Arithmetical, geometrical, and categorical forays into particle physics

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Declaration

This Thesis is based on original research carried out while the author was a graduate student in the Department of Physics, University of Cambridge, from October 2018 to May 2021.

The material in Chapters 3, 4, 5, Section 1.1, and Appendix A is based on work done with Ben Allanach and Ben Gripaios published in Refs. [17, 18, 16, 19]. The material in Chapter 6, Sections 1.2 and 2.3, and Appendix C is based on work done with Joe Davighi and Ben Gripaios published in Ref. [58]. The material in Chapter 7, Sections 1.3 and 2.4, and Appendix D is based on work done with Ben Gripaios published in Ref. [88].

The material in Ref. [58] (Chapter 6, Sections 1.2 and 2.3, and Appendix C) has previously been submitted for a degree of doctor of philosophy by Joe Davighi at the University of Cambridge. No other part of this work has been submitted, or is being concurrently submitted, for a degree or other qualification at the University of Cambridge or any other university or similar institution.

This Thesis contains fewer than 60,000 words including the abstract, tables, footnotes and appendices, but excluding references.

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Abstract

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This Thesis will focus on three different forays into particle physics using pure mathematics.

Our first foray studies anomaly free gauge algebras. Using geometric methods, we reproduce a solution given to the anomaly cancellation conditions associated with a pure $\mathfrak{u}(1)$-gauge theory in Costa et al. [Phys. Rev. Lett.123 (2019) 151601]. Using similar techniques, the general solution to the anomaly cancellation conditions associated with a $\mathfrak{u}(1)$-extension of the Standard Model gauge algebra when the chiral fermion content is that of the Standard Model plus three singlets, is found for the first time. For the same Standard Model set up, a computational approach is used to catalogue all semisimple extensions.

The second foray studies quantum mechanics in magnetic backgrounds. For such problems, it is known that a global lagrangian need not exist, and even if it does, it may shift by a total derivative under the action of the symmetry group. These two facts pose an obstruction to the standard techniques of harmonic analysis. We show that these obstructions can be overcome by passing to a redundant description with the particle moving on a $U(1)$-principal bundle of the original configuration space, and the symmetry replaced with an associated $U(1)$-central extension. We demonstrate the power of this technique using a series of examples.

For the final foray we look at the inverse Higgs phenomenon which is important for the study of Goldstone bosons. We take holonomic constraints as our starting point giving them a categoric construction. The dual of this construction leads to a new type of constraint we call a coholonomic constraint. Coholonomic constraints like holonomic ones, are equivalent to unconstrained systems. We show that every instance of the inverse Higgs phenomenon in the literature can be treated as a coholonomic constraint, or a slight generalisation thereof we call a comeronomic constraint. In this framework, the essential Goldstone bosons of the inverse Higgs phenomenon correspond to the degrees of freedom of the unconstrained system.

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To Ron Smith and Stan Tooby,
neither of whom saw the completion of this journey...

Table of contents

List of figures

List of tables

Nomenclature

Acronyms / Abbreviations

ACCs	Anomaly cancellation conditions
CDF	Costa, Dobrescu and Fox (authors of [51])
irrep	Irreducible representation
ODE	Ordinary differential equation
PDE	Partial differential equation
RHN	Right-handed neutrinos
SE	Schrödinger equation
SM	Standard Model
unirrep	Unitary irreducible representations

Chapter 1

Introduction

The current state of particle physics has at its heart the Standard Model (SM) containing every interaction (bar gravity) and particle we know to exist with some degree of confidence. However, some experimental results disagree with the predictions of the SM indicating its incompleteness. These include the recent (at the time of writing) B-anomaly results in the quantity R_K from the LHCb experiment [3] and the muon $g - 2$ results from Fermilab [6].

Several complementary approaches attempt to aid in the understanding of why there are discrepancies between the SM and experiments. These aids include model building techniques and effective field theory methods. In this Thesis, we take another approach; the *modus operandi* being to study a facet of particle physics (directly or indirectly related to the SM) formally using techniques from (pure) mathematics. The critic may ask why such an approach is useful? The reason is simple; understanding concepts formally allows one to identify assumptions made, identify possible generalisations, and derive new results aiding model building or more generic phenomenological studies. Furthermore, the use of pure mathematics here allows new, currently unexplored, perspectives to be found.

In order to make three different forays into particle physics within the context of the above *modus operandi*, four areas of mathematics will be used within this Thesis. These areas of mathematics are group theory, differential geometry, arithmetic geometry and category theory.

The reader should not be surprised by the presence of group theory, as it is very much intertwined with the study of particle physics. After all, a particle in the SM is just a field living in an irreducible representation of the SM gauge group. The same gauge group determines what is and is not forbidden in Gell-Mann's totalitarian principle; 'Everything not forbidden is compulsory'. Examples from the Author's work excluded from this Thesis and demonstrating traditional uses of group theory in particle physics are [59, 49].

Also, the reader may not be surprised by our inclusion of differential geometry, which has a similar intertwining with particle physics as group theory. For instance, the formal way to study a gauge theory is using a principle bundle over the space-time of the theory. Different connections of this principal bundle correspond to different values of the gauge field, and the concept of gauge symmetry (or more properly 'gauge redundancy') results from different choices of sections of this principle bundle. Thus, if we study gauge theories properly we need to use differential geometry. Our use of differential geometry will, however, go beyond its use for gauge theories.

The presence of arithmetic geometry and category theory may however surprise (and mildly annoy) the reader, which are both more usually restricted to the ivory towers of pure mathematicians and beyond the usual particle physicists curriculum. The despondent reader is reassured that many benefits arise from their use. Arithmetic geometry concerns itself with finding rational solutions to polynomial equations, and is related to number theory and algebraic geometry (the more generic study of zeros of polynomials). Whereas category theory concerns itself with objects (*e.g.* sets) and morphisms between these objects (*e.g.* functions) and the properties thereof.

As previously stated, these four branches of mathematics will be used to make three different forays into particle physics. The first foray uses group theory and arithmetic geometry specifically utilising techniques in projective (algebraic) geometry to consider anomaly free extensions of gauge theories. (The last part of this first foray will also use some non-technical category theory.) The second foray uses group theory plus differential geometry to solve quantum mechanical problems in the presence of magnetic backgrounds. The third foray also uses group theory, in addition to differential geometry, and category theory to study constraints in quantum field theories with specific emphasis on the inverse Higgs phenomenon. The three forays and the areas of mathematics they relate to are given in Fig. 1.1.

Let us introduce the three forays in more detail:

1.1 Anomaly free algebras

When considering a gauge theory, there are two bits of data of particular interest; the gauge algebra and the representations of particles under this gauge algebra. It is a requirement that the gauge theory is invariant under gauge transformations. If the theory is gravitational, it must also be invariant under orientation preserving

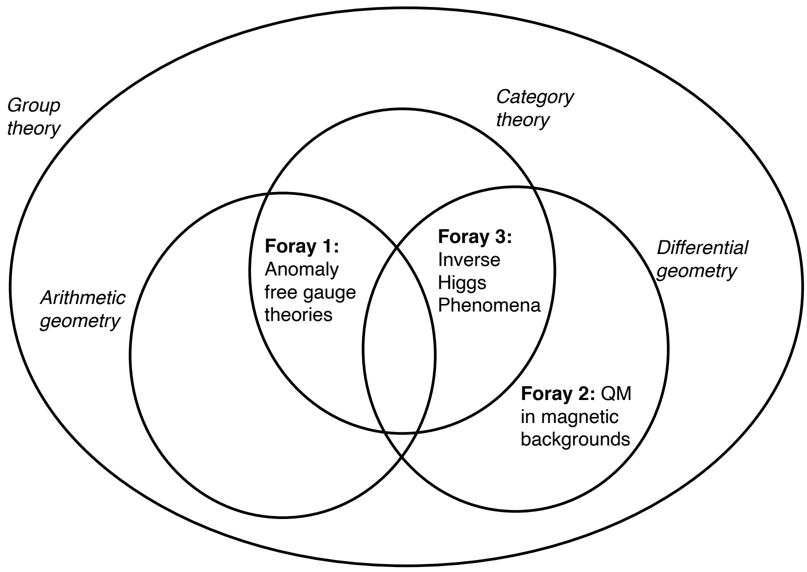

Fig. 1.1 The forays into particle physics in this Thesis, and their different uses of mathematics.

diffeomorphisms and the local Lorentz symmetry. Specifically it is the quantum partition function that must be invariant under such transformations. Roughly, if the fermionic path integral 'measure' shifts under these transformations, and this shift is not counteracted by another mechanism, it is said that we have an 'anomaly' in the theory.

Anomalies in a theory can be split into two classes: global and local. In this Thesis, we only consider gauge algebras, and not their underlying gauge groups. Statements about global anomalies generically rely on the full gauge group, and thus will not be discussed here (see [57]). Local anomalies can be studied using index theorems in topology, and are related to the famous triangle diagrams [26] (see also [121, 33]). It is widely accepted that a ($4d$) theory is free of local anomalies if for every X in the image of the gauge algebra fermionic representation, X and X^3 are traceless. These conditions will be called the anomaly cancellation conditions (ACCs).[1]

The simplest possible gauge theory to ask for anomaly cancellation is a pure $\mathfrak{u}(1)$-theory. If there are n-Weyl fermions, all taken as left-handed and with charges z_i, then

[1]Even if the ACCs are not satisfied, it is possible that the theory is absent of local anomalies via different mechanisms, for example, the Green-Schwartz mechanism [84]. Throughout we will assume anomaly free means the ACCs are satisfied.

the ACCs are

$$\sum_{i=1}^{n} z_i = 0, \tag{1.1}$$

$$\sum_{i=1}^{n} z_i^3 = 0. \tag{1.2}$$

The assumption of compactness, so that our gauge group is $U(1)$, implies commensurate charges, z_i's *i.e.* rational ratios of each other. (When referring to $\mathfrak{u}(1)$-gauge theories compactness will be assumed.)

Solutions to the ACCs for a pure $\mathfrak{u}(1)$-gauge theory were first found in Ref. [51]. Our first step (which is derived from [17]) in this Thesis is to provide a geometric interpretation to this solution, using techniques from arithmetic geometry.[2] This geometric interpretation allows the full solution to be written down with ease, and similar solutions to be found.[3]

By using arithmetic geometry we can also tackle more complicated gauge theories. Extensions of the SM gauge algebra by $\mathfrak{u}(1)$ are perhaps the most interesting from the point of view of phenomenology.

These are phenomenologically interesting since such a $\mathfrak{u}(1)$ may be related to a gauged, spontaneously broken $U(1)$ subgroup. This leads to a massive SM-neutral spin-1 particle corresponding to a Z'. Models with Z's have been studied exhaustively in the literature: to explain dark matter [130, 120, 11, 127, 126, 128, 8, 129], the anomalous magnetic moment of the muon [91], axions [29] or leptogenesis [47], proton stabilisation [44], supersymmetry breaking [95], fermion masses and mixing (via the Froggatt-Nielsen mechanism) [75], and, most recently [78, 39, 41, 24, 40, 54, 53, 27, 11, 55, 45, 85, 25, 70, 48, 30, 35, 36, 100, 21, 22, 142, 46, 69, 77, 32, 5, 31, 20, 13, 9, 64, 79, 96, 12, 65, 10, 23, 42], apparent lepton family non-universality (FNU) in certain rare neutral current B-meson decays [1, 2, 93].

The study of $\mathfrak{u}(1)$ extensions of the SM are also of interest since more generic extensions tend to have $\mathfrak{u}(1)$ extensions as subalgebras.

To consider the ACCs, we take 3 SM families of quarks and leptons, together with 3 right-handed neutrinos, whose charges we label $Q_i, U_i, D_i, L_i, E_i, N_i$ respectively, with $i \in \{1, 2, 3\}$. We consider this to be the most plausible scenario with regard to

[2] Arithmetic geometry has previously been applied to anomaly equations in [109], which showed that for the SM hypercharge ACCs, the assumption of commensurate charges means that gauge anomaly cancellation implies gravitational anomaly cancellation.

[3] Further developments related to $\mathfrak{u}(1)$ anomaly cancellation can be found in [52, 63].

aesthetics and observation (*e.g.* the fit to neutrino oscillation data). The ACCs become

$$0 = \sum_{i=1}^{3} (6Q_i + 3U_i + 3D_i + 2L_i + E_i + N_i)\,, \tag{1.3a}$$

$$0 = \sum_{i=1}^{3} (3Q_i + L_i)\,, \tag{1.3b}$$

$$0 = \sum_{i=1}^{3} (2Q_i + U_i + D_i)\,, \tag{1.3c}$$

$$0 = \sum_{i=1}^{3} (Q_i + 8U_i + 2D_i + 3L_i + 6E_i)\,, \tag{1.3d}$$

$$0 = \sum_{i=1}^{3} \left(Q_i^2 - 2U_i^2 + D_i^2 - L_i^2 + E_i^2\right)\,, \tag{1.3e}$$

$$0 = \sum_{i=1}^{3} \left(6Q_i^3 + 3U_i^3 + 3D_i^3 + 2L_i^3 + E_i^3 + N_i^3\right)\,. \tag{1.3f}$$

where $Q_i, U_i, D_i, L_i, E_i, N_i$ are all commensurate assuming the compactness of the gauge group.

A numerical scan of the solutions to these equations was given in [14] up to a maximum charge of 10. But, before our paper [16] (the subject of the second part of our first foray), no generic solution to these equations was known.

The holy grail is to find all anomaly free extensions of the SM for a fixed fermionic particle content. Without assumptions, like the restriction to $\mathfrak{u}(1)$-extensions above, however, this task seems unsurmountable. Instead of considering $\mathfrak{u}(1)$-extensions, one may consider semisimple extensions. Finding all of these is a feasible task, not least because, up to a physical equivalence, the list of such extensions is guaranteed to be finite. The simple ratios of the SM hypercharge also hints at the phenomenological relevance of such extensions.

The third and final part of our first foray (based on [19]) is concerned with the above problem. To be specific, let $\mathfrak{sm}$ be the SM gauge algebra, and $\gamma : \mathfrak{sm} \to \mathfrak{su}(48)$ the embedding of the SM into $\mathfrak{su}(48)$ via the fermion representation. We will find all commuting diagrams of the form

$$\begin{array}{ccccc} & & \mathfrak{g} & & \\ & \overset{\alpha}{\nearrow} & & \overset{\beta}{\searrow} & \\ \mathfrak{sm} & & \xrightarrow{\gamma} & & \mathfrak{su}(48) \end{array} \tag{1.4}$$

up to some notion of physical equivalence (to be defined). Here $\mathfrak{g}$ is semisimple, and α and β are embeddings of Lie algebras. Physically $\mathfrak{g}$ would correspond to a GUT (or more properly a PUT, for *petite unified theory*). The map α tells us how the SM fits into the GUT and β tells us how the GUT acts on the fermionic particle content. The embedding β is taken to be anomaly free. We will show that are 340 such diagrams which we find with a computer program, 26 of which were maximal and 6 minimal.

1.2 Quantum mechanics in magnetic backgrounds

Consider a particle moving on a smooth, connected, manifold M in the presence of some background magnetic field. Suppose furthermore that the dynamics is invariant under some, connected, Lie group G of global symmetries acting smoothly on M.

The study of the quantum mechanics of such a system, the subject of our second foray, which is based on [58], is complicated by two well-known facts. The first complication is that it is, in general, not possible to write down a term in the lagrangian representing the magnetic field that is valid globally on M. Instead, the best that one can do is to cover M by overlapping patches and to use multiple lagrangians, each of which is valid only locally on some patch. The most famous example, due to Dirac [61] and solved by Tamm [141] (see also [151, 150]), is given by the motion of an electrically-charged particle in the presence of a magnetic monopole, but we will see that there exists an example that is arguably even simpler (and certainly more prevalent in everyday life!), given by the motion of a rigid body which happens to be a fermion.

This latter example is interesting for another reason, which is that it shows that our set-up includes systems in which there is no apparent magnetic field, but rather a vector potential is being used to encode a global topological effect – spin, in the case at hand – in a manifestly local way. Thus, we will be able to write a local term in the lagrangian that accounts for the extra factor of -1 that the state of the fermion acquires when it undergoes a complete rotation, rather than arbitrarily assigning it by hand, as is usually done. This is desirable, given our prejudice that physics should be local.

The second complication is that the corresponding lagrangian (or lagrangians) will not be invariant under the action of G, but rather will shift by a total derivative. Perhaps the simplest example, made famous by Landau [106], is given by the motion of a particle in a plane in the presence of a uniform magnetic field, where there is no choice of gauge such that the lagrangian is invariant under translations in more than one direction.

At the classical level, neither of these complications causes any problems, since they disappear once we pass from the lagrangian to the classical equations of motion. Indeed, the equations of motion are both globally valid and invariant (or rather covariant) under G. Thus, we can attempt to solve for the classical dynamics using our usual arsenal of techniques. But this is not the case at the quantum level. There, our usual technique is to convert the hamiltonian into an operator on $L^2(M)$ and to exploit the conserved charges corresponding to G to solve, at least partially, the resulting Schrödinger equation. Here though, we do not have a unique hamiltonian, but rather several; even if we did have a unique hamiltonian, we would, in general, find that the naïve operators corresponding to the conserved charges of G do not commute with it. The last problem is often remedied by redefining the conserved charges, but then one finds that the new charges do not form a Lie algebra, unless we add further charges.

These two complications are apparently unrelated, at least as we have presented them. But they are related in the sense that neither could occur in the first place, were it not for a basic tenet of quantum mechanics, namely that physical states are represented by rays in a Hilbert space. Thus, the overall phase of a vector in a Hilbert space is not physical. This is what makes it possible, ultimately, to resolve the apparent paradox that, at a point in M where two patches overlap, we have multiple, distinct lagrangians, but each of them gives rise to the same physics. Similarly, it allows us to absorb extra phases that arise from boundary contributions in the path integral under a G transformation, when the lagrangian is not strictly invariant.

In this foray we show that, by exploiting this basic property, one can formulate and solve (or at least, attempt to solve) such quantum systems in a unified way, using methods from harmonic analysis. In a nutshell, the idea is as follows. A magnetic field defines a connection on a $U(1)$-principal bundle P over M. From G (which acts on M), we can construct a central extension $\tilde{G}$ of G by $U(1)$ (which depends on the connection and on P, and which acts on P). We reformulate the original dynamical system on M in terms of an equivalent system (with a redundant degree of freedom) of a particle moving on P. This reformulation allows us to circumvent both of the complications discussed above: not only do we have a unique, globally-valid, local lagrangian on P, but also the Hilbert space carries a *bona fide* representation of $\tilde{G}$ (in contrast to the original theory, in which the Hilbert space carries a projective representation of G which commutes with the hamiltonain, corresponding to the fact that a quantum state is represented by a ray in a Hilbert space). As a result, we can attempt a solution using harmonic analysis, with respect to the group $\tilde{G}$.

It should be remarked that neither the formulation nor the method of solution that we describe here can really be considered new. The formulation via central extensions has appeared in a number of places in the literature, mainly with applications to symplectic geometry and geometric quantisation (see *e.g.*, [113, 144]) and the use of harmonic analysis to solve quantum systems in the absence of magnetic fields (and hence without the complications described above) was described in [87]. What is new, we hope, is the synthesis of these ideas, which leads to a uniform approach to solving quantum-mechanical systems, including cases with magnetic fields (a type of topological interaction due to its independence from the worldvolume metric) or other non-trivial topological terms.[4]

The methods we present are most powerful in cases where G acts transitively on M (meaning that any point in M can be reached from any other via the action of G) corresponding to a special case ($0+1$ spacetime dimensions) of the usual non-linear sigma model of quantum field theory on a homogeneous space G/H. The constraint that G acts transitively is a strong one; it implies, in particular, that any potential term in the lagrangian must be a constant. We thus have a 'free' particle, in the sense that, in the absence of the magnetic field (and ignoring possible higher-derivative terms), the classical trajectories are given by the geodesics of some G-invariant metric. Despite the strong restrictions, one finds that a large class of interesting quantum mechanical models fall into this class and can be solved in this way. Examples discussed in the sequel include the systems considered by Landau (which, in contrast with Landau, we solve by keeping a transitive group of symmetries - either translations or the full Euclidean group - manifest) and Dirac (where we constrain the particle to move on the surface of a sphere, so that the rotation group acts transitively).

In cases where G does not act transitively, the methods typically provide only a partial solution, in that they allow us to reduce the Schrödinger equation to one on

[4]We remark in passing that some of the systems we study are superintegrable, offering a complementary way of understanding their exact solvability.

the space of orbits of G. But even here we find interesting examples where a complete solution is possible.

1.3 The inverse Higgs phenomenon

In our third foray, which is based on [88], we want to describe constraints in field theories with symmetry in a general way, using the language of differential geometry. Of particular interest is the special case in which the symmetry group acts transitively on the space carrying the fields. This includes theories of Goldstone bosons exhibiting the so-called 'inverse Higgs phenomenon', in which the presence of constraints involving derivatives of the fields implies that Goldstone's theorem no longer holds, leading to richer possibilities for dynamics [94]. Such constraints are generic, due to the simple fact that no symmetry can act transitively on the fields and their derivatives, once we include enough derivatives.

A well-known example of the inverse Higgs phenomenon are the phonons occurring in crystalline media. Here, we can roughly think of the breaking pattern as breaking nine symmetry generators; the rotations and translations of the crystal, and space-time boosts. Thus from Goldstones theorem, we would expect nine Goldstone bosons, but in actual fact we only have three. This is due to the fact that we can write invariant equations (or 'constraints') which express the Goldstone bosons of the rotations and boosts, in terms of the derivatives of the three Goldstone bosons of translations. This removes the rotation, and boost Goldstone bosons from consideration, leaving only three essential Goldstone bosons - the phonons. This removal of Goldstone bosons is an instance of the 'inverse Higgs phenomenon'.

Another example of this phenomenon is the non-relativistic particle in which both position and velocity are *a priori* treated as separate Goldstone bosons. The symmetry group then allows you to relate the Goldstone bosons of velocity to the derivatives of the Goldstone bosons of position (in the usual way). This example of the inverse Higgs phenomenon allows you to derive the kinetic energy of a non-relativistic particle via a topological term.

Our main motivation for the foray is not the pursuit of generality for its own sake, but rather to show that many of the apparently *ad hoc* constructions existing in the literature on the inverse Higgs phenomenon are, in fact, very natural, when viewed with a sufficient level of abstraction. Doing so also makes it easier to see which of the various assumptions made are necessary for physical consistency and which are merely convenient.

Perhaps the most important insight we obtain is the following: In the special case where the symmetry acts transitively, any constraint is necessarily nonholonomic. Such constraints are notoriously difficult to deal with in general, even in classical mechanics (an infamous example being the motion of a bicycle). By suitably reformulating the more familiar notion of a holonomic constraint in our framework, we will see that there exists a special class of nonholonomic constraints that are dual (in the sense of category theory) to holonomic constraints, which we thus call *co*holonomic constraints. The duality is somewhat fiddly at the level of the aforementioned 'space carrying the fields' (which is, mathematically, a fibred manifold), but it reduces to the following simple statement at the level of the kinematic degrees of freedom of the physical theory: a system with a holonomic constraint is equivalent to an unconstrained system defined on a subobject, while a system with a coholonomic constraint is equivalent to an unconstrained system defined on a quotient object. The first part of the statement (which is, mathematically, a theorem about sheaves) corresponds, at an elementary level, to the notion of 'solving the constraint to eliminate redundant degrees of freedom', while its dual corresponds to the familiar notion that one can consider just 'essential Goldstone bosons'. Because theories constrained in such ways are kinematically equivalent to unconstrained ones, no new issues of physical consistency arise and no new difficulties are encountered in formulating and studying dynamics (unlike for bicycle motion).

Remarkably, it turns out that every example of the inverse Higgs phenomenon that we have been able to find in the literature involves the dual of either a holonomic constraint or, in just a few cases, of a slight generalisation thereof, which we call (co)meronomic constraints. Systems with (co)meronomic constraints are not obviously equivalent to unconstrained systems and so we must worry about issues of physical consistency. Here, we content ourselves with establishing just two basic consistency properties enjoyed by such constrained systems, namely that they satisfy basic locality requirements and that local degrees of freedom exist at every spacetime point (in the language of sheaf theory, we require that the degrees of freedom form a sheaf whose stalks are not empty).

To describe the other features of our approach, it is perhaps easiest to sketch the basic ingredients. Our foray will begin by describing constraints in field theories without regard to symmetry. Rather than using local coordinates, as in the physics literature, we use a coordinate free approach, which not only allows us to take global considerations into account, but also clarifies exactly which mathematical structures are being made use of.

In the most basic examples of field theories, the fields are smooth maps from some 'spacetime' manifold to some 'target' manifold, so the 'space carrying the fields' can be taken to be simply the product of the two manifolds. We generalise by replacing this product by a fibred manifold. Fibred manifolds are the most general objects that (locally) admit smooth sections, which can serve as the local degrees of freedom (*i.e.* the 'fields' of the field theory). Fibred manifolds generalise the more familiar notion of fibre bundles, in that over each point in spacetime there is a well-defined fibre. But unlike fibre bundles, the fibres over different points in spacetime may not even have the same homotopy type, let alone diffeomorphism class, so there is no meaningful notion, even locally, of a 'target space'.

Fibred manifolds form a category and we will see that many of the constructions required for dynamics are conveniently understood using the language of category theory. For example, there is a functor – the rth-jet functor – which sends a fibred manifold to its rth-jet manifold, which is itself a fibred manifold encoding the notion of the derivatives of sections of order up to r, in a coordinate-free way. Consistent dynamical constraints may be described as certain subobjects of the jet manifold, and we show how holonomic and meronomic constraints (and their duals) can be understood in this way. Consistency, for us, amounts to insisting that the sections that are compatible with the constraint form a sheaf (such that locality is obeyed) whose stalks are non-empty (meaning that local degrees of freedom exist at every point in spacetime).

We then introduce the notion of symmetry, via a Lie group action on the fibred manifold. A great deal of simplification arises in the special case where the action is transitive and equivariant with respect to the projection onto spacetime, which we call a fibrewise action (an example is the galilean symmetry of a non-relativistic particle). In such a case, both the fibred manifold and its jet manifolds take the form of fibre bundles associated to the L-principal bundle $G \to G/L$, for Lie groups $L \subset G$. The category of such bundles (called homogeneous bundles in the mathematical literature) is equivalent to the category of manifolds equipped with an action of the group L. This simple statement extends and makes rigorous the physicist's vague notion (put forward in [50, 43]) that 'in studying sigma models based on a target space G/L, L invariance implies G invariance'. It also shows that some constructions used in the literature on sigma models, such as connections and vielbeins, are unnecessary. We describe a number of examples with group actions of this type.

More generally, it is an unavoidable fact that starting from a group action on a fibred manifold, in general only a partial group action is induced on its jet manifolds (the Poincaré symmetry of a relativistic particle is an example). It therefore makes sense to work with partial group actions from the off in the general case, which we do. Though the resulting mathematics is technically rather cumbersome, the results are conceptually straightforward, thanks to the category-theoretic nature of our earlier constructions. We also discuss a number of examples with partial group actions.

This foray will be purely at the level of kinematics; in particular, we do not discuss how dynamics can be specified in the form of an action (in the physics sense of the word). In all examples we study, this is, however, straightforward: the action is determined by choosing a differential form on the constraint manifold (which is a submanifold of the rth jet manifold) whose degree coincides with the dimension of the spacetime manifold. The action is then evaluated on a section (*i.e.* a field) by pulling back the differential form along the section and integrating over spacetime. One complication is that many such forms yield actions that are trivial in the sense that they are either identically zero or do not contribute to the equations of motion. In the presence of symmetry, this makes the classification of invariant dynamical theories tricky, because the set of such theories includes those whose action is not invariant under the group transformations, but rather shifts by such a trivial action.

1.4 Thesis layout

Before delving into the finer details of each of the different forays, Chapter 2 will introduce some of the more elementary mathematical constructions used in this Thesis. This Chapter may be skipped without loss of continuity.

Chapters 3, 4 and 5 are related to our first foray, and are respectively extracted from the papers [17, 16, 19]. The first of these Chapters will look at a geometric solution to the ACCs of a pure $\mathfrak{u}(1)$-gauge theory. Associated with this Chapter is Appendix A. Here we will prove a theorem stated in the text, as well as providing an alternative solution to the ACCs for the case with an even number of Weyl fermions. Chapter 4 contains the first ever solution to the ACCs associated with $\mathfrak{u}(1)$-extensions of the SM, with three right handed neutrinos. Chapter 5 is slightly distinct from the other two dedicated to this foray. Here we will find, using a brute force approach, all anomaly free semisimple extensions of the SM contained in $\mathfrak{su}(48)$. Appendix B will look at the more formal aspects of this search.

The second foray will be restricted to a single Chapter, Chapter 6. Here we will discuss quantum mechanics in magnetic backgrounds. This Chapter is extracted from the paper [58]. Part of our discussion will use harmonic analysis, the rudiments of which we describe in Appendix C.

The third foray is again restricted to a single Chapter, Chapter 7, and is extracted from the paper [88]. We will discuss in this Chapter how category theory can be used to form constraints dual to holonomic ones, and how these, after a slight generalisation, contain every example of the inverse Higgs phenomenon we could find. This Chapter is the heaviest mathematically, thus we will do two things to ease the burden on the reader: Firstly, we will adopt the mathematical 'Theorem', 'Lemma', 'Definition' format. Elsewhere in this Thesis, bar one exception in Chapter 3, theorems, lemmas and definitions have been placed within the flow of the main text. The second thing we have done to ease the burden, is to place many of the more involved proofs into Appendix D.

After completing all our forays, this Thesis will conclude in Chapter 8, where we will review what has been discussed in this Thesis. We will also discuss possible future directions of study.

Throughout this Thesis, when a term is first introduced it will be placed in *italics*. All such definitions appear in the index of this Thesis, with reference to the page of first introduction. A word that may cause specific confusion throughout this Thesis is 'geometry'. This is due to its presence in both the algebric and differential guise. It is hoped it will be clear from context which one we are referring to, but as a general rule; 'geometry' in Chapters 3 and 4 will refer to algebraic geometry, whilst elsewhere to differential geometry.

Chapter 2

Mathematical prerequisites

This Chapter contains basic descriptions of the mathematical prerequisites for this Thesis.

We start in §2.1 by explaining the algebraic concepts including a definition of projective spaces which will be used in Chapters 3 and 4. In §2.2 we look at Lie algebras, going into detail defining root systems and studying embeddings. A detailed understanding of these points is not so important for the rest of this Thesis, but are included for completeness and as an aid to Chapter 5. The next Section, §2.3, will look at ideas in topology and differential geometry relevant to Chapters 6 and 7. The last Section of this Chapter, §2.4, is devoted to category theory. This is of primary use in Chapter 7 although it underlies many of the concepts used in this Thesis.

2.1 Algebra

The material in this Section forms the foundation for many definitions which follow.

Perhaps the proper place to start is by defining a set. We content ourselves with a rough definition: A *set* is a collection of objects, where repetitions do not occur and order does not matter.

We start our more formal definitions with that of a field. Common fields include: the rational numbers, $\mathbb{Q}$; the real numbers, $\mathbb{R}$; and the complex numbers, $\mathbb{C}$. Formally, a *field* is a set $\mathbb{F}$, with two operators, addition $+ : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$ and multiplication $\cdot : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$ such that:

1. $(\mathbb{F}, +)$ forms an abelian group,

2. $(\mathbb{F}\backslash\{0\}, \cdot)$ forms an abelian group where 0 is the identity of $(\mathbb{F}, +)$,

3. $a \cdot (b + c) = (b + c) \cdot a = a \cdot b + a \cdot c$ for all $a, b, c \in \mathbb{F}$.

A *vector space* over a field $\mathbb{F}$ consists of a set V and two operators, addition $+ : V \times V \to V$, and scalar multiplication $\mathbb{F} \times V \to V$, such that:

1. $(V, +)$ is an abelian group,

2. $a(\vec{v} + \vec{w}) = a\vec{v} + a\vec{w}$ for all $a \in \mathbb{F}$, and $\vec{v}, \vec{w} \in \mathbb{F}$,

3. $(a + b)\vec{v} = a\vec{v} + b\vec{v}$ for all $a, b \in \mathbb{F}$, and $\vec{v} \in V$,

4. $(ab)\vec{v} = a(b\vec{v})$ for all $a, b \in \mathbb{F}$, and $\vec{v} \in V$,

5. $1\vec{v} = \vec{v}$ for all $\vec{v} \in V$ where 1 is the identity of the group $(\mathbb{F}/\{0\}, \cdot)$.

An *affine space* over a vector space V is a set A, and a operator $+ : A \times V \to V$, such that

1. $a + 0 = a$ for all $a \in A$ where 0 is the identity of $(V, +)$,

2. $(a + \vec{v}) + \vec{w} = a + (\vec{v} + \vec{w})$ for all $a \in A$ and $\vec{v}, \vec{w} \in V$,

3. the map $V \to A : \vec{v} \mapsto a + \vec{v}$ is a bijection for all $a \in A$.

Intuitively, affine spaces can be thought of as vector spaces where one forgets about the zero vector.

Given a field $\mathbb{F}$, the space $\mathbb{F}^n$ can be given the structure of a vector space over $\mathbb{F}$. The *projective space* $\mathrm{P}\mathbb{F}^{n-1}$ (note the change in superscript) is defined to be the set of all lines through the origin of $\mathbb{F}^n$. Formally, if $\vec{0}$ is the zero vector in $\mathbb{F}^n$, then $\mathrm{P}\mathbb{F}^{n-1} = (\mathbb{F}^n \backslash \{\vec{0}\})/\sim$, where $\sim$ is the equivalence relation defined by the condition that $\vec{v}_1 \sim \vec{v}_2$ with $\vec{v}_1, \vec{v}_2 \in \mathbb{F}^n$ if and only if a $\lambda \in \mathbb{F}$ exists such that $\vec{v}_1 = \lambda \vec{v}_2$. We will be most interested in the projective space $\mathrm{P}\mathbb{Q}^{n-1}$ for some n.

2.2 Lie algebras

We now introduce the basics of the theory of Lie algebras. Suitable, more detailed, sources for the material in this Section include [66, 60, 110, 152, 132, 97, 76], which form the main references for the material here.

A *Lie algebra* is a pair $(\mathfrak{g}, [\cdot, \cdot])$ of a vector space $\mathfrak{g}$ and an anti-symmetric, bilinear operator (called the *Lie bracket*) $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying the Jacobi identity.

A Lie algebra is *simple* if and only if it has no non-trivial ideals. A Lie algebra is *semisimple* if and only if it is the direct sum of simple ideals. A Lie algebra is *reductive* if and only if it is the direct sum of a semisimple Lie algebra and an abelian Lie algebra. We have that

$$\begin{matrix}\text{simple} \\ \text{algebras}\end{matrix} \subset \begin{matrix}\text{semisimple} \\ \text{algebras}\end{matrix} \subset \begin{matrix}\text{reductive} \\ \text{algebras}\end{matrix}$$

As an example, both $\mathfrak{su}(2)$ and $\mathfrak{su}(3)$ are simple. The algebra $\mathfrak{su}(2) \oplus \mathfrak{su}(3)$ is semisimple but not simple. The algebra $\mathfrak{su}(2) \oplus \mathfrak{su}(3) \oplus \mathfrak{u}(1)$ is reductive but not semisimple. We will use the phrase *non-simple* to refer to a Lie algebra which is semisimple but not simple.

A Lie algebra is said to be *compact* if it is the Lie algebra of a compact group. The main effect of this is to change the representations we allow for the Lie algebra.

A *Cartan subalgebra*, $\mathfrak{h}$, of $\mathfrak{g}$ is a self-normalising nilpotent subalgebra. The Cartan subalgebra is not generically unique, however, if $\mathfrak{g}$ is defined on an algebraically closed field of characteristic 0, then for any two Cartan subalgebras $\mathfrak{h}_1$ and $\mathfrak{h}_2$ of $\mathfrak{g}$, there exists an inner automorphism (which we shall shortly define formally) u such that $u(\mathfrak{h}_1) = \mathfrak{h}_2$ (see *e.g.* [60, Th. 3.5.1]). From here out, we consider all our algebras to be complexified, and therefore over the algebraically closed field $\mathbb{C}$ of characteristic 0.

2.2.1 Root systems

We now introduce the notion of a root system, in an abstract sense, and then will return to Lie algebras with root systems at our disposal. Given a real vector space E with inner product $(\cdot, \cdot)$, a (crystallographic) *root system* is a set of non-zero vectors, Φ, in E satisfying the following conditions:

1. the vectors in Φ span E,
2. if $\alpha \in \Phi$, and $k \in \mathbb{R}$, then $k\alpha \in \Phi$ if and only if $k = \pm 1$,
3. for $\alpha \in \Phi$, defining $s_\alpha : E \to E : \beta \mapsto \beta - 2\frac{(\alpha,\beta)}{(\alpha,\alpha)}\alpha$, then Φ is closed under the action of s_α,
4. for $\alpha, \beta \in \Phi$ then $2(\alpha, \beta)/(\alpha, \alpha) \in \mathbb{Z}$.

Two root systems, Φ in E and Φ' in E' are said to be isomorphic, if there exists a linear transformation $\phi : E \to E'$ such that $2(\phi(\alpha), \phi(\beta))/(\phi(\alpha), \phi(\alpha)) = 2(\alpha, \beta)/(\alpha, \alpha)$.

A root system is said to be *irreducible* if it can be written as the orthogonal disjoint union of two other root systems Φ_1, Φ_2. Explicitly, an orthogonal disjoint union is a union $\Phi_1 \cup \Phi_2$ with $\Phi_1 \cap \Phi_2 = \emptyset$ and $(\alpha, \beta) = 0$ for all $\alpha \in \Phi_1$ and $\beta \in \Phi_2$.

Associated with the root system Φ are the orthogonal transformations of E, we denoted s_α above. The subgroup of all orthogonal transformations of E generated by the s_α's for all $\alpha \in \Phi$ is called the *Weyl group*.

A subset Φ^+ of Φ is called a set of *positive roots* if it satisfies the following conditions:

1. for each $\alpha \in \Phi$ exactly one of the pair $\{\alpha, -\alpha\}$ is in Φ^+,

2. if $\alpha, \beta \in \Phi^+$ and $\alpha + \beta \in \Phi$ then $\alpha + \beta \in \Phi^+$.

Given a Φ there are many Φ^+, but they are all related by the action of the Weyl group. A root $\alpha \in \Phi^+$ is called a *simple root* if there does not exist a $\beta, \gamma \in \Phi^+$ such that $\alpha = \beta + \gamma$. The set of simple roots will be denoted Δ. The set of vectors Δ forms a basis of E.

Given Δ, one can define its *Dynkin diagram* as the graph with a node for each $\alpha \in \Delta$, and

$$\frac{4(\alpha, \beta)^2}{(\alpha, \alpha)(\beta, \beta)} \tag{2.1}$$

edges between the node specified by $\alpha, \beta \in \Delta$. If $(\beta, \beta) > (\alpha, \alpha)$ we direct the edge from α to β. Two root systems are isomorphic if and only if they have identical Dynkin diagrams. Fig. 2.1 shows the classification of Dynkin diagrams of irreducible root systems.

Each root system has a corresponding *dual root system*. For $\alpha \in \Phi$ let $\alpha^\vee = 2\alpha/(\alpha, \alpha)$. The dual root system $\Phi^\vee$ is formed by the set of elements $\alpha^\vee$. The set, $\Phi^\vee$ is indeed a root system of the inner product space E. The set $(\Phi^+)^\vee$ is a set of positive roots, and $\Delta^\vee$ the corresponding set of simple roots. Since, $\Delta^\vee$ forms a basis of E, we can form a basis $\{(\alpha^\vee, \cdot)\}$ of E^* (the dual of E). The dual of this basis, is again a (different) basis of E, which we denote $\tilde{\Delta}$. The elements of $\tilde{\Delta}$ are called the *fundamental weights*. The *weight lattice* is then defined to be

$$\Lambda = \left\{ \sum_i a_i \omega_i \in E \mid a_i \in \mathbb{Z}, \omega_i \in \tilde{\Delta} \right\}. \tag{2.2}$$

Given a root system Φ a *subsystem* is a subset $\Gamma \subseteq \Phi$ satisfying the following two conditions:

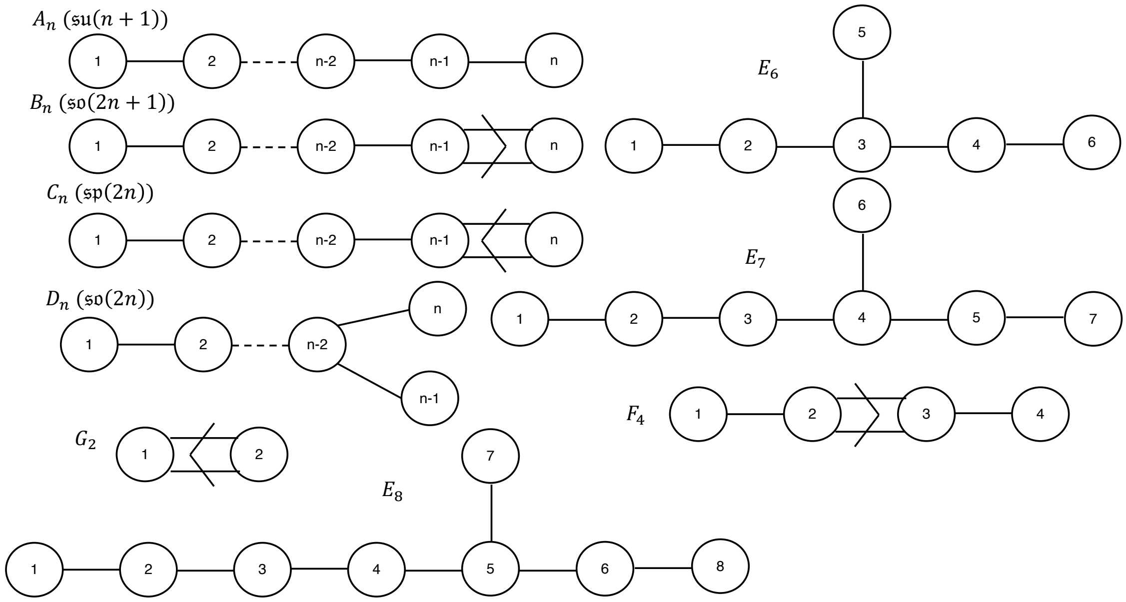

Fig. 2.1 The complete classification of Dynkin diagrams, and consequently Lie algebras.

1. if $\alpha \in \Gamma$ then $-\alpha \in \Gamma$,
2. if $\alpha, \beta \in \Gamma$, and $\alpha + \beta \in \Phi$, then $\alpha + \beta \in \Gamma$.

Lastly we define a Π-*system* which is a subset of vectors $\Pi \subseteq \Phi$, such that

1. the vectors in Π are linearly independent,
2. if $\alpha, \beta \in \Pi$, then $\alpha - \beta \notin \Phi$.

2.2.2 Lie algebras and root systems

Returning to Lie algebras. Given a Lie algebra $\mathfrak{g}$, and a Cartan subalgebra, $\mathfrak{h}$, a *root* is an element $\alpha \in \mathfrak{h}^*$ (the dual of $\mathfrak{h}$) such that there exists an $X \in \mathfrak{g}$ with $\mathrm{ad}_h X = \alpha(h)X$ for all $h \in \mathfrak{h}$. We denote the set of roots as $\Phi(\mathfrak{g}, \mathfrak{h})$, and $E(\mathfrak{g}, \mathfrak{h})$ the vector space over $\mathbb{R}$ spanned by the vectors $\Phi(\mathfrak{g}, \mathfrak{h})$. If $\mathfrak{g}$ is semisimple we can endow $E(\mathfrak{g}, \mathfrak{h})$ with an inner product so that $\Phi(\mathfrak{g}, \mathfrak{h})$ is a root system in $E(\mathfrak{g}, \mathfrak{h})$. This inner product is defined via the *killing form*, which is the map $\kappa : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C} : (X, Y) \mapsto \mathrm{tr}(\mathrm{ad}_X \cdot \mathrm{ad}_Y)$. Specifically, for every $\lambda \in E(\mathfrak{g}, \mathfrak{h})$, there exists a unique $h_\lambda \in \mathfrak{h}$ such that $\kappa(h_\lambda, h) = \lambda(h)$ for all $h \in \mathfrak{h}$ (see *e.g.* [97]). The inner product is then defined by $(\cdot, \cdot) : E(\mathfrak{g}, \mathfrak{h}) \times E(\mathfrak{g}, \mathfrak{h}) \to \mathbb{R} : (\lambda_1, \lambda_2) \mapsto \kappa(h_{\lambda_1}, h_{\lambda_2})$.

Different choices of Cartan subalgebras of $\mathfrak{g}$ lead to isomorphic root systems. Furthermore, two complex semisimple Lie algebras with isomorphic root systems are themselves isomorphic. A semisimple algebra is simple if and only if its root system is irreducible. These facts alone, and the classification of the Dynkin diagrams in Fig. 2.1, allows for a classification of semisimple Lie algebras. The simple Lie algebra associated with each Dynkin diagram is included in Fig. 2.1.

2.2.3 Embeddings

We now turn to embeddings of Lie algebras. An *embedding* of the Lie algebra $\mathfrak{g}'$, into $\mathfrak{g}$ is an injective map compatible with the Lie brackets. An embedding is called an *isomorphism* if it is also a surjection. An embedding is called an *automorphism* if $\mathfrak{g}' = \mathfrak{g}$. The group of automorphisms of $\mathfrak{g}$ is denoted $\mathrm{Aut}(\mathfrak{g})$. The subgroup, $\mathrm{Int}(\mathfrak{g}) \subseteq \mathrm{Aut}(\mathfrak{g})$, of *inner automorphisms* is that generated by $\exp(\mathrm{ad}_X)$, where $X \in \mathfrak{g}$ such that ad_X is nilpotent (see *e.g.* [38]). The group $\mathrm{Out}(\mathfrak{g}) = \mathrm{Aut}(\mathfrak{g})/\mathrm{Int}(\mathfrak{g})$ is called the group of *outer automorphisms*.

Two embeddings $f_1 : \mathfrak{g}' \to \mathfrak{g}$ and $f_2 : \mathfrak{g}' \to \mathfrak{g}$ are said to be *equivalent* if a $u \in \mathrm{Int}(\mathfrak{g})$ exists such that $f_1 = u \circ f_2$. If $\mathfrak{g}'$ and $\mathfrak{g}$ are semisimple, with chosen Cartan subalgebras $\mathfrak{h}'$ and $\mathfrak{h}$, then for any $f : \mathfrak{g}' \to \mathfrak{g}$, an equivalent one exists such that $f(\mathfrak{h}') \subseteq \mathfrak{h}$. This allows us to define $f^* : \mathfrak{h}^* \to \mathfrak{h}'^* : \lambda \mapsto \lambda \circ f$. If Λ is the weight lattice associated with $(\mathfrak{g}, \mathfrak{h})$ (which is independent of possible other choices) and Λ' is the weight lattice associated with $(\mathfrak{g}', \mathfrak{h}')$, then $f^*(\Lambda) \subseteq \Lambda'$. We denote the corresponding function $\Lambda f : \Lambda \to \Lambda'$. The function Λf written in a basis of fundamental weights, is called the *projection matrix*. (We will, where appropriate, extend the action of Λf to the corresponding Euclidian spaces).

If f is an automorphism, then we get $\Lambda f : \Lambda \to \Lambda$, but in addition $\Lambda f(\Phi) = \Phi$ for the corresponding root system (if it preserves the Cartan subalgebra). The function Λf is then an automorphism of the root system Φ. Taking the automorphisms of Φ and taking the quotient with respect to the Weyl group we are left with $\mathcal{O}(\mathfrak{g})$, the group of graph automorphisms of the Dynkin diagram corresponding to $\mathfrak{g}$. We then have a short exact sequence

$$1 \longrightarrow \mathrm{Int}(\mathfrak{g}) \longrightarrow \mathrm{Aut}(\mathfrak{g}) \longrightarrow \mathcal{O}(\mathfrak{g}) \longrightarrow 1 \tag{2.3}$$

which, furthermore splits. Thus $\mathrm{Out}(\mathfrak{g}) \cong \mathcal{O}(\mathfrak{g})$, and an automorphism $f \in \mathrm{Aut}(\mathfrak{g})$ which fixes $\mathfrak{h}$ is inner if and only if its Λf is in the Weyl group. The elements of $\mathcal{O}(\mathfrak{g})$

will sometimes be called 'the outer automorphisms' (although we admit that this is an abuse of terminology).

