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The quantum convolution product

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Abstract. In classical statistical mechanics, physical states (probability measures) are embedded in the Banach algebra of complex Borel measures on phase space, where the algebra product is realized by convolution. Convolution is state-preserving; namely, the convolution of two classical states is a state too. This is a special case of the convolution algebra of all complex measures on a locally compact group. A natural problem is whether an analogous structure may emerge in the quantum setting. By resorting to a group-theoretical construction, a quantum counterpart of the convolution of probability measures — the twirled product, or *quantum convolution* — can be introduced, yielding a group-covariant, associative binary operation on the states of a quantum system, that preserves the convex structure of this set. The analogy with the classical setting becomes striking in the case where the symmetry group is abelian. We focus, in particular, on the quantum convolution product stemming from the group of phase-space translations.

1 Introduction

It is known that some ideas of Arthur Eddington had a profound influence on the work of Paul Dirac [1]. One of the main distinguishing features of Eddington's program, as a scientist and a philosopher of science, is the so-called "Principle of Identification" [2], according to which the mathematics of a physical theory should be developed at an *a priori* stage, *before* the identification of the physically relevant quantities may eventually take place. This guiding principle led Dirac to believe that the progress of physics requires a mathematics that gets more and more advanced; otherwise stated, that the most powerful method for progressing in modern theoretical physics is to exploit all available resources of pure mathematics in order to perfect the mathematical formalisms at the basis of physical models, and only *after* each success in this direction, to try to provide an interpretation of the new mathematical structures as suitable physical entities [1]. These ideas of Eddington and Dirac remain influential and inspiring still today. Accordingly, our aim in the present contribution is to illustrate an interesting example of how an abstract mathematical notion — the *convolution* of functions or, more generally, of measures — can be extended to the realm of quantum mechanics and, consequently, admits an interpretation within quantum measurement theory.

The paper is organized as follows. In section 2, we introduce some basic ideas. Next, in section 3, we discuss some mathematical tools that allow us, in section 4, to define a quantum analog of the convolution of probability measures, i.e., a binary operation on the density operators. The connection of the quantum convolution with quantum measurement theory is analyzed in section 5, and, in section 6, we show that the quantum convolution can be defined in every Hilbert space dimension. Convolution is a group-theoretical operation, and the analogy between the classical and the quantum setting becomes more evident in the case where the relevant group is abelian. This point is clarified by realizing the quantum convolution in terms of Wigner distributions; see section 7. Finally, in section 8, a few conclusions are drawn.

2 Prologue: operator algebras, quantum states, convolution

The theory of operator algebras and its manifold applications play a central role in quantum mechanics and quantum field theory; e.g., quantum *states* can be regarded as *normalized positive functionals* on the C*-algebra $\mathcal{B}(\mathcal{H})$ of all bounded *observables* on a separable complex Hilbert space $\mathcal{H}$ [3]. In quantum mechanics, one usually restricts to considering a distinct class of physical states, the *normal states* (i.e., the quantum analog of the σ*-additive* probability measures). These states can be implemented as normalized, positive trace class operators (density operators) on the Hilbert space $\mathcal{H}$, that form a *convex subset* $\mathcal{D}(\mathcal{H})$ of the Banach space $\mathcal{T}(\mathcal{H})$ of trace class operators on $\mathcal{H}$. By endowing the *selfadjoint part* $\mathcal{B}(\mathcal{H})_{\mathsf{s}}$ of the C*-algebra $\mathcal{B}(\mathcal{H})$ with the pair of non-associative binary operations

$$\begin{aligned} A \bowtie B &:= (AB + BA)/2, \quad \text{(the symmetric Jordan product)} \\ A \diamond B &:= (AB - BA)/2\mathrm{i}, \quad \text{(the antisymmetric Lie product)} \end{aligned} \tag{1}$$

one further obtains a *Jordan-Lie Banach algebra* [3]. As mere functionals, normal states play, within the algebraic framework of quantum observables, a rather indirect role; in particular, for any $\rho, \sigma \in \mathcal{D}(\mathcal{H})$, we have:

- The Jordan product $\rho \bowtie \sigma \in \mathcal{B}(\mathcal{H})_{\mathsf{s}}$ is a density operator iff $\rho = \sigma \equiv P$, where P is a *pure state* [4].
- The Lie product $\rho \diamond \sigma \in \mathcal{B}(\mathcal{H})_{\mathsf{s}}$ is *not* a density operator (indeed, $\mathrm{tr}(\rho \diamond \sigma) = 0$).

