_m[R_m(-1)^F]$, where the product over m is over all the n directions, and R_m is a reflection in the mth direction. This is indeed a symmetry of the theory and hence a good symmetry with which to project. So we have that T–duality takes the orientifold groups into one another:

$$\{\Omega\}\leftrightarrow\{1,\Omega\textstyle\prod_m[R_m(-1)^F]\}\ . \tag{367}$$

This is a rather trivial example of an orientifold group, since it takes type II strings on the torus T^n and simply gives a theory which is simply related to type I string theory on T^n by n T–dualities. Nevertheless, it is illustrative of the general constructions of orientifold backgrounds made by using more complicated orientifold groups. This is a useful piece of technology for constructing string backgrounds with interesting gauge groups, with fewer symmetries, as a starting point for phenomenological applications.

8.1.3 T–duality for the Heterotic Strings

As we noticed in section 7.7, there is a T–duality equivalence between the heterotic strings once we compactify on a circle. Let us uncover it carefully. We can begin by compactifying the $SO(32)$ string on a circle of radius R, with Wilson line:

$$A_9^i=\frac{1}{2\pi R}\mathrm{diag}\left\{\frac{1}{2},\cdots\frac{1}{2},0,\cdots,0\right\}\ , \tag{368}$$

with eight $\frac{1}{2}$'s and eight 0's breaking down the gauge group to $SO(16)\times SO(16)$. We can compute the mass spectrum of the nine–dimensional theory which results from this reduction, in the presence of the Wilson line. This is no harder than the computations which we did in section 4. The Wilson line simply shifts the contribution to the spectrum coming from the p_L^i momenta. We can focus on the sector which is uncharged under the gauge group, *i.e.* we switch off the p_L^i. The mass formula is:

$$p_{\substack{L\\R}}=\frac{(n+2m)}{R}\pm\frac{2mR}{\alpha'}\ ,$$

where we see that the allowed windings (coming in units of two) are controlled by the integer m, and the momenta are controlled by m and n in the combination $n+2m$.

We could instead have started from the $E_8\times E_8$ string on a circle of radius R', with Wilson line

$$A_9^i=\frac{1}{2\pi R'}\mathrm{diag}\{1,0\cdots0,1,0,\cdots,0\}\ , \tag{369}$$

again in two equal blocks of eight. This also breaks down the gauge group to $SO(16)\times SO(16)$. A computation of the spectrum of the neutral states gives:

$$p'_{\substack{L\\R}}=\frac{(n'+2m')}{R'}\pm\frac{2m'R'}{\alpha'}\ ,$$

for integers n' and m'. We see that if we exchange $n+2m$ with m' and m with $n'+2m'$ then the spectrum is invariant if we do the right handed parity identification $p_L\leftrightarrow p'_L$, $p_R\leftrightarrow -p'_R$, provided that the circles' radii are inversely related as $R'=\alpha'/(2R)$.

We shall see that this relation will result in some very remarkable connections between non–perturbative string vacua much later, in section 11.

8.2 D–Branes as BPS Solitons

Let us return to the type II strings, and the D–branes which we can place in them. While there is type II string theory in the bulk, (*i.e.,* away from the branes and orientifolds), notice that the open string boundary conditions are invariant under only one supersymmetry. In the original Type I theory, the left–moving world–sheet current for spacetime supersymmetry $j_\alpha(z)$ flows into the boundary and the right–moving current $\tilde{j}_\alpha(\bar{z})$ flows out, so only the total charge $Q_\alpha + \tilde{Q}_\alpha$ of the left- and right-movers is conserved. Under T–duality this becomes

$$Q_\alpha + \left(\prod_m P_m\right) \tilde{Q}_\alpha \ , \tag{370}$$

where the product of reflections P_m runs over all the dualised dimensions, that is, over all directions orthogonal to the D–brane. Closed strings couple to open, so the general amplitude has only one linearly realized supersymmetry. That is, the vacuum without D–branes is invariant under $N = 2$ supersymmetry, but the state containing the D–brane is invariant under only $N = 1$: *it is a BPS state.*[180, 89]

BPS states must carry conserved charges. In the present case there is only one set of charges with the correct Lorentz properties, namely the antisymmetric R-R charges. The world volume of a p–brane naturally couples to a $(p+1)$–form potential $C_{(p+1)}$, which has a $(p+2)$-form field strength $G_{(p+2)}$. This identification can also be made from the g_s^{-1} behaviour of the D–brane tension: this is the behaviour of an R–R soliton, as can be seen directly from writing solutions of the supergravity equations of motion. we won't have time to develop that here[90, 91].

The IIA theory has Dp–branes for $p = 0$, 2, 4, 6, and 8. The vertex operators (335) describe field strengths of all even ranks from 0 to 10. The n–form and $(10-n)$–form field strengths are Hodge dual to one another[6], so a p–brane and $(6-p)$–brane are sources for the same field, but one magnetic and one electric. The field equation for the 10-form field strength allows no propagating states, but the field can still have a physically significant energy density[180, 92, 93].

The IIB theory has Dp–branes for $p = -1$, 1, 3, 5, 7, and 9. The vertex operators (335) describe field strengths of all odd ranks from 1 to 9, appropriate to couple to all but the 9–brane. The 9–brane does couple to a nontrivial *potential,* as we will see below.

A (-1)–brane is a Dirichlet instanton, defined by Dirichlet conditions in the time direction as well as all spatial directions.[94] Of course, it is not clear that T–duality in the time direction has any meaning, but one can argue for the presence of (-1)–branes as follows. Given 0–branes in the IIA theory, there should be virtual 0–brane world–lines that wind in a purely spatial direction. Such world–lines are required by quantum mechanics, but note that they are essentially instantons, being localized in time. A T–duality in the winding direction then gives a (-1)–brane. One of the first clues to the relevance of D–branes,[28] was the observation that D–instantons, having action g_s^{-1}, would contribute effects of order e^{-1/g_s} as expected from the behaviour of large orders of string perturbation theory.[95]

The D–brane, unlike the fundamental string, carries R–R charge. This is consistent with the fact that they are BPS states, and so there must be a conserved charge. A more careful argument, involving the R–R vertex operators, can be used to show that they *must* couple thus, and furthermore that fundamental strings cannot carry R–R charges.

8.2.1 A Summary of Forms and Branes

Common to both type IIA and IIB are the NS–NS sector fields

$$\Phi \ , G_{\mu\nu} \ , B_{\mu\nu} \ .$$

The latter is a rank two antisymmetric tensor potential, and we have seen that the fundamental closed string couples to it electrically by the coupling

$$\nu_1 \int_{\mathcal{M}_2} B_{(2)} \ ,$$

[6] This works at the level of vertex operators *via* a Γ–matrix identity.

where $\nu_1 = (2\pi\alpha')^{-1}$, $\mathcal{M}_2$ is the world sheet, with coordinates ξ^a, $a = 0, 1$. $B_{(2)} = B_{ab} d\xi^a d\xi^b$, and B_{ab} is the pullback of $B_{\mu\nu}$ *via* (253).

By ten dimensional Hodge duality, we can also construct a six form potential $B_{(6)}$, by the relation $dB_{(6)} = *dB_{(2)}$. There is a natural electric coupling $\nu_5 \int_{\mathcal{M}_6} B_{(6)}$, to the world–volume $\mathcal{M}_6$ of a five dimensional extended object. This NS–NS charged object, which is commonly called the "NS5–brane" is the magnetic dual of the fundamental string.[68, 69] It is in fact, in the ten dimensional sense, the monopole of the $U(1)$ associated to $B_{(2)}$. We shall be forced to discuss it by strong coupling considerations in section 11.3.

The string theory has other potentials, from the R–R sector:

$$\begin{array}{ll} \text{type IIA}: & C_{(1)}\ , C_{(3)}\ , C_{(5)}\ , C_{(7)} \\ \text{type IIB}: & C_{(0)}\ , C_{(2)}\ , C_{(4)}\ , C_{(6)}\ , C_{(8)} \end{array}$$

where in each case the last two are Hodge duals of the first two, and $C_{(4)}$ is self dual. (A p–form potential and a rank q–form potential are Hodge dual to one another in D dimensions if $p+q = D-2$.) Dp–branes are the basic p–dimensional extended sources which couple to all of these *via* an electric coupling of the form:

$$\mu_p \int_{\mathcal{M}_{p+1}} C_{(p+1)}$$

to their $p+1$–dimensional world volumes $\mathcal{M}_{p+1}$.

8.3 The D–Brane Charge and Tension

The discussion of section 5.3 will supply us with the world–volume action (266) for the bosonic excitations of the D–branes in this supersymmetric context. Now that we have seen that Dp–branes are BPS states, and couple to R–R sector $(p+1)$–form potential, we ought to compute the values of their charges and tensions.

Focusing on the R–R sector for now, supplementing the spacetime supergravity action with the D–brane action we must have at least (recall that the dilaton will not appear here, and also that we cannot write this for $p = 3$):

$$S = -\frac{1}{2\kappa_0^2} \int G_{(p+2)}{}^* G_{(p+2)} + \mu_p \int_{\mathcal{M}_{p+1}} C_{(p+1)}, \qquad (371)$$

where μ_p is the charge of the Dp–brane under the $(p+1)$–form $C_{(p+1)}$. $\mathcal{M}_{p+1}$ is the world–volume of the Dp–brane.

Now the same vacuum cylinder diagram as in the bosonic string, as we did in section 6. With the fermionic sectors, our trace must include a sum over the NS and R sectors, and furthermore must include the GSO projection onto even fermion number. Formally, therefore, the amplitude looks like:[180]

$$\mathcal{A} = \int_0^\infty \frac{dt}{2t} \mathrm{Tr}_{\mathrm{NS+R}} \left\{ \frac{1 + (-1)^F}{2} e^{-2\pi t L_0} \right\} \ . \qquad (372)$$

Performing the traces over the open superstring spectrum gives

$$\begin{aligned} \mathcal{A} \ = \ & 2V_{p+1} \int \frac{dt}{2t} (8\pi^2\alpha' t)^{-(p+1)/2} e^{-t\frac{Y^2}{2\pi\alpha'}} \\ & \frac{1}{2} f_1^{-8}(q) \left\{ -f_2(q)^8 + f_3(q)^8 - f_4(q)^8 \right\}, \end{aligned} \qquad (373)$$

where again $q = e^{-2\pi t}$, and we are using the definitions given in section 4, when we computed partition functions of various sorts.

The three terms in the braces come from the open string R sector with $\frac{1}{2}$ in the trace, from the NS sector with $\frac{1}{2}$ in the trace, and the NS sector with $\frac{1}{2}(-1)^F$ in the trace; the R sector with $\frac{1}{2}(-1)^F$ gives no net contribution. In fact, these three terms sum to zero by Jacobi's abstruse identity, $0 = -f_2(q)^8 + f_3(q)^8 -$

$f_4(q)^8$, as they ought to since the open string spectrum is supersymmetric, and we are computing a vacuum diagram.

What does this result mean? Recall that this vacuum diagram also represents the exchange of closed strings between two identical branes. the result $\mathcal{A} = 0$ is simply a restatement of the fact that D–branes are BPS states: The net forces from the NS–NS and R–R exchanges cancel. $\mathcal{A} = 0$ has a useful structure, nonetheless, and we can learn more by identifying the separate NS–NS and R–R pieces. This is easy, if we look at the diagram afresh in terms of closed string: In the terms with $(-1)^F$, the world–sheet fermions are *periodic* around the cylinder thus correspond to R–R exchange. Meanwhile the terms without $(-1)^F$ have *anti–periodic* fermions and are therefore NS–NS exchange.

Obtaining the $t \to 0$ behaviour as before (use the limits in equations (217)) gives

$$\begin{aligned}\mathcal{A}_{\rm NS} \;=\; -\mathcal{A}_{\rm R} \;&\sim\; \frac{1}{2}V_{p+1}\int\frac{dt}{t}(2\pi t)^{-(p+1)/2}(t/2\pi\alpha')^4 e^{-t\frac{Y^2}{8\pi^2\alpha'^2}} \\ &=\; V_{p+1}2\pi(4\pi^2\alpha')^{3-p}G_{9-p}(Y^2).\end{aligned} \tag{374}$$

Comparing with field theory calculations runs just as it did in section 6, with the result:[180]

$$2\kappa_0^2\mu_p^2 = 2\kappa^2\tau_p^2 = 2\pi(4\pi^2\alpha')^{3-p}. \tag{375}$$

Finally, using the explicit expression (352) for κ in terms of string theory quantities, we get an extremely simple form for the charge:

$$\mu_p = (2\pi)^{-p}\alpha'^{-\frac{(p+1)}{2}}\ , \quad \text{and} \quad \tau_p = g_s^{-1}\mu_p\ . \tag{376}$$

(For consistency with the discussion in the bosonic case, we shall still use the symbol T_p to mean $\tau_p g_s$, in situations where we write the action with the dilaton present. It will be understood then that $e^{-\Phi}$ contains the required factor of g_s^{-1}.)

It is worth updating our bosonic formula (272) for the coupling of the Yang–Mills theory which appears on the world–volume of Dp–branes with our superstring result above, to give:

$$g^2_{{\rm YM},p} = \tau_p^{-1}(2\pi\alpha')^{-2} = (2\pi)^{p-2}\alpha'^{(p-3)/2}\ , \tag{377}$$

a formula we will use a lot in what is to follow.

Note that our formula for the tension (376) gives for the D1–brane

$$\tau_1 = \frac{1}{2\pi\alpha' g_s}\ , \tag{378}$$

which sets the ratios of the tension of the fundamental string, $\tau_1^{\rm F} \equiv T = (2\pi\alpha')^{-1}$, and the D–string to be simply the string coupling g_s. This is a very elegant normalization and is quite natural.

D–branes that are not parallel feel a net force since the cancellation is no longer exact. In the extreme case, where one of the D–branes is rotated by π, the coupling to the dilaton and graviton is unchanged but the coupling to the R–R tensor is reversed in sign. So the two terms in the cylinder amplitude add, instead of canceling, as Jacobi cannot help us. The result is:

$$\mathcal{A} = V_{p+1}\int\frac{dt}{t}(2\pi t)^{-(p+1)/2}e^{-t(Y^2-2\pi\alpha')/8\pi^2\alpha'^2}f(t) \tag{379}$$

where $f(t)$ approaches zero as $t \to 0$. Differentiating this with respect to Y to extract the force per unit world–volume, we get

$$F(Y) = Y\int\frac{dt}{t}(2\pi t)^{-(p+3)/2}e^{-t(Y^2-2\pi\alpha')/8\pi^2\alpha'^2}f(t)\ . \tag{380}$$

The point to notice here is that the force diverges as $Y^2 \to 2\pi\alpha'$. This is significant. One would expect a divergence, of course, since the two oppositely charged objects are on their way to annihilating.[96] The interesting feature is that the divergence begins when their separation is of order the string length. This is where the physics of light fundamental strings stretching between the two branes begins to take over. Notice that the argument of the exponential is tU^2, where $U = Y/(2\alpha')$ is the energy of the lightest open string connecting the branes. A scale like U will appear again, as it is a useful guide to new variables to D–brane physics at "substringy" distances[97, 98, 99] in the limit where α' and Y go to zero.

8.4 The Orientifold Charge and Tension

Orientifold planes also break half the supersymmetry and are R–R and NS–NS sources. In the original Type I theory the orientation projection keeps only the linear combination $Q_\alpha + \tilde{Q}_\alpha$. In the T–dualised theory this becomes $Q_\alpha + (\prod_m P_m)\tilde{Q}_\alpha$ just as for the D–branes. The force between an orientifold plane and a D–brane can be obtained from the Möbius strip as in the bosonic case; again the total is zero and can be separated into NS–NS and R–R exchanges. The result is similar to the bosonic result (297),

$$\mu'_p = \mp 2^{p-5}\mu_p, \qquad \tau'_p = \mp 2^{p-5}\tau_p \ , \tag{381}$$

where the plus sign is correlated with $SO(n)$ groups and the minus with $USp(n)$. Since there are 2^{9-p} orientifold planes, the total O–plane charge is $\mp 16\mu_p$, and the total fixed-plane tension is $\mp 16\tau_p$.

8.5 Type I from Type IIB, Revisited

A non–zero total tension represents a source for the graviton and dilaton, for which the response is simply a time dependence of these background fields[100]. A non–zero total R–R source is more serious, since this would mean that the field equations are inconsistent: There is a violation of Gauss' Law, as R–R flux lines have no place to go in the compact space T^{9-p}. So our result tells us that on T^{9-p}, we need exactly 16 D–branes, with the SO projection, in order to cancel the R–R $G_{(p+2)}$ form charge. This gives the T–dual of $SO(32)$, completing our simple orientifold story.

The spacetime anomalies for $G \neq SO(32)$ (see also section 7.2.1) are thus accompanied by a divergence[101] in the full string theory, as promised, with inconsistent field equations in the R–R sector: As in field theory, the anomaly is related to the ultraviolet limit of a (open string) loop graph. But this ultraviolet limit of the annulus/cylinder ($t \to \infty$) is in fact the infrared limit of the closed string tree graph, and the anomaly comes from this infrared divergence. From the world–sheet point of view, as we have seen in the bosonic case, inconsistency of the field equations indicates that there is a conformal anomaly that cannot be canceled. This is associated to the presence of a "tadpole" which is simply an amplitude for creating quanta out of the vacuum with a one–point function, which is a sickness of the theory which must be cured.

The prototype of all of this is the original $D = 10$ Type I theory[34]. The N D9–branes and single O9–plane couple to an R–R 10-form, and we can write its action formally as

$$(32 \mp N)\frac{\mu_{10}}{2}\int C_{10}. \tag{382}$$

The field equation from varying C_{10} is just $G = SO(32)$.

8.6 Dirac Charge Quantization

We are of course studying a quantum theory, and so the presence of both magnetic and electric sources of various potentials in the theory should give some cause for concern. We should check that the values of the charges are consistent with the appropriate generalisation of[108] the Dirac quantisation condition. The field strengths to which a Dp–brane and D$(6-p)$–brane couple are dual to one another, $G_{(p+2)} = *G_{(8-p)}$.

We can integrate the field strength $*G_{(p+2)}$ on an $(8-p)$–sphere surrounding a Dp–brane, and using the action (371), we find a total flux $\Phi = \mu_p$. We can write $*G_{(p+2)} = G_{(8-p)} = dC_{(7-p)}$ everywhere except on a Dirac "string" (see also insert 9.2.1; here it is really a sheet), at the end of which lives the D$(6-p)$ "monopole". Then

$$\Phi = \frac{1}{2\kappa_0^2}\int_{S_{8-p}} *G_{(p+2)} = \frac{1}{2\kappa_0^2}\int_{S_{7-p}} C_{(7-p)} \ . \qquad (383)$$

where we perform the last integral on a small sphere surrounding the Dirac string. A $(6-p)$–brane circling the string picks up a phase $e^{i\mu_{6-p}\Phi}$. The condition that the string be invisible is

$$\mu_{6-p}\Phi = 2\kappa_0^2\mu_{6-p}\mu_p = 2\pi n. \qquad (384)$$

The D–branes' charges (375) satisfy this with the minimum quantum $n = 1$.

While this argument does not apply directly to the case $p = 3$, as the self–dual 5–form field strength has no covariant action, the result follows by the T–duality recursion relation (256) and the BPS property.

8.7 D–Branes in Type I

As we saw in section 7.2.1, the only R–R potentials available in type I theory are the 2–form and its dual, the 6–form, and so we can have D1–branes in the theory, and D5–branes, which are electromagnetic duals of each other. The overall 16 d9–branes carry an $SO(32)$ gauge group, as we have seen from many points of view. Let us remind ourselves of how this gauge group came about, since there are important subtleties of which we should be mindful[123].

The action of Ω has representation γ_Ω, which acts on the Chan–Paton indices, as discussed in section 4:

$$\Omega: \quad |\psi, ij> \longrightarrow (\gamma_\Omega)_{ii'}|\Omega\psi, j'i'> (\gamma_\Omega^{-1})_{j'j} \ ,$$

where ψ represents the vertex operator which makes the state in question, and $\Omega\psi$ is the action of Ω on it. The reader should recall that we transposed the indices because Ω exchanges the endpoints of the string. We can consider the square of Ω:

$$\Omega^2: \qquad |\psi, ij> \longrightarrow \left[\gamma_\Omega(\gamma_\Omega^T)^{-1}\right]_{ii'}|\psi, i'j'> \left[\gamma_\Omega^T\gamma_\Omega\right]_{j'j} \ , \qquad (385)$$

and so we see that we have the choice

$$\gamma_\Omega^T = \pm\gamma_\Omega \ .$$

If γ_Ω is symmetric, the with n branes we can write it as $\mathbf{I}_{2n}$, the $2n\times 2n$ identity matrix. Since the 99 open string vertex operator is $\partial_t X^\mu$, it has (as we have seen a lot in section 4) $\Omega = -1$. Therefore we do have the symmetric choice since, as we tacitly assumed in equation (385) $\Omega^2 = 1$, and so we conclude that the Chan–Paton wave–function is antisymmetric. Since $n = 16$, we have gauge group $SO(32)$.

