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Total Momentum and Other Noether Charges for Particles Interacting in a Quantum Spacetime

Giovanni Amelino-Camelia *, Giuseppe Fabiano and Domenico Frattulillo

Dipartimento di Fisica Ettore Pancini, Università di Napoli "Federico II" and INFN, Sezione di Napoli, Complesso Univ. Monte S. Angelo, I-80126 Napoli, Italy; giuseppe.fabiano@unina.it (G.F.); domenico.frattulillo@unina.it (D.F.)

* Correspondence: giovanni.amelinocamelia@unina.it

Abstract: There has been strong interest in the fate of relativistic symmetries in some quantum spacetimes, partly because of its possible relevance for high-precision experimental tests of relativistic properties. However, the main technical results obtained so far concern the description of suitably deformed relativistic symmetry transformation rules, whereas the properties of the associated Noether charges, which are crucial for the phenomenology, are still poorly understood. Here, we tackle this problem focusing on first-quantized particles described within a Hamiltonian framework and using as a toy model the so-called "spatial kappa-Minkowski noncommutative spacetime", where all the relevant conceptual challenges are present but, as here shown, in technically manageable fashion. We derive the Noether charges, including the much-debated total momentum charges, and we reveal a strong link between the properties of these Noether charges and the structure of the laws of interaction among particles.

Keywords: quantum gravity; quantum spacetime; relativity; Hopf algebras

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1. Introduction

The structure of the quantum gravity problem invites us to contemplate the possibility that spacetime itself might be affected by one form or another of quantization [1–6]. Besides (some remnant of) the light-cone structure specified by the speed-of-light scale, a quantum spacetime would inevitably also have additional structure, concerning its quantization, which in most models is specified in terms of a length scale. There has been strong interest [7–12] in understanding how this additional structure would affect some relevant spacetime symmetries, an issue which, besides its conceptual appeal, might also bear some relevance for phenomenology, since some spacetime symmetries can be tested with very high accuracy. Among the most studied models that fit this profile, there are some noncommutative spacetimes which are known to require a deformation of spacetime symmetries such that the noncommutativity length scale plays the role of second relativistically invariant scale [13–15], in addition to the speed-of-light scale. The quantum spacetimes which have been most studied from this perspective are Lie-algebra noncommutative spacetimes ($[x_\mu, x_\nu] = i\Gamma^\rho_{\mu\nu} x_\rho$ with $\Gamma^\rho_{\mu\nu}$ coefficients of length dimension), and in particular, some authors have argued [16] that certain aspects of the quantum gravity problem could motivate investigations of the κ-Minkowski noncommutative spacetime

$$[x^0, x^i] = i\ell x^i, \qquad [x^i, x^j] = 0 \tag{1}$$

where $i, j = 1, 2, 3$ and ℓ is a length scale usually assumed to be of the order of the Planck length. It is also noteworthy that several studies in 2+1-dimensional quantum gravity (see, e.g., Ref. [17]) exhibit the possibility to reabsorb the gravitational (topological) degrees of freedom into a Lie-algebra noncommutativity of the coordinates of particles, of the type often labeled as "spinning spacetime" [18]:

$$[x^\mu, x^\nu] = i\ell\epsilon^{\mu\nu}_\rho x^\rho \tag{2}$$

where $\epsilon^{\mu\nu}_\rho$ denotes the Levi-Civita tensor.

A description of the symmetries of these quantum spacetimes based on Hopf algebras has been studied extensively, establishing several robust mathematical properties, but the associated phenomenology has been stagnating because of the limitations of our understanding of the relationship between deformed spacetime symmetries and conserved charges in theories describing particles interacting in such quantum spacetimes (It has been established [19,20] that there is a generalization of the Noether theorem which applies to free theories formulated in some noncommutative spacetimes; however, a free theory has no phenomenology and it cannot even provide intuition for how the charges of different particles should be combined in conservation laws relevant for particle reactions). A key issue for phenomenology is how the charges of different particles should be combined in conservation laws relevant for particle reactions, and so far, this was based only on one or another heuristic "naturalness argument", guessing what happens when particles interact relying exclusively on results for free particles.

Here, we attempt to address this long-standing challenge by introducing a novel strategy of analysis. While previous attempts all focused on action/Lagrangian formulations of theories in noncommutative spacetimes [19–22], here we investigate the Noether charge issue within a Hamiltonian setup, finding that this provides several advantages. And our approach is also empowered by using as an illustrative example of Noether charge analysis in a quantum spacetime the case of the so-called "spatial 2D kappa-Minkowski noncommutative spacetime", where all the relevant conceptual challenges are present but, as shown here, in technically manageable fashion. We exhibit some examples of Hamiltonians describing two-particle and three-particle interactions for which the Noether charges can be constructively derived. A key take-home message is that within a given description of (deformed) relativistic symmetries for the free-particle case, the total charges that are then conserved when one allows multiparticle interactions depend strongly on the form of the Hamiltonian, also exposing the weakness of previous "naturalness arguments" used for guessing the Noether charges.

2. Preliminaries

Before getting to our novel results, we devote this section to a short review of properties of spatial 2D κ-Minkowski and to a general perspective on possible noncommutative spacetime generalizations of harmonic-oscillator-type Hamiltonians (the type of Hamiltonians for which, in the following sections, we shall derive Noether charges).