2.1 A natural problem: the existence of a suitable state-preserving product

Considering the importance of the algebraic methods in quantum mechanics, a natural problem is whether it is possible to endow the Banach space $\mathcal{T}(\mathcal{H})$ with a suitable *binary operation,* in such a way to obtain an *algebra* structure that is *state-preserving* (the product of two states is a state too). In order to achieve a noteworthy solution to this problem, we will impose some additional conditions [5–8]:

1. This product should be a *genuinely binary* operation (it should depend on both its entries).
2. We get, in particular, an *associative* algebra on the trace class $\mathcal{T}(\mathcal{H})$.
3. The product is *continuous* wrt some suitable topology.
4. This product is group-theoretical: It enjoys some nice *covariance* property wrt a *symmetry action* of a given abstract *group* G.
5. The state-preserving algebra is *commutative* in the case where G is *abelian* (non-commutativity being a natural feature of an algebra of *observables*).

Actually, all these requirements are modeled on the properties of a binary operation that can be defined in a *classical setting.* In fact, take the *convolution product* $\mu \circledast \nu$ of two complex (Radon) measures μ, ν on a *locally compact group* G; for the reader's convenience, the precise definition is briefly recalled below. The Banach space $\mathcal{M}(G)$ of all such measures, endowed with this product, becomes a *Banach algebra.* Moreover, if μ, ν are *classical states* (i.e., probability measures), then $\mu \circledast \nu$ is a classical state too.

2.2 Convolution of measures and functions of positive type

Recall that the *convolution* [9] $\mu \circledast \nu$ of a pair of *probability measures* $\mu, \nu \in \mathcal{M}(G)$, which is defined — as a positive functional on $\mathsf{C_c}(G)$ (the continuous $\mathbb{C}$-valued functions with compact support on G) — by

$$\int_G \varphi(g)\, \mathrm{d}\mu \circledast \nu(g) := \int_G \int_G \varphi(gh)\, \mathrm{d}\mu(g) \mathrm{d}\nu(h), \quad \forall\, \varphi \in \mathsf{C_c}(G), \tag{2}$$

is a probability measure too. Endowed with convolution the convex set $\mathscr{P}(G)$ of all Radon *probability measures* on G becomes a *semigroup*, with *identity* δ_e (the Dirac measure at the identity e of G). In the case where G is *abelian*, to the convolution of two probability measures $\mu, \nu \in \mathscr{P}(G)$ corresponds — via the *Fourier-Stieltjes transform* $(\mu, \nu) \mapsto (\chi_\mu \equiv \widehat{\mu}, \chi_\nu \equiv \widehat{\nu})$ — the *point-wise multiplication* $\chi_\mu\, \chi_\nu$ of the corresponding *characteristic functions* χ_μ and χ_ν. For $G = \mathbb{R}^n \times \mathbb{R}^n$ (the vector group of phase-space translations), we can take the *symplectic Fourier-Stieltjes transform*

$$\chi_\mu(q,p) \equiv \widehat{\mu}(q,p) := \int_{\mathbb{R}^n \times \mathbb{R}^n} \mathrm{e}^{\mathrm{i}\omega(q,p\,;\tilde{q},\tilde{p})}\, \mathrm{d}\mu(\tilde{q},\tilde{p}), \quad \omega(q,p\,;\tilde{q},\tilde{p}) := q \cdot \tilde{p} - p \cdot \tilde{q}, \quad (q,p) \in \mathbb{R}^n \times \mathbb{R}^n. \tag{3}$$

Note that $\chi_\mu \equiv \widehat{\mu}$ is a *function of positive type* [9] on $\mathbb{R}^n \times \mathbb{R}^n$; see the general definition below. Thus, the point-wise product $\chi_\mu\, \chi_\nu$ of two functions of positive type on the group $\mathbb{R}^n \times \mathbb{R}^n$ of phase-space translations is a function of positive type too. As it will be clear soon, the normalization condition $\mu(\mathbb{R}^n \times \mathbb{R}^n) = 1$ translates into the condition that $\chi_\mu(0) = \|\chi_\mu\|_\infty = 1$; i.e., into *the normalization of* χ_μ *as a functional.*

Indeed, we also recall [9] that the Banach space $\mathrm{L}^1(G)$ of $\mathbb{C}$-valued functions on a locally compact group G, integrable wrt the (say, left) *Haar measure* μ_G, once endowed with the *convolution product*

$$(\varphi_1 \circledast \varphi_2)(g) := \int_G \varphi_1(h)\varphi_2(h^{-1}g)\, \mathrm{d}\mu_G(h) \tag{4}$$