If γ_Ω was antisymmetric, then we could have written it as

$$\gamma_\Omega = \begin{pmatrix} 0 & i\mathbf{I}_n \\ -i\mathbf{I}_n & 0 \end{pmatrix} \ ,$$

and we would have been able to have gauge group $USp(2n)$. In fact, we shall have to make this choice for D5–branes. Let us see why. Let us place the D5–branes so that they are point–like in the directions X^m, $m = 6, 7, 8, 9$, and aligned in the directions X^μ, $\mu = 0, 1, \ldots, 5$.

Consider the 5–5 sector, *i.e.* strings beginning and ending on D5–branes. Again we have $\Omega = -1$ for the vectors $\partial_t X^\mu$, and the opposite sign for the transverse scalars $\partial_n X^m$. In general, other sectors can have different mode expansions. Generically the mode for a fermion is ψ_r and Ω acts on this as $\pm(-1)^r = \pm e^{i\pi r}$. In the NS sector they are half–integer and since GSO requires them to act in pairs in vertex operators, their individual $\pm i$'s give $\Omega = \pm 1$, with a similar result in the R sector by supersymmetry.

The 59 sector is more subtle[123]. The X^m are now half–integer moded and the ψ^m are integer moded. The ground states of the latter therefore form a representation of the Clifford algebra and we can bosonise them into a spin field, as we did in section 7 in a similar situation: $e^{iH_3} \sim \psi^6 + i\psi^7$, and $e^{iH_4} \sim \psi^8 + i\psi^9$. In fact, the vertex operator (the part of it relevant to this discussion) in that sector is

$$V_{59} \sim e^{i(H_3+H_4)/2} \ .$$

Now consider the square of this operator. It has parts which are either in the 55 sector or the 99 sector, and is of the form

$$V_{59}^2 \sim e^{i(H_3+H_4)} \sim (\psi^6 + i\psi^7)_{-1/2}(\psi^8 + i\psi^9)_{-1/2}|0> \ .$$

So it has $\Omega = -1$, since each $\psi_{-1/2}$ gives $\pm i$. So $\Omega^2 = -1$ for V_{59} for consistency.

Returning to our problem of the choices to make for the Chan–Paton factors we see that we have an extra sign in equation 385, and so must choose the antisymmetric condition $\gamma_\Omega^T = -\gamma_\Omega$. Therefore, in type I string theory, n D5–branes have gauge group $USp(2n)$. Notice that this means that a single one has $SU(2)$, and the Chan–Paton wave function can be chosen as the Pauli matrices. The Chan–Paton wave–function for the scalars for transverse motion must simply be δ^{ij}, since we have another sign. This simply means that the two D5–branes (corresponding to the 2 index choices) are forced to move with each other as one unit.

Notice that this fits rather nicely with our charge quantisation computation of the last section.[123] The orientifold projection will halve the force between D1–branes and between D5–branes in the charge calculation, and so their effective charges would be reduced by $\sqrt{2}$, violating the Dirac quantisation condition by a factor of a half. However, the fact that the D5–branes are forced to move as a pair restores a factor of two in the quantisation condition, and so we learn that D–branes are still the smallest consistent charge carries of the R–R sector.

We can augment the argument above for Dp branes in type I in general, and obtain[123]

$$\Omega^2 = (\pm i)^{\frac{9-p}{2}} \ .$$

For $p = 3$ and $p = 7$, we see that it simply gives an inconsistency, which is itself consistent with the fact that there is no R–R 4–forms or 8–form for a stable D3–brane or D7–brane to couple to. For $p = 1$ we recover the naively expected result that they have an $SO(2n)$ gauge group.

9 World–Volume Curvature Couplings

We've now seen that we can construct D–branes which, in superstring theory, have important extra properties. Much of what we have learned about them in the bosonic theory is still true here of course, a key result being that the world–volume dynamics is governed by the dynamics of open strings, *etc.* Still relevant is the Dirac–Born–Infeld action (266) for the coupling to the background NS–NS fields,

$$S_{DBI} = -\tau_p \int_{\mathcal{M}_{p+1}} d^{p+1}\xi \, e^{-\Phi} \det{}^{1/2}(G_{ab} + B_{ab} + 2\pi\alpha' F_{ab}) \ . \tag{386}$$

and the non–Abelian extensions mentioned later in section 5.

As we have seen in the previous section, for the R–R sector, they are sources of $C_{(p+1)}$. We therefore also have the Wess–Zumino–like term

$$S_{WZ} = \mu_p \int_{\mathcal{M}_{p+1}} C_{(p+1)} \ . \tag{387}$$

Perhaps not surprisingly, there are other terms of great importance, and this section will uncover a number of them. In fact, there are many ways of deducing that there *must* be other terms, and one way is to use the fact that D–branes turn into each other under T–duality.

9.1 Tilted D–Branes and Branes within Branes

There are additional terms in the action involving the D–brane gauge field. Again these can be determined from T–duality. Consider, as an example, a D1–brane in the 1–2 plane. The action is

$$\mu_1 \int dx^0\, dx^1\, \left(C_{01} + \partial_1 X^2 C_{02}\right)\ . \tag{388}$$

Under a T–duality in the x^2–direction this becomes

$$\mu_2 \int dx^0 dx^1 dx^2\, \left(C_{012} + 2\pi\alpha' F_{12} C_0\right)\ . \tag{389}$$

We have used the T–transformation of the C fields as discussed in section 8.1.1, and also the recursion relation (256) between D–brane tensions.

This has an interesting interpretation. As we saw before in section 5.2.1, a Dp–brane tilted at an angle θ is equivalent to a D$(p+1)$–brane with a constant gauge field of strength $F = (1/2\pi\alpha')\tan\theta$. Now we see that there is additional structure: the flux of the gauge field couples to the R–R potential $C^{(p)}$. In other words, the flux acts as a source for a D$(p-1)$–brane living in the world–volume of the D$(p+1)$–brane. In fact, given that the flux comes from an integral over the whole world–volume, we cannot localize the smaller brane at a particular place in the world–volume: it is "smeared" or "dissolved" in the world–volume.

In fact, we shall see when we come to study supersymmetric combinations of D–branes that supersymmetry requires the D0–brane to be completely smeared inside the D2–brane. It is clear here how it manages this, by being simply T–dual to a tilted D1–brane.

9.2 Anomalous Gauge Couplings

The T–duality argument of the previous section can be generalized to discover more terms in the action, but we shall take another route to discover such terms, exploiting some important physics in which we already have invested considerable time.

Let us return to the type I string theory, and the curious fact that we had to employ the Green–Schwarz mechanism (see section 7.3, where we mixed a classical and a quantum anomaly in order to achieve consistency). Focusing on the gauge sector alone for the moment, the classical coupling which we wrote in equation (343) implies a mixture of the 2–form $C_{(2)}$ with gauge field strengths:

$$S = \frac{1}{3\times 2^6 (2\pi)^5 \alpha'} \int C_{(2)} \left(\frac{\mathrm{Tr}_{\mathrm{adj}}(F^4)}{3} - \frac{[\mathrm{Tr}_{\mathrm{adj}}(F^2)]^2}{900}\right)\ . \tag{390}$$

We can think of this as an interaction on the world–volume of the D9–branes showing a coupling to a D1–brane, analogous to that which we saw for a D0–brane inside a D2–brane in equation (389). This might seem a bit of a stretch, but let us write it in a different way:

$$\begin{aligned} S &= \mu_9 \int \frac{(2\pi\alpha')^4}{3\times 2^6} C_{(2)} \left(\frac{\mathrm{Tr}_{\mathrm{adj}}(F^4)}{3} - \frac{[\mathrm{Tr}_{\mathrm{adj}}(F^2)]^2}{900}\right) \\ &= \mu_9 \int \frac{(2\pi\alpha')^4}{4!} C_{(2)} \mathrm{Tr}(F^4)\ , \end{aligned} \tag{391}$$

where, crucially, in the last line we have used the properties (347) of the traces for $SO(32)$ to rewrite things in terms of the trace in the fundamental.

Another exhibit we would like to consider is the kinetic term for the modified 3–form field strength, $\widetilde{G}_{(3)}$, which is

$$S = -\frac{1}{4\kappa_0^2} \int \widetilde{G}_{(3)} \wedge^* \widetilde{G}_{(3)}\ . \tag{392}$$

Since $d\omega_{3Y} = \mathrm{Tr}(F\wedge F)$ and $d\omega_{3L} = \mathrm{Tr}(R\wedge R)$, this gives, after integrating by parts and, dropping the parts with R for now:

$$\begin{aligned} S &= \frac{\alpha'}{4\kappa^2}\int C_{(6)}\wedge\left(\frac{1}{30}\mathrm{Tr}_{\rm adj}(F\wedge F)\right) \\ &= \mu_9\int\frac{(2\pi\alpha')^2}{2}C_{(6)}\wedge\left(\mathrm{Tr}(F\wedge F)\right)\ . \end{aligned} \tag{393}$$

again, we have converted the traces using (347), we've used the relation (352) for κ_0 and we've recalled the definition (346).

Upon consideration of the three examples (389), (391), and (393), it should be apparent that a pattern is forming. The full answer for the gauge sector is the result[110, 111]

$$\mu_p\int_{\mathcal{M}_{p+1}}\left[\sum_p C_{(p+1)}\right]\wedge\mathrm{Tr}\,e^{2\pi\alpha' F+B}\ , \tag{394}$$

(We have included non–trivial B on the basis of the argument given in section 5.2.) So far, the gauge trace (which is in the fundamental) has the obvious meaning. We note that there is the possibility that in the full non–Abelian situation, the C can depend on *non–commuting* transverse fields X^i, and so we need something more general[54]. The expansion of the integrand (394) involves forms of various rank; the notation means that the integral picks out precisely the terms whose rank is $(p+1)$, the dimension of the Dp–brane's world–volume.

Looking at the first non–trivial term in the expansion of the exponential in the action we see that there is the term that we studied above corresponding to the dissolution of a D$(p-2)$–brane into the sub 2–plane in the Dp–brane's world volume formed by the axes X^i and X^j, if field strength components F_{ij} are turned on.

At the next order, we have a term which is quadratic in F which we could rewrite as:

$$S = \frac{\mu_{p-4}}{8\pi^2}\int C_{(p-3)}\wedge\mathrm{Tr}(F\wedge F)\ . \tag{395}$$

We have used the fact that $\mu_{p-4}/\mu_p = (2\pi\sqrt{\alpha'})^4$. Recall that there are non–Abelian field configurations called "instantons" for which the quantity $\int\mathrm{Tr}(F\wedge F)/8\pi^2$ gives integer values.

Interestingly, we see that if we excite an instanton configuration on a 4 dimensional sub–space of the Dp–brane's world–volume, it is equivalent to precisely one unit of D$(p-4)$–brane charge, which is remarkable.

In trying to understand what might be the justification (other than T–duality) for writing the full result (394) for all branes so readily, the reader might recognize something familiar about the object we built the action out of. The quantity $\exp(iF/(2\pi))$, using a perhaps more familiar normalization, generates polynomials of the Chern classes of the gauge bundle of which F is the curvature. It is called the Chern character.

In the Abelian case we first studied, we had non–vanishing first Chern class $\mathrm{Tr}F/(2\pi)$, which after integrating over the manifold, gives a number which is in fact quantized. For the non–Abelian case, the second Chern class $\mathrm{Tr}(F\wedge F)/(8\pi^2)$ computes the integer known as the instanton number, and so on. These numbers, being integers, are topological invariants of the gauge bundle. By the latter, we mean the fibre bundle of the gauge group over the world–volume, for which the gauge field A is a connection.

A fibre bundle is a rule for assigning a copy of a certain space (the fibre: in this case, the gauge group G) to every point of another space (the base: here, the world–volume). The most obvious case of this is simply a product of two manifolds (since one can be taken as the base and then the product places a copy of the other at every point of the base), but this is awfully trivial. More interesting is to have only a product space locally.

Then, the whole structure of the bundle is given by a collection of such local products glued together in an overlapping way, together with a set of transition functions which tell one how to translate from one local

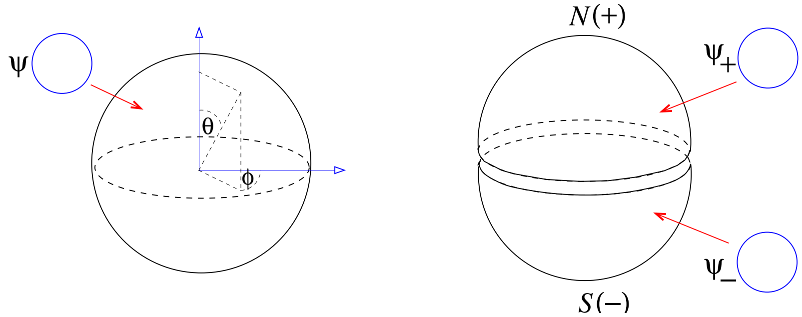


Figure 30: Constructing the monopole bundle

patch to another on the overlap. In the case of a gauge theory, this is all familiar. The transition rule is simply a G gauge transformation, and we are allowed to use the term "vector bundle" in this case. For the connection or gauge field this is: $A \to gAg^{-1} + gdg^{-1}$. So the gauge field is not globally defined. Perhaps the most familiar gauge bundle is the monopole bundle corresponding to a Dirac monopole. See the next section.

9.2.1 The Dirac Monopole as a Gauge Bundle

A gauge bundle is sometimes called a principal fibre bundle. Perhaps everybody's favourite gauge bundle is the Dirac monopole. Take a sphere S^2 as our base. We will fibre a circle over it. Recall that S^2 cannot be described by a global set of coordinates, but we can use two, the Northern and the Southern hemisphere, with overlap in the vicinity of the Equator. Put standard polar coordinates (θ, ϕ) on S^2, where $\theta = \pi/2$ is the Equator. Put an angular coordinate $e^{i\psi}$ on the circle. We will use ϕ_+ in the North and ψ_- in the South.

So our bundle is a copy of two patches which are locally $S^2 \times S^1$,

$$+\text{Patch}: \quad \{\theta, \phi, e^{i\psi_+}\}\ ; \qquad -\text{Patch}: \quad \{\theta, \phi, e^{i\psi_-}\}\ ,$$

together with a transition function which relates them.

The relation between the two can be chosen to be

$$e^{i\psi_-} = e^{in\phi} e^{i\psi_+}\ ,$$

where n is an integer, since as we go around the equator, $\phi \to \phi + 2\pi$, the gluing together of the fibres must still make sense.

The boring case $n = 0$ is sensible, but it simply gives the trivial bundle $S^2 \times S^1$. The case $n = 1$ is the familiar Hopf fibration, which describes the manifold S^3 as a circle bundle over S^2. It is a Dirac monopole of unit charge. Higher values of n give charge n monopoles. The integer n is characteristic of the bundle. It is in fact (minus) the integral of the first Chern class.

Let us compute the first Chern number. A natural choice for the connection 1–form (gauge potential) in each patch is simply

$$+\text{Patch}: \quad A_+ + d\psi_+\ ; \qquad -\text{Patch}: \quad A_- + d\psi_-\ ,$$

so that the transition function defined above allows us to connect the two patches, defining the standard $U(1)$ gauge transformation

$$A_+ = A_- + n\phi\ .$$

Here are the gauge potentials which are standard in this example:

$$A_\pm = n\frac{(\pm 1 - \cos\theta)}{2} d\phi\ ,$$

which, while being regular almost everywhere, clearly have a singularity (the famous Dirac string) in the $\mp$ patch. The curvature 2–form is simply

$$F = dA = \frac{n}{2}\sin\theta d\theta\wedge d\phi \ .$$

This is a closed form, but it is not exact, since there is not a unique answer to what A can be over the whole manifold. We can compute the first Chern number by integrating the first Chern class to get:

$$\int_{S^2}\frac{F}{2\pi} = \int_{+}\frac{F}{2\pi} + \int_{-}\frac{F}{2\pi} = n \ .$$

9.3 Characteristic Classes and Invariant Polynomials

The topology of a particular fibration can be computed by working out just the right information about its collection of transition functions. For a gauge bundle, the field strength or curvature 2–form $F = dA + A\wedge A$ is a nice object with which to go and count, since it is globally defined over the whole base manifold. When the group is Abelian, $F = dA$ and so $dF = 0$. If the bundle is not trivial, then we can't write F as dA everywhere and so F is closed but not exact. Then F is said to be an element of the cohomology group $H^2(\mathcal{B},\mathbb{R})$ of the base, which we'll call $\mathcal{B}$. The first Chern class $F/2\pi$ defines an integer when integrated over $\mathcal{B}$, telling us to which topological class F belongs; this integer is a topological invariant.

For the non–Abelian case, F is no longer closed, and so we don't have the first Chern class. However, the quantity $\mathrm{Tr}(F\wedge F)$ is closed, since as we know from section 7.1.3, it is actually $d\omega_{3Y}$.

So if the Chern–Simons 3–form ω_{3Y} is not globally defined, we have a non–trivial bundle, and $\mathrm{Tr}(F\wedge F)$, being closed but not exact, defines an element of the cohomology group $H^4(\mathcal{B},\mathbb{R})$. The second Chern class $\mathrm{Tr}(F\wedge F)/8\pi^2$ integrated over $\mathcal{B}$ gives an integer which says to which topological class F belongs. Physicists call such configurations with non–zero values of this integer "instantons", and this number is the "instanton number".

As we have said above, D–branes appear to compute certain topological features of the gauge bundle on their world–volumes, corresponding here to the Chern classes of the cohomology. As we shall see, they compute other topological numbers as well, and so let us pause to appreciate a little of the tools that they employ, in order to better be able to put them to work for us.

The first and second Chern classes shall be denoted $c_1(F)$ and $c_2(F)$ and so on, $c_j(F)$ for the jth Chern class. Let us call the gauge group G, and keep in mind $U(n)$ (we will make appropriate modifications to our statements to include $O(n)$ later). We'd like to know how to compute the $c_j(F)$. The remarkable thing is that they arise from forming polynomials in F which are invariant under G. Forget that F is a two–form for now, and just think of it as an $n{\times}n$ matrix. The $c_j(F)$ are found by expanding a natural invariant expression in F as a series in t:

$$\det\left(t\mathbf{I} + \frac{iF}{2\pi}\right) = \sum_{j=0}^{n} c_{n-j}(F)t^j \ . \qquad (396)$$

(Here, we use the i in F to keep the expression real, since $U(N)$ generators are anti–Hermitean.) The great thing about this is that there is an excellent trick for finding explicit expressions for the c_j's which will allow us to manipulate them and relate them to other quantities. Assume that the matrix $iF/2\pi$ has been diagonalised. Call this diagonal matrix X, with n distinct non–vanishing eigenvalues x_i, $i = 1\ldots n$. Then we have

$$\det(t\mathbf{I} + X) = \prod_{i=1}^{n}(t + x_i) = \sum_{j=0}^{n} c_{n-j}(x)t^j \ , \qquad (397)$$

and we find by explicit computation that the c_j's are symmetric polynomials:

$$\begin{aligned} c_0 &= 1\ , \quad c_1=\sum_i^n x_i\ , \quad c_2=\sum_{i_1<i_2}^n x_{i_1}x_{i_2}\ ,\cdots \\ c_j &= \sum_{i_1<i_2<\cdots i_j}^n x_{i_1}x_{i_2}\cdots x_{i_j}\ , \quad c_n=x_1x_2\cdots x_n\ . \end{aligned} \tag{398}$$

Now rewrite the expressions on the eigenvalues back as matrix expressions in terms of X, and then replace X by $iF/2\pi$, to get:

$$\begin{aligned} c_0(F) &= 1\ , \quad c_1(F)=\frac{i}{2\pi}\mathrm{Tr}F\ , \\ c_2(F) &= \frac{1}{2}\left(\frac{i}{2\pi}\right)^2[\mathrm{Tr}F\wedge\mathrm{Tr}F-\mathrm{Tr}(F\wedge F)]\ , \\ c_n(F) &= \left(\frac{i}{2\pi}\right)^n\det F\ . \end{aligned} \tag{399}$$

In the case of $SU(N)$, the generators are traceless, and so

$$c_2(F)=\frac{1}{8\pi^2}\mathrm{Tr}(F\wedge F)\ ,$$

the expression we saw before. The $c_j(F)$ are rank $2j$ forms, so of course, the largest one that is gives a meaningful quantity is the one for which $\dim(B)=2j$.