2.1. Spatial 2D κ-Minkowski

The most studied variant of κ-Minkowski noncommutativity is a case of space/time noncommutativity (spatial coordinates commute among themselves but do not commute with the time coordinate), which in the 2D case is characterized by the following commutator between time and spatial coordinate [23]:

$$[x^0, x^1] = i\ell x^1 \tag{3}$$

where ℓ (often rewritten as $1/\kappa$) is a length scale usually assumed to be of the order of the Planck length. It is well established [23–25] that the symmetries of 2D space/time κ-Minkowski noncommutativity are described by the 2D κ-Poincaré Hopf Algebra.

In this study, we follow Ref. [26] by focusing on a scenario with a time coordinate which is fully commutative and two spatial coordinates governed by κ-Minkowski noncommutativity

$$[x_2, x_1] = i\ell x_1 \,. \tag{4}$$

All the results established in a wide literature on the 2D space/time κ-Minkowski of Equation (3) and its Hopf-algebra symmetries are easily converted into results for our 2D spatial κ-Minkowski of Equation (4) and its Hopf-algebra symmetries, by the replacement of coordinates $x^0 \to ix_2$, a replacement of noncommutativity parameter $\ell \to i\ell$, and then replacing the time-translator generator with a suitable generator of translations along the x_2 direction, $P_0 \to -iP_2$ while the boost generator of 2D space/time κ-Minkowski is replaced by the rotation generator of 2D spatial κ-Minkowski, $N \to -iR$. This leads to a description of the translation and rotation symmetries of 2D spatial κ-Minkowski such that

$$[P_2, P_1] = 0 \qquad [R, P_2] = -iP_1 \qquad [R, P_1] = \frac{i}{2\ell}(1 - e^{-2\ell P_2}) + i\frac{\ell}{2}P_1^2 \tag{5}$$

which is a deformation of the Euclidean algebra in two dimensions. A central element of this algebra, which will be a crucial ingredient for the construction of our Hamiltonians, is given by

$$\mathcal{C} = \frac{4}{\ell^2}\sinh^2(\ell P_2/2) + e^{\ell P_2} P_1^2 \tag{6}$$

This is a deformation of the $P_1^2 + P_2^2$ Casimir element of the Euclidean algebra.

We shall introduce interactions among particles within a Hamiltonian setup and be satisfied showing our results to order ℓ^2. We note here some commutation relations which shall be valuable in those Hamiltonian analyses:

$$[x_1, P_1] = i \quad [x_1, P_2] = 0 \quad [x_2, P_1] = -i\ell P_1 \quad [x_2, P_2] = i \tag{7}$$

$$\begin{aligned} &[R, x_1] = ix_2 \\ &[R, x_2] = -i\left(x_1 - \ell x_1 P_2 + \frac{\ell}{2}x_2 P_1 + \frac{\ell}{2}P_1 x_2 + \ell^2 x_1 P_2^2 + \frac{\ell^2}{4}(x_1 P_1^2 + P_1^2 x_1)\right) \end{aligned} \tag{8}$$

which satisfy Jacobi identities.

The nonlinearity of the commutators (5), typical of Hopf-algebra symmetries, produce the difficulties for Noether charges, which are the main focus of this study. For free particles, it has been shown [19,20] that the charges associated to P_1, P_2 and R are conserved (but of course, any nonlinear function of a conserved quantity is also conserved). For interacting particles, it is unclear which combinations of the charges should be conserved in particle reactions. In particular, for a process $A + B \to C + D$, it is clear that $P_1^A + P_1^B = P_1^C + P_1^D$ is not an acceptable conservation law because of the nonlinearity of $[R, P_1]$ (i.e., $P_1^A + P_1^B = P_1^C + P_1^D$ would not be covariant). So it is clear that the total momentum of a system composed of particles A and B cannot have the component $P_1^A + P_1^B$, but it is not clear which nonlinear combination of the momenta gives the total momentum of a system (and would be therefore conserved in particle reactions). A popular way to guess the momentum composition formula is based on the so-called "coproduct" [27–29], which for our purposes, is sufficient to introduce in terms of the properties of suitably ordered products of plane waves; for two plane waves of momenta k and q, one has that, as a result of the noncommutativity (4),

$$e^{ik_1x^1}e^{ik_2x^2}e^{iq_1x^1}e^{iq_2x^2} = e^{i(k\oplus_\kappa q)_1x^1}e^{i(k\oplus_\kappa q)_2x^2} \tag{9}$$

where

$$\begin{aligned}(k\oplus_\kappa q)_1 &= k_1 + e^{-\ell k_2}q_1\\ (k\oplus_\kappa q)_2 &= k_2 + q_2\end{aligned} \tag{10}$$

In order for these quantities to close the single-particle algebra (5), rotations should also combine non-linearly:

$$(R_k \oplus_\kappa R_q) = R_k + e^{-\ell k_2}R_q \tag{11}$$

Alternative ways for guessing the momentum composition formula have also been proposed. As an alternative to the "κ-coproduct composition law" of Equations (10) and (11), we shall also consider the "proper-dS composition law",

$$\begin{aligned}\mathcal{P}_1 = (p^A \oplus_{dS} p^B)_1 &= p_1^A + p_1^B - \ell(p_2^A p_1^B + p_1^A p_2^B) +\\ &\quad + \frac{\ell^2}{2}\left[(p_2^A p_1^B + p_1^A p_2^B)(p_2^A + p_2^B) - p_1^A (p_1^B)^2 - (p_1^A)^2 p_1^B\right]\\ \mathcal{P}_2 = (p^A \oplus_{dS} p^B)_2 &= p_2^A + p_2^B + \ell p_1^A p_1^B +\\ &\quad - \frac{\ell^2}{2}\left[-p_1^B p_1^A (p_2^B + p_2^A) + p_2^A (p_1^B)^2 + (p_1^A)^2 p_2^B\right]\\ \mathcal{R} = (R^A \oplus_{dS} R^B) &= R^A + R^B\end{aligned} \tag{12}$$

which was motivated using some geometric arguments (one can show that with these choices of composition laws, momentum space acquires the geometrical structure of de Sitter space [30]).