— which is coherent with the corresponding definition (2) of convolution of measures [9] (when considering the complex measures on G that are absolutely continuous wrt μ_G), and satisfies the covariance property

$$(L_g\varphi_1) \circledast \varphi_2 = L_g(\varphi_1 \circledast \varphi_2), \quad \text{where } \big(L_g\varphi\big)(h) := \varphi(g^{-1}h), \tag{5}$$

— and with the *involution*

$$\mathscr{I} : \varphi \mapsto \varphi^*, \quad \varphi^*(g) := \Delta_G(g^{-1})\, \overline{\varphi(g^{-1})}, \tag{6}$$

where Δ_G is the *modular function* on G, is a *Banach $*$-algebra.*

Definition 1. A *positive bounded linear functional* on the Banach $*$-algebra $(\mathrm{L}^1(G), \circledast, \mathscr{I})$ — realized as a function in the Banach space $\mathrm{L}^\infty(G)$ of all μ_G-essentially bounded functions — is called a *function of positive type* on G. Namely, a function $\chi \in \mathrm{L}^\infty(G)$ is said to be of positive type if, for all $\varphi \in \mathrm{L}^1(G)$,

$$\int_G \chi(g)\, (\varphi^* \circledast \varphi)(g)\, \mathrm{d}\mu_G(g) \geq 0. \quad \text{(the PTF condition)} \tag{7}$$

Every function of positive type $\chi \in \mathrm{L}^\infty(G)$ agrees μ_G-almost everywhere with a (bounded) *continuous* function [9–11], and then, assuming (without loss of generality) that χ is continuous, we have:

$$\|\chi\|_\infty := \mu_G\text{-ess}\ \sup_{g\in G} |\chi(g)| = \sup_{g\in G} |\chi(g)| = \chi(e). \tag{8}$$

For a *bounded continuous* function $\chi \colon G \to \mathbb{C}$ the following facts are *equivalent* [9–11]:

(P1) χ is a function of positive type.

(P2) χ satisfies the PTF condition (7), for all $\varphi \in \mathsf{C_c}(G) \subset \mathrm{L}^1(G)$.

(P3) χ satisfies the condition

$$\int_G \int_G \chi(g^{-1}h)\, \overline{\varphi(g)}\, \varphi(h)\, \mathrm{d}\mu_G(g)\, \mathrm{d}\mu_G(h) \geq 0, \tag{9}$$

for all $\varphi \in \mathsf{C_c}(G)$.

(P4) χ is a *positive definite function*, i.e.,

$$\sum_{j,k} \chi(g_j^{-1} g_k)\, \overline{c_j}\, c_k \geq 0, \tag{10}$$

for every finite set $\{g_1, \ldots, g_m\} \subset G$ and arbitrary complex numbers $c_1, \ldots, c_m$.

Let G be *abelian*, and let $\widehat{G}$ be its *Pontryagin dual* group, i.e., the abelian group of all *unitary characters* of G [9]. Denoting by $\mathcal{M}(\widehat{G})$ the Banach space of *complex Radon measures* on $\widehat{G}$, by *Bochner's theorem* [9] we can add another point to the previous list; i.e., conditions (P1)–(P4) are equivalent to the following:

(P5) $\chi \equiv \chi_\mu$ is the Fourier transform of a *positive* measure $\mu \in \mathcal{M}(\widehat{G})$.

3 Mathematical tools: stochastic products, square integrable representations

To define a quantum analog of convolution, we first need to introduce two mathematical tools: the notion of (quantum) stochastic product and the square integrable representations of a locally compact group.

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3.1 Stochastic maps and stochastic products

A *convex-linear* map on the density operators $\mathfrak{S}\colon \mathcal{D}(\mathcal{H}) \to \mathcal{D}(\mathcal{H})$ is called *stochastic*. It can always be extended to a *trace-preserving, positive linear map* $\mathfrak{S}_{\mathrm{ext}}\colon \mathcal{T}(\mathcal{H}) \to \mathcal{T}(\mathcal{H})$ [5]. Analogously, we can define a *stochastic product* on $\mathcal{D}(\mathcal{H})$ as a map (a binary operation)

$$(\cdot) \odot (\cdot)\colon \mathcal{D}(\mathcal{H}) \times \mathcal{D}(\mathcal{H}) \to \mathcal{D}(\mathcal{H}) \tag{11}$$

that is *convex-linear* wrt both its arguments: for all $\rho, \sigma, \tau, \upsilon \in \mathcal{D}(\mathcal{H})$ and $\alpha, \epsilon \in [0,1]$,

$$\begin{aligned}(\alpha\rho + (1-\alpha)\sigma) \odot (\epsilon\tau + (1-\epsilon)\upsilon) = \alpha\epsilon\, \rho \odot \tau + \alpha(1-\epsilon)\, \rho \odot \upsilon \\ + (1-\alpha)\epsilon\, \sigma \odot \tau + (1-\alpha)(1-\epsilon)\, \sigma \odot \upsilon.\end{aligned} \tag{12}$$