If two embeddings $f_1 : \mathfrak{g}' \to \mathfrak{g}$ and $f_2 : \mathfrak{g}' \to \mathfrak{g}$ branch every representation of $\mathfrak{g}$ in the same way, they are called *linearly equivalent*. Two representations are linearly equivalent if and only if Λf_1 and Λf_2 differ by the Weyl group [66, Theorem. 1.1]. Unfortunately, linear equivalence is in general coarser then equivalence. However, for the classical Lie algebras the linear equivalence f_1 and f_2 implies the existence of an automorphism relating f_1 and f_2. This is not the case for exceptional Lie algebras [116]. Nevertheless, we content ourselves with working with projection matrices (as it is all we will need in the sequel).

We now turn to maximal embeddings. An embedding $f : \mathfrak{g}' \to \mathfrak{g}$ of semisimple algebras is said to be *maximal* if there is no two embeddings (but not isomorphisms) of semisimple algebras $f_1 : \mathfrak{g}' \to \mathfrak{g}''$, and $f_2 : \mathfrak{g}'' \to \mathfrak{g}$ such that $f = f_2 \circ f_1$. Once maximal embeddings are found generic embeddings can then be found via a recursive approach.

If $\mathfrak{g} = \mathfrak{su}(2)$ then there is a single maximal embedding $f : \emptyset \to \mathfrak{su}(2)$. If $\mathfrak{g}$ is non-simple with a decomposition into simple algebras $\bigoplus_i \mathfrak{g}_i$ the maximal embeddings are of two types [66, Th 15.1]. The first type of maximal embedding is of the form

$$\mathfrak{g}' = \bigoplus_{k<i} \mathfrak{g}_k \bigoplus \mathfrak{g}'_i \bigoplus_{k>i} \mathfrak{g}_k, \quad f = \bigoplus_{k<i} \iota_k \bigoplus \iota_i \circ f_i \bigoplus_{k>i} \iota_k, \tag{2.4}$$

where $\iota_k : \mathfrak{g}_k \mapsto \mathfrak{g}$, is the natural inclusion, and $f_i : \mathfrak{g}'_i \mapsto \mathfrak{g}$ is a maximal embedding of simple algebras (which we will discuss shortly). The second type is of the form

$$\mathfrak{g}' = \bigoplus_{k\neq j} \mathfrak{g}_k, \quad f = \bigoplus_{k<i} \iota_k \bigoplus (\iota_i + \iota_j \circ f_{ij}) \bigoplus_{k>i} \iota_k, \tag{2.5}$$

where $f_{ij} : \mathfrak{g}_i \to \mathfrak{g}_j$ is an isomorphism.

For a simple Lie algebra $\mathfrak{g}$, a *regular maximal embedding*, $f : \mathfrak{g}' \to \mathfrak{g}$ is one for which, up to an equivalence, $[\mathfrak{h}, f(\mathfrak{g}')] \subseteq f(\mathfrak{g}')$, where $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$.

All maximal regular embeddings may be constructed as follows: Letting Δ denote the set of simple roots of $\mathfrak{g}$, we define a partial ordering on Φ defined by $\lambda_1 \geq \lambda_2$ if $\lambda_1 - \lambda_2 = \sum_{\alpha\in\Delta} k_\alpha \alpha$ with $k_\alpha \geq 0 \ \forall \alpha \in \Delta$. The simple algebra $\mathfrak{g}$ has a unique maximal root with respect to this ordering, which we call δ [132]. We form all sets $\Delta \cup \{-\delta\}/\{\alpha\}$ for $\alpha \in \Delta$ and $\Delta/\{\alpha\}$ for $\alpha \in \Delta$. Denoting one of these sets Δ', then Δ' forms a Π-system of Φ. A subset Γ is then formed by taking all integer combinations of elements in Δ' such that the result is in Φ. The subsystem Γ is itself a root system, and defines a semisimple subalgebra, which is $\mathfrak{g}'$. The projection matrix Λf can be determined from the embedding of Γ into Φ.

Non-regular maximal embeddings are called S-embeddings. They can be split into simple and non-simple S-embeddings. For the exceptional algebras these can be determined explicitly. The simple S-embeddings for the classical algebras are associated with irreducible representations. Generically (there are some exceptions), a d-dimensional irreducible representation (irrep) ϕ' of $\mathfrak{g}'$ defines a maximal S-subalgebra of: $\mathfrak{su}(d)$ if it does not possess a bilinear invariant; $\mathfrak{so}(d)$ if it possesses a symmetric bilinear invariant; and $\mathfrak{sp}(d)$ if it possesses a anti-symmetric bilinear invariant. In these cases the weights, ω_i, of the defining representation of $\mathfrak{g}$ project under Λf into those, ω_i', defined by ϕ'. Thus we have

$$\Lambda f \circ \omega_{\sigma(i)} = \omega_i', \tag{2.6}$$

for some permutation $\sigma \in S_d$. Some choices of permutations will not be valid, for example they would imply $\Lambda f \omega_1 \neq -\Lambda f(-\omega_1)$, violating the linearity on Λf. Those which are consistent are related by automorphisms. Using Eq. 2.6 and psudoinverses of matrices, one can determine Λf (as done in [71, 72]).

Lastly we come to non-simple S-subalgebras for the classical algebras. The classification of these algebras is much simpler than in other cases and is given in [67, Th. 1.3 & 1.4]. They are related to certain embeddings of the defining representations of *e.g.* $Sp(2s) \times Sp(2t)$ into $SO(2st)$. In a similar way to Eq. 2.6 the explicit matrices Λf can be found.

2.3 Topology and differential geometry

We now move onto the mathematical prerequisites relating to topology and differential geometry. The main references for this Section are [119, 118, 101, 122].

A *topological space* is a set T with a family of subsets τ such that[1]

1. $T \in \tau$ and $\emptyset \in \tau$ where $\emptyset$ is the empty set,

2. any union of subsets in τ is also into τ,

3. any intersection of finitely many subsets in τ is in τ.

[1] Topology is used implicitly in particle physics, for example when one 'adds a point at infinity', they are really adding a point to the set underlying a topological space, and specifying new open subsets that contain it.

The subsets in τ are called the *open subsets*, and their complements in T are called the *closed subsets*. Given two topological spaces T and T', a map $f : T \to T'$ is called *continuous*, if for every open $X \subseteq T'$ the subset $f^{-1}(X)$ is open in T. A *homeomorphism* is a continuous bijection with a continuous inverse.

A (smooth) *manifold* is a topological space M, with a family of pairs $\{(U_i, \phi_i)\}$, where U_i is open in M, and $\bigcup_i U_i = M$. The quantity ϕ_i is a map from U_i to an open subset $V_i \subseteq \mathbb{R}^n$ for fixed n, which is a homeomorphism. For any i, j such that $U_i \cap U_j \neq \emptyset$ then $\phi_i \circ \phi_j^{-1}$ acting on $\phi_j(U_i \cap U_j)$ is infinitely differentiable, in the usual sense on $\mathbb{R}^n$. The family $\{(U_i, \phi_i)\}$ is called an *atlas*, and each member (U_i, ϕ_i) is called a *chart*. Two atlases on the same topological space are said to be equivalent, and thought of as defining the same manifold, if their union is also a smooth atlas. The equivalence class of atlases defined by this relation is called the *smooth structure* on M.

A *smooth map* between two manifolds M and M' is a map $f : M \to M'$, such that for any suitable choice of charts ϕ_i on M and ϕ'_j on M', $\phi'_j \circ f \circ \phi_i^{-1}$ is infinitely differentiable.

Let (U_i, ϕ_i) be a chart containing the point p. This chart defines local coordinates around p, which we will denote x^μ. Let $\gamma(t) : \mathbb{R} \to M$ be a smooth map (such a map is called a *curve*). Then

$$\left.\frac{dx^\mu(\gamma(t))}{dt}\right|_p \frac{\partial}{\partial x^\mu} \tag{2.7}$$

is called the *tangent vector* of $\gamma(t)$ at p. The set of all possible tangent vectors at p is the *tangent space* at p and denoted $T_p(M)$. The set $T_p(M)$ has the structure of a vector space. Given a map $f : M \to M'$, the map $T_p f : T_p M \to T_{f(p)} M'$ is defined by

$$\left.\frac{dx^\mu(\gamma(t))}{dt}\right|_p \frac{\partial}{\partial x^\mu} \mapsto \left.\frac{dx'^\mu(f \circ \gamma(t))}{dt}\right|_p \frac{\partial}{\partial x^\mu}. \tag{2.8}$$

Elements of the dual vector space $T_p^*(M)$ are called *differential forms*. Anti-symmetric tensors made up of r differential forms are called the r-forms.

We now look at different (smooth) maps between manifolds. A *submersion* is a map $f : M \to M'$ such that $T_p f$ is a surjection for all $p \in M$. An *immersion* is map $f : M \to M'$ such that $T_p f$ is an injection for all $p \in M$. An *embedding* is a map $f : M \to M'$ which is an injective immersion and a homeomorphism onto its image. The image of an embedding is called a *embedded submanifold* (or *submanifold* for short).

A *fibred manifold* consists of two manifolds, the *total space* Y, a *base space* X and a surjective submersion $\pi : Y \to X$. The inverse image $\pi^{-1}(x)$ for any point in $x \in X$ is an embedded submanifold of Y. It is called the *fibre* of the fibred manifold at x. For a fibre manifold, the submanifolds $\pi^{-1}(x)$ at distinct x need not be of the same homotopy type, or diffeomorphism class. A *fibre bundle* is a special type of fibred manifold. For a fibre bundle with *typical fibre* F, for every $x \in X$ there exists a neighbourhood U with $x \in U \subseteq X$ and a diffeomorphism $\theta : \pi^{-1}(U) \to U \times F$ with $\pi \circ \theta^{-1}(m, f) = m$. In a fibre bundle, every fibre is manifestly diffeomorphic (recall our discussion in §1.3, where we related this to a target space).

Given a fibre bundle over X, we can equip X with an open covering $\{U_i\}$ and a collection of $\{\theta_i\}$, where each θ_i is a diffeomorphism between $\pi^{-1}(U_i)$ and $U_i \times F$ as above. This is called a *local trivialisation*. On double intersections of the open sets in the cover, one may have *transition functions*, $t_{ij} : U_i \cap U_j \to G$, to some group G, known as the *structure group*. There is a left-action of G on the fibre F defined such that $\theta_j^{-1}(m, f) = \theta_i^{-1}(m, t_{ij}(m)f)$.

For either a fibred manifold or a fibre bundle, a *local section* is a map $\sigma : U \to Y$ where U is open in X and $\pi \circ \sigma$ is the identity on U. A *global section* is a local section for $U = X$. Given a local trivialisation on a fibre bundle, a global section may be described by a series of maps $s_i : U_i \to F$ such that $s_i = t_{ij} s_j$.

A specific type of fibre bundle is a principal bundle. In a *principal bundle* the structure group G is a Lie group, which as a manifold, is diffeomorphic to the typical fibre F. In addition, the Lie group G has a right action, which we will denote R_g, on P such that $\pi \circ R_g = \pi$ which acts both freely and transitively on each fibre.

On a principal bundle $\pi : P \to X$, we can define a *principal connection* 1-form (or simply *connection* for short). This is a 1-form on P with value in the Lie algebra, $\mathfrak{g}$, of the Lie group G. A connection must satisfy the following conditions

$$\begin{aligned} A(X^{\#}) &= X, \\ R_g^* A &= \mathrm{Ad}_{g^{-1}} A, \end{aligned} \tag{2.9}$$

where X is in the Lie algebra $\mathfrak{g}$, and the vector field $X^{\#}$ on P is defined by

$$X^{\#} f(p) = \frac{d}{dt} f\left(R_{e^{itX}} \cdot p\right)\bigg|_{t=0} \tag{2.10}$$

for $p \in P$ and $f : P \to \mathbb{R}$.

Given a connection, a *horizontal lift* of a curve $\gamma : [0, 2\pi] \to M$ is a curve $\gamma_{hl} : [0, 2\pi] \to P$ such that $\gamma = \pi \circ \gamma_{hl}$ and the tangent vector at each point $p \in \mathrm{Im}(\gamma_{hl})$, which we call Y_p, satisfies $A(Y_p) = 0$. In other words, γ_{hl} is horizontal with respect to the connection. A horizontal lift of a curve is unique, up to specifying the start point in the fibre above $\gamma(0)$.

Using a horizontal lift we can define the holonomy. The *holonomy* of a loop $\gamma : [0, 2\pi] \to M$ is defined as the element $g \in G$ such that

$$\gamma_{hl}(2\pi) = R_g \gamma_{hl}(0). \tag{2.11}$$

Let $\tilde{\gamma} : [0, 2\pi] \to P$ be an arbitrary loop which projects down to γ under π. The horizontal lift is related to $\tilde{\gamma}$ by

$$\gamma_{hl} = R_{\left(e^{-i\int_0^t \tilde{\gamma}^* A}\right)} \circ \tilde{\gamma}. \tag{2.12}$$

Using Eq. 2.11 and Eq. 2.12, one finds that the holonomy of γ (with respect to the connection A) is equal to $e^{-i\int_0^{2\pi} \tilde{\gamma}^* A}$.

2.4 Category Theory

We finally come to category theory. Suitable, introductory, references for this material are [134, 146].

A *category* $\mathtt{C}$ is a collection of objects and morphisms between those objects satisfying a series of axioms. Namely, for each object C there is an identity morphism $\mathrm{id}_C : C \to C$ and we can compose any morphism from C with any morphism to C, subject to the rules that composition is associative and that pre- or post-composing a morphism with the identity morphism returns the original morphism. Examples based on previous discussions are the category $\mathtt{Set}$, whose objects are sets and whose morphisms are functions; the category $\mathtt{Top}$, whose objects are topological spaces and whose morphisms are continuous maps; and the category $\mathtt{Man}$, whose objects are smooth manifolds and whose morphisms are smooth maps.

Given a pair of categories $\mathtt{C}, \mathtt{C}'$, a *functor* $F : \mathtt{C} \to \mathtt{C}'$ is a mapping of each object C in $\mathtt{C}$ to an object $F(C)$ in $\mathtt{C}'$ and a mapping of each morphism $f : C \to \tilde{C}$ in $\mathtt{C}$ to a morphism $F(f) : F(C) \to F(\tilde{C})$ in $\mathtt{C}'$ that preserves identities and composition. We have, for instance, functors $\mathtt{Man} \to \mathtt{Top}$ and $\mathtt{Top} \to \mathtt{Set}$ that simply forget the extra structure.

Given a category $\mathtt{C}$, its *opposite category* $\mathtt{C}^{\mathrm{op}}$ has the same objects as $\mathtt{C}$, but all morphisms have their sources and targets swapped. A functor from $\mathtt{C}^{\mathrm{op}}$ to $\mathtt{C}'$ is often called a *contravariant functor* from $\mathtt{C}$ to $\mathtt{C}'$.

Given a pair of functors $F, F' : \mathtt{C} \to \mathtt{C}'$, a *natural transformation* $\eta : F \Rightarrow F'$ is, for every object C in $\mathtt{C}$ a morphism $\eta_C : F(C) \to F'(C)$, such that, for every morphism $f : C \to \tilde{C}$, the diagram

$$\begin{array}{ccc} F(C) & \xrightarrow{\eta_C} & F'(C) \\ {\scriptstyle F(f)}\big\downarrow & & \big\downarrow{\scriptstyle F'(f)} \\ F(\tilde{C}) & \xrightarrow[\eta_{\tilde{C}}]{} & F'(\tilde{C}) \end{array} \tag{2.13}$$

commutes. A *natural isomorphism* is a natural transformation for which each morphism η_C is an isomorphism in $\mathtt{C}'$.

An *equivalence of categories* $\mathtt{C}, \mathtt{C}'$ is a pair of functors $F : \mathtt{C} \to \mathtt{C}'$ and $F' : \mathtt{C}' \to \mathtt{C}$, such that there exist natural isomorphisms between $F' \circ F$ and the identity functor on $\mathtt{C}$ and between $F \circ F'$ and the identity functor on $\mathtt{C}'$.

We now want to introduce the notion of a limit. To do so, we first need to define diagrams and cones. A *diagram* $\mathtt{D}$ in the category $\mathtt{C}$ is a collection of objects $\{D_i\}_{i\in I}$ and morphisms $\{g_a : D_i \to D_j\}_{a\in I'}$ between them.[2] A *cone* of $\mathtt{D}$ is a tuple $(C, \{f_i\}_{i\in I})$ containing an object C and morphisms $f_i : C \to D_i$, such that for each $g_a : D_i \to D_j$ the diagram (which really is a diagram, in the sense of our definition)

$$\begin{array}{ccccc} & & C & & \\ & {\scriptstyle f_i}\swarrow & & \searrow{\scriptstyle f_j} & \\ D_i & & \xrightarrow[g_a]{\qquad\qquad} & & D_j \end{array} \tag{2.14}$$

commutes.[3] A *limit* of $\mathtt{D}$ is a cone $(C, \{f_i\}_{i\in I})$ of $\mathtt{D}$ that is universal in the sense that any other cone $(C', \{f'_i\}_{i\in I})$ of $\mathtt{D}$ factors through it via a unique *mediating morphism* $u : C' \to C$. In other words, $f'_i = f_i \circ u$ for all $i \in I$. A limit need not exist for a given diagram, but if it does it is guaranteed to be unique up to unique isomorphism. It is therefore common to abuse terminology and talk about 'the' limit of a diagram, and we will do so too.

For example, a *pullback* is the limit of the diagram

$$D_1 \xrightarrow{g_1} D_0 \xleftarrow{g_2} D_2 \ . \tag{2.15}$$

It exists in $\mathtt{Top}$ and the limiting object C is given by the set $D_1 \times_{D_0} D_2 := \{(d_1, d_2) \in D_1 \times D_2 | g_1(d_1) = g_2(d_2) \in D_0\}$, with the subspace topology, and the maps $f_{1,2}$ given by

[2]Equivalently, a diagram is a functor from an indexing category to $\mathtt{C}$.

[3]The fact that this diagram commutes means that to uniquely specify a cone, we do not need to specify all morphisms f_i, since some can be deduced. In what follows, we shall only write down those morphisms which can not be deduced from commutative diagrams.

the restrictions to $D_1 \times_{D_0} D_2$ of the projections $D_1 \times D_2 \rightrightarrows D_{1,2}$. It does not exist, in general, in `Man` or related categories. It does, however, exist in `Man` when one morphism, g_2 say, is either a surjective submersion or an open embedding, in which case f_1 enjoys the same property.

A special case of a pullback is an *inverse image*, in which one morphism, g_2 say, is a monomorphism (roughly equivalent to an injection). In `Set`, this is the usual inverse image and so it is common to denote the limiting object by $g_1^{-1}(D_2)$, with the other data often left implicit. As for a general pullback, the inverse image is not guaranteed to exist in `Man` or related categories. Though, it does exist in `Man` in the special case where g_2 is not just a monomorphism but is an open embedding. Another case where it exists is the limit of $Y \xrightarrow{\pi} X \xleftarrow{x} *$, where Y is a fibred manifold and $x : * \to X$ is the inclusion of a point at $x \in X$. Here x is a monomorphism, but is not an open embedding, but the limit nevertheless exists because the map π is a surjective submersion, the limiting object being precisely the manifold given by the fibre $\pi^{-1}(x)$ which we introduced in §2.3.

As another example, the *equaliser* is the limit of the diagram

$$D_1 \underset{g_1}{\overset{g_2}{\rightrightarrows}} D_0 \ . \tag{2.16}$$

Equalisers always exist in `Top`, but like pullbacks may not exist in `Man` or its cousins.

2.4.1 Presheaves, sheaves and étalé spaces

Let $\mathcal{O}_X$ be the category whose objects are the open subsets of the topological space X, and whose morphisms are inclusions. A functor $\Gamma : \mathcal{O}_X^{op} \to$ `Set` is called a *presheaf*. The category whose objects are these functors and whose morphisms are natural transformations between them is called the *category of presheaves*.

A *sheaf* is a presheaf, Γ, satisfying the following conditions:

1. if $\{U_i\}$ is an open cover of U, and $a, b \in \Gamma(U)$ such that $\Gamma\iota_{U_i,U}(a) = \Gamma\iota_{U_i,U}(b)$ for every inclusion $\iota_{U_i,U} : U_i \mapsto U$, then $a = b$,
2. if there exists $a_i \in \Gamma(U_i)$ such that on $U_i \cap U_j \neq \emptyset$ we have $\Gamma\iota_{U_i\cap U_j,U_i}(a_i) = \Gamma\iota_{U_i\cap U_j,U_i}(a_j)$, then an $a \in \Gamma(U)$ exists such that $\Gamma\iota_{U_i,U}(a) = a_i$.

The category formed with sheaves as objects, and natural transformations between them as morphisms, is called the *category of sheaves*.

Related to sheaves are étalé spaces. These are founded in topology and so a pure classifier would have put them in §2.3. An *étalé space* over a topological space X is a pair (E, p) of a topological space E and a local homeomorphism $p : E \to X$. An *étalé morphism* between two étalé spaces (E, p) and (E', p'), is a map $f : E \to E'$ such that $p' \circ f = p$. The *category of étalé spaces*, has étalé spaces as objects and étalé morphisms as morphisms. The category of sheaves and the category of étalé spaces are equivalent categories.

Chapter 3

Solving the anomaly equations pure $\mathfrak{u}(1)$-gauge theory

Our first foray is concerned with the study of anomaly free gauge theories and extensions thereof. A pure $\mathfrak{u}(1)$-gauge theory is the simplest example where one can study anomalies. This can, alternatively, be thought of as a theory where the particles involved are assumed to carry no other gauge representations (*e.g.* in a dark sector). The task of solving the ACCs (Eqs. 1.1) associated with such a theory using techniques well established in geometry, is the topic of this Chapter and the first part in this three-part foray into anomaly free gauge theories.

We start in §3.1 by explaining a solution to these ACCs given in [51]. In §3.2, we give a geometric reformulation of this solution, in the process, generalising a number theoretic result of Mordell to dimensions higher than three. We conclude in §3.3.

There is one potential inconvenience in our parameterisation, in that there are special solutions generated differently from others, which we circumvent in Appendix A.1. We present a different form of the general solution for an even number of Weyl fermions in Appendix A.2.

3.1 The CDF solution

The local anomaly cancellation equations for a $\mathfrak{u}(1)$-gauge theory with n left-handed chiral fermions of charge z_i, which may be taken to be integers (by an appropriate rescaling, and by assumption of compactness of the gauge group), are given in Eq. 1.1, and are repeated here for convenience

$$\sum_{i=1}^{n} z_i = 0, \tag{3.1}$$

$$\sum_{i=1}^{n} z_i^3 = 0. \tag{3.2}$$

The first of these, Eq. 3.1, comes from a one-loop triangle diagram with two external gravitons and one external $\mathfrak{u}(1)$-gauge boson [68], whilst Eq. 3.2 comes from the similar diagram with three external $\mathfrak{u}(1)$-gauge bosons [7, 28, 37, 89, 82]. Although written for left-handed chiral fermions, these equations are general for a theory with both left-handed and right-handed chiral fermions since we can charge conjugate any right-handed representation, reversing the sign of its charge and giving a left-handed representation. Eq. 3.2 is a cubic diophantine equation in n variables; since it is not yet known how to solve a generic such equation even in 2 variables (corresponding to an elliptic curve [90]), one might expect that finding the general solution to Eqs. 3.1-3.2 is a difficult problem. However, a paper by Costa, Dobrescu and Fox (CDF) [51] managed to do so, in the following way.

CDF observed that given two integer solutions

$$\underline{x} := (x_1, \ldots, x_n) \text{ and } \underline{y} := (y_1, \ldots, y_n), \tag{3.3}$$

of Eq. 3.1, and Eq. 3.2, a third could be constructed from a 'merger' operation, which they denoted '$\oplus$'

$$\underline{x} \oplus \underline{y} := \left(\sum_{i=1}^{n} x_i y_i^2 \right) \underline{x} - \left(\sum_{i=1}^{n} x_i^2 y_i \right) \underline{y}. \tag{3.4}$$

Some solutions to Eq. 3.1 and Eq. 3.2 are easy to find, having for each charge z_i another charge $z_j = -z_i$. Using solutions of this form, which we call vector-like solutions, and the merger CDF showed that one can construct chiral sets of charges, namely those where $z_i + z_j \neq 0$ for all i and j. They then showed (via rather lengthy algebra) that any solution can be constructed from these chiral sets of charges by permutation of charges or concatenation with each other or with vector-like solutions. For n even the specific mergers they considered were

$$(l_1, k_1, \ldots, k_m, -l_1, -k_1, \ldots, -k_m) \oplus (0, 0, l_1, \ldots, l_m, -l_1, \ldots, -l_m), \tag{3.5}$$

where $m = n/2 - 1 \geq 2$ and $k_i, l_i \in \mathbb{Z},\ i \in \{1, \dots, m\}$. Whilst for n odd they were

$$(0, k_1, \dots, k_{m+1}, -k_1, \dots, -k_{m+1}) \oplus (l_1, \dots, l_m, k_1, 0, -l_1, \dots, -l_m, -k_1), \qquad (3.6)$$

where $m = (n-3)/2 \geq 1$. CDF showed that if one wants to avoid zero charges or vector-like copies of charges then conditions have to be applied to k_i's and l_i's.

In this Chapter, we show that the ingenious methods of CDF have a simple geometric interpretation, corresponding to elementary constructions long known to number theorists [117]. Viewing them in this context allows a fully general solution to be written down in one fell swoop. The geometric interpretation allows us to give a variety of other, qualitatively similar, parameterisations of the general solution, as well as a qualitatively different form of the general solution for even n. It also allows us to show that to generate all solutions from CDF's parameterisation only requires permutations and not the other operations.

3.2 Geometric Method

By way of motivation, consider the $n = 6$ solution $(0, -9, 7, -1, 8, -5)$ to Eq. 3.1, and Eq. 3.2. The only way to get this solution using the method outlined in CDF is by permutation. Our geometric solution will, on the other hand, be able to generate such solutions without resorting to permutations.[1] The reasoning behind this, as we shall see later, lies in our use of a geometrical approach, namely that of projective geometry over the field $\mathbb{Q}$ of rational numbers.

Within the projective space $\mathrm{P}\mathbb{F}^{n-1}$ over a field $\mathbb{F}$ (as defined in §2.1) we can define d-planes. By a *d-plane* (for $d < n-1$) we mean a d-dimensional projective subspace of $\mathrm{P}\mathbb{F}^{n-1}$, which can be written as

$$\Gamma = \sum_{i=1}^{d+1} \alpha_i p_i, \qquad (3.7)$$

where $[\alpha_1 : \cdots : \alpha_{d+1}] \in \mathrm{P}\mathbb{F}^d$ parameterise the d-plane and $p_i \in \mathrm{P}\mathbb{F}^{n-1}$ are fixed. A 1-plane, for example, is just a (projective) line, homeomorphic to a circle.

To motivate the use of projective space on physical grounds, we note that the Lie algebra $\mathfrak{u}(1)$ is isomorphic to $\mathbb{R}$. Given our assumption of compactness, this implies that our charges z_i are not only real-valued, but also commensurate, meaning that if $z_j \neq 0$, then z_i/z_j is rational for all i. We can scale every z_i by a single real parameter

[1] Though, as we indicate, utilising permutations can be useful.

without changing the physics, as long as the coupling constant is also appropriately scaled. This, along with the fact that the z_i's are commensurate, allows us to undertake a scaling such that all charges are rational, *viz.* $z_i \in \mathbb{Q}$.[2] It also tells us that we should think of the set of all charges as living in projective space, specifically $P\mathbb{Q}^{n-1}$ and indeed, Eq. 3.1, and Eq. 3.2, being homogeneous, define loci therein.

It is convenient for us to eliminate z_n in our equations from the cubic equation in Eq. 3.2 to get

$$\sum_{i=1}^{n-1} z_i^3 - \left(\sum_{i=1}^{n-1} z_i\right)^3 = 0. \tag{3.8}$$

This equation is homogenous, meaning it is well defined on our equivalence classes in $P\mathbb{Q}^{n-2}$, and as such it defines a cubic hypersurface (given it is co-dimension 1) of $P\mathbb{Q}^{n-2}$. In order to make progress in solving this equation, we review some geometric methods used in diophantine analysis.

3.2.1 The method of chords

Consider a homogenous cubic in n-variables, with rational coefficients, defining a locus in $\mathbb{Q}^n$. Let a and b be two points in $\mathbb{Q}^n$ on the locus. A result from antiquity[3] tells us that a chord between a and b will intersect the surface at a third point in $\mathbb{Q}^n$. One can understand this result as follows, let $L(t) = a + t(b - a)$ be the chord joining a and b. Points both lying on this chord and in the cubic surface must satisfy the equation $kt(t-1)(t-t_0) = 0$ where $k, t_0 \in \mathbb{Q}$. This result comes from considering the cubic along the chord and noting that a cubic has one or three (possibly degenerate) real roots. Hence within $\mathbb{Q}^n$, there is a third point of intersection, corresponding to $t = t_0$ and given by $L(t_0)$. We note that this result is equally valid in projective space, $P\mathbb{Q}^n$. We will call this construction the 'method of chords'.

Further, a rather more recent (though equally elementary) result of Mordell [117] states that *all* rational points in a cubic surface in $P\mathbb{Q}^2$ can be constructed from chords in this way, starting from a projective line, L, and a point, $p \notin L$ that both lie in the surface. It follows from the realisation that in fact *any* point in $P\mathbb{Q}^2$ (*ergo* any point on the cubic) is on a chord from p_1 to a point in L. As we will see, this result generalises in a straightforward way to $P\mathbb{Q}^n$, but there is no analogous result in $\mathbb{Q}^n$. In

[2]In the end, we can scale them all so they are integer, as we previously claimed. But working with the field $\mathbb{Q}$, rather than the ring $\mathbb{Z}$, allows us to do geometry.

[3]The result certainly goes back at least to Fermat and Newton in the 17th century and may go back even further to Diophantus in the 3rd century. A historical account is given in [140].

$\mathbb{Q}^3$ for example, the analogous result would have to involve two skew lines, L_1 and L_2. However, points forming a plane with L_2 which is parallel to L_1 will be missed. In $\mathrm{P}\mathbb{Q}^2$, there is no concept of parallel lines – pairs of lines are either disjoint or intersecting – and indeed the aforementioned points all lie on a chord connecting a point on L to p.

This simple observation, when generalised to higher n, underlies the fact that the point $(0, -9, 7, -1, 8, -5)$ is missing from CDFs $n = 6$ parameterisation, but is included when we work in projective space, as we will discuss in detail in §3.2.4.

Before actually using any of these results, we note that our general method will not work in the cases for $n = 1$, and $n = 2$. This is because for $n = 1$ and $n = 2$ it would require a notion of a (-1)-plane! Part of the discussion, namely that in Appendix A.1, is also valid only for $n \geq 4$. Happily, the solutions to the $n = 1, 2, 3$ cases can be found directly, allowing us to restrict our general discussion to $n \geq 4$. Namely for $n = 1$ the solution is $z_1 = 0$. For $n = 2$, Eq. 3.8 results in no effective constraint (one obtains that the left-hand side is identically zero for any z_1) and so the solution of Eq. 3.1, 3.2 is the point $[z_1 : z_2] = [1 : -1] \in \mathrm{P}\mathbb{Q}$. As a result of Fermat's last theorem, we have three solutions for $n = 3$: $[1 : 0 : -1]$, $[0 : 1 : -1]$ and $[1 : -1 : 0]$. Eqs. 3.1 and 3.2 are invariant under permutations of the z_i and so these three solutions are all in one equivalence class under such permutations.

We now consider higher n where the results above are more useful. For illustrative purposes, we will start with a rather explicit discussion of the case $n = 4$.

3.2.2 Application for $n = 4$

Let us consider the cubic anomaly-free surface in $\mathrm{P}\mathbb{Q}^2$,

$$z_1^3 + z_2^3 + z_3^3 - (z_1 + z_2 + z_3)^3 = 0, \tag{3.9}$$

corresponding to the $n = 4$ case of our problem, where we remember that $z_4 = -(z_1 + z_2 + z_3)$ from the gravitational mixed anomaly constraint. Using Mordell's result within this surface we take the line $\Gamma_1 = [k_1 : k_2 : -k_1]$ and the point $\Gamma_2 = [0 : l_1 : -l_1]$ in $\mathrm{P}\mathbb{Q}^2$, which are easily seen to lie on the cubic. Using the overall scaling of projective space, we could rescale such that $l_1 = 1$. At this stage, however, we refrain from doing so, preferring a slightly redundant parameterisation in order to stay closer to our analysis of the higher n cases below. We then construct a line passing through a generic point on each of Γ_1 and Γ_2 as $L_1 = \alpha_1[k_1 : k_2, : -k_1] + \alpha_2[0 : l_1 : -l_1]$, where $k_{1,2}, l_1 \in \mathbb{Q}$. The homogeneous parameter $[\alpha_1 : \alpha_2] \in \mathrm{P}\mathbb{Q}^1$ parameterises L_1, which must intersect the cubic surface at a third point, assuming that L_1 is not wholly in the cubic surface. On substituting the chord into Eq. 3.9 we obtain the constraint on α_1 and α_2 at intersections of the line and the cubic surface:

$$-3(k_1 - k_2)l_1\alpha_1\alpha_2 \left[(k_1 + k_2)\alpha_1 + l_1\alpha_2\right] = 0. \tag{3.10}$$

If L_1 were entirely in the cubic surface, the left-hand side would have evaluated to zero independently of the values of α_1 or α_2. The third point of intersection is specified by setting the square bracket in Eq. 3.10 to zero, i.e.

$$[\alpha_1 : \alpha_2] = [l_1 : -(k_1 + k_2)], \tag{3.11}$$

a rational point.

Now consider an arbitrary point $[a_1 : a_2 : a_3] \in \mathrm{P}\mathbb{Q}^2$ *not* in Γ_2. We can define a line between this point and one on Γ_2: $L_2 = \beta_1[0 : l_1 : -l_1] + \beta_2[a_1 : a_2 : a_3]$. It can be seen that this line intersects Γ_1 at $[\beta_1 : \beta_2] = [a_3 - a_1 : l_1]$. This, combined with Eq. 3.11, tells us that every such rational solution to the cubic equation can be found by considering lines between points on Γ_1 and Γ_2. What we have done here is apply Mordell's result to solve the $n = 4$ case of our problem.

3.2.3 Arbitrary $n \geq 4$

To consider arbitrary values of $n \geq 4$ we must generalise Mordell's result to an arbitrary cubic *hyper*surface X in $\mathrm{P}\mathbb{Q}^{n-2}$. The generalisation is immediate and gives the following **Theorem**: Let $\Gamma_1, \Gamma_2 \subset X$ be disjoint planes of dimensions $d_1, d_2 = m_o := (n-3)/2$, if n is odd and of dimensions $d_1 = m_e := (n-2)/2$ and $d_2 = m_e - 1$ if n is even. Every rational point $p \in \mathrm{P}\mathbb{Q}^{n-2}$ (*ergo* every $p \in X$) lies on a chord joining a point in Γ_1 to a point in Γ_2.

Proof: The result is obvious if $p \in \Gamma_2$. If $p \notin \Gamma_2$, then p and Γ_2 define a (d_2+1)-plane, which intersects Γ_1 in a point p^1. The line through p and p^1 intersects Γ_2 in a point p^2, yielding a chord. □

In the case of interest, the (projective) line $L = \alpha_1 p^1 + \alpha_2 p^2$ through $p^{1,2}$ with homogeneous parameter $[\alpha_1 : \alpha_2] \in \mathrm{P}\mathbb{Q}^1$ intersects the cubic hypersurface X defined by (3.8) when

$$3\alpha_1\alpha_2 \sum_{i=1}^{n-1} \left(\alpha_1 p_i^2 P_i^1 + \alpha_2 p_i^1 P_i^2\right) = 0, \;\; P_i^a := (p_i^a)^2 - \left(\sum_{j=1}^{n-1} p_j^a\right)^2.$$

Thus, along with the points $p^{1,2}$ (corresponding to $\alpha_{2,1} = 0$) we get either a third rational point on X at

$$[\alpha_1 : \alpha_2] = \left[\sum_{i=1}^{n-1} p_i^1 P_i^2 : -\sum_{i=1}^{n-1} p_i^2 P_i^1 \right], \tag{3.12}$$

or, if the terms on the right-hand side both vanish, we have that every rational point on L is on X. Lines which lie in X may be regarded as slightly awkward to deal with. Happily, it is possible, as we show in Appendix A.1, to find every solution on such a line by a permutation of the coordinates of a solution arising as the unique third point of intersection on a line not lying in X. A comparison of (3.12) with (3.4) shows that the 'merger' operation is really nothing but the finding of the third rational point starting from two others.

To get a fully general solution, we just need to find suitable Γ_1, Γ_2. To wit,

$$\begin{aligned}
\Gamma_1^e &= [k_1 : \cdots : k_{m_e} : k_{m_e+1} : -k_1 : \cdots : -k_{m_e}] \\
\Gamma_2^e &= [0 : l_1 : \cdots : l_{m_e} : -l_1 : \cdots : -l_{m_e}] \\
\Gamma_1^o &= [k_1 : \cdots : k_{m_o+1} : -k_1 : \cdots : -k_{m_o+1}] \\
\Gamma_2^o &= [l_2 : \cdots : l_{m_o} : l_{m_o+1} : 0 : -l_1 : \cdots : -l_{m_o} : -l_{m_o+1}].
\end{aligned} \tag{3.13}$$

These planes are disjoint (only meeting at the origin, which is not in $\mathrm{P}\mathbb{Q}^{n-2}$), so by the Theorem they yield all rational solutions of (3.1).

3.2.4 Comparison with CDF

The parameterisations of CDF, in contrast to ours, have $k_{m_e+1} = -l_1$ and $l_{m_o+1} = k_1$. We have already discussed that CDF's solution misses the point $(0, -9, 7, -1, 8, -5)$, for $n = 6$ and that for them this has to be found by permuting another solution, for example that generated with $k_1 = 14$, $k_2 = 2$, $l_1 = -18$, $l_2 = -9$ after scaling. In our parameterisation $(0, -9, 7, -1, 8, -5)$ can be obtained directly with, for example, $k_3 = 0$, $k_1 = 3$, $k_2 = -2$, $l_1 = 1$, and $l_2 = -1$ in (3.13), giving $p^1 = [3, -2, 0, -3, 2]$ and $p^2 = [0 : 1 : -1 : -1 : 1]$ and the correct third point of intersection.

It is easy to see why CDF's parameterisation misses this point; they cannot set k_3 to zero and l_1 not. Viewing things in the affine space $\mathbb{Q}^5$, the geometric nature of such missed points becomes manifest. The planes for $n = 6$ in (3.13) can be seen as corresponding to

$$\tilde{\Gamma}_1^e = (k_1, k_2, 1, -k_1, -k_2) \tag{3.14}$$
$$\tilde{\Gamma}_2^e = (0, l_1, l_2, -l_1, l_2). \tag{3.15}$$

in $\mathbb{Q}^5$. The $3-d$ plane defined by $\tilde{\Gamma}_2^e$ and the point $(-9, 7, -1, 8, -5)$ does not intercept the $2-d$ plane $\tilde{\Gamma}_1^e$, which is the same reason why Mordell's result fails to catch all the points in $\mathbb{Q}^3$. CDF go halfway to allowing such points, but by fixing $k_3 = l_1$ they don't quite catch them all.

We can be more specific and ask: given the planes in (3.13) where we force $k_{m_e+1} = -l_1$ and $l_{m_0+1} = k_1$ to retrieve CDF's solution, what points don't lie on lines between them? It is easy to see that for even n this would require either k_{m_e+1} or l_1 to be zero and for odd n either l_{m_0+1} or k_1, but not both. Thus, for the point $[a_1 : \cdots : a_n]$ to not lie on such a line, we need, for even n,

$$a_1 + \cdots + a_{n-1} = 0 \text{ or } a_1 + a_{n-2} = 0, \tag{3.16}$$

or, for odd n,

$$a_1 + \cdots + a_{n-1} = 0 \text{ or } a_{m_0+2} = 0. \tag{3.17}$$

For a non-zero solution we can always rearrange the charges so that none of these conditions are satisfied.

The only other points CDF miss are those where the line between the two planes in (3.13) lies within X. For example for $n = 4$ setting $k_2 = k_1$ gives a line L which lies in X. As an explicit example, consider $k_{1,2}, l_1 = 1$. This line is given by

$$L = \alpha_1[1 : 1 : -1] + \alpha_2[0 : 1 : -1]. \tag{3.18}$$

For CDF, points on this line correspond to solutions of the form $(A, -A, B, -B)$ for $A, B \in \mathbb{Z}$. However, CDF's $n = 4$ parameterisation

$$(-l_1^3(k_1 + l_1), -k_1 l_1^2(k_1 + l_1), k_1 l_1^2(k_1 + l_1), l_1^3(k_1 + l_1)) \tag{3.19}$$

can never land on such solutions. Nevertheless, CDF's parameterisation can get these by permutations, for the same reason that the parameterisation given here can, as we discuss in Appendix A.1.

The above two points not only show when CDF's parameterisation fails to reach a specific point but also proves that their parameterisation produces every point up to permutations.

3.3 Closing remarks

The pioneering work of CDF finds solutions to the local $\mathfrak{u}(1)$ anomaly cancellation constraints. This allows the construction of the general solution, provided one allows permutations. Our geometric method provides the general solution directly without having to perform additional steps. The geometric method also explains how some of the otherwise obscure features of CDF's construction (particularly the 'merging' procedure of two solutions) come about. Due to an immediate generalisation of a theorem by Mordell, the geometric method is guaranteed to find *all* rational solutions for a fixed number of charges n. Therefore (after clearing all denominators), it finds all integer solutions.

Two further remarks are in order. Firstly, as we have seen, our parameterisation of the general solution is somewhat distasteful, in that occasionally the chord L joining points on $\Gamma_{1,2}$ lies in X, and so yields not one, but infinitely many solutions. Another way to find these solutions is to permute the coordinates z_i of solutions arising as the unique third intersection of a line L which is not in X, as shown in Appendix A.1. Secondly, in the case where n is even, a completely different, and arguably even simpler, construction of a general solution is possible. Indeed, in such cases, the cubic hypersurface has double points, where both the left-hand side of (3.8) and its partial derivatives vanish (e.g. the rational point $[+1 : -1 : +1 : -1 : \ldots : +1 : -1 : +1]$). A line through such a double point intersects the cubic in one other rational point (or the line lies entirely in X) and thus all solutions can be obtained by constructing all lines through just a single double point, as it were. This is worked through explicitly in Appendix A.2, and is related to the method used in the next Chapter.

Chapter 4

Gauge rank extensions of the standard model

In the previous Chapter we gave a geometric solution to the anomaly equations of a pure $\mathfrak{u}(1)$-gauge theory. In this Chapter we move onto the second part of our first foray, which is to solve for the first time the anomaly equations associated with a $\mathfrak{u}(1)$-extension of the SM gauge algebra. These solutions inform models where the rank of the SM is increased, since the extra $\mathfrak{u}(1)$ may be a sub-algebra of some larger additional gauge extension, as well as future phenomenological Z' studies.

The layout of this Chapter is as follows: Firstly in §4.1 we will reintroduce the ACCs associated with this theory. In §4.2, we give a sketch of our solution. The details of the solution, as well as the solution itself will be given in §4.3. We give the closing remarks in §4.4.

4.1 The anomaly cancellation conditions

We introduced the ACCs associated with a $\mathfrak{u}(1)$ extension of the gauge algebra of the SM in Eq. 1.3. They take the form of homogeneous polynomial equations, which are repeated below

$$0 = \sum_{i=1}^{3} (6Q_i + 3U_i + 3D_i + 2L_i + E_i + N_i)\,, \tag{4.1a}$$

$$0 = \sum_{i=1}^{3} (3Q_i + L_i)\,, \tag{4.1b}$$

$$0 = \sum_{i=1}^{3} (2Q_i + U_i + D_i) \,, \tag{4.1c}$$

$$0 = \sum_{i=1}^{3} (Q_i + 8U_i + 2D_i + 3L_i + 6E_i) \,, \tag{4.1d}$$

$$0 = \sum_{i=1}^{3} \left(Q_i^2 - 2U_i^2 + D_i^2 - L_i^2 + E_i^2\right) , \tag{4.1e}$$

$$0 = \sum_{i=1}^{3} \left(6Q_i^3 + 3U_i^3 + 3D_i^3 + 2L_i^3 + E_i^3 + N_i^3\right) . \tag{4.1f}$$

Here $Q_i, U_i, D_i, L_i, E_i, N_i$, with $i \in \{1, 2, 3\}$, are respectively the charges of 3 SM families of quarks and leptons, together with 3 right-handed neutrinos. We have taken all fermions to be left-handed which can be achieved via charge conjugation. We will assume that the charges are commensurate, corresponding to the expectation that the gauge group is compact, and have assumed that there are three right-handed neutrinos, to fit aesthetically with neutrino oscillation data. We note that we can solve for the non-commensurate case, and for other (odd) numbers of neutrinos, which we discuss briefly in the closing remarks.

Finding *any* solutions to diophantine equations (or even establishing their existence or otherwise) is, in general, a notoriously difficult problem in number theory (very roughly, the state of the art is a single cubic in 3 unknowns). Surprisingly, we will see that one can, in fact, find *all* solutions to Eqs. 4.1a-4.1f.

4.2 Sketch of the solution

The keys to solving Eqs. 4.1a-4.1f are twofold. The first is to convert it to a problem in geometry by observing that one can equivalently seek rational solutions (since any integer solution trivially defines a rational solution and since, by clearing denominators, every rational solution defines an integer solution). The rational numbers form a field, allowing one to carry out division and hence various basic geometrical constructions. The 18 charges appearing in Eqs. 4.1a-4.1f then form coordinates for the affine space $\mathbb{Q}^{18}$. In fact, given that scaling all charges by a common multiple leads to the same physics (as we have remarked, the scaling can be absorbed in a redefinition of the gauge coupling), it is convenient to consider not the charges themselves, but the equivalence classes under such a scaling, which define the projective space $P\mathbb{Q}^{17}$ (whose points, in this Chapter, we sometimes call *rational points* for emphasis). The homogeneous

	Q_1	Q_2	Q_3	U_1	U_2	U_3	D_1	D_2	D_3	L_1	L_2	L_3	E_1	E_2	E_3	N_1	N_2	N_3
A	0	0	1	0	0	-4	0	0	2	0	0	-3	0	0	6	0	0	0
B	1	1	1	-1	-1	-1	-1	-1	-1	-3	-3	-3	3	3	3	3	3	3
C	-1	0	1	-1	0	1	-1	0	1	-1	0	1	-1	0	1	0	0	0

Table 4.1 Sample solutions of Eqs. 4.1a-4.1f. Point A corresponds to the 'Third Family Hypercharge Model' [9], while B is the combination of baryon minus lepton number.

polynomials Eqs. 4.1a-4.1f define a projective variety in $P\mathbb{Q}^{17}$ whose points, which we call *rational solutions*, we seek.

The second key to solving the problem is that it is easy enough to find *some* rational solutions, (*e.g.* by means of a numerical scan [14]); 3 such points, A, B, and C, are defined in Table 4.1. These can be used as the starting point for geometric constructions, similar to in the last Chapter. To give an example, consider just the quadratic (Eq. 4.1e) and suppose we know one rational point on the quadratic, C say. Ignoring degenerate cases for now, a line L through C intersects the quadratic at 1 other rational point R and moreover every rational point on the quadratic (indeed every point in the ambient space!) lies on a line through C. Thus, by parameterising all such lines, all rational points on the quadratic may be found. [1]

To solve the full set of Eqs. 4.1a-4.1f will require a more elaborate construction, as follows. Firstly, we note that the 4 linear equations 4.1a-4.1d simply define a projective subspace of $P\mathbb{Q}^{17}$ isomorphic to $P\mathbb{Q}^{13}$, to which we restrict our attention in what follows. Secondly, we exploit the fact that B is a singular point (namely a point at which the underlying variety in real space is not a smooth manifold). In fact it is unique (up to the addition of a multiple of the hypercharge) [2] among such points in that it is a double point of both the quadratic (Eq. 4.1e) and the cubic (Eq. 4.1f). Particle physics *cognoscenti* will instantly recognise point B as the combination of baryon number minus lepton number. (As we describe in [15], which studies how such singular points arise in gauge theories in general, this turns out to be no surprise.)

The utility of the point B is the following. Since it is a double point of the *cubic*, lines through it will have similar properties to the lines through the (regular) point C of the quadratic that we have already discussed: generically, a line M through B will intersect the cubic in at 1 other rational point, X say, and moreover every rational

[1] These arguments are standard ones in elementary number theory [117], but skeptical readers will hopefully be convinced by the explicit discussion that follows.