The natural object which D–branes seem to have on their world–volume is in fact the Chern character[172], $ch(F)=\mathrm{Tr}\exp(iF/2\pi)$. This computes a specific combination of the Chern classes, and we can compute this by using our symmetric polynomial expressions in (398). Working with the diagonal X again we have

$$\begin{aligned} ch(x) &= \sum_i e^{x_i}=\sum_i\left(1+x_i+\frac{x_i^2}{2}+\cdots\right) \\ &= n+c_1+\frac{1}{2}(c_1^2-2c_2)+\cdots\ , \quad \text{and so we have:} \\ ch(F) &= n+c_1(F)+\frac{1}{2}(c_1^2(F)-2c_2(F))+\cdots \end{aligned} \tag{400}$$

The Chern character is to be thought of as an important generating function of the Chern classes and in fact it is a powerful tool, in that it is well behaved in the sense that for bundle E and a bundle F, the relations

$$ch(E\oplus F)=ch(E)+ch(F)\ , \quad \text{and} \quad ch(E\otimes F)=ch(E)\wedge ch(F) \tag{401}$$

are true. This is part of an important technology to doing "algebra" on bundles allowing one to perform operations which compare them to each other, *etc.*

For the case $G=O(n)$, the characteristic classes are called Pontryagin classes. We may think of the bundle as a real vector bundle. Now we have

$$\det\left(t\mathbf{I}+\frac{F}{2\pi}\right)=\sum_{j=0}^n p_{n-j}(F)t^j\ . \tag{402}$$

Again, consider having diagonalised to X. We can't quite diagonalize, but can get it into the block diagonal form:

$$X=\begin{pmatrix} 0 & x_1 & & & \\ -x_1 & 0 & & & \\ & & 0 & x_2 & \\ & & -x_2 & 0 & \\ & & & & \ddots \end{pmatrix}\ . \tag{403}$$

Now we have the relation:

$$\det\left(t\mathbf{I}+X\right)=\det\left(t\mathbf{I}+X^T\right)=\det\left(t\mathbf{I}-X\right)\ ,$$

and so we see that the $p_j(F)$ must be even in F. A bit of work similar to that which we did above for the Chern classes gives:

$$\begin{aligned}
p_1(F) &= -\frac{1}{2}\left(\frac{1}{2\pi}\right)^2 \mathrm{Tr}F^2\ ,\\
p_2(F) &= \frac{1}{8}\left(\frac{1}{2\pi}\right)^4\left[(\mathrm{Tr}F^2)^2-2\mathrm{Tr}F^4\right]\ ,\cdots\ , etc.\\
p_{[\frac{n}{2}]}(F) &= \left(\frac{1}{2\pi}\right)^n \det F\ ,
\end{aligned} \tag{404}$$

where $[n/2]=n/2$ if n is even or $(n-1)/2$ otherwise.

Now an important case of orthogonal groups is of course the tangent bundle to a manifold of dimension n. Using the vielbiens formalism of section 2.14, the structure group is $O(n)$. The 2–form to use is the curvature 2–form R. Then we have *e.g.*,

$$p_1(R)=-\frac{1}{8\pi^2}\mathrm{Tr}R\wedge R\ . \tag{405}$$

The Euler class is naturally defined here too. For an orientable even dimensional $n=2k$ manifold M, the Euler class class $e(M)$ is defined *via*

$$e(X)e(X)=p_k(X)\ .$$

We write X here and not the 2–form R, since we would have a $4k$–form which vanishes on M. However, $e(R)$ makes sense as a form since its rank is n, which is the dimension of M.

Let's test this out for the two–sphere S^2. Using the formalism of section 2.14, the curvature two–form can be computed as $R_{\theta\phi}=\sin\theta d\theta\wedge d\phi$. Then we can compute

$$p_1(S^2)=-\frac{1}{8\pi^2}\mathrm{Tr}R\wedge R=\left(\frac{1}{2\pi}\sin\theta d\theta\wedge d\phi\right)^2\ .$$

So we see that

$$e(S^2)=\frac{1}{2\pi}\sin\theta d\theta\wedge d\phi\ .$$

The integral of this from over the manifold given the Euler number:

$$\chi=\int_{S^2}e(S^2)=2\ ,$$

a result we know well and have used extensively.

Returning to our story, two other remarkable generating functions of importance are the $\hat{A}$ ("A–roof") or Dirac genus:

$$\begin{aligned}
\hat{A} &= \prod_{j=1}^n\frac{x_j/2}{\sinh x_j/2}=\prod_{j=1}^n\left(1+\sum_{n\geq 1}(-1)^n\frac{2^{2n}-2}{(2n)!}B_n x_j^{2n}\right)\\
&= 1-\frac{1}{24}p_1+\frac{1}{5760}(7p_1^2-4p_2)+\cdots\ ,
\end{aligned} \tag{406}$$

and the Hirzebruch $\hat{\mathcal{L}}$–polynomial

$$\begin{aligned}\mathcal{L} &= \prod_{j=1}^{n} \frac{x_j}{\tanh x_j} = \prod_{j=1}^{n}\left(1+\sum_{n\geq 1}(-1)^{n-1}\frac{2^{2n}}{(2n)!}B_n x_j^{2n}\right) \\ &= 1+\frac{1}{3}p_1+\frac{1}{45}(7p_2-p_1^2)+\cdots\ ,\end{aligned} \tag{407}$$

where the B_n are the Bernoulli numbers, $B_1 = 1/6, B_2 = 1/30, B_3 = 1/42, \ldots$. These are very important characteristics as well, and again have useful algebraic properties for facilitating the calculus of vector bundles along the lines given by (401). As we shall see, they also play a very natural role in our story here.

9.4 Anomalous Curvature Couplings

So we seem to have wandered away from our story somewhat, but in fact we are getting closer to a big part of the answer. If the above formula (394) is true, then D–branes evidently know how to compute the topological properties of the gauge bundle associated to their world–volumes. This is in fact a hint of a deeper mathematical structure underlying the structure of D–branes and their charge, and we shall see it again later.

There is another strong hint of what is going on based on the fact that we began to deduce much of this structure using the terms we discovered were needed to cancel anomalies. So far we have only looked at the terms involving the curvature of the gauge bundle, and not the geometry of the brane itself which might have non–trivial R associated to its tangent bundle. Indeed, since the gauge curvature terms came from anomalies, we might expect that the tangent bundle curvatures do too. Since these are so closely related, one might expect that there is a very succinct formula for those couplings as well. Let us look at the anomaly terms again. The key terms are the curvature terms in (343) and the curvature terms arising from the modification (346) of the field strength of $C_{(3)}$ to achieve gauge invariance. The same deduction we made to arrive at (393) will lead us to $\text{Tr}R^2$ terms coupling to $C_{(6)}$. Also, if we convert to the fundamental representation, we can see that there is a term

$$-\frac{1}{3\times 2^6(2\pi)^5}\int C_{(2)}\text{Tr}F^2\text{tr}R^2.$$

This mixed anomaly type term can be generated in a number of ways, but a natural guess[104, 105, 106] (motivated by remarks we shall make shortly) is that there is a $\sqrt{\hat{A}}$ term on the world volume, multiplying the Chern characteristic. In fact, the precise term, written for all branes, is:

$$\mu_p\int_{\mathcal{M}_{p+1}}\sum_i C_{(i)}\left[e^{2\pi\alpha' F+B}\right]\sqrt{\hat{\mathcal{A}}(4\pi^2\alpha' R)}\ . \tag{408}$$

Working with this expression, using the precise form given in (406) will get the mixed term precisely right, but the $C_{(6)}\text{tr}R^2$ will not have the right coefficient, and also the remaining fourth order terms coupling to $C_{(2)}$ are incorrect, after comparing the result to (343).

The reason why they are not correct is because there is another crucial contribution which we have not included. There is an orientifold O9–plane of charge $-32\mu_9$ as well. As we saw, it is crucial in the anomaly cancellation story of the previous section and it must be included here for precisely the same reasons. While it doe not couple to the $SO(32)$ gauge fields (open strings), it certainly has every right to couple to gravity, and hence source curvature terms involving R. Again, as will be clear shortly, the precise term for Op–planes of this type is: [117]

$$\tilde{\mu}_p\int_{\mathcal{M}_{p+1}}\sum_i C_{(i)}\sqrt{\hat{\mathcal{L}}(\pi^2\alpha' R)}\ , \tag{409}$$

where $\hat{\mathcal{L}}(R)$ is defined above in equation (407). Remarkably, expanding this out will repair the pure curvature terms so as to give all of the correct terms in X_8 to reproduce (343), and the $C_{(6)}$ coupling is precisely:

$$\begin{aligned} S &= \mu_9 \int \frac{(2\pi\alpha')^2}{2} C_{(6)} \wedge (\mathrm{Tr}(F\wedge F) - \mathrm{Tr}R\wedge R) \\ &= \mu_9 \int \frac{(2\pi\alpha')^2}{2} C_{(6)} \wedge Y_4 \ . \end{aligned} \tag{410}$$

Beyond just type I, it is worth noting that the $R\wedge R$ term will play an important role on the world volumes of branes. It can be written in the form:

$$\frac{\mu_p(4\pi^2\alpha')^2}{48} \int_{\mathcal{M}_{p+1}} C_{(p-3)} \wedge p_1(R) \ . \tag{411}$$

By straightforward analogy with what we have already observed about instantons, another way to get a D$(p-4)$–brane inside the world–volume of a Dp–brane is to wrap the brane on a four dimensional surface of non–zero $p_1(R)$. Indeed, as we saw in equation (362), the K3 surface has $p_1 = -2\chi = -48$, and so wrapping a Dp–brane on K3 gives D$(p-4)$–brane charge of precisely -1. This will be important to us later.[109, 113]

9.5 A Relation to Anomalies

There is one last amusing fact that we should notice, which will make it very clear that the curvature couplings that we have written above are natural for branes and O–planes of all dimensionalities. The point is that the curvature terms just don't accidentally resemble the anomaly polynomials we saw before, but are built out of the very objects which can be used to generate the anomaly polynomials that we listed in section 7.1.4.

In fact, we can use them to generate anomaly polynomials for dimension $D = 4k+2$. We can pick out the appropriate powers of the curvature two forms by using the substitution

$$\sum_{i=1}^{2k+1} x_i^{2m} = \frac{1}{2}(-1)^m \mathrm{tr}R^{2m} \ .$$

Then in fact the polynomial $\hat{I}_{1/2}$ is given by the $\hat{A}$ genus:

$$\begin{aligned} \hat{I}_{1/2} &= \hat{A} = \prod_{j=1}^{2k+1} \frac{x_j/2}{\sinh x_j/2} \\ &= \prod_{j=1}^{2k+1} \left(1 + \frac{y_j^2}{3!} + \frac{y_j^4}{5!} + \cdots\right)^{-1} \\ &= \prod_{j=1}^{2k+1} \left(1 - \frac{1}{6}y_j^2 + \frac{7}{360}y_j^4 - \frac{31}{15120}y_j^6 + \cdots\right) \\ &= 1 - \frac{1}{6}\mathcal{Y}_2 + \frac{1}{180}\mathcal{Y}_4 + \frac{1}{72}\mathcal{Y}_2^2 \\ &\quad -\frac{1}{2835}\mathcal{Y}_6 - \frac{1}{1080}\mathcal{Y}_2\mathcal{Y}_4 - \frac{1}{1296}\mathcal{Y}_2^3 + \cdots \end{aligned} \tag{412}$$

where

$$\mathcal{Y}_{2m} = \sum_{i=1}^{2k+1} y_i^{2m} = \frac{1}{2}\left(-\frac{1}{4}\right)^m \mathrm{tr}R^{2m} \ .$$

The trick is then to simply pick out the piece of the expansion which fits the dimension of interest, remembering that the desired polynomial is of rank $D+2$. So for example, picking out the order 12 terms will give precisely the 12–form polynomial in section 7.1.4, *etc.*

The gravitino polynomials come about in a similar way. In fact,

$$\begin{aligned} I_{3/2} &= I_{1/2}\left(-1+2\sum_{j=1}^{2k+1}\cosh x_i\right) \\ &= I_{1/2}\left(D-1+4\mathcal{Y}_2+\frac{4}{3}\mathcal{Y}_4+\frac{8}{45}\mathcal{Y}_6+\cdots\right). \end{aligned} \tag{413}$$

Also, the polynomials for the antisymmetric tensor comes from

$$\begin{aligned} I_A &= -\frac{1}{8}\hat{\mathcal{L}}(R) = -\frac{1}{8}\sum_{j=1}^{2k+1}\frac{x_j}{\tanh x_j} \\ &= -\frac{1}{8}-\frac{1}{6}\mathcal{Y}_2+\left(\frac{7}{45}-\frac{1}{9}\mathcal{Y}_2^2\right) \\ &\quad +\frac{1}{2835}\left(-496\mathcal{Y}_6+588\mathcal{Y}_2\mathcal{Y}_4-140\mathcal{Y}_2^3+\cdots\right) \ . \end{aligned} \tag{414}$$

Finally, it is easy to work out the anomaly polynomial for a charged fermion. One simply multiplies by the Chern character:

$$I_{1/2}(F,R) = \mathrm{Tr}e^{iF}I_{1/2}(R) \ . \tag{415}$$

Now it is perhaps clearer what must be going on[105, 106]. The D–branes and O–planes, and any intersections between them all define sub–spacetimes of the ten dimensional spacetime, where potentially anomalous theories live. This is natural, since as we have already learned, and shall explore much more, there are massless fields of various sorts living on them, possibly charged under any gauge group they might carry.

As the world–volume intersections may be thought of as embedded in the full ten dimensional theory, there is a mechanism for canceling the anomaly which generalizes that which we have already encountered. For example, since the Dp–brane is also a source for the R–R sector field $G^{(p+2)}$, it modifies it according to

$$G_{(p+2)} = dC_{(p+1)} - \mu_p\delta(x_0)\cdots\delta(x_p)dx_0\wedge\cdots dx_p\mathcal{F}(R,F) \ , \tag{416}$$

where the delta functions are chosen to localize the contribution to the world–volume of the brane, extended in the directions $x_0, x_1, \ldots x_p$. Also μ_p is the Dp–brane (or Op–plane) charge, and the polynomial $\mathcal{F}$ must be chosen so that the classically anomalous variation $\delta C_{(p+1)}$ required to keep $G^{(p+2)}$ gauge invariant can cancel the anomaly on the branes' intersection. Following this argument to its logical conclusion, and using the previously mentioned fact that the possible anomalies are described in terms of the characteristic classes $\exp(iF)$, $\hat{A}(R)$ and $\hat{\mathcal{L}}(R)$, reveals that $\mathcal{F}$ takes the form of the couplings that we have already written. We see that the Green–Schwarz mechanism from type I is an example of something much more general, involving the various geometrical objects which can appear embedded in the theory, and not just the fundamental string itself. Arguments along these lines also uncover the feature that the normal bundle also contributes to the curvature couplings as well. The full expressions, for completeness, are:

$$\mu_p\int_{\mathcal{M}_{p+1}}\sum_i C_{(i)}\left[e^{2\pi\alpha' F+B}\right]\sqrt{\frac{\hat{\mathcal{A}}(4\pi^2\alpha' R_T)}{\hat{\mathcal{A}}(4\pi^2\alpha' R_N)}} \ , \tag{417}$$

and

$$\tilde{\mu}_p\int_{\mathcal{M}_{p+1}}\sum_i C_{(i)}\sqrt{\frac{\hat{\mathcal{L}}(\pi^2\alpha' R_T)}{\hat{\mathcal{L}}(\pi^2\alpha' R_N)}} \ , \tag{418}$$

where the subscripts T, N denotes curvatures of the tangent and the normal frame, respectively.

9.6 D–branes and K–Theory

In fact, the sort of argument above is an independent check on the precise normalization of the D–brane charges, which we worked out by direct computation in previous sections. As already said before, the close relation to the topology of the gauge and tangent bundles of the branes suggests a connection with tools which might uncover a deeper classification. This tool is called *"K–theory"*. K–theory should be thought of as a calculus for working out subtle topological differences between vector bundles, and as such makes a natural physical appearance here[107, 22, 24].

This is because there is a means of constructing a D–brane by a mechanism known as "tachyon condensation" on the world volume of higher dimensional branes. Recall that in section 8 we observed that a Dp–brane and an anti–Dp–brane will annihilate. Indeed, there is a tachyon in the spectrum of $p\bar{p}$ strings. Let us make the number of brane be N, and the number of anti–branes as $\bar{N}$. Then the tachyon is charged under the gauge group $U(N)\times U(\bar{N})$. The idea is that a suitable choice of the tachyon can give rise to topology which must survive even if all of the parent branes annihilate away. For example, if the tachyon field is given a topologically stable kink as a function of one of the dimensions inside the brane, then there will be a $p-1$ dimensional structure left over, to be identified with a D$(p-1)$–brane. This idea is the key to seeing how to classify D–branes, by constructing all branes in this way.

Most importantly, we have two gauge bundles, that of the Dp–branes, which we might call E, and that of the D$\bar{p}$–branes, called F. To classify the possible D–branes which can exist in the world volume, one must classify all such bundles, defining as equivalent all pairs which can be reached by brane creation or annihilation: If some number of Dp–branes annihilate with D$\bar{p}$–branes, (or if the reverse happens, *i.e.* creation of Dp–D$\bar{p}$ pairs), the pair (E,F) changes to $(E\oplus G, F\oplus G)$, where G is the gauge bundle associated to the new branes, identical in each set. These two pairs of bundles are equivalent. The group of distinct such pairs is (roughly) the object called $K(X)$, where X is the spacetime which the branes fill (the base of the gauge bundles). Physically distinct pairs have non–trivial differences in their tachyon configurations which would correspond to different D–branes after complete annihilation had taken place. So K–theory, defined in this way, is a sort of more subtle or advanced cohomology which goes beyond the more familiar sort of cohomology we encounter daily.

The technology of K–theory is beyond that which we have room for here, but it should be clear from what we have seen in this section that it is quite natural, since the world–volume couplings of the charge of the branes is *via* the most natural objects with which one would want to perform sensible operations on the gauge bundles of the branes like addition and subtraction: the characteristic classes, $\exp(iF)$, $\hat{A}(R)$ and $\hat{\mathcal{L}}(R)$. Actually, this might have been enough to simply get the result that D–brane charges were classified by cohomology. The fact that it is K–theory (which can compute differences between bundles that cohomology alone would miss) is probably related to a very important physical fact about the underlying theory which will be more manifest one day.

9.7 Further Non–Abelian Extensions

One can use T–duality to go a bit further and deduce a number of non–Abelian extensions of the action, being mindful of the sort of complications mentioned at the beginning of section (5.5). In the absence of geometrical curvature terms it turns out to be:[54, 55]

$$\mu_p\int_{p-\mathrm{brane}}\mathrm{Tr}\left(\left[e^{2\pi\alpha'\mathbf{i}_\Phi\mathbf{i}_\Phi}\sum_p C_{(p+1)}\right]e^{2\pi\alpha' F+B}\right)\ . \tag{419}$$

Here, we ascribe the same meaning to the gauge trace as we did previously (see section (5.5)). The meaning of $\mathbf{i}_X$ is as the "interior product" in the direction given by the vector Φ^i, which produces a form of one degree fewer in rank. For example, on a two form $C_{(2)}(\Phi)=(1/2)C_{ij}(\Phi)dX^idX^j$, we have

$$\mathbf{i}_\Phi C_{(2)}=\Phi^iC_{ij}(\Phi)dX^j\ ;\quad \mathbf{i}_\Phi\mathbf{i}_\Phi C_{(2)}(\Phi)=\Phi^j\Phi^iC_{ij}(\Phi)=\frac{1}{2}[\Phi^i,\Phi^j]C_{ij}(\Phi)\ , \tag{420}$$

where we see that the result of acting twice is non–vanishing when we allow for the non–Abelian case, with C having a nontrivial dependence on Φ.

9.8 Further Curvature Couplings

We deduced geometrical curvature couplings to the R–R potentials a few subsections ago. In particular, such couplings induce the charge of lower p branes by wrapping larger branes on topologically non–trivial surfaces.

In fact, as we saw before, if we wrap a Dp–brane on K3, there is induced precisely -1 units of charge of a D$(p-4)$–brane. This means that the charge of the effective $(p-4)$–dimensional object is

$$\mu = \mu_p V_{\rm K3} - \mu_{p-4} \ , \tag{421}$$

where $V_{\rm K3}$ is the volume of the K3. However, we can go further and notice that since this is a BPS object of the six dimensional $\mathcal{N}=2$ string theory obtained by compactifying on K3, we should expect that it has a tension which is

$$\tau = \tau_p V_{\rm K3} - \tau_{p-4} = g_s^{-1}\mu \ . \tag{422}$$

If this is indeed so, then there must be a means by which the curvature of K3 induces a shift in the tension in the world–volume action. Since the part of the action which refers to the tension is the Dirac–Born–Infeld action, we deduce that there must be a set of curvature couplings for that part of the action as well. Some of them are given by the following:[114, 120]

$$S = -\tau_p \int d^{p+1}\xi \ e^{-\Phi} {\rm det}^{1/2}(G_{ab}+\mathcal{F}_{ab})\Bigg(1 - \frac{1}{3\times 2^8\pi^2} \times \\ \Big(\mathcal{R}_{abcd}\mathcal{R}^{abcd} - \mathcal{R}_{\alpha\beta ab}R^{\alpha\beta ab} + 2\hat{\mathcal{R}}_{\alpha\beta}\hat{\mathcal{R}}^{\alpha\beta} - 2\hat{\mathcal{R}}_{ab}\hat{\mathcal{R}}^{ab}\Big) + O(\alpha'^4)\Bigg) \ , \tag{423}$$

where $\mathcal{R}_{abcd} = (4\pi^2\alpha')R_{abcd}$, *etc.*, and a,b,c,d are the usual tangent space indices running along the brane's world volume, while α,β are normal indices, running transverse to the world–volume.