Proposition 1 ([5])**.** *Every stochastic product is automatically continuous wrt the topologies on $\mathcal{D}(\mathcal{H})$ and on $\mathcal{D}(\mathcal{H}) \times \mathcal{D}(\mathcal{H})$ that are induced, respectively, by the metrics*

$$\mathsf{d}_1(\rho,\sigma) := \|\rho - \sigma\|_1 \quad \textit{and} \quad \mathsf{d}_{1,1}((\rho,\tau),(\sigma,\upsilon)) := \max\{\|\rho-\sigma\|_1, \|\tau-\upsilon\|_1\}. \tag{13}$$

For every stochastic product $(\cdot) \odot (\cdot)\colon \mathcal{D}(\mathcal{H}) \times \mathcal{D}(\mathcal{H}) \to \mathcal{D}(\mathcal{H})$, there exists a unique extension to a stochastic map $(\cdot) \boxdot (\cdot)\colon \mathcal{T}(\mathcal{H}) \times \mathcal{T}(\mathcal{H}) \to \mathcal{T}(\mathcal{H})$, i.e., a unique state-preserving bilinear map on $\mathcal{T}(\mathcal{H})$ such that

$$\rho \odot \sigma = \rho \boxdot \sigma, \quad \forall\, \rho, \sigma \in \mathcal{D}(\mathcal{H}). \tag{14}$$

The notion of *stochastic product* of quantum states is naturally related to an algebraic notion. The Banach space $\mathcal{T}(\mathcal{H})$, endowed with a map

$$(\cdot) \boxdot (\cdot)\colon \mathcal{T}(\mathcal{H}) \times \mathcal{T}(\mathcal{H}) \to \mathcal{T}(\mathcal{H}) \tag{15}$$

that is both *stochastic* (bilinear and state-preserving) and an *associative* binary operation, is called a *stochastic algebra*. By the second assertion of Proposition 1, every stochastic product $(\cdot) \odot (\cdot)$ on $\mathcal{D}(\mathcal{H})$ can be regarded as the restriction of a unique stochastic bilinear map $(\cdot)\boxdot(\cdot)$ on $\mathcal{T}(\mathcal{H})$, that is *associative* iff $(\cdot) \odot (\cdot)$ is. Thus, one may equivalently define a stochastic algebra as a Banach space of trace class operators $\mathcal{T}(\mathcal{H})$ equipped with an *associative* stochastic product on the convex subset $\mathcal{D}(\mathcal{H})$.

Let us consider the Banach space $\mathsf{BL}(\mathcal{H})$ of *bounded bilinear maps* on $\mathcal{T}(\mathcal{H})$, endowed with its standard norm

$$\|(\cdot) \boxdot (\cdot)\|_{(1)} := \sup\{\|A \boxdot B\|_1 \mid A, B \in \mathcal{T}(\mathcal{H})\colon \|A\|_1, \|B\|_1 \le 1\}. \tag{16}$$

Noting that every stochastic map $(\cdot) \boxdot (\cdot)$ is *adjoint-preserving* — $A^* \boxdot B^* = (A \boxdot B)^*$ — hence, it can be restricted to the selfadjoint part $\mathcal{T}(\mathcal{H})_{\mathsf{s}}$ of $\mathcal{T}(\mathcal{H})$, we have:

Proposition 2 ([5])**.** *Every stochastic map $(\cdot) \boxdot (\cdot)\colon \mathcal{T}(\mathcal{H}) \times \mathcal{T}(\mathcal{H}) \to \mathcal{T}(\mathcal{H})$ is bounded and its norm satisfies $\|(\cdot) \boxdot (\cdot)\|_{(1)} \le 2$; while, its restriction $(\cdot) \boxminus (\cdot)$ to a bilinear map on $\mathcal{T}(\mathcal{H})_{\mathsf{s}}$ is such that $\|(\cdot) \boxminus (\cdot)\|_{(1)} = 1$. Thus, if the stochastic map is associative, then $(\mathcal{T}(\mathcal{H})_{\mathsf{s}}, (\cdot) \boxminus (\cdot))$ is a Banach algebra, because $\|A \boxminus B\|_1 \le \|A\|_1\, \|B\|_1$, $A, B \in \mathcal{T}(\mathcal{H})_{\mathsf{s}}$.*