2.2. Deformations of Harmonic Oscillator Hamiltonians

Our next task is to introduce the class of Hamiltonians on which we shall focus our search for Noether charges. Their core ingredient is the harmonic oscillator potential in two spatial dimensions. We shall consider deformations of the Hamiltonian

$$H_0^{AB} = \frac{(\vec{p}^A)^2}{2m} + \frac{(\vec{p}^B)^2}{2m} + \frac{1}{2}g(\vec{q}^A - \vec{q}^B)^2 \tag{13}$$

where g is the coupling constant, the labels A and B refer to the two particles interacting, $\vec{q}^J$ ($J \in \{A, B\}$) are ordinary commutative spatial coordinates, and $\vec{p}^J$ are the corresponding momenta, with standard Heisenberg commutators ($[q_j^J, p_k^K] = i\,\delta^{JK}\delta_{jk}$, with $J, K \in \{A, B\}$ and $j, k = 1, 2$). The total momentum and total angular momentum defined through

$$\vec{P} = \vec{p}^A + \vec{p}^B \qquad R_0 = R_0^A + R_0^B \tag{14}$$

are conserved charges since they commute with the Hamiltonian, $[H_0^{AB}, \vec{P}] = 0$ and $[H_0^{AB}, R_0] = 0$. Both the total generators $\{P_i, R_0\}$ and the single-particle generators $\{p_i^I, R_0^I\}$ close the un-deformed Galilean algebra.

For reasons which shall soon be clear, we also want to test our approach for interactions among more than two particles, and for that purpose, our starting point is the three-particle Hamiltonian

$$H_0^{ABC} = \frac{(\vec{p}^A)^2}{2m} + \frac{(\vec{p}^B)^2}{2m} + \frac{(\vec{p}^C)^2}{2m} + \frac{1}{2}g(\vec{q}^A - \vec{q}^B)^2 + \frac{1}{2}g(\vec{q}^A - \vec{q}^C)^2 + \frac{1}{2}g(\vec{q}^B - \vec{q}^C)^2 \tag{15}$$

This is of interest to us particularly because the interacting potential $V_3(\vec{q}^A, \vec{q}^B, \vec{q}^C)$ can be split into the sum $V_2(\vec{q}^A, \vec{q}^B) + V_2(\vec{q}^A, \vec{q}^C) + V_2(\vec{q}^B, \vec{q}^C)$ with V_2 having the same functional form for each pair of particles; in the case studies for which we performed our Noether charge analyses, this property cannot be maintained in the presence of noncommutativity of coordinates.

Evidently, the Hamiltonian (15) commutes with the total charges defined as $\vec{P} = \vec{p}^A + \vec{p}^B + \vec{p}^C$ and $R_0 = R_0^A + R_0^B + R_0^C$.

A key ingredient of our deformed Hamiltonians will be of course the kinetic term, for which we adopt the form

$$H_K \equiv \frac{\mathcal{C}}{2m} \approx \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + \ell \frac{p_1^2 p_2}{2m} + \ell^2 \frac{p_1^2 p_2^2}{4m} + \ell^2 \frac{p_2^4}{24m} \tag{16}$$

obtained from the Casimir element $\mathcal{C}$ of our Equation (5) (to order ℓ^2).

We will look for suitable interaction potentials within some rather broad parameterizations. We parameterize the two-particle case as follows:

$$V^{AB} = V(\vec{x}^A, \vec{x}^B) = \frac{1}{2} g (\vec{x}^A - \vec{x}^B)^2 + \ell g \sum \alpha_{ijk}^{IJK} p_i^I x_j^J x_k^K + \ell^2 g \sum \beta_{ijkh}^{IJKH} p_i^I p_j^J x_k^K x_h^H \tag{17}$$

where α_{ijk}^{IJK} and β_{ijkh}^{IJKH} are numerical coefficients and the sum extends both to spatial indices (lower-case letters) and particle indices (upper-case letters).

Similarly, for the three-particle case, our *ansatz* is given by

$$\begin{aligned} V^{ABC} = V(\vec{x}^A, \vec{x}^B, \vec{x}^C) =& \frac{1}{2} g (\vec{x}^A - \vec{x}^B)^2 + \frac{1}{2} g (\vec{x}^B - \vec{x}^C)^2 + \frac{1}{2} g (\vec{x}^C - \vec{x}^A)^2 + \\ &+ \ell g \sum \tilde{\alpha}_{ijk}^{IJK} p_i^I x_j^J x_k^K + \ell^2 g \sum \tilde{\beta}_{ijkh}^{IJKH} p_i^I p_j^J x_k^K x_h^H \end{aligned} \tag{18}$$

where $\tilde{\alpha}_{ijk}^{IJK}$ and $\tilde{\beta}_{ijkh}^{IJKH}$ are other sets of numerical coefficients and the particle indices run over $\{A, B, C\}$.