Some explanation is needed. Recall that the embedding of the brane into D–dimensional spacetime is achieved with the functions $X^\mu(\xi^a)$, $(a=0,\ldots,p;\mu=0,\ldots,D-1)$ and the pullback of a spacetime field F_μ is performed by soaking up spacetime indices μ with the local "tangent frame" vectors $\partial_a X^\mu$, to give $F_a = F_\mu\partial_a X^\mu$. There is another frame, the "normal frame", with basis vectors ζ^μ_α, $(\alpha = p+1,\ldots,D-1)$. Orthonormality gives $\zeta^\mu_\alpha\zeta^\nu_\beta G_{\mu\nu} = \delta_{\alpha\beta}$ and also we have $\zeta^\mu_\alpha\partial_a X^\nu G_{\mu\nu}=0$.

We can pull back the spacetime Riemann tensor $R_{\mu\nu\kappa\lambda}$ in a number of ways, using these different frames, as can be seen in the action. $\hat{R}$ with two indices are objects which were constructed by contraction of the *pulled–back* fields. They are *not* the pull back of the bulk Ricci tensor, which vanishes at this order of string perturbation theory anyway.

In fact, for the case of K3, it is Ricci flat and everything with normal space indices vanishes and so we get only $R_{abcd}R^{abcd}$ appearing, which alone computes the result (362) for us, and so after integrating over K3, the action becomes:

$$S = -\int d^{p-3}\xi \ e^{-\Phi}\left[\tau_p V_{\rm K3} - \tau_{p-4}\right]{\rm det}^{1/2}(G_{ab}+\mathcal{F}_{ab}) \ , \tag{424}$$

where again we have used the recursion relation between the D–brane tensions. So we see that we have correctly reproduced the shift in the tension that we expected on general grounds for the effective D$(p-4)$–brane.

10 Multiple D–Branes

In section 5, we saw a number of interesting terms arise in the D–brane world–volume action which had interpretations as smaller branes. For example, a $U(1)$ flux was a D$(p-2)$–brane fully delocalized in the world–volume, while for the non–Abelian case, we saw a D$(p-4)$–brane arise as an instanton in the world–volume gauge theory. Interestingly, while the latter breaks half of the supersymmetry again, as it ought to, the former is still half BPS, since it is T–dual to a tilted D$(p+1)$–brane.

It is worthwhile trying to understand this better back in the basic description using boundary conditions and open string sectors, and this is the first goal of this section. After that, we'll have a closer look at the nature of the BPS bound and the superalgebra, and study various key illustrative examples.

10.1 Dp and Dp' From Boundary Conditions

Let us consider two D–branes, Dp and Dp', each parallel to the coordinate axes. (We can of course have D–branes at angles, [121] but we will not consider this here.) An open string can have both ends on the same D–brane or one on each. The $p-p$ and $p'-p'$ spectra are the same as before, but the $p-p'$ strings are new. Since we are taking the D–branes to be parallel to the coordinate axes, there are four possible sets of boundary conditions for each spatial coordinate X^i of the open string, namely NN (Neumann at both ends), DD, ND, and DN. What really will matter is the number ν of ND plus DN coordinates. A T–duality can switch NN and DD, or ND and DN, but ν is invariant. Of course ν is even because we only have p even or p odd in a given theory in order to have a chance of preserving supersymmetry.

The respective mode expansions are

$$\begin{aligned}
\text{NN:}\qquad & X^\mu(z,\bar z) = x^\mu - i\alpha' p^\mu \ln(z\bar z) + i\sqrt{\frac{\alpha'}{2}}\sum_{m\neq 0}\frac{\alpha^\mu_m}{m}(z^{-m}+\bar z^{-m}),\\
\text{DN, ND:}\qquad & X^\mu(z,\bar z) = i\sqrt{\frac{\alpha'}{2}}\sum_{r\in\mathbb{Z}+1/2}\frac{\alpha^\mu_r}{r}(z^{-r}\pm\bar z^{-r}),\\
\text{DD:}\qquad & X^\mu(z,\bar z) = -i\frac{\delta X^\mu}{2\pi}\ln(z/\bar z) + i\sqrt{\frac{\alpha'}{2}}\sum_{m\neq 0}\frac{\alpha^\mu_m}{m}(z^{-m}-\bar z^{-m})\ .
\end{aligned} \tag{425}$$

In particular, the DN and ND coordinates have half–integer moding. The fermions have the same moding in the Ramond sector (by definition) and opposite in the Neveu–Schwarz sector. The string zero point energy is 0 in the R sector as always, and using (70) we get:

$$(8-\nu)\left(-\frac{1}{24}-\frac{1}{48}\right)+\nu\left(\frac{1}{48}+\frac{1}{24}\right) = -\frac{1}{2}+\frac{\nu}{8} \tag{426}$$

in the NS sector.

The oscillators can raise the level in half–integer units, so only for ν a multiple of 4 is degeneracy between the R and NS sectors possible. Indeed, it is in this case that the Dp–Dp' system is supersymmetric. We can see this directly. As discussed in sections 8.1.1 and 8.2, a D–brane leaves unbroken the supersymmetries

$$Q_\alpha + P\tilde Q_\alpha\ , \tag{427}$$

where P acts as a reflection in the direction transverse to the D–brane. With a second D–brane, the only unbroken supersymmetries will be those that are also of the form

$$Q_\alpha + P'\tilde Q_\alpha = Q_\alpha + P(P^{-1}P')\tilde Q_\alpha\ . \tag{428}$$

with P' the reflection transverse to the second D–brane. Then the unbroken supersymmetries correspond to the $+1$ eigenvalues of $P^{-1}P'$. In DD and NN directions this is trivial, while in DN and ND directions it is a net parity transformation. Since the number ν of such dimensions is even, we can pair them as we did in section 7.1.1, and write $P^{-1}P'$ as a product of rotations by π,

$$e^{i\pi(J_1+\ldots+J_{\nu/2})} \ . \tag{429}$$

In a spinor representation, each $e^{i\pi J}$ has eigenvalues $\pm i$, so there will be unbroken supersymmetry only if ν is a multiple of 4 as found above. [7]

For example, Type I theory, besides the D9–branes, will have D1–branes and D5–branes. This is consistent with the fact that the only R-R field strengths are the three-form and its Hodge–dual seven–form. The D5–brane is required to have two Chan–Paton degrees of freedom (which can be thought of as images under Ω) and so an $SU(2)$ gauge group.[122, 123]

When $\nu=0$, $P^{-1}P'=1$ identically and there is a full ten-dimensional spinor of supersymmetries. This is the same as for the original Type I theory, to which it is T–dual. In $D=4$ units, this is $\mathcal{N}=4$, or sixteen supercharges. For $\nu=4$ or $\nu=8$ there is $D=4$ $\mathcal{N}=2$ supersymmetry.

Let us now study the spectrum for $\nu=4$, saving $\nu=8$ for later. Sometimes it is useful to draw a quick table showing where the branes are located. Here is one for the (9,5) system, where the D5–brane is pointlike in the x^6, x^7, x^8, x^9 directions and the D9–brane is (of course) extended everywhere:

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9
D9	–	–	–	–	–	–	–	–	–	–
D5	–	–	–	–	–	–	•	•	•	•

A dash under x^i means that the brane is extended in that direction, while a dot means that it is pointlike there. Continuing with our analysis, we see that the NS zero–point energy is zero. There are four periodic world–sheet fermions ψ^i, namely those in the ND directions. The four zero modes generate $2^{4/2}$ or four ground states, of which two survive the GSO projection. In the R sector the zero–point energy is also zero; there are four periodic transverse ψ, from the NN and DD directions not counting the directions $\mu=0,1$. Again these generate four ground states of which two survive the GSO projection. The full content of the p–p' system is then is half of an $N=2$ hypermultiplet. The other half comes from the p'–p states, obtained from the orientation reversed strings: these are distinct because for $\nu\neq 0$ the ends are always on different D–branes.

Let us write the action for the bosonic $p-p'$ fields χ^A, starting with $(p,p')=(9,5)$. Here A is a doublet index under the $SU(2)_R$ of the $N=2$ algebra. The field χ^A has charges $(+1,-1)$ under the $U(1)\times U(1)$ gauge theories on the branes, since one end leaves, and the other arrives. The minimally coupled action is then

$$\int d^6\xi \, \left(\sum_{a=0}^{5} |(\partial_a + iA_a - iA'_a)\chi|^2 + \left(\frac{1}{4g^2_{\mathrm{YM},p}} + \frac{1}{4g^2_{\mathrm{YM},p'}}\right) \sum_{I=1}^{3} (\chi^\dagger \tau^I \chi)^2 \right) \ , \tag{430}$$

with A_a and A'_a the brane gauge fields, $g_{\mathrm{YM},p}$ and $g_{\mathrm{YM},p'}$ the effective Yang–Mills couplings (377), and τ^I the Pauli matrices. The second term is from the $N=2$ D–terms for the two gauge fields. It can also be written as a commutator $\mathrm{Tr}\,[\phi^i,\phi^j]^2$ for appropriately chosen fields ϕ^i, showing that its form is controlled by the dimensional reduction of an F^2 pure Yang–Mills term.

The integral is over the 5–brane world-volume, which lies in the 9–brane world-volume. Under T–dualities in any of the ND directions, one obtains $(p,p')=(8,6)$, $(7,7)$, $(6,8)$, or $(5,9)$, but the intersection of the branes remains $(5+1)$-dimensional and the p–p' strings live on the intersection with action (430). In the present case the D-term is non–vanishing only for $\chi^A=0$, though more generally (say when there are several coincident p and p'–branes), there will be additional massless charged fields and flat directions arise.

[7] We will see that there are supersymmetric *bound states* when $\nu=2$.

Under T–dualities in r NN directions, one obtains $(p,p') = (9-r, 5-r)$. The action becomes

$$\int d^{6-r}\xi \left(\sum_{a=0}^{5-r} |(\partial_a + iA_a - iA'_a)\chi|^2 \ + \frac{\chi^\dagger\chi}{(2\pi\alpha')^2}\sum_{a=6-r}^{5}(X_a - X'_a)^2 \right.$$
$$\left. + \ \left(\frac{1}{4g^2_{\mathrm{YM},p}} + \frac{1}{4g^2_{\mathrm{YM}p'}}\right)\sum_{i=1}^{3}(\chi^\dagger\tau^I\chi)^2\right) \ . \qquad (431)$$

The second term, proportional to the separation of the branes, is from the tension of the stretched string.

10.2 The BPS Bound for the Dp–Dp' System

The ten dimensional $\mathcal{N}=2$ supersymmetry algebra (in a Majorana basis) is

$$\begin{aligned}\{Q_\alpha, Q_\beta\} &= 2(\Gamma^0\Gamma^\mu)_{\alpha\beta}(P_\mu + Q^{\mathrm{NS}}_\mu/2\pi\alpha')\\ \{\tilde{Q}_\alpha, \tilde{Q}_\beta\} &= 2(\Gamma^0\Gamma^\mu)_{\alpha\beta}(P_\mu - Q^{\mathrm{NS}}_\mu/2\pi\alpha')\\ \{Q_\alpha, \tilde{Q}_\beta\} &= 2\sum_p \frac{\tau_p}{p!}(\Gamma^0\Gamma^{m_1}\dots\Gamma^{m_p})_{\alpha\beta}Q^{\mathrm{R}}_{m_1\dots m_p} \ .\end{aligned} \qquad (432)$$

Here Q^{NS} is the charge to which the NS–NS two–form couples, it is essentially the winding of a fundamental string stretched along $\mathcal{M}_1$:

$$Q^{\mathrm{NS}}_\mu \equiv \frac{Q^{\mathrm{NS}}}{v_1}\int_{\mathcal{M}_1} dX^\mu \ , \quad \text{with} \quad Q^{\mathrm{NS}} = \frac{1}{\mathrm{Vol}\,S^7}\int_{S^7} e^{-2\Phi}\, {}^*H^{(3)} \qquad (433)$$

and the charge Q^{NS} is normalised to one per unit spatial world–volume, $v_1 = L$, the length of the string. It is obtained by integrating over the S^7 which surrounds the string. The Q^{R} are the R–R charges, defined as a generalisation of winding on the space $\mathcal{M}_p$:

$$Q^{\mathrm{R}}_{\mu_1\dots\mu_p} \equiv \frac{Q^{\mathrm{R}}_p}{v_p}\int_{\mathcal{M}_p} dX^{\mu_1}\wedge\cdots dX^{\mu_p} \ , \quad \text{with} \quad Q^{\mathrm{R}}_p = \frac{1}{\mathrm{Vol}\,S^{8-p}}\int_{S^{8-p}} {}^*G^{(p+2)} \ . \qquad (434)$$

The sum in (432) runs over all orderings of indices, and we divide by p! Of course, p is even for IIA or odd for IIB. The R–R charges appear in the product of the right- and left–moving supersymmetries, since the corresponding vertex operators are a product of spin fields, while the NS-NS charges appear in right–right and left–left combinations of supercharges.

As an example of how this all works, consider an object of length L, with the charges of p fundamental strings ("F–strings", for short) and q D1–branes ("D–strings) in the IIB theory, at rest and aligned along the direction X^1. The anticommutator implies

$$\frac{1}{2}\left\{\begin{bmatrix} Q_\alpha \\ \tilde{Q}_\alpha\end{bmatrix}, \begin{bmatrix} Q_\beta \ \tilde{Q}_\beta\end{bmatrix}\right\} = \begin{bmatrix} 1 & 0 \\ 0 & 1\end{bmatrix} M\delta_{\alpha\beta} + \begin{bmatrix} p & q/g_s \\ q/g_s & -p\end{bmatrix}\frac{L(\Gamma^0\Gamma^1)_{\alpha\beta}}{2\pi\alpha'} \ . \qquad (435)$$

The eigenvalues of $\Gamma^0\Gamma^1$ are ± 1 so those of the right–hand side are $M \pm L(p^2+q^2/g^2)^{1/2}/2\pi\alpha'$. The left side is a positive matrix, and so we get the "BPS bound" on the tension [124]

$$\frac{M}{L} \geq \frac{\sqrt{p^2+q^2/g_s^2}}{2\pi\alpha'} \equiv \tau_{p,q} \ . \qquad (436)$$

Quite pleasingly, this is saturated by the fundamental string, $(p,q) = (1,0)$, and by the D–string, $(p,q) = (0,1)$.

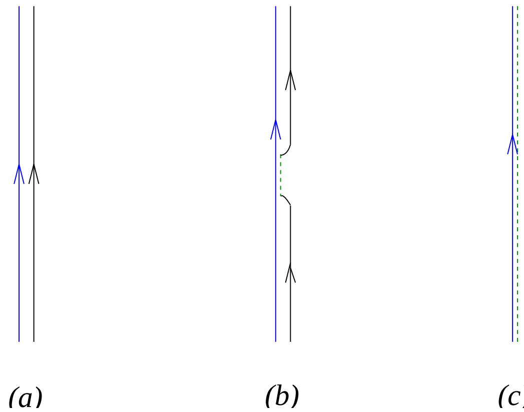


Figure 31: *(a)* A parallel D–string and F–string, which is not supersymmetric. *(b)* The F–string breaks, its ends attaching to the D-string, resulting in *(c)* the final supersymmetric state, a D–string with flux.

It is not too hard to extend this to a system with the quantum numbers of Dirichlet p and p' branes. The result for ν a multiple of 4 is

$$M \geq \tau_p v_p + \tau_{p'} v_{p'} \tag{437}$$

and for ν even but not a multiple of 4 it is [8]

$$M \geq \sqrt{\tau_p^2 v_p^2 + \tau_{p'}^2 v_{p'}^2} \ . \tag{438}$$

The branes are wrapped on tori of volumes v_p and v_p' in order to make the masses finite.

The results (437) and (438) are consistent with the earlier results on supersymmetry breaking. For ν a multiple of 4, a separated p–brane and p'–brane do indeed saturate the bound (437). For ν not a multiple of four, they do not saturate the bound (438) and cannot be supersymmetric.

10.3 Bound States of Fundamental Strings and D–Strings

Consider a parallel D1–brane (D–string) and a fundamental string (F–string) lying along X^1. The total tension

$$\tau_{D1} + \tau_{F1} = \frac{g_s^{-1} + 1}{2\pi\alpha'} \tag{439}$$

exceeds the BPS bound (436) and so this configuration is not supersymmetric. However, it can lower its energy[29] as shown in figure 31. The F–string breaks, its endpoints attached to the D–string. The endpoints can then move off to infinity, leaving only the D–string behind. Of course, the D–string must now carry the charge of the F-string as well. This comes about because the F–string endpoints are charged under the D–string gauge field, so a flux runs between them; this flux remains at the end.

Thus the final D–string carries both the NS–NS and R–R two–form charges. The flux is of order g_s, its energy density is of order g_s, and so the final tension is $(g_s^{-1} + O(g_s))/2\pi\alpha'$. This is below the tension of the separated strings and of the same form as the BPS bound (436) for a $(1,1)$ string. A more detailed calculation shows that the final tension saturates the bound,[110] so the state is supersymmetric. In effect, the F–string has dissolved into the D–string, leaving flux behind.

We can see quite readily that this is a supersymmetric situation using T–duality. We can choose a gauge in which the electric flux is $F_{01}{=}\dot{A}_1$. T–dualising along the x^1 direction, we ought to get a D0–brane, which we do, except that it is moving with constant velocity, since we get $\dot{X}^1 = 2\pi\alpha'\dot{A}_1$. This clearly has the same supersymmetry as a stationary D0–brane, having been simply boosted.

[8]The difference between the two cases comes from the relative sign of $\Gamma^M(\Gamma^{M'})^T$ and $\Gamma^{M'}(\Gamma^M)^T$.

To calculate the number of BPS states we should put the strings in a box of length L to make the spectrum discrete. For the $(1,0)$ F-string, the usual quantization of the ground state gives eight bosonic and eight fermionic states moving in each direction for $16^2 = 256$ in all.

This is the ultrashort representation of supersymmetry: half the 32 generators annihilate the BPS state and the other half generate $2^8 = 256$ states. The same is true of the $(0,1)$ D-string and the $(1,1)$ bound state just found, as will be clear from the later duality discussion of the D–string.

It is worth noting that the $(1,0)$ F–string leaves unbroken half the supersymmetry and the $(0,1)$ D–string leaves unbroken a different half of the supersymmetry. The $(1,1)$ bound state leaves unbroken not the intersection of the two (which is empty), but yet a different half. The unbroken symmetries are linear combinations of the unbroken and broken supersymmetries of the D–string.

All the above extends immediately to p F–strings and one D–string, forming a supersymmetric $(p,1)$ bound state. The more general case of p F–strings and q D–strings is more complicated. The gauge dynamics are now non–Abelian, the interactions are strong in the infrared, and no explicit solution is known. When p and q have a common factor, the BPS bound makes any bound state only neutrally stable against falling apart into subsystems. To avoid this complication let p and q be relatively prime, so any supersymmetric state is discretely below the continuum of separated states. This allows the Hamiltonian to be deformed to a simpler supersymmetric Hamiltonian whose supersymmetric states can be determined explicitly, and again there is one ultrashort representation, 256 states. It is left to the reader to consult the literature[29, 1] for the details.

10.4 The Three–String Junction

Let us consider further the BPS saturated formula derived and studied in the two previous subsections, and write it as follows:

$$\tau_{p,q} = \sqrt{(p\tau_{1,0})^2 + (q\tau_{0,1})^2} \ . \tag{440}$$

An obvious solution to this is

$$\tau_{p,q} \sin\alpha = q\tau_{0,1} \ , \quad \tau_{p,q}\cos\alpha = p\tau_{1,0} \ . \tag{441}$$

with $\tan\alpha = q/(pg_s)$. Recall that these are tensions of strings, and therefore we can interpret the equations (441) as balance conditions for the components of forces. In fact, it is the required balance for three strings,[128, 126] and we draw the case of $p = q = 1$ in figure 32.

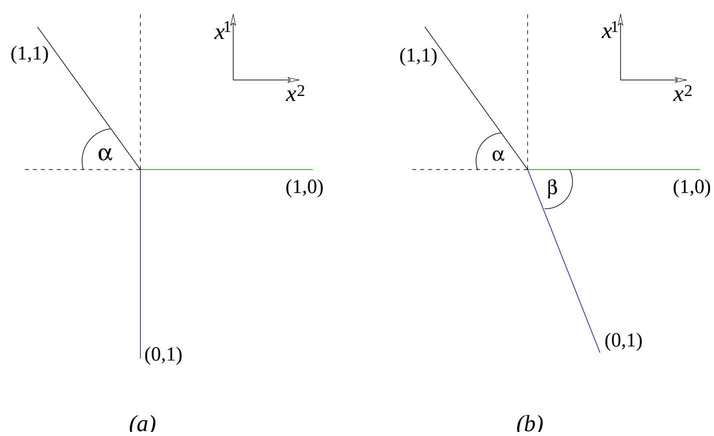


Figure 32: *(a)* When an F–string ends on a D–string it causes it to bend at an angle set by the string coupling. On the other side of the junction is a (1,1) string. This is in fact a BPS state. *(b)* Switching on some amount of the R–R scalar can vary the other angle, as shown.