3.2 Square integrable representations

Let $U\colon G \to \mathcal{U}(\mathcal{H})$ be an irreducible (in general, *projective* [12]) representation of the group G, acting in a separable complex Hilbert space $\mathcal{H}$. Here, $\mathcal{U}(\mathcal{H})$ denotes the unitary group of $\mathcal{H}$, and G is supposed to be a locally compact, second countable Hausdorff topological group (in short, a l.c.s.c. group). Given vectors $\psi, \phi \in \mathcal{H}$, and denoting by $\langle\cdot,\cdot\rangle$ the scalar product in $\mathcal{H}$ (linear, say, in its *second* argument), consider the associated *coefficient*

$$c_{\psi\phi}\colon G \ni g \mapsto \langle U(g)\,\psi, \phi\rangle \in \mathbb{C}, \tag{17}$$

which is a bounded Borel function. The coefficient functions allow us to define the set

$$\mathscr{A}(U) := \{\psi \in \mathcal{H} \mid \exists \phi \in \mathcal{H}\colon\ \phi \ne 0,\ c_{\psi\phi} \in \mathrm{L}^2(G, \mu_G; \mathbb{C})\} \tag{18}$$

of *U-admissible vectors* in $\mathcal{H}$. The representation U is called *square integrable* if $\mathscr{A}(U) \ne \{0\}$.

Assume that G is *unimodular* ($\Delta_G \equiv 1$, i.e., the Haar measure μ_G is both left and right invariant) and U is *square integrable*. In this case, it turns out that $\mathscr{A}(U) = \mathrm{L}^2(G, \mu_G; \mathbb{C})$, all coefficients of U are square integrable and the *orthogonality relations* hold, i.e.,

$$\int_G \mathrm{d}\mu_G(g)\ \langle \eta, U(g)\phi\rangle \langle U(g)\psi, \chi\rangle = \mathsf{c}_U\, \langle \eta, \chi\rangle \langle \psi, \phi\rangle, \quad \forall\, \eta, \chi, \psi, \phi \in \mathcal{H}, \tag{19}$$

where c_U is a (strictly) positive constant, depending on U and on the normalization of μ_G. It turns out that the orthogonality relations are more complicated in the non-unimodular case; see Theorem 1 below. Remarkable examples of square integrable representations of a unimodular group are the irreducible unitary representations of *compact* groups [9] and the *Weyl system* [13], i.e., an irreducible projective representation of the abelian group $\mathbb{R}^n \times \mathbb{R}^n$.

More generally, we have (see [14–19], and references therein):

Theorem 1. *Let $U\colon G \to \mathcal{U}(\mathcal{H})$ be square integrable. Then, the set $\mathscr{A}(U)$ of U-admissible vectors is a dense linear subspace of $\mathcal{H}$, that is stable under the action of U, and, for any pair of vectors $\phi \in \mathcal{H}$ and $\psi \in \mathscr{A}(U)$, the coefficient function $c_{\psi\phi}\colon G \to \mathbb{C}$ is square integrable wrt the left Haar measure μ_G. There exists a unique positive selfadjoint, injective linear operator D_U in $\mathcal{H}$ — the* Duflo-Moore operator *associated with U — such that*

$$\mathscr{A}(U) = \mathrm{dom}\big(D_U\big), \tag{20}$$

and, moreover, the following generalized orthogonality relations hold:

$$\int_G \overline{c_{\psi_1\phi_1}(g)}\; c_{\psi_2\phi_2}(g)\; \mathrm{d}\mu_G(g) = \langle \phi_1, \phi_2\rangle\, \langle D_U\, \psi_2, D_U\, \psi_1\rangle, \quad \forall\, \phi_1, \phi_2 \in \mathcal{H},\ \forall\, \psi_1, \psi_2 \in \mathscr{A}(U). \tag{21}$$

The Duflo-Moore operator D_U is bounded iff G is unimodular and, in such case, it is a multiple of the identity, i.e., $D_U = \mathsf{d}_U\, I$, for some $\mathsf{d}_U > 0$.

An important problem is that of classifying the square integrable representations of G. Assume that G is a *semidirect product* of an *abelian*, closed normal subgroup $\mathbb{A}$ by a closed subgroup H: $G = \mathbb{A} \rtimes H$. As is well known, by the *Mackey machine* [9, 12] (or 'little group method'), under mild hypotheses one can obtain all the irreducible unitary representations of G as *induced representations*. Next, by a classical result [15], one can achieve a complete classification of the *square integrable* unitary representations of G.