[2] If one were to add multiples of hypercharge to any solution, one would obtain another solution. This redundancy could be removed, resulting in the projective dimension of the variety being one fewer.

point on the cubic (indeed every point in the ambient space) will lie on a line through B. [3]

Now let us consider the cubic and the quadratic in tandem. If B were merely a regular point of the quadratic, we would face the difficulty that the point X on the cubic would not normally lie on the quadratic. But because B is also a double point of the quadratic, we are guaranteed that the line either lies entirely in the quadratic, or has no point in the quadratic other than B. On its own, this fact is not particularly useful, since it is the latter type of line which is generic (consider, *e.g.*, the variety in $\mathrm{P}\mathbb{Q}^2$ defined using coordinates $(x, y, z) \in \mathbb{Q}^3$ by $xy = 0$, which has a double point at $(0, 0, 1)$). What is needed is a construction which generically spits out lines of the former type. But this is easy: we use the original construction of rational points R of the quadratic, and then consider, for each such R, the line M joining B to R. Generically, R is distinct from B, in which case the line lies entirely in the quadratic (since it has a point on the quadratic, *viz.* R, which is not B, every point on it must be on the quadratic) and by finding the line's other intersection with the cubic, we get a new rational solution. A moment's consideration shows that all rational solutions of Eqs. 4.1a-4.1f can be obtained in this way.

In summary, we have the following construction, which is shown schematically in Fig. 4.1. Starting from a rational point on the quadratic (we take C, but almost any point on the quadratic distinct from B would do), we construct the line L joining C to an arbitrary point S in $\mathrm{P}\mathbb{Q}^{13}$. This line generically hits the quadratic at a point R and the line M joining R to the singular point B (which lies in the quadratic) generically hits the cubic at a point X, which is a solution of Eqs. 4.1a-4.1f. Varying the position of the point S generates all solutions, so $S \in \mathrm{P}\mathbb{Q}^{13}$ parameterises the space of solutions.

Before delving into the nitty-gritty of the parameterisation, a couple of remarks are in order. One is that we must, at some point, deal with the non-generic cases. In the construction of solutions to the quadratic, we may find that the line L either lies entirely in the quadratic, or is tangent to it at C, meaning no further solution is obtained. The same situation may arise for the line M. As we will see, they do not cause any serious headaches. The other remark is that our parameterisation of the general solution via points $S \in P\mathbb{Q}^{13}$ is clearly redundant. For example, many points S will specify the same line L. As we shall discuss, these redundancies could easily be removed, but would result in uglier formulæ.

[3] This observation goes back at least to Fermat and probably all the way to the diophantine school [140].

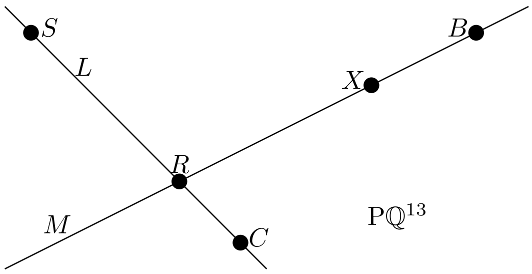

Fig. 4.1 Sketch of the geometric construction. S is any point in the space $\mathrm{P}\mathbb{Q}^{13}$ defined by the linear anomaly cancellation equations, C is any point in $\mathrm{P}\mathbb{Q}^{13}$ satisfying the quadratic equation, and B is the double point of both the quadratic and the cubic equation. L is the line CS, which generically intersects the quadratic at R. M is the line BR which lies in the quadratic and generically intersects the cubic at X, yielding a solution to all anomaly cancellation equations.

4.3 Nitty-gritty of the solution

Given 3 points P, P', P'' in $\mathrm{P}\mathbb{Q}^{17}$ whose homogeneous coordinates are

$$(Q_i, U_i, D_i, L_i, N_i, E_i),\ (Q_i', U_i', D_i', L_i', E_i', N_i'),\ \text{and}\ (Q_i'', U_i'', D_i'', L_i'', E_i'', N_i'') \tag{4.2}$$

respectively, it will be useful to define

$$q(P, P') := \sum_{i=1}^{3} (Q_i Q_i' - 2U_i U_i' + D_i D_i' - L_i L_i' + E_i E_i'), \tag{4.3}$$

and

$$c(P, P', P'') := \sum_{i=1}^{3} (6Q_i Q_i' Q_i'' + 3U_i U_i' U_i'' + 3D_i D_i' D_i'' \\ +2L_i L_i' L_i'' + E_i E_i' E_i'' + N_i N_i' N_i''). \tag{4.4}$$

Now, to find the point R, we take a general point on the line SC, parameterised using homogeneous coordinates as $L = \alpha C + \beta S$, where $\alpha, \beta \in \mathbb{Q}$, and substitute into Eq. 4.1e, yielding

$$\beta(2q(C, S)\alpha + q(S, S)\beta) = 0. \tag{4.5}$$

Cancelling the factor of β (which appears because the point C is a solution) the general solution to this equation is

$$R = q(S,S)C - 2q(C,S)S + \delta_{q(S,S),0}\delta_{q(C,S),0}(aC + bS), \tag{4.6}$$

where the Kronecker deltas (defined as $\delta_{x,y} = 1$ if $x = y$ and $\delta_{x,y} = 0 \forall x \neq y$) encode the cases where the line lies entirely within the quadratic, with $a, b \in \mathbb{Q}$ being arbitrary parameters.

To find the point X, we repeat the procedure, substituting the parameterisation $M = \epsilon R + \gamma B$, where $\epsilon, \gamma \in \mathbb{Q}$, into the cubic Eq. 4.1f, yielding

$$\epsilon^2(3c(B,R,R)\gamma + c(R,R,R)\epsilon) = 0. \tag{4.7}$$

Cancelling the factor of ϵ^2 (which reflects the fact that B is a double point of the cubic) yields

$$X = c(R,R,R)B - 3c(B,R,R)R + \delta_{c(B,R,R),0}\delta_{c(R,R,R),0}(rB + tR), \tag{4.8}$$

with $r, t \in \mathbb{Q}$ being arbitrary parameters.

Denoting by S_{Q_i} the value of Q_i, *etc.*, at the point S; the restriction of S to the sub-space $P\mathbb{Q}^{13}$ defined by the linear equations Eqs. 4.1a-4.1d can be achieved by fixing S_{Q_3}, S_{U_3}, S_{L_3} and S_{E_3} by the relations

$$\begin{aligned}
S_{Q_3} &= \frac{1}{2}\left[-2S_{Q_1} - 2S_{Q_2} + \sum_{i=1}^{3}(S_{D_i} + S_{N_i})\right], \\
S_{U_3} &= -\left[S_{U_1} + S_{U_2} + \sum_{i=1}^{3}(2S_{D_i} + S_{N_i})\right], \\
S_{L_3} &= -\frac{1}{2}\left[2S_{L_1} + 2S_{L_2} + 3\sum_{i=1}^{3}(S_{D_i} + S_{N_i})\right], \\
S_{E_3} &= -S_{E_1} - S_{E_2} + \sum_{i=1}^{3}(3S_{D_i} + 2S_{N_i}).
\end{aligned} \tag{4.9}$$

Our solution is then given in terms of the 18 parameters [4]

$$S_{Q_1}, S_{Q_2}, S_{U_1}, S_{U_2}, S_{D_1}, S_{D_2}, S_{D_3}, S_{L_1}, S_{L_2}, S_{E_1}, S_{E_2}, S_{N_1}, S_{N_2}, S_{N_3}, a, b, r, t \in \mathbb{Q}, \quad (4.10)$$

where the algebraic parameterisation of the solution is as in Eq. 4.8 and R is defined in Eq. 4.6. All that remains to write the parameterisation explicitly is to substitute the charges of B and C from Table 4.1. The rational solution X is then given by

$$\begin{aligned}
Q_1 &= \Gamma - \Sigma + \Lambda S_{Q_1}, \\
Q_2 &= \Gamma + \Lambda S_{Q_2}, \\
Q_3 &= \Gamma + \Sigma + \Lambda S_{Q_3}, \\
U_1 &= -\Gamma - \Sigma + \Lambda S_{U_1}, \\
U_2 &= -\Gamma + \Lambda S_{U_2}, \\
U_3 &= -\Gamma + \Sigma + \Lambda S_{U_3}, \\
D_1 &= -\Gamma - \Sigma + \Lambda S_{D_1}, \\
D_2 &= -\Gamma + \Lambda S_{D_2}, \\
D_3 &= -\Gamma + \Sigma + \Lambda S_{D_3}, \\
L_1 &= -3\Gamma - \Sigma + \Lambda S_{L_1}, \\
L_2 &= -3\Gamma + \Lambda S_{L_2}, \\
L_3 &= -3\Gamma + \Sigma + \Lambda S_{L_3}, \\
E_1 &= 3\Gamma - \Sigma + \Lambda S_{E_1}, \\
E_2 &= 3\Gamma + \Lambda S_{E_2}, \\
E_3 &= 3\Gamma + \Sigma + \Lambda S_{E_3}, \\
N_1 &= 3\Gamma + \Lambda S_{N_1}, \\
N_2 &= 3\Gamma + \Lambda S_{N_2}, \\
N_3 &= 3\Gamma + \Lambda S_{N_3},
\end{aligned} \quad (4.11)$$

[4] A comment on the parameter count is in order. Generically, since we start with 18 affine parameters and have 6 equations, we might expect the solution to have only 12 affine parameters. The 4 parameters a, b, r, t appear only in degenerate cases. Furthermore, one can show that it suffices to restrict to points S corresponding to vectors orthogonal to both B and C, which brings us down to the expected number. We refrain from doing so, since it complicates the (already baroque) formulæ.

where

$$\begin{aligned}
\Gamma &= c(R,R,R) + r\delta_{c(B,R,R),0}\delta_{c(R,R,R),0},\\
\Sigma &= (-3c(B,R,R) + t\delta_{c(B,R,R),0}\delta_{c(R,R,R),0})(q(S,S) + a\delta_{q(S,S),0}\delta_{q(C,S),0}),\\
\Lambda &= (-3c(B,R,R) + t\delta_{c(B,R,R),0}\delta_{c(R,R,R),0})(-2q(C,S) + b\delta_{q(S,S),0}\delta_{q(C,S),0}). \qquad (4.12)
\end{aligned}$$

This solution is provided in the ancillary directory of the `arXiv` preprint of the paper [16] in the form of a `Mathematica` notebook.

One way to check that the above parameterisation captures all solutions is to show that it can be inverted, in the following way. For a known solution T an inverse is a set of the 18 parameters in Eq. 4.10 which return T when substituted into Eq. 4.11. One choice of parameters which achieves this is $S = T$ and, $a = 0$, $b = 1$, $r = 0$ and $t = 1$ (a, b, r and t are only needed when T corresponds to one of the exceptional cases). This inverse has been successfully checked on the 21 549 920 solutions obtained by a scan in [14], which includes all integral solutions (up to permutations) with a maximum absolute charge up to 10.

4.4 Closing Remarks

The general solution Eq. 4.11 to Eqs. 4.1a-4.1f is presented for the first time. It was found by exploiting the presence of a singular point, namely the one corresponding to baryon minus lepton number, which is unique (up to the addition of a multiple of the hypercharge) in that it is a double point of both the quadratic (Eq. 4.1e) and the cubic (Eq. 4.1f). As such, one cannot expect the method to be of general applicability in studying anomaly cancellation in gauge theories. But it nevertheless generalises to some situations that may be of phenomenological interest. A first generalisation is to consider an arbitrary number n of right-handed neutrinos (RHN). Here, it turns out that our method can be applied provided that n is odd and $n \neq 1$, with the charges of the extra neutrinos at the required singular point being given by $N_{2i} = +3, N_{2i+1} = -3$, for $i \geq 2$. It also generalises to an odd number of SM families with an odd number of RHN equal to or exceeding the number of families, though this is probably of lesser phenomenological interest.

Other cases require other methods, but are not without hope. In Ref. [18], for example, a related but different method was used (following the material in the last Chapter) to find a complete solution of the 1 SM family case (with an arbitrary number of RHN) along with a number of existence results for 3 families with a variety of numbers of RHN.

Our solution generalises to *real* charges, corresponding to the case where the gauge group is not compact. The only change in our solution method would be changing rationals to reals everywhere, and as a consequence all parameters in Eq. 4.11 should be taken as real. Unlike in the *one-family* SM with floating real hypercharges where anomaly cancellation enforces them to be commensurate [147] here solutions exist with non-commensurate charges, for example let every SM field's charge be equal to its hypercharge and $N_1 = \sqrt{3}$, $N_2 = 0$, $N_3 = -\sqrt{3}$.

Chapter 5

Semisimple extensions of the Standard Model

The previous two Chapters considered ACCs related to $\mathfrak{u}(1)$ algebras. In Chapter 3 we looked at a pure $\mathfrak{u}(1)$-gauge theory. In Chapter 4 we looked at $\mathfrak{u}(1)$-extensions of the SM. In this Chapter, as the third part of our first foray, we will find all anomaly free semisimple extensions of the SM for a fixed fermionic particle content. We use a brute force approach using a computer.

We will start in §5.1 with a discussion of the motivation and results. In §5.2 we will give the background theory, and in §5.3 describe the computation. Closing remarks will be given in §5.4.

Appendix B gives a more formal discussion of the program outlined in this Chapter, using the mathematical language introduced in §2.2.

5.1 Motivation and results

In searching for theories of physics beyond the Standard Model (SM), it is of interest to ask how the gauge Lie algebra $\mathfrak{sm} := \mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)$ could be extended to a larger Lie algebra $\mathfrak{g} \supset \mathfrak{sm}$. To give a useful answer to this question requires us to make some plausible assumptions, not least because there are, *a priori*, infinitely many such algebras, and because the question is anyway meaningless if we do not specify how $\mathfrak{g}$ acts on the physical degrees of freedom.

The list of possible $\mathfrak{g}$ becomes not only finite, but also can be determined explicitly with the help of a computer, once we specify that $\mathfrak{g}$ is semisimple, as is indicated by the fact that ratios of hypercharges are simple fractions and by the fact that gauge couplings appear to unify, and that $\mathfrak{g}$ acts by a unitary (respectively orthogonal) representation on some given complex (respectively real) matter fields, so as to preserve the lagrangian kinetic terms. (Strictly speaking, to get a finite list we must discard the largest summand of $\mathfrak{g}$ that acts trivially on the matter, which is anyway of no interest, and identify algebras that lead to equivalent physical theories, as we discuss below.) The list may be further curtailed by insisting that $\mathfrak{g}$ be free of local anomalies (global anomalies require us to specify the gauge group, in general, and will not concern us here) with respect to fermionic matter, so that it can be gauged.[1] Such a list can serve as a *vade mecum* for model builders.

In this Chapter, we find all such $\mathfrak{g}$ in the case where the matter fields are taken to be the 3 generations of quarks and leptons of the SM along with 3 $\mathfrak{sm}$-singlet fermions (invoked to give neutrinos their observed masses), bringing the total number of Weyl fermions to 48. A valid $\mathfrak{g}$ is then given by an anomaly-free semisimple algebra that contains $\mathfrak{sm}$ and is contained in $\mathfrak{su}(48)$. Two such algebras will lead to physical theories that are equivalent if they are related by an inner automorphism of $\mathfrak{su}(48)$, since such an automorphism can be effected by a linear change of variables of the fermion fields, which leaves the path integral invariant. They will also lead to equivalent theories if they are related by an outer automorphism of $\mathfrak{g}$, since applying such a transformation does not change the image of $\mathfrak{g}$ in $\mathfrak{su}(48)$.[2]

Although we study just one example, the methods we use can be generalised at whim. For example, one could easily include the scalar Higgs fields of the SM (in which case one seeks all $\mathfrak{g}$ containing $\mathfrak{sm}$ and contained in $\mathfrak{su}(48) \oplus \mathfrak{so}(4)$) or indeed with n additional fermions and m additional scalars (in which case the containing algebra is $\mathfrak{su}(48+n) \oplus \mathfrak{so}(4+m)$).

To illustrate the results we obtain, it is useful to begin with the simpler case with just a single generation of quarks and leptons together with a single $\mathfrak{sm}$-singlet fermion. Here we already know that there are at least 3 possibilities, corresponding to the unification algebras $\mathfrak{ps} := \mathfrak{su}(4) \oplus \mathfrak{su}(2)^{\oplus 2}$ (*i.e.* $\mathfrak{su}(4) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2)$) [133], $\mathfrak{su}(5)$ [81], and $\mathfrak{so}(10)$ [74, 80] (which are all subalgebras of $\mathfrak{su}(16)$; without the extra $\mathfrak{sm}$-singlet, we have just $\mathfrak{su}(5) \subset \mathfrak{su}(15)$); the 'new result' with just a single generation, then, is that there are, perhaps unsurprisingly, no other possibilities.[3]

[1] If we discard the requirement that $\mathfrak{g}$ be semisimple, the list becomes infinite, even if we add only a single anomaly-free $\mathfrak{u}(1)$, as Ref. [16] shows.

[2] Algebras related by other automorphisms are also likely to give identical theories, although we will not explore this point in more detail here.

[3] In each case there is a single non-trivial outer automorphism of $\mathfrak{g}$, so without this equivalence we would find 6 possibilities, corresponding to the fact that one could assign the fermions to carry either of the inequivalent 16-dimensional spinor representations of $\mathfrak{so}(10)$, say.

Things become more interesting with more generations, because of the possibility of interplay between gauge and flavour symmetries. With two generations, for example, there are 45 possible algebras, up to equivalence. Some of these are easily guessed, such as the algebra $\mathfrak{so}(10) \oplus \mathfrak{su}(2)$, with the right hand factor acting as a flavour symmetry mixing the 2 generations, along with $\mathfrak{so}(10) \oplus \mathfrak{su}(5)$, with each summand acting non-trivially only on a single generation. But there also exist possibilities that are less easy to guess and which are interesting in that they mix up flavour and gauge symmetries in an essential way. One of these has algebra $\mathfrak{su}(8) \oplus \mathfrak{su}(2)^{\oplus 2}$, which generalises with N generations to $\mathfrak{su}(4N) \oplus \mathfrak{su}(2)^{\oplus 2}$. This construction relies on the obvious embedding $\mathfrak{su}(4) \oplus \mathfrak{su}(N) \subset \mathfrak{su}(4N)$, showing that it provides a generalisation of the usual Pati-Salam symmetry $\mathfrak{ps}$ for $N = 1$ containing an $\mathfrak{su}(N)$ flavour symmetry. Thus, whereas in the usual Pati-Salam setup lepton number is interpreted as the fourth colour, here flavour is to be interpreted as the remaining $4N - 4$ colours!

A qualitatively different generalisation of the Pati-Salam model with two generations can be obtained as follows. Since the fermion fields in the one-generation version form two irreducible representations, there is a possible $\mathfrak{su}(2)^{\oplus 2}$ flavour symmetry when we go to two generations, giving the algebra $\mathfrak{su}(4) \oplus \mathfrak{su}(2)^{\oplus 4}$, with the 32 fermion fields arranging themselves into the representation $(\mathbf{4}, \mathbf{2}, \mathbf{2}, \mathbf{1}, \mathbf{1}) \oplus (\overline{\mathbf{4}}, \mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{2})$. This is anomaly free because $\mathfrak{su}(2)$ has no anomalous representations. Noting that $\mathfrak{su}(2)^{\oplus 2} \cong \mathfrak{so}(4) \subset \mathfrak{so}(5) \cong \mathfrak{sp}(4)$ and that the defining representation of $\mathfrak{sp}(4)$ restricts to the $(\mathbf{2}, \mathbf{2})$ of $\mathfrak{su}(2)^{\oplus 2}$, it follows that this can be enlarged even further to $\mathfrak{su}(4) \oplus \mathfrak{sp}(4)^{\oplus 2}$, again leading to an essential mixing of flavour symmetry with $\mathfrak{sm}$.

Since this last construction relies on 'accidental' isomorphisms of low-dimensional Lie algebras, we do not expect it to generalise to $N > 2$ generations. Two qualitatively new algebras do appear, however. One uses the embeddings $\mathfrak{su}(16N) \supset \mathfrak{su}(4) \oplus \mathfrak{sp}(2N)^{\oplus 2} \subset \mathfrak{su}(4) \oplus (\mathfrak{so}(N) \oplus \mathfrak{sp}(2))^{\oplus 2}$ along with the isomorphism $\mathfrak{sp}(2) \cong \mathfrak{su}(2)$ to produce an algebra containing $\mathfrak{ps}$ along with an $\mathfrak{so}(N)^{\oplus 2}$ flavour symmetry. The other uses the embeddings $\mathfrak{su}(16N) \supset \mathfrak{su}(4) \oplus \mathfrak{sp}(2N) \oplus \mathfrak{so}(2N) \subset \mathfrak{su}(4) \oplus \mathfrak{so}(N) \oplus \mathfrak{sp}(2) \oplus \mathfrak{so}(2) \oplus \mathfrak{so}(N)$, to produce an algebra containing not $\mathfrak{ps}$, but rather its subalgebra $\mathfrak{su}(4) \oplus \mathfrak{su}(2) \oplus \mathfrak{so}(2) \supset \mathfrak{sm}$, which is not only not semisimple, but is also not left-right symmetric. Again we find a flavour symmetry isomorphic to $\mathfrak{so}(N)^{\oplus 2}$, but now embedded differently in the SM flavour symmetry. These constructions rely on the embeddings $\mathfrak{so}(2) \oplus \mathfrak{so}(N) \subset \mathfrak{so}(2N)$ and $\mathfrak{sp}(2) \oplus \mathfrak{so}(N) \subset \mathfrak{sp}(2N)$. Note that in these examples, flavour symmetry is unified with the electroweak symmetry rather than with the strong symmetry, and in a variety of ways.[4]

The upshot is that with 3 generations we get many more algebras (340, up to equivalence) but all of them can be regarded as variations on the themes already described. This shows that the model building possibilities are in fact extremely limited, unless we include additional fermion fields. Nevertheless, we find a small number of interesting possibilities which mix gauge and flavour symmetries in an essential way. In particular, if such symmetries are gauged, the corresponding gauge bosons can change both flavour and colour/electroweak charges of matter fields.

The algebras organise themselves into 26 (respectively 6) equivalence classes of semisimple anomaly free algebras that are maximal (respectively minimal) with respect to inclusion (note that $\mathfrak{su}(48)$ is not anomaly free, and $\mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)$ is not semisimple, so these definitions are cromulent). We list these in Table 5.1. The full list of 340 algebras can be found in a supplementary file to [19].

5.2 Theory

We now describe the mathematical formulation of the problem. Because of the need to track automorphisms, this is most easily done using the language of category theory. A suitable category has objects, labelled $(\mathfrak{g}, \alpha, \beta, \gamma)$, given by commuting diagrams of the form

$$\begin{tikzcd} & \mathfrak{g} \arrow[dr, "\beta"] & \\ \mathfrak{sm} \arrow[ur, "\alpha"] \arrow[rr, "\gamma"] & & \mathfrak{su}(48) \end{tikzcd} \tag{5.1}$$

where $\mathfrak{g}$ is a semisimple Lie algebra, and α, β, γ are embeddings, *i.e.* injective maps that preserve Lie brackets. A morphism, labelled (j, i), from $(\mathfrak{g}', \alpha', \beta', \gamma')$ to $(\mathfrak{g}, \alpha, \beta, \gamma)$ is then a commuting diagram of the form

$$\begin{tikzcd} & \mathfrak{g} \arrow[dr, "\beta"] & \\ \mathfrak{sm} \arrow[ur, "\alpha"] \arrow[rr, "\gamma"] & \mathfrak{g}' \arrow[u, "j"] \arrow[dr, "\beta'"] & \mathfrak{su}(48) \\ \mathfrak{sm} \arrow[u, "\mathrm{id}"] \arrow[ur, "\alpha'"] \arrow[rr, "\gamma'"] & & \mathfrak{su}(48) \arrow[u, "i"] \end{tikzcd} \tag{5.2}$$

[4] For an even number of generations, we also have an embedding $\mathfrak{su}(16N) \supset \mathfrak{su}(4) \oplus \mathfrak{so}(2N)^{\oplus 2} \supset \mathfrak{su}(4) \oplus (\mathfrak{sp}(2) \oplus \mathfrak{sp}(N))^{\oplus 2}$, using the embedding $\mathfrak{sp}(2) \oplus \mathfrak{sp}(N) \subset \mathfrak{so}(2N)$.

where j is an embedding and i is an inner automorphism. We call a morphism (j, i) an equivalence if j is an isomorphism (*i.e.* if j also surjects). We say such a diagram 5.1 is maximal (resp. minimal) if the only morphisms out of (resp. into) it are equivalences. Roughly speaking, $(\mathfrak{g}, \alpha, \beta, \gamma)$ is maximal if there is no larger semisimple algebra containing $\mathfrak{g}$ and contained within $\mathfrak{su}(48)$, such that the embeddings are consistent (as per the diagram above). It is minimal, if no smaller semisimple algebra containing the $\mathfrak{sm}$ and contained within $\mathfrak{g}$ with consistent embeddings, exists. For example, the usual $\mathfrak{su}(5)$ GUT is minimal, but the $\mathfrak{so}(10)$ GUT is not since it contains the $\mathfrak{su}(5)$ GUT.

Our goal is then to find all inequivalent diagrams $(\mathfrak{g}, \alpha, \beta, \gamma)$ for which γ corresponds to a SM embedding. To do so, we choose Cartan subalgebras of each algebra appearing in diagram 5.1, which we denote $\mathfrak{h}_{\mathfrak{sm}}$, $\mathfrak{h}$ and $\mathfrak{h}_{48}$ for $\mathfrak{sm}$, $\mathfrak{g}$ and $\mathfrak{su}(48)$ respectively. Up to equivalence, we choose α, β, and γ such that these get embedded into one another.

Equivalently we can seek a pair of diagrams

$$\begin{array}{ccccc} & & \mathfrak{g} & & \\ & \overset{\kappa}{\nearrow} & & \overset{\beta}{\searrow} & \\ \mathfrak{su}(3) \oplus \mathfrak{su}(2) & & \xrightarrow{\quad\rho\quad} & & \mathfrak{su}(48) \end{array} \qquad \begin{array}{ccccc} & & \mathfrak{h} & & \\ & \overset{\tilde{\kappa}}{\nearrow} & & \overset{\beta|}{\searrow} & \\ \mathfrak{u}(1) & & \xrightarrow{\quad\tilde{\rho}\quad} & & \mathfrak{h}_{48} \end{array} \tag{5.3}$$

where $\gamma = \rho \oplus \tilde{\rho}$ corresponds to a SM embedding, $\alpha = \kappa \oplus \tilde{\kappa}$, and $\beta|$ on the right hand side denotes the obvious restriction map. One has to take care to ensure that the image of κ and $\tilde{\kappa}$ commute.

5.3 Computation

In rough terms, our approach to the computation is as follows. The first step is to evaluate all embeddings β. Up to inner automorphisms of $\mathfrak{su}(48)$ these are, of course, inequivalent representations of semisimple Lie algebras of dimension 48 and so can be found using standard representation theory techniques. We keep only those which are anomaly free. For every $\mathfrak{g}$ for which a β exists, we then find all embeddings κ up to outer automorphisms using the theory of maximal embeddings. We then find all κ and β such that there exists a ρ equivalent to the SM, which form the left-hand diagram in Eq. 5.3. For a given diagram, we then determine if compatible $\tilde{\kappa}$ and $\tilde{\rho}$ exist (taking account of possible inner-automorphisms which may need to be applied). By finding all embeddings $j : \mathfrak{g}' \to \mathfrak{g}$ for algebras which appear in our final list, the program then checks which are maximal and which are minimal.

The program itself uses projection matrices rather than embeddings. Projection matrices describe how the weights project from one algebra to the other. For a given $\mathfrak{g}$, the projection matrices corresponding to potentially allowed κ's can be found using the theory of maximal embeddings [112, 67, 66, 110, 152]. (We use the `Mathematica` program `LieART` [71, 72] to generate the maximal projection matrices themselves).

The output of the program (provided in a supplementary file to [19]) consists of the highest weights of the representation specified by β and the projection matrix of the embedding κ, to which we have appended a final row specifying $\tilde{\kappa}$ (to wit, acting on the weights of $\mathfrak{g}$, this row returns the corresponding $\mathfrak{u}(1) \subset \mathfrak{sm}$ charges). It is explicitly demonstrated in Appendix B, that our list of outputs catalogues the equivalence classes described in §5.2.

This approach results in a number of practical issues when it comes to carrying out the computation, which we now describe, along with their resolutions, in rough order of importance.

(i) Since $\mathfrak{su}(2)$ has anomaly-free irreducible representations of every dimension, there are many possible anomaly-free embeddings of ideals of $\mathfrak{g}$ made up of $\mathfrak{su}(2)$s in $\mathfrak{su}(48)$. For example, there are $\mathcal{O}(10^5)$ for $\mathfrak{g} = \mathfrak{su}(2)$ and $\mathcal{O}(10^7)$ for $\mathfrak{g} = \mathfrak{su}(2)^{\oplus 2}$. We reduce this by first finding all possible β for $\mathfrak{g}$ without an $\mathfrak{su}(2)$ ideal and then requiring that they contain $\mathfrak{su}(3) \subset \mathfrak{sm}$ (here we use the fact that the restriction of κ to $\mathfrak{su}(3)$ has to map trivially into any $\mathfrak{su}(2)$ ideal of $\mathfrak{g}$). To these $\mathfrak{g}$ we add all possible ideals made up of $\mathfrak{su}(2)$s and retest to see if a full κ exists.

(ii) Even after ignoring ideals made up of $\mathfrak{su}(2)$s, there are still $\mathcal{O}(10^6)$ anomaly free embeddings of $\mathfrak{g}$. We determine these in a bottom-up fashion by first finding the $\mathcal{O}(10^3)$ anomaly-free representations of dimension 48 of the $\mathcal{O}(10^2)$ possible simple $\mathfrak{g}$ (*e.g.* $\mathbf{5} \oplus \overline{\mathbf{10}}$ plus 33 singlets of $\mathfrak{su}(5)$) and then using these to find all anomaly-free representations of possible semisimple $\mathfrak{g}$ of dimension 48. Here, we use the fact that a representation of a semisimple algebra is anomaly-free iff. its restriction to any simple ideal is anomaly-free. For example, the representations of $\mathfrak{su}(4) \oplus \mathfrak{sp}(4)$ given by

1. $(\mathbf{4}, \mathbf{4}) \oplus (\overline{\mathbf{4}}, \mathbf{1})^{\oplus 4} \oplus (\mathbf{1}, \mathbf{1})^{\oplus 16}$
2. $(\overline{\mathbf{4}}, \mathbf{4}) \oplus (\mathbf{4}, \mathbf{1})^{\oplus 4} \oplus (\mathbf{1}, \mathbf{1})^{\oplus 16}$
3. $(\mathbf{4}, \mathbf{1})^{\oplus 4} \oplus (\overline{\mathbf{4}}, \mathbf{1})^{\oplus 4} \oplus (\mathbf{1}, \mathbf{4})^{\oplus 4}$

are anomaly-free because they all reduce to the $\mathbf{4}^{\oplus 4} \oplus \overline{\mathbf{4}}^{\oplus 4}$ of $\mathfrak{su}(4)$ and the $\mathbf{4}^{\oplus 4}$ of $\mathfrak{sp}(4)$ (plus the appropriate number of singlets). This bottom-up method is also used later to find all representations when ideals made up of $\mathfrak{su}(2)$s are included.

Maximal		
	Algebra	Fermion representations corresponding to β
1	$\mathfrak{so}(10) \oplus \mathfrak{su}(2)$	$(\mathbf{16}, \mathbf{3})$
2	$\mathfrak{so}(10)^{\oplus 3}$	$(\mathbf{16},\mathbf{1},\mathbf{1}) \oplus (\mathbf{1},\mathbf{16},\mathbf{1}) \oplus (\mathbf{1},\mathbf{1},\mathbf{16})$
3	$\mathfrak{so}(10)^{\oplus 2} \oplus \mathfrak{su}(2)$	$(\mathbf{16},\mathbf{1},\mathbf{1}) \oplus (\mathbf{1},\mathbf{16},\mathbf{2})$
4	$\mathfrak{su}(4) \oplus \mathfrak{sp}(6)^{\oplus 2}$	$(\bar{\mathbf{4}},\mathbf{6},\mathbf{1}) \oplus (\mathbf{4},\mathbf{1},\mathbf{6})$
5	$\mathfrak{su}(4)^{\oplus 2} \oplus \mathfrak{sp}(6)$	$(\bar{\mathbf{4}},\mathbf{6},\mathbf{1}) \oplus (\mathbf{4},\mathbf{1},\mathbf{6})$
6	$\mathfrak{su}(12) \oplus \mathfrak{su}(2)^{\oplus 2}$	$(\overline{\mathbf{12}},\mathbf{2},\mathbf{1}) \oplus (\mathbf{12},\mathbf{1},\mathbf{2})$
7	$\mathfrak{su}(4) \oplus \mathfrak{sp}(4)^{\oplus 2} \oplus \mathfrak{so}(10)$	$(\bar{\mathbf{4}},\mathbf{4},\mathbf{1},\mathbf{1}) \oplus (\mathbf{4},\mathbf{1},\mathbf{4},\mathbf{1}) \oplus (\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{16})$
8	$\mathfrak{su}(5) \oplus \mathfrak{su}(2)^{\oplus 3}$	$(\bar{\mathbf{5}},\mathbf{3},\mathbf{1},\mathbf{1}) \oplus (\mathbf{10},\mathbf{1},\mathbf{3},\mathbf{1}) \oplus (\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2}) \oplus (\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$
9	$\mathfrak{su}(5) \oplus \mathfrak{su}(2)^{\oplus 3}$	$(\bar{\mathbf{5}},\mathbf{3},\mathbf{1},\mathbf{1}) \oplus (\mathbf{10},\mathbf{1},\mathbf{3},\mathbf{1}) \oplus (\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{3})$
10	$\mathfrak{su}(5) \oplus \mathfrak{su}(2)^{\oplus 3}$	$(\bar{\mathbf{5}},\mathbf{1},\mathbf{1},\mathbf{1}) \oplus (\bar{\mathbf{5}},\mathbf{2},\mathbf{1},\mathbf{1}) \oplus (\mathbf{10},\mathbf{1},\mathbf{3},\mathbf{1}) \oplus (\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2}) \oplus (\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$
11	$\mathfrak{su}(5) \oplus \mathfrak{su}(2)^{\oplus 3}$	$(\bar{\mathbf{5}},\mathbf{1},\mathbf{1},\mathbf{1}) \oplus (\bar{\mathbf{5}},\mathbf{2},\mathbf{1},\mathbf{1}) \oplus (\mathbf{10},\mathbf{1},\mathbf{3},\mathbf{1}) \oplus (\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{3})$
12	$\mathfrak{su}(5) \oplus \mathfrak{su}(2)^{\oplus 3}$	$(\mathbf{10},\mathbf{1},\mathbf{1},\mathbf{1}) \oplus (\bar{\mathbf{5}},\mathbf{3},\mathbf{1},\mathbf{1}) \oplus (\mathbf{10},\mathbf{1},\mathbf{2},\mathbf{1}) \oplus (\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2}) \oplus (\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$
13	$\mathfrak{su}(5) \oplus \mathfrak{su}(2)^{\oplus 3}$	$(\mathbf{10},\mathbf{1},\mathbf{1},\mathbf{1}) \oplus (\bar{\mathbf{5}},\mathbf{3},\mathbf{1},\mathbf{1}) \oplus (\mathbf{10},\mathbf{1},\mathbf{2},\mathbf{1}) \oplus (\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{3})$
14	$\mathfrak{su}(5)^{\oplus 2} \oplus \mathfrak{so}(10) \oplus \mathfrak{su}(2)$	$(\bar{\mathbf{5}},\mathbf{1},\mathbf{1},\mathbf{1}) \oplus (\mathbf{10},\mathbf{1},\mathbf{1},\mathbf{1}) \oplus (\mathbf{1},\bar{\mathbf{5}},\mathbf{1},\mathbf{1}) \oplus (\mathbf{1},\mathbf{10},\mathbf{1},\mathbf{1}) \oplus (\mathbf{1},\mathbf{1},\mathbf{16},\mathbf{1})$ $\oplus(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2})$
15	$\mathfrak{su}(5)^{\oplus 3} \oplus \mathfrak{su}(2)$	$(\bar{\mathbf{5}},\mathbf{1},\mathbf{1},\mathbf{1}) \oplus (\mathbf{10},\mathbf{1},\mathbf{1},\mathbf{1}) \oplus (\mathbf{1},\bar{\mathbf{5}},\mathbf{1},\mathbf{1}) \oplus (\mathbf{1},\mathbf{10},\mathbf{1},\mathbf{1}) \oplus (\mathbf{1},\mathbf{1},\bar{\mathbf{5}},\mathbf{1})$ $\oplus(\mathbf{1},\mathbf{1},\mathbf{10},\mathbf{1}) \oplus (\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{3})$
16	$\mathfrak{su}(8) \oplus \mathfrak{so}(10) \oplus \mathfrak{su}(2)^{\oplus 2}$	$(\mathbf{1},\mathbf{16},\mathbf{1},\mathbf{1}) \oplus (\bar{\mathbf{8}},\mathbf{1},\mathbf{2},\mathbf{1}) \oplus (\mathbf{8},\mathbf{1},\mathbf{1},\mathbf{2})$
17	$\mathfrak{su}(4) \oplus \mathfrak{sp}(4) \oplus \mathfrak{so}(10) \oplus \mathfrak{su}(2)^{\oplus 2}$	$(\bar{\mathbf{4}},\mathbf{4},\mathbf{1},\mathbf{1},\mathbf{1}) \oplus (\mathbf{1},\mathbf{1},\mathbf{16},\mathbf{1},\mathbf{1}) \oplus (\mathbf{4},\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{2})$
18	$\mathfrak{su}(4) \oplus \mathfrak{sp}(4) \oplus \mathfrak{so}(10) \oplus \mathfrak{su}(2)^{\oplus 2}$	$(\bar{\mathbf{4}},\mathbf{4},\mathbf{1},\mathbf{1},\mathbf{1}) \oplus (\mathbf{1},\mathbf{1},\mathbf{16},\mathbf{1},\mathbf{1}) \oplus (\mathbf{4},\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{2})$
19	$\mathfrak{su}(4) \oplus \mathfrak{sp}(6) \oplus \mathfrak{su}(2)^{\oplus 3}$	$(\bar{\mathbf{4}},\mathbf{6},\mathbf{1},\mathbf{1},\mathbf{1}) \oplus (\mathbf{4},\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{1}) \oplus (\mathbf{4},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2})$
20	$\mathfrak{su}(4) \oplus \mathfrak{sp}(6) \oplus \mathfrak{su}(2)^{\oplus 3}$	$(\bar{\mathbf{4}},\mathbf{6},\mathbf{1},\mathbf{1},\mathbf{1}) \oplus (\mathbf{4},\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{1}) \oplus (\mathbf{4},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2})$
21	$\mathfrak{su}(4)^{\oplus 2} \oplus \mathfrak{su}(2)^{\oplus 3}$	$(\bar{\mathbf{4}},\mathbf{6},\mathbf{1},\mathbf{1},\mathbf{1}) \oplus (\mathbf{4},\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{1}) \oplus (\mathbf{4},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2})$
22	$\mathfrak{su}(5) \oplus \mathfrak{so}(10) \oplus \mathfrak{su}(2)^{\oplus 3}$	$(\mathbf{1},\mathbf{16},\mathbf{1},\mathbf{1},\mathbf{1}) \oplus (\bar{\mathbf{5}},\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1}) \oplus (\mathbf{10},\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1}) \oplus (\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2})$
23	$\mathfrak{su}(5)^{\oplus 2} \oplus \mathfrak{su}(2)^{\oplus 3}$	$(\mathbf{1},\bar{\mathbf{5}},\mathbf{1},\mathbf{1},\mathbf{1}) \oplus (\mathbf{1},\mathbf{10},\mathbf{1},\mathbf{1},\mathbf{1}) \oplus (\bar{\mathbf{5}},\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1}) \oplus (\mathbf{10},\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1})$ $\oplus(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2}) \oplus (\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$
24	$\mathfrak{su}(5)^{\oplus 2} \oplus \mathfrak{su}(2)^{\oplus 3}$	$(\mathbf{1},\bar{\mathbf{5}},\mathbf{1},\mathbf{1},\mathbf{1}) \oplus (\mathbf{1},\mathbf{10},\mathbf{1},\mathbf{1},\mathbf{1}) \oplus (\bar{\mathbf{5}},\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1}) \oplus (\mathbf{10},\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1})$ $\oplus(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{3})$
25	$\mathfrak{su}(4) \oplus \mathfrak{so}(10) \oplus \mathfrak{su}(2)^{\oplus 4}$	$(\mathbf{1},\mathbf{16},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1}) \oplus (\mathbf{4},\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{1}) \oplus (\bar{\mathbf{4}},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{2})$
26	$\mathfrak{su}(4)^{\oplus 2} \oplus \mathfrak{sp}(4)^{\oplus 2} \oplus \mathfrak{su}(2)^{\oplus 2}$	$(\bar{\mathbf{4}},\mathbf{1},\mathbf{4},\mathbf{1},\mathbf{1},\mathbf{1}) \oplus (\mathbf{4},\mathbf{1},\mathbf{1},\mathbf{4},\mathbf{1},\mathbf{1}) \oplus (\mathbf{1},\bar{\mathbf{4}},\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1}) \oplus (\mathbf{1},\mathbf{4},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2})$
Minimal		
27	$\mathfrak{su}(5)$	$(\bar{\mathbf{5}})^{\oplus 3} \oplus (\mathbf{10})^{\oplus 3} \oplus (\mathbf{1})^{\oplus 3}$
28	$\mathfrak{su}(4) \oplus \mathfrak{su}(2)^{\oplus 2}$	$(\bar{\mathbf{4}},\mathbf{2},\mathbf{1})^{\oplus 3} \oplus (\mathbf{4},\mathbf{1},\mathbf{2})^{\oplus 3}$
29	$\mathfrak{su}(4)^{\oplus 2} \oplus \mathfrak{su}(2)$	$(\bar{\mathbf{4}},\mathbf{6},\mathbf{1}) \oplus (\mathbf{4},\mathbf{1},\mathbf{2})^{\oplus 3}$
30	$\mathfrak{su}(4) \oplus \mathfrak{su}(5) \oplus \mathfrak{su}(2)^{\oplus 2}$	$(\mathbf{1},\bar{\mathbf{5}},\mathbf{1},\mathbf{1})^{\oplus 2} \oplus (\mathbf{1},\mathbf{10},\mathbf{1},\mathbf{1})^{\oplus 2} \oplus (\bar{\mathbf{4}},\mathbf{1},\mathbf{2},\mathbf{1}) \oplus (\mathbf{4},\mathbf{1},\mathbf{1},\mathbf{2}) \oplus (\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})^{\oplus 2}$
31	$\mathfrak{su}(4) \oplus \mathfrak{su}(5) \oplus \mathfrak{su}(2)^{\oplus 2}$	$(\mathbf{1},\bar{\mathbf{5}},\mathbf{1},\mathbf{1}) \oplus (\mathbf{1},\mathbf{10},\mathbf{1},\mathbf{1}) \oplus (\bar{\mathbf{4}},\mathbf{1},\mathbf{2},\mathbf{1})^{\oplus 2} \oplus (\mathbf{4},\mathbf{1},\mathbf{1},\mathbf{2})^{\oplus 2} \oplus (\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$
32	$\mathfrak{su}(4)^{\oplus 3} \oplus \mathfrak{su}(2)^{\oplus 2}$	$(\bar{\mathbf{4}},\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1}) \oplus (\mathbf{1},\bar{\mathbf{4}},\mathbf{1},\mathbf{2},\mathbf{1}) \oplus (\mathbf{1},\mathbf{1},\bar{\mathbf{4}},\mathbf{2},\mathbf{1}) \oplus (\mathbf{4},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2}) \oplus (\mathbf{1},\mathbf{4},\mathbf{1},\mathbf{1},\mathbf{2})$ $\oplus(\mathbf{1},\mathbf{1},\mathbf{4},\mathbf{1},\mathbf{2})$

Table 5.1 All maximal and minimal anomaly-free algebras for exactly 3 generations of SM fermions plus 3 right-handed neutrinos.

(iii) Finding the possible representations of semisimple algebras above requires considering a large number of permutations of a list (as do other steps in the calculation, *e.g.* finding κ from maximal embeddings). For example, to find the anomaly-free representations 1-3 above requires consideration of around 500 different permutations. The computation is greatly expedited by the use of an algorithm that determines which permutations can be skipped based on previous cases.

Two more minor improvements are as follows: *(iv)* the fact that one can discard β whose corresponding representation has a non-trivial part of dimension fewer than 45 (or 36 before we include $\mathfrak{su}(2)$ ideals), since these cannot lead to a valid α; *(v)* in a similar vein, no α exists for those β whose corresponding representations have more than 3 vanishing weights, or weights in negative pairs, since such weights must be associated with $\mathfrak{sm}$-singlets of which there are just 3.

The program took less than an hour to run on a personal computer. As such, model-builders should find it easy to adapt it to other cases of interest.[5]

5.4 Closing Remarks

We have produced, for the first time, a definitive list of semisimple Lie algebras that contain the SM Lie algebra, are free of local anomalies, and act by a unitary representation on SM fermions plus 3 singlet neutrinos. Such extensions can mix flavour, colour, and electroweak symmetries in non-trivial ways. There are 340 physically-inequivalent algebras in total; whilst these are provided in a supplementary file to [19], the minimal and maximal ones are listed in Table 5.1. No exceptional Lie algebras appear, since they require fermions beyond those in the SM plus 3 singlet neutrinos. Many of the symmetries listed are semi-familiar, being variations on the theme of well-known unification and flavour symmetries. A few of the symmetries in our catalogue have the novel feature of combining unification and flavour symmetries in an essential way, motivating their further study. For example, we have $\mathfrak{su}(12) \oplus \mathfrak{su}(2)^{\oplus 2}$, $\mathfrak{su}(8) \oplus \mathfrak{su}(2)^{\oplus 2}$, $\mathfrak{su}(4) \oplus \mathfrak{sp}(6)^{\oplus 2}$, $\mathfrak{su}(4) \oplus \mathfrak{sp}(6) \oplus \mathfrak{so}(6)$, and $\mathfrak{su}(4) \oplus \mathfrak{sp}(4)^{\oplus 2}$.[6]

Adding additional matter fields changes the list, but is straightforward to carry out, in principle. An interesting example to investigate would be to add a Dirac fermion in the same representation as the Higgs boson, since it constitutes a viable candidate for weakly-interacting thermal relic dark matter.

[5] The programs and instructions on their use can be found in the ancillary information attached to the `arXiv` preprint version of the paper [19].

[6] The $\mathfrak{su}(12) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ example appeared in a machine-learning scan of a subset of Type IIA orientifolds on $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ with intersecting D6-branes [108]

Chapter 6

Quantum mechanics in magnetic backgrounds

The previous three Chapters were related to the study of anomaly free algebras and our first foray. In this Chapter we study our second foray, which we introduced in §1.2. The aim is to overcome the two obstructions to using harmonic analysis in solving quantum mechanics problems with 'magnetic backgrounds' (corresponding to a type of topological terms). The two obstructions are the fact it may not be possible to write down a globally defined lagrangian and, even when it is, the lagrangian may shift by a total derivative under the action of the symmetry group. The method to overcome these obstructions uses principal bundles and central extensions.

The pertinent mathematical definitions for this Chapter were introduced in §2.3. We start this Chapter by illustrating the ideas with elementary (but incomplete) discussions of the examples of planar motion in a uniform magnetic field (§6.1.1) and of rigid body rotation (§6.1.2). These examples are particularly transparent because, for the former, the principal bundle is (topologically) trivial, meaning that the effects come from the magnetic field, while for the latter, the magnetic field vanishes (though the vector potential does not) so all effects arise from the topology of the bundle.

After this, in §6.2, we give full mathematical details of the method. We then complete the discussion of rigid body rotation (§6.3.1) and give a series of other examples which illustrate the method: the Dirac monopole (§6.3.2), a charged particle in the electromagnetic field of a dyon (§6.3.3), a repeat of Landau levels on a plane (but using the full Euclidean group (§6.3.4)), motion on the Heisenberg group manifold (§6.3.5), and motion in a uniform magnetic field with a mass that varies with position (§6.3.6). All the examples considered in this Chapter are summarised in Table 6.1. Our conclusions are presented in §6.4. Appendix C contains a discussion of the rudiments of harmonic analysis in the presence of constraints, which is used throughout this Chapter.

6.1 Prototypes

6.1.1 Planar motion in a uniform magnetic field

Our first example is one made famous by Landau, in which a particle moves in the xy-plane with a uniform magnetic field $B \in \mathbb{R}$ in the z-direction. In this example, the subtleties are entirely due to the presence of the magnetic field. In particular, no matter what gauge is chosen, the usual lagrangian shifts by a non-vanishing total derivative under the action of the symmetry group, which for the purposes of the present discussion we take to be translations in $\mathbb{R}^2$. As a result, the usual quantum hamiltonian does not commute with the momenta and one cannot solve via a Fourier transform (which corresponds to harmonic analysis with respect to the group $\mathbb{R}^2$).