Is this at all consistent with what we already know? The answer is yes. An F–string is allowed to end on a D–string by definition, and a (1,1) string is produced, due to flux conservation, as we discussed above. The issue here is just how we see that there is bending. The first thing to notice is that the angle α goes to $\pi/2$ in the limit of zero string coupling, and so the D–string appears in that case to run straight. This had better be true, since it is then clear that we simply were allowed to ignore the bending in our previous weakly coupled string analysis. (This study of the bending of branes beyond zero coupling has important consequences for the study of one–loop gauge theory data.[130])

Parenthetically, it is nice to see that in the limit of infinite string coupling, α goes to 0. The diagram is better interpreted as a D–string ending on an F–string with no resulting bending. This fits nicely with the fact that the D– and F–strings exchange roles under the strong/weak coupling duality ("S–duality") of the type IIB string theory, as we shall see in section 11.

When we wrote the linearized BIon equations in section 5.7, we ignored the 1+1 dimensional case. Let us now include that part of the story here as a 1+1 dimensional gauge theory discussion. There is a flux F_{01} on the world–volume, and the end of the F–string is an electric source. Given that there is only one spatial dimension, the F–string creates a discontinuity on the flux, such that *e.g*:[131, 63]

$$F_{01} = \begin{cases} g_s \ , & x_1 > 0 \\ 0 \ , & x_1 < 0 \end{cases} \ , \tag{442}$$

so we can choose a gauge such that

$$A_0 = \begin{cases} g_s x^1 \ , & x_1 > 0 \\ 0 \ , & x_1 < 0 \end{cases} \ . \tag{443}$$

Just as in section 5.7, this is BPS if one of the eight scalars Φ^m is also switched on so that

$$\Phi^2(x^1) = A_0 \ . \tag{444}$$

How do we interpret this? Since $(2\pi\alpha')\Phi^2$ represents the x^2 position of the D–string, we see that for $x^1 < 0$ the D–string is lying along the x^1 axis, while for $x^1 > 0$, it lies on a line forming an angle $\tan^{-1}(1/g_s)$ with the x^1, axis.

Recall the T_1–dual picture we mentioned in the previous section, where we saw that the flux on the D–string (making the (1,1) string) is equivalent to a D0–brane moving with velocity $(2\pi\alpha')F_{01}$. Now we see that the D0–brane loses its velocity at $x^1 = 0$. This is fine, since the apparent impulse is accounted for by the momentum carried by the F–string in the T–dual picture. (One has to tilt the diagram in order to T–dualise along the (1,1) string in order to see that there is F–string momentum.)

Since as we have seen many times that the presence of flux on the world–volume of a Dp–brane is equivalent to having a dissolved D$(p-2)$–brane, *i.e.,* non–zero $C_{(p-1)}$ source, we can modify the flux on the $x^1 < 0$ part of the string this way by turning on the R–R scalar C_0. This means that $\Phi^2(x^1)$ will be linear there too, and so the angle β between the D– and F–strings can be varied too (see figure 32*(b)*). It is interesting to derive the balance conditions from this, and then convert it into a modified tension formula, but we will not do that here.[131]

It is not hard to imagine that given the presence we have already deduced of a general (p,q) string in the theory that there are three–string junctions to be made out of any three strings such that the (p,q)–charges add up correctly, giving a condition on the angles at which they can meet. This is harder to do in the full non–Abelian gauge theories on their world–volumes, but in fact a complete formula can be derived using the underlying $SL(2,\mathbb{Z})$ symmetry of the type IIB string theory. We will have more to say about this symmetry later.

General three–string junctions have been shown to be important in a number of applications (see e.g. refs.[128, 129]), and there is a large literature on the subject which we are unfortunately not able to review here.

10.5 Aspects of D–Brane Bound States

Bound states of p–branes and p'–branes have many applications, and so it is worth listing some of the results here. Here we focus on $p' = 0$, since we can always reach it from a general (p, p') using T–duality.

10.5.1 0–0 bound states

The BPS bound for the quantum numbers of n 0–branes is $n\tau_0$, so any bound state will be at the edge of the continuum. What we would like to know is if there is actually a true bound state wave function, *i.e.,* a wavefunction which is normalisable. To make the bound state counting well defined, compactify one direction and give the system momentum m/R with m and n relatively prime.[132] The bound state now lies discretely below the continuum, because the momentum cannot be shared evenly among unbound subsystems.

This bound state problem is T–dual to the one just considered. Taking the T–dual, the n D0–branes become D1–branes, while the momentum becomes winding number, corresponding to m F-strings. There is therefore one ultrashort multiplet of supersymmetric states when m and n are relatively prime.[132] This bound state should still be present back in infinite volume, since one can take R to be large compared to the size of the bound state. There is a danger that the size of the wavefunction we have just implicitly found might simply grow with R such that as $R \to \infty$ it becomes non–normalisable again. More careful analysis is needed to show this. It is sufficient to say here that the bound states for arbitrary numbers of D0–branes are needed for the consistency of string duality, so this is an important problem. Some strong arguments have been presented in the literature, ($n = 2$ is proven) but the general case is not yet proven[133].

10.5.2 0–2 bound states

Now the BPS bound (438) puts any bound state discretely below the continuum. One can see a hint of a bound state forming by noticing that for a coincident D0–brane and D2–brane the NS 0–2 string has a negative zero–point energy (426) and so a tachyon (which survives the GSO projection), indicating instability towards something. In fact the bound state (one short representation) is easily described: the D0–brane dissolves in the D2–brane, leaving flux, as we have seen numerous times. The brane R–R action (394) contains the coupling $C_{(1)}F$, so with the flux the D2–brane also carries the D0–brane charge.[134] There is also one short multiplet for n D0–branes. This same bound state is always present when $\nu = 2$.

10.5.3 0–4 bound states

The BPS bound (437) makes any bound state marginally stable, so the problem is made well–defined as in the 0–0 case by compactifying and adding momentum.[135] The interactions in the action (431) are relevant in the infrared so this is again a hard problem, but as before it can be deformed into a solvable supersymmetric system. Again there is one multiplet of bound states.[135] Now, though, the bound state is invariant only under $\frac{1}{4}$ of the original supersymmetry, the intersection of the supersymmetries of the D0–brane and of the D4–brane. The bound states then lie in a short (but not ultrashort) multiplet of 2^{12} states.

For 2 D0–branes and one D4–brane, one gets the correct count as follows.[136] Think of the case that the volume of the D4–brane is large. The 16 supersymmetries broken by the D4–brane generate 256 states that are delocalized on the D4–brane. The 8 supersymmetries unbroken by the D4–brane and broken by the D0–brane generate 16 states (half bosonic and half fermionic), localized on the D0–brane. The total number is the product 2^{12}. Now count the number of ways two D0–branes can be put into their 16 states on the D4–brane: there are 8 states with both D0–branes in the same (bosonic) state and $\frac{1}{2}16 \cdot 15$ states with the D–branes in different states, for a total of $8 \cdot 16$ states. But in addition, the two–branes can bind, and there are again 16 states where the bound state binds to the D4–brane. The total, tensoring again with the D4–brane ground states, is $9 \cdot 16 \cdot 256$.

For n D0–branes and one D4–brane, the degeneracy D_n is given by the generating functional [136]

$$\sum_{n=0}^{\infty} q^n D_n \;=\; 256 \prod_{k=1}^{\infty}\left(\frac{1+q^k}{1-q^k}\right)^8 \ , \tag{445}$$

where the term k in the product comes from bound states of k D0–branes then bound to the D4–brane. Some discussion of the D0–D4 bound state, and related issues, can be found in the references[137].

10.5.4 0–6 bound states

The relevant bound is (438) and again any bound state would be below the continuum. The NS zero-point energy for 0–6 strings is positive, so there is no sign of decay. One can give D0–brane charge to the D6–brane by turning on flux, but there is no way to do this and saturate the BPS bound. So it appears that there are *no* supersymmetric bound states. Incidentally, and unlike the 0–2 case, the 0–6 interaction is repulsive, both at short distance and at long.

10.5.5 0–8 bound states

The case of the D8–brane is special, since it is rather big. It is a domain wall, since there is only one spatial dimension transverse to it. In fact, the D8–brane on its own is not really a consistent object. Trying to put it into type IIA runs into trouble, since the string coupling blows up a finite distance from it on either side because of the nature of its coupling to the dilaton. To stop this happening, one has to introduce a pair of O8–planes, one on each side, since they (for SO groups) have negative charge (-8 times that of the D8–brane) and can soak up the dilaton. We therefore should have 16 D8–branes for consistency, and so we end up in the type I$'$ theory, the T–dual of Type I. The bound state problem is now quite different, and certain details of it pertain to the strong coupling limit of certain string theories, and their "matrix"[148, 149] formulation.[138, 139] We shall revisit this in section 11.5.

11 Strings at Strong Coupling

One of the most striking results of the middle '90's was the realization that all of the superstring theories are in fact dual to one another at strong coupling.[140]

This also brought eleven dimensional supergravity into the picture and started the search for M–theory, the dynamical theory within which all of those theories would fit as various effective descriptions of perturbative limits. All of this is referred to as the "Second Superstring Revolution". Every revolution is supposed to have a hero or heroes. We shall consider branes to be cast in that particular role, since they (and D–branes especially) supplied the truly damning evidence of the strong coupling fate of the various string theories. We shall discuss aspects of this in the present section. We simply study the properties of D–branes in the various string theories, and then trust to that fact that as they are BPS states, many of these properties will survive at strong coupling.

11.1 Type IIB/Type IIB Duality

11.1.1 D1–Brane Collective Coordinates

Let us first study the D1–brane. This will be appropriate to the study of type IIB and the type I string by Ω–projection. Its collective dynamics as a BPS soliton moving in flat ten dimensions is captured by the 1+1 dimensional world–volume theory, with 16 or 8 supercharges, depending upon the theory we are in. (See figure 33*(a)*.)

It is worth first setting up a notation and examining the global symmetries. Let us put the D1–brane to lie along the x^1 direction, as we will do many times in what is to come. This arrangement of branes breaks the Lorentz group up as follows:

$$SO(1,9) \supset SO(1,1)_{01} \times SO(8)_{2-9} \ , \tag{446}$$

Accordingly, the supercharges decompose under (446) as

$$\mathbf{16} = \mathbf{8}_{+} \oplus \mathbf{8}_{-} \tag{447}$$

where $\pm$ subscripts denote a chirality with respect to $SO(1,1)$.

For the 1–1 strings, there are 8 Dirichlet–Dirichlet (DD) directions, the Neveu–Schwarz (NS) sector has zero point energy $-1/2$. The massless excitations form vectors and scalars in the 1+1 dimensional model. For the vectors, the Neumann–Neumann (NN) directions give a gauge field A^μ. Now, the gauge field has no local dynamics, so the only contentful bosonic excitations are the transverse fluctuations. These come from the 8 Dirichlet–Dirichlet (DD) directions x^m, $m = 2, \cdots, 9$, and are

$$\phi^m(x^0, x^1): \quad \lambda_\phi \psi^m_{-\frac{1}{2}}|0> \ . \tag{448}$$

The fermionic states ξ from the Ramond (R) sector (with zero point energy 0, as always) are built on the vacua formed by the zero modes $\psi^i_0, i{=}0, \ldots, 9$. This gives the initial **16**. The GSO projection acts on the vacuum in this sector as:

$$(-1)^F = e^{i\pi(S_0+S_1+S_2+S_3+S_4)} \ . \tag{449}$$

A left or right–moving state obeys $\Gamma^0\Gamma^1\xi_\pm = \pm\xi_\pm$, and so the projection onto $(-1)^F\xi{=}\xi$ says that left and right moving states are odd and (respectively) even under $\Gamma^2 \ldots \Gamma^9$, which is to say that they are either in the $\mathbf{8_s}$ or the $\mathbf{8_c}$. So we see that the GSO projection simply correlates world sheet chirality with spacetime chirality: ξ_- is in the $\mathbf{8}_c$ of $SO(8)$ and ξ_+ is in the $\mathbf{8}_s$.

So we have seen that for a D1–brane in type IIB string theory, the right–moving spinors are in the $\mathbf{8_s}$ of $SO(8)$, and the left-moving spinors in the $\mathbf{8_c}$. These are the same as the fluctuations of a fundamental IIB string, in static gauge[29], and here spacetime supersymmetry is manifest. (It is in "Green–Schwarz" form[102].) There, the supersymmetries Q_α and $\tilde{Q}_\alpha$ have the same chirality. Half of each spinor annihilates the F–string and the other half generates fluctuations. Since the supersymmetries have the same $SO(9,1)$ chirality, the $SO(8)$ chirality is correlated with the direction of motion.

So far we have been using the string metric. We can switch to the Einstein metric, $g^{(\rm E)}_{\mu\nu} = e^{-\Phi/2}g^{(\rm S)}_{\mu\nu}$, since in this case gravitational action has no dependence on the dilaton, and so it is an invariant under duality. The tensions in this frame are:

$$\begin{aligned} &\text{F–string:} \qquad g_s^{1/2}/2\pi\alpha' \\ &\text{D–string:} \qquad g_s^{-1/2}/2\pi\alpha' \ . \end{aligned} \tag{450}$$

Since these are BPS states, we are able to trust these formulae at arbitrary values of g_s.

Let us see what interpretation we can make of these formulae: At weak coupling the D–string is heavy and the F–string tension is the lightest scale in the theory. At strong coupling, however, the D–string is the lightest object in the theory, (a dimensional argument shows that the lowest–dimensional branes have the lowest scale[141]), and it is natural to believe that the theory can be reinterpreted as a theory of weakly coupled D–strings, with $g'_s = g_s^{-1}$. One cannot prove this without a non–perturbative definition of the theory, but quantizing the light D–string implies a large number of the states that would be found in the dual theory, and self–duality of the IIB theory seems by far the simplest interpretation—given that physics below the Planck energy is described by some specific string theory, it seems likely that there is a unique extension to higher energies. This agrees with the duality deduced from the low energy action and other considerations.[140, 154] In particular, the NS–NS and R–R two-form potentials, to which the D– and F–strings respectively couple, are interchanged by this duality.

This duality also explains our remark about the strong and weak coupling limits of the three string junction depicted in figure 32. The roles of the D– and F–strings are swapped in the $g_s \to 0, \infty$ limits, which fits with the two limiting values $\alpha \to \pi/2, 0$.

11.1.2 S–Duality and $SL(2,\mathbb{Z})$

The full duality group of the $D = 10$ Type IIB theory is expected to be $SL(2,\mathbb{Z})$.[142, 144] This relates the fundamental string not only to the R–R string but to a whole set of strings with the quantum numbers of p F–strings and q D–strings for p and q relatively prime.[124] The bound states found in section 10.3 are just what is required for $SL(2,\mathbb{Z})$ duality.[29] As the coupling and the R–R scalar are varied, each of these strings becomes light at the appropriate point in moduli space. It is worth noting here that there is a beautiful subject called "F–theory", which is a tool for generating very complicated type IIB backgrounds by geometrizing the $SL(2,\mathbb{Z})$ symmetry[173, 174]. We won't be able to cover this here due to lack of time, sadly.

11.2 $SO(32)$ Type I/Heterotic Duality

11.2.1 D1–Brane Collective Coordinates

Let us now consider the D1–brane in the Type I theory. We must modify our previous analysis in two ways. First, we must project onto Ω–even states. As in section 2.6, the $U(1)$ gauge field A is in fact projected out, since ∂_t is odd under Ω. The normal derivative ∂_n, is even under Ω, and hence the Φ^m survive. Turning to the fermions, we see that Ω acts as $e^{i\pi(S_1+S_2+S_3+S_4)}$ and so the left–moving $\mathbf{8_c}$ is projected out and the right-moving $\mathbf{8_s}$ survives.

Recall that D9–branes must be introduced after doing the Ω projection of the type IIB string theory. These are the $SO(32)$ Chan–Paton factors. This means that we must also include the massless fluctuations due to strings with one end on the D1-brane and the other on a D9–brane. (See figure 33*(b)*) The zero point energy in the NS sector for these states is 1/2, and so there is way to make a massless state. The R sector has zero point energy zero, as usual, and the ground states come from excitations in the x^0, x^1 direction, since it is in the NN sector that the modes are integer. The GSO projection $(-)^F = \Gamma^0\Gamma^1$ will project out

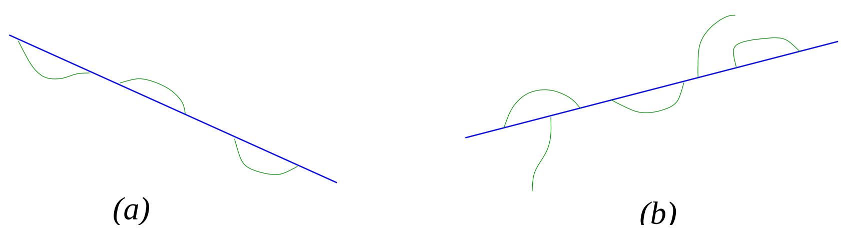


Figure 33: D1–branes *(a)* in Type IIB theory its fluctuations are described by 1–1 strings. *(b)* in Type I string theory, there are additional contributions from 1–9 strings.

one of these, λ_-, while the right moving one will remain. The Ω projection simply relates 1–9 strings to 9–1 strings, and so places no constraint on them. Finally, we should note that the 1–9 strings, as they have one end on a D9–brane, transform as vectors of $SO(32)$.

Now, by the argument that we saw in the case of the type IIB string, we should deduce that this string becomes a light fundamental string in some dual string theory at strong coupling. We have seen such a string before in section 7.4. It is the "heterotic" string with has $(0, 1)$ spacetime supersymmetry, and a left moving family of 32 fermions transforming as the **32** of $SO(32)$. They carry a current algebra which realizes the $SO(32)$ as a spacetime gauge symmetry. The other ten dimensional heterotic string, with gauge group $E_8\times E_8$, has a strong coupling limit which we will examine shortly, using the fact that upon compactifying on a circle, the two heterotic string theories are perturbatively related by T–duality (see section 8.1.3).[163, 164]

We have obtained the $SO(32)$ string here with spacetime supersymmetry and with a left–moving current algebra $SO(32)$ in fermionic form[153]. As we learned in section 7.4, we can bosonise these into the 16 chiral bosons which we used to construct the heterotic string in the first instance. This also fits rather well with the fact that we had already notice that we could have deduced that such a string theory might exist just by looking at the supergravity sector in section 7.6. This is just how type I/heterotic duality was deduced first[144, 154] and then D–brane constructions were used to test it more sharply[153]. We shall see that considerations of the strong coupling limit of various other string theories will again point to the existence of the heterotic string. We have already seen hints of that in section 7.9.5. Of course, the heterotic strings were discovered by direct perturbative construction, but it is amusing to thing that in another world, they may be discovered by string duality.

We end with a brief remark about some further details that we shall not pursue. Recall that it was mentioned at the end of section 7.4, the fermionic $SO(32)$ current algebra requires a GSO projection. By considering a closed D1–brane we see that the Ω projection removes the $U(1)$ gauge field, but in fact allows a discrete gauge symmetry: a holonomy ±1 around the D1–brane.

This discrete gauge symmetry is the GSO projection, and we should sum over all consistent possibilities. The heterotic strings have spinor representations of $SO(32)$, and we need to be able to make them in the Type I theory, in order for duality to be correct. In the R sector of the discrete D1–brane gauge theory, the 1–9 strings are periodic. The zero modes of the fields Ψ^i, representing the massless 1–9 strings, satisfy the Clifford algebra $\{\Psi^i_0, \Psi^j_0\} = \delta^{ij}$, for $i, j = 1, \cdots, 32$, and so just as for the fundamental heterotic string we get spinors $\mathbf{2^{31}} + \overline{\mathbf{2^{31}}}$. One of them is removed by the discrete gauge symmetry to match the spectrum with a single massive spinor which we uncovered directly using lattices in section 7.4.1.

11.3 Dual Branes from 10D String–String Duality

There is an instructive way to see how the D–string tension turns into that of an F–string. In terms of supergravity fields, part of the duality transformation (354) involves

$$G_{\mu\nu} \to e^{-\tilde{\Phi}}\tilde{G}_{\mu\nu} \ , \quad \Phi \to -\tilde{\Phi} \ , \tag{451}$$

where the quantities on the right, with tildes, are in the dual theory. This means that in addition to $g_s = \tilde{g}_s^{-1}$, for the relation of the string coupling to the dual string coupling, there is also a redefinition of the string length, *via*

$$\alpha' = \tilde{g}_s^{-1}\tilde{\alpha}' \ ,$$

which is the same as

$$\alpha' g_s^{-1} = \tilde{\alpha}' \ .$$

Starting with the D–string tension, these relations give:

$$\tau_1 = \frac{1}{2\pi\alpha' g_s} \to \frac{1}{2\pi\tilde{\alpha}'} = \tau_1^{\rm F} \ ,$$

precisely the tension of the fundamental string in the dual string theory, measured in the correct units of length.