The class of semidirect products admitting square integrable unitary representations includes the following remarkable cases [19]:

- The (unimodular) *reduced Heisenberg-Weyl group* $\overline{\mathbb{H}}_n = \mathbb{H}_n/2\pi\mathbb{Z}$.
- The (non-unimodular) one-dimensional *affine groups* $\mathbb{R}\rtimes\mathbb{R}_*^+$ and $\mathbb{R}\rtimes\mathbb{R}_*$ — where $\mathbb{R}_*^+$ is the subgroup of *dilations*, acting on the group of translations on the real line, while $\mathbb{R}_*$ is the group of nonzero real numbers (dilations and the reflection wrt the origin of $\mathbb{R}$) — that are related to wavelet analysis. In particular, the affine group $G = \mathbb{R} \rtimes \mathbb{R}_*^+$ admits two (unitarily inequivalent) square integrable representations. These representations suitably generate the *continuous wavelet transform*.
- The (non-unimodular) *shearlet groups* — i.e., $\mathbb{R}^n \rtimes (\mathbb{R}^{n-1} \times \mathbb{R}_*^+)$ and $\mathbb{R}^n \rtimes (\mathbb{R}^{n-1} \times \mathbb{R}_*)$ — that give rise to the *shearlet transform*.
- The (non-unimodular) *similitude group* $\mathbb{R}^n \rtimes (\mathrm{SO}(n) \times \mathbb{R}_*^+)$.

On the other hand, the *euclidean* group $\mathbb{R}^n \rtimes \mathrm{SO}(n)$ and the (universal cover of the) *Poincaré* group $\mathbb{R}^4 \rtimes \mathrm{SL}(2;\mathbb{C})$ *do not* admit square integrable representations [19].

4 The quantum convolution as a covariant stochastic product

As anticipated, the main ingredients of our recipe for defining a class of covariant stochastic products are the following:

1. We suitably choose an irreducible *projective representation* $U\colon G \to \mathcal{U}(\mathcal{H})$ of a locally compact (i.e., l.c.s.c.) group G in a separable complex Hilbert space $\mathcal{H}$. For the sake of simplicity, we assume that G is *unimodular*, and U is supposed to be *square integrable*; thus, the orthogonality relations (19) hold (i.e., $\mathscr{A}(U) = \mathcal{H}$, and, for a suitable normalization of the Haar measure, $D_U = I$ in (21)). U gives rise to an *isometric representation* of G in the Banach space $\mathcal{T}(\mathcal{H})$:

$$\mathsf{S}_U(g)A := U(g)\, A\, U(g)^*, \quad A \in \mathcal{T}(\mathcal{H}). \tag{22}$$

This is a *symmetry action* of G on the space where the quantum states live (the trace class of $\mathcal{H}$). Although the representation U behaves, in general, *projectively* — i.e.,

$$U(gh) = \gamma(g,h)\, U(g)\, U(h), \tag{23}$$

with *multiplier* [12] $\gamma(g,h) \in \mathbb{T}$ — the symmetry action S_U behaves like an ordinary representation:

$$\mathsf{S}_U(gh) = \mathsf{S}_U(g)\; \mathsf{S}_U(h). \tag{24}$$

2. We pick a *fiducial density operator* υ in $\mathcal{D}(\mathcal{H})$ — the 'probe state' — and a *Borel probability measure* $\varpi\colon \mathscr{B}(G) \to \mathbb{R}^+$ on G, where $\mathscr{B}(G)$ is the Borel σ-algebra of G; in particular, the standard choice is the point mass measure $\varpi = \delta \equiv \delta_e$.

The stochastic product we are going to define — called *twirled product* or *quantum convolution* [5–8] — will therefore depend on the symmetry group G, like the classical convolution, but also on our (arbitrary) choice of the square integrable representation U, of the probe state υ and of the probability measure ϖ.

The next step consists in establishing two fundamental technical facts.

Proposition 3 ([5])**.** *For every Borel probability measure μ on G, the mapping*

$$\mathcal{T}(\mathcal{H}) \ni A \mapsto \int_G \mathrm{d}\mu(g)\; (\mathsf{S}_U(g)\, A) =: \mu[U]\, A \in \mathcal{T}(\mathcal{H}) \tag{25}$$

defines a stochastic map $\mu[U]$; in particular, $\mu[U]\,\mathcal{D}(\mathcal{H}) \subset \mathcal{D}(\mathcal{H})$.