To circumvent this we write the action, contributing to the action phase e^{iS}, as

$$S = \int dt \left(\frac{1}{2}\dot{x}^2 + \frac{1}{2}\dot{y}^2 - \dot{s} - By\dot{x} \right), \tag{6.1}$$

with an additional degree of freedom $s \in \mathbb{R}$, with $s \sim s + 2\pi$, which shall be redundant. The advantage of doing so is that, unlike the lagrangian without s, which shifts by a total derivative proportional to $B\dot{x}$ under a translation in y, the lagrangian in Eq. 6.1 is genuinely invariant under a central extension by $U(1)$ of the translation group. Note that we have chosen to work in the Landau gauge here, although our method, as we shall soon discuss in detail, is independent of gauge choice.

This central extension is the Heisenberg group, Hb, defined as the equivalence classes of $(x, y, s) \in \mathbb{R}^3$ under the equivalence relation $s \sim s + 2\pi$, with multiplication law

$$[(x', y', s')] \cdot [(x, y, s)] = [(x + x', y + y', s + s' - By'x)], \tag{6.2}$$

and corresponding to $\mathbb{R}^2 \times S^1$ as a manifold. Notice that the group $\mathbb{R}^2$ of translations appears not as a *sub*group of Hb, but rather as the *quotient* group of Hb with respect to the central $U(1)$ subgroup $\{[(0, 0, s)]\}$. Thus we have a homomorphism Hb $\to \mathbb{R}^2$, given explicitly by $[(x, y, s)] \mapsto (x, y)$, whose kernel is the central $U(1)$. Notice that our definition of the group multiplication law depends on $B \in \mathbb{R}$, reflecting the fact that even though the groups with distinct values of B are isomorphic as groups, they are not isomorphic as central extensions.

Given Eq. 6.1, the momentum p_s conjugate to s satisfies the constraint $p_s + 1 = 0$. We take care of this in the usual way, by forming the total hamiltonian (see *e.g.* [92])

$$H = \frac{1}{2}\left(p_x + By\right)^2 + \frac{1}{2}p_y^2 + v(t)\left(p_s + 1\right), \tag{6.3}$$

with p_x and p_y being the momenta conjugate to x and y respectively, and with $v(t)$ being a Lagrange multiplier. Upon quantising (something we will later define formally), we obtain the hamiltonian operator

$$\hat{H} = \frac{1}{2}\left(-i\frac{\partial}{\partial x} + By\right)^2 - \frac{1}{2}\frac{\partial^2}{\partial y^2} + v(t)\left(-i\frac{\partial}{\partial s} + 1\right), \tag{6.4}$$

which has a natural action on the space of square integrable functions on the Heisenberg group, $L^2(\text{Hb})$. The physical Hilbert space $\mathcal{H}$ must take account of the constraint (or, equivalently, the redundancy in our description), so we define it to be not $L^2(\text{Hb})$, but rather the subspace

$$\mathcal{H} = \left\{\Psi(x,y,s) \in L^2(\text{Hb}) \,\middle|\, \left(-i\frac{\partial}{\partial s} + 1\right)\Psi(x,y,s) = 0\right\}. \tag{6.5}$$

Note that this subspace of $L^2(\text{Hb})$ is closed under the action of the Heisenberg group and under the action of $\hat{H}$, implying that it is also closed under time evolution.

We then want to solve the time-independent Schrödinger equation (from hereon 'SE') $\hat{H}\Psi = E\Psi$. To solve the SE, we decompose Ψ into unitary irreducible representations (henceforth 'unirreps') of Hb:[1]

$$\Psi(x,y,s) = \int drdt \frac{|B|}{2\pi}\pi^B(r,t;x,y,s)f(r,t), \tag{6.6}$$

where $r, t \in \mathbb{R}$ are real numbers. Here,

$$\pi^k(r,t;x,y,s) = e^{ik(xr - s/B)}\delta(r + y - t), \qquad k/B \in \mathbb{Z}, \tag{6.7}$$

which denote the matrix elements of the infinite-dimensional unirreps of Hb, which act on the vector space $L^2(\mathbb{R}, dt)$. The fact that only the unirrep with $k = B$ appears in the decomposition in Eq. 6.6, follows from enforcing the constraint in Eq. 6.5, as we show in Appendix C.

[1] To say we are 'decomposing Ψ into unirreps of Hb' is a slight abuse of terminology; what we mean, precisely, is discussed in §6.2.1.

Notice that with this decomposition $\Psi(x, y, s)$ may not be square integrable (as the matrix elements of π^B themselves are not). As such, once we have found our 'solutions' to the SE with this decomposition we must check that they are square integrable (or more generally the limit of a Weyl sequence). This subtlety will be omitted here due to the familiar form our final solutions will take.

Substituting the decomposition in Eq. 6.6 into the SE, and using the constraint to eliminate the Lagrange multiplier, yields

$$\frac{|B|}{2\pi}\int drdt\left(\frac{1}{2}\left(-i\frac{\partial}{\partial x}+By\right)^2-\frac{1}{2}\frac{\partial^2}{\partial y^2}-E\right)f(r,t)e^{i(Bxr-s)}\delta(r+y-t)=0. \quad (6.8)$$

After some straightforward manipulation, this reduces to

$$\left(\frac{1}{2}B^2t^2-\frac{1}{2}\frac{\partial^2}{\partial t^2}-E\right)f(r,t)=0. \quad (6.9)$$

This differential equation, which we recognise as the SE for the simple harmonic oscillator, has the solutions

$$f(r,t)=H_n\left(\sqrt{|B|}t\right)e^{-|B|t^2/2}g(r), \quad E=|B|(n+1/2), \quad (6.10)$$

where $H_n(x)$ are the Hermite polynomials and $g(r)$ is an arbitrary function of r. The corresponding eigenfunctions are thus

$$\Psi_n(x,y,s)=\frac{|B|}{2\pi}\int drdtH_n\left(\sqrt{|B|}t\right)e^{-|B|t^2/2}g(r)e^{i(Bxr-s)}\delta(r+y-t). \quad (6.11)$$

We can of course eliminate our redundant degree of freedom, by setting $s=0$ for example, to obtain corresponding wavefunctions living in $L^2(\mathbb{R}^2)$ (more precisely, the wavefunction is described by a section of a Hermitian line bundle). In the above expression $g(r)$ accounts for the degeneracy in the Landau levels. On choosing $g(r)=\delta(r-\alpha/B)$ for $\alpha\in\mathbb{R}$ (and setting $s=0$) we arrive at familiar solutions to this system, of the form

$$\Psi_{n,\alpha}(x,y)=e^{i\alpha x}H_n\left(\sqrt{|B|}(y+\alpha/B)\right)e^{-\frac{|B|}{2}(y+\alpha/B)^2}. \quad (6.12)$$

Now let us recap what we have achieved. Certainly, our result for the spectrum is not new; nor are our observations regarding the momentum generators. Rather, what is new is the observation that we can reformulate the problem via a redundant description, in which a central extension of G by $U(1)$ acts on the configuration space of that redundant description, in a way that allows us to solve for the spectrum using methods of harmonic analysis. While this may seem like overkill, it is important to realise that Landau's original method of solution [106] only works for this specific system of a particle on $\mathbb{R}^2$ in a magnetic background, and moreover works only in a particular gauge (the 'Landau gauge'). It is not at all clear how such an approach could be generalised to other target spaces (or gauges). In contrast, as we shall soon see in §6.2, using harmonic analysis on a central extension can be generalised to any group G acting on any target space manifold M, since it exploits the underlying group-theoretic structure of the system.

6.1.2 Bosonic versus fermionic rigid bodies

Our second prototypical example illustrates the approach in a case where one cannot form a globally-defined lagrangian without extending the configuration space by a redundant degree of freedom. This prototype also provides an example where the relation to magnetic fields is not immediately apparent.

To wit, we consider the quantum mechanics of a rigid body in three space dimensions, whose configuration space is $SO(3)$, with dynamics invariant under the rotation group. Evidently, such a rigid body could be either a boson or a fermion (it could, for example, be a composite made up of either an even or odd number of electrons and protons). If it is a fermion, then its wavefunction should acquire a factor of -1 when the body undergoes a complete rotation about some axis and we expect, on general physical grounds, that we can represent this effect via a local lagrangian term. To see how it can be done, we first note that the term should be both $SO(3)$ invariant and topological. It is thus reasonable to guess that it can be written in terms of a magnetic field, or more precisely, a connection on some $U(1)$-principal bundle over $SO(3)$.[2] Confirmation that this is indeed the case comes from the fact that (up to equivalence), there are just two $U(1)$-principal bundles over $SO(3)$ (to see this, note that such bundles are classified by the first Chern class, which is a cohomology class in $H^2(SO(3), \mathbb{Z}) \cong \mathbb{Z}/2$). Thus we have the trivial bundle $SO(3) \times U(1)$ and a non-trivial bundle, which we may take to be $U(2)$, the group of 2×2 unitary matrices. Clearly, these are not only $U(1)$-principal bundles, but also they have the structure of central extensions of $SO(3)$ by $U(1)$, which we need for our construction. The trivial bundle admits the zero

[2]For those readers unfamiliar with principal bundles, we note that a technical understanding should not be necessary to follow the discussion in this Section. Nonetheless, since the notion of a principal bundle shall be central to the general formalism which we shall set out in §6.2, we provide a more-or-less self-contained introduction to the relevant concepts in § 2.3.

connection and describes the boson, while the non-trivial bundle admits a non-zero (but nevertheless flat) connection, which accounts for the fermionic phase.

Let us now see this more clearly by means of an explicit construction. An element $U \in U(2)$ projects down to an element $O \in SO(3)$ by projecting out its ($U(1)$-valued) overall phase. We parameterise a matrix $U \in U(2)$ by

$$U = e^{i\chi} \begin{pmatrix} e^{i(\psi+\phi)/2}\cos(\theta/2) & e^{-i(\psi-\phi)/2}\sin(\theta/2) \\ -e^{i(\psi-\phi)/2}\sin(\theta/2) & e^{-i(\psi+\phi)/2}\cos(\theta/2) \end{pmatrix}, \tag{6.13}$$

where $\theta \in [0,\pi]$, $\phi \in [0,2\pi)$, $\psi \in [0,4\pi)$ and $\chi \in [0,2\pi)$ with the equivalence relation $(\theta,\phi,\psi,\chi) \sim (\theta,\phi,\psi+2\pi,\chi+\pi)$. Now, consider the curve $\gamma'(t)$ in $U(2)$ defined by

$$\gamma'(t) = \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix}, \quad t \in [0,\pi], \tag{6.14}$$

and define the curve $\gamma(t)$ to be the projection of $\gamma'(t)$ to $SO(3)$, which one might think of as the particle worldline in the original configuration space. The curve $\gamma'(t)$ is a horizontal lift of $\gamma(t)$ with respect to the connection, which in our coordinates can be represented by $A = d\chi$. For our purposes here, this simply means that the tangent vector $X_{\gamma'}$ to the curve $\gamma'(t)$ satisfies $A(X_{\gamma'}) = 0$, *i.e.* it has no component in the χ direction.

Notice that in $U(2)$ we have $\gamma'(0) = I$ and $\gamma'(\pi) = -I$, and that these two points, while distinct in $U(2)$, both project to the identity in $SO(3)$. The relative phase of π between $\gamma'(0)$ and $\gamma'(\pi)$ is called the holonomy of $\gamma(t)$. This implies that the rigid body is in this case a fermion, because the loop $\gamma(t)$ in $SO(3)$ corresponds to a 2π-rotation about the z-axis in $\mathbb{R}^3$. If we had instead equipped the rigid body with the trivial choice of bundle $SO(3) \times U(1)$, instead of $U(2)$, then the phase returns to zero upon traversing any closed loop in $SO(3)$, thus corresponding to a boson.

This fermionic versus bosonic nature is furthermore manifest in the differing representation theory of the Lie groups $U(2)$ and $SO(3) \times U(1)$. This shall be important when we solve for the spectrum of this quantum mechanical system in §6.3.1. While the unirreps of $SO(3) \times U(1)$ are all odd-dimensional (as we would expect for the integral angular momentum eigenstates of a bosonic rigid body), $U(2)$ also contains unirreps of even dimension (for example, the defining 2-d representation), leading to the possibility of eigenstates with half-integral angular momentum, which is exactly what we expect for a fermionic rigid body, via the spin-statistics theorem.

For our purposes, it will be useful to consider a different path $\tilde{\gamma}(t)$ in $U(2)$ that also projects down to γ in $SO(3)$, defined by

$$\tilde{\gamma}(t) = \begin{pmatrix} e^{2it} & 0 \\ 0 & 1 \end{pmatrix}, \quad t \in [0, \pi]. \tag{6.15}$$

While this path $\tilde{\gamma}$ is not a horizontal lift of the worldline γ, it nonetheless still projects down to γ, but is now a closed loop in $U(2)$ with the property that the exponential of the integral over $\tilde{\gamma}$ of the connection $A = d\chi$ is equal to the holonomy, *viz.* $e^{-i\int_{\tilde{\gamma}} A} = e^{-i\int_0^\pi dt} = -1$. This means that we can represent the holonomy (which is the contribution to the action phase from the topological term) in terms of a local action, namely the integral of the connection over an appropriately chosen loop $\tilde{\gamma}$. Given the existence of the horizontal lift, the fact that $U(1)$ is connected means such a loop always exists. As we might expect from the fact that there is a redundancy in our description, the choice of loop is, however, not unique. Nevertheless, the integral is of course independent of this choice.

The upshot is that this topological phase, which results in fermionic statistics of the rigid body, can be obtained from the integral of a lagrangian (the connection) on the principal bundle, here $U(2)$, which is both globally-defined and manifestly local. Due to the topological twisting of the bundle, there is no corresponding globally-defined lagrangian on the original configuration space, here $SO(3)$.

In this Section we have discussed two quantum mechanical prototypes, which are at first sight very different from a physical perspective. What both examples have in common is the possibility of a topological term in the action phase. In our first example of quantum mechanics on the plane (§6.1.1), this topological term corresponded to the familiar coupling of our particle to a magnetic field transverse to the plane of motion. We saw that, in order to identify a symmetry group that commutes with the hamiltonian, it was necessary to pass to an equivalent description on an extended space, with that symmetry group being the Heisenberg group. We then saw how one could obtain the Landau level spectrum by using harmonic analysis on the Heisenberg group, a method that works in any gauge. In contrast, in our second example of a rigid body (in this subsection), the topological term corresponded to a vanishing magnetic field, but we nonetheless saw that the term can have interesting effects, in this case leading to either fermionic or bosonic character of the rigid body.

Mathematically, both examples admit a common description: the topological term in the action phase is the holonomy of a connection on a $U(1)$-principal bundle P over the configuration space M. Such a topological term may not correspond to any globally-defined lagrangian on M (as in §6.1.2), or may not be invariant under the action of the group G which acts on M (as in §6.1.1); or, indeed, both (interconnected) issues may arise. Having demonstrated in our two prototypes that these problems can be remedied by passing to an equivalent description on an extended space (namely, the principal bundle P) with an action by a central extension of G, we are now ready to explain the general formalism.

6.2 Formalism

We shall consider quantum mechanics of a point particle whose configuration space is a smooth, connected manifold M. This can be described by an action whose degrees of freedom are maps ϕ from the 1-dimensional worldline, Σ, to the target space M, *viz.* $\phi : \Sigma \to M$. We consider the smooth action $\alpha : G \times M \to M$ of a connected Lie group G on M, which shall define the (global) symmetries of the system. Since, in the path integral approach to quantum mechanics, it is only the *relative* action phase between pairs of worldlines that is physical, we are free are to consider only worldlines which are closed, without loss of generality.

6.2.1 Quantum mechanics in magnetic backgrounds

We will now define the dynamics of the particle on M by specifying a G-invariant action phase, $e^{iS[\phi]}$, defined on all closed worldlines, or equivalently on all piecewise-smooth loops in M.

The action consists of two pieces (ignoring potential and higher-derivative terms). The first piece is the kinetic term, constructed out of a G-invariant metric on M. The second piece in the action couples the (electrically charged) particle to a background magnetic field. This is a topological term in the action phase (in the sense that it does not require the metric), equal to the holonomy of a connection A on a $U(1)$-principal bundle P over M (see §2.3), evaluated over the loop ϕ. It is shown in [56] that for this term in the action phase to be invariant under the action α of the Lie group G, we require that the contraction of each vector field X generating α with the curvature 2-form ω is an exact 1-form. That is, we require

$$\iota_X \omega = df_X \quad \forall X \in \mathfrak{g}, \tag{6.16}$$

where each f_X is a globally-defined function (equivalently, a 0-form) on M, and ι_X denotes the contraction of a differential form and a vector field (the details don't concern us). This condition, which we shall refer to as the Manton condition, is necessary for the G-invariance of the topological term evaluated on all piecewise-smooth loops in M (provided that G is connected, as we are assuming). This Manton condition is analogous to the moment map formula for a group action to be hamiltonian with respect to a given symplectic structure. The difference here, mathematically, is that the field strength ω need not be a non-degenerate 2-form.

It will be of use later, when we end up constructing an equivalent action on P, to specify a local trivialisation of P over a suitable set of coordinate charts $\{U_\alpha\}$ on M. We let $s_\alpha \in [0, 2\pi)$ be the $U(1)$-phase in this local trivialisation and define the transition functions $t_{\alpha\beta} = e^{i(s_\alpha - s_\beta)}$. Technically speaking, we need two coordinate charts on P, denote them $V_{\alpha,1}$ ($s_\alpha \neq \pi$) and $V_{\alpha,2}$ ($s_\alpha \neq 0$), for each U_α, to cover the S^1 fibre. In what follows, we will often gloss over this technicality; from hereon, s_α should be assumed to be written locally in one of these coordinate charts, which we shall denote collectively by V_α to avoid drowning in a sea of indices. Following this ethos, we will also tend to drop the α subscript on s_α when we turn to solving the examples in §6.3.

Our objective is to solve the SE corresponding to this G-invariant quantum mechanics, which we shall ultimately achieve by passing to a central extension of G by $U(1)$, and using harmonic analysis on that central extension.

To motivate our method, we shall first review how harmonic analysis can be used to solve the corresponding (time-independent) SE in the *absence* of the magnetic background, by exploiting the group-theoretic structure of the system [87]. Solving the SE amounts to finding the spectrum of an appropriate hamiltonian operator $\hat{H}$, which in this case can be quantised as the Laplace-Beltrami operator corresponding to the choice of G-invariant metric on M, on an appropriate Hilbert space. In the absence of a magnetic field, the Hilbert space can be taken to be $L^2(M)$. We can endow this Hilbert space with a highly reducible, unitary representation of G, namely the left-regular representation defined by

$$\rho(g)\Psi(m) := \Psi(\alpha_{g^{-1}}m) \text{ for } m \in M,\ g \in G, \text{ and } \Psi \in L^2(M). \tag{6.17}$$

The action of ρ allows us to decompose the vector space $L^2(M)$ into a direct sum (or, more generally, a direct integral) of vector spaces $V^{\lambda,t}$, such that the restriction of ρ to each $V^{\lambda,t}$ yield a unirrep of G, which we label by its equivalence class $\lambda \in \Lambda$. Each unirrep may, of course, appear more than once in the decomposition of $L^2(M)$ and so we index these by $t \in T^\lambda$. We will fix a basis for each vector space $V^{\lambda,t}$, which we denote by $e_r^{\lambda,t}$, where $r \in R^\lambda$ indexes the (possibly infinite-dimensional) basis, which does not depend on t.

In our examples we often specify the operator in the unirrep λ by its form in the chosen basis, which we denote $\pi^\lambda(s,q)$, where s and q index the basis. In many cases, as in §6.1.1, it will transpire that we can set $e_r^{\lambda,t} = \pi^\lambda(r,t)$. In other instances were this is not the case, one can nonetheless infer a suitable form for the $e_r^{\lambda,t}$ from $\pi^\lambda(s,q)$.

It is then a consequence of Schur's lemma that if

$$\hat{H}\rho(g)f(m) = \rho(g)\hat{H}f(m), \tag{6.18}$$

then the operator $\hat{H}$ will be diagonal in both λ and r, and can only mix $e_r^{\lambda,t}$ in the index t and not r or λ, *i.e.* it only mixes between equivalent unirreps. In most cases this simplifies the SE by reducing the number of different types of partial derivatives present, often resulting in a family of ordinary differential equations (ODEs) [87].

6.2.2 An equivalent action with manifest symmetry and locality

Interestingly, coupling our particle on M to a magnetic background, in the manner described in §6.2.1, may prevent one from constructing a local hamiltonian that satisfies Eq. 6.18. As elucidated by our pair of prototypes in §6.1, there are two obstructions to this method.

Firstly, as demonstrated by our prototypical example (§6.1.2), it may not be possible to form a globally-valid lagrangian on M. Secondly, as demonstrated by our prototypical example (§6.1.1), even when the construction of a globally-valid lagrangian is possible (*i.e.* when ω, the magnetic field strength, is the exterior derivative of a globally-defined 1-form), the lagrangian may vary by a total derivative under the action of G. This means that Eq. 6.18 will fail to hold, and the hamiltonian will not act only between equivalent unirreps of G.

It is possible to overcome both these problems by considering an equivalent dynamics on the principal bundle $\pi : P \to M$, instead of on M, as we shall now explain.

The topological term, which is just the holonomy of the connection A on P, can be written as the integral of A over any loop $\tilde{\phi}$ in P which projects down to our original loop ϕ on M, *i.e.* one that satisfies $\pi \circ \tilde{\phi} = \phi$ (see §2.3). Pulling back A to the worldline using $\tilde{\phi}$, we obtain on a patch V_α of P

$$\tilde{\phi}^* A = \left(\dot{s}_\alpha(t) + A_{\alpha,i}\left(x^k(t)\right)\dot{x}^i(t)\right) dt, \tag{6.19}$$

where $x^i(t) := x^i(\pi \circ \tilde{\phi}(t))$ denote local coordinates in M (with $i = 1, \ldots, \dim M$), $s_\alpha(t) := s_\alpha(\tilde{\phi}(t))$, $\dot{s}_\alpha := ds_\alpha/dt$ &c, and $A|_{V_\alpha} := ds_\alpha + A_{\alpha,i}dx^i$ is the connection restricted to the patch V_α. Given that we can also pull back the metric, and thus the kinetic term, from M to P, we can 'lift' our original definition of the action from M to the principal bundle P. The contribution to the action from a local patch V_α is then

$$S[\tilde{\phi}]\Big|_{V_\alpha} = \int dt \, \left\{ g_{ij}\dot{x}^i\dot{x}^j - \dot{s}_\alpha - A_{\alpha,i}\dot{x}^i \right\}, \tag{6.20}$$

where $g_{ij}dx^i dx^j$ will henceforth denote the pullback of the metric to P.

As we have anticipated, this reformulation of the dynamics on P has two important virtues. Firstly, there is a globally-defined lagrangian 1-form on P for the topological term, namely the connection A. Secondly, this lagrangian is strictly invariant under the Lie group central extension $\tilde{G}$ of G by $U(1)$, defined to be the set

$$\tilde{G} = \{(g, \varphi) \in G \times \mathrm{Aut}(P, A) \mid \pi \circ \varphi = \alpha_g \circ \pi\}, \tag{6.21}$$

endowed with the group action $(g, \varphi) \cdot (g', \varphi') = (gg', \varphi \circ \varphi')$ [123, 144], which as a manifold is the pullback bundle of $\pi : P \to M$ by the orbit map $\phi_m : G \to M$, $g \mapsto g \cdot m$, for any $m \in M$ [123]. Here, $\mathrm{Aut}(P, A)$ denotes the group of principal bundle automorphisms of P (*i.e.* diffeomorphisms which commute with the right action of the structure group on P) which preserve A, *i.e.* for $\varphi \in \mathrm{Aut}(P, A)$ we have $\varphi^* A = A$. There is a short exact sequence

$$0 \longrightarrow U(1) \xrightarrow{\ \iota\ } \tilde{G} \xrightarrow{\ \pi'\ } G \longrightarrow 0, \tag{6.22}$$

with the subgroup $\mathrm{Im}(\iota)$ central in $\tilde{G}$, thus exhibiting $\tilde{G}$ as a central extension of G by $U(1)$. Here $\iota : U(1) \ni e^{i\theta} \mapsto (\mathrm{id}, R_{e^{i\theta}}) \in \tilde{G}$, where $R_g \in \mathrm{Aut}(P, A)$ indicates the right action of $U(1)$ on the bundle P, and $\pi' : \tilde{G} \ni (g, \phi) \mapsto g \in G$. This group has a natural action on the principal bundle P, which we denote by $\tilde{\alpha} : \tilde{G} \times P \to P$, defined by $\tilde{\alpha}_{(g,\varphi)}p = \varphi(p)$, for $p \in P$.

The price to pay for these two virtues is that we have introduced a redundancy (which locally comes in the form of an extra coordinate s_α) into our description. We must account for this redundancy with an appropriate definition of the Hilbert space, to which we turn in the next Subsection.

6.2.3 Quantisation

Equipped with this reformulation of the dynamics on P, and the extended Lie group $\tilde{G}$, we are now in a position to construct a local hamiltonian operator and solve for its spectrum by decomposing into unirreps of $\tilde{G}$.

To do this, we first form the classical hamiltonian by taking the Legendre transform of the lagrangian, defined on the 'extended phase space' T^*P. At this stage the redundancy in our description becomes apparent, with the momentum p_{s_α} conjugate to the (local) fibre coordinate s_α being constant, *viz.* $p_{s_\alpha} + 1 = 0$, as we saw in §6.1.1. We can enforce this constraint by quantising the so-called 'total hamiltonian'

$$H|_{V_\alpha} = \frac{1}{2}(p_i + A_{\alpha,i})g^{ij}(p_j + A_{\alpha,j}) + v(t)(p_{s_\alpha} + 1), \tag{6.23}$$

where p_i is the momentum conjugate to the coordinate x^i, and $v(t)$ is an arbitrary function of t which plays the role of a Lagrange multiplier. This hamiltonian is naturally quantised as the magnetic analogue of the Laplace-Beltrami operator, in which the covariant derivative ∇ on M is replaced by $\nabla + A$, giving

$$\hat{H}\Big|_{V_\alpha} = \frac{1}{2}\left(-i\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\sqrt{g} + A_{\alpha,i}\right) g^{ij}\left(-i\frac{\partial}{\partial x^j} + A_{\alpha,j}\right) + v(t)\left(-i\frac{\partial}{\partial s_\alpha} + 1\right), \tag{6.24}$$

which is a Hermitian operator acting on the Hilbert space

$$\mathcal{H} = \left\{\Psi \in L^2(P, \tilde{\mu}) \,\middle|\, \left(-i\frac{\partial}{\partial s_\alpha} + 1\right)\Psi = 0 \text{ on } V_\alpha\right\} \tag{6.25}$$

where locally the measure is given by $\tilde{\mu} = \sqrt{g}\; ds dx^1 \ldots dx^n$. The Hilbert space $\mathcal{H}$ is isomorphic to the space of square integrable sections on the hermitian line bundle associated with P with respect to the measure $\mu = \sqrt{g}\; dx^1 \ldots dx^n$ [143, 150].

6.2.4 Method of solution: harmonic analysis on central extensions

Because the local hamiltonian commutes with the left regular representation of $\tilde{G}$, we expect to be able to use harmonic analysis on $\tilde{G}$ (when it exists!) to solve for the spectrum of Eq. 6.24. The Hilbert space $\mathcal{H}$ is endowed with the left-regular representation ρ of $\tilde{G}$, under which a wavefunction $\Psi \in \mathcal{H}$ transforms as

$$\tilde{\rho}(\tilde{g})\Psi(p) := \Psi(\tilde{\alpha}_{\tilde{g}^{-1}}p) \qquad \forall p \in P,\ \tilde{g} \in \tilde{G}. \tag{6.26}$$

We use harmonic analysis to decompose this representation into unirreps of $\tilde{G}$, in analogy with how we decomposed into unirreps of G in the absence of a magnetic background, above. Thus, let $e_r^{\lambda,t}(p \in P)$ now denote a basis for this decomposition, which schematically takes the form

$$\Psi = \sum_{\lambda} \int \mu(\lambda, r, t) f^{\lambda}(r, t) e_r^{\lambda,t}(p) \in L^2(P, \tilde{\mu}) \tag{6.27}$$

for an appropriate measure $\mu(\lambda, r, t)$. Note that the basis functions may not be square integrable; if this is not the case one may check that the solutions are the limit of an appropriate Weyl sequence (see *e.g.* [87]). In the presence of the magnetic background, we have passed to a redundant formulation of the dynamics on P, and the crucial difference is that we must now account for this redundancy when using harmonic analysis. It turns out (see Appendix C) that this redundancy can often be accounted for by restricting the decomposition in Eq. 6.27 to the subspace of unirreps which satisfy the constraint $(-i\partial_s + 1)e_r^{\lambda,t}(p) = 0$, which we can moreover equip with an appropriate completeness relation. In the examples that follow in §6.3, this decomposition into a restricted subspace of unirreps will serve as our starting point for harmonic analysis.

Then, exactly as above, the fact that the hamiltonian commutes with the left-regular representation (of $\tilde{G}$, not G) means that the action of $\hat{H}$ will only mix equivalent representations (that is, it can mix between different values of the t index, but not the r index or λ label). Thus, the SE will be simplified, often to a family of ODEs, as we shall see explicitly in a plethora of examples in the following Section.

It is important to acknowledge that performing harmonic analysis in the manner we have described, for the general setup of interest in which a (possibly non-compact) general Lie group acts non-transitively on the underlying manifold, is far from being a solved problem in mathematics. For example, it is not known under what conditions the integrals denoted in Eq. 6.27 actually exist, and whether the functions $f^{\lambda}(r, t)$ can be extracted from Ψ by appropriate integral transform methods. Thus, much of what has been said should be taken with a degree of caution. Fortunately, in the examples that we consider in §6.3, all of the required properties follow from properties of the usual Fourier transform, and in all cases the method that we have outlined in this Section works satisfactorily.

6.3 Examples

In §6.1.1 and §6.1.2 we explained the use of our method for planar motion in a magnetic field, then pointed out the existence of a topological term for the quantum mechanical rigid body, and explained how this term can endow the rigid body with fermionic statistics. We will start this Section where §6.1.2 left off, by solving for the spectrum of this fermionic rigid body using harmonic analysis on the group $U(2)$. After this we will look at a series of other examples where our method is of use. Some of these are well known systems, *e.g.* charged particle motion in the field of a Dirac monopole, whilst others are new, *e.g.* the motion of a particle on the Heisenberg manifold. The results of all the examples considered in this Chapter are summarised in Table 6.1.

Section	M $[G]$	P $[\tilde{G}]$	Lagrangian on P	Spectrum
6.1.1 Landau levels	$\mathbb{R}^2$ $[\mathbb{R}^2]$	$\mathbb{R}^2 \times U(1)$ [Hb]	$\frac{1}{2}\dot{x}^2 + \frac{1}{2}\dot{y}^2 - \dot{s} - By\dot{x}$	$\lvert B\rvert(n+1/2)$, $n \in \mathbb{N}_0$
6.3.1 Fermionic rigid body	$\mathbb{R}P^3$ $[SO(3)]$	$U(2)$ $[U(2)]$	$\frac{1}{2}\left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2(\theta) + \left(\dot{\psi} + \dot{\phi}\cos(\theta)\right)^2\right) - \dot{s}$	$j(j+1)/2$, $j \in \mathbb{N}_0 + 1/2$
6.3.2 Dirac monopole	S^2 $[SU(2)]$	$L(g,1)$ $[SU(2) \times U(1)]$	$\frac{1}{2}\left(\dot{\theta}^2 + \sin^2(\theta)\dot{\phi}^2\right) - \frac{1}{2}\dot{\chi} - \frac{g}{2}\cos(\theta)\dot{\phi}$	$\frac{1}{8}(4j^2+4j-g^2)$, $j \in \mathbb{N}_0+g/2$
6.3.3 Dyon	$\mathbb{R}_+ \times S^2$ $[SU(2)]$	$\mathbb{R}_+ \times L(g,1)$ $[SU(2) \times U(1)]$	$\frac{1}{2}\left(\dot{\theta}^2 + \sin^2(\theta)\dot{\phi}^2\right) - \frac{q}{r} - \frac{1}{2}\dot{\chi} - \frac{g}{2}\cos(\theta)\dot{\phi}$	$-q^2/(2(n+a))$, $n \in \mathbb{N}_{>0}$, $a = \frac{1}{2}(1+((2j+1)^2-g^2)^{1/2})$
6.3.4 Landau levels	$\mathbb{R}^2$ $[ISO(2)]$	$\mathbb{R}^2 \times U(1)$ $[\widetilde{\mathrm{ISO}}(2)]$	$\frac{1}{2}(\dot{x}^2 + \dot{y}^2) - \dot{s} - \partial_x h(x,y)\dot{x} - \partial_y h(x,y)\dot{y} - By\dot{x}$	$\lvert B\rvert(n+1/2)$, $n \in \mathbb{N}_0$
6.3.5	$\mathbb{R}^3$ [Hb]	$\mathbb{R}^4$ $[\widetilde{\mathrm{Hb}}]$	$\frac{1}{2}(\dot{x}^2 + \dot{y}^2 + (\dot{z} - x\dot{y})^2) - \dot{s} - x\dot{z} + \frac{x^2}{2}\dot{y}$	Anharmonic oscillator
6.3.6	$\mathbb{R}^3$ $[\mathbb{R}^2]$	$\mathbb{R}^3 \times U(1)$ [Hb]	$\frac{1}{2}\left(\frac{1}{a+z^2}\dot{x}^2 + \frac{1}{a+z^2}\dot{y}^2 + \dot{z}^2\right) - \dot{s} - By\dot{x}$	$\sqrt{\lvert B\rvert(2n+1)}(m+1/2)$ $+a\lvert B\rvert(n+1/2)$, $n, m \in \mathbb{N}_0$

Table 6.1 Summary of examples presented in Chapter 6. The particle lives on the manifold M, with dynamics invariant under G. Coupling to a magnetic background defines a $U(1)$-principal bundle $\pi : P \to M$, on which we form a lagrangian strictly invariant under a $U(1)$-central extension of G, denoted $\tilde{G}$.

6.3.1 Back to the rigid body

We resume the example discussed in §6.1.2. On a local coordinate patch on $P = U(2)$, we define a $U(2)$-invariant action incorporating a kinetic term by

$$S = \int dt \left(\frac{1}{2}\dot{\theta}^2 + \frac{1}{2}\dot{\phi}^2 \sin^2\theta + \frac{1}{2}\left(\dot{\psi} + \dot{\phi}\cos\theta\right)^2 - \dot{s} \right). \tag{6.28}$$

The total hamiltonian on this patch is

$$H = \frac{1}{2}p_\theta^2 + \frac{1}{2\sin^2\theta}\left(p_\phi^2 + p_\psi^2 - 2\cos\theta\ p_\phi p_\psi\right) + v(t)(p_s + 1), \tag{6.29}$$

which we quantise as the operator

$$\hat{H} = -\frac{1}{2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) - \frac{1}{2\sin^2\theta}\left(\frac{\partial^2}{\partial\psi^2} + \frac{\partial^2}{\partial\phi^2} - 2\cos\theta\frac{\partial^2}{\partial\phi\partial\psi}\right) + v(t)\left(-i\frac{\partial}{\partial s} + 1\right), \tag{6.30}$$

acting on wavefunctions $\Psi(\theta,\phi,\psi,s) \in L^2(U(2))$ satisfying $\left(-i\frac{\partial}{\partial s} + 1\right)\Psi = 0$. The unirreps whose matrix elements satisfy this condition when considered as functions on $U(2)$, are given by

$$\pi^j_{m,m'}(\theta,\phi,\psi,s) = e^{-is}D^j_{m'm}(\theta,\phi,\psi), \tag{6.31}$$

where j is a positive half-integer, m, $m' \in \{-j, -j+1, \dots, j\}$, and $D^j_{m'm}$ is a Wigner D-matrix, defined (in our local coordinates) by

$$D^j_{m'm}(\theta,\phi,\psi) = \left(\frac{(j+m)!(j-m)!}{(j+m')!(j-m')!}\right)^{1/2} (\sin(\theta/2))^{m-m'}(\cos(\theta/2))^{m+m'} P^{(m-m',m+m')}_{j-m}(\cos\theta)e^{-im'\psi}e^{-im\phi}. \tag{6.32}$$

These are matrix elements of an unirrep of $U(2)$ and, as was the case in §6.1.1, transform in the corresponding conjugate representation when the left-regular representation is applied. The Wigner D-matrices satisfy the completeness relation

$$\sum_{m'\in\mathbb{Z}+1/2}\ \sum_{m\in\mathbb{Z}+1/2}\ \sum_{j=\max(|m|,|m'|)}^{\infty} \frac{2j+1}{8\pi^2}\left(D^j_{m'm}(\theta',\phi',\psi')\right)^* D^j_{m'm}(\theta,\phi,\psi) = \delta_{2\pi}(\phi-\phi')\delta_{2\pi}(\psi-\psi')\delta(\cos\theta-\cos\theta'), \tag{6.33}$$

where $\delta_{2\pi}(\cdots)$ represents a Dirac delta comb with periodicity 2π, and the sum over j is over half-integers.

Following the formalism set out in §6.2, we decompose Ψ into a basis $\{e_m^{j,m'}\}$ for $L^2(U(2))$, which in this case can be chosen to be $e_m^{j,m'} = \pi^j_{m,m'}$, the matrix elements of unirreps of $U(2)$ introduced above, giving us

$$\Psi = \sum_{m'\in\mathbb{Z}+1/2}\sum_{m\in\mathbb{Z}+1/2}\sum_{j=\max(|m|,|m'|)}^{\infty}\frac{2j+1}{8\pi}e^{-is}D^j_{m'm}(\theta,\phi,\psi)f^j_{m'm}, \tag{6.34}$$

with inverse

$$f^j_{m'm} = \int d\left(\cos(\theta')\right)d\psi' d\phi'\left(D^j_{m'm}(\theta',\phi',\psi')e^{-is}\right)^*\Psi(\theta',\phi',\psi',s). \tag{6.35}$$

The SE then reduces to

$$\sum_{m'\in\mathbb{Z}+1/2}\sum_{m\in\mathbb{Z}+1/2}\sum_{j=\max(|m|,|m'|)}^{\infty}\frac{2j+1}{8\pi}\left\{\frac{j(j+1)}{2}-E\right\}e^{-is}D^j_{m'm}(\theta,\phi,\psi)f^j_{m'm}=0, \tag{6.36}$$

yielding the energy levels

$$E^j_{m'm} = \frac{1}{2}j(j+1), \qquad \text{for } j \text{ half-integer.} \tag{6.37}$$

The corresponding wavefunctions, on our local coordinate patch, can be written

$$\Psi^j_{m'm}(\theta,\phi,\psi,s) = e^{-is}D^j_{m'm}(\theta,\phi,\psi). \tag{6.38}$$

Setting the fibre coordinate s to zero defines, a section on the hermitian line bundle associated with the principal bundle $U(2)$, in other words a physical wavefunction. On traversing a double intersection of coordinate charts on $SO(3)$, the above expression for the section will shift by a transition function.

We note in passing that on setting $s = 0$ the $U(2)$ representations appearing in this decomposition reduce to representations of $SU(2)$. This occurs due to a well-known happy accident, namely that the projective representations of a Lie group G (here $SO(3)$) whose second Lie algebra cohomology vanishes (as is the case for every semisimple Lie group) in fact correspond to *bona fide* representations of the universal cover of G (here $SU(2)$). That is, under these conditions, familiar to most physicists, we may decompose the Hilbert space into unirreps of the universal cover of G, without technically needing to pass to a central extension. It is, however, important to point out that even in an example such as this, one cannot write down a local action for the topological term on the universal cover $SU(2)$, but must pass to the central extension, $U(2)$.

6.3.2 The Dirac monopole

Here we consider the $G = SU(2)$-invariant dynamics of a particle moving on the 2-sphere. We may embed $M = S^2$ in $\mathbb{R}^3$, parametrised by the standard spherical coordinates $(\theta \sim \theta + \pi, \phi \sim \phi + 2\pi)$. We cover S^2 with two charts U_+ and U_-, which exclude the South and North poles respectively. At the centre sits a magnetic monopole of charge $g \in \mathbb{Z}$. This background magnetic field specifies a particular $U(1)$-principal bundle P_g over S^2 with connection A, which we may write in our coordinates as

$$\begin{aligned} A|_{U_+} &= ds_+ - \frac{g}{2}(1 - \cos\theta)\, d\phi \\ A|_{U_-} &= ds_- - \frac{g}{2}(-1 - \cos\theta)\, d\phi, \end{aligned} \tag{6.39}$$

where $s_\pm$ denotes a local coordinate in the $U(1)$ fibre. This can be conveniently written as

$$A = \frac{1}{2}d\chi + \frac{g}{2}\cos\theta d\phi, \tag{6.40}$$

where $\frac{1}{2}\chi = s_+ - \frac{g}{2}\phi$ on U_+ and $\frac{1}{2}\chi = s_- + \frac{g}{2}\phi$ on U_-. The transition functions over a trivialisation on $\{U_+, U_-\}$ are specified via the choice

$$(p, e^{i\delta}) \in U_+ \times U(1) \mapsto (p, e^{i\delta}e^{ig\phi}) \in U_- \times U(1). \tag{6.41}$$

For general g, this bundle P_g is in fact the lens space $L(g,1)$, which is a particular quotient of S^3 by a $\mathbb{Z}/g\mathbb{Z}$ action. When $g = 1$, the bundle is simply $P_1 \cong S^3$, described via the Hopf fibration and when $g = 2$, the bundle is simply $\mathbb{R}P^3$.[3]

As was the case in the previous example, it is here not possible to write down a global 1-form lagrangian on S^2. Rather, as was first demonstrated by Wu & Yang [151], one must write the action on S^2 as a sum of line integrals on different charts, together with the insertion of 0-forms (the transition functions) evaluated at points in double intersections of charts. Thus, it is not possible to use the usual hamiltonian formalism to solve for the spectrum of the corresponding quantum mechanics problem.

[3] The lens spaces $L(g,1)$ make another appearance in physics as the possible vacuum manifolds for the electroweak interaction [86].

Following our formalism, we should instead reformulate the problem by writing down an equivalent, globally-defined lagrangian on the $U(1)$-principal bundle $P_g = L(g,1)$ defined above. The action is

$$S = \int dt \left\{ \frac{1}{2}\left(\dot{\theta}^2 + \sin^2\theta\, \dot{\phi}^2\right) - \frac{1}{2}\dot{\chi} - \frac{g}{2}\cos\theta\, \dot{\phi} \right\}. \tag{6.42}$$

This lagrangian is invariant under $\tilde{G} = SU(2)\times U(1)$, the unique (up to Lie group isomorphisms) $U(1)$-central extension of $SU(2)$, with uniqueness following from the fact that $SU(2)$ is a simple and simply-connected Lie group [144]. We parametrise an element $\tilde{g} \in \tilde{G}$ by

$$\tilde{g} = \left(\begin{pmatrix} e^{i(\psi+\phi)/2}\cos\frac{\theta}{2} & e^{-i(\psi-\phi)/2}\sin\frac{\theta}{2} \\ -e^{i(\psi-\phi)/2}\sin\frac{\theta}{2} & e^{-i(\psi+\phi)/2}\cos\frac{\theta}{2} \end{pmatrix}, \ e^{i(g\psi-\chi)/2} \right) \in SU(2)\times U(1). \tag{6.43}$$

The corresponding total hamiltonian is

$$\hat{H} = \frac{1}{2}p_\theta^2 + \frac{1}{2\sin^2\theta}\left(p_\phi + \frac{g}{2}\cos\theta\right)^2 + v(t)\left(p_\chi + \frac{1}{2}\right), \tag{6.44}$$

which when quantised gives

$$\hat{H} = -\frac{1}{2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{2\sin^2\theta}\left(-i\frac{\partial}{\partial\phi} + \frac{g}{2}\cos\theta\right)^2 + v(t)\left(-i\frac{\partial}{\partial\chi} + \frac{1}{2}\right), \tag{6.45}$$

where the Hilbert space $\mathcal{H}$ is the subspace of square integrable functions on $L(g,1)$ for which the last term in Eq. 6.45 vanishes.

We now wish to solve for the spectrum of this hamiltonian using harmonic analysis on the Lie group $\tilde{G} = SU(2)\times U(1)$. Matrix elements of unirreps of $SU(2)\times U(1)$ which are annihilated by the constraint $\left(-i\frac{\partial}{\partial\chi} + \frac{1}{2}\right)\pi^j_{m,m'} = 0$ are given by

$$\pi^j_{m,m'}(\theta,\phi,\psi,\chi) = e^{i(g\psi-\chi)/2} D^j_{m'm}(\theta,\phi,\psi). \tag{6.46}$$

Here $D^j_{m'm} := e^{-im'\psi - im\phi} d^j_{m'm}(\theta)$ are the same Wigner D-matrices as defined in Eq. 6.32, and the matrices $d^j_{m'm}(\theta)$ are conventionally referred to as 'Wigner d-matrices'. The subspace of these unirreps with $m' = g/2$ do not depend on the coordinate ψ, and provide a suitable basis for decomposing square-integrable functions on the lens space $L(g,1)$. We denote these basis functions by $e^{j,g/2}_m(\theta,\phi,\chi) = \pi^j_{m,g/2}(\theta,\phi,\psi,\chi)$, which satisfy the constraint condition and which transform as unirreps of $SU(2)\times U(1)$. This subspace of $\mathcal{H}$ carries the completeness relation

$$\sum_{m+g/2\in\mathbb{Z}} \sum_{j=\max(|m|,g/2)}^{\infty} \frac{2j+1}{4\pi} \left(e_m^{j,g/2}(\theta',\phi',\chi')\right)^* e_m^{j,g/2}(\theta,\phi,\chi) = e^{-i(\chi-\chi')/2}\delta_{2\pi}(\phi-\phi')\delta(\cos\theta-\cos\theta'), \quad (6.47)$$

which allows us to decompose any wavefunction in $\Psi \in \mathcal{H}$ into unirreps as follows

$$\Psi(\theta,\phi,\chi) = e^{-i\chi/2} \sum_{m+g/2\in\mathbb{Z}} \sum_{j=\max(|m|,g/2)}^{\infty} \frac{2j+1}{4\pi} f_m^j e^{-im\phi} d^j_{g/2,m}(\theta), \quad (6.48)$$

where

$$f_m^j = \int d(\cos\theta')d\phi'\; e^{im\phi'+i\chi'/2} d^j_{g/2,m}(\theta')\Psi(\theta',\phi',\chi'). \quad (6.49)$$

If we now substitute the decomposition in Eq. 6.48 into the SE, after simplification, we get

$$\sum_{m+g/2\in\mathbb{Z}} \sum_{j=\max(|m|,g/2)}^{\infty} \frac{2j+1}{4\pi} \left(\frac{1}{8}(4j^2+4j-g^2)-E\right) e^{-i\chi/2}e^{-im\phi}d^j_{g/2,m}(\theta) = 0. \quad (6.50)$$

Thus the solution to the SE is

$$\Psi_m^j(\theta,\phi,\chi) = e^{-i\chi/2-im\phi}d^j_{g/2,m}(\theta), \quad E_m^j = \frac{1}{8}(4j^2+4j-g^2). \quad (6.51)$$

Notice that the eigenstates are labeled by two quantum numbers j and m, but that for a given j the eigenstates with different values of m are degenerate in energy due to the rotational invariance of the problem.

To write our solution in terms of a section on a hermitian line bundle associated with P_g, we set $s_+ = 0$ on U_+ and $s_- = 0$ on U_-, corresponding to $\chi = -g\phi$ and $\chi = g\phi$ respectively. This yields

$$\begin{aligned} \Psi^j_{m,+}(\theta,\phi) &= e^{i\frac{g}{2}\phi-im\phi}d^j_{g/2,m}(\theta),\\ \Psi^j_{m,-}(\theta,\phi) &= e^{-i\frac{g}{2}\phi-im\phi}d^j_{g/2,m}(\theta). \end{aligned} \quad (6.52)$$

These solutions agree with the solutions of Wu and Yang [150], who solved this system by considering local hamiltonians on U_+ and U_- separately.

6.3.3 Charged particle orbiting a dyon

In the previous Section we found the spectrum of an electrically charged particle in the presence of a magnetic monopole. Within our formalism, it is straightforward to generalise this to study an electrically charged particle in the background field of a dyon, and use harmonic analysis to reduce the corresponding SE to an ODE.

The required modification is to include an r-dependent kinetic term, where r is the radial distance from a dyon located at the origin, together with an r-dependent potential term, in the action in Eq. 6.42. We have

$$S = \int dt \left\{ \frac{1}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta\, \dot{\phi}^2\right) - \frac{q}{r} - \frac{1}{2}\dot{\chi} - \frac{g}{2}\cos\theta\, \dot{\phi} \right\}. \tag{6.53}$$

where q is the electric charge of the dyon, and $g \in \mathbb{Z}$ is the (quantised) magnetic charge of the dyon as before. The original configuration space M of the system is $\mathbb{R}_+ \times S^2$, whilst this action is written on the $U(1)$-principal bundle $P_{q,g} = \mathbb{R}_+ \times L(g,1)$ where $L(g,1)$ is the lens space as in §6.3.2. This action is invariant under a *non-transitive* action of $SU(2) \times U(1)$, as defined in the previous Section.