One might understandably ask the question about the fate of other branes under S–dualities[155]. For the type IIB's D3–brane:

$$\tau_3 = \frac{1}{(2\pi)^3\alpha'^2 g_s} \to \frac{1}{(2\pi)^3\tilde{\alpha}'^2\tilde{g}_s} = \tau_3 \ ,$$

showing that the dual object is again a D3–brane. For the D5–brane, in either type IIB or type I theory:

$$\tau_5 = \frac{1}{(2\pi)^5\alpha'^3 g_s} \to \frac{1}{(2\pi)^5\tilde{\alpha}'^3\tilde{g}_s^2} = \tau_5^{\rm F} \ ,$$

This is the tension of a fivebrane which is *not* a D5–brane. This is interesting, since for both dualities, the R–R 2–form $C^{(2)}$ is exchanged for the NS–NS 2–form $B^{(2)}$, and so this fivebrane is magnetically charged under the latter. It is in fact the magnetic dual of the fundamental string. Its g_s^{-2} behaviour identifies it as a soliton of the NS–NS sector.

So we conclude that there exists in both the type IIB and $SO(32)$ heterotic theories such a brane, and in fact such a brane can be constructed directly as a soliton solution[68]. They should perhaps be called

"F5–branes", since they are magnetic duals to fundamental strings or "F1–branes", but this name never stuck. They go by various names like "NS5–brane", since they are made of NS–NS sector fields, or "solitonic fivebrane", and so on. As they are constructed completely out of closed string fields, T–duality along a direction parallel to the brane does not change its dimensionality, as would happen for a D–brane. We conclude therefore that they also exist in the T–dual Type IIA and $E_8 \times E_8$ string theories. Let us study them a bit further.

11.3.1 The Heterotic NS–Fivebrane

For the heterotic cases, the soliton solution also involves a background gauge field, which is in fact an instanton. This follows from the fact that in type I string theory, the D5–brane is an instanton of the D9–brane gauge fields as we saw with dramatic success in section 9.2. As we saw there and in section 7, through equation (346), $\mathrm{Tr}(F \wedge F)$ and $\mathrm{tr}(R \wedge R)$ both magnetically source the two–form potential $C_{(2)}$, since by taking one derivative:

$$d\tilde{G}^{(3)} = -\frac{\alpha'}{4}\left[\mathrm{Tr}F^2 - \mathrm{tr}R^2\right] \ .$$

By strong/weak coupling duality, this must be the case for the NS–NS two form $B_{(2)}$. To leading order in α', we can make a solution of the heterotic low energy equation of motion with these clues quite easily as follows. Take for example an $SU(2)$ instanton embedded in an $SU(2)$ subgroup of the $SO(4)$ in the natural decomposition: $SO(32) \supset SO(28) \times SO(4)$. As we said, this will source some dH, which in turn will source the metric and the dilaton. In fact, to leading order in α', the corrections to the metric away from flat space will not give any contribution to $\mathrm{tr}(R \wedge R)$, which has more derivatives than $\mathrm{Tr}(F \wedge F)$, and is therefore subleading in this discussion. The result should be an object which is localized in $\mathbb{R}^4$ with a finite core size (the "dressed" instanton), and translationally invariant in the remaining $3+1$ directions. This deserves to be called a fivebrane.

Once we have deduced the existence of this object in the $SO(32)$ heterotic string, it is straightforward to see that it must exist in the $E_8 \times E_8$ heterotic string too. We simply compactify on a circle in a world–volume direction where there is no structure at all. Shrinking it away takes us to the other heterotic theory, with an NS5–brane of precisely the same sort of structure as above. Alternatively, we could have just constructed the fivebrane directly using the ideas above without appealing to T–duality at all.

11.3.2 The Type IIA and Type IIB NS5–brane

As already stated, similar reasoning leads one to deduce that there must be an NS5–brane in type II string theory[9].

For the same reasons as for the heterotic string, once we have made an NS5–brane for the type IIB string, it is easy to see that we can use T–duality along a world–volume direction (where the solution is trivial) in order to make one in the type IIA string theory as well.

A feature worth considering is the world–volume theory describing the low energy collective motions of these type II branes. This can be worked out directly, and string duality is consistent with the answer: From the duality, we can immediately deduce that the type IIB's NS5–brane must have a vector multiplet, just like the D5–brane. Also as with D5–branes, there is enhanced $SU(N)$ gauge symmetry when N coincide[151], the extra massless states being supplied by light D1–branes stretched between them. (See figure 34.) The vector multiplet can be read off from table 2 as $(\mathbf{2},\mathbf{2}) + 4(\mathbf{1},\mathbf{1}) + 2(\mathbf{1},\mathbf{2}) + 2(\mathbf{2},\mathbf{1})$. There are four scalars, which are the four transverse positions of the brane in ten dimensions. The fermionic content can be seen to be manifestly non–chiral giving a $(1,1)$ supersymmetry on the world–volume.

For the type IIA NS5–brane, things are different. Following the T–duality route mentioned above, it can be seen that the brane actually must have a chiral (0,2) supersymmetry. So it cannot have a vector multiplet any more, and instead there is a six dimensional tensor multiplet on the brane. So there is a two–form

[9]In the older literature, it is sometimes called a "symmetric fivebrane", after its left–right symmetric σ–model description, in constrast to that of the heterotic NS5–brane[68].

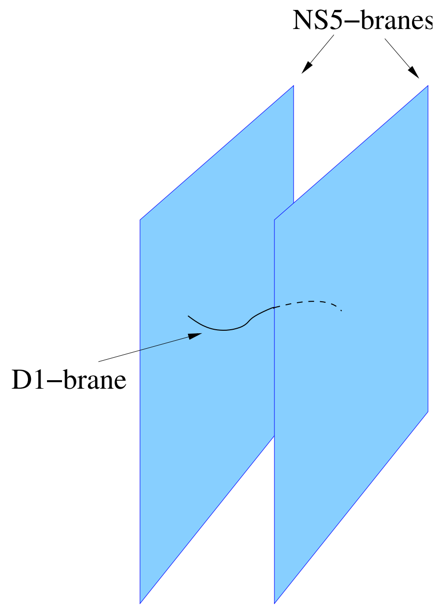


Figure 34: D1–branes stretched between NS5–branes in type IIB string theory will give extra massless vectors when the NS5–branes coincide.

potential instead of a 1–form potential, which is interesting. The tensor multiplet can be read off from table 2 as $(\mathbf{1},\mathbf{3})+5(\mathbf{1},\mathbf{1})+4(\mathbf{2},\mathbf{1})$, with a manifestly chiral fermionic content. There are now *five* scalars, which is suggestive, since in their interpretation as collective coordinates for transverse motions of the brane, there is an implication of an *eleventh* direction. This extra direction will become even more manifest in section 11.4.

There is an obvious $U(1)$ gauge symmetry under the transformation $B^+_{(2)}\to B^+_{(2)}+d\Lambda_{(1)}$, and the question arises as to whether there is a non–Abelian generalisation of this when many branes coincide. On the D–brane side of things, it is clear how to construct the extra massless states as open strings stretched between the branes whose lengths can shrink to zero size in the limit. Here, there is a similar, but less well–understood phenomenon. The tensor potential on the world–volume is naturally sourced by six–dimensional strings, which are in fact the ends of open D2–branes ending on the NS5–branes. The mass or tension of these strings is set by the amount that the D2–branes are stretched between two NS–branes, by precise analogy with the D–brane case. So we are led to the interesting case that there are tensionless strings when many NS5–branes coincide, forming a generalized enhanced gauge tensor multiplet. (See figure 35.) These strings are not very well understood, it must be said.

They are not sources of a gravity multiplet, and they appear not to be weakly coupled in any sense that is understood well enough to develop an intrinsic perturbation theory for them[10].

However, the theory that they imply for the branes is apparently well–defined. The information about how it works fits well with the dualities relating it to better understood things, as we have seen here. Furthermore, it can be indirectly defined using the AdS/CFT correspondence[181, 182].

It should be noted that we do not have to use D–branes or duality to deduce a number of the features mentioned above for the world–volume theories on the NS5–branes. That there is either a (1,1) vector multiplet or a (0,2) tensor multiplet was first uncovered by direct analysis of the collective dynamics of the NS5–branes as supergravity solitons in the type II theories[150].

[10]The cogniscenti will refer to theories of non–Abelian "gerbes" at this point. The reader should know that these are not small furry pets, but well–defined mathematical objects. They are (apparently) a generalisation of the connection on a vector bundle, appropriate to 2–form gauge fields.[86]

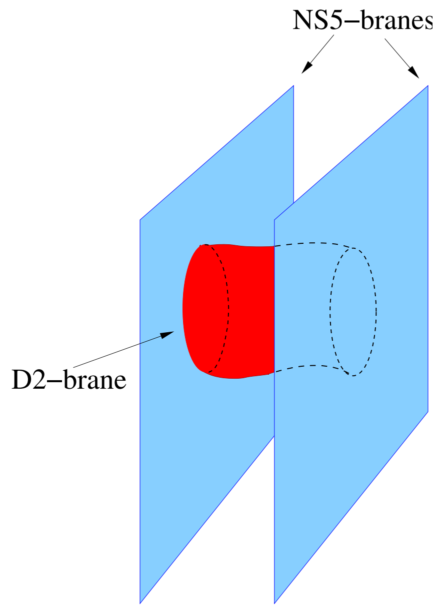


Figure 35: D2–branes stretched between NS5–branes in type IIA string theory will give extra massless self–dual tensors when the NS5–branes coincide.

11.4 Type IIA/M–Theory Duality

Let us turn our attention to the type IIA theory and see if at strong coupling we can see signs of a duality to a useful weakly coupled theory. In doing this we will find that there are even stranger dualities than just a string/string duality (which is strange and beautiful enough as it is!), but in fact a duality which points us firmly in the direction of the unexplored and the unknown.

11.4.1 A Closer Look at D0–branes

Notice that in the IIA theory, the D0–brane has a mass $\tau_0 = \alpha'^{-1/2} g_s$, as measured in the string metric. As $g_s \to \infty$, this mass becomes light, and eventually becomes the lightest scale in the theory, lighter even than that of the fundamental string itself.

We can trust the extrapolation of the mass formula done in this way because the D0–brane is a BPS object, and so the formula is protected from *e.g.*, leveling off to some still not–too–light scale by loop corrections, *etc.* So we are being shown a new feature of the theory here, and it would be nice to make sense of these new feature. Notice that in addition, we have seen in section 10.5 that n D0–branes have a single supersymmetric bound state with mass $n\tau_0$. So in fact, these are genuine physical particles, charged under the $U(1)$ of the R–R one–form $C_{(1)}$, and forming an evenly spaced tower of mass states which is become light as we go further to strong coupling. How are we to make sense of this in ten dimensional string theory?

In fact, the spectrum we just described is characteristic of the appearance of an additional dimension, where the momentum (Kaluza–Klein) states have masses n/R and form a continuum as $R \to \infty$. Here, $R = \alpha'^{1/2} g_s$, so weak coupling is small R and the theory is effectively ten dimensional, while strong coupling is large R, and the theory is eleven dimensional. We saw such Kaluza–Klein behaviour in section 4.2. The charge of the nth Kaluza–Klein particle corresponds to n units of momentum $1/R$ in the hidden dimension. In this case, this $U(1)$ is the R–R one form of type IIA, and so we interpret D0–brane charge as eleven dimensional momentum.

11.4.2 Eleven Dimensional Supergravity

In this way, we are led to consider eleven dimensional supergravity as the strong coupling limit of the type IIA string. This is only for *low energy*, of course, and the issue of the complete description of the short distance physics at strong coupling to complete the "M–theory", is yet to be settled. It cannot be simply eleven dimensional supergravity, since that theory (like all purely field theories of gravity) is ill–defined at short distances. A most widely examined proposal for the structure of the short distance physics is "Matrix Theory"[148, 149], which we unfortunately don't have time to discuss here.

In the absence of a short distance theory, we have to make do with the low-energy effective theory, which is a graviton, and antisymmetric 3–form tensor gauge field $A_{(3)}$, and their superpartners. Notice that this theory has the same number of bosonic and fermionic components as the type II theory. Take type IIA and note that the NS–NS sector has 64 bosonic components as does the R–R sector, giving a total of 128. Now count the number of physical components of a graviton, together with a 3–form in eleven dimensions. The answer is $9 \times 10/2 - 1 = 44$ for the graviton and $9 \times 8 \times 7/(3 \times 2) = 84$ for the 3–form. The superpartners constitute the same number of fermionic degrees of freedom, of course, giving an $\mathcal{N} = 1$ supersymmetry in eleven dimensions, equivalent to 32 supercharges, counting in four dimensional units. In fact, a common trick to be found in many discussions for remembering how to write the type IIA Lagrangian (see *e.g.* ref.[2]) is to simply dimensionally reduce the 11 dimensional supergravity Lagrangian. Now we see that a physical reason lies behind it. The bosonic part of the action is:

$$S_{\rm IID} = \frac{1}{2\kappa_{11}^2}\int d^{11}x\sqrt{-G}\left(R - \frac{1}{48}(F_{(4)})^2\right) - \frac{1}{12\kappa_{11}^2}\int A_{(3)} \wedge F_{(4)} \wedge F_{(4)} \ , \qquad (452)$$

and we shall work out $2\kappa_{11}^2 = 16\pi G_N^{11}$ shortly.

To relate the type IIA string coupling to the size of the eleventh dimension we need to compare the respective Einstein–Hilbert actions,[144] ignoring the rest of the actions for now:

$$\frac{1}{2\kappa_0^2 g_s^2}\int d^{10}x\,\sqrt{-G_{\rm s}}R_{\rm s} = \frac{2\pi R}{2\kappa_{11}^2}\int d^{10}x\,\sqrt{-G_{11}}R_{11} \ . \qquad (453)$$

The string and eleven dimensional supergravity metrics are equal up to an overall rescaling,

$$G_{{\rm s}\mu\nu} = \zeta^2 G_{11\mu\nu} \qquad (454)$$

and so $\zeta^8 = 2\pi R\kappa_0^2 g_s^2/\kappa_{11}^2$. The respective masses are related $n/R = m_{11} = \zeta m_{\rm s} = n\zeta\tau_0$ or $R = \alpha'^{1/2}g_s/\zeta$. Combining these with the result (352) for κ_0, we obtain

$$\zeta = g_s^{1/3}\left[2^{7/9}\pi^{8/9}\alpha'\kappa_{11}^{-2/9}\right] \qquad (455)$$

and the radius in eleven dimensional units is:

$$R = g_s^{2/3}\left[2^{-7/9}\pi^{-8/9}\kappa_{11}^{2/9}\right] \ . \qquad (456)$$

In order to emphasize the basic structure we hide in braces numerical factors and factors of κ_{11} and α'. The latter factors are determined by dimensional analysis, with κ_{11} having units of (11D supergravity length$^{9/2}$) and α' (string theory length^2). We are free to set $\zeta = 1$, using the same metric and units in M–theory as in string theory. In this case

$$\kappa_{11}^2 = g_s^3\left[2^7\pi^8\alpha'^{9/2}\right] \ , \quad \text{and then} \quad R = g_s\ell_s \ . \qquad (457)$$

The reason for not always doing so is that when we have a series of dualities, as below, there will be different string metrics. For completeness, let us note that if we define Newton's constant *via* $2\kappa_{11}^2 = 16\pi G_N^{11}$, then we have:

$$\kappa_{11}^2 = 2^7\pi^8\ell_p^9 \ ; \quad \ell_p = g_s^{1/3}\sqrt{\alpha'} = g_s^{1/3}\ell_s \ . \qquad (458)$$

11.5 $E_8 \times E_8$ Heterotic String/M–Theory Duality

We have deduced the duals of four of the five ten dimensional string theories. Let us study the final one, the $E_8 \times E_8$ heterotic string, which is T–dual to the $SO(32)$ string.[163, 164]

Compactify on a large radius $R_{\rm HA}$ and turn on a Wilson line which breaks $E_8 \times E_8$ to $SO(16) \times SO(16)$. As we learned in section 8.1.3, this is T–dual to the $SO(32)$ heterotic string, again with a Wilson line breaking the group to $SO(16) \times SO(16)$. The couplings and radii are related

$$\begin{aligned} R_{\rm HB} &= \frac{\ell_s^2}{R_{\rm HA}} \ , \\ g_{s,\rm HB} &= g_{s,\rm HA}\frac{\ell_s}{R_{\rm HA}} \ . \end{aligned} \tag{459}$$

Now use type I/heterotic duality to write this as a type I theory with [144]

$$\begin{aligned} R_{\rm IB} &= g_{s,\rm HB}^{-1/2}R_{\rm HB} \ = \ g_{s,\rm HA}^{-1/2}\frac{\ell_s^{3/2}}{R_{\rm HA}^{1/2}} \ , \\ g_{s,\rm IB} &= g_{s,\rm HB}^{-1} \ = \ g_{s,\rm HA}^{-1}\frac{R_{\rm HA}}{\ell_s} \ . \end{aligned} \tag{460}$$

The radius is very small, so it is useful to make another T–duality, to the "type I′" or "type IA" theory.

The compact dimension is then a segment of length $\pi R_{\rm IA}$ with eight D8–branes and O8–planes at each end, and

$$\begin{aligned} R_{\rm IA} &= \frac{\ell_s^2}{R_{\rm IB}} \ = \ g_{s,\rm HA}^{1/2}R_{\rm HA}^{1/2}\ell_s^{1/2} \ , \\ g_{s,\rm IA} &= g_{s,\rm IB}\frac{\ell_s}{R_{\rm IB}\sqrt{2}} = g_{s,\rm HA}^{-1/2}\frac{R_{\rm HA}^{3/2}}{\ell_s^{3/2}\sqrt{2}} \ . \end{aligned} \tag{461}$$

It is worth drawing a picture of this arrangement, and it is displayed in figure 36. Notice that since the charge of an O8–plane is precisely that of 8 D8–branes, the charge of the R–R sector is locally canceled at each end. There is therefore no R–R flux in the interior of the interval and so crucially, we see that the physics between the ends of the segment is given locally by the IIA string. Now we can take $R_{\rm HA} \to \infty$ to recover the original ten–dimensional theory (in particular the Wilson line is irrelevant and the original $E_8 \times E_8$ restored). Both the radius and the coupling of the Type IA theory become large. Since the bulk physics is locally that of the IIA string[11], the strongly coupled limit is eleven dimensional.

Taking into account the transformations (454) and (456), the radii of the two compact dimensions in M–theory units are

$$\begin{aligned} R_9 &= \zeta_{\rm IA}^{-1}R_{\rm IA} \ = \ g_s^{2/3}\left[2^{-11/18}\pi^{-8/9}\kappa_{11}^{2/9}\right] \\ R_{10} &= g_{s,\rm IA}^{2/3}\left[2^{-7/9}\pi^{-8/9}\kappa_{11}^{2/9}\right] \ = \ g_{s,\rm HA}^{-1/3}R_{\rm HA}\left[2^{-10/9}\pi^{-8/9}\alpha'^{-1/2}\kappa_{11}^{2/9}\right] \ . \end{aligned} \tag{462}$$

Again, had we chosen $\zeta_{\rm IA} = 1$, we would have

$$R_{10} = R_{\rm HA}2^{-1/3} \ , \quad R_9 = g_s\ell_s 2^{1/6} \ . \tag{463}$$

[11] Notice that this is not the case if the D8–branes are placed in a more general arrangement where the charges are not canceled locally. For such arrangements, the dilaton and R–R 9–form is allowed to vary piecewise linearly between neighbouring D8–branes. The supergravity between the branes is the "massive" supergravity considered by Romans[92]. This is a very interesting topic in its own right, which we shall not have room to touch upon here. A review of some aspects, with references, is given in the bibliography[171].

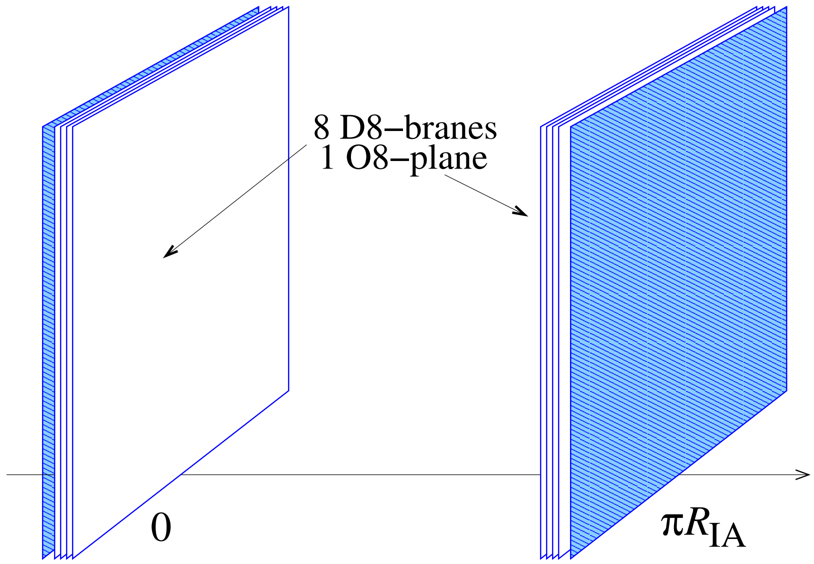


Figure 36: The type IA configuration of two groups of 8 D8–branes and O8–planes resulting from a $SO(16)\times SO(16)$ Wilson line.