3. Charges with Proper-dS Composition

The debate on the alternative ways to combine charges in a κ-Minkowski setup has mainly relied on naturalness arguments based on the properties of free particles in κ-Minkowski. As announced in our opening remarks, we intend to show here that there is no notion of "naturalness" at stake here; how charges should combine depends on the form of the laws of interaction among particles (and so evidently goes beyond the scopes of the description of free particles) and different composition laws can emerge from different descriptions of the interactions. We shall establish our case relying on Hamiltonian theories within first-quantized quantum mechanics, where the relevant issues can be seen in particularly vivid fashion.

We choose as our first task the one of exhibiting a Hamiltonian (within first-quantized quantum mechanics) which selects uniquely the proper-dS composition law, which we already reviewed in Equation (12) and we show again here for convenience:

$$\begin{aligned} \mathcal{P}_1 = (p^A \oplus_{dS} p^B)_1 =& p_1^A + p_1^B - \ell (p_2^A p_1^B + p_1^A p_2^B) + \\ &+ \frac{\ell^2}{2} \left[(p_2^A p_1^B + p_1^A p_2^B)(p_2^A + p_2^B) - p_1^A (p_1^B)^2 - (p_1^A)^2 p_1^B \right] \\ \mathcal{P}_2 = (p^A \oplus_{dS} p^B)_2 =& p_2^A + p_2^B + \ell p_1^A p_1^B + \\ &- \frac{\ell^2}{2} \left[- p_1^B p_1^A (p_2^B + p_2^A) + p_2^A (p_1^B)^2 + (p_1^A)^2 p_2^B \right] \\ \mathcal{R} = (R^A \oplus_{dS} R^B) =& R^A + R^B \end{aligned} \tag{19}$$

One can easily verify that $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{R}$ close the algebra (5) up to order ℓ^2, which we also rewrite here for convenience:

$$[\mathcal{P}_2, \mathcal{P}_1] = 0 \qquad [\mathcal{R}, \mathcal{P}_2] = -i\mathcal{P}_1 \qquad [\mathcal{R}, \mathcal{P}_1] = i(\mathcal{P}_2 - \ell\mathcal{P}_2^2 + \frac{\ell}{2}\mathcal{P}_1^2 + \frac{2\ell^2\mathcal{P}_2^3}{3}) \tag{20}$$

We start by showing that for the case of two particles interacting, there is a Hamiltonian H_{dS}^{AB}, deformation of the H_0^{AB} of Equation (13), such that $[\vec{\mathcal{P}}, H_{dS}^{AB}] = 0$ and $[\mathcal{R}, H_{dS}^{AB}] = 0$. As anticipated in Section 2.2, our Hamiltonian H_{dS}^{AB} will be of the form

$$H_{dS}^{AB} = H_K^A + H_K^B + V_{dS}^{AB} \tag{21}$$

where H_K is fixed to be that of Equation (16), while V_{dS}^{AB} must be specified consistently with Equation (17), for some choice of the parameters that Equation (17) leaves to be determined.

We work partly by reverse engineering; we use $[\vec{\mathcal{P}}, H_{dS}^{AB}] = 0$ and $[\mathcal{R}, H_{dS}^{AB}] = 0$ as conditions that must be satisfied by the parameters of Equation (17), and then, once we have such an acceptable V_{dS}^{AB}, we show that the resulting Hamiltonian H_{dS}^{AB} uniquely selects the proper-dS charges (19) as its conserved charges.

We find that in particular, the following choice of V_{dS}^{AB}:

$$\begin{aligned}
V_{dS}^{AB} =& \frac{g}{2}\Big[(\vec{x}^A - \vec{x}^B)^2 + 2\ell\Big(- p_2^A(x_1^A)^2 + \frac{1}{2}p_1^A x_1^A x_2^A + \\
&+\frac{1}{2}x_2^A x_1^A p_1^A + p_2^A x_1^A x_1^B - x_2^A p_1^A x_1^B + (A \leftrightarrow B)\Big) + \\
&+\frac{1}{2}\ell^2\Big((p_1^B)^2(-2(x_2^A)^2 + 6x_2^A x_2^B - 2(x_2^B)^2) + 4p_1^B p_2^B x_1^A x_2^A - p_1^A p_1^B x_2^A x_2^B + \\
&-6p_1^B x_1^A x_2^B p_2^B - 2p_1^B x_1^B(p_2^A x_2^A - x_2^B p_2^B) - 2(p_2^B)^2((x_1^A)^2 - x_1^A x_1^B - (x_1^B)^2) + \\
&+p_2^B p_1^A x_2^A x_1^B - 2p_2^B p_2^A x_1^A x_1^B - 3p_2^B x_2^A x_1^B p_1^B + 2x_1^A p_1^A(p_1^A x_1^A - p_1^A x_1^B + \\
&+p_2^A x_2^A - \frac{3}{2}p_2^A x_2^B + \frac{3}{2}x_2^B p_2^B) + x_2^A p_2^A x_1^B p_1^B + p_1^B p_2^A x_1^A x_2^B + (A \leftrightarrow B)\Big)\Big]
\end{aligned} \tag{22}$$

is indeed such that $[\vec{\mathcal{P}}, H_K^A + H_K^B + V_{dS}^{AB}] = 0$ and $[\mathcal{R}, H_K^A + H_K^B + V_{dS}^{AB}] = 0$.