Proposition 4 ([5])**.** *For a suitable normalization of the Haar measure μ_G — determined by setting $D_U = I$ in the orthogonality relations (21) — and for every density operator $\rho \in \mathcal{D}(\mathcal{H})$, the mapping*

$$\nu_{\rho,\upsilon}\colon \mathscr{B}(G) \ni \mathcal{E} \mapsto \int_{\mathcal{E}} \mathrm{d}\mu_G(g)\; \mathrm{tr}\big(\rho\, (\mathsf{S}_U(g)\,\upsilon)\big) \in [0,1] \tag{26}$$

is a Borel probability measure on G.

Taking into account these facts, we can define a *binary operation* in $\mathcal{D}(\mathcal{H})$ as follows:

$$(\rho,\sigma) \mapsto \rho \overset{\upsilon}{\underset{\varpi}{\odot}} \sigma := \underbrace{\big(\overbrace{(\nu_{\rho,\upsilon} \circledast \varpi)}^{\text{convolution}}[U]\big)}_{\text{stochastic map } \mu[U]} \sigma = \int_G \mathrm{d}(\nu_{\rho,\upsilon} \circledast \varpi)(g)\; (\mathsf{S}_U(g)\sigma). \tag{27}$$

Here, note that, by Proposition 4 (which relies on the *square-integrability* of the projective representation U), $\nu_{\rho,\upsilon} \circledast \varpi$ is a probability measure; hence, by Proposition 3, $\mu[U] \equiv (\nu_{\rho,\upsilon} \circledast \varpi)[U]$ is a stochastic map; explicitly:

$$\rho \overset{\upsilon}{\underset{\varpi}{\odot}} \sigma = \int_G \mathrm{d}\mu_G(g) \int_G \mathrm{d}\varpi(h)\; \mathrm{tr}\big(\rho\, (\mathsf{S}_U(g)\,\upsilon)\big)\, (\mathsf{S}_U(gh)\,\sigma). \tag{28}$$

For the sake of notational conciseness, in the following result we set $\rho_g \equiv \mathsf{S}_U(g)\,\rho = U(g)\,\rho\, U(g)^*$.

Theorem 2 ([5])**.** *The binary operation*

$$(\,\cdot\,) \overset{\upsilon}{\underset{\varpi}{\odot}} (\,\cdot\,)\colon \mathcal{D}(\mathcal{H}) \times \mathcal{D}(\mathcal{H}) \to \mathcal{D}(\mathcal{H}) \quad \text{(the quantum convolution induced by the triple } (U,\upsilon,\varpi)) \tag{29}$$

is an associative stochastic product, left-covariant wrt the isometric representation S_U, i.e.,

$$\rho_g \overset{\upsilon}{\underset{\varpi}{\odot}} \sigma = \Big(\rho \overset{\upsilon}{\underset{\varpi}{\odot}} \sigma\Big)_g . \quad \text{(compare with (5))} \tag{30}$$

In the case where the locally compact group G is abelian, the quantum convolution product — like the convolution of probability measures — is commutative.

Two further properties (of families) of quantum convolutions are called *invariance* and *equivariance* [5]. Let X be a G-*space* under some group *action* [12]

$$(\,\cdot\,)[\,\cdot\,]\colon G \times X \ni (g,x) \mapsto g[x] \in X. \tag{31}$$

Suppose that the points of X label a *family* of stochastic products:

$$\Big\{ (\,\cdot\,) \overset{x}{\odot} (\,\cdot\,)\colon \mathcal{D}(\mathcal{H}) \times \mathcal{D}(\mathcal{H}) \to \mathcal{D}(\mathcal{H}) \,\Big|\, x \in X \Big\}. \tag{32}$$

We say that this family of products is *invariant* wrt the action $(\,\cdot\,)[\,\cdot\,]$ if

$$\rho \overset{x}{\odot} \sigma = \rho \overset{g[x]}{\odot} \sigma, \quad \forall\, x \in X,\ \forall\, g \in G,\ \forall\, \rho,\sigma \in \mathcal{D}(\mathcal{H}). \tag{33}$$

We say that the family of stochastic products (32) is *right inner equivariant* wrt the pair $((\,\cdot\,)[\,\cdot\,], \mathsf{S}_U)$ if

$$\rho \overset{x}{\odot} (\mathsf{S}_U(g^{-1})\,\sigma) = \rho \overset{g[x]}{\odot} \sigma, \quad \forall\, x \in X,\ \forall\, g \in G,\ \forall\, \rho,\sigma \in \mathcal{D}(\mathcal{H}). \tag{34}$$