The quantised total hamiltonian corresponding to the action in Eq. 6.53 is given by

$$\hat{H} = -\frac{1}{2r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{1}{2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) - \frac{1}{2r^2\sin^2\theta}\left(-i\frac{\partial}{\partial\phi} + \frac{g}{2}\cos\theta\right)^2 + \frac{q}{r} + v(t)\left(-i\frac{\partial}{\partial\chi} + \frac{1}{2}\right) \tag{6.54}$$

which acts on the physical Hilbert space. The decomposition of a wavefunction $\Psi(r,\theta,\phi,\chi)$ in this Hilbert space is completely analogous to the decomposition in Eq. 6.48, however this time the f^j_m, which were previously constants, should be replaced with functions $f^j_m(r)$. On substituting this decomposition into the SE, we arrive at the following differential equation for $f^j_m(r)$,

$$\left(-\frac{1}{2r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{8r^2}(4j^2 + 4j - g^2) + \frac{q}{r} - E\right) f^j_m(r) = 0. \tag{6.55}$$

The bounded solutions to this ODE were derived in [34], giving the spectrum

$$E_n = -\frac{q^2}{2(n+a)^2}, \quad n \in \mathbb{N}_{>0}, \tag{6.56}$$

where $a = \frac{1}{2}\left(1 + ((2j+1)^2 - g^2)^{1/2}\right)$.

6.3.4 Planar motion in a uniform magnetic field (take two)

In §6.1.1 we solved for the spectrum of a particle on $\mathbb{R}^2$ in the presence of a uniform magnetic field perpendicular to the plane, by considering the group $\mathbb{R}^2$ of translations in the plane, and passing to its central extension, the Heisenberg group Hb. Of course, the symmetry group of this system is larger than $\mathbb{R}^2$, because both the kinetic term and the magnetic coupling are invariant not just under translations, but also under rotations. Thus, in this Section, we revisit this problem (and solve it again) using a different implementation of our general method, by instead considering the particle as living on the quotient space $M = \mathrm{ISO}(2)/SO(2) \cong \mathbb{R}^2$, with $G = \mathrm{ISO}(2)$ being the Euclidean group in two dimensions. Thus, our solution here shall involve the representation theory of a central extension of $G = \mathrm{ISO}(2)$, which will be a four-dimensional group, rather than the representation theory of Hb which was used in §6.1.1.

As usual, we formulate the action on a $U(1)$-principal bundle P over the target space $M = \mathrm{ISO}(2)/SO(2) \cong \mathbb{R}^2$. Using coordinates (x, y, s), where $(x, y) \in \mathbb{R}^2$ provide global coordinates on the base space, and s denotes a local coordinate in the $U(1)$ fibre, the action is

$$S = \int \left(\frac{1}{2}(\dot{x}^2 + \dot{y}^2) - \dot{s} - \frac{\partial h}{\partial x}\dot{x} - \frac{\partial h}{\partial y}\dot{y} - By\dot{x} \right) dt, \tag{6.57}$$

where $h(x, y)$ is an arbitrary smooth function of x and y, which corresponds to a choice of gauge for the magnetic vector potential. Note that in all the examples in this Chapter, there is a choice of gauge made in writing down the magnetic vector potential which appears in the action. While different choices of gauge will in general result in different central extensions $\tilde{G}$, gauge-equivalent vector potentials nonetheless correspond to central extensions which are isomorphic as Lie groups. In this sense, the choice of gauge has little affect on the representation theory used in our calculations. For this example, we have chosen to make this gauge-dependence (or, rather, independence) explicit, by formulating the action in a general gauge from the outset.

As usual, the lagrangian is not invariant under the isometry group $G = \mathrm{ISO}(2)$, but rather it shifts by a total derivative under the translation subgroup. The lagrangian is, however, genuinely invariant under a $U(1)$-central extension of ISO(2), which we will denote by $\widetilde{\mathrm{ISO}}(2)$, which is a four-dimensional group defined by

$$\left\{\xi'_x, \xi'_y, \xi'_c, \xi'_s\right\} \cdot \left\{\xi_x, \xi_y, \xi_c, \xi_s\right\} = \Big\{\xi'_x + \xi_x \cos\xi'_c + \xi_y \sin\xi'_c,\ \xi'_y + \xi_y \cos\xi'_c - \xi_x \sin\xi'_c,$$
$$\xi_c + \xi'_c, \xi_s + \xi'_s - \frac{B}{2}\left((\xi_x \cos\xi'_c + \xi_y \sin\xi'_c)\xi'_y - (\xi_y \cos\xi'_c - \xi_x \sin\xi'_c)\xi'_x\right) \Big\}. \tag{6.58}$$

This group acts on the principal bundle P via

$$\tilde{\alpha}_{(\xi'_x,\xi'_y,\xi'_c,\xi'_s)} \cdot (x,y,s) = \Big\{x', y', \xi_s + \xi'_s - \frac{B}{2}\left((x\cos\xi'_c + y\sin\xi'_c)\xi'_y\right.$$
$$\left. -(y\cos\xi'_c - x\sin\xi'_c)\xi'_x\right) + \left(\frac{B}{2}xy - \frac{B}{2}x'y'\right) + (h(x,y) - h(x',y'))\Big\}, \quad (6.59)$$

where $x' = \xi'_x + x\cos\xi'_c + y\sin\xi'_c$ and $y' = \xi'_y + y\cos\xi'_c - x\sin\xi'_c$.

The corresponding total hamiltonian is

$$H = \frac{1}{2}\left(p_x + \frac{\partial h}{\partial x} + By\right)^2 + \frac{1}{2}\left(p_y + \frac{\partial h}{\partial y}\right)^2 + v(t)(p_s + 1), \quad (6.60)$$

which we quantise as the Hermitian operator

$$\hat{H} = \frac{1}{2}\left(-i\frac{\partial}{\partial x} + \frac{\partial h}{\partial x} + By\right)^2 + \frac{1}{2}\left(-i\frac{\partial}{\partial y} + \frac{\partial h}{\partial y}\right)^2 + v(t)\left(-i\frac{\partial}{\partial s} + 1\right). \quad (6.61)$$

The Hilbert space $\mathcal{H}$ is the subspace of square integrable functions on the bundle P which are annihilated by the constraint $\left(-i\frac{\partial}{\partial s} + 1\right) = 0$. We shall now solve the SE for this system by decomposing this Hilbert space into unirreps of the group $\widetilde{\mathrm{ISO}}(2)$ defined above. We start from the following unirreps [115]

$$\pi^{\lambda}_{m\geq n}(\xi_x,\xi_y,\xi_c,\xi_s) = e^{-i(\mathrm{Sgn}(B)n+\lambda+\tilde{\delta})\xi_c} e^{-i\xi_s} \left(\frac{n!}{m!}\right)^{\frac{1}{2}} e^{i\mathrm{Sgn}(B)(m-n)\tan^{-1}\left(\frac{\xi_y}{\xi_x}\right)} e^{-\frac{|B|(\xi_x^2+\xi_y^2)}{4}}$$
$$\left(-i\sqrt{\xi_x^2+\xi_y^2}\left|\frac{B}{2}\right|^{1/2}\right)^{m-n} L_n^{m-n}\left(\frac{|B|}{2}(\xi_x^2+\xi_y^2)\right), \quad (6.62)$$

$$\pi^{\lambda}_{m\leq n}(\xi_x,\xi_y,\xi_c,\xi_s) = e^{-i(\mathrm{Sgn}(B)n+\lambda+\tilde{\delta})\xi_c} e^{-i\xi_s} \left(\frac{m!}{n!}\right)^{\frac{1}{2}} e^{i\mathrm{Sgn}(B)(m-n)\tan^{-1}\left(\frac{\xi_y}{\xi_x}\right)} e^{-\frac{|B|(\xi_x^2+\xi_y^2)}{4}}$$
$$\left(-i\sqrt{\xi_x^2+\xi_y^2}\left|\frac{B}{2}\right|^{1/2}\right)^{n-m} L_m^{n-m}\left(\frac{|B|}{2}(\xi_x^2+\xi_y^2)\right), \quad (6.63)$$

where $\lambda \in \mathbb{Z}$, $m,n \in \mathbb{N}_0$, $\tilde{\delta} = 1$ if $B > 0$ and $\tilde{\delta} = 0$ otherwise, and L_n^{m-n} are the associated Laguerre polynomials. A set of functions in the Hilbert space which transform under these representations can be inferred by comparing the multiplication rule in $\widetilde{\mathrm{ISO}}(2)$ with the group action on the principal bundle P. We thus obtain the following basis of functions on P:

$$e_n^{\lambda_0,m}|_{m\geq n}(x,y,s) = e^{-i\left(s+h+\frac{B}{2}xy\right)} \left(\frac{n!}{m!}\right)^{\frac{1}{2}} e^{i\mathrm{Sgn}(B)(m-n)\tan^{-1}\left(\frac{y}{x}\right)} e^{-\frac{|B|(\xi_x^2+\xi_y^2)}{4}}$$
$$\left(-i\sqrt{\xi_x^2+\xi_y^2}\left|\frac{B}{2}\right|^{1/2}\right)^{m-n} L_n^{m-n}\left(\frac{|B|}{2}(\xi_x^2+\xi_y^2)\right), \quad (6.64)$$

$$e_n^{\lambda_0,m}|_{m\leq n}(x,y,s) = e^{-i\left(s+h+\frac{B}{2}xy\right)} \left(\frac{m!}{n!}\right)^{\frac{1}{2}} e^{i\mathrm{Sgn}(B)(m-n)\tan^{-1}\left(\frac{y}{x}\right)} e^{-\frac{|B|(x^2+y^2)}{4}}$$
$$\left(-i\sqrt{x^2+y^2}\left|\frac{B}{2}\right|^{1/2}\right)^{n-m} L_m^{n-m}\left(\frac{|B|}{2}(x^2+y^2)\right). \quad (6.65)$$

where $\lambda_0 = -\mathrm{Sgn}(B) - \tilde{\delta}$. When acted on by the left regular representation of $\widetilde{\mathrm{ISO}}(2)$ these functions transform under the unirrep corresponding to the conjugate of the $\lambda = \lambda_0$ unirrep defined in Eqs. 6.62, 6.63 above. We know it is sufficient to consider only these unirreps since they satisfy a completeness relation given by

$$\frac{|B|}{2\pi}\sum_{m,n}\left(e_n^{\lambda_0,m}(x',y',s')\right)^* e_n^{\lambda_0,m}(x,y,s) = e^{-i(s-s')}\delta(x-x')\delta(y-y'). \quad (6.66)$$

Thus, we can decompose a wavefunction in our Hilbert space into unirreps of $\widetilde{\mathrm{ISO}}(2)$ as

$$\Psi(x,y,s) = \frac{|B|}{2\pi}\sum_{m,n} e_n^{\lambda_0,m}(x,y,s) f_{m,n}, \quad (6.67)$$

where the inverse transform is given by

$$f_{m,n} = \int dxdy (e_n^{\lambda_0,m}(x',y',s'))^* \Psi(x,y,s). \quad (6.68)$$

After substituting the decomposition in Eq. 6.67 into the SE, we obtain

$$\frac{|B|}{2\pi}\sum_{m,n}\left(|B|(n+1/2)-E\right) e_n^{\lambda_0,m}(z,ys) f_{m,n} = 0. \quad (6.69)$$

Thus, we arrive at the familiar Landau level spectrum

$$E_{m,n} = |B|(n+1/2), \quad \Psi_{m,n} = e_n^{\lambda_0,m}(x,y,s), \quad (6.70)$$

where setting $s = 0$ in $e_n^{\lambda_0,m}$ gives us a suitable set of eigenfunctions on $\mathbb{R}^2$.

6.3.5 Quantum mechanics on the Heisenberg group

In this Section, we turn to a new example not previously considered in the literature, of particle motion on the Heisenberg group. We equip $M = \mathrm{Hb}$ with a left-invariant metric, and thus take $G = \mathrm{Hb}$ also. We shall couple the particle to a background magnetic field, corresponding to an Hb-invariant closed 2-form on Hb, for which the magnetic vector potential which appears in the lagrangian shifts by a total derivative under the action of the group Hb on itself.

While a version of the Heisenberg group appeared in §6.1.1 (as the central extension of the translation group $\mathbb{R}^2$), for our purposes in this Section we shall redefine the Heisenberg group to be the set of triples $(x, y, z) \in \mathbb{R}^3$ equipped with multiplication law

$$(x', y', z') \cdot (x, y, z) = (x + x', y + y', z + z' + yx'). \qquad (6.71)$$

To avoid any possible confusion, we emphasise that in this Section the Heisenberg group is taken as the original configuration space of our particle dynamics, which we shall reformulate as an equivalent dynamics *on a central extension of the Heisenberg group.* This central extension will be a four-dimensional Lie group which we shall denote $\widetilde{\mathrm{Hb}}$.

Before we proceed with writing down the action for this system (and eventually solving for the spectrum using harmonic analysis on $\widetilde{\mathrm{Hb}}$), we first pause to offer a few words of motivation for considering this system, since it does not correspond to any physical quantum mechanics system (although there are indirect links to the anharmonic oscillator, see *e.g.* [99]). In any case, our motivation is entirely mathematical. Firstly, we wanted a new example where the central extension of Lie groups $0 \to U(1) \to \tilde{G} \to G$ is non-trivial, *i.e.* $\tilde{G}$ is not just a direct product, and moreover that it corresponds to a non-trivial central extension of Lie algebras $0 \to \mathbb{R} \to \tilde{\mathfrak{g}} \to \mathfrak{g}$. The requirement that a Lie algebra $\mathfrak{g}$ admits a non-trivial central extension requires, by a theorem of Whitehead [149, 148], that the Lie algebra $\mathfrak{g}$ cannot be semisimple. Of course, abelian Lie groups provide a source of such non-trivial central extensions, because their Lie algebra cohomology is in a sense maximal (noting that the second Lie algebra cohomology of $\mathfrak{g}$ is isomorphic to the group of inequivalent (up to Lie algebra isomorphisms) central extensions of $\mathfrak{g}$). However, we sought a more interesting example where the original group G is non-abelian. To that end, non-abelian nilpotent Lie groups provide a richer source of suitable central extensions, because the second Lie algebra cohomology of any nilpotent $\mathfrak{g}$ is at least two-dimensional [62]. The Heisenberg Lie algebra, and the corresponding Lie group Hb, provides the simplest such example.

Since we are taking the Heisenberg group to be topologically just $\mathbb{R}^3$, we can cover the target space with a single patch and write the lagrangian using globally-defined coordinates (x, y, z). The action on Hb, including the topological term, is

$$S = \int dt \left(\frac{1}{2} \left(\dot{x}^2 + \dot{y}^2 + (\dot{z} - x\dot{y})^2 \right) - x\dot{z} + \frac{x^2}{2}\dot{y} \right). \tag{6.72}$$

The kinetic term corresponds to a left-Hb-invariant metric on Hb, as mentioned above, and we have chosen a normalisation for the (real-valued) coefficient of the topological term $-x\dot{z} + \frac{x^2}{2}\dot{y}$.[4] This topological term in the lagrangian shifts by a total derivative under the group action in Eq. 6.71. Following our now-familiar procedure, we thus reformulate the action on a $U(1)$-principal bundle P over Hb, on which s provides a local coordinate in the fibre. The action on P is written

$$S = \int dt \left(\frac{1}{2} \left(\dot{x}^2 + \dot{y}^2 + (\dot{z} - x\dot{y})^2 \right) - \dot{s} - x\dot{z} + \frac{x^2}{2}\dot{y} \right), \tag{6.73}$$

where the only difference is the $\dot{s}$ term. By adding this redundant degree of freedom to the action it becomes strictly invariant under the $U(1)$-central extension of Hb defined by the multiplication law

$$(x', y', z', s') \cdot (x, y, z, s) = \left(x + x', y + y', z + z' + yx', s + s' - zx' - y\frac{x'^2}{2} \right), \tag{6.74}$$

which we denote by $\tilde{G} = \widetilde{\mathrm{Hb}}$.

The total hamiltonian corresponding to the action in Eq. 6.72 is given by

$$H = \frac{1}{2}p_x^2 + \frac{1}{2}\left(p_z + x\right)^2 + \frac{1}{2}\left(p_y - \frac{x^2}{2} + x\left(p_z + x\right) \right)^2 + v(t)\left(p_s + 1\right), \tag{6.75}$$

which quantises to

$$\hat{H} = -\frac{1}{2}\frac{\partial^2}{\partial x^2} + \frac{1}{2}\left(-i\frac{\partial}{\partial z} + x \right)^2 + \frac{1}{2}\left(-i\frac{\partial}{\partial y} - \frac{x^2}{2} + x\left(-i\frac{\partial}{\partial z} + x \right) \right)^2 + v(t)\left(-i\frac{\partial}{\partial s} + 1 \right). \tag{6.76}$$

[4] Note that this is not the most general Hb-invariant topological term we can write down.

acting on the Hilbert space of square integrable functions on $\widetilde{\text{Hb}}$ that are annihilated by $\left(-i\frac{\partial}{\partial s}+1\right)$.

Because the group $\widetilde{\text{Hb}}$ defined in Eq. 6.74 has a nilpotent Lie algebra, its representation theory can be found via Kirillov's orbit method [98]. The unirrep matrix elements that we are interested in, which in this case are functions on $\widetilde{\text{Hb}}$, are infinite-dimensional, given by

$$\pi^q(r,t;x,y,z,s)=\delta(t-r-x)e^{i(-s+zr+\frac{1}{2}yr^2)+q/2y}, \tag{6.77}$$

which satisfy the completeness relation

$$\int\frac{dqdrdt}{2(2\pi)^2}\left(\pi^q(r,t;x',y',z',s')\right)^*\pi^q(r,t;x,y,z,s)=e^{-i(s-s')}\delta(x-x')\delta(y-y')\delta(z-z'). \tag{6.78}$$

We thus decompose a wavefunction into unirreps using these functions as our basis elements, $e_r^{q,t}(x,y,z,s)=\pi^q(r,t;x,y,z,s)$, giving us

$$\Psi(x,y,z,s)=\int\frac{dqdrdt}{2(2\pi)^2}e_r^{q,t}(x,y,z,s)f_q(r,t), \tag{6.79}$$

where

$$f_q(r,t)=\int dx'dy'dz'\left(e_r^{q,t}(x',y',z',s')\right)^*\Psi(x',y',z',s'). \tag{6.80}$$

Using this decomposition, and the expression in Eq. 6.76 for the hamiltonian, the SE reduces to

$$-\frac{1}{4(2\pi)^3}\int dqdrdt\ e_r^{q,t}(x,y,z,s)\left(\frac{\partial^2 f_q(r,t)}{\partial t^2}+2Ef_q(r,t)-\frac{1}{4}\left((t^2+q)^2+4t^2\right)f_q(r,t)\right)=0. \tag{6.81}$$

The ODE in the parentheses coincides with the SE for an anharmonic oscillator. This differential equation can be solved order-by-order in perturbation theory (in the parameter q), as is discussed in numerous sources, for example [114]. If the SE of this problem could be solved using other means, this decomposition would allow one to study the eigenstates of the anharmonic oscillator.

6.3.6 Trapped particle in a magnetic field

Our last example will demonstrate our method in a case where the group action $\alpha:G\times M\to M$ is non-transitive (we saw another such non-transitive example, that of a particle orbiting a dyon, in §6.3.3). In particular, we will consider particle dynamics on $M = \mathbb{R}^3$, invariant under the action of a subgroup $G = \mathbb{R}^2 \subset \mathbb{R}^3$ corresponding to translations in x and y. We will begin this Section by formulating the problem, and introducing the necessary representation theory, to describe a generic such action. We will then consider a special case, in which the components of the inverse metric on $\mathbb{R}^3$ vary quadratically in the z direction. This corresponds, physically, to a z-dependent effective mass. In this special case, we shall find that the solutions to the SE become localised (or 'trapped') around the $z = 0$ plane.

Consider the action

$$S = \int dt \left(\frac{1}{2} \left(a_x(z)\dot{x}^2 + a_y(z)\dot{y}^2 + a_z(z)\dot{z}^2 \right) + V(z) - By\dot{x} - yf'(z)\dot{z} \right), \tag{6.82}$$

for a particle moving on $\mathbb{R}^3$. Here $a_x(z)$, $a_y(z)$, $a_z(z)$, $V(z)$, and $f(z)$ are (for now) arbitrary smooth functions of z, with $a_x(z)$, $a_y(z)$, and $a_z(z)$ necessarily non-vanishing. This action is quasi-invariant under the non-transitive action of translations in x and y, but is not invariant under translations in the z direction. We thus consider an equivalent action on a $U(1)$-principal bundle over $\mathbb{R}^3$, which has to be the trivial one, $P = \mathbb{R}^3 \times U(1)$, with coordinates $(x, y, z, s \sim s + 2\pi)$. The action is given by

$$S = \int dt \left(\frac{1}{2} \left(a_x(z)\dot{x}^2 + a_y(z)\dot{y}^2 + a_z(z)\dot{z}^2 \right) + V(z) - \dot{s} - By\dot{x} - yf'(z)\dot{z} \right), \tag{6.83}$$

which is strictly invariant under $\tilde{G} = \text{Hb}$, the Heisenberg group (the unique $U(1)$-central extension of $\mathbb{R}^2$ up to isomorphism), which in this Section we parametrise by $(\zeta_x, \zeta_y, \zeta_s)$, with its group action on the bundle $\mathbb{R}^3 \times U(1)$ defined by

$$\tilde{\alpha}_{(\zeta'_x, \zeta'_y, \zeta'_s)} \circ (x, y, z, s) = (x + \zeta'_x, y + \zeta'_y, z, s + \zeta'_s - \zeta'_y(Bx + f(z))). \tag{6.84}$$

The total hamiltonian corresponding to the above action is given by

$$H = \frac{1}{2a_x(z)} (p_x + By)^2 + \frac{1}{2a_y(z)} p_y^2 + \frac{1}{2a_z(z)} (p_z + yf'(z))^2 + V(z) + v(t)(p_s + 1), \tag{6.85}$$

which we quantise as the operator

$$\hat{H} = \frac{1}{2a_x(z)} \left(-i\frac{\partial}{\partial x} + By \right)^2 - \frac{1}{2a_y(z)} \frac{\partial^2}{\partial y^2} + \frac{1}{2a_z(z)} \left(-i\frac{\partial}{\partial z} + yf'(z) \right)^2 + V(z) + v(t) \left(-i\frac{\partial}{\partial s} + 1 \right). \tag{6.86}$$

We decompose a wavefunction into unirreps of Hb, exactly as in §6.1.1. The difference in this non-transitive case is that the coefficients of the unirreps will depend on z, *viz.*

$$\Psi(x,y,z,s) = \frac{2\pi}{|B|}\int drdte_r^{B,t}(x,y,s)f(r,t;z), \tag{6.87}$$

where as before

$$e_r^{B,t}(x,y,s) = e^{iBxr-is}\delta(r+y-t). \tag{6.88}$$

This however, now transforms under the unirrep of Hb defined by

$$\tilde{\pi}^{-B}(r,t;\zeta_x,\zeta_y,\zeta_z) = \left(\exp\left(if(z)\zeta_y\right)e_r^{B,t}(\zeta_x,\zeta_y,\zeta_s)\right)^*, \tag{6.89}$$

which takes account of the transformation of s which is not the same as ζ_s, as was the case in our previous examples. This can be seen from

$$\begin{aligned}\rho((\zeta'_x,\zeta'_y,\zeta'_s))\cdot e^{i(Bxr-s)}\delta(r+y-t)&\\ &= e^{i(B(x-\zeta'_x)r-i(s-(\zeta'_s+B\zeta'_y\zeta'_x)+\zeta'_y(Bx+f'(z))}\delta(r+y-\zeta'_y-t),\\ &= \int dq\left(e^{if'(z)\zeta_y}e^{i(B\zeta_x q-\zeta_s)}\delta(q+\zeta_y-r)\right)^* e^{i(Bxq-s)}\delta(q+y-t).\end{aligned} \tag{6.90}$$

Upon this decomposition, the SE reduces to the following partial differential equation (PDE)

$$\left(\frac{B^2t^2}{2a_x(z)} - \frac{\partial_t^2}{2a_y(z)} + \frac{\left(-i\partial_z+(t-r)f'(z)\right)^2}{2a_z(z)} + V(z)\right)f(r,t;z) = Ef(r,t;z). \tag{6.91}$$

Even in this case where G acts non-transitively on M, we see that using harmonic analysis (on a central extension) has removed derivatives with respect to the two variables x and y, and replaced them with derivatives with respect to the single variable t, which labels distinct copies of the unirrep in Eq. 6.89 that appears in the Hilbert space.

As a specific example where this PDE can be solved analytically, we take $f'(z)=0$, $V(z)=0$, $a_z(z)=1$, and $a_x(z)=a_y(z)=(a+z^2)^{-1}$ with $a\in\mathbb{R}_+$. That is, we do not consider the addition of a z-dependent potential, but we do consider a (specific) z-dependent metric on $\mathbb{R}^3$. This equation admits solutions by separation of variables, *viz.* $f(r,t;z)=f(r,t)g(z)$, after which $f(r,t)$ is found to satisfy a simple harmonic oscillator equation (with quantum number $n\in\mathbb{Z}$) analogous to Eq. 6.9. Likewise, $g(z)$

is then found to satisfy

$$\left(-\frac{1}{2}\frac{\partial^2}{\partial z^2}g(z) + |B|(n+1/2)g(z)(a+z^2)\right) = Eg(z), \quad n \in \mathbb{Z}, \tag{6.92}$$

which is simply the harmonic oscillator equation again. As such the z-dependence may be written in the form

$$g(z) = H_m\left((|B|(2n+1))^{1/4}\, z\right) e^{-\sqrt{|B|(2n+1)}z^2/2}, \qquad m \in \mathbb{Z}. \tag{6.93}$$

We can obtain an expression for the eigenstates by inverting the decomposition in Eq. 6.87 and setting $s=0$, to obtain functions on $\mathbb{R}^3$. Following a similar procedure to that in §6.1.1, we arrive at the eigenstates

$$\begin{aligned}\Psi_{m,n,\alpha}&(x,y,z)\\ &= H_m\left((|B|(2n+1))^{1/4}\, z\right) e^{-\sqrt{|B|(2n+1)}z^2/2} e^{i\alpha x} H_n(\sqrt{|B|}(y+\alpha/B))e^{-\frac{|B|}{2}(y+\alpha/B)^2},\end{aligned} \tag{6.94}$$

where $\alpha \in \mathbb{R}$. The energy levels depend only on the two quantum numbers n and m, both in $\mathbb{Z}$, and are given by

$$E_{m,n,\alpha} = \sqrt{|B|(2n+1)}(m+1/2) + a|B|(n+1/2). \tag{6.95}$$

Thus, interestingly, the eigenstates for this system appear to be trapped in the z-direction (even though naïvely one may expect the opposite).

6.4 Closing remarks

We have formulated the quantum mechanics of a particle moving on a manifold M, with dynamics invariant under the action of a Lie group G, in the presence of a background magnetic field. The coupling to a magnetic background, which is included via a topological term in the action, defines a $U(1)$-principal bundle P over M with connection. We suggest that such dynamics should be recast using an equivalent action on this principal bundle P, for two reasons. Firstly, a globally-defined lagrangian is guaranteed to exist only on P, but not on M itself. Secondly, even if a lagrangian were to be defined (locally) on M, this lagrangian would not in general be invariant under the action of G; rather, due to the presence of the topological term, it might shift by a total derivative. Once reformulated on P, we have shown that the lagrangian will be strictly invariant, not under G, but under a larger symmetry group $\tilde{G}$, which is a $U(1)$-central extension of G. We show how to construct this central extension $\tilde{G}$, which is a *bona fide* symmetry group of the system, in the general case.

We have discussed a plethora of examples in which these two (related) complications arise in coupling a particle to a magnetic background, and in every case show explicitly how reformulating the dynamics on the principal bundle P remedies the issues. To highlight just one example, we have revisited the seemingly humble problem of quantising a rotating rigid body in three dimensions, a system that is familiar from every undergraduate quantum mechanics course, which is equivalent to particle motion on the configuration space $SO(3)$. What is perhaps less familiar, and which is of interest to us in this Chapter, is that there is in fact a topological term in this theory. This topological term, whose existence stems from the non-vanishing cohomology group $H^2(SO(3), \mathbb{Z}) \cong \mathbb{Z}/2$, can only be written as a globally-defined term in the lagrangian if we pass to a principal bundle over $SO(3)$. There are two choices of such a bundle, both of which are isomorphic to central extensions of $SO(3)$; the bundle is either $U(2)$, or $SO(3) \times U(1)$. We show that the former choice corresponds to a term in the action phase that evaluates to -1 upon traversing closed loops in the configuration space, and thus has the affect of ascribing fermionic character to the rigid body.

The second main feature of this Chapter is the introduction of a new method for *solving* the Schrödinger equation for such quantum mechanical systems with magnetic backgrounds. Our method exploits the group-theoretic structure of the problem, by decomposing the Hilbert space into unitary irreducible representations of the central extension $\tilde{G}$. The method is thus very general; indeed, we show that it is a suitable match for the generality of the problem which we are attempting to solve. Because the Hilbert space carries a *bona fide* representation of the group $\tilde{G}$ (but not the group G, in which the Hilbert space carries only a projective representation), we expect that such a decomposition should yield a solution for the spectrum of the corresponding hamiltonian. In the example of the fermionic rigid body mentioned above, we immediately see the appearance of spin-$\frac{1}{2}$ representations in the spectrum by decomposing into representations of $\tilde{G} = U(2)$, thus exhibiting the non-trivial connection between topological terms in the action and representation theory.

We proceed to illustrate in all our examples how methods from harmonic analysis can be used to decompose the Hilbert space into representations of a central extension $\tilde{G}$, and in all cases this decomposition is found to be fruitful, typically reducing the SE to a family of ODEs whose solutions might be known. Our chosen examples range over some much-loved problems in quantum mechanics, including that of a particle moving on a plane in a uniform perpendicular magnetic field, a charged particle moving in the field of a magnetic monopole, and a charged particle moving in the field of a dyon. This last example illustrates the virtues of our method even in cases where the group G acts non-transitively on M, in reducing the problem to one on the space of orbits of G. We also study some new examples, including a particle moving on the Heisenberg group in the presence of a magnetic background, for which the Schrödinger equation is found to reduce, after decomposing into irreducible representations of a central extension of the Heisenberg group, to that of an anharmonic oscillator.

We anticipate that there are many more quantum mechanics problems which can be described by dynamics on a manifold with invariance under a Lie group action, and a coupling to a magnetic field, because this setup is a very general one. For example, the cases where $M = \mathbb{R}^n$ or $SO(n)$ appear ubiquitously in physics and chemistry, and one might describe more realistic molecular systems moving in magnetic fields, for example, by using a perturbative analysis around these simple cases. Another possible source of examples, of interest to condensed matter physicists and particle theorists, might be provided by quantum field theories admitting instanton solutions, in which great insight can be gained by solving for quantum mechanics on the instanton moduli space. Since such theories typically also contain topological terms in the action, the method of solution we have outlined in this Chapter, in which we first construct the *bona fide* symmetry group using central extensions and then bring to bear the heavy machinery of harmonic analysis, would be applicable.

Finally, we observe that all the quantum mechanical problems studied in this Chapter have had topological terms that are linear in time derivatives. This is not, however, the only possibility for lagrangians which are quasi-invariant under the action of a symmetry Lie group G. For an example where this is not the case, consider a free non-relativistic particle. This can be described in terms of motion in space which has a transitive action by the Galileo group, but is such that the lagrangian is not invariant, but shifts by a total derivative under a boost. It turns out that the familiar kinetic term for such a non-relativistic free particle, *viz.* $\frac{1}{2}m\dot{x}^2$, which is *quadratic* in time derivatives rather than linear, is nonetheless the result of a topological term in the action. To formulate and solve this example using the methods employed here, likely requires the use of so-called 'inverse Higgs phenomenon'. The inverse Higgs phenomenon, although not in the context outlined here, is the subject of the next Chapter.

Chapter 7

Inverse Higgs phenomena as duals of holonomic constraints

We have now studied two different forays into particle physics using techniques from mathematics. The previous forays were related to anomaly free gauge algebras and quantum mechanics problems in magnetic backgrounds. We will now proceed to our third and final foray. We will study constraints and symmetries in quantum field theory. This will lead us to formal definitions of new constraints which encapsulate the inverse Higgs phenomenon, as presented in the literature. The main mathematical techniques used in this Chapter are derived from category theory (see §2.4) and differential geometry (see §2.3).

The layout of this Chapter is as follows: In §7.1 we give the motivation of what is needed for our approach and why. In §7.2 we will give a more formal discussion of the categoric constructions needed. Then, in §7.3 we will categorically define holonomic constraints, showing they have a dual we call coholonomic constraints. We will define in a similar manor (co)meronomic constraints.

We first introduce symmetry in §7.4 where we consider a transitive symmetry group that maps equivariantly down to the space time. In this case a simplification arises due to the use of homogenous bundles. In this Section we will also make connection with the inverse Higgs phenomenon literature, showing that many examples there can be treated formally in terms of coholonomic and comeronomic constraints. We will deal with more generic symmetries in §7.5. We will need the somewhat technical theory of partial group actions. Two examples where partial group actions are needed will be given, corresponding to a $(1+1)$-d non-relativistic particle and a string in a plane. We will conclude in §7.6.

This Chapter is more mathematically involved than those that proceed it in this Thesis. Thus, as an aid to the reader we will use theorem-like environments. Furthermore, many of the more involved mathematical proofs in this Chapter have been relegated to Appendix D.

7.1 Motivating ideas

Since the required mathematical machinery for this Chapter goes somewhat beyond the usual physicist's curriculum, we begin by describing in an informal way what it is, and why it is needed. More formal descriptions are in what follows and Chapter 2.

Since physics is based upon local measurements in spacetime, it is natural to work using explicit local coordinates x^μ in spacetime. But since the specific choice of such coordinates is made at the observer's whim, the physics itself should not depend upon the choice. Coupled with the desire to be able to describe spacetimes that are not contractible, we are naturally led to the concept of a spacetime manifold X, which should moreover have a smooth structure so that we can define a dynamical action involving derivatives. (In what follows, almost everything will be taken to be smooth, so we omit reference to it unless there is a risk of confusion.)

A manifold comes naturally equipped with open sets and it is perhaps helpful to visualise these as 'laboratories without walls' in which observers can carry out their local measurements. The 'without walls' condition, or more precisely the condition that a set be open, ensures that observers whose laboratories intersect can compare measurements without having to worry about annoyances such as boundary conditions, *&c.*

Now that we have our mathematical model of spacetime, we may consider the degrees of freedom, or fields, of a field theory living on it. In the approach using explicit local coordinates, these take the form of maps $x^\mu \mapsto y^a(x^\mu)$, but there are several reasons why, in the approach using manifolds, we should not simply replace this by a map from X to some other manifold representing an internal or 'target' space. One is that there are known examples in physics, namely gauge theories, where this is not the case (there, the matter fields are instead sections of a fibre bundle). A second reason is that this construction amounts to the assertion that the internal spaces at each spacetime point can be canonically identified with one another, which seems inconsistent with the general expectation that physics should not feature 'action at a distance'. A third reason is that this structure is anyway not preserved once we take derivatives into account, as we shall see below.

We instead take the fields of a field theory (at least in the unconstrained case) to be local sections of a fibred manifold (see §2.3). Recall, that a fibred manifold consists of a pair of manifolds, X – the base – and Y – the total space – together with a surjective submersion $\pi : Y \to X$ and a local section is a smooth map $\alpha : U \to Y$ on some open subset $U \subseteq X$ which is a right inverse to π.[1]

A fibred manifold is perhaps best viewed as a generalisation of the more familiar notion of a fibre bundle (also defined in §2.3). Indeed, just as for a fibre bundle, the inverse image $\pi^{-1}(x)$ of a point $x \in X$ in the base is itself a manifold, which we call the fibre at x. As previously mentioned, unlike a fibre bundle, the fibres over different points generically don't have the same homotopy type or diffeomorphism class.[2] Since we interpret the fibre in physics as the internal space over the spacetime point x, we see that fibred manifolds allow for dramatically different field theories than those we are used to.

Nevertheless, such theories are compatible with the usual consistency requirements that we impose on physical theories. Indeed, just as for a fibre bundle, the fact that α is a right inverse to π guarantees that the sections collectively form a sheaf on X and so satisfy basic locality requirements. Most of these conditions (*i.e.* those for a presheaf) seem almost too obvious to mention;[3] for example, we require that sections (*i.e.* fields) defined on an open set (*i.e.* in a laboratory) restrict to fields defined on an open subset (*i.e.* in a smaller laboratory contained in the original one). But one – the sheaf condition – is not so trivial: it requires that given sections agreeing on the intersection of some collection of open sets, there exists a unique section on the union of that collection. It is thus a necessary precondition on kinematics for different observers to be able to compare measurements.

Moreover, just as for fibre bundles, the fact that $\pi : Y \to X$ is a surjective submersion guarantees that a local section exists in some neighbourhood of every point of X. Because of the presheaf condition, local sections will then exist on all subneigbourhoods and we interpret this as capturing the physically-reasonable requirement that local degrees of freedom should exist in a sufficiently small neighbourhood of each spacetime point.

In fact, a stronger statement is possible: a fibred manifold admits a local section not just at every point in X, but through every point in Y. Indeed, it is possible to choose *adapted coordinates* (x^μ, y^α) in a neighbourhood of every point of Y such that

[1] Suitable references are [107, 102, 138, 137, 131].

[2] An example is given by $Y = \mathbb{R}^2 - \{0\}$, and $X = \mathbb{R}$, with the projection onto the first factor. The fibre at $x = 0$ does not have the same homotopy type as elsewhere.

[3] The details are given in §2.4.1.

π restricts to $(x^\mu, y^\alpha) \mapsto x^\mu$, whose sections are equivalent to functions $x^\mu \mapsto y^a(x^\mu)$. This brings us back to our starting point, showing that fibred manifolds give us a global, coordinate-free notion of (unconstrained) fields that is compatible with locality.

The introduction of constraints will require us to reexamine this picture. Indeed, a constraint will restrict us to a subset of the local sections, namely those that satisfy the constraint. We will need to check that our basic physical requirements are still satisfied and this will require heavy use of the theory of sheaves (see §2.4.1). In particular, we need to ensure that locality is preserved, *i.e.* that the sections still form a sheaf, since the existence part of the sheaf condition is no longer obviously satisfied. Moreover, it is also obviously the case that sections will no longer exist through every point of Y (consider the case of a holonomic constraint, part of the data of which is a submanifold of Y) and so we will need to ensure that local sections exist at least at every point of X, as we required before. This is equivalent to the requirement that the stalks (to be defined shortly) of the sheaf are non-empty.

Mostly, we will not actually work with sheaves, but rather with the equivalent notion of étalé spaces, since they simplify the discussion of stalks as well as group actions. An étalé space can be given a physical motivation as follows. Imagine an observer at $x \in X$, whose laboratory is arbitrarily small. Such an observer will not be able to distinguish local sections $\alpha : U \to Y$, and $\beta : V \to Y$, for $U, V \ni x$, if there is an open subset W with $x \in W \subseteq U \cap V$, such that $\alpha \circ \iota_{W,U} = \beta \circ \iota_{W,V}$, where $\iota_{W,U}$ and $\iota_{W,V}$, are the inclusion maps. Thus, the observer is sensitive only to the equivalence class $[\alpha]_x$ of local sections, where $[\alpha]_x = [\beta]_x$ if α and β agree in the way just described. An equivalence class at x is called a *germ at x* and the set of such germs is called the *stalk at x*. The étalé space $(\Gamma Y, \Gamma\pi)$ is then defined as follows. The topological space ΓY is, as a set, the disjoint union over $x \in X$ of the stalks, equipped with the unique topology making the map $\Gamma\pi : \Gamma Y \to X : [\alpha]_x \mapsto x$ into a local homeomorphism.[4] In physics terms, the étalé space encodes the totality of information available to observers with arbitrarily small laboratories.

Evidently, the germs making up the points of ΓY remember all the derivatives (in some adapted coordinates) of local sections so contain at least enough information to allow us to define constraints involving any finite number of derivatives (as well as an action to any finite order in some effective field theory expansion).[5] But the topological

[4] in this topology, given $U \in X$ and a local section $\alpha : U \to Y$, the set $\{[\alpha]_x | x \in U\} \subset \Gamma Y$ is open and the set of such open sets obtained by varying U and α forms a basis for the topology.

[5] In fact they contain more information, as the following example shows: let $Y = \mathbb{R}^2$ and $X = \mathbb{R}$, with the standard projection. Then $\alpha(x) = (x, e^{-1/x^2})$ for $x \neq 0$ and $\alpha(0) = 0$, has the same Taylor expansion as $\beta(x) = 0$ at $x = 0$ but $[\alpha]_0 \neq [\beta]_0$.

space ΓY is not even Hausdorff in general, so cannot be given a smooth structure. To apply the full power of differential geometry to the discussion of constraints, we need to recover such a structure. This can be done by defining coarser equivalence classes, denoted $j^r_x\alpha$, with $j^r_x\alpha = j^r_x\beta$ if and only if the derivatives of α and β (computed in some adapted coordinates, the choice of which does not affect the result) agree up to and including the rth order. The set of all equivalence classes $j^r_x\alpha$ for all $x \in X$ is denoted J^rY. The set J^rY can be given a smooth structure making it into a manifold, called the *rth-jet manifold*, and making the map $\pi^r : J^rY \to X : j^r_x\alpha \mapsto x$ a surjective submersion (an observation which is vital for our discussion). If (x^μ, y^a) are adapted coordinates on Y, and locally $\alpha : x^\mu \mapsto (x^\mu, y^a(x^\mu))$, J^rY admits *induced coordinates*, which for J^1Y take the form (x^μ, y^a, y^a_μ) such that $j^r_x\alpha$ corresponds to the point $(x^\mu, y^a(x^\mu), \partial_\mu y^a(x^\mu))$, with an obvious generalisation to $J^{r>1}Y$. It is these induced coordinates that physicists use to write down lagrangians, but the approach using jet bundles has the advantage of being coordinate free. We remark that, even if one starts from a fibred manifold in the form of a product $Y = F \times X$, the jet manifold need not take the form of a product $J^rY = F' \times X$. This shows, as we vaguely alluded to earlier, that even for physical theories whose degrees of freedom are maps from spacetime to a target, one must pass to the more general fibred manifold picture once derivatives are included.

7.2 Categorical constructions

The categories `Set`, `Man`, and `Top` that we saw in §2.4 will play only a supporting rôle in our story. The main character will be a category of fibred manifolds over a fixed base, which we now define.

Definition 7.2.1. *Given a smooth base manifold X, let* $\texttt{Fib}_X$ *denote the* category of fibred manifolds over X*, whose objects are fibred manifolds (Y, π), where Y is a smooth manifold and $\pi : Y \to X$ is a smooth surjective submersion. A morphism, called a* fibred morphism*, between objects (Y, π) and (Y', π') is a smooth map $f : Y \to Y'$ such that $\pi' \circ f = \pi$.*

We will omit the adjective smooth in what follows, unless there is a risk of confusion.

Along with $\texttt{Fib}_X$, we will need a variety of other categories, defined as follows. Let $\mathcal{O}_X$ be the category whose objects are open subsets of X, and whose morphisms are the inclusions of subsets. We then have the usual category $\texttt{Pre}_X$ of presheaves on X given by the functor category $\texttt{Set}^{\mathcal{O}_X^{op}}$, together with its full subcategory $\texttt{She}_X$ of sheaves on X whose objects are those presheaves satisfying the sheaf condition.

Finally, we need the category $\texttt{Eta}_X$ of étalé spaces on X, as introduced in §2.4.1, an object of which is an étalé space (E, p), consisting of a topological space E and a local homeomorphism $p : E \to X$, and a morphism of which, called an étalé morphism is a continuous map $f : E \to E'$ such that $p' \circ f = p$. There is a functor $\texttt{Pre}_X \to \texttt{Eta}_X$ whose restriction to $\texttt{She}_X$ forms, together with the functor which sends an étalé space to its sheaf of sections, an equivalence of categories (see *e.g.* [146]). Thus we are free to work either with $\texttt{She}_X$, or $\texttt{Eta}_X$ and we will see that the latter is mainly convenient for our purposes.

Having introduced the necessary categories, we now consider functors between them. In §7.1 we saw how to construct both an étalé space and the rth jet manifolds, using the local sections of a fibred manifold. Unsurprisingly, these constructions are functorial.

Definition 7.2.2. *The* local sections functor $\Gamma : \texttt{Fib}_X \to \texttt{Eta}_X$ *sends a fibred manifold* (Y, π) *to the étalé space* $(\Gamma Y, \Gamma\pi)$ *and sends a fibred morphism* $f : Y \to Y'$ *to the étalé morphism* $\Gamma f : \Gamma Y \to \Gamma Y' : [\alpha]_x \mapsto [f \circ \alpha]_x$.

Definition 7.2.3. *The* rth-jet functor $J^r : \texttt{Fib}_X \to \texttt{Fib}_X$ *sends a fibred manifold* (Y, π) *to* $(J^r Y, \pi^r)$ *and sends a fibred morphism* $f : Y \to Y'$ *to* $J^r f : J^r Y \to J^r Y' : j^r_x \alpha \mapsto j^r_x (f \circ \alpha)$.

The functors Γ and J^r are well-behaved with respect to special classes of morphisms, as the following two theorems show.

Lemma 7.2.4. *The functor* Γ *sends an injection to an open topological embedding, but does not necessarily send surjections to surjections. (Proof: Appendix D.1)*

Lemma 7.2.5. *The functor* J^r *preserves submersions, surjective submersions, immersions, injective immersions, and embeddings, but does not necessarily preserve surjections or injections. (Proof: Appendix D.1)*

Finally we introduce two sets of natural transformations involving Γ and J^r, obtained either by forgetting the derivatives of sections or by prolonging sections to higher-jet manifolds.

Definition 7.2.6. *For* $r \geq l \geq 0$, *the* forget derivatives map *is the natural transformation* $J^r \Rightarrow J^l$ *defined on* (Y, π) *by the surjective submersion (in fact, affine bundle map for* $l \geq r - 1$*)* $\pi^{r,l} : J^r Y \to J^l Y : j^r_x \alpha \mapsto j^l_x \alpha$.

Definition 7.2.7. *For* $r > 0$, *the* prolong sections map *is the natural transformation* $\Gamma \Rightarrow \Gamma J^r$ *defined on* (Y, π) *by* $j^r : \Gamma Y \to \Gamma J^r Y : [\alpha]_x \mapsto [j^r \alpha]_x$, *where* $[\alpha]_x$ *is the germ at* x *of the local section* α *on* $U \ni x$.

7.3 Constraints

7.3.1 Holonomic and higher-degree constraints

In the physicist's world of local coordinates (x^μ, y^a), a holonomic constraint is usually defined as a set of smooth relations of the form $f(x^\mu, y^a) = 0$. The inadequacy of this definition can easily be seen by considering examples from classical mechanics in the plane (so $\pi : Y \to X$ is the map $\mathbb{R}^3 \to \mathbb{R} : (x^0, y^1, y^2) \mapsto x^0$), such as $y^1 y^2 = 0$ or $(y^1)^2 + (y^1)^2 + (x^0)^2 - 1 = 0$. Ills of the kind observed in the first example can be cured by insisting that a holonomic constraint be an embedded submanifold Z of Y and those in the second example by insisting that Z itself be a fibred manifold over X, embedded in Y via a fibred morphism [103, 105, 104]. Thus we make the following

Definition 7.3.1.1. *A* fibred submanifold *(resp.* open fibred submanifold*) of a fibred manifold (Y, π) is a fibred manifold (Z, ζ) together with a fibred morphism $\iota_Z : Z \to Y$ that is an embedding (resp. open embedding).*

A holonomic constraint as defined in [103, 105, 104] then amounts to a choice of fibred submanifold of (Y, π) and we will use this as a working definition for now (later we will make an equivalent definition that appears rather perverse, but turns out to be much more useful for finding more general constraints). The local degrees of freedom of the field theory can then obviously be taken to be the local sections of (Z, ζ). Since these form a sheaf whose stalks are non-empty (since ζ is a surjective submersion), we obtain a theory which is consistent with locality and in which local degrees of freedom exist.