We observe that our V_{dS}^{AB} is symmetric under exchange of the particles (this is not always the case; see later). Most importantly, we find that indeed the Hamiltonian $H_K^A + H_K^B + V_{dS}^{AB}$ uniquely selects the proper-dS charges (19) as its conserved charges. In order to see this, we start from a general parameterization of the two-particle charges

$$\begin{aligned}
P_1^{tot} &= \sum p_1^I + \ell\gamma_{ij}^{IJ} p_i^I p_j^J + \ell^2\Gamma_{ijk}^{IJK} p_i^I p_j^J p_k^K \\
P_2^{tot} &= \sum p_2^I + \ell\theta_{ij}^{IJ} p_i^I p_j^J + \ell^2\Theta_{ijk}^{IJK} p_i^I p_j^J p_k^K \\
R^{tot} &= \sum R^I + \ell\phi_i^{IJ} p_i^I R^J + \ell^2\Phi_{ij}^{IJK} p_i^I p_j^J R^K
\end{aligned} \tag{23}$$

where $\gamma, \theta, \phi, \Gamma, \Theta, \Phi$ are sets of real coefficients and the sum is intended over particle indices I, J, K (which take values in $\{A, B\}$) and over the spatial indices i, j, k. We also require that no terms with all particle indices equal to each other are present, so that we recover the definition of single-particle charge when the charges of the other particles are zero.

By requesting that these charges commute with $H_K^A + H_K^B + V_{dS}^{AB}$, the parameters in Equation (23) are fully fixed, giving indeed the proper-dS charges (19).

Next, we turn to the corresponding three-particle case, for which the proper-dS composition leads to the following formulas for the charges:

$$
\begin{aligned}
\tilde{\mathcal{P}}_1 &= ((p^A \oplus_{dS} p^B) \oplus_{dS} p^C)_1 = p_1^A + p_1^B + p_1^C - \ell\big(p_2^B(p_1^C + p_1^A) + p_2^C(p_1^B + p_1^A) + \\
&\quad + p_2^A(p_1^C + p_1^B)\big) + \frac{\ell^2}{2}\big(-2p_1^A p_1^B p_1^C - (p_1^B)^2 p_1^A + 2p_1^A p_2^B p_2^C - p_1^B (p_1^A)^2 + \\
&\quad - (p_1^C)(p_1^B + p_1^A)^2 + 2p_2^C p_1^B p_2^A + (p_2^B + p_2^A)(p_2^B p_1^A + p_2^A p_1^B) + \\
&\quad + (p_2^A + p_2^B + p_2^C)(p_2^C(p_1^B + p_1^A) + p_1^C(p_2^A + p_2^B) - (p_1^C)^2(p_1^B + p_1^A))\big) \\
\tilde{\mathcal{P}}_2 &= ((p^A \oplus_{dS} p^B) \oplus_{dS} p^C)_2 = p_2^A + p_2^B + p_2^C + \ell\big(p_1^B p_1^A + p_1^C p_1^B + p_1^A p_1^C\big) + \\
&\quad - \frac{\ell^2}{2}\big(p_2^C(p_1^A + p_1^B)^2 - p_1^C(p_2^C(p_1^B + p_1^A) + (p_1^B - p_1^A)(p_2^B - p_2^A)) + \\
&\quad + (p_1^C)^2(p_2^B + p_2^A) + (p_1^B - p_1^A)(-p_2^B p_1^A + p_1^B p_2^A)\big) \\
\tilde{\mathcal{R}} &= (R^A \oplus_{dS} R^B) \oplus_{dS} R^C = R^A + R^B + R^C
\end{aligned}
\tag{24}
$$

Evidently, we must find a Hamiltonian H_{dS}^{ABC}, deformation of the H_0^{ABC} of Equation (15), such that $[\vec{\tilde{\mathcal{P}}}, H_{dS}^{ABC}] = 0$ and $[\tilde{\mathcal{R}}, H_{dS}^{ABC}] = 0$. As anticipated in Section 2.2, our Hamiltonian H_{dS}^{ABC} will be of the form

$$
H_{dS}^{ABC} = H_K^A + H_K^B + H_K^C + V_{dS}^{ABC} \tag{25}
$$

where H_K is again fixed to be that of Equation (16), while V_{dS}^{ABC} must be specified consistently with Equation (18), for some choice of the parameters that Equation (18) leaves to be determined.

A natural first guess is that the three-particle potential V_{dS}^{ABC} is given (see Equation (15)) by a combination of our two-particle potentials given in Equation (22), i.e., $V_{dS}^{ABC} = V_{dS}^{AB} + V_{dS}^{BC} + V_{dS}^{AC}$, but one can easily check that this does not commute with the three-particle proper-dS charges (24). What does work is adding an extra term:

$$
V_{dS}^{ABC} = V_{dS}^{AB} + V_{dS}^{BC} + V_{dS}^{AC} + V_{dS(\bigstar)}^{ABC} \tag{26}
$$

with

$$
\begin{aligned}
V_{dS(\bigstar)}^{ABC} = \frac{g\ell^2}{2}\Big(& p_1^C p_2^B (x_1^C x_2^A - x_2^C x_1^A) - p_1^C p_2^A x_2^C x_1^B - p_2^C p_1^B (x_1^C x_2^A - x_2^C x_1^A) + \\
& + p_1^B p_1^A x_2^C (2x_2^C - x_2^A - x_2^B) - p_1^B p_2^A x_2^C (2x_1^C - x_1^B) - 2p_2^B p_1^A x_1^C x_2^C + \\
& + p_2^B p_2^A x_1^C (2x_1^C - x_1^A - x_1^B) + p_1^A x_1^C p_2^B x_2^B + x_1^C p_1^C p_2^A x_2^B + \\
& + x_2^C p_2^C p_1^A x_1^B + x_1^A p_1^A p_2^B x_2^C + x_2^A p_2^A p_1^B x_1^C - p_2^C p_1^A x_1^C x_2^B\Big)
\end{aligned}
\tag{27}
$$