As $R \to \infty$, $R_{10} \to \infty$ also, while R_9 remains fixed and (for g_s large) large compared to the Planck scale. This suggests that in the strongly coupled limit of the ten–dimensional $E_8 \times E_8$ heterotic string an, eleventh dimension again appears: It is a line segment of length R_9, with one E_8 factor on each endpoint.[159]

We have not fully completed the argument, since we only have argued for $SO(16)$ at each end. One way to see how the E_8 arises is to start from the other end and place eleven dimensional supergravity on a line segment. This theory is anomalous, but the anomaly can be canceled by having 248 vector fields on each ten dimensional boundary. So the **120** of $SO(16)$ is evidently joined by 128 new massless states at strong coupling. As we saw in section 7.4 in the decomposition of E_8 to $SO(16)$, the adjoint breaks up as $\mathbf{248} = \mathbf{120} \oplus \mathbf{128}$, where the **128** is the spinor representation of $SO(16)$. Now we see why we could not construct this in perturbative type IA string theory. Spinor representations of orthogonal groups cannot be made with Chan–Paton factors. However, we can see these states as massive D0–D8 bound states, T–dual to the D1–D9 spinors we were able to make in the $SO(32)$ case in section 11.2. Now, with $SO(16)$ at each end, we can make precisely the pair of **128**'s we need.

11.6 M2–branes and M5–branes

Just as in the other superstring theories, we can make extended objects in the theory. The most natural one to consider first, given what we have displayed as the content of the theory is one which carries the charge of the higher rank gauge field, $A_{(3)}$. This is a two dimensional brane (a membrane) which we shall call the M2–brane[156]:

By eleven dimensional Hodge duality, it is easy to see that there is another natural object, a fivebrane which is magnetically dual to the M2–brane, called the M5–brane[157]. The tensions of the single M2– and M5–branes of 11 dimensional supergravity are:

$$\tau_2^{\rm M} = (2\pi)^{-2}\ell_p^{-3} \ ; \qquad \tau_5^{\rm M} = (2\pi)^{-5}\ell_p^{-6} \ . \qquad (464)$$

The product of the M–branes' tensions gives

$$\tau_2^{\rm M}\tau_5^{\rm M} = 2\pi(2\pi)^{-8}\ell_p^{-9} = \frac{2\pi}{2\kappa_{11}^2} \qquad (465)$$

and so is the minimum allowed by the quantum theory, in close analogy with what we know for D–branes from equation (384).

11.6.1 From D–Branes and NS5–branes to M–Branes and Back

It is interesting to track the eleven–dimensional origin of the various branes of the IIA theory.[145] The D0–branes are, as we saw above, are Kaluza–Klein states. The F1–branes, the IIA strings themselves, are wrapped M2–branes of M–theory. The D2–branes are M2–branes transverse to the eleventh dimension X^{10}. The D4–branes are M5–theory wrapped on X^{10}, while the NS5–branes are M5–branes transverse to X^{10}. The D6–branes, being the magnetic duals of the D0–branes, are Kaluza–Klein monopoles[143, 158] As mentioned in the previous section, the D8–branes have a more complicated fate. To recapitulate, the point is that the D8–branes cause the dilaton to diverge within a finite distance,[153] and must therefore be a finite distance from an orientifold plane, which is essentially a boundary of spacetime as we saw in section 4.11.1. As the coupling grows, the distance to the divergence and the boundary necessarily shrinks, so that they disappear into it in the strong coupling limit: they become part of the gauge dynamics of the nine–dimensional boundary of M–theory,[159] used to make the $E_8 \times E_8$ heterotic string, as discussed in more detail above. This raises the issue of the strong coupling limit of orientifolds in general.

There are various results in the literature, but since the issues are complicated, and because the techniques used are largely strongly coupled field theory deductions, which take us well beyond the scope of these notes, so we will have to refer the reader to the literature[176].

One can see further indication of the eleventh dimension in the world–volume dynamics of the various branes. We have already seen this in section 11.3.2 where we saw that the type IIA NS5–brane has a chiral tensor multiplet on its world–volume, the five scalars of which are indicative of an eleven dimensional origin. We saw in the above that this is really a precursor of the fact that it lifts to the M5–brane with the same world–volume tensor multiplet, when type IIA goes to strong coupling. The world–volume theory is believed to be a 5+1 dimensional fixed point theory. Consider as another example the D2–brane. In $2+1$ dimensions, the vector field on the brane is dual to a scalar, through Hodge duality of the field strength, $*F_2 = d\phi$. This scalar is the eleventh embedding dimension.[146] It joins the other seven scalars already defining the collective modes for transverse motion to show that there are *eight* transverse dimensions. Carrying out the duality in detail, the D2–brane action is therefore found to have a hidden eleven–dimensional Lorentz invariance. So we learn that the M2–brane, which it becomes, has a 2+1 dimensional theory with eight scalars on its worldvolume. The existence of this theory may be inferred in purely field theory terms as being an infra–red fixed point of the 2+1 dimensional gauge theory.[170]

11.7 U–Duality

A very interesting feature of string duality is the enlargement of the duality group under further toroidal compactification. There is a lot to cover, and it is somewhat orthogonal to most of what we want to do for the rest of the book, so we will err on the side of brevity (for a change). The example of the Type II string on a five–torus T^5 is useful, since it is the setting for the simplest black hole state counting, and we have already started discussing it in section 7.8.

11.7.1 Type II Strings on T^5 and $E_{6(6)}$

As we saw in section 7.8, the T–duality group is $O(5,5,\mathbb{Z})$. The 27 gauge fields split into $10+16+1$ where the middle set have their origin in the R–R sector and the rest are NS–NS sector fields. The $O(5,5;\mathbb{Z})$ representations here correspond directly to the **10**, **16**, and **1** of $SO(10)$. There are also 42 scalars.

The crucial point here is that there is a larger symmetry group of the supergravity, which is in fact $E_{6,(6)}$. It generalizes the $SL(2,\mathbb{R})$ $(SU(1,1))$ S–duality group of the type IIB string in ten dimensions. In that case

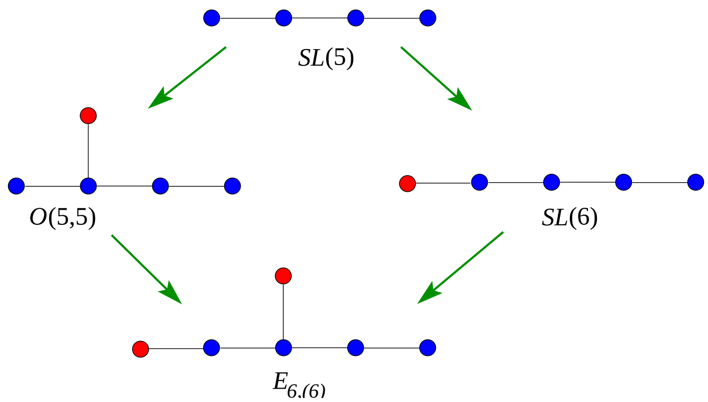


Figure 37: The origins of $E_{6(6)}$ for Type II strings on T^5.

there are two scalars, the dilaton Φ and the R–R scalar $C_{(0)}$, and they take values on the coset space

$$\frac{SL(2,\mathbb{R})}{U(1)} \simeq \frac{SU(1,1)}{SO(2)} \ .$$

The low energy supergravity theory for this compactification has a continuous symmetry, $E_{6(6)}$ which is a non–compact version of E_6.[166]

One way of seeing roughly where $E_{6,(6)}$ comes from is as follows: The naive symmetry resulting from a T^5 compactification would be $SL(5,\mathbb{R})$, the generalisation of the $SL(2,\mathbb{R})$ of the T^2 to higher dimensional tori. There are two things which enlarge this somewhat. The first is an enlargement to $SL(6,\mathbb{R})$, which ought to be expected, since the type IIB string already has an $SL(2,\mathbb{R})$ in ten dimensions. This implies the existence of an an extra circle, enlarging the naive torus from T^5 to T^6. This is of course something we have already discovered in section 11.4: at strong coupling, the type IIA string sees an extra circle. Below ten dimensions, T–duality puts both type II strings on the same footing, and so it is most efficient to simply think of the problem as M–theory (at least in its 11 dimensional supergravity limit) compactified on a T^6. Another enlargement is due to T–duality. As we have learned, the full T–duality group is $O(5,5,\mathbb{Z})$, and so we should expect a classical enlargement of the naive $SL(5,\mathbb{R})$ to $O(5,5)$. That $E_{6,(6)}$ contains these two enlargements can be seen quite efficiently[171] in the Dynkin diagrams in figure 37:

Actually, the above embedding is not unique, but we are not attempting a proof here; we are simply showing that $E_{6,(6)}$ is not unreasonable, given what we already know. The notation $E_{6,(6)}$ means that by analytic continuation of some of the generators, we make a non–compact version of the group (much as in the same way as we get $SL(2,\mathbb{R})$ from $SU(2)$). The maximal number of generators for which this is possible is the relevant case here.

The gauge bosons are in the **27** of $E_{6(6)}(\mathbb{Z})$, which is the same as the **27** of $E_{6(6)}$. The decomposition under $SO(10) \sim O(5,5;\mathbb{Z})$ is familiar from grand unified model building: $\mathbf{27} \to \mathbf{10} + \mathbf{16} + \mathbf{1}$. Another generalisation is that the 42 scalars live on the coset

$$\frac{E_{6,(6)}}{USp(8)} \ .$$

In the light of string duality, just as the various branes in type IIB string theory formed physical realizations of multiplets of $SL(2,\mathbb{Z})$, so do the branes here. A discrete subgroup $E_{6(6)}(\mathbb{Z})$ is the "U–duality" symmetry. The particle excitations carrying the **10** charges are just the Kaluza–Klein and winding strings. The U–duality requires also states in the **16**. These are just the various ways of wrapping Dp–branes to give D–particles (10 for D2, 5 for D4 and 1 for D0). Finally, the state carrying the **1** charge is the NS5–brane, wrapped entirely on the T^5.

In fact, the U–duality group for the type II strings on T^d is $E_{d+1,(d+1)}$, where for $d = 4, 3, 2, 1, 0$ we have that the definition of the appropriate E–group is $SO(5,5), SL(5), SL(2) \times SL(3), SL(2) \times \mathbb{R}_+, SL(2)$. These groups can be seen with similar embedding of Dynkin diagrams we we have done above.

12 Concluding Remarks

I hope that these notes have been a useful starting point for you. There was a lot to cover, so no doubt I have disappointed nearly everyone, since their favourite topic was not covered. On the other hand, you now have a basic spring–board from which you can launch yourself into studying some of the exciting topics covered in the other lecture courses, seeing how the ideas you've learned fit into more advanced areas of string theory, branes, field theory, cosmology, *etc.* Have fun!

References

[1] J. Polchinski, *"String Theory"*, Vols. 1 and 2; Cambridge Univ. Pr. (1998) (UK) (Cambridge Monographs on Mathematical Physics).

[2] M. B. Green, J. H. Schwarz and E. Witten, *"Superstring Theory"*, Vols. 1 and 2; Cambridge Univ. Pr. (1987) (UK) (Cambridge Monographs on Mathematical Physics).

[3] This is an example:
E. Kiritsis, *"Introduction to Superstring Theory"*, hep-th/9709062. Leuven Univ. Pr. (1998) 315 p, (Belgium) (Leuven Notes in Mathematical and Theoretical Physics. B9).

[4] C. V. Johnson, *"D–Branes"*, Cambridge Univ. Pr. (2003) (UK) (Cambridge Monographs on Mathematical Physics).

[5] J. Polchinski, S. Chaudhuri and C. V. Johnson, *"Notes on D–Branes"*, hep-th/9602052.

[6] J. Polchinski, *"TASI Lectures on D–Branes"*, hep-th/9611050.

[7] C.V. Johnson, *"D-Brane primer"*, in TASI 1999, *"Strings, Branes and Gravity"*, World Scientific (2001), hep-th/0007170.

[8] C. V. Johnson, *"Etudes on D-branes"*, hep-th/9812196.

[9] Two very useful treatments are: P. Ginsparg, *"Applied Conformal Field Theory"*, Les Houches, France, Jun 28 - Aug 5, 1988, eds. E. Brezin and J. Zinn-Justin, North–Holland, (1990); P. Di Francesco, P. Mathieu and D. Senechal, *"Conformal Field Theory"*, New York, Springer (1997).

[10] P. H. Ginsparg and G. W. Moore, *'Lectures on 2D Gravity and 2D String Theory'*, hep-th/9304011;
P. Di Francesco, P. H. Ginsparg and J. Zinn-Justin, *'2D Gravity and Random Matrices'*, Phys. Rept. **254** (1995) 1, hep-th/9306153;
I. R. Klebanov, *'String Theory in Two Dimensions'*, hep-th/9108019;
S. Mukhi, *'Topological Matrix Models, Liouville Matrix Model and* $c = 1$ *String Theory'*, hep-th/0310287.

[11] J. McGreevy, H. Verlinde, JHEP 0401:039,2004, hep-th/0304224;
I. R. Klebanov, J. Maldacena, N. Seiberg, JHEP 0307:045,2003, hep-th/0305159;
ibid., hep-th/0309168;
C. V. Johnson, JHEP 0403:041,2004, hep-th/0311129;

[12] J. Dai, R. G. Leigh and J. Polchinski, Mod. Phys. Lett. **A4** (1989) 2073.

[13] A. Chodos and C. B. Thorn, Nucl. Phys. **B72** (1974) 509; W. Siegel, Nucl. Phys. **B109** (1976) 244; S. M. Roy and V. Singh, Pramana **26** (1986) L85; Phys. Rev. **D35** (1987) 1939; J. A. Harvey and J. A. Minahan, Phys. Lett. **B188** (1987) 44.

[14] N. Ishibashi and T. Onogi, Nucl. Phys. **B318** (1989) 239; G. Pradisi and A. Sagnotti, Phys. Lett. **B216** (1989) 59; A. Sagnotti, Phys. Rept. **184** (1989) 167; P. Horava, Nucl. Phys. **B327** (1989) 461.

[15] J. H. Schwarz, Nucl. Phys. **B65** (1973), 131; E. F. Corrigan and D. B. Fairlie, Nucl. Phys. **B91** (1975) 527; M. B. Green, Nucl. Phys. **B103** (1976) 333; M. B. Green and J. A. Shapiro, Phys. Lett. **64B** (1976) 454; A. Cohen, G. Moore, P. Nelson, and J. Polchinski, Nucl. Phys. **B267**, 143 (1986); **B281**, 127 (1987).

[16] M. Dine, P. Huet, and N. Seiberg, Nucl. Phys. **B322** (1989) 301.

[17] P. Horava, Phys. Lett. **B231** (1989) 251; M. B. Green, Phys. Lett. **B266** (1991) 325.

[18] K. Kikkawa and M. Yamanaka, Phys. Lett. **B149** (1984) 357; N. Sakai and I. Senda, Prog. Theor. Phys. **75** (1986) 692.

[19] V.P. Nair, A. Shapere, A. Strominger, and F. Wilczek, Nucl. Phys. **B287** (1987) 402.

[20] P. Ginsparg and C. Vafa, Nucl. Phys. **B289** (1987) 414.

[21] T. H. Buscher, Phys. Lett. **B194B** (1987) 59; **B201** (1988) 466.

[22] A. Sen, JHEP **9806**, 007 (1998), hep-th/9803194; JHEP **9808**, 010 (1998), hep-th/9805019; JHEP **9808**, 012 (1998), hep-th/9805170;JHEP **9809**, 023 (1998), hep-th/9808141; JHEP **9810**, 021 (1998), hep-th/9809111;
Reviews can be found in: A. Sen, *"Non-BPS states and branes in string theory"*, hep-th/9904207; A. Lerda and R. Russo, Int. J. Mod. Phys. **A15**, 771 (2000), hep-th/9905006.

[23] D. J. Gross, J. A. Harvey, E. Martinec and R. Rohm, Phys. Rev. Lett. **54**, 502 (1985); Nucl. Phys. **B256**, 253 (1985); Nucl. Phys. **B267**, 75 (1986).

[24] Two useful reviews are: J. H. Schwarz, *"TASI lectures on non-BPS D-brane systems"*, in TASI 1999: *"Strings, Branes and Gravity"*, World Scientific (2001), hep-th/9908144; K. Olsen and R. Szabo, *"Constructing D–branes From K–theory"*, hep-th/9907140.

[25] J. Paton and Chan Hong–Mo, Nucl. Phys. **B10** (1969) 519.

[26] L. Dixon, J.A. Harvey, C. Vafa and E. Witten, Nucl. Phys. **B261**, 678 (1985).

[27] J. H. Schwarz, in *"Lattice Gauge Theory, Supersymmetry and Grand Unification"*, 233, Florence 1982, Phys. Rept. **89** (1982) 223;
N. Marcus and A. Sagnotti, Phys. Lett. **119B** (1982) 97.

[28] J. Polchinski, Phys. Rev. **D50** (1994) 6041, hep-th/9407031.

[29] E. Witten, Nucl. Phys. **B460** (1996) 335, hep-th/9510135.

[30] A. Sagnotti, in *"Non-Perturbative Quantum Field Theory"*, eds. G. Mack et. al. (Pergamon Press, 1988) 521; V. Periwal, unpublished; J. Govaerts, Phys. Lett. **B220** (1989) 77.

[31] A. Dabholkar, *"Lectures on orientifolds and duality"*, hep-th/9804208.

[32] S.P. de Alwis, *"A Note on Brane Tension and M Theory"*, hep-th/9607011.

[33] C. Lovelace, Phys. Lett. **B34** (1971) 500; L. Clavelli and J. Shapiro, Nucl. Phys. **B57** (1973) 490; M. Ademollo, R. D' Auria, F. Gliozzi, E. Napolitano, S. Sciuto, and P. di Vecchia, Nucl. Phys. **B94** (1975) 221; C. G. Callan, C. Lovelace, C. R. Nappi, and S. A. Yost, Nucl. Phys. **B293** (1987) 83.

[34] J. Polchinski and Y. Cai, Nucl. Phys. **B296** (1988) 91; C. G. Callan, C. Lovelace, C. R. Nappi and S.A. Yost, Nucl. Phys. **B308** (1988) 221.

[35] A. Abouelsaood, C. G. Callan, C. R. Nappi and S. A. Yost, Nucl. Phys. **B280**, 599 (1987).

[36] This is a vast subject by now. Ref.[34], and the last of ref.[33] are some of the originals, but there are many more. Some good reviews are:
P. Di Vecchia and A. Liccardo, *"D branes in string theory. I"*, hep-th/9912161; P. Di Vecchia and A. Liccardo, *"D-branes in string theory. II"*, hep-th/9912275. I. V. Vancea, *"Introductory lectures to D-branes"*, hep-th/0109029.

[37] R. G. Leigh, Mod. Phys. Lett. **A4** (1989) 2767.

[38] S. Coleman and E. Weinberg, Phys. Rev. **D7** (1973) 1888.

[39] J. Polchinski, Comm. Math. Phys. **104** (1986) 37.

[40] M. Douglas and B. Grinstein, Phys. Lett. **B183** (1987) 552; (E) **187** (1987) 442; S. Weinberg, Phys. Lett. **B187** (1987) 278; N. Marcus and A. Sagnotti, Phys. Lett. **B188** (1987) 58.

[41] See refs[42, 43, 44].

[42] C. Bachas, Phys. Lett. **B374** (1996) 37, hep-th/9511043.

[43] C. Bachas, *"Lectures on D–branes"*, hep-th/9806199.

[44] E. Bergshoeff, M. de Roo, M. B. Green, G. Papadopoulos, and P.K. Townsend, Nucl. Phys. **B470** (1996) 113, hep-th/9601150; E. Alvarez, J. L. F. Barbon, and J. Borlaf, Nucl. Phys. **B479**, 218 (1996), hep-th/9603089; E. Bergshoeff and M. De Roo, Phys. Lett. **B380** (1996) 265, hep-th/9603123.

[45] E. S. Fradkin and A. A. Tseytlin, Phys. Lett. **B163** (1985) 123.

[46] A. A. Tseytlin, Nucl. Phys. **B501**, 41 (1997), hep-th/9701125.

[47] See refs.[49, 50, 51, 52, 53]

[48] See for example refs.[46, 49, 53, 54]

[49] D. Brecher and M. J. Perry, Nucl. Phys. **B527**, 121 (1998), hep-th/9801127.

[50] D. Brecher, Phys. Lett. **B442**, 117 (1998), hep-th/9804180.

[51] M. R. Garousi and R. C. Myers, Nucl. Phys. **B542**, 73 (1999), hep-th/9809100.

[52] A. Hashimoto and W. I. Taylor, Nucl. Phys. **B503**, 193 (1997), hep-th/9703217; P. Bain, hep-th/9909154.

[53] A. A. Tseytlin, *"Born-Infeld action, supersymmetry and string theory"*, hep-th/9908105.

[54] R. C. Myers, JHEP **9912**, 022 (1999), hep-th/9910053.

[55] W. I. Taylor and M. Van Raamsdonk, Nucl. Phys. **B573**, 703 (2000), hep-th/9910052; Nucl. Phys. **B558**, 63 (1999), hep-th/9904095.