At some level, this corresponds to the physicist's notion that holonomic constraints are easily dealt with, because one can simply eliminate redundant degrees of freedom. But it is important to note that our working definition of a holonomic constraint is much more than just a coordinate independent reformulation of the usual physicist's notion. Not only does it remove pathological examples such as those already discussed, but it also includes constraints which would be considered nonholonomic by the physicist, in that they cannot be expressed locally in terms of relations $f(x^\mu, y^a) = 0$. For example, in classical mechanics in the plane, our working definition includes the fibred submanifold defined by $(y^1)^2 + (y^2)^2 > 1$.

Now let us turn our attention to constraints which are nonholonomic in the sense that they include derivatives of order $r > 0$ and below of the fields, in local coordinates. An obvious guess is to consider a fibred submanifold not of $(Y, \pi) \cong (J^0 Y, \pi^0)$, but rather of $(J^r Y, \pi^r)$. Denoting the fibred morphism embedding by $\iota_Q : Q \to J^r Y$, the degrees of freedom of the field theory would then correspond to the local sections of (Y, π) whose prolongation to J^rY lies in $\iota_Q(Q) \subset J^rY$. We now encounter two potential difficulties. One is that it is not obvious, *a priori*, that the constraint is consistent with locality, in that the degrees of freedom form a sheaf. Even if they do, it is not obvious that degrees of freedom exist at every spacetime point in X, or in other words that the stalks of the sheaf are not empty. In fact, it will turn out that the first condition is automatically satisfied, but this will require some work to show, so let us return to it shortly. The second condition is not automatcially satisfied, as the following counterexample from classical mechanics in the plane shows. The first jet manifold there is given by $(J^1Y, \pi^1) = (\mathbb{R}^5, (x^0, y^1, y^2, y^1_0, y^2_0) \mapsto x^0)$; letting $(Q, \nu) = (\mathbb{R}^3, (x^0, y^1, y^1_0) \to x^0)$ with $\iota_Q : (x^0, y^1, y^1_0) \mapsto (x^0, y^1, 0, y^1_0, 1)$, we see that there are no local sections at all!

Now let us return to the first condition. The statement that the degrees of freedom form a sheaf is equivalent to the following

Theorem 7.3.1.2. *The pull-back of* $\Gamma\iota_Q : \Gamma Q \to \Gamma J^rY$*, and* $j^r : \Gamma Y \to \Gamma J^rY$ *in* $\texttt{Eta}_X$ *exists and we denote it by* (E^Q, p^Q)*. (Proof: Appendix D.3)*

These considerations motivate the following

Definition 7.3.1.3. *A* consistent constraint of order r *on the fibred manifold* (Y, π) *is a subfibred manifold* $Q \subset J^rY$ *such that the stalks of the pullback* E^Q*, whose existence was shown in the previous theorem, are non-empty.*

The difficulty with nonholonomic constraints, at least those defined by a submanifold $Q \subset J^rY$, thus reside in establishing that the stalks are non-empty. The rest of this Section will be devoted to finding ways in which this can be achieved.

To do so, it is useful to re-examine the notion of a holonomic constraint, our working definition of which identifies it with a consistent constraint of order 0. The following argument shows, however, that we are also free to regard it as a consistent constraint of any order r. Firstly, Lemma 7.2.5 has shown that the functor J^r sends a subfibred manifold $\iota_Z : Z \to Y$ to a subfibred manifold $J^r\iota_Z : J^rZ \to J^rY$. Moreover, the resulting étalé spaces (E^Q, p^Q) are isomorphic (to $(\Gamma Z, \Gamma\zeta)$) for all r, so define consistent constraints of order r that lead to field theories with equivalent degrees of freedom.

The notion of different constraints leading to theories that are physically the same, in the sense of having equivalent degrees of freedom, leads us to make the following

Definition 7.3.1.4. *Consistent constraints (of any order) are* kinematically equivalent *if their corresponding étalé spaces are isomorphic.*

Going further, let us make the following, apparently rather perverse, definition of a holonomic constraint.

Definition 7.3.1.5. *A* holonomic constraint of degree r for (Z, Ω) *is a limit in* $\mathtt{Fib}_X$ *of the diagram*

$$\begin{array}{ccc} J^r Z & \xrightarrow{J^r \iota_Z} & J^r Y \\ {\scriptstyle \zeta^{r,0}}\big\downarrow & \overset{\Omega}{\searrow} & \big\downarrow{\scriptstyle \pi^{r,0}} \\ Z & \xrightarrow[\iota_Z]{} & Y \end{array} \tag{7.1}$$

where (Z, ζ) *is a fibred submanifold of* (Y, π) *with embedding* ι_Z *and the fibred morphism* Ω *is such that the lower triangle commutes.*

The definition is perverse for more than one reason. Firstly, the requirement that the lower triangle commutes evidently shows that given (Z, ι_Z) there exists a unique map Ω, namely $\iota_Z \circ \zeta^{r,0}$, so there is no data associated to Ω. Secondly, the fact that the square commutes shows that the limiting object is (uniquely isomorphic to) $(J^r Z, \zeta^r)$, with the fibred morphism to $J^r Z$ in the diagram being the identity and with all other fibred morphism being fixed by the commutativity of the diagram. Nevertheless, it is clear that our new definition is equivalent to our old working definition, in that it yields a kinematically equivalent constraint.

The beauty (if it can be called that) of our new definition is that it admits a non-trivial dual, to which we now turn.

7.3.2 Coholonomic constraints

We begin with a preliminary definition that is the dual of 7.3.1.1.

Definition 7.3.2.1. *A* fibred quotient of the fibred manifold (Y, π) *is a fibred manifold* (Z, ζ) *together with a fibred morphism* $\tau_Z : Y \to Z$ *that is a surjective submersion.*

Dualising our new definition of a holonomic constraint, we have the following

Definition 7.3.2.2. *A* coholonomic constraint of degree r for (Z, Ω) *is a limit in* $\mathtt{Fib}_X$ *of the diagram*

$$\begin{array}{ccc} J^r Y & \xrightarrow{J^r \tau_Z} & J^r Z \\ {\scriptstyle \pi^{r,0}}\big\downarrow & \overset{\Omega}{\swarrow} & \big\downarrow{\scriptstyle \zeta^{r,0}} \\ Y & \xrightarrow[\tau_Z]{} & Z \end{array} \tag{7.2}$$

where (Z, ζ) *is a fibred quotient of* (Y, π) *whose surjective submersion is* τ_Z *and the fibred morphism* Ω *is such that the lower triangle commutes.*

A number of remarks are now in order. Firstly, we remark that our 'dual' construction is not obtained by dualising everything. Rather, we simply replace the notion of a fibred submanifold, namely a fibred manifold together with a fibred morphism from it to (Y, π) that is an embedding, by the dual notion of a fibred quotient. We have not changed the direction of the map Ω, and nor have we replaced the limit by a colimit.

Secondly, we remark that the map Ω, which now takes the form of a lift of $\zeta^{r,0}$ through τ_Z, no longer necessarily exists; nor, if it does, is it necessarily unique. As we shall see, this opens the door to a rather rich notion of a constraint, which will capture, in particular, the essence of the inverse Higgs phenomenon.

Thirdly, we remark that if we were to remove the datum of the map Ω from the definition, we would not obtain anything interesting. The limit in that case is simply J^rY, so we recover the unconstrained field theory on Y.

A fourth remark is that it is not obvious that the limit we have defined exists. In fact we have the following

Proposition 7.3.2.3. *The limit of Diagram 7.2 exists; denoting it by* $((Q, \nu), \{\iota_Q : Q \to J^rY\})$*,* ι_Q *is an embedding. (Proof: Appendix D.3)*

Because Q is embedded, we are furthermore guaranteed, by Theorem 7.3.1.2 above, that the degrees of freedom form a sheaf, so are consistent with locality. But in fact much more is true.

Theorem 7.3.2.4. *The étalé space* (E^Q, p^Q) *for a coholonomic constraint of degree* r *for* (Z, Ω) *is isomorphic to* $(\Gamma Z, \Gamma\zeta)$*. (Proof: Appendix D.3)*

So not only are coholonomic constraints of degree r for (Z, Ω) consistent constraints, but, just as for holonomic constraints, we find that they are kinematically equivalent to the unconstrained theory on the fibred manifold Z. Comparing with the physics literature, we see that our theorem corresponds to the notion of 'essential Goldstone bosons'. Indeed, these are to be interpreted precisely as the local description in adapted coordinates of the local sections of $\zeta : Z \to X$.

Moreover, our theorem shows that, even though we started from a definition of coholonomic constraint which was not the exact categorical dual of a holonomic constraint, we end up with a duality at the level of field theories which is satisfyingly precise: a holonomic constraint is kinematically equivalent to an unconstrained theory on a fibred submanifold, while a coholonomic constraint is kinematically equivalent to an unconstrained theory on a fibred quotient.

7.3.3 Meronomic constraints

To describe all of the examples of the inverse Higgs phenomenon in the literature within our formalism requires us to slightly generalise the notion of coholonomic constraints. This is most conveniently done by first generalising holonomic constraints and then dualising as before.

Locally, meronomic constraints look like holonomic constraints and so we call them *meronomic* constraints (from the greek for 'part' and 'law', in much the same way that holonomic is from 'whole' and 'law'). Compared with holonomic constraints, we have an extra datum in the form of an open fibred submanifold of J^rY.

Definition 7.3.3.1. *A* meronomic constraint of degree r for (Z, R, Ω) *is a limit in* $\mathtt{Fib}_X$ *of the diagram*

$$\begin{array}{ccc} R & & \\ \downarrow{\scriptstyle \iota_R} & \searrow^{\Omega} & \\ J^rZ & \xrightarrow{J^r\iota_Z} & J^rY \\ \downarrow{\scriptstyle \zeta^{r,0}} & \searrow & \downarrow{\scriptstyle \pi^{r,0}} \\ Z & \xrightarrow[\iota_Z]{} & Y \end{array} \tag{7.3}$$

where (Z, ζ) is a fibred submanifold of (Y, π) with embedding ι_Z, (R, ρ) is an open fibred submanifold of (J^rZ, ζ^r) with open embedding ι_R, and the fibred morphism Ω is such that the diagram commutes.

Just as for holonomic constraints, the datum of the map Ω adds nothing here, since it must equal $\iota_Z \circ \zeta^{r,0} \circ \iota_R$, but is present so that we obtain something more general when we dualise.[6]

Completely analogously to a holonomic constraint, the limit in the definition exists and is given by $((R, \rho), \{\mathrm{id} : R \to R\})$, up to unique isomorphism.

The fact that ι_R is an open embedding is what makes a meronomic constraint locally look like a holonomic constraint. Due to this, the étalé space (E^Q, p^Q) is guaranteed to have non-empty stalks, since, roughly, for any $x \in X$ there will be a local section of Z, β with $J^r\beta(x)$ lying in the open set R, we can then just restrict the domain of β so that $J^r\beta$ lies wholly in R. $[J^r\beta]_x$ then defines a point in $(p^Q)^{-1}(x)$.

We recover the special case of a holonomic constraint by choosing ι_R to be an isomorphism.

[6] Amusingly, if we dualise without the map Ω, we obtain not a trivial unconstrained theory (as we did in the holonomic case), but rather a class of constraints that are equivalent to a subclass of meronomic constraints. This fact is proven and made use of in Theorem 7.3.4.2.

7.3.4 Comeronomic constraints

Turning the handle, we now obtain the dual notion corresponding to a meronomic constraint, which is relevant for certain physical examples of the inverse Higgs phenomenon.

Definition 7.3.4.1. *A* comeronomic constraint of degree r for (Z, R, Ω) *is a limit in* $\mathtt{Fib}_X$ *of the diagram*

$$\begin{array}{ccc} & & R \\ & & \downarrow{\scriptstyle \iota_R} \\ J^rY & \xrightarrow{J^r\tau_Z} & J^rZ \\ {\scriptstyle \pi^{r,0}}\downarrow & & \downarrow{\scriptstyle \zeta^{r,0}} \\ Y & \xrightarrow[\tau_Z]{} & Z \end{array} \qquad (\Omega : R \to Y) \tag{7.4}$$

where (Z, ζ) is a fibred quotient of (Y, π) whose surjective submersion is τ_Z, (R, ρ) is an open fibred submanifold of (J^rZ, ζ^r) with open embedding ι_R, and the fibred morphism Ω is such that the diagram commutes.

Proposition 7.3.4.2. *The limit of the Diagram 7.4 exists, denoting it by $((Q, \nu), \{\iota_Q : Q \to J^rY, f^R_Q : Q \to R\})$, then ι_Q is an embedding. (Proof: Appendix D.3)*

As with holonomic constraints and meronomic constraints, a coholonomic constraint is a special instance of a comeronomic constraint, corresponding to the case where ι_R is an isomorphism.

For a holonomic constraint we had that the étalé space (E^Q, p^Q) was isomorphic to $(\Gamma Z, \Gamma\zeta)$. For comeronomic constraints we have the following

Theorem 7.3.4.3. *The étalé space (E^Q, p^Q) associated with a comeronomic constraint is isomorphic to the étalé space (E^R, p^R) associated with the embedding of R into J^rZ. (Proof: Appendix D.3)*

For the same reason that meronomic constraints lead to non-empty stalks and hence consistent constraints, the stalks of (E^R, p^R) will be non-empty and, due to the isomorphism, so will those of (E^Q, p^Q). Even more importantly, we learn that a comeronomic constraint is kinematically equivalent to a meronomic constraint on the fibred quotient (Z, ζ).

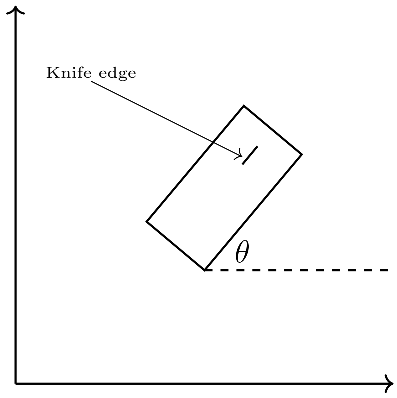

Fig. 7.1 The Chaplygin sleigh

7.3.5 An example from classical mechanics: the Chaplygin sleigh

Here we give an example of a comeronomic constraint in classical mechanics, showing that, despite their abstract definition, they occur in remarkably simple examples. The example is based on the famous example of a Chaplygin sleigh, with the minor tweak that we forbid the sleigh from being translationally at rest, thus deleting a single point from the space of possible translational velocities of the sleigh.

Recall that a Chaplygin sleigh is a rigid body sliding in the plane, with motion that is frictionless apart from a 'knife edge' at a point on the object that prevents motion at that point perpendicular to the edge of the knife, as in Fig. 7.1.

The fibred manifold Y over $\mathbb{R}$ is thus $\mathbb{R}^3 \times S^1$ with local adapted coordinates (t, x, y, θ) representing the time, position of the knife edge in the plane, and orientation of the sleigh, and with fibering map $(t, x, y, \theta) \mapsto t$. The jet bundle J^1Y is thus $\mathbb{R}^3 \times S^1 \times \mathbb{R}^3$ with local adapted coordinates $(t, x, y, \theta, x_t, y_t, \theta_t)$. To describe the system as a comeronomic constraint, we start with the fibred quotient of Y obtained by projecting out the S^1, which admits global coordinates (t, x, y) and consider the open fibred submanifold R of $J^1Z = \{(t, x, y, x_t, y_t)\}$ obtained by deleting the points with $x_t = y_t = 0$ (enforcing the constraint that the sleigh is not allowed to be translationally at rest). This allows us to define a fibred morphism $\Omega : R \to Y$ which acts as the identity on (t, x, y) but sends (x_t, y_t) to the point $(x_t/\sqrt{x_t^2 + y_t^2}, y_t/\sqrt{x_t^2 + y_t^2})$ on the unit circle $S^1 \subset \mathbb{R}^2$. This has precisely the effect of enforcing the constraint that the sleigh may not move perpendicularly to the knife edge at the knife edge.

As above, this theory is kinematically equivalent to a theory with a meronomic constraint defined by R embedded into Z; the explicit isomorphism takes the stalk whose section is defined by $t \mapsto (t, x(t), y(t), \theta(t))$, (where $x(t)$, $y(t)$ and $\theta(t)$ are required to satisfy the constraint), to the stalk defined by the section $t \mapsto (t, x(t), y(t))$ in E^R.

7.4 Fibrewise group actions and homogeneous bundles

Having described the consistent constraints that appear in theories featuring the inverse Higgs phenomenon, we now discuss the rôle played by symmetry, in the form of a Lie group G acting smoothly on Y. Things simplify greatly in the case where G also acts on X such that the fibering map $\pi : Y \to X$ is G-equivariant, for the simple reason that a well-defined group action is then induced on each r-jet manifold J^rY, and this action is such that the maps $\pi^{r,l}$ are G-equivariant. We call such an action a *fibrewise group action.* For more general G actions on Y, one induces at best a partial group action on J^rY and we will defer the somewhat technical study of this situation to the next Section.

When the G action is fibrewise, it is possible to define a number of subgroups of G that are familiar to physicists (it is important to remark that none of these subgroups are defined in the case of more general group actions). For each $x \in X$ we define the *internal symmetry group at* x, as the stabiliser G_x of $x \in X$. The *internal symmetry group* G_X can then be defined as $\cap_x G_x$; equivalently, G_X is the subgroup of G that acts trivially on X. G_X is a normal subgroup of G, and we can define the *spacetime symmetry group* as the group G/G_X.

Of most interest to us (since we are interested in theories of Goldstone bosons) is the case where G acts, in addition, transitively on Y, such that Y is diffeomorphic to G/K for some Lie subgroup $K \subseteq G$. Because π is surjective and G-equivariant, it follows that G also acts transitively on X, so we have that X is diffeomorphic to G/H for some Lie subgroup $H \subseteq G$ such that $H \supseteq K$. Moreover, the fibred manifold $\pi : Y \to X$ is isomorphic (in $\mathtt{Fib}_X$) to $G/K \to G/H$, which has the structure of a fibre bundle with fibre H/K associated to the H-principal bundle $G \to G/H$. This, along with the corresponding jet manifolds, is an example of a *homogeneous bundle* and the theory of such bundles can be brought to bear.

To give a simple example that allows us to make contact with the typical situation encountered in physical theories, suppose that $G = A \times B$ for some Lie groups A and B, and let $K \subseteq B$ and $H = A \times K$, so that $K \subseteq H \subseteq G$ as required. Recalling that $Y \cong G/K \cong A \times B/K$ and $X \cong G/H \cong B/K$, we have that the internal symmetry group at $bK \in X$ is $A \times K_{bK}$, where K_{bK} is the subgroup of K given by $\{k \in K | kbK = bK\}$. If K is, say, the Lorentz group and B the Poincaré group, we have that $G_X = A$ and $G/G_X = B$. In other words, the internal symmetry is A and the spacetime symmetry is B. We stress that this simple result will not obtain in more general situations, even when the G action is fibrewise.

We now wish to go further and discuss the group actions that are induced on jet manifolds and their interplay with coholonomic and comeronomic constraints. A first observation is that, even if we start with a transitive group action on Y, for sufficiently large r the group action induced on J^rY will not be transitive. Indeed, since a manifold with a transitive action of G is diffeomorphic to a homogeneous space of G, the dimension of such a manifold is bounded above by the dimension of G. But the dimension of J^rY increases without bound with r. It it is this simple fact that both allows for, and exhibits the generic nature of, the inverse Higgs phenomenon: once we include enough derivatives in a field theory, G cannot act transitively and subsets of the orbits G can be used to define non-trivial constraints that are nevertheless compatible with the action of G. Since they necessarily involve derivatives (G acts transitively on $Y \cong J^0Y$, so there are no constraints that are compatible with the G action) the constraints are necessarily nonholonomic, according to the usual definition, leading to possible problems with consistency. But all constraints in the literature on the inverse Higgs phenomenon turn out to be either coholonomic or comeronomic, so consistency is guaranteed.

To explore this in more detail requires us to first review the theory of homogeneous bundles based on the principal L-bundle $G \to G/L$, for $L \subseteq G$. The key observation here is that these form a category and that that category is equivalent to the category of manifolds with an L action. This equivalence of categories is a rigorous statement of the physicist's vague notion that, in sigma models, G invariance follows from L invariance alone.

Some of the discussion in this Section requires results extending the results of the previous Section to the case where a group acts. The proofs of these results are subsumed into the proofs for the more general case of a partial group action, given in the next Section and Appendix D.

7.4.1 The category of homogeneous bundles

We now review the theory of homogeneous bundles. For more details, see *e.g.* [145].

Let G be a Lie group, and L a Lie subgroup of G. A *homogeneous bundle over the homogeneous space* G/L is a triple $(Y, \pi, \mathcal{Y})$ consisting of a smooth manifold Y equipped with a smooth action $\mathcal{Y} : G \times Y \to Y$ of G and a smooth bundle map $\pi : Y \to G/L$ that is equivariant with respect to $\mathcal{Y}$ and the usual action $\mathcal{L} : G \times G/L \to G/L$ of G given by $\mathcal{L}_g : g'L \mapsto gg'L$.

The homogeneous bundles over G/L form the objects of a category, which we now define.

Definition 7.4.1.1. *Let* $\mathtt{HBun}_{G/L}$ *be the category whose objects are homogeneous bundles over the homogeneous space* G/L*, and whose morphisms from* $(Y, \pi, \mathcal{Y})$ *to* $(Y', \pi', \mathcal{Y}')$ *are smooth maps* $f : Y \to Y'$ *such that* $\pi' \circ f = \pi$ *and* $\mathcal{Y}'_g \circ f = f \circ \mathcal{Y}_g$ *for all* $g \in G$*.*

The category $\mathtt{HBun}_{G/L}$ is equivalent to the category defined as follows.

Definition 7.4.1.2. *Let* L-$\mathtt{Man}$ *be the category whose objects are pairs* $(M, \mathcal{M})$*, consisting of a smooth manifold* M *equipped with a smooth action* $\mathcal{M} : L \times M \to M$ *of* L*, which we call an* L-manifold*, and whose morphisms between* $(M, \mathcal{M})$ *and* $(M', \mathcal{M}')$ *are smooth maps* $f : M \to M'$ *such that* $\mathcal{M}'_l \circ f = f \circ \mathcal{M}_l$ *for all* $l \in L$*, which we call* L-maps.

We will not give the functors defining this equivalence, which we denote by

$$\Pi : \mathtt{HBun}_{G/L} \to L\text{-}\mathtt{Man} \quad \text{and} \quad \hat{\Pi} : L\text{-}\mathtt{Man} \to \mathtt{HBun}_{G/L},$$

explicitly (the reader is directed to [145] for an explicit form), but simply record the following lemma.

Lemma 7.4.1.3. *The functors* Π *and* $\hat{\Pi}$ *send open embeddings to open embeddings. (Proof: Follows manifestly from the definitions of* $\Pi, \hat{\Pi}$*, and the quotient and subspace topologies.)*

7.4.2 Constructing constraints

To specify a comeronomic constraint with fibrewise group actions requires the following data:

1. a fibred manifold (Y, π), a fibred quotient (Z, ζ) of (Y, π), an open fibred submanifold (R, ρ) of $(J^r Z, \zeta^r)$, and a suitable fibred morphism $\Omega : R \to Y$ (as per the definition with no group acting given in 7.3.4.1);
2. fibrewise group actions $\mathcal{Y}$, $\mathcal{Z}$, and $\mathcal{R}$ of G on Y, Z, and R such that: the surjective submersion $\tau_Z : Y \to Z$ is equivariant with respect to the actions $\mathcal{Y}$ and $\mathcal{Z}$; the

open embedding $\iota_R : R \to J^r Z$ is equivariant with respect to the action $\mathcal{R}$ and the action $J^r\mathcal{Z}$ of G on $J^r Z$ induced by $\mathcal{Z}$; [7] the fibred morphism Ω is equivariant with respect to $\mathcal{R}$ and $\mathcal{Y}$.

Specifying this data becomes simpler in the case of most physical interest, namely when G acts transitively on Y, where we have the following

Theorem 7.4.2.1. *Let a comeronomic constraint be defined by a diagram as in 7.4, where all objects have a G-action and all morphisms are G-equivariant, and let G act transitively on Y. Then $Z \cong G/L$ for some $L \subseteq G$ and all objects and morphisms in the diagram lie in the subcategory* $\mathtt{HBun}_{G/L}$.

Thus we can describe everything in terms of homogeneous bundles or, via the equivalence of categories, in terms of manifolds with an L-action.

Proof. The maps τ_Z and π are required to be G-equivariant, so it follows that G also acts transitively on Z and X, so we can write $Y \cong G/K$, $X \cong G/H$, and $Z \cong G/L$, with $K \subseteq L \subseteq H \subseteq G$. Moreover, the maps τ_Z and ζ are G-equivariant bundle maps and so we have that (Y, τ_Z) and (Z, ζ) define objects in $\mathtt{HBun}_{G/L}$ whose typical fibres are the L-manifolds given by L/K and a point, respectively. Further, since the map $\zeta^{r,0}$ is a G-equivariant bundle map, we have that $(J^r Z, \zeta^{r,0})$ also defines an object in $\mathtt{HBun}_{G/L}$.

Now consider the open fibred submanifold R in $J^r Z$. Because G acts transitively on Z, the equivariant map $\zeta^{r,0} \circ \iota_R$ must be a bundle map. The argument goes as follows. Because G acts transitively, the map must be a surjection and because both $\zeta^{r,0}$ and ι_R are submersions, it must also be a submersion. But then the same arguments given in [145] to derive the equivalence of categories between homogeneous bundles and L manifolds show that the map is isomorphic to a bundle map. (In particular, it is clear that the fibres of the fibred manifold are all diffeomorphic to one another, since any one can be reached from another by a diffeomorphism corresponding to some $g \in G$.) So $(R, \zeta^{r,0} \circ \iota_R)$ also defines an object in $\mathtt{HBun}_{G/L}$. All the morphisms in Diagram 7.4 are equivariant by assumption and commutativity of the diagram ensures that they define morphisms in $\mathtt{HBun}_{G/L}$. □

So we can carry the discussion over to L-`Man`, where Z is represented by a point and Y is represented by the homogeneous space L/K. Suppose that R is represented by the L-manifold ΠR. For generic r, an explicit description of $J^r Z \to Z$ as an L-manifold is

[7] In adapted local coordinates, this action can be deduced using the chain rule; §7.5 gives a formal definition.

somewhat unpleasant; we content ourselves with giving a description for $r = 1$ where, since $J^1Z \to Z$ is an affine bundle, we obtain an affine space with an action of L. This covers all examples in the literature, bar one, corresponding to the Galileid [124], where one needs $r = 2$.

Proposition 7.4.2.2. *For a fibred manifold $Z \cong G/L \to X \cong G/H$, a typical fibre of the affine bundle $J^1Z \to Z$ is given by the L-affine space $A(\mathfrak{g}/\mathfrak{h}, \mathfrak{g}/\mathfrak{l})$ over $\mathrm{Hom}(\mathfrak{g}/\mathfrak{h}, \mathfrak{h}/\mathfrak{l})$ of linear sections of the linear map $\mathfrak{g}/\mathfrak{l} \to \mathfrak{g}/\mathfrak{h}$, where $\mathfrak{g}$ denotes the Lie algebra of G, &c.; the action of L is by pre- or post-composition with the actions on $\mathfrak{g}/\mathfrak{l}$, and $\mathfrak{h}/\mathfrak{l}$ induced by the adjoint action of $L \subseteq G$ on $\mathfrak{g}$. (Proof: Follows from [138, Lemma 4.1.3].)*

In all, we have the following

Theorem 7.4.2.3. *When G acts transitively on Y, the required data for a comeronomic constraint of order 1 can be specified by*

1. *a chain of inclusions of 4 Lie groups, $K \subseteq L \subseteq H \subseteq G$, which define Y, Z, X in Diagram 7.4;*
2. *an open L-submanifold ΠR of $A(\mathfrak{g}/\mathfrak{h}, \mathfrak{g}/\mathfrak{l})$, which defines (R, ι_R);*
3. *an L-map $\Pi\Omega : \Pi R \to L/K$, which defines Ω.*

One checks that by Lemma 7.4.1.3 we get an open embedding ι_R if and only if we start from an open embedding in L-`Man` and that the fibred morphism Ω is such that the diagram commutes.

We now go on to describe a number of examples.

7.4.3 Examples

We will now list examples of inverse Higgs phenomena taken from the literature [83, 125, 124]. For each of the examples, we will specify all the data indicated in the previous Subsection required to specify a comeronomic constraint. In cases where the constraint is in fact coholonomic, we will simply not mention ΠR.

Example 7.4.3.1 (1-d Non-relativistic point particle)**.** We have that $G = \mathrm{Gal}(0+1, 1)$, which corresponds to the Heisenberg group. We label a set of Lie algebra generators of G as $\{T, X, V\}$ with $[V, T] = X$, and all other commutators zero. The other Lie groups involved correspond to $H = \mathbb{R}^2 = \{\exp(xX + vV)\}$, $L = \mathbb{R} = \{\exp(vV)\}$, and $K = \{\mathrm{id}\}$. The space $A(\mathfrak{g}/\mathfrak{h}, \mathfrak{g}/\mathfrak{l})$ has elements given by maps of the form

$f_a(cT + \mathfrak{h}) = cT + caX + \mathfrak{l}$. The element $\exp(v'V) \in L$ acts on f_a as $f_a \mapsto f_{a+v'}$. The map $\Pi\Omega : f_a \mapsto \exp(aV) \in L/K$ is a valid L-map.

Example 7.4.3.2 (3-d Non-relativistic point particle)**.** Now consider the 3-d version of the previous example. We take as P the time translation generator, C_I ($I = 1, 2, 3$) the spatial translations, B_I the boosts, and J_I the rotations, closely following the notation of [83]. The symmetry group corresponds to $G = \mathrm{SGal}(0+1, 3) = \{e^{tP} e^{\rho^I C_I} e^{v^I B_I} e^{\theta^I J_I}\}$. The other groups take the form $H = \mathbb{R}^3 \rtimes \mathrm{ISO}^+(3) = \{e^{\rho^I C_I} e^{v^I B_I} e^{\theta^I J_I}\}$, $L = \mathbb{R}^3 \rtimes SO(3) = \{e^{v^I B_I} e^{\theta^I J_I}\}$, and $K = SO(3) = \{e^{\theta^I J_I}\}$. The affine space $A(\mathfrak{g}/\mathfrak{h}, \mathfrak{g}/\mathfrak{l})$ has elements given by $f_{a^I}(cP + \mathfrak{h}) = cP + ca^I C_I + \mathfrak{l}$. Under the action of $e^{v^I B_I} e^{\theta^I J_I} \in L$, $a_I \mapsto (e^{\theta^K J_K})_I{}^J a_J + v_I$. The map $\Pi\Omega : f_{a^I} \mapsto e^{a^I V_I} K$ is an L-map.

Example 7.4.3.3 ($(1+1)$-d, $N = 1$ Galileon)**.** The symmetry group here is $G = \mathrm{SGal}(1+1, 1)$. The group G has Lie algebra generators $\{P_0, P_1, K_1\}$, which generate the $(1+1)$-d Poincaré subalgebra and $\{B^0, B^1, C\}$, which have the non-zero commutators $[B^\mu, P_\nu] = \eta^\mu{}_\nu C$, $[K_1, B^0] = -B^1$, and $[K_1, B^1] = -B^0$. We can then write the group G as $\{e^{x^\mu P_\mu} e^{\chi C} e^{\rho_\mu B^\mu} e^{\eta K_1}\}$, the group H as $\{e^{\chi C} e^{\rho_\mu B^\mu} e^{\eta K_1}\}$, the group L as $\{e^{\rho_\mu B^\mu} e^{\eta K_1}\}$, and the group K as $\{e^{-\eta K_1}\}$. The affine space $A(\mathfrak{g}/\mathfrak{h}, \mathfrak{g}/\mathfrak{l})$ has elements given by $f_{a_\mu}(c^\mu P_\mu + \mathfrak{h}) = c^\mu P_\mu + c^\mu a_\mu C + \mathfrak{l}$. Under the action of $e^{b'_\mu B^\mu} e^{\eta' K_1} \in L$, denoting $\Lambda(\eta') = e^{\eta' K_1}$, we get $a_\mu \mapsto \Lambda_\mu{}^\nu(\eta') a_\nu + b'_\mu$. The map $\Pi\Omega : f_{a_\mu} \mapsto e^{a_\mu B^\mu} K$ is an L-map.

Example 7.4.3.4 ($(3+1)$-d, $N = 1$ Galileon)**.** We now repeat the previous example in the $(3+1)$-d case, so that $G = \mathrm{SGal}(3+1, 1)$. The group G has the Lie algebra generators, $\{P_\mu, K_i, J_i\}$ which generate the $(3+1)$-d Poincaré subalgebra, and $\{B^\mu, P_\nu\}$ which have the non-zero commutators $[P_\nu, B^\mu] = -\eta_\nu{}^\mu C$, $[K_i, B^0] = -B^i$, $[K_i, B^j] = -\eta^j{}_i B^0$, and $[J_i, B^j] = -\epsilon_{ijk} B^k$. We can then write the Lie groups involved as $G = \{e^{x^\mu P_\mu} e^{\chi C} e^{b_\mu B^\mu} e^{\eta^i K_i} e^{\theta^i J_i}\}$, $H = \{e^{\chi C} e^{b_\mu B^\mu} e^{\eta^i K_i} e^{\theta^i J_i}\}$, $L = \{e^{b_\mu B^\mu} e^{\eta^i K_i} e^{\theta^i J_i}\}$, and $K = \{e^{\eta^i K_i} e^{\theta^i J_i}\}$ which, except for the addition of a rotation, have an identical form to the $(1+1)$-d case. Analogous to what we found above, the affine space $A(\mathfrak{g}/\mathfrak{h}, \mathfrak{g}/\mathfrak{l})$ has elements given by $f_{a_\mu}(c^\mu P_\mu + \mathfrak{h}) = c^\mu P_\mu + c^\mu a_\mu C + \mathfrak{l}$. Under the action of $e^{b'_\mu B^\mu} e^{\eta'^i K_i} e^{\theta'^i J_i} \in L$, denoting $\Lambda' = e^{\eta'^i K_i} e^{\theta'^i J_i}$, we get $a_\mu \mapsto (\Lambda')_\mu{}^\nu a_\nu + b'_\mu$, exactly as above. Again, a valid L-map is $\Pi\Omega : f_{a_\mu} \mapsto e^{a_\mu B^\mu} K$.

Example 7.4.3.5 ($(1+1)$-d Type-1 Superfluid)**.** Here G is the product of the $(1+1)$-d Poincaré group and $U(1)$. The group G has the Lie algebra generators $\{P_0, P_1, K_1\}$ which generate the Poincaré subalgebra and Q which generates the subalgebra associated with $U(1)$. We can write G as $G = \{e^{x^\mu P_\mu} e^{\theta Q} e^{\eta K_1}\}$ for $\mu \in \{0, 1\}$. The relevant subgroups correspond to $H = \{e^{\theta Q} e^{\eta K_1}\}$, $L = \{e^{\eta K_1}\}$, and $K = \{\mathrm{id}\}$. The space

$A(\mathfrak{g}/\mathfrak{h}, \mathfrak{g}/\mathfrak{l})$ has elements given by $f_{a_\mu}(c\mu P_\mu + \mathfrak{h}) = c^\mu P_\mu + c^\mu a_\mu Q + \mathfrak{l}$. In a similar way to the Galileon example above, under the action of $\Lambda(\eta') \equiv e^{\eta' K_1} \in L$, we have that $a_\mu \mapsto \Lambda_\mu{}^\nu(\eta') a_\nu$.

We define the open subset ΠR as the set of f_{a_μ} with a_μ future time-like. A valid choice in L-map is then $\Pi\Omega : f_{a_\mu} \mapsto \exp\left(-\operatorname{arctanh} \frac{a_1}{a_0} K_1\right)$.

Example 7.4.3.6 ((3 + 1)-d Type-I Superfluid)**.** Turning to the (3 + 1)-d version of the previous example, our group is now the product of the (3 + 1)-d Poincaré group and a $U(1)$. The Lie algebra generators of the (3 + 1)-d Poincaré Lie subalgebra, as before, take the form $\{P_\mu, K_i, J_i\}$. The generator of the $U(1)$ Lie algebra is Q. We can write the relevant Lie groups as $G = \{e^{x^\mu P_\mu} e^{\phi Q} e^{\eta^i K_i} e^{\theta^i J_i}\}$, $H = \{e^{\phi Q} e^{\eta^i K_i} e^{\theta^i J_i}\}$, $L = \{e^{\eta^i K_i} e^{\theta^i J_i}\}$, and $K = \{e^{\theta^i J_i}\}$. The affine space $A(\mathfrak{g}/\mathfrak{h}, \mathfrak{g}/\mathfrak{l})$ has elements given by $f_{a_\mu}(c^\mu P_\mu + \mathfrak{h}) \mapsto c^\mu P_\mu + c^\mu a_\mu Q + \mathfrak{l}$. Under the action of $\Lambda' \in L$, $a_\mu \mapsto \Lambda'_\mu{}^\nu a_\nu$.

We again need to restrict to an open subset of $A(\mathfrak{g}/\mathfrak{h}, \mathfrak{g}/\mathfrak{l})$, ΠR. We define ΠR by the condition of a future time-like a_μ, the action $\Pi\mathcal{R}$ is that induced by this embedding. The map $\Pi\Omega$ then takes the form $f_{a_\mu} \mapsto \exp\left(\frac{1}{|\vec{a}|} \operatorname{arctanh}\left(\frac{|\vec{a}|}{a^0}\right)(a^i K_i)\right)$, where $|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$. One can demonstrate the equivariant property of $\Pi\Omega$ using a slightly technical prescription relying on Thomas-Wigner rotations and related ideas.

Example 7.4.3.7 ((3 + 1)-d Solid)**.** Our last example from the literature corresponds to the (3 + 1)-d Solid. Here G is the product of the (3 + 1)-d Poincaré group and the 3-d Euclidean group, $\mathrm{ISO}^+(3)$. The generators of the Poincaré subalgebra take the form $\{P_i, K_i, J_i\}$ and those of the $\mathrm{ISO}^+(3)$ subalgebra the form $\{Q_i, \tilde{Q}_i\}$. Here, Q_i correspond to the translations and $\tilde{Q}_i$ the rotations. We can write the groups involved as $G = \{e^{x^\mu P_\mu} e^{\rho^i Q_i} e^{\phi^i \tilde{Q}_i} e^{\eta^i K_i} e^{\theta^i J_i}\}$, $H = \{e^{\rho^i Q_i} e^{\phi^i \tilde{Q}_i} e^{\eta^i K_i} e^{\theta^i J_i}\}$, $L = \{e^{\phi^i \tilde{Q}_i} e^{\eta^i K_i} e^{\theta^i J_i}\}$, and $K = \{e^{\theta^i (J_i + \tilde{Q}_i)}\}$. The affine space $A(\mathfrak{g}/\mathfrak{h}, \mathfrak{g}/\mathfrak{l})$ has elements given by $f_{a^i_\mu}(c^\mu P_\mu + \mathfrak{h}) = c^\mu P_\mu + c^\mu a^i_\mu Q_i + \mathfrak{l}$. The group L is the product of the Lorentz group and $SO(3)$ and we can write an element of L as $(\Lambda', R') \in L$. The L-action on $A(\mathfrak{g}/\mathfrak{h}, \mathfrak{g}/\mathfrak{l})$ then takes the form $a^i_\mu \mapsto R'^j{}_i \Lambda'_\nu{}^\mu a^i_\mu$.

Following [125], we define $S^\mu = \epsilon^{\mu\alpha\beta\gamma} a^1_\alpha a^2_\beta a^3_\gamma$, and $N_k{}^i = (\Lambda_S)^\mu{}_k a^i_\mu$, where

$$\Lambda_S = \exp\left(\frac{1}{|\vec{S}|} \operatorname{arctanh}\left(\frac{|\vec{S}|}{S^0}\right)(S^i K_i)\right), \tag{7.5}$$

analogous to the above. We define the open subset ΠR of $A(\mathfrak{g}/\mathfrak{h}, \mathfrak{g}/\mathfrak{l})$ by requiring S^μ to be future time-like and by requiring $\det(N) > 0$. The map $\Pi\Omega$ then takes $f_{a^i_\mu}$ to $(\Lambda_S, \sqrt{N^T N} N^{-1})K$. Again, one can show the equivariant property of $\Pi\Omega$ using Thomas-Wigner rotations.

7.5 Partial actions and constraints

7.5.1 Formalities

When the group action is not fibrewise, we need to consider partial actions. Since we will need to consider partial actions in both the topological and smooth contexts we give definitions for both, as follows [4, 136].

Definition 7.5.1.1. *A partial action of the topological (resp. Lie) group G on the topological space (resp. manifold) Y is a pair $\mathcal{Y} = (\{Y_g\}_{g\in G}, \{\mathcal{Y}_g\}_{g\in G})$ such that:*

1. *for all $g \in G$, Y_g are topological spaces (resp. manifolds) that are open topological embeddings (resp. open smooth embeddings), embedded in Y via maps $\mathscr{Y}_g : Y_g \to Y$, $\mathcal{Y}_g : Y_{g^{-1}} \to Y_g$ are homeomorphisms (resp. diffeomorphisms) with inverse $\mathcal{Y}_{g^{-1}} = \mathcal{Y}_g^{-1}$, and $Y_e = Y$;*
2. *the set $\mathcal{U}_Y = \{(g, y) \in G \times Y \mid g \in G, y \in Y_{g^{-1}}\}$ is an open subset of $G \times Y$ and the map $\bar{\mathcal{Y}} : \mathcal{U}_Y \to Y : (g, y) \mapsto \mathcal{Y}_g(y)$ is continuous (resp. smooth);*
3. *the action of $\mathcal{Y}_{g_1g_2}$ extends that of $\mathcal{Y}_{g_2} \circ \mathcal{Y}_{g1}$ acting on $(\mathcal{Y}_{g_1})^{-1}(Y_{g_2^{-1}})$.*

When $Y_g = Y$ for all $g \in G$, we return to the usual definition of a continuous (resp. smooth) group action. We used these global actions in §7.4. We next generalise the definitions of the categories $\texttt{Fib}_X$ and $\texttt{Eta}_X$ in §7.2 to form new categories with a partial action present.

Definition 7.5.1.2. *For a Lie group G, the category G-$\texttt{Fib}_X$ is defined to be the category whose objects are triples $(Y, \pi, \mathcal{Y})$, where (Y, π) is a fibred manifold over X and $\mathcal{Y} = (\{Y_g\}_{g\in G}, \{\mathcal{Y}_g\}_{g\in G})$ is a partial action of G on Y, and whose morphisms between $(Y, \pi, \mathcal{Y})$ and $(Y', \pi', \mathcal{Y}')$, are fibred morphisms $f : Y \to Y'$ for which $f(Y_g) \subseteq Y'_g$ and for which the diagram*

$$\begin{array}{ccc} Y_{g^{-1}} & \xrightarrow{f} & Y'_{g^{-1}} \\ {\scriptstyle \mathcal{Y}_g}\downarrow & & \downarrow{\scriptstyle \mathcal{Y}'_g} \\ Y_g & \xrightarrow[f]{} & Y'_g \end{array} \tag{7.6}$$

commutes, for all $g \in G$.

Definition 7.5.1.3. *For a topological group G, the category G-$\texttt{Eta}_X$ is defined to be the category whose objects are triples $(E, p, \mathcal{E})$ where (E, p) is an étalé space over X and $\mathcal{E} = (\{E_g\}_{g\in G}, \{\mathcal{E}_g\}_{g\in G})$ is a partial action of G on E, and whose morphisms between $(E, p, \mathcal{E})$ and $(E', p', \mathcal{E}')$ are étalé morphisms $f : E \to E'$ which satisfy $f(E_g) \subseteq E'_g$ and the analogous commutative diagram to 7.6.*

We now let G be a Lie group, corresponding to the symmetry group of our system. The corresponding category of étalé spaces is G^d-$\mathtt{Eta}_X$, where the topological group G^d is the group G equipped with the discrete topology. Our functors Γ and J^r can then be modified to account for partial actions as follows.[8]

Definition 7.5.1.4. *The* equivariant local sections functor $\Gamma : G\text{-}\mathtt{Fib}_X \to G^d\text{-}\mathtt{Eta}_X$ *takes* $(Y, \pi, \mathcal{Y})$ *to* $(\Gamma Y, \Gamma\pi, \Gamma\mathcal{Y})$*, with* $\Gamma\mathcal{Y} := (\{\Gamma Y_g\}_{g\in G^d}, \{\Gamma\mathcal{Y}_g\}_{g\in G^d})$ *and*

$$\Gamma Y_g := \{[\beta]_x \in \Gamma Y \mid \beta(x) \in Y_g \ \forall x \in \mathrm{dom}(\beta), \pi \circ Y_{g^{-1}} \circ \beta \text{ is an open embedding}\}, \tag{7.7}$$

$$\Gamma\mathcal{Y}_g : \Gamma Y_{g^{-1}} \to \Gamma Y_g : [\beta]_g \mapsto [Y_g \circ \beta \circ h_{g,\beta}^{-1}]_{h_{g,\beta}(x)}, \tag{7.8}$$

where $h_{g,\beta}$ *is the map defined by* $\pi \circ Y_g \circ \beta$*, but with its codomain restricted to be its image. The functor* Γ *takes the morphism* $f : Y \to Y'$ *to* $f : \Gamma Y \to \Gamma Y' : [\alpha]_x \mapsto [f \circ \alpha]_x$.

Definition 7.5.1.5. *The* equivariant rth-jet functor $J^r : G\text{-}\mathtt{Fib}_X \to G\text{-}\mathtt{Fib}_X$ *takes* $(Y, \pi, \mathcal{Y})$ *to* $(J^rY, \pi^r, J^r\mathcal{Y})$*, with* $J^r\mathcal{Y} := (\{J^rY_g\}_{g\in G}, \{J^r\mathcal{Y}_g\}_{g\in G})$ *and*

$$J^rY_g = \{j^r_x\beta \in J^rY \mid \beta(x) \in Y_g \ \forall x \in \mathrm{dom}(\beta), \pi \circ Y_{g^{-1}} \circ \beta \text{ is an open embedding}\}, \tag{7.9}$$

$$J^r\mathcal{Y}_g : J^rY_{g^{-1}} \to J^rY_g : j^r_x\beta \mapsto j^r_{h_{g,\beta}(x)}(Y_g \circ \beta \circ h_{g,\beta}^{-1}), \tag{7.10}$$

where again, $h_{g,\beta}$ *is the map defined by* $\pi \circ Y_g \circ \beta$*, but with its codomain restricted to its image. The functor* J^r *takes the morphism* $f : Y \to Y'$ *to* $J^rf : J^rY \to J^rY' : j^r_x\alpha \mapsto j^r_x(f \circ \alpha)$.

The functors Γ and J^r preserve the same properties listed in Lemmas 7.2.4 and 7.2.5. In addition, we have the following

Lemma 7.5.1.6. *Say a morphism between* $(Z, \zeta, \mathcal{Z})$ *and* $(Y, \pi, \mathcal{Y})$ *in* G-$\mathtt{Fib}_X$ *is an* embedding of partial actions *if the underlying fibred morphism* $\iota : Z \to Y$ *is an embedding such that* $Z_g = \iota^{-1}(Y_g)$ *for all* $g \in G$*, along with the analogous statement for* G-$\mathtt{Eta}_X$*. The functors* J^r *and* Γ *preserve embeddings of partial actions. (Proof: Appendix D.2)*

Returning to natural transformations, we have the following

Proposition 7.5.1.7. *The maps* $\pi^{r,l} : J^rY \to J^lY$ *form a natural transformation of functors* G-$\mathtt{Fib}_X \to G$-$\mathtt{Fib}_X$*. The maps* $j^r : \Gamma Y \to \Gamma J^rY$ *form a natural transformation in of functors* G-$\mathtt{Fib}_X \to G^d$-$\mathtt{Eta}_X$*. (Proof: Appendix D.2)*

[8] The proof that these functors are well defined is given in Appendix D.2.

We now go on to constraints, which we express by a single Theorem:

Theorem 7.5.1.8. *The results given in §7.3 hold with the categories* $\texttt{Fib}_X$ *and* $\texttt{Eta}_X$ *replaced with* G-$\texttt{Fib}_X$ *and* G^d-$\texttt{Eta}_X$*, with the functors replaced by their corresponding equivariant versions defined in 7.5.1.4 and 7.5.1.5, and with 'embeddings' replaced with 'embeddings of partial actions'. (Proof: Appendix D.3)*

7.5.2 Examples

We now examine three physical examples using the framework of partial actions.