One can easily check that the H_{dS}^{ABC} of Equations (25)–(27) commutes with the proper-dS charges (24). Most importantly, we find that indeed, our Hamiltonian H_{dS}^{ABC} uniquely selects the proper-dS charges (24) as its conserved charges. In order to see this, we start from a general parameterization of the three-particle charges

$$
\begin{aligned}
\tilde{P}_1^{tot} &= \sum p_1^I + \ell\tilde{\gamma}_{ij}^{IJ} p_i^I p_j^J + \ell^2 \tilde{\Gamma}_{ijk}^{IJK} p_i^I p_j^J p_k^K \\
\tilde{P}_2^{tot} &= \sum p_2^I + \ell\tilde{\theta}_{ij}^{IJ} p_i^I p_j^J + \ell^2 \tilde{\Theta}_{ijk}^{IJK} p_i^I p_j^J p_k^K \\
\tilde{R}^{tot} &= \sum R^I + \ell\tilde{\phi}_i^{IJ} p_i^I R^J + \ell^2 \tilde{\Phi}_{ij}^{IJK} p_i^I p_j^J R^K
\end{aligned}
\tag{28}
$$

which shares the same properties outlined for the two-particle ansatz (23) (the particle indices run over $\{A, B, C\}$ and $\tilde{\gamma}, \tilde{\theta}, \tilde{\phi}, \tilde{\Gamma}, \tilde{\Theta}, \tilde{\Phi}$ are sets of real coefficients).

We find that by requesting that these charges commute with our $H_K^A + H_K^B + V_{dS}^{AB}$, the parameters in Equation (28) are fully fixed, giving indeed the proper-dS charges (24).

We leave to future studies the task of exploring the meaning of the extra term $V_{dS(\bigstar)}^{ABC}$. Whereas the potential in the original three-particle Hamiltonian H_0^{ABC} of Equation (15) was just a sum of two-particle potentials, we found that the potential in its correct "proper-dS deformation" H_{dS}^{ABC} must include the extra term $V_{dS(\bigstar)}^{ABC}$, which is cubic in the observables of the three particles and is made of all terms involving simultaneously observables of all three particles.

Also noteworthy is that for the three-particle case, the proper-dS composition gives charges which are not symmetric under particle exchange (see (24)) and accordingly, our Hamiltonian H_{dS}^{ABC} is also not symmetric under particle exchange. We do not see any objective problem with this lack of particle exchange symmetry, but still, it is a bit unsettling. This made us interested in investigating which charges would be conserved if we adopted a particle exchange-symmetrized version of our Hamiltonian H_{dS}^{ABC}

$$H_{dS(sym)}^{ABC} = H_K^A + H_K^B + H_K^C + V_{dS}^{AB} + V_{dS}^{BC} + V_{dS}^{AC} + \frac{1}{6} \sum_{\pi(A,B,C)} V_{dS(\bigstar)}^{\pi(ABC)} \tag{29}$$

i.e., the Hamiltonian obtained by summing over all the possible particle permutations, $\pi(ABC)$, of the extra term.

We then ask for which choices of the parameters of our Equation (28) the Hamiltonian $H_{dS(sym)}^{ABC}$ commutes with the charges parameterized in our Equation (28), and we find that $H_{dS(sym)}^{ABC}$ uniquely selects as its conserved charges the following ones:

$$\begin{aligned}
\mathcal{P}_1^{dS(sym)} &= \frac{1}{3}[(p^A \oplus_{dS} p^B) \oplus_{dS} p^C + p^A \oplus_{dS} (p^B \oplus_{dS} p^C) + (p^A \oplus_{dS} p^C) \oplus_{dS} p^B]_1 = \\
&= p_1^A + p_1^B + p_1^C - \ell(p_1^A p_2^B + p_1^B p_2^A + p_1^A p_2^C + p_1^C p_2^A + p_1^B p_2^C + p_1^C p_2^B) + \\
&+ \frac{\ell^2}{2}(p_1^A((p_2^B)^2 + (p_2^C)^2) + (p_1^B + p_1^C)(p_2^A)^2 - p_1^A (p_1^B)^2 - p_1^B (p_1^A)^2 + \\
&- p_1^C (p_1^B)^2 - p_1^B (p_1^C)^2 + p_1^B (p_2^C)^2 + p_1^C (p_2^B)^2 + + p_1^C p_2^C p_2^B + p_1^B p_2^B p_2^C + \\
&- p_1^A (p_1^C)^2 - p_1^C (p_1^A)^2 - 4 p_1^A p_1^B p_1^C + \frac{8}{3}(p_1^A p_2^B p_2^C + p_1^B p_2^A p_2^C + p_1^C p_2^B p_2^A) + \\
&+ p_2^A (p_1^C p_2^C + p_1^B p_2^C + p_1^A p_2^C + p_1^A p_2^B)) \\
\mathcal{P}_2^{dS(sym)} &= \frac{1}{3}[(p^A \oplus_{dS} p^B) \oplus_{dS} p^C + p^A \oplus_{dS} (p^B \oplus_{dS} p^C) + (p^A \oplus_{dS} p^C) \oplus_{dS} p^B]_2 = \\
&= p_2^A + p_2^B + p_2^C + \ell(p_1^C (p_1^B + p_1^A) + p_1^B p_1^A) + \\
&- \frac{1}{6}\ell^2 (3(p_1^C)^2 (p_2^B + p_2^A) + \\
&- p_1^C (3 p_2^C (p_1^B + p_1^A) + 3 p_1^B p_2^B - 4 p_1^B p_2^A - 4 p_2^B p_1^A + 3 p_1^A p_2^A) + \\
&+ p_2^C (3(p_1^B)^2 + 4 p_1^B p_1^A + 3(p_1^A)^2) + 3(p_1^B - p_1^A)(p_1^B p_2^A - p_2^B p_1^A)) \\
\mathcal{R}^{dS(sym)} &= R^A + R^B + R^C
\end{aligned} \tag{30}$$