[56] There are many good treatments of anomalies. The field theory treatment should begin with a good modern text. See vol. 2 of ref.[57], vol. 2 of ref.[2], and: S. B. Treiman, E. Witten, R. Jackiw and B. Zumino, *"Current Algebra and Anomalies"*, World Scientific, Singapore, (1985).

[57] S. Weinberg, *"The Quantum Theory Of Fields"*, Vols. 1, 2, and 3, Cambridge Univ. Pr. (2000) (UK) (Cambridge Monographs on Mathematical Physics)

[58] See refs.[59, 60, 61, 62, 63]

[59] G. W. Gibbons, Nucl. Phys. **B514**, 603 (1998), hep-th/9709027.

[60] C. G. Callan and J. M. Maldacena, Nucl. Phys. **B513**, 198 (1998), hep-th/9708147.

[61] P. S. Howe, N. D. Lambert and P. C. West, Nucl. Phys. **B515**, 203 (1998), hep-th/9709014; S. Lee, A. Peet and L. Thorlacius, Nucl. Phys. **B514**, 161 (1998), hep-th/9710097.

[62] R. Emparan, Phys. Lett. **B423**, 71 (1998), hep-th/9711106.

[63] J. P. Gauntlett, J. Gomis and P. K. Townsend, JHEP **9801**, 003 (1998), hep-th/9711205.

[64] E. B. Bogomolny, Sov. J. Nucl. Phys. **24**, 449 (1976).

[65] M. K. Prasad and C. M. Sommerfield, Phys. Rev. Lett. **35**, 760 (1975).

[66] J. A. Shapiro and C. B. Thorn, Phys. Rev. **D36** (1987) 432;
J. Dai and J. Polchinski, Phys. Lett. **B220** (1989) 387.

[67] F. Gliozzi, J. Scherk and D. Olive, Nucl. Phys. **B122**, 253 (1977); Phys. Lett. **B65**, 282 (1976).

[68] A. Strominger, Nucl. Phys. **B343**, (1990) 167; *Erratum: ibid.*, **353** (1991) 565; S–J. Rey, in *"Superstrings and Particle Theory: Proceedings"*, eds. L. Clavelli and B. Harms, (World Scientific, 1990); S–J. Rey, Phys. Rev. **D43** (1991) 526; I. Antoniades, C. Bachas, J. Ellis and D. Nanopoulos, Phys. Lett. **B211** (1988) 393; *ibid.,* Nucl. Phys. **328** (1989) 117; C. G. Callan, J.A. Harvey and A. Strominger, Nucl. Phys. **B359** (1991) 611.

[69] C. G. Callan, J.A. Harvey and A. Strominger, in Trieste 1991, *"String Theory and Quantum Gravity"*, hep-th/9112030.

[70] D. Friedan, E. Martinec, and S. Shenker, Nucl. Phys. **B271** (1986) 93.

[71] For example, see the conventions in: E. Bergshoeff, C. Hull and T. Ortin, Nucl. Phys. **B451**, 547 (1995), hep-th/9504081.

[72] M. B. Green, C. M. Hull and P. K. Townsend, Phys. Lett. **B382**, 65 (1996), hep-th/9604119.

[73] P. Meessen and T. Ortin, Nucl. Phys. **B541**, 195 (1999), hep-th/9806120.

[74] See refs.[75, 76, 77, 84, 85]

[75] D. N. Page, Phys. Lett. **B80**, 55 (1978).

[76] M. A. Walton, Phys. Rev. **D37**, 377 (1988).

[77] A very useful reference for the properties of string theory on ALE spaces is: D. Anselmi, M. Billó, P. Fré, L. Girardello and A. Zaffaroni, Int. J. Mod. Phys. **A9** (1994) 3007, hep-th/9304135.

[78] An excellent reference for various relevant geometrical facts is:
T. Eguchi, P. B. Gilkey and A. J. Hanson, *"Gravitation, Gauge Theories And Differential Geometry"*, Phys. Rept. **66**, 213 (1980).

[79] G. W. Gibbons and S. W. Hawking, Commun. Math. Phys. **66** 291, (1979).

[80] T. Eguchi and A. J. Hanson, Ann. Phys. **120** (1979) 82.

[81] N. J. Hitchin, *"Polygons and gravitons"*, in Gibbons, G.W. (ed.), Hawking, S.W. (ed.): *"Euclidean quantum gravity"*, World Scientific (1993), pp. 527–538.

[82] F. Klein, *"Vorlesungen Über das Ikosaeder und die Auflösung der Gleichungen vom fünften Grade"*, Teubner, Leipzig 1884; F. Klein, *"Lectures on the Icosahedron and the Solution of an Equation of Fifth Degree"*, Dover, New York, (1913).

[83] J. Mckay, Proc. Symp. Pure Math. **37** (1980) 183. Amer. Math. Soc.

[84] N. Seiberg, Nucl. Phys. **B303**, 286 (1988).

[85] P. S. Aspinwall and D. R. Morrison, *"String theory on K3 surfaces"*, in Greene, B. (ed.), Yau, S.T. (ed.): Mirror symmetry II* 703-716, hep-th/9404151.

[86] P. Aspinwall, *"K3 Surfaces and String Duality"*, in TASI 1996, World Scientific 1997, hep-th/9611137; *"Compactification, Geometry and Duality: $N = 2$"*, in TASI 1999: *"Strings, Branes and Gravity"*, World Scientific (2001), hep-th/0001001.

[87] G. W. Gibbons and S. W. Hawking, Comm. Math. Phys. **66** (1979) 291.

[88] M. K. Prasad, Phys. Lett. **B83**, 310 (1979).

[89] M. B. Green, Phys. Lett. **B329** (1994) 435, hep-th/9403040.

[90] G. T. Horowitz and A. Strominger, Nucl. Phys. **B360** (1991) 197.

[91] For a review of string solitons, see: M.J. Duff, Ramzi R. Khuri and J.X. Lu, *"String Solitons"*, Phys. Rept. **259** (1995) 213, hep-th/9412184.

[92] L. J. Romans, Phys. Lett. **B169** (1986) 374.

[93] J. Polchinski and A. Strominger, Phys. Lett. **B388**, 736 (1996), hep-th/9510227.

[94] M. B. Green, Phys. Lett. **B69** (1977) 89; **B201** (1988) 42; **B282** (1992) 380.

[95] S. H. Shenker, *"The Strength of Non–Perturbative Effects in String Theory"*, in Cargese 1990, Proceedings: *"Random Surfaces and Quantum Gravity"* (1990) 191.

[96] T. Banks and L. Susskind, *"Brane - Anti-Brane Forces"*, hep-th/9511194.

[97] S. H. Shenker, *"Another Length Scale in String Theory?"*, hep-th/9509132.

[98] D. Kabat and P. Pouliot, Phys. Rev. Lett. **77**, 1004 (1996), hep-th/9603127; U. H. Danielsson, G. Ferretti and B. Sundborg, Int. J. Mod. Phys. **A11**, 5463 (1996), hep-th/9603081.

[99] M. R. Douglas, D. Kabat, P. Pouliot and S. H. Shenker, Nucl. Phys. **B485**, 85 (1997), hep-th/9608024.

[100] W. Fischler and L. Susskind, Phys. Lett. **B171** (1986) 383; **173** (1986) 262.

[101] M. B. Green and J. H. Schwarz, Phys. Lett. **B149** (1984) 117; **B151** (1985) 21.

[102] M. B. Green and J. H. Schwarz, Phys. Lett. B **136** (1984) 367, Nucl. Phys. B **243** (1984) 285.

[103] M. B. Green, J. H. Schwarz and P. C. West, Nucl. Phys. B **254**, 327 (1985).

[104] C. G. Callan and J. A. Harvey, Nucl. Phys. **B250**, 427 (1985); S. G. Naculich, Nucl. Phys. **B296**, 837 (1988); J. M. Izquierdo and P. K. Townsend, Nucl. Phys. **B414**, 93 (1994), hep-th/9307050; J. D. Blum and J. A. Harvey, Nucl. Phys. **B416**, 119 (1994), hep-th/9310035.

[105] M.B. Green, J.A. Harvey and G. Moore, Class. Quant. Grav. **14** (1997) 47, hep-th/9605033.

[106] Y.E. Cheung and Z. Yin, Nucl. Phys. **B517** (1998) 69, hep-th/9710206.

[107] R. Minasian and G. Moore, JHEP **9711**, 002 (1997), hep-th/9710230; E. Witten, JHEP **9812**, 019 (1998), hep-th/9810188; P. Horava, Adv. Theor. Math. Phys. **2**, 1373 (1999), hep-th/9812135; D. Diaconescu, G. Moore and E. Witten, hep-th/0005091 and hep-th/0005090.

[108] R. I. Nepomechie, Phys. Rev. **D31** (1985) 1921; C. Teitelboim, Phys. Lett. **B167** (1986) 63, 69.

[109] M. Bershadsky, C. Vafa, and V. Sadov, Nucl. Phys. **B463** (1996) 420, hep-th/9511222.

[110] M. Li, Nucl. Phys. **B460** (1996) 351, hep-th/9510161.

[111] M. R. Douglas, *"Branes within Branes"*, hep-th/9512077.

[112] A. A. Belavin, A. M. Polyakov, A. S. Shvarts and Y. S. Tyupkin, Phys. Lett. B **59**, 85 (1975).

[113] See also the very useful refs.[117, 116, 118, 119, 115].

[114] K. Dasgupta, D. P. Jatkar and S. Mukhi, Nucl. Phys. **B523**, 465 (1998), hep-th/9707224.

[115] K. Dasgupta and S. Mukhi, JHEP **9803**, 004 (1998), hep-th/9709219. C. A. Scrucca and M. Serone, Nucl. Phys. **B556**, 197 (1199), hep-th/9903145.

[116] B. Craps and F. Roose, Phys. Lett. **B445**, 150 (1998), hep-th/9808074; B. Craps and F. Roose, Phys. Lett. **B450**, 358 (1999), hep-th/9812149.

[117] J. F. Morales, C. A. Scrucca and M. Serone, Nucl. Phys. **B552**, 291 (1999), hep-th/9812071; B. Stephanski,Nucl. Phys. **B548**, 275 (1999), hep-th/9812088.

[118] S. Mukhi and N. V. Suryanarayana, JHEP **9909**, 017 (1999), hep-th/9907215.

[119] J. F. Ospina Giraldo, hep-th/0006076; hep-th/0006149.

[120] C. P. Bachas, P. Bain and M. B. Green, JHEP **9905**, 011 (1999), hep-th/9903210.

[121] M. Berkooz, M. R. Douglas and R. G. Leigh, Nucl. Phys. **B480**, 265 (1996), hep-th/9606139.

[122] E. Witten, Nucl. Phys. **B460** (1996) 541, hep-th/9511030.

[123] E. G. Gimon and J. Polchinski, Phys. Rev. **D54** (1996) 1667, hep-th/9601038.

[124] J. H. Schwarz, Phys. Lett. **B360** (1995) 13; (E) **B364** (1995) 252, hep-th/9508143.

[125] P. S. Aspinwall, Nucl. Phys. Proc. Suppl. **46** (1996) 30, hep-th/9508154.

[126] J. H. Schwarz, Nucl. Phys. Proc. Suppl. **55B**, 1 (1997), hep-th/9607201.

[127] P. K. Townsend, *"M-theory from its superalgebra"*, hep-th/9712004.

[128] O. Aharony, J. Sonnenschein and S. Yankielowicz, Nucl. Phys. **B474**, 309 (1996), hep-th/9603009. M. R. Gaberdiel and B. Zwiebach, Nucl. Phys. B **518** (1998) 151, hep-th/9709013.

[129] A. Sen, JHEP **9803** (1998) 005, hep-th/9711130.

[130] E. Witten, Nucl. Phys. **B500**, 3 (1997), hep-th/9703166.

[131] K. Dasgupta and S. Mukhi, Phys. Lett. **B423**, 261 (1998), hep-th/9711094.

[132] A. Sen, Phys. Rev. **D54**, 2964 (1996), hep-th/9510229.

[133] Here is a selection of papers in this topic: J. Froehlich and J. Hoppe, Commun. Math. Phys. **191**, 613 (1998), hep-th/9701119; P. Yi, Nucl. Phys. **B505**, 307 (1997), hep-th/9704098; S. Sethi and M. Stern, Commun. Math. Phys. **194**, 675 (1998), hep-th/9705046; M. Porrati and A. Rozenberg, Nucl. Phys. **B515**, 184 (1998), hep-th/9708119; M. B. Green and M. Gutperle, JHEP **9801**, 005 (1998), hep-th/9711107; M. B. Halpern and C. Schwartz, Int. J. Mod. Phys. **A13**, 4367 (1998), hep-th/9712133; G. Moore, N. Nekrasov and S. Shatashvili, Commun. Math. Phys. **209**, 77 (2000), hep-th/9803265; N. A. Nekrasov, *"On the size of a graviton"*, hep-th/9909213; S. Sethi and M. Stern, Adv. Theor. Math. Phys. **4** (2000) 487, hep-th/0001189.

[134] P .K. Townsend, Phys. Lett. **B373** (1996) 68, hep-th/9512062.

[135] A. Sen, Phys. Rev. **D53**, 2874 (1996), hep-th/9511026.

[136] C. Vafa, Nucl. Phys. **B463** (1996) 415, hep-th/9511088.

[137] S. Sethi and M. Stern, Phys. Lett. B398 (1997) 47, hep-th/9607145; Nucl. Phys. **B578**, 163 (2000), hep-th/0002131.

[138] G. Papadopoulos and P. K. Townsend, Phys. Lett. **B393**, 59 (1997), hep-th/9609095.

[139] U. H. Danielsson and G. Ferretti, Int. J. Mod. Phys. **A12** (1997) 4581, hep-th/9610082; S. Kachru and E. Silverstein, Phys. Lett. **B396** (1997) 70, hep-th/9612162; D. Lowe, Nucl. Phys. **B501** (1997) 134, hep-th/9702006; T. Banks, N. Seiberg, E. Silverstein, Phys. Lett. **B401** (1997) 30, hep-th/9703052; T. Banks and L. Motl, J.H.E.P. **12** (1997) 004, hep-th/9703218; D. Lowe, Phys. Lett. **B403** (1997) 243, hep-th/9704041; S-J. Rey, Nucl. Phys. **B502** (1997) 170, hep-th/9704158.

[140] See refs.[142, 143, 144]. There are also excellent reviews available, some of which are listed in refs.[147, 126, 127].

[141] C. M. Hull, Nucl. Phys. **B468** (1996) 113, hep-th/9512181.

[142] C. M. Hull and P. K. Townsend, Nucl. Phys. **B438**, 109 (1995), hep-th/9410167.

[143] P. K. Townsend, Phys. Lett. **B350** (1995) 184, hep-th/9501068.

[144] E. Witten, Nucl. Phys. **B443** (1995) 85, hep-th/9503124.

[145] See for example refs.[134, 126, 147].

[146] See refs.[160, 161, 134].

[147] For other reviews, see: M. J. Duff, *"M–Theory (the Theory Formerly Known as Strings)"*, Int. J. Mod. Phys. **A11** (1996) 5623, hep-th/9608117; A. Sen, *"An Introduction to Non–perturbative String Theory"*, hep-th/9802051.

[148] T. Banks, W. Fischler, S. H. Shenker and L. Susskind, Phys. Rev. **D55** (1997) 5112, hep-th/9610043

[149] For reviews, see: T. Banks, *"TASI lectures on matrix theory"*, in TASI 1999, *"Strings, Branes and Gravity"*, World Scientific (2001), hep-th/9911068. T. Banks, *"Matrix Theory"*, Nucl. Phys. Proc. Suppl. **B67** (1998) 180, hep-th/9710231; D. Bigatti and L. Susskind, *"Review of Matrix Theory"*, hep-th/9712072; H. Nicolai and R. Helling, *"Supermembranes and M(atrix) Theory"*, hep-th/9809103; W. I. Taylor, *"The M(atrix) model of M-theory,"* hep-th/0002016; A. Bilal, *"M(atrix) theory: A pedagogical introduction,"* Fortsch. Phys. **47**, 5 (1999), hep-th/9710136.

[150] C. G. Callan, J. A. Harvey, and A. Strominger, Nucl. Phys. **B367** (1991) 60.

[151] E. Witten, in the proceedings of *"Strings 95"*, USC, 1995, hep-th/9507121.

[152] P. S. Aspinwall, Phys. Lett. B **357**, 329 (1995), hep-th/9507012.

[153] J. Polchinski and E. Witten, Nucl. Phys. **B460** (1996) 525, hep-th/9510169.

[154] A. Dabholkar, Phys. Lett. **B357** (1995) 307; C. M. Hull, Phys. Lett. **B357** (1995) 545.

[155] C. V. Johnson, N. Kaloper, R. R. Khuri and R. C. Myers, Phys. Lett. **B368** (1996) 71, hep-th/9509070.

[156] E. Bergshoeff, E. Sezgin and P. K. Townsend, Phys. Lett. **B189** (1987) 75; M. J. Duff and K. S. Stelle, Phys. Lett. **B253** (1991) 113.

[157] R. Güven, Phys. Lett. **B276** (1992) 49.

[158] R. d. Sorkin, Phys. Rev. Lett. **51**, 87 (1983); D. J. Gross and M. J. Perry, Nucl. Phys. **B226**, 29 (1983).

[159] P. Horava and E. Witten, Nucl. Phys. **B460** (1996) 506, hep-th/9510209.

[160] M. J. Duff and J. X. Lu, Nucl. Phys. **B390** (1993) 276, hep-th/9207060; S. P. de Alwis and K. Sato, Phys. Rev. **D53** (1996) 7187, hep-th/9601167; A. A. Tseytlin, Nucl. Phys. **B469** (1996) 51, hep-th/9602064.

[161] C. Schmidhuber, Nucl .Phys. **B467** (1996) 146, hep-th/9601003.

[162] K. Hori, Nucl. Phys. **B539**, 35 (1999), hep-th/9805141; K. Landsteiner and E. Lopez, Nucl. Phys. **B516**, 273 (1998), hep-th/9708118; E. Witten, JHEP **9802**, 006 (1998), hep-th/9712028; E. G. Gimon, *"On the M-theory interpretation of orientifold planes"*, hep-th/9806226; C. Ahn, H. Kim and H. S. Yang, Phys. Rev. **D59**, 106002 (1999), hep-th/9808182; S. Sethi, JHEP **9811**, 003 (1998), hep-th/9809162; C. Ahn, H. Kim, B. Lee and H. S. Yang, Phys. Rev. **D61**, 066002 (2000), hep-th/9811010; A. Hanany, B. Kol and A. Rajaraman, JHEP **9910**, 027 (1999), hep-th/9909028; A. M. Uranga, JHEP **0002**, 041 (2000), hep-th/9912145; A. Hanany and B. Kol, JHEP **0006**, 013 (2000), hep-th/0003025.

[163] K. S. Narain, Phys. Lett. **169B** (1986) 41.

[164] P. Ginsparg, Phys. Rev. **D35** (1987) 648.

[165] K. S. Narain, M. H. Sarmadi and E. Witten, Nucl. Phys. **B279**, 369 (1987).

[166] B. Julia, in *"Supergravity and Superspace"*, ed. by S. W. Hawking and M. Rocek (Cambridge U. P., Cambridge UK, 1981).

[167] C. Vafa and E. Witten, Nucl. Phys. **B431** (1994) 3, hep-th/9408074.

[168] C. Vafa, Nucl. Phys. **B463** (1996) 435, hep-th/9512078.

[169] A. Strominger, Phys. Lett. **B383** (1996) 44, hep-th/9512059.

[170] S. Sethi and L. Susskind, Phys. Lett. B **400** (1997) 265, hep-th/9702101.

[171] An excellent review can be found in D. R. Morrison, *"TASI Lectures on Compactification and Duality"*, in TASI 1999, *"Strings, Branes and Gravity"*, World Scientific (2001).

[172] M. R. Douglas, J. Geom. Phys. **28**, 255 (1998), hep-th/9604198.

[173] C. Vafa, Nucl. Phys. **B469**, 403 (1996), hep-th/9602022.

[174] D. R. Morrison and C. Vafa, Nucl. Phys. **B473**, 74 (1996), hep-th/9602114; Nucl. Phys. **B476**, 437 (1996), hep-th/9603161.

[175] M. J. Duff, *"TASI lectures on branes, black holes and anti-de Sitter space"*, in TASI 1999: *"Strings, Branes and Gravity"*, World Scientific (2001), hep-th/9912164.

[176] See refs.[177, 178, 179, 162].

[177] N. Seiberg, Phys. Lett. **B384**, 81 (1996), hep-th/9606017.

[178] A. Sen, JHEP **9709**, 001 (1997), hep-th/9707123; JHEP **9710**, 002 (1997) hep-th/9708002.

[179] N. Seiberg and E. Witten, in *"Saclay 1996, The mathematical beauty of physics"*, hep-th/9607163.

[180] J. Polchinski, Phys. Rev. Lett. **75**, 4724 (1995), hep-th/9510017.

[181] J. Maldacena, Adv. Theor. Math. Phys. **2** (1988) 231, hep-th/9711200.

[182] This is a review: O. Aharony, S. S. Gubser, J. Maldacena, H. Ooguri and Y. Oz, Phys. Rept. **323**, 183 (2000), hep-th/9905111.