Example 7.5.2.1 ((1 + 1)-d Type-I superfluid)**.** As a warm up, we re-examine the Type-I superfluid in Example 7.4.3.5. Here all our group actions will in fact be global. The symmetry group, G, corresponds to the product of $U(1)$ and the $(1+1)$-d Poincaré group. A general element of this group will be specified by $(x'^{\mu}, \eta', \theta') \in \mathbb{R}^2 \times S^1$. With $X = \mathbb{R}^2$, we first specify the fibred quotient in G-$\texttt{Fib}_X$ given by the fibred manifolds

$$\begin{aligned}(Y, \pi, \mathcal{Y}) &= (\mathbb{R}^3 \times S^1, (x^{\mu}, \eta, \theta) \mapsto x^{\mu}, (\{Y\}_{g\in G}, \\ &\qquad \{\mathcal{Y}_g : (x^{\mu}, \eta, \theta) \mapsto (x'^{\mu} + \Lambda^{\mu}{}_{\nu}(\eta')x^{\nu}, \eta + \eta', \theta + \theta')\}_{g\in G})), \\ (Z, \zeta, \mathcal{Z}) &= (\mathbb{R}^2 \times S^1, (x^{\mu}, \theta) \mapsto x^{\mu}, (\{Z\}_{g\in G}, \\ &\qquad \{\mathcal{Z}_g : (x^{\mu}, \theta) \mapsto (x'^{\mu} + \Lambda^{\mu}{}_{\nu}(\eta')x^{\nu}, \theta + \theta')\}_{g\in G})).\end{aligned} \tag{7.11}$$

and the surjective submersion $\tau_Z : Y \to Z : (x^{\mu}, \eta, \theta) \mapsto (x^{\mu}, \theta)$. The first jet manifolds of Y and Z are

$$\begin{aligned}(J^1Y, \pi^1, J^1\mathcal{Y}) &= (\mathbb{R}^3 \times S^1 \times \mathbb{R}^4, (x^{\mu}, \eta, \theta, \eta_{\mu}, \theta_{\mu}) \mapsto x^{\mu}, \\ &\qquad (\{J^1Y\}_{g\in G}, \{J^1\mathcal{Y}_g : (\cdots, \eta_{\mu}, \theta_{\mu}) \mapsto (\cdots, \Lambda_{\mu}{}^{\nu}(\eta')\eta_{\nu}, \Lambda_{\mu}{}^{\nu}(\eta')\theta_{\nu})\}_{g\in G})), \\ (J^1Z, \zeta^1, J^1\mathcal{Z}) &= (\mathbb{R}^2 \times S^1 \times \mathbb{R}^2, (x^{\mu}, \theta, \theta_{\mu}) \mapsto x^{\mu}, \\ &\qquad (\{J^1Z\}_{g\in G}, \{J^1\mathcal{Z}_g : (\cdots, \theta_{\mu}) \mapsto (\cdots, \Lambda_{\mu}{}^{\nu}(\eta')\theta_{\nu})\}_{g\in G})).\end{aligned} \tag{7.12}$$

We want to form a comeronomic constraint, so introduce another fibred manifold

$$\begin{aligned}(R, \rho, \mathcal{R}) &= (\mathbb{R}^2 \times S^1 \times \mathbb{R}^2, (x^{\mu}, \theta, z_1, z_2) \mapsto x^{\mu}, \\ &\qquad (\{R\}_{g\in G}, \{\mathcal{R}_g : (x^{\mu}, \theta, z_1, z_2) \mapsto (\Lambda^{\mu}{}_{\nu}(\eta')x^{\nu} + x'^{\mu}, \theta + \theta', z_1 + \eta', z_2)\}_{g\in G})),\end{aligned} \tag{7.13}$$

with the open embedding of partial actions

$$\iota_R : R \to J^1Z : (x^\mu, \theta, z_1, z_2) \mapsto (x^\mu, \theta, \cosh z_1 e^{z_2}, -\sinh z_1 e^{z_2}). \tag{7.14}$$

Physically R manifests the condition of restricting to θ_μ (as introduced in the definition of J^1Z) that are future time-like vectors. We then can choose $\Omega : R \to Y : (x^\mu, \theta, z_1, z_2) \mapsto (x^\mu, z_1, \theta)$. This choice gives

$$(Q, \nu, \mathcal{Q}) = (\mathbb{R}^2 \times S^1 \times \mathbb{R}^4, (x^\mu, \theta, z_1, z_2, \eta_\mu) \mapsto x^\mu, (\{Q\}_{g\in G}, \{\mathcal{Q}_g\}_{g\in G})), \tag{7.15}$$

where $\mathcal{Q}_g$ is as suggested by the notation. A point in E^Q considered as a subspace of ΓZ is of the form $[x^\mu \mapsto (x^\mu, -\operatorname{arctanh}(\partial_1\theta(x^\mu)/\partial_0\theta(x^\mu)), \theta(x^\mu)))]_{x^\mu}$, for future time-like $\partial_\mu\theta$. Under the isomophrism of étalé spaces this gets mapped to the point $[x^\mu \mapsto (x^\mu, \theta(x^\mu))]_{x^\mu}$ in E^R, considered as a subspace of ΓZ.

Example 7.5.2.2 ((1 + 1)-d relativistic particle)**.** Here the symmetry group G is the (1+1)-d Poincaré group, an element of which we specify by $(x'^\mu, \eta') \in \mathbb{R}^3$, for $\mu \in \{0, 1\}$. We have $X = \mathbb{R}$, and the fibred quotient in G-$\texttt{Fib}_X$ defined by

$$\begin{aligned}(Y, \pi, &\mathcal{Y}) \\ &= (\mathbb{R}^3, (x^\mu, \eta) \mapsto x^0, (\{Y\}_{g\in G}, \{\mathcal{Y}_g : (x^\mu, \eta) \mapsto (x'^\mu + \Lambda^\mu{}_\nu(\eta')x^\nu, \eta + \eta')\}_{g\in G})), \\ (Z, \zeta, &\mathcal{Z}) = (\mathbb{R}^2, (x^\mu) \mapsto x^0, (\{Z\}_{g\in G}, \{\mathcal{Z}_g : x^\mu \mapsto x'^\mu + \Lambda^\mu{}_\nu(\eta')x^\nu\}_{g\in G})).\end{aligned} \tag{7.16}$$

and the surjective submersion $\tau_Z : Y \to Z : (x^\mu, \eta) \mapsto x^\mu$. The corresponding first jet manifolds are

$$\begin{aligned}&(J^1Y, \pi^1, J^1\mathcal{Y}) = \\ &\Bigg(\mathbb{R}^5, (x^\mu, \eta, x^1_0, \eta_0) \mapsto x^0, \Bigg(\{\{(x^\mu, \eta, x^1_0, \eta_0) \mid \Lambda^0{}_0(\eta') \neq \Lambda^0{}_1(\eta')x^1_0\}\}_{g\in G}, \Big\{J^1\mathcal{Y}_g : \\ &(\cdots, x^1_0, \eta_0) \mapsto \left(\cdots, \frac{\Lambda^1{}_0(\eta') + \Lambda^1{}_1(\eta')x^1_0}{\Lambda^0{}_0(\eta') + \Lambda^0{}_1(\eta')x^1_0}, \frac{\eta_0}{\Lambda^0{}_0(\eta') + \Lambda^0{}_1(\eta')x^1_0}\right)\Big\}_{g\in G}\Bigg)\Bigg),\end{aligned} \tag{7.17}$$

and

$$(J^1Z, \zeta^1, J^1\mathcal{Z}) = \Bigg(\mathbb{R}^3, (x^\mu, \eta, x^1_0) \mapsto x^0, \Bigg(\{\{(x^\mu, x^1_0) \mid {\Lambda^0}_0(\eta') \neq {\Lambda^0}_1(\eta')x^1_0\}\}_{g\in G}, \\ \left\{J^1\mathcal{Z}_g : (\cdots, x^1_0) \mapsto \left(\cdots, \frac{{\Lambda^1}_0(\eta') + {\Lambda^1}_1(\eta')x^1_0}{{\Lambda^0}_0(\eta') + {\Lambda^0}_1(\eta')x^1_0}\right)\right\}_{g\in G}\Bigg)\Bigg). \tag{7.18}$$

Notice that although the group action on Y and Z is the global one, the action on J^1Y and J^1Z is strictly partial. We then choose $\Omega : J^1Z \to Y : (x^\mu, x^1_0) \mapsto (x^\mu, \operatorname{arctanh} x^1_0)$, which leads to the coholonomic constraint

$$(Q, \nu, \mathcal{Q}) = (\mathbb{R}^5, (x^\mu, x^1_0, \eta_0) \mapsto x^0, \\ (\{\{(x^0, x^1_0, \eta_0) \mid {\Lambda^0}_0(\eta') \neq {\Lambda^0}_1(\eta')x^1_0\}\}_{g\in G}, \{\mathcal{Q}_g\}_{g\in G})) \tag{7.19}$$

where $\mathcal{Q}_g$ is as suggested by notation. The manifold Q is embedded into J^1Y via $\iota_Q : (x^\mu, x^1_0, \eta_0) \mapsto (x^\mu, \operatorname{arctanh} x^1_0, x^1_0, \eta_0)$. An element of E^Q, as a subspace of ΓY, is $[x^0 \mapsto (x^0, x^1(x^0), \operatorname{arctanh} \partial_0 x^1(x^0))]_{x^0}$, under the isomorphism of étalé spaces, this gets mapped into $[x^0 \mapsto (x^0, x^1(x^0))]_{x^0}$ in ΓZ.

Example 7.5.2.3 (String in a plane)**.** This follows the same pattern as the $(1+1)$-d relativistic particle discussed above. It was previously studied in [111], and in fact the mathematical set-up coincides with a system studied as an example in [131] in the context of symmetries of differential equations. Here G is the Euclidean group in 2-d, a general element of which we label by $(x', y', \theta') \in \mathbb{R}^2 \times S^1$. We have $X = \mathbb{R}$, and the fibred quotient in G-$\mathtt{Fib}_X$ given by

$$(Y, \pi, \mathcal{Y}) = (\mathbb{R}^2 \times S^1, (x, y, \theta) \mapsto x, (\{Y\}_{g\in G}, \\ \{\mathcal{Y}_g : (x, y, \theta) \mapsto (x' + x\cos\theta' + y\sin\theta', y' + y\cos\theta' - x\sin\theta', \theta + \theta')\}_{g\in G})), \\ (Z, \zeta, \mathcal{Z}) = (\mathbb{R}^2 \times S^1, (x, y) \mapsto x, (\{Z\}_{g\in G}, \\ \{\mathcal{Z}_g : (x, y) \mapsto (x' + x\cos\theta' + y\sin\theta', y' + y\cos\theta' - x\sin\theta')\}_{g\in G})). \tag{7.20}$$

and the surjective submersion $\tau_Z : Y \to Z : (x, y, \theta) \mapsto (x, y)$. The corresponding first jet manifolds are

$$(J^1Y, \pi^1, J^1\mathcal{Y}) = \Bigg(\mathbb{R}^2 \times S^1 \times \mathbb{R}^2, (x, y, \theta, y_x, \theta_x) \mapsto x, \\ \Bigg(\{\{(x, y, \theta, y_x, \theta_x) \mid \cos\theta' \neq -y_x \sin\theta'\}\}_{g\in G}, \\ \left\{J^1\mathcal{Y}_g : (\cdots, y_x, \theta_x) \mapsto \left(\cdots, \frac{\sin\theta' + y_x\cos\theta'}{\cos\theta' - y_x\sin\theta'}, \frac{\theta_x}{\cos\theta' - y_x\sin\theta'}\right)\right\}_{g\in G}\Bigg)\Bigg),$$

and

$$(J^1Z, \zeta^1, J^1\mathcal{Z}) = \Bigg(\mathbb{R}^3, (x, y, y_x) \mapsto x, \Bigg(\{\{(x, y, y_x) \mid \cos\theta' \neq -y_x \sin\theta'\}\}_{g\in G}, \\ \left\{J^1\mathcal{Z}_g : (\cdots, y_x) \mapsto \left(\cdots, \frac{\sin\theta' + y_x\cos\theta'}{\cos\theta' - y_x\sin\theta'}\right)\right\}_{g\in G}\Bigg)\Bigg). \tag{7.21}$$

Forming the coholonomic constraint associated with $\Omega : (x, y, y_x) \mapsto (x, y, \arctan y_x)$, we get

$$(Q, \nu, \mathcal{Q}) = (\mathbb{R}^4, (x, y, y_x, \theta_x) \mapsto x^0, \\ (\{\{(x, y, y_x, \theta_x) \mid \cos\theta' \neq -y_x \sin\theta'\}\}_{g\in G}, \{\mathcal{Q}_g\}_{g\in G}\})), \tag{7.22}$$

where, as above, $\mathcal{Q}_g$ is as suggested by the notation. The embedding of Q into J^1Y is given by $\iota_Q : (x, y, y_x, \theta_x) \mapsto (x, y, \arctan y_x, y_x, \theta_x)$. A typical element of E^Q, as a subspace of ΓY, is of the form $[x \mapsto (x, y(x), \arctan \partial_x y(x))]_x$, which maps to $[x \mapsto (x, y(x))]_x$ in ΓZ under the isomorphism of étalé spaces.

7.6 Closing remarks

In this Chapter we studied constraints in quantum field theories in the presence of symmetries. We used both category theory and differential geometry (in particular the theory of jet manifolds) to achieve this.

We showed how a holonomic constraint can be treated as the limit of a diagram within category theory. This, rather redundant description, had the benefit of permitting a categoric dual, which was also a type of constraint which we called coholonomic.

Due to their kinematical equivalence to unconstrained systems, coholonomic constraints obey the most basic physical requirements as the constrained fields form a sheaf with non-empty stalks.

A slight generalisation of holonomic constraints was then developed, which we called meronomic constraints. These were defined by open embeddings into holonomic constraints. They, as such, locally look like holonomic constraints and obey our basic physical requirements. They also permit a categoric dual, which we termed comeronomic constraints which are kinematically equivalent to particularly simple types of meronomic constraints.

We started our initial discussion in the absence of symmetries. When it came to adding symmetries, two different routes were taken. Firstly, in the simple case where we have transitive group actions on certain spaces, our theory reduces to a much simpler one in the category of homogenous bundles. More generically, however, one must resort to the use of partial actions. Despite their arguable complexity, partial group actions were required for some naïvely simple systems, for example the relativistic particle.

The main result of this Chapter is that every instance of the inverse Higgs phenomenon we could find in the literature could be treated as either a coholonomic or comeronomic constraint. The kinematic equivalences for coholonomic and comeronomic constraints correspond to the reduction to systems of essential Goldstone bosons. Going forward, it is hoped that this result will be of use in the study and classification of the inverse Higgs phenomenon. It is also hoped that related ideas can be studied in a manor similar to the one we have presented in this Chapter. This would allow interrelationships between phenomenon to be seen in a more rigorous way.

Chapter 8

Conclusion

This Thesis started with a discussion of the current state of particle physics and the approach we would take to help drive it forward. This approach was based on the use of areas of (pure) mathematics to solve problems in particle physics.

It is hoped that the three forays in this Thesis have helped to convince the reader that taking such an approach is both useful and fruitful. However, this approach cannot sit in isolation, since many of the ideas presented in this Thesis will only be brought to their full fruition after more direct phenomenological studies.

In our first foray we looked at anomaly free gauge algebras. In particular, Chapter 3 looked at the anomaly cancelation conditions for a pure $\mathfrak{u}(1)$-gauge theory. Here we used planes in a projective space to give a geometric interpretation to the solution first presented in [51]. Chapter 4 looked at the anomaly cancelation conditions for $\mathfrak{u}(1)$-extensions of the SM with 3 right-handed neutrinos. We solved these cancelation conditions for the first time, again using a geometric approach, this time based around the special properties of the point corresponding to baryon number minus lepton number. Chapter 5 focused on semisimple extensions of the SM. We discussed a computational method which allowed all such extensions to be found for fixed fermionic particle content, which we again took to be the SM particle content plus 3 right-handed neutrinos. We found that there were 340 extensions. The gauge algebras studied in each of these three parts may be used in model building endeavours, and exhaustive studies of anomaly free gauge algebras, for example, related to proton decay.

Our second foray, the focus of Chapter 6, concerned quantum mechanical problems in magnetic backgrounds. Quantum mechanical problems are usually solved using harmonic analysis (a generalisation of the Fourier transform). However, in the presence of a magnetic field (a type of topological term), harmonic analysis may not work. This is for two reasons: Firstly it may not be possible to write down a globally defined lagrangian. Secondly, the lagrangian may shift by a total derivative under the action of the symmetry group meaning that the conserved charges do not commute with the hamiltonian, and the simplification which occurs as a result of harmonic analysis is not present. Both these obstructions can be overcome in the same brushstroke, by simultaneously considering the dynamics on the $U(1)$-principle bundle defined by the magnetic field, and replacing the original symmetry group with a related $U(1)$-central extension. The method deployed in this foray seems a promising approach for use in real world applications, both in particle physics, and, for example, condensed matter physics and chemistry.

Our last foray, Chapter 7, was related to the inverse Higgs phenomenon. We formulated the usual notion of a holonomic constraint as the limit of a diagram in category theory. We then showed that this definition had a categorical dual forming what we call a coholonomic constraint which, like holonomic constraints, are kinematically equivalent to unconstrained systems (which should be compared with essential Goldstone bosons). A small generalisation was then made to meronomic and comeronomic constraints both of which obeyed our basic physical requirements. With the help of homogenous bundles, which provided a simplification when certain transitive group actions were present, we showed that every instance of the inverse Higgs phenomenon can be considered as an instance of a coholonomic or a comeronomic constraint. It is hoped that our new framework for considering the inverse Higgs phenomenon will lead to new phenomenological examples of the phenomenon, as well as drive forward theoretical developments in this area.

The three forays in this Thesis are of course only examples of the *modus operandi* of formal mathematical explorations into particle physics. They are, however, examples which despite there specificity, can be used as the bedrock for further similar studies of the same vein. One may, for example, wish to combine forays 2 and 3 to study quantum mechanics problems in the presence of the inverse Higgs phenomenon. One may also wish to develop further the use of jet manifolds in particle physics.

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Appendix A

Additional material for Chapter 3

A.1 Any solution via permutations

Here, we give a proof of the statement that any solution in Chapter 3 sitting on a line in X between the d-planes defined in Eq. 3.13 can be found by the permutation of the coordinates of a solution which is on a line not in X. The proof of this statement follows similar reasoning to the proof regarding permutations of solutions in [51]. We must distinguish between n odd and even, so we do them each in turn.

A.1.1 Even $n \geq 4$

We redefine variables such that

$$\begin{aligned} x_i &= z_1, && \text{for } i = 1, \\ x_i &= z_i + z_{m_e+i}, && \text{for } i = 2, \cdots, m_e + 1, \\ y_i &= z_i + z_{m_e+1+i}, && \text{for } i = 1, \cdots, m_e. \end{aligned}$$

The d-planes in Eq. 3.13 are defined in our new variables by $y_i = 0$ for Γ_1^e and $x_i = 0$ for Γ_2^e. Consider a point $p = [x_i : y_i] \notin \Gamma_1^e \cup \Gamma_2^e$. There is a unique line

$$L_p = \alpha p^1 + \beta p^2,$$

through p, $p^1 \in \Gamma_1^e$ and $p^2 \in \Gamma_2^e$. Under the permutation $\phi^e : z_{m_e+1} \leftrightarrow z_{2m_e+1}$, only y_{m_e} changes and

$$L_{\phi^e(p)} = \alpha p^1 + \beta \phi^e(p^2).$$

A necessary condition for L_p to be in X is that

$$-3y_{m_e}x_{m_e+1}\left(2\sum_{i=1}^{m_e}x_i + x_{m_e+1}\right) + \cdots = 0 \Leftrightarrow$$
$$-3(z_{m_e}+z_{2m_e+1})(z_{m_e+1}+z_{2m_e+1})\left(2\sum_{i=1}^{2m_e+1}z_i - z_{m_e+1} - z_{2m_e+1}\right) + \cdots = 0,$$

where the dots indicate terms which are independent of y_{m_e}.

Thus if L_p is in X, for a solution p with coordinates permuted such that

$$|z_{m_e+1}| \neq |z_{2m_e+1}| \quad \text{and} \quad z_{m_e+1} + z_{2m_e+1} - \sum_{i=1}^{2m_e+1} z_i \neq 0,$$

then $L_{\phi^e(p)}$ will not be in X. The only case where this cannot be done is where all $|z_i|$ are equal, but such solutions already occur in Γ_1^e after permutations of the z_i.

A.1.2 Odd $n \geq 4$

Here,

$$\begin{aligned} x_i &= z_{m_0+1}, && \text{for } i = 1, \\ x_i &= z_{i-1} + z_{m_0+1+i}, && \text{for } i = 2, \cdots, m_o+1, \\ y_i &= z_i + z_{m_o+1+i}, && \text{for } i = 1, \cdots, m_o+1. \end{aligned}$$

Again, Γ_1^o is simply defined by $y_i = 0$ and Γ_2^o is defined by $x_i = 0$. Similar to the even n case, we take a point $p = [x_i : y_i] \notin \Gamma_1^o \cup \Gamma_2^o$. There is again a unique line

$$L_p = \alpha p^1 + \beta p^2,$$

through p, where $p^1 \in \Gamma_1^o$ and $p^2 \in \Gamma_2^o$. Taking $\phi^o : z_1 \leftrightarrow z_{m_o+2}$, only x_2 changes, where

$$L_{\phi^o(p)} = \alpha\phi^o(p^1) + \beta p^2.$$

A necessary condition for L_p to be in X is then

$$-3x_2y_1\left(2\sum_{i=2}^{m_o+1} z_i + y_1\right) + \cdots = 0 \Leftrightarrow$$
$$-3(z_1 + z_{m_o+3})(z_1 + z_{m_o+2})\left(2\sum_{i=1}^{2m_0+2} z_i - z_1 - z_{m_0+2}\right) + \cdots = 0,$$

where now the dots indicate terms which are independent x_2.

If L_p is in X for a solution p with coordinates permuted such that

$$|z_1| \neq |z_{m_o+2}| \quad \text{and} \quad z_1 + z_{m_o+2} - 2\sum_{i=1}^{2m_o+2} z_i \neq 0,$$

then $L_{\phi^o(p)}$ will not be in X. We may use this construction for all solutions and n odd.

A.2 Alternative solution for n-even

For even n, the cubic Eq. 3.8 has double points; that is points where all of the partial derivatives of the left-hand side vanish, as well as the left-hand side itself. An example of such a double point is

$$d = [+1 : -1 : +1 : -1 : \ldots : +1 : -1 : +1] \in \mathrm{P}\mathbb{Q}^{n-2}. \qquad \text{(A.1)}$$

So for *e.g.* $n = 6$, we have $[+1 : -1 : +1 : -1 : +1]$.

Consider a line through our double point d, $L = \gamma_1 d + \gamma_2 r$, for $r \in \mathrm{P}\mathbb{Q}^{n-2}$ a fixed point and $[\gamma_1 : \gamma_2]$ specifying the position along the line. Any point in $\mathrm{P}\mathbb{Q}^{n-2}$ lies on such a line, and further every such line is either in the hypersurface X (defined by Eq. 3.8) or passes through that hypersurface at exactly one other point.

This other point of intersection can be found by substituting L into Eq. 3.8:

$$\gamma_2^2\left(3\gamma_1\sum_{i=1}^{n-1} d_iR_i + \gamma_2\sum_{i=1}^{n-1} r_iR_i\right) = 0, \quad R_i := r_i^2 - \left(\sum_{j=1}^{n-1} r_j\right)^2. \qquad \text{(A.2)}$$

Either $\gamma_2 = 0$ (the original point d), the left hand side is zero independently of γ_1 and γ_2 (corresponding to L being in X) or

$$[\gamma_1 : \gamma_2] = \left[\sum_{i=1}^{n-1} r_i R_i : -3\sum_{i=1}^{n-1} d_i R_i\right], \tag{A.3}$$

giving the second point of intersection. As such we can see that the lines L can be used to find all solutions to Eq. 3.8 parameterised by r_i, and if L is in X by $[\gamma_1 : \gamma_2]$.

Continuing our example, for $n = 6$, we have that Eq. A.2 becomes

$$\begin{aligned} &3\gamma_1(r_1^2 - r_2^2 + r_3^2 - r_4^2 + r_5^2 - (r_1 + r_2 + r_3 + r_4 + r_5)^2) \\ &+\gamma_2(r_1^3 + r_2^3 + r_3^3 + r_4^3 + r_5^3 - (r_1 + r_2 + r_3 + r_4 + r_5)^3) = 0 \end{aligned} \tag{A.4}$$

implying the second point of intersection is at

$$\begin{aligned} [\gamma_1 : \gamma_2] =&[(r_1^3 + r_2^3 + r_3^3 + r_4^3 + r_5^3 - (r_1 + r_2 + r_3 + r_4 + r_5)^3) \\ &: -3(r_1^2 - r_2^2 + r_3^2 - r_4^2 + r_5^2 - (r_1 + r_2 + r_3 + r_4 + r_5)^2)]. \end{aligned} \tag{A.5}$$

Appendix B

Formalities of Chapter 5

In this Appendix we will look into the ideas of Chapter 5 in more detail. In particular, we will specify the equivalence classes produced by the computer program, and show that there exists a bijective map between these and the equivalence classes given in the theory section.

We start by reviewing some basic results and definitions.

B.1 Basic results and definitions

Idempotents: Let $\mathfrak{g}$ be a semisimple Lie algebra, and $\mathfrak{h}_\mathfrak{g}$ a Cartan subalgebra thereof. Associated with this set up we have the Euclidian space $E(\mathfrak{g}, \mathfrak{h}_\mathfrak{g})$. For each $\lambda \in E(\mathfrak{g}, \mathfrak{h}_\mathfrak{g})$, we have an associated $h_\lambda \in \mathfrak{h}_\mathfrak{g}$, the set of all such h_λ forms the *idempotent* of $\mathfrak{h}_\mathfrak{g}$.

Theorem B.1.1. *If $\iota : \mathfrak{g}' \to \mathfrak{g}$ is an embedding of semisimple Lie algebras, which embeds the Cartan subalgebra $\mathfrak{h}_{\mathfrak{g}'}$ of $\mathfrak{g}'$ into $\mathfrak{h}_\mathfrak{g}$ of $\mathfrak{g}$ then $\iota(X)$ is in the idempotent of $\mathfrak{h}_\mathfrak{g}$ if and only if X is in the idempotent of $\mathfrak{h}_{\mathfrak{g}'}$.*

Proof. The embedding $\iota : \mathfrak{g}' \to \mathfrak{g}$ defines a map $\Lambda\iota : E(\mathfrak{g}, \mathfrak{h}_\mathfrak{g}) \mapsto E(\mathfrak{g}', \mathfrak{h}_{\mathfrak{g}'})$. From this we can define $\Lambda\iota^\dagger : E(\mathfrak{g}', \mathfrak{h}_{\mathfrak{g}'}) \to E(\mathfrak{g}, \mathfrak{h}_\mathfrak{g})$ such that

$$(\Lambda\iota(\lambda), \lambda') = (\lambda, \Lambda\iota^\dagger(\lambda')) \quad \forall \lambda \in E(\mathfrak{g}, \mathfrak{h}_\mathfrak{g}), \lambda' \in E(\mathfrak{g}', \mathfrak{h}_{\mathfrak{g}'}). \tag{B.1}$$

It is easy to see that for $h'_{\lambda'}$ within the idempotent of $\mathfrak{h}_{\mathfrak{g}'}$ then $\iota(h'_{\lambda'}) = h_{\Lambda\iota^\dagger(\lambda')}$, which is in the idempotent of $h_{\mathfrak{g}'}$. This proves the *if* part.

For the *only if* part, note that any $h' \in E(\mathfrak{g}', \mathfrak{h}_{\mathfrak{g}'})$ can be written as $h' = h_{\lambda'_1} + ih_{\lambda'_2}$. From the linearity of ι, this maps to the idempotent of $\mathfrak{h}_\mathfrak{g}$ if and only if $\iota(h_{\lambda'_2}) = 0$.

Since ι is an injection, we must have that $h_{\lambda'_2} = 0$ and therefore $h' = h_{\lambda'_1}$. No element outside of $\mathfrak{h}_{\mathfrak{g}'}$ can map under ι into $\mathfrak{h}_{\mathfrak{g}}$. This proves the *only if* part. □

Cartan embeddings: Consider an embedding $\alpha : \mathfrak{s} \oplus \mathfrak{u}(1) \to \mathfrak{g}$ for $\mathfrak{s}$ and $\mathfrak{g}$ semisimple. We call such an embedding a *Cartan embedding* if there exists a Cartan subalgebra $\mathfrak{h}_{\mathfrak{s}}$ of $\mathfrak{s}$ and one $\mathfrak{h}_{\mathfrak{g}}$ of $\mathfrak{g}$ such that $\alpha(\mathfrak{h}_{\mathfrak{s}} \oplus \mathfrak{u}(1)) \subseteq \mathfrak{h}_{\mathfrak{g}}$, and that there exists a non-zero $X \in \mathfrak{u}(1)$ such that $\alpha(X)$ is in the idempotent of $\mathfrak{h}_{\mathfrak{g}}$. We will call such an X a *Cartan element* of $\mathfrak{u}(1)$. If $i : \mathfrak{g} \to \mathfrak{g}'$ is an embedding of semisimple Lie algebras and α is a Cartan embedding, then it is clear that $i \circ \alpha$ is also a Cartan embedding.

For a given Cartan embedding α and a chosen Cartan element X we define the map $\alpha^*_X : E(\mathfrak{g}, \mathfrak{h}_{\mathfrak{g}}) \to E(\mathfrak{s}, \mathfrak{h}_{\mathfrak{s}}) \times \mathbb{R} : v \mapsto (\Lambda\alpha|(v), \alpha(X)v)$, where $\Lambda\alpha|$ is the projection matrix formed by restricting α to $\mathfrak{s}$.

As a generalisation of Theorem 1.1. of [66] we have

Theorem B.1.2. *Let α_1 and α_2 be two Cartan embeddings so that we can choose the same $\mathfrak{h}_{\mathfrak{s}}$ and $\mathfrak{h}_{\mathfrak{g}}$ for both. Assume further that they are related by $\alpha_1 = j \circ \alpha_2$ for some inner automorphism j (this then implies we can choose the same Cartan element X). Then $(\alpha_1)^*_X = (\alpha_2)^*_X \circ w$ for some element w in the Weyl group of $\mathfrak{g}$.*

Simply laced algebras: Let $\mathfrak{g}$ be a semisimple Lie algebra for which we choose a Cartan subalgebra $\mathfrak{h}_{\mathfrak{g}}$ and a set of simple roots $\Delta(\mathfrak{g}, \mathfrak{h}_{\mathfrak{g}})$. The algebra $\mathfrak{g}$ is said to be *simply laced* if all its roots are of the same length. The algebra $\mathfrak{g}$ then has a basis $\{h_{\tilde{\lambda}_i}\}_{\lambda_i \in \Delta(\mathfrak{g}, \mathfrak{h}_{\mathfrak{g}})}$, and $\{E_{\tilde{\lambda}}\}_{\tilde{\lambda} \in \Phi(\mathfrak{g}, \mathfrak{h}_{\mathfrak{g}})}$ which have the commutators [73, 135, 139]

$$\begin{aligned} [h_{\lambda_i}, h_{\lambda_j}] &= 0 \\ [h_{\lambda_i}, E_{\tilde{\lambda}}] &= (\lambda_i, \tilde{\lambda}) E_{\tilde{\lambda}} \\ [E_{\tilde{\lambda}}, E_{-\tilde{\lambda}}] &= -h_{\tilde{\lambda}} \\ [E_{\tilde{\lambda}_1}, E_{\tilde{\lambda}_2}] &= (-1)^{B(\tilde{\lambda}_1, \tilde{\lambda}_2)} E_{\tilde{\lambda}_1 + \tilde{\lambda}_2}, \quad \text{if } \tilde{\lambda}_1 + \tilde{\lambda}_2 \in \Phi(\mathfrak{g}), \\ [E_{\tilde{\lambda}_1}, E_{\tilde{\lambda}_2}] &= 0, \quad \text{otherwise} \end{aligned} \tag{B.2}$$

where $B(\tilde{\lambda}_1, \tilde{\lambda}_2)$ is a bilinear form on $E(\mathfrak{g}, \mathfrak{h}_{\mathfrak{g}})$ such that on the simple roots $B(\lambda_i, \lambda_j)$ is 0 if $i < j$, is $\frac{1}{2}(\lambda_i, \lambda_j)$ if $i = j$ and is (λ_j, λ_i) otherwise. The space spanned by $\{h_{\tilde{\lambda}_i}\}_{\lambda_i \in \Delta(\mathfrak{g}, \mathfrak{h}_{\mathfrak{g}})}$ is $\mathfrak{h}_{\mathfrak{g}}$. Throughout this Appendix, the convention above has been used.

The Standard Model embedding: To be explicit, we write down a valid embedding γ, of $\mathfrak{sm}$ into $\mathfrak{su}(48)$ here

$$
\begin{aligned}
\gamma(h^{\mathfrak{su}(3)}_{\lambda_1}) &= \sum_{q=0}^{2}\left(h^{\mathfrak{su}(48)}_{\lambda_{1+6q}} + 2h^{\mathfrak{su}(48)}_{\lambda_{2+6q}} + h^{\mathfrak{su}(48)}_{\lambda_{3+6q}}\right) + \sum_{i=0}^{i=5} h^{\mathfrak{su}(48)}_{\lambda_{20+3i}},\\
\gamma(h^{\mathfrak{su}(3)}_{\lambda_2}) &= \sum_{q=0}^{2}\left(h^{\mathfrak{su}(48)}_{\lambda_{3+6q}} + 2h^{\mathfrak{su}(48)}_{\lambda_{4+6q}} + h^{\mathfrak{su}(48)}_{\lambda_{5+6q}}\right) + \sum_{i=0}^{i=5} h^{\mathfrak{su}(48)}_{\lambda_{19+3i}},\\
\gamma(E^{\mathfrak{su}(3)}_{\pm\lambda_1}) &= \sum_{q=0}^{2}\left(E^{\mathfrak{su}(48)}_{\pm(\lambda_{1+6q}+\lambda_{2+6q})} + E^{\mathfrak{su}(48)}_{\pm(\lambda_{2+6q}+\lambda_{3+6q})}\right) + \sum_{i=0}^{i=5} E^{\mathfrak{su}(48)}_{\pm\lambda_{20+3i}},\\
\gamma(E^{\mathfrak{su}(3)}_{\pm\lambda_2}) &= \sum_{q=0}^{2}\left(E^{\mathfrak{su}(48)}_{\pm(\lambda_{3+6q}+\lambda_{4+6q})} + E^{\mathfrak{su}(48)}_{\pm(\lambda_{4+6q}+\lambda_{5+6q})}\right) - \sum_{i=0}^{i=5} E^{\mathfrak{su}(48)}_{\pm\lambda_{19+3i}},\\
\gamma(E^{\mathfrak{su}(3)}_{\pm(\lambda_1+\lambda_2)}) &= \sum_{q=0}^{2}\left(E^{\mathfrak{su}(48)}_{\pm(\lambda_{1+6q}+\lambda_{2+6q}+\lambda_{3+6q}+\lambda_{4+6q})} + E^{\mathfrak{su}(48)}_{\pm(\lambda_{2+6q}+\lambda_{3+6q}+\lambda_{4+6q}+\lambda_{5+6q})}\right)\\
&\quad + \sum_{i=0}^{i=5} E^{\mathfrak{su}(48)}_{\pm(\lambda_{19+3i}+\lambda_{20+3i})},\\
\gamma(h^{\mathfrak{su}(2)}_{\lambda}) &= \sum_{q=0}^{2}\left(h^{\mathfrak{su}(48)}_{\lambda_{1+6q}} + h^{\mathfrak{su}(48)}_{\lambda_{3+6q}} + h^{\mathfrak{su}(48)}_{\lambda_{5+6q}}\right) + \sum_{i=0}^{l=2} h^{\mathfrak{su}(48)}_{\lambda_{37+2i}}\\
\gamma(E^{\mathfrak{su}(2)}_{\pm\lambda}) &= \sum_{q=0}^{2}\left(E^{\mathfrak{su}(48)}_{\pm\lambda_{1+6q}} + E^{\mathfrak{su}(48)}_{\pm\lambda_{3+6q}} + E^{\mathfrak{su}(48)}_{\pm\lambda_{5+6q}}\right) + \sum_{i=0}^{l=2} E^{\mathfrak{su}(48)}_{\pm\lambda_{37+2i}}\\
\gamma(h_X) &= \sum_{q=1}^{18} q h^{\mathfrak{su}(48)}_{\lambda_q} + \sum_{u=0}^{9}(18-4u)h^{\mathfrak{su}(48)}_{\lambda_{18+u}} + \sum_{d=0}^{9}(-18+2d)h^{\mathfrak{su}(48)}_{\lambda_{27+d}}\\
&\quad + \sum_{l=0}^{6}(-3l)h^{\mathfrak{su}(48)}_{\lambda_{36+l}} + \sum_{e=0}^{9}(-18+6e)h^{\mathfrak{su}(48)}_{\lambda_{42+e}}
\end{aligned}
\tag{B.3}
$$

Note that γ is a Cartan embedding with respect to the obvious Cartan subalgebras, and with h_X as a Cartan element (one could say h_X is defined by this, up to a constant).

B.2 Theory

Let $D_\circ$ be the set of all pairs of embeddings (α, β) such that for our γ above, there exists an inner automorphism i of $\mathfrak{su}(48)$ with[1]

$$\begin{array}{ccccc} & & \mathfrak{g} & & \\ & \overset{\alpha}{\nearrow} & & \overset{\beta}{\searrow} & \\ \mathfrak{sm} & & \xrightarrow{\quad i\circ\gamma \quad} & & \mathfrak{su}(48) \end{array} \tag{B.4}$$

commuting.

Theorem B.2.1. *Every α is a Cartan embedding, and h_X is a Cartan element.*

Define on $D_\circ$ the equivalence relation that $(\alpha, \beta) \sim (\alpha', \beta')$ if and only if $(\alpha', \beta') = (O \circ \alpha, j \circ \beta \circ O^{-1})$ for some inner automorphism j of $\mathfrak{su}(48)$ and a general automorphism O of $\mathfrak{g}$. Let $D := D_\circ / \sim$.

B.3 The output of the computer program

For each non-isomorphic semisimple Lie algebra we fix a Cartan subalgebra $\mathfrak{h}_\mathfrak{g}$. For $\mathfrak{su}(3) \oplus \mathfrak{su}(2)$ and $\mathfrak{su}(48)$ it is fixed by that implicitly used in the definition of γ. Let $\Lambda(\mathfrak{g}' \to \mathfrak{g})$ denote the set of projection matrices corresponding to embeddings from $\mathfrak{g}'$ to $\mathfrak{g}$ which preserve the corresponding Cartan subalgebras.

Let $\Gamma := \bigcup_\mathfrak{g} (\Lambda(\mathfrak{su}(3) \oplus \mathfrak{su}(2) \to \mathfrak{g}) \times \mathfrak{H}_\mathfrak{g} \times \Lambda(\mathfrak{g} \to \mathfrak{su}(48)))$ where $\mathfrak{H}_\mathfrak{g}$ is the idempotent of $\mathfrak{h}_\mathfrak{g}$. Define $S_0 \subseteq \Gamma$ to be all those elements (M, v, N) such that there exists a $w \in W_{48}$ (the Weyl group of $\mathfrak{su}(48)$) such that

$$\begin{array}{ccccc} & & \Lambda\mathfrak{g} & & \\ & \overset{M\otimes v}{\swarrow} & & \overset{N}{\nwarrow} & \\ \Lambda(\mathfrak{su}(3) \oplus \mathfrak{su}(2)) \times \mathbb{R} & & \xleftarrow[\quad \gamma^*_{h_X} \circ w \quad]{} & & \Delta(\mathbf{48}) \end{array} \tag{B.5}$$

commutes where $\Delta(\mathbf{48})$ is the set of weights of the fundamental representation of $\mathfrak{su}(48)$.

We define on S_0 the equivalence relation that $(M, v, N) \sim_1 (M', v', N')$ if and only if for each $n \leq 48$ dimensional representation ϕ of $\mathfrak{g}$ (taken to map the relevant Cartan subalgebras into one another) a $w_\phi \in W_{\mathfrak{su}(n)}$ exists such that $M \circ \Lambda\phi \circ w_\phi = M' \circ \Lambda\phi$ and, furthermore, a $w \in W_{\mathfrak{su}(48)}$ exists such that $N \circ w = N'$. We let $S_1 := S_0 / \sim_1$.

[1]The introduction of this inner automorphism is not strictly needed, however, we find it convenient.

We now define an equivalence relation on S_1, given by the condition that

$$[(M, v, N)]_1 \sim_2 [(M', v', N')]_1 \tag{B.6}$$

if and only if there is an automorphism $\sigma \in \mathcal{O}(\mathfrak{g})$, such that

$$[(M \circ \sigma, v \circ \sigma, \sigma^{-1} \circ N)]_1 = [(M', v', N')]_1. \tag{B.7}$$

We let $S_2 := S_1/\sim_2$.

The program outputs S_2.

Let us define another equivalence relation on S_0, to match more closely that on D. We say $(M, v, N) \sim (M, v, N)$ if and only if $(M', v', N') = (M' \circ o, v \circ o, o^{-1} \circ M \circ w)$ for a $w \in W_{\mathfrak{su}(48)}$ and a root system automorphism o of $\Phi(\mathfrak{g}, \mathfrak{h}_{\mathfrak{g}})$. Let $S = S_0/\sim$.

Theorem B.3.1. *The space $S = S_2$.*

Proof. Follows from the explicit M's appearing in our list and Theorems 1.1 and 1.3 of [66]. □

B.4 The index map

The claim is that the space S catalogues D. Put more formally, there is a bijection between the sets S and D, which we will call the index map. Before defining our bijection we note the following theorem

Theorem B.4.1. *Each $[(\alpha, \beta)] \in D$ has a representative $(\alpha_\circ, \beta_\circ)$ which maps the algebra $\mathfrak{h}_{\mathfrak{su}(3)\oplus\mathfrak{su}(2)} \oplus \mathfrak{u}(1)$ into $\mathfrak{h}_{\mathfrak{g}}$ and $\mathfrak{h}_{\mathfrak{g}}$ into $\mathfrak{h}_{\mathfrak{su}(48)}$.*

We then define our index map as

$$R : D \to S : [(\alpha, \beta)] \mapsto [(\alpha_\circ^*, \Lambda\beta_\circ)] \tag{B.8}$$

where $\Lambda\beta$ is the projection matrix of β. Here we are dropping an implicit h_X on $\alpha_\circ^*$, which should be assumed throughout.

Theorem B.4.2. *The map R is well defined.*

Proof. Let $(\alpha_\circ, \beta_\circ)$ and $(O \circ \alpha_\circ, j \circ \beta_\circ \circ O^{-1})$ be two valid representatives of $[(\alpha, \beta)]$. Split O into $\iota \circ \tilde{O}$ where ι is an inner automorphism and $\tilde{O}$ preserves $\mathfrak{h}_{\mathfrak{g}}$. Then using the latter representative we have $((\iota \circ \alpha)^* \circ \Lambda\tilde{O}, \Lambda\tilde{O}^{-1}\Lambda(\tilde{j} \circ \beta))$ for some new inner automorphism $\tilde{j}$. Furthermore, $(i \circ \alpha)^* = \alpha^* \circ w$ for some $w \in W_{\mathfrak{g}}$, and likewise for $\Lambda(j \circ \beta) = \Lambda\beta \circ w'$ for a $w' \in W_{\mathfrak{g}}$. It is easy to then see that using either $(\alpha_\circ, \beta_\circ)$ and $(O \circ \alpha_\circ, j \circ \beta_\circ \circ O^{-1})$ gives the same image under R. □

Theorem B.4.3. *The map R is surjective.*

Proof. The ethos of the proof is to take a (M, v, N) for each $[(M, v, N)]$ in our list and show that there exists an α, and β such that $R([(\alpha, \beta)]) = [(M, v, N)]$. We only need to do this for the minimal algebras. The non-minimal ones follow suit by composing with a valid embedding from a minimal case.

The existence of a β with the correct properties is guaranteed, thus we focus on α.