which are indeed symmetric under particle exchange. Moreover, these charges $\mathcal{P}_1^{dS(sym)}$, $\mathcal{P}_2^{dS(sym)}$, $\mathcal{R}^{dS(sym)}$ close the algebra (5).

4. Charges with κ-Coproduct Composition

We now move on to applying the same strategy of the analysis to the coproduct composition law of Equations (10) and (11), which we rewrite here (at order ℓ^2) for convenience:

$$\begin{aligned}\mathcal{P}_1 &= (p^A \oplus_\kappa p^B)_1 = p_1^A + p_1^B - \ell p_2^A p_1^B + \frac{\ell^2}{2}(p_2^A)^2 p_1^B \\ \mathcal{P}_2 &= (p^A \oplus_\kappa p^B)_2 = p_2^A + p_2^B \\ \mathcal{R} &= R^A \oplus_\kappa R^B = R^A + R^B - \ell p_2^A R^B + \frac{\ell^2}{2}(p_2^A)^2 R^B\,,\end{aligned} \tag{31}$$

As performed for the proper-dS case, our first objective is to find a Hamiltonian H_κ^{AB}, deformation of the H_0^{AB} of Equation (13), such that $[\vec{\mathcal{P}}, H_\kappa^{AB}] = 0$ and $[\mathcal{R}, H_\kappa^{AB}] = 0$. Applying the same strategy as the previous section, we find that the Hamiltonian $H_\kappa^{AB} = H_K^A + H_K^B + V_\kappa^{AB}$ with

$$\begin{aligned}V_\kappa^{AB} = \frac{g}{2}\Bigg[&(\vec{x}^A - \vec{x}^B)^2 + \\ 2\ell&\left(-p_2^A (x_1^A)^2 + x_1^A p_2^A x_1^B + \frac{1}{2}x_2^A p_1^A x_1^A + \frac{1}{2}x_2^A x_1^A p_1^A - x_2^B p_1^A x_1^A - \frac{1}{2}x_2^B p_1^A x_1^A\right) \\ 2\ell^2&\left((p_2^A)^2 (x_1^A)^2 + \frac{1}{2}x_1^A (p_1^A)^2 x_1^A - \frac{1}{2}x_1^A (p_2^A)^2 x_1^B\right)\Bigg]\end{aligned} \tag{32}$$

is such that indeed, $[\vec{\mathcal{P}}, H_K^A + H_K^B + V_\kappa^{AB}] = 0$ and $[\mathcal{R}, H_K^A + H_K^B + V_\kappa^{AB}] = 0$. And we find that the Hamiltonian $H_K^A + H_K^B + V_\kappa^{AB}$ uniquely selects the κ-coproduct charges (31) as its conserved charges. This is easily shown by starting again from the general charge ansatz (23) and requiring that they commute with $H_K^A + H_K^B + V_\kappa^{AB}$; this requirement fully fixes all the parameters in Equation (23), giving indeed the κ-coproduct charges (31).

It is noteworthy that the κ-coproduct charges (31) are not symmetric under the exchange of particles A and B, and accordingly, our Hamiltonian H_κ^{AB} is also not symmetric (because the potential V_κ^{AB} of (32) is not symmetric). We found that the analogous issue of lacking particle exchange symmetry that we encountered in our analysis of the proper-dS composition law could be "fixed" by resorting to a symmetrized version of the Hamiltonian, but for the κ-coproduct composition law, this is not the case; if one considers the symmetrized Hamiltonian

$$H_{\kappa(sym)}^{AB} = \frac{H_\kappa^{AB} + H_\kappa^{BA}}{2} \tag{33}$$

then one finds that no choice of the parameters in (23) leads to charges that commute with $H_{AB}^{\kappa(sym)}$.

For the three-particle case, the κ-coproduct composition law gives

$$\begin{aligned}\tilde{\mathcal{P}}_1 = (p^A \oplus_\kappa p^B \oplus_\kappa p^C)_1 &= p_1^A + p_1^B + p_1^C + \ell\left(-p_1^B p_2^A - p_1^C(p_2^A + p_2^B)\right) + \\ &+ \ell^2\left(\frac{p_1^B (p_2^A)^2}{2} + \frac{1}{2}p_1^C (p_2^A + p_2^B)^2\right)\end{aligned} \tag{34}$$

$$\tilde{\mathcal{P}}_2 = (p^A \oplus_\kappa p^B \oplus_\kappa p^C)_2 = p_2^A + p_2^B + p_2^C$$

$$\begin{aligned}\tilde{\mathcal{R}} = (R^A \oplus_\kappa R^B \oplus_\kappa R^C) &= R^A + R^B + R^C + \ell\left(-p_2^A\left(R^B\right) - R^C\left(p_2^A + p_2^B\right)\right) + \\ &+ \ell^2\left(\frac{1}{2}R^C\left(p_2^A + p_2^B\right)^2 + \frac{1}{2}(p_2^A)^2 R^B\right)\end{aligned}$$

Using the same procedure as Section 3, one finds that the Hamiltonian

$$H_\kappa^{ABC} = H_K^A + H_K^B + H_K^C + V_\kappa^{AB} + V_\kappa^{BC} + V_\kappa^{AC} + V_{\kappa(\bigstar)}^{ABC}, \tag{35}$$

with

$$\begin{aligned} V_{\kappa(\bigstar)}^{ABC} =& g\ell\Big(p_1^B(x_1^C x_2^A - x_2^C x_1^A + x_1^A x_2^B - x_1^C x_2^B) + p_2^B x_1^B(x_1^C - x_1^A)\Big) + \\ &+ g\frac{\ell^2}{2}\Big((p_1^B)^2(x_1^C x_1^A - x_1^B x_1^C + x_1^B x_1^A) - (p_2^B)^2(x_1^C x_1^A - 2x_1^A x_1^B) + \\ &+ p_1^B x_1^C(p_1^A x_1^A - p_2^B x_2^B) + p_1^B p_2^A(x_2^C x_1^A - x_1^A x_2^B) + p_2^B(p_1^B x_1^A x_2^C + p_2^A x_1^A x_1^B)\Big), \end{aligned} \tag{36}$$

commutes with $\vec{\tilde{\mathcal{P}}}$ and $\tilde{\mathcal{R}}$. It is noteworthy that the κ-coproduct extra term $V_{\kappa(\bigstar)}^{ABC}$, besides involving terms that depend simultaneously on observables of all three particles, also involves terms that depend only on two of the particles (and these additional terms cannot be re-absorbed in a redefinition of the potentials $\tilde{V}_\kappa^{IJ}$ since they are different for different pairs of particles).

Also, in this case, we find that the Hamiltonian H_κ^{ABC} of our Equation (35) uniquely selects the κ-coproduct charges (34) as its conserved charges; by requesting that the parameterized charges of Equation (28) commute with H_κ^{ABC}, the parameters in Equation (28) are fully fixed, giving indeed the κ-coproduct charges (34).

H_κ^{ABC} is not symmetric under particle exchange, and its symmetrized version,

$$H_{\kappa(sym)}^{ABC} = H_K^A + H_K^B + H_K^C + \frac{1}{6}\sum_{\pi(A,B,C)} V_\kappa^{\pi(ABC)}, \tag{37}$$

is not a viable alternative since it does not have any conserved charges; there is no choice of the parameters in Equation (28) such that the parameterized charges of Equation (28) commute with $H_{\kappa(sym)}^{ABC}$.

5. Closing Remarks

Inevitably, the physics community is approaching the challenge of understanding the deformed relativistic symmetries of some quantum spacetimes from a perspective which is mainly informed by our experience with special relativity, but a price can be paid when we unknowingly make inferences based on the linearity of most special-relativistic laws. In particular, the way in which special relativity governs how free-particle charges combine in conservation laws applicable when particles interact is completely governed by the linearity of transformation laws, so that charges inevitably combine linearly. Working within special relativity, one does not even fully appreciate how the chosen form of interaction could affect the conservation laws, because the linearity of transformation laws imposes that in all cases, charges combine linearly, independently of the type of interactions being considered. This is probably the reason why, before this study, the debate on total charges for quantum spacetimes had not contemplated a possible role for the interactions, and instead relied on one or another "naturalness argument" based on the form of the relativistic properties of free particles.

We showed here, using the toy model of spatial 2D κ-Minkowski, that the nonlinearity of deformed-relativistic transformation laws is such that the correct notion of total charge depends strongly on how one introduces interactions among particles. We found that, starting from the same description of free particles, for interacting particles, one can have at least three different ways for obtaining total charges: the one based on the proper-dS composition law, the one based on the κ-coproduct composition law, and the one obtained by symmetrizing the proper-dS composition law. Interestingly, we also found that it is instead not possible to introduce interactions such that conservation laws are obtained by symmetrizing the κ-coproduct composition law. To our knowledge, ours are the first results establishing in such a tangible way a tight connection between how one introduces particle interactions and a suitable notion of conserved total charges. This realization also raises some interesting conceptual issues, since of course, the way in which particles interact is fixed by Nature (using which "criteria"?). The Hamiltonians we focused on here appear to be unpleasantly complex, and it would be surprising (though of course possible) that Nature would choose such complex ways to describe interactions among particles. It is legitimate to wonder if some ways to quantize spacetime with deformed spacetime symmetries could produce simpler descriptions of interactions among particles. If such an aspect of simplicity was found for a certain scheme of spacetime quantization, it might provide encouragement for studies of other aspects of that quantum spacetime.

We conjecture that it should also be possible to apply to other Lie-algebra noncommutative spacetimes (see Section 1) the lessons learned here within the spatial 2D spatial κ-Minkowski toy model. Instead, our approach does not apply directly to other types of spacetime noncommutativity, such as the canonical noncommutativity relevant for string theory [2,31], but our study raises issues which might also deserve investigation in studies of other attempted formulations of quantum spacetime.

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