We start with algebra 27 in our list, corresponding to the well-known algebra $\mathfrak{su}(5)$. Each simple root for each simple ideal will be labelled by λ_i, using standard ordering as specified by the Dynkin diagrams. A valid α (in that it returns the M and v in our list, and is well-defined) is given by

$$\begin{aligned}\alpha(h^{\mathfrak{su}(3)}_{\lambda_1}) = h^{\mathfrak{su}(5)}_{\lambda_4},\ \alpha(h^{\mathfrak{su}(3)}_{\lambda_2}) = h^{\mathfrak{su}(5)}_{\lambda_5},\ \alpha(E^{\mathfrak{su}(3)}_{\pm\lambda_1}) = E^{\mathfrak{su}(5)}_{\pm\lambda_4},\ \alpha(E^{\mathfrak{su}(3)}_{\pm\lambda_2}) = E^{\mathfrak{su}(5)}_{\pm\lambda_5},\\ \alpha(E^{\mathfrak{su}(3)}_{\pm(\lambda_1+\lambda_2)}) = E^{\mathfrak{su}(5)}_{\pm(\lambda_4+\lambda_5)},\\ \alpha(h^{\mathfrak{su}(2)}_{\lambda_1}) = h^{\mathfrak{su}(5)}_{\lambda_1},\ \alpha(E^{\mathfrak{su}(2)}_{\pm\lambda_1}) = E^{\mathfrak{su}(5)}_{\pm\lambda_1},\\ \alpha(h^{\mathfrak{u}(1)}) = 3h^{\mathfrak{su}(5)}_{\lambda_1} + 6h^{\mathfrak{su}(5)}_{\lambda_2} + 4h^{\mathfrak{su}(5)}_{\lambda_3} + 2h^{\mathfrak{su}(5)}_{\lambda_4}.\end{aligned} \tag{B.9}$$

For algebra 28 in our list, which we label $\mathfrak{su}(4) \oplus \mathfrak{su}_L(2) \oplus \mathfrak{su}_R(2)$, a valid α is given by

$$\begin{aligned}\alpha(h^{\mathfrak{su}(3)}_{\lambda_1}) = h^{\mathfrak{su}(4)}_{\lambda_2},\ \alpha(h^{\mathfrak{su}(3)}_{\lambda_2}) = h^{\mathfrak{su}(4)}_{\lambda_1},\ \alpha(E^{\mathfrak{su}(3)}_{\pm\lambda_1}) = -E^{\mathfrak{su}(4)}_{\pm\lambda_2},\ \alpha(E^{\mathfrak{su}(3)}_{\pm\lambda_2}) = E^{\mathfrak{su}(4)}_{\pm\lambda_1},\\ \alpha(E^{\mathfrak{su}(3)}_{\pm(\lambda_1+\lambda_2)}) = E^{\mathfrak{su}(4)}_{\pm(\lambda_2+\lambda_1)},\\ \alpha(h^{\mathfrak{su}(2)}_{\lambda_1}) = h^{\mathfrak{su}_L(2)}_{\lambda_1},\ \alpha(E^{\mathfrak{su}(2)}_{\pm\lambda_1}) = E^{\mathfrak{su}_L(2)}_{\pm\lambda_1},\\ \alpha(h^{\mathfrak{u}(1)}) = -h^{\mathfrak{su}(4)}_{\lambda_1} - 2h^{\mathfrak{su}(4)}_{\lambda_2} - 3h^{\mathfrak{su}(4)}_{\lambda_3} + 3h^{\mathfrak{su}_R(2)}_{\lambda_1}.\end{aligned} \tag{B.10}$$

For algebra 29, which we label by $\mathfrak{su}(4) \oplus \mathfrak{su}_R(4)\mathfrak{su}_L(2)$, a valid α is

$$\begin{aligned}\alpha(h^{\mathfrak{su}(3)}_{\lambda_1}) = h^{\mathfrak{su}(4)}_{\lambda_1},\ \alpha(h^{\mathfrak{su}(3)}_{\lambda_2}) = h^{\mathfrak{su}(4)}_{\lambda_2},\ \alpha(E^{\mathfrak{su}(3)}_{\pm\lambda_1}) = E^{\mathfrak{su}(4)}_{\pm\lambda_1},\ \alpha(E^{\mathfrak{su}(3)}_{\pm\lambda_2}) = E^{\mathfrak{su}(4)}_{\pm\lambda_2},\\ \alpha(E^{\mathfrak{su}(3)}_{\pm(\lambda_1+\lambda_2)}) = E^{\mathfrak{su}(4)}_{\pm(\lambda_2+\lambda_1)},\\ \alpha(h^{\mathfrak{su}(2)}_{\lambda_1}) = h^{\mathfrak{su}_L(2)}_{\lambda_1},\ \alpha(E^{\mathfrak{su}(2)}_{\pm\lambda_1}) = E^{\mathfrak{su}_L(2)}_{\pm\lambda_1},\\ \alpha(h^{\mathfrak{u}(1)}) = h^{\mathfrak{su}(4)}_{\lambda_1} + 2h^{\mathfrak{su}(4)}_{\lambda_2} + 3h^{\mathfrak{su}(4)}_{\lambda_3} + \frac{9}{2}h^{\mathfrak{su}_R(4)}_{\lambda_1} + 3h^{\mathfrak{su}_R(4)}_{\lambda_2} + \frac{3}{2}h^{\mathfrak{su}_R(4)}_{\lambda_3}.\end{aligned} \tag{B.11}$$

For both algebras 30 and 31 corresponding to $\mathfrak{su}(4) \oplus \mathfrak{su}(5) \oplus \mathfrak{su}_L(2) \oplus \mathfrak{su}_R(2)$ a valid α is

$$\begin{aligned}
\alpha(h^{\mathfrak{su}(3)}_{\lambda_1}) = h^{\mathfrak{su}(4)}_{\lambda_2} + h^{\mathfrak{su}(5)}_{\lambda_4},\ \alpha(h^{\mathfrak{su}(3)}_{\lambda_2}) = h^{\mathfrak{su}(4)}_{\lambda_1} + h^{\mathfrak{su}(5)}_{\lambda_5},\\
\alpha(E^{\mathfrak{su}(3)}_{\pm\lambda_1}) = -E^{\mathfrak{su}(4)}_{\pm\lambda_2} + E^{\mathfrak{su}(5)}_{\pm\lambda_4},\ \alpha(E^{\mathfrak{su}(3)}_{\pm\lambda_2}) = E^{\mathfrak{su}(4)}_{\pm\lambda_1} + E^{\mathfrak{su}(5)}_{\pm\lambda_5},\\
\alpha(E^{\mathfrak{su}(3)}_{\pm(\lambda_1+\lambda_2)}) = E^{\mathfrak{su}(4)}_{\pm(\lambda_2+\lambda_1)} + E^{\mathfrak{su}(5)}_{\pm(\lambda_4+\lambda_5)},\\
\alpha(h^{\mathfrak{su}(2)}_{\lambda_1}) = h^{\mathfrak{su}_L(2)}_{\lambda_1} + h^{\mathfrak{su}(5)}_{\lambda_1},\ \alpha(E^{\mathfrak{su}(2)}_{\pm\lambda_1}) = E^{\mathfrak{su}_L(2)}_{\pm\lambda_1} + E^{\mathfrak{su}(5)}_{\pm\lambda_1},\\
\alpha(h^{\mathfrak{u}(1)}) = -h^{\mathfrak{su}(4)}_{\lambda_1} - 2h^{\mathfrak{su}(4)}_{\lambda_2} - 3h^{\mathfrak{su}(4)}_{\lambda_3} + 3h^{\mathfrak{su}_R(2)}_{\lambda_1} + 3h^{\mathfrak{su}(5)}_{\lambda_1} + 6h^{\mathfrak{su}(5)}_{\lambda_2}\\
+4h^{\mathfrak{su}(5)}_{\lambda_3} + 2h^{\mathfrak{su}(5)}_{\lambda_4}.
\end{aligned} \tag{B.12}$$

Lastly, for algebra 32 in our list, which we write as $\mathfrak{su}_1(4) \oplus \mathfrak{su}_2(4) \oplus \mathfrak{su}_3(4) \oplus \mathfrak{su}_L(2) \oplus \mathfrak{su}_R(2)$, a valid α is

$$\begin{aligned}
\alpha(h^{\mathfrak{su}(3)}_{\lambda_1}) = \sum_i h^{\mathfrak{su}_i(4)}_{\lambda_2},\ \alpha(h^{\mathfrak{su}(3)}_{\lambda_2}) = \sum_i h^{\mathfrak{su}_i(4)}_{\lambda_1},\ \alpha(E^{\mathfrak{su}(3)}_{\pm\lambda_1}) = -\sum_i E^{\mathfrak{su}_i(4)}_{\pm\lambda_2},\\
\alpha(E^{\mathfrak{su}(3)}_{\pm\lambda_2}) = \sum_i E^{\mathfrak{su}_i(4)}_{\pm\lambda_1},\ \alpha(E^{\mathfrak{su}(3)}_{\pm(\lambda_1+\lambda_2)}) = \sum_i E^{\mathfrak{su}_i(4)}_{\pm(\lambda_2+\lambda_1)},\\
\alpha(h^{\mathfrak{su}(2)}_{\lambda_1}) = h^{\mathfrak{su}_L(2)}_{\lambda_1},\ \alpha(E^{\mathfrak{su}(2)}_{\pm\lambda_1}) = E^{\mathfrak{su}_L(2)}_{\pm\lambda_1},\\
\alpha(h^{\mathfrak{u}(1)}) = \sum_i(-h^{\mathfrak{su}_i(4)}_{\lambda_1} - 2h^{\mathfrak{su}_i(4)}_{\lambda_2} - 3h^{\mathfrak{su}_i(4)}_{\lambda_3}) + 3h^{\mathfrak{su}_R(2)}_{\lambda_1}.
\end{aligned} \tag{B.13}$$

□

Theorem B.4.4. *The map R is injective.*

Proof. The ethos of this proof is as follows: If $[(\alpha, \beta)]$ maps to $[(M, v, N)]$ under R, then there exists an $(\alpha_\circ, \beta_\circ) \in [(\alpha, \beta)]$ such that $\Lambda\alpha_\circ| = M$ and $\Lambda\beta_\circ = N$. We work with such a representative of $[(\alpha, \beta)]$. We then show that any other $(\alpha'_\circ, \beta'_\circ)$ with $\Lambda\alpha'_\circ| = M$ and $\Lambda\beta'_\circ = N$ is in the same equivalence class as $(\alpha_\circ, \beta_\circ)$.

This has to be done for each case individually. To illustrate the proof, we take the example of $\mathfrak{so}(10)$. Here there are two choices of $\alpha_\circ(h_X)$ which are given by

$$\begin{pmatrix} 3 & 6 & 4 & 0 & 0 & 2 \end{pmatrix},\quad \begin{pmatrix} -3 & -6 & -8 & -6 & -3 \end{pmatrix}. \tag{B.14}$$

Choose $\alpha_\circ$ to be

$$\alpha_\circ(h^{\mathfrak{su}(3)}_{\lambda_1}) = h^{\mathfrak{so}(10)}_{\lambda_3},\ \alpha_\circ(h^{\mathfrak{su}(3)}_{\lambda_2}) = h^{\mathfrak{so}(10)}_{\lambda_5},\ \alpha_\circ(E^{\mathfrak{su}(3)}_{\pm\lambda_1}) = E^{\mathfrak{so}(10)}_{\pm\lambda_3},$$
$$\alpha_\circ(E^{\mathfrak{su}(3)}_{\pm\lambda_2}) = E^{\mathfrak{so}(10)}_{\pm\lambda_5},\ \alpha_\circ(E^{\mathfrak{su}(3)}_{\pm(\lambda_1+\lambda_2)}) = E^{\mathfrak{so}(10)}_{\pm(\lambda_3+\lambda_5)},$$
$$\alpha_\circ(h^{\mathfrak{su}(2)}_{\lambda_1}) = h^{\mathfrak{so}(10)}_{\lambda_1},\ \alpha_\circ(E^{\mathfrak{su}(2)}_{\pm\lambda_1}) = E^{\mathfrak{so}(10)}_{\pm\lambda_1},$$
$$\alpha_\circ(h^{\mathfrak{u}(1)}) = 4h^{\mathfrak{so}(10)}_{\lambda_1} + 6h^{\mathfrak{so}(10)}_{\lambda_2} + 4h^{\mathfrak{so}(10)}_{\lambda_3} + 2h^{\mathfrak{so}(10)}_{\lambda_5}. \tag{B.15}$$

Define the inner automorphisms (which respectively can be interpreted as an action of the Weyl group and a translation) for $\lambda \in \Phi$ and $a \in \mathbb{C}\backslash\{0\}$

$$t_\lambda(a) := \exp\left(a\ \mathrm{ad}E_\lambda\right)\exp\left(\frac{1}{a}\ \mathrm{ad}E_{-\lambda}\right)\exp\left(a\ \mathrm{ad}E_\lambda\right),$$
$$T_\lambda(a) := t_\lambda(a)t_\lambda(1). \tag{B.16}$$

Letting $\tilde{\lambda} = \lambda_1 + 2\lambda_2 + 2\lambda_3 + \lambda_4 + \lambda_5$, then every other choice of $\alpha_\circ$ which has $\Lambda\alpha_\circ| = M$ can be reached by the subgroup of inner automorphisms generated by [66, Thm. 4.2].

$$\langle T_{\lambda_3}(a), T_{\lambda_5}(b), T_{\lambda_1}(c), t_{\tilde{\lambda}}(1) \mid a, b, c \in \mathbb{C}\backslash\{0\}\rangle. \tag{B.17}$$

The first two generators change how the embedding α acts on $\mathfrak{su}(3)$, the third how it acts on $\mathfrak{su}(2)$, and the last generator swaps between the two $\alpha_\circ(h_X)$'s.

Furthermore, any $\Lambda\beta$ defines a unique β up to inner automorphism [66, No. 1]. This is enough to prove our statement above. All other algebras work in a similar fashion.

To show that this holds for $\mathfrak{g}$ with a $\mathfrak{su}(8)$ or $\mathfrak{su}(12)$ ideal at a purely algebraic level, is fairly involved. We write a set of inner automorphisms that will work for $\mathfrak{su}(8)$ with

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{B.18}$$

The generators of the inner automorphisms are

$$\langle T_{\lambda_2}(\beta_1), T_{\lambda_5}(\beta_2), \exp(\beta_3\ \mathrm{ad}E_{\lambda_3+\lambda_4+\lambda_5}), \exp(\beta_4\ \mathrm{ad}E_{-\lambda_3-\lambda_4-\lambda_5}),$$
$$T_{\lambda_1}(\gamma_1), T_{\lambda_4}(\gamma_2), \exp(\gamma_3\ \mathrm{ad}E_{\lambda_2+\lambda_3+\lambda_4}), \exp(\gamma_4\ \mathrm{ad}E_{-\lambda_2-\lambda_3-\lambda_4}) \mid \beta_i, \gamma_i \in \mathbb{C}\backslash\{0\}\rangle. \tag{B.19}$$

□

B.5 Maximal and minimal algebras

We say that $[(M', v', N')] \in S$ related to $\mathfrak{g}'$ embeds into $[(M, v, N)] \in S$ related to $\mathfrak{g}$ if and only if there exists a (non-isomorphic) $K \in \Lambda(\mathfrak{g}' \to \mathfrak{g})$ and a $w \in W_{48}$ such that $(M' \circ K, v' \circ K, N' \circ w) = (M, v, K \circ N)$. Likewise, we say $[(\alpha', \beta')] \in D$ related to $\mathfrak{g}'$ embeds into $[(\alpha, \beta)]$ related to $\mathfrak{g}$ if and only if there exists a (non-isomorphic) embedding $k : \mathfrak{g}' \to \mathfrak{g}$ and an inner automorphisms $i : \mathfrak{su}(48) \to \mathfrak{su}(48)$ such that $(k \circ \alpha', i \circ \beta') = (\alpha, \beta \circ k)$. It is easy to show that both these definitions are independent of representatives.

Theorem B.5.1. *We have that* $[(\alpha', \beta')] \in D$ *embeds into* $[(\alpha, \beta)] \in D$ *if and only if* $R([(\alpha', \beta')]) \in S$ *embeds into* $R([(\alpha, \beta)]) \in S$.

Proof. For the *if* part, suppose $[(M', v', N')]$ embeds into $[(M, v, N)]$ through K. Let α' be an embedding that projects down to (M', v'), α one that projects to (M, v) and k one that projects to K. Then $k \circ \alpha'$ and α must be related by at most an inner automorphism (seen explicitly from our list), which we include in k such that $k \circ \alpha' = \alpha$. Let β' be an embedding that projects down to N' and β one that projects down to N. Then β' and $\beta \circ k$ are related by an inner automorphism i on $\mathfrak{su}(48)$ so that $i \circ \beta' = \beta \circ k$. Since (α, β) and (α', β') must also be in $D_\circ$, this proves the *if* statement.

For the *only if* part, let $(k \circ \alpha', \beta' \circ i) = (\alpha, \beta \circ k)$. Then $k = j \circ \tilde{k}$ where j is an inner automorphism of $\mathfrak{g}$ and $\tilde{k}$ preserves Cartan subalgebras. Then $(k \circ \alpha')^* = (j \circ \tilde{k} \circ \alpha')^* = (\tilde{k} \circ \alpha')^* \circ w = (\alpha')^* \circ \Lambda k \circ w$. Let $K = \Lambda(j' \circ k)$ where j' is an inner automorphism preserving Cartan subalgebras with $\Lambda j' = w$. For β parts, we then have that $\Lambda(\beta \circ k) = K \circ \Lambda\beta \circ w'$. Furthermore, $\Lambda(i \circ \beta') = \Lambda\beta \circ w''$. From this we see that the conditions for $[((\alpha')^*, \Lambda\beta')]$ to embed into $[(\alpha^*, \Lambda\beta)]$ are satisfied. □

Appendix C

Rudiments of harmonic analysis with constraints

In this Appendix we will review, by way of an example, the form of harmonic analysis used throughout Chapter 6. The example we will use is that of planar motion in a magnetic field, as discussed in §6.1.1.

In all the examples in Chapter 6, we decompose the left-regular representation of $\tilde{G}$, which recall is a central extension by $U(1)$ of the original group G (constructed in §6.2), into unirreps of $\tilde{G}$. In our prototypical example, we have $G = M = \mathbb{R}^2$ and $\tilde{G} = \text{Hb}$, and the left-regular representation of Hb is defined by

$$\rho((x', y', s')) \cdot \Psi(x, y, s) = \Psi(x - x', y - y', s - s' - Bx'y' + By'x). \tag{C.1}$$

for $\Psi(x, y, s) \in \mathcal{H}$, where the Hilbert space $\mathcal{H}$ was defined in Eq. 6.5.

In this example we first decompose a general $\tilde{\Psi}(x, y, s) \in L^2(\text{Hb})$ into unirreps of Hb, following [87]:

$$\tilde{\Psi}(x, y, s) = \sum_k \int drdt \frac{|k|}{4\pi^2} \pi^k(r, t; x, y, s) g^k(r, t) \in L^2(\text{Hb}), \tag{C.2}$$

where recall the unirreps π^k are

$$\pi^k(r, t; x, y, s) = e^{ik(xr - s/B)} \delta(r + y - t), \qquad k/B \in \mathbb{Z}, \tag{C.3}$$

which transform under the left-regular representation as

$$\rho((x', y', s')) \cdot \pi^B(r, t; x, y, s) = \int \pi^{-B}(q, r; x', y', s') \pi^B(q, t; x, y, s) dq, \tag{C.4}$$

i.e. in the unirrep π^{-B}. The inverse transform is

$$g^k(r,t) = \int dxdyds \left(\pi^k(r,t;x,y,s)\right)^* \Psi(x,y,s). \tag{C.5}$$

These unirreps satisfy the Schur orthogonality relation

$$\int dxdyds \left(\pi^k(r,t;x,y,s)\right)^* \pi^{k'}(r',t';x,y,s) = \frac{4\pi^2}{|k|}\delta_{\frac{k}{B},\frac{k'}{B}}\delta(r-r')\delta(t-t'). \tag{C.6}$$

Enforcing the constraint $(-i\partial_s + 1)\tilde{\Psi} = 0$, and using the orthogonality relation in Eq. C.6, immediately implies $g^k(r,t) = 0,\ \forall k \neq B$. We can then write

$$\Psi(x,y,s) = \int drdt \frac{|B|}{2\pi}\pi^B(r,t;x,y,s)f(r,t) \in \mathcal{H}, \tag{C.7}$$

thus recovering the decomposition in Eq. 6.6, where $g^k(r,t) = 2\pi\delta_{\frac{k}{B},1}f(r,t)$, and the inverse of this decomposition is given by

$$f(r,t) = \int dx'dy' \left(\pi^B(r,t;x',y',s')\right)^* \Psi(x',y',s'). \tag{C.8}$$

In other words, we may restrict our decomposition to those unirreps which satisfy the constraint. This restricted subspace of unirreps (which satisfy the constraint) inherits the following completeness relation

$$\int drdt \frac{|B|}{2\pi} \left(\pi^B(r,t;x',y',s')\right)^* \pi^B(r,t;x,y,s) = e^{-i(s-s')}\delta(x-x')\delta(y-y'). \tag{C.9}$$

It seems plausible that, under suitably general assumptions, one may decompose a general state $\Psi \in \mathcal{H}$ into a basis of unirreps of $\tilde{G}$ which satisfy the constraint, following a similar procedure to that used in this example. We have indeed found this to be the case in all examples considered, as can be verified on a case-by-case basis by obtaining a completeness relation on the Hilbert space $\mathcal{H}$, analogous to Eq. C.9.

Appendix D

Proofs for Chapter 7

D.1 Proofs for §7.2

Throughout this Appendix we let (Y, π) and (Z, ζ) be fibred manifolds in $\texttt{Fib}_X$, and f a morphism in this category.

Proof of Lemma 7.2.4

Injections: Let $f : Z \to Y$ be an injection. Recalling that $\Gamma f[\alpha]_x = \Gamma f[\beta]_x \Leftrightarrow [f \circ \alpha]_x = [f \circ \beta]_x$, then there is a $\tilde{\alpha} \in [\alpha]_x$ and a $\tilde{\beta} \in [\beta]_x$ such that $f \circ \tilde{\alpha} = f \circ \tilde{\beta}$, but since f is an injection this implies $\tilde{\alpha} = \tilde{\beta}$ so $[\alpha]_x = [\beta]_x$. Thus if f is an injection, so is Γf. But Γf is an open map, as can be seen from its explicit form, and the topology on the étalé spaces, and an injective open map is a topological embedding, thus Γf is a topological embedding.

Counterexample for surjections: Let $X = \mathbb{R}$ and $(Y, \pi) = (\mathbb{R}^2, (x, y) \mapsto x)$, and let $f : Y \to Y : (x, y) \mapsto (x, y^3)$, which is a surjection. Let $\alpha : x \mapsto (x, x)$, then $[\alpha]_{x=0}$ is not in the image of Γf, since $x^{1/3}$ is not smooth at the origin.

Proof of Lemma 7.2.5

Submersions: Let $f : Y \to Z$ be a submersion. Then around every $y \in Y$ there is a neighbourhood U_y which has coordinates (x^μ, z^i, y^a) such that $f(U_y)$ (a submersion is open) has coordinates (x^μ, z^i) with $f : (x^\mu, z^i, y^a) \mapsto (x^\mu, z^i)$. The open subset $(\pi^{r,0})^{-1}(U_y)$ then has coordinates $(x_\mu, z^i, y^a, z^i_I, y^a_I)$, for multi-indices I, whist $(\zeta^{r,0})^{-1}(f(U_y))$ has coordinates (x_μ, z^i, z^i_I), where $J^r f : (x_\mu, z^i, y^a, z^i_I, y^a_I) \to (x^\mu, z^i, z^i_I)$. This map is clearly also a submersion.

Surjective submersions: Let $f : Y \to Z$ be a surjective submersion. This follows directly from the case of a submersion, by noting that every point $z \in Z$ sits in a neighbourhood $f(U_y)$ as constructed above.

Embeddings: Let $f : Z \to Y$ be an embedding. For every $f(z)$ there is a neighbourhood of Y, $U_{f(z)}$, which has coordinates (x^μ, z^a, y^i), such that $f^{-1}(U_{F(z)})$ has coordinates (x^μ, z^a) with $f : (x^\mu, z^a) \mapsto (x^\mu, z^a, 0)$. In the corresponding induced coordinates, $J^r f : (x^\mu, z^a, z^a_I) \mapsto (x^\mu, z^a, 0, z^a_I, 0)$. Since $J^r f$ in these coordinates maps an open subset to an open subset in the induced topology of its image, it is manifestly an immersion in these coordinates, and since it injectively maps the fibre above (x^μ, z^a) to the fibre above $(x^\mu, z^a, 0)$, it is an embedding.

Immersions: An immersion is equivalent to a local embedding, and thus this follows from the above.

Injective immersions: Let $f : Z \to Y$ be an injective immersion. For every $z \in Z$ there is a neighbourhood V_z, such that there is a neighbourhood around $f(z)$, $U_{f(z)}$ and coordinates on these neighbourhoods with $f : (x_\mu, z^i) \mapsto (x_\mu, z^i, y^a = 0)$. Using the induced coordinates on $(\zeta^{r,0})^{-1}(V_z)$ and $(\pi^{r,0})^{-1}(U_{F(z)})$, we get $J^r f : (x_\mu, z^i, z^i_I) \mapsto (x_\mu, z^i, 0, z^i_I, 0)$. Each of these coordinates covers its respective fibers of *e.g.* $J^r Z \to Z$ and, since the map between Z and Y is an injection, we can see that $J^r f$ is an injection. The form of $J^r f$ in these local coordinates also indicates that it is an immersion.

Counterexamples for injections and surjections: Let $X = \mathbb{R}$ and $(Y, \pi) = (\mathbb{R}^2, (x, y) \mapsto x)$, and let $f : Y \to Y : (x, y) \mapsto (x, y^3)$, which is a bijection. We have $J^1 f : (x, y, y_x) \mapsto (x, y^3, 3y_x y^2)$, which is neither surjective (since *e.g.* $(x, 0, 1)$ is not in the image) nor injective (since *e.g.* $J^1 f((x, 0, 1)) = J^1 f((x, 0, 2))$).

D.2 Proofs for §7.5

Throughout this Appendix, $(Y, \pi, \mathcal{Y})$ is an object in G-$\texttt{Fib}_X$. Further, the statement that, *e.g.*, $[\beta]_x \in \Gamma Y_g$ will be understood to imply that we are taking β to have a small enough domain that it satisfies the conditions in the definition of ΓY_g, and similarly for $j^r_x \beta$.

Proof that 7.5.1.4 and 7.5.1.5 are well defined

We must show that the partial actions in the definitions of Γ and J^r are indeed partial actions and Γ and J^r yield *bona fide* morphisms in the codomain. We deal with them in turn.

The partial action $\Gamma\mathcal{Y}$*:* We check that the list of properties in Def. 7.5.1.1 hold for $\Gamma\mathcal{Y}$.

1. From their definition, ΓY_g are open in ΓY. The maps $\Gamma\mathcal{Y}_g$ are manifestly open maps.

   We now want to show that $\Gamma\mathcal{Y}_g$ and $\Gamma\mathcal{Y}_{g^{-1}}$ are mutually inverse. Since both $\Gamma\mathcal{Y}_g$ and $\Gamma\mathcal{Y}_{g^{-1}}$ are open, this will also show not only that they are both continuous, but also that the image of $\Gamma\mathcal{Y}_g$, say, really is ΓY_g. Let $[\beta]_x \in \Gamma Y_{g^{-1}}$, and $\beta' = \mathcal{Y}_g \circ \beta \circ h_{g,\beta}^{-1}$. Then

   $$\pi \circ \mathcal{Y}_{g^{-1}} \circ \beta' = \pi \circ \mathcal{Y}_{g^{-1}} \circ \mathcal{Y}_g \circ \beta \circ h_{g,\beta}^{-1} = \pi \circ \beta \circ h_{g,\beta}^{-1} = h_{g,\beta}^{-1}. \tag{D.1}$$

   Hence, $\pi \circ \mathcal{Y}_{g^{-1}} \circ \beta'$ is an open embedding and have $h_{g^{-1},\beta'} = h_{g,\beta}^{-1}$. Acting on $[\beta]_x \in \Gamma Y_{g^{-1}}$ with $\Gamma\mathcal{Y}_{g^{-1}} \circ \Gamma\mathcal{Y}_g$ we get

   $$\Gamma\mathcal{Y}_{g^{-1}} \circ \Gamma\mathcal{Y}_g([\beta]_x) = [\mathcal{Y}_{g^{-1}} \circ \mathcal{Y}_g \circ \beta \circ h_{g,\beta}^{-1} \circ h_{g,\beta}]_{h_{g,\beta}^{-1} \circ h_{g,\beta}(x)} = [\beta]_x, \tag{D.2}$$

   so $\Gamma\mathcal{Y}_{g^{-1}}$ and $\Gamma\mathcal{Y}_g$ are indeed mutually inverse.

   Next we turn our attention to the case when $g = e$ (the identity of G). Looking at the definition of ΓY_g, for $g = e$, for any local section $\beta(x) \in Y_e = Y$, and $\pi \circ \mathcal{Y}_e \circ \beta = \iota_{U,X}$, which is an open embedding, thus $\Gamma Y_e = \Gamma Y$. For each $\beta \in \mathcal{Y}_e$, $h_{e,\beta} = \mathrm{id}_U$, and thus, from its definition, $\Gamma\mathcal{Y}_e$ is indeed the identity on ΓY.

2. The condition that $\Gamma\mathcal{U}_Y$ is open, and that $\Gamma\bar{\mathcal{Y}}$ is continuous, follows trivially from the fact we chose the discrete topology on G (recall that $\Gamma : G\text{-}\mathtt{Fib}_X \to G^d\text{-}\mathtt{Eta}_X$). If we had not done so, then generically $\Gamma\bar{\mathcal{Y}}$ would not be continuous.

3. For an arbitrary point, $[\beta]_x \in (\Gamma\mathcal{Y}_{g_1})^{-1}(\Gamma Y_{g_2^{-1}})$, we can take $\beta : U \to Y$ such that $\beta(x) \in (\mathcal{Y}_{g_1})^{-1}(Y_{g_2^{-1}})$. For such a β, we have that $\pi \circ \mathcal{Y}_{g_1 g_2} \circ \beta = \pi \circ \mathcal{Y}_{g_1} \circ \mathcal{Y}_{g_2} \circ \beta$, the left hand side of which, given the form of β, must be an open embedding, and therefore the right hand side must be too. This tells us that $(\Gamma\mathcal{Y}_{g_1})^{-1}(\Gamma Y_{g_2^{-1}}) \subseteq \Gamma Y_{(g_1 g_2)^{-1}}$. From their explict actions, it can then be seen that the action of $\Gamma\mathcal{Y}_{g_1 g_2}$ extends the action of $\Gamma\mathcal{Y}_{g_1}$ followed by $\Gamma\mathcal{Y}_{g_2}$.

The partial action $J^r\mathcal{Y}$: As for $\Gamma\mathcal{Y}$ we follow the list given in Def. 7.5.1.1, but now the proof is somewhat more involved, since we must check smoothness in addition.

1. Firstly we need to show that J^rY_g are open. Let (x^μ, y^a, y^a_μ) be some induced coordinates of $U \subseteq J^1Y$, so that $\pi^{1,0}(U) \in Y_g$. Let (x'^μ, y'^a) be some adapted coordinates of the image of $\pi^{1,0}(U)$ under $\mathcal{Y}_g$ (which can be made to exist by making U small enough). Then the condition on whether a point (x^μ, y^a, y^a_μ) is in (J^rY_g) can be expressed in terms of the Jacobian function defined locally on U by

$$\text{jac} : p \to \det(D_\mu(x^{\mu\prime} \circ \mathcal{Y}_g \circ \pi^{1,0})|_{(x^\mu, y^a, y^a_\mu)}) \text{ where } D_\mu = \frac{\partial}{\partial x^\mu} + y^a_\mu \frac{\partial}{\partial y^a}. \quad \text{(D.3)}$$

The points in $U \cap J^1Y_g$ correspond to those in $\text{jac}^{-1}(\mathbb{R} - \{0\})$ which is open, since $\mathbb{R} - \{0\}$ is open and jac is continuous. The union of all such open subsets for all U is J^1Y_g, which is therefore open in J^1Y. For generic r, $J^rY_g = (\pi^{r,0})^{-1}(J^1Y_g)$ are open for all $g \in G$ since $\pi^{r,0}$ is continuous.

We now need to show the smoothness of $J^r\mathcal{Y}_g$. In our local coordinates above, for $(x^\mu, y^a, y^a_\mu) \in U \cap J^1Y_g$ we define the matrix

$$M_\nu{}^\mu = D_\nu(x'^\mu \circ \mathcal{Y}_g \circ \pi^{1,0})|_{(x^\mu, y^a, y^a_\mu)}, \quad \text{(D.4)}$$

which is essentially the Jacobian matrix, which given our definition of $J^r\mathcal{Y}_g$ is invertible on this space. To determine the smoothness of $J^1\mathcal{Y}_g$ we can look at its value in the induced coordinates associated with (x'^μ, y'^a) on J^1Y, (x'^μ, y'^a, y'^a_μ). The smoothness in the coordinates x'^μ and y'^a follows directly from that of $\mathcal{Y}_g$. For y'^a_μ we have

$$y'^a_\mu \circ J^1\mathcal{Y}_g(x^\mu, y^a, y^a_\mu) = (M^{-1})_\nu{}^\mu D_\mu(y'^a \circ \mathcal{Y}_g \circ \pi^{1,0})|_{(x^\mu, y^a, y^a_\mu)}, \quad \text{(D.5)}$$

which is indeed smooth. For generic r, the smoothness of $J^r\mathcal{Y}_g$ follows from the smoothness of $J^1 \cdots J^1\mathcal{Y}_g$, noting that J^rY is embedded, via an embedding of partial actions, into $J^1 \cdots J^1Y$ (for r, J^1's) and one can pick out the appropriate coordinates to show smoothness.

The property that $J^r\mathcal{Y}_g$ and $J^r\mathcal{Y}_{g^{-1}}$ are mutually inverse follows in the same way as for $\Gamma\mathcal{Y}_g$ and $\Gamma\mathcal{Y}_{g^{-1}}$.

2. The argument that $J^r\mathcal{U}_Y$ is open and $J^r\bar{\mathcal{Y}}$ is smooth follows in exactly the same way as our arguments showing that J^rY_g are open and that $J^r\mathcal{Y}_g$ is smooth. In effect it follows from the smoothness of $\bar{\mathcal{Y}}$ and the (locally defined) Jacobian.

3. The property that $J^r\mathcal{Y}_{g_1g_2}$ extends the combined action of $J^r\mathcal{Y}_{g_1}$ and $J^r\mathcal{Y}_{g_2}$, follows in the same way as for Γ.

Target morphisms of Γ *and* J^r*:* We want to show, for f a morphism in $G\text{-}\mathtt{Fib}_X$, that Γf is a morphism in $G^d\text{-}\mathtt{Eta}_X$ and that J^rf is a morphism in $G\text{-}\mathtt{Fib}_X$. We show it for Γf, noting that for J^rf the proof works analogously. Let $(Y,\pi,\mathcal{Y})$ and $(Y',\pi',\mathcal{Y}')$ be two objects in $G\text{-}\mathtt{Fib}_X$ and $f : Y \to Y'$ be a morphism between these two objects. We need to show that Γf interacts with our partial actions correctly (the other required properties hold trivially). Thus, let $[\beta]_x \in \Gamma Y_g$. We need that $\Gamma f[\beta]_x = [f\circ\beta]_x \in \Gamma Y'_g$. Since f is a morphism in $G\text{-}\mathtt{Fib}_X$, we have $f\circ\beta(x) \in Y'_g$ for all $x \in U$. Then

$$\pi'\circ\mathcal{Y}'_{g^{-1}}\circ f\circ\beta = \pi'\circ f\circ\mathcal{Y}_{g^{-1}}\circ\beta = \pi\circ\mathcal{Y}_{g^{-1}}\circ\beta. \tag{D.6}$$

Thus we have that $\pi'\circ\mathcal{Y}'_{g^{-1}}\circ f\circ\beta$ is an open embedding and have $h_{g^{-1},f\circ\beta} = h_{g^{-1},\beta}$. This means that $\Gamma f(\Gamma Y_g) \subseteq \Gamma Y'_g$. We now need to check that Γf obeys the commuting diagram 7.6. So letting, $[\beta]_x \in \Gamma Y_{g^{-1}}$, (for convenience we have swapped g and g^{-1}), we have

$$\Gamma f\circ\Gamma\mathcal{Y}_g[\beta]_x = [f\circ\mathcal{Y}_g\circ\beta\circ h^{-1}_{g,\beta}] = [\mathcal{Y}'_g\circ f\circ\beta\circ h^{-1}_{g,f\circ\beta}] = \Gamma\mathcal{Y}'_g\circ\Gamma f[\beta]_x. \tag{D.7}$$

This shows that Γf is indeed a morphism in $G^d\text{-}\mathtt{Eta}_X$. As mentioned, the analogous arguments apply for J^rf.

Proof of Lemma 7.5.1.6

We want to show that Γ and J^r preserve embeddings of partial actions, in accordance with Lemma 7.5.1.6. Let us do this for Γ, noting that the proof for J^r is analogous. Let $\iota : Q \to Y$ be an embedding of partial actions, meaning that it is an embedding and that $\mathcal{Q}_g = \iota^{-1}(Y_g)$. We want to show that $\Gamma Q_g = (\Gamma\iota)^{-1}(\Gamma Y_g)$. Let $[\beta]_x \in \Gamma\mathcal{Y}_g$, such that $\Gamma\iota_Q[\tilde{\beta}]_x = [\beta]_x$ for some $[\tilde{\beta}]_x \in \Gamma Q$. We, first, want to show that $[\tilde{\beta}]_x \in \Gamma Q_g$. Since $\Gamma\iota_Q$ is an injection, $[\tilde{\beta}]_x$ is the unique element mapping into $[\beta]_x$. Explicitly we let $\beta = \iota\circ\tilde{\beta}$, meaning that $\iota\circ\tilde{\beta}(x) \in Y_g$ for all $x \in U$. Thus $\tilde{\beta}(x) \in Q_g$, since ι is an embedding of partial actions. Finally, we use that $h_{g^{-1},\tilde{\beta}} = h_{g^{-1},\iota\circ\tilde{\beta}}$ to show that $h_{g^{-1},\tilde{\beta}}$ must be an open embedding. From this we can deduce that $[\tilde{\beta}]_x \in \Gamma Q_g$, and hence that $\mathcal{Q}_g = \iota^{-1}(Y_g)$.

Proof of Proposition 7.5.1.7

The only non-trivial thing to check here is that the claimed morphisms j^r are indeed morphisms.

For $j^r : \Gamma Y \to \Gamma J^r Y : [\alpha]_x \mapsto [j^r \alpha]_x$, we, firstly, need to show that if $[\alpha]_x \in \Gamma Y_g$ then $[j^r\alpha]_x \in \Gamma J^r Y_g$. Assuming then that $[\alpha]_x \in \Gamma Y_g$, we have $j^r\alpha(x) = j^r_x(\alpha) \in J^r Y_g$, by the similarities in the definitions of $J^r Y_g$ and ΓY_g. We then have

$$\pi^r \circ J^r \mathcal{Y}_{g^{-1}} \circ j^r \alpha = \pi^r \circ j^r(\mathcal{Y}_{g^{-1}} \circ \alpha \circ h^{-1}_{g^{-1},\alpha}) \circ h_{g^{-1},\alpha} = \pi \circ \mathcal{Y}_{g^{-1}} \circ \alpha \circ h^{-1}_{g^{-1},\alpha} \circ h_{g^{-1},\alpha}$$
$$= \pi \circ \mathcal{Y}_{g^{-1}} \circ \alpha \quad \text{(D.8)}$$

Since $\pi \circ \mathcal{Y}_{g^{-1}} \circ \alpha$ is an open embedding, so is $\pi^r \circ J^r \mathcal{Y}_{g^{-1}} \circ j^r \alpha$, and thus $[j^r\alpha]_x \in \Gamma J^r Y_g$. We also have that $h_{g,j^r\alpha} = h_{g,\alpha}$. We now need to show that j^r is such that the Diagram 7.6 commutes. Let $[\alpha]_x \in \Gamma Y_g$, then

$$\Gamma J^r \mathcal{Y}_g \circ j^r[\alpha]_x = [J^r \mathcal{Y}_g \circ j^r \alpha \circ h^{-1}_{g,j^r\alpha}]_{h_{g,j^r\alpha}(x)} = [j^r(\mathcal{Y}_{g^{-1}} \circ \alpha \circ h^{-1}_{g,\alpha}) \circ h_{g,\alpha} \circ h^{-1}_{g,j^r\alpha}]_{h_{g,j^r\alpha}(x)}$$
$$= [j^r(\mathcal{Y}_{g^{-1}} \circ \alpha \circ h^{-1}_{g,\alpha})]_{h_{g,\alpha}(x)} = j^r \circ \Gamma \mathcal{Y}_{g^{-1}}[\alpha]_x \quad \text{(D.9)}$$

Thus, j^r is indeed a morphism in G^d-$\texttt{Eta}_X$.

D.3 Proofs for §7.3 in conjunction with §7.5

We give proofs in the most general case of cormeronomic constraints with a group action present.

Proof of Theorem 7.3.1.2

We want to show that the pullback of $\Gamma\iota_Q : \Gamma Q \to \Gamma J^r Y$ and $j^r : \Gamma Y \to \Gamma J^r Y$ exists in G^d-$\texttt{Eta}$. We define the cone $((E^Q, p^Q, \mathcal{E}^Q), \{P^Q_Y : E^Q \to \Gamma Y, P^Q_Q : E^Q \to \Gamma Q\})$ and show it is the limit of this pullback.

We let E^Q be the topological space defined by the pullback in $\texttt{Top}$, which exists. As a set

$$E^Q = \{([\alpha]_x, [\beta]_x) \in \Gamma Y \times \Gamma Q \mid j^r[\alpha]_x = \Gamma\iota_Q[\beta]_x\}. \quad \text{(D.10)}$$

and $P^Q_Y : ([\alpha]_x, [\beta]_x) \mapsto [\alpha]_x$ and $P^Q_Q : ([\alpha]_x, [\beta]_x) \mapsto [\beta]_x$. We let $E^Q_g = (P^Q_Y)^{-1}(\Gamma Y_g)$ and we let $\mathcal{E}^Q_g$ be the unique maps such that $P^Q_Y \circ \mathcal{E}^Q_g = \Gamma Y_g \circ P^Q_Y$, which combined form a valid partial action, $\mathcal{E}^Q = (\{E^Q_g\}_{g\in G^d}, \{\mathcal{E}^Q_g\}_{g\in G^d})$. Explicitly $\mathcal{E}_g : ([\alpha]_x, [\beta]_x) \mapsto (\Gamma\mathcal{Y}_g[\alpha]_x, \Gamma\mathcal{Q}_g[\beta]_x)$, which works since ι_Q is an embedding of partial actions.

Now let us show that it is indeed the limit of the pullback. Let $((E', p', \mathcal{E}'), \{P'_Y : E' \to \Gamma Y, P'_Q : E' \to \Gamma Q\})$ be another cone. We define the map of sets $u : E' \to E^Q : e' \mapsto (P'_Y(e'), P'_Q(e'))$. We then have that $P^Q_Y \circ u = P'_Y$ and $P^Q_Q \circ u = P'_Q$. Since P^Q_Y is an embedding, from the first of these equations we get that u is continuous and it is unique. It can also, trivially, be used to show that u is an étalé morphism. Finally, to make sure it is actually in G^d-`Eta`$_X$, we need to ensure it interacts correctly with the partial actions. Since $P^Q_Y \circ u(E'_g) = P'_Y(E'_g) \subseteq \Gamma Y_g$, we have that $u(E'_g) \subseteq E^Q_g$. The explicit form of $\mathcal{E}^Q_g$, and the fact that P'_Y and P'_Q are morphisms in G^d-`Eta`$_X$, tells us that so too is u.

Proof of Propositions 7.3.2.3 and 7.3.4.2

We now turn our attention to proving Proposition 7.3.4.2 and consequently Proposition 7.3.2.3. We split this proof into a series of Lemmas.

Lemma. *The limit of the pullback diagram*

$$\begin{array}{ccc} & & R \\ & & \big\downarrow{\scriptstyle \iota_R} \\ J^rY & \xrightarrow{J^r\tau_Z} & J^rZ \end{array} \tag{D.11}$$

exists in G-`Fib`$_X$; denoting it by the cone $((S, \psi, \mathcal{S}), \{\iota_S : S \to J^rY, \kappa : S \to R\})$, then ι_S is an embedding and κ is a surjective submersion.

Proof. In `Man` the limit of this diagram exists, the ι_S defined by this pullback is an embedding, and the κ is a surjective submersion. We take S, ι_S and κ as defined by this pullback in `Man`. As a set, we have that

$$S = \{(j^r_x\alpha, r) \in J^rY \times R | J^r\tau_Z(j^r_x\alpha) = \iota_R(r)\} \tag{D.12}$$

with $\iota_s : (j^r_x\alpha, r) \mapsto j^r_x\alpha$, and $\kappa : (j^r_x\alpha, r) \mapsto r$, which are both fibred morphisms. We define $\mathcal{S}$ in the same way in which we defined $\mathcal{E}^Q$ above. That is, let $S_g = \iota_S^{-1}(J^rY_g)$. Let $\mathcal{S}_g$ the unique map such that $\iota_S \circ \mathcal{S}_g = J^rY_g \circ \mathcal{S}_g$. Explicitly, $\mathcal{S}_g : (j^r_x\alpha, r) \mapsto (\Gamma\mathcal{Y}_g j^r_x\alpha, \mathcal{R}_g r)$, which is valid since ι_R is an embedding of partial actions and $J^r\tau_Z$ is a morphism of partial actions. This makes ι_S an embedding of partial actions, and κ a morphism of partial actions.

The fact that this construction indeed leads to a limit, follows from the same arguments as for $(E^Q, p^Q, \mathcal{E}^Q)$ above. □

Lemma. *The limit of the equaliser diagram*

$$S \overset{\pi^{r,0}\circ\iota_S}{\underset{\Omega\circ\kappa}{\rightrightarrows}} Y \tag{D.13}$$

exists in G-$\mathtt{Fib}_X$; denoting it by the cone $((Q, \nu, \mathcal{Q}), \{\iota_S^Q : Q \to S\})$, then we have that ι_S^Q is an embedding of partial actions.

Proof. Let $\langle \pi^{r,0} \circ \iota_S, \Omega \circ \kappa \rangle : S \to Y \times Y : s \mapsto (\pi^{r,0} \circ \iota_S(s), \Omega \circ \kappa(s))$. The map $\pi^{r,0} \circ \iota_S$ is a submersion, meaning $\langle \pi^{r,0} \circ \iota_S, \Omega \circ \kappa \rangle$ is transverse to the diagonal map $\Delta_Y : Y \to Y \times Y$. Thus the inverse image $Q = \langle \pi^{r,0} \circ \iota_S, \Omega \circ \kappa \rangle^{-1}(\Delta_Y(Y))$ exists, with a corresponding embedding $\iota_Q^S : Q \to S$ of Q into S.

Let $\nu := \pi^r \circ \iota_Q$, let $y \in J^rY$, and let $U_y \in J^rY$ be a neighbourhood of y, with coordinates $(x^\mu, z^a, y^i, z_I^a, y_I^i)$. In these coordinates, Q is described by $(x^\mu, z^a, f^i(x^\mu, z^a, z_I^a), z_I^a, y_I^i)$, for some smooth f^i. From this, we see that $\nu : (x^\mu, z^a, f^i(x^\mu, z^a, z_I^a), z_I^a, y_I^i) \mapsto (x^\mu)$ is a surjective submersion.

As a set, we have that

$$Q = \{s \in S \mid \pi^{r,0} \circ \iota_S(s) = \Omega \circ \kappa(s)\}. \tag{D.14}$$

To define $\mathcal{Q}$ we first define $Q_g = (\iota_Q^S)^{-1}(S_g)$. Then, as before, we let $\mathcal{Q}_g$ be the unique map such that $\iota_Q^S \circ \mathcal{Q}_g = \mathcal{S}_g \circ \iota_Q^S$. The fact that such $\mathcal{Q}_g$ exist can be seen from the form of Q and the fact that $\pi^{r,0}$, ι_S, Ω, and κ are all morphisms in G-$\mathtt{Fib}_X$. This then makes ι_S^Q an embedding of partial actions.

The universality property then follows that of $(E^Q, p^Q, \mathcal{E}^Q)$. □

Let $\iota_Q := \iota_S \circ \iota_S^Q$; since both maps in the composition are embeddings of partial actions, so is ι_Q. Let $f_Q^R := \kappa \circ \iota_S^Q$, we then have

Lemma. *The triple $((Q, \nu, X, \mathcal{Q}), \{\iota_Q, f_Q^R\})$ forms the limit of Diagram 7.4*

Proof. First let us show that $((Q, \nu, X, \mathcal{Q}), \{\iota_Q, f_Q^R\})$ is indeed a cone of Diagram 7.4. For this to hold we need $\pi^{r,0} \circ \iota_Q = \Omega \circ f_Q^R$, which follows from the equaliser in Diagram D.13, and $J^r\tau_Z \circ \iota_Q = \iota_R \circ f_Q^r$ which follows from the pullback D.11.

To show that this cone is a limit, suppose we have another cone

$$((Q', \nu', X, \mathcal{Q}), \{f_{\hat{\pi}^r}, f_R\}). \tag{D.15}$$

From Eqs. D.12 and D.14, we can write Q as

$$Q = \{(j^r_x\alpha, r) \in J^r\hat{\pi} \times R \mid J^r\tau_Z(j^r_x\alpha) = \iota_r(r), \Omega(r) = \pi^{r,0}(j^r_x\alpha)\}. \tag{D.16}$$

We let $u : Q' \to Q : q' \mapsto (f_{\hat{\pi}^r}(q'), f_R(q'))$; the standard argument shows that this is a mediating morphism. □

Proof of Theorems 7.3.2.4 and 7.3.4.3

We want to prove Theorem 7.3.4.3 and consequently Theorem 7.3.2.4. Namely, we want to show that there exists an isomorphism between the étalé spaces

$$(E^Q, p^Q, \mathcal{E}^Q) \quad \text{and} \quad (E^R, p^R, \mathcal{E}^R). \tag{D.17}$$

This follows from the structure of a series of cones and limits. We start by noting that

$$((\Gamma Q, \Gamma\nu, \mathcal{Q}), \{\Gamma f^r_Q, \Gamma\iota_Q\}) \tag{D.18}$$

is the limit of Diagram 7.4 in G^d-`Eta`, something which can be shown explicitly following the standard arguments used previously. But,

$$((E^R, \tilde{p}^R, \mathcal{E}^R), \{P^R_R, j^r \circ \Gamma\Omega \circ P^R_R\}) \tag{D.19}$$

is manifestly a cone of this diagram since, for instance,

$$\begin{aligned} \Gamma J^r\tau_Z \circ j^r \circ \Gamma\Omega \circ P^R_R = j^r \circ \Gamma(\tau_Z \circ \Omega) = j^r \circ \Gamma\zeta^{r,0} \circ \Gamma\iota_R \circ P^R_R \\ = j^r \circ P^R_Z = \Gamma\iota_R \circ P^R_R. \end{aligned} \tag{D.20}$$

We denote the corresponding mediating morphism, $\mathcal{N} : \Gamma R \to \Gamma Q$. Since,

$$\Gamma\iota_Q \circ \mathcal{N} = j^r \circ \Gamma\Omega \circ P^R_R, \tag{D.21}$$

$((E^R, p^R, \mathcal{E}^R), \{\mathcal{N}, \Gamma\Omega \circ P^R_R\})$ is a cone of the diagram defining E^Q. This means we have a mediating morphism $\mathcal{I} : E^R \to E^Q$. In a similar vein, $((E^Q, p^Q, \mathcal{E}^Q), \{\Gamma f^R_Q \circ P^Q_Q, \Gamma\tau_Z \circ P^Q_Y\})$ is a cone of the diagram defining E^R since

$$\Gamma\iota_R \circ \Gamma f^R_Q \circ P^Q_Q = \Gamma J^r\tau_Z \circ \Gamma\iota_Q \circ P^Q_Q = \Gamma J^r\tau_Z \circ j^r \circ P^Q_Y = j^r \circ \Gamma\tau_Z \circ P^Q_Y \tag{D.22}$$

and thus we have a mediating morphism $\tilde{\mathcal{I}} : E^Q \to \tilde{E}^R$. With this we can show that $\tilde{\mathcal{I}} \circ \mathcal{I}$ is the identity, since

$$P_Z^R \circ \tilde{\mathcal{I}} \circ \mathcal{I} = \Gamma\tau_Z \circ P_Y^Q \circ \mathcal{I} = \Gamma\tau_Z \circ \Gamma\Omega \circ P_R^R = \Gamma\zeta^{r,0} \circ \Gamma\iota_R \circ P_R^R = P_Z^R, \tag{D.23}$$

which, since P_Z^R is an embedding, shows that $\tilde{\mathcal{I}} \circ \mathcal{I}$ is the identity. The statement that $\mathcal{I} \circ \tilde{\mathcal{I}}$ is the identity holds in a similar vein, since

$$P_Y^Q \circ \mathcal{I} \circ \tilde{\mathcal{I}} = \Gamma\Omega \circ P_R^R \circ \tilde{\mathcal{I}} = \Gamma\Omega \circ \Gamma f_Q^R \circ P_Q^Q = \Gamma\pi^{r,0} \circ \Gamma\iota_Q \circ P_Q^Q = P_Y^Q. \tag{D.24}$$

Thus $\mathcal{I}$ and $\tilde{\mathcal{I}}$ are mutually inverse and form isomorphisms.

Index