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Proposition 5 ([5]). *The family of quantum convolutions*

$$\Big\{(\cdot)\underset{\varpi}{\overset{\upsilon}{\odot}}(\cdot)\ \Big|\ \upsilon\in\mathcal{D}(\mathcal{H}),\ \varpi\in\mathscr{P}(G)\Big\} \tag{35}$$

is invariant wrt the group action

$$g[(\upsilon,\varpi)] := (\upsilon_g \equiv \mathsf{S}_U(g)\upsilon, \varpi^g), \quad \varpi^g(\mathcal{E}) := \varpi(g^{-1}\mathcal{E}); \tag{36}$$

namely,

$$\rho \underset{\varpi}{\overset{\upsilon}{\odot}} \sigma = \rho \underset{\varpi^g}{\overset{\upsilon_g}{\odot}} \sigma. \tag{37}$$

It is right inner equivariant wrt the pair $((\cdot)[\,\cdot\,], \mathsf{S}_U)$, *where* $(\cdot)[\,\cdot\,]$ *is the group action*

$$g[(\upsilon,\varpi)] := (\upsilon, \varpi_g), \quad \varpi_g(\mathcal{E}) := \varpi(\mathcal{E}g); \tag{38}$$

namely,

$$\rho \underset{\varpi}{\overset{\upsilon}{\odot}} \sigma_{g^{-1}} = \rho \underset{\varpi_g}{\overset{\upsilon}{\odot}} \sigma. \tag{39}$$

5 Adopting Eddington's principle: the quantum convolution and quantum measurement

In the spirit of Eddington's Principle of Identification, we now provide an interpretation of the quantum convolution product within quantum measurement theory. Recall that a quantum measurement can be described mathematically — at two different levels of detail — by means of a *positive operator-valued measure* (or quantum observable), or, in a more complete formulation, by a (quantum) *instrument* [20].

A *positive operator-valued measure* (in short, a POVM) is an effect-valued map of the form

$$\mathcal{A} \ni \mathcal{E} \mapsto \mathsf{E}(\mathcal{E}) \in \mathcal{B}(\mathcal{H}), \quad 0 \le \mathsf{E}(\mathcal{E}) \le I, \tag{40}$$

where $\mathcal{A}$ is a σ-algebra of subsets of a certain set S (the *sample space*) and I is the identity operator, such that

1. $\mathsf{E}(\varnothing) = 0$;
2. $\mathsf{E}(S) = I$;
3. $\mathsf{E}(\cup_k \mathcal{E}_k) = \sum_k \mathsf{E}(\mathcal{E}_k)$ (in the weak sense), for any sequence $\{\mathcal{E}_k\}$ of *mutually disjoint subsets* in $\mathcal{A}$.

Therefore, for every state $\rho \in \mathcal{D}(\mathcal{H})$, the mapping $\mathcal{A} \ni \mathcal{E} \mapsto \mathrm{tr}(\rho\,\mathsf{E}(\mathcal{E})) \in [0,1]$, induced by the POVM, is a probability measure on the sample space S, i.e., the measurement outcome statistics when measuring the state ρ.

A *quantum instrument* is a map $\mathcal{A} \ni \mathcal{E} \mapsto \mathcal{I}_{\mathcal{E}}$, where $\mathcal{I}_{\mathcal{E}}\colon \mathcal{T}(\mathcal{H}) \to \mathcal{T}(\mathcal{H})$ is a *quantum operation* [20], such that, for every state $\rho \in \mathcal{D}(\mathcal{H})$,

$$\mathcal{I}_{\varnothing}(\rho) = 0, \quad \mathrm{tr}(\mathcal{I}_S(\rho)) = 1, \quad \mathrm{tr}\big(\mathcal{I}_{\cup_k \mathcal{E}_k}(\rho)\big) = \textstyle\sum_k \mathrm{tr}\big(\mathcal{I}_{\mathcal{E}_k}(\rho)\big). \tag{41}$$

The quantum instrument $\mathcal{E} \mapsto \mathcal{I}_{\mathcal{E}}$ determines a *unique* POVM $\mathcal{E} \mapsto \mathsf{E}(\mathcal{E})$ via $\mathrm{tr}(\rho\,\mathsf{E}(\mathcal{E})) = \mathrm{tr}(\mathcal{I}_{\mathcal{E}}(\rho))$ [20] (but, in general, more instruments induce the same POVM). Thus, it describes both the outcome statistics $\mathcal{E} \mapsto \mathrm{tr}(\mathcal{I}_{\mathcal{E}}(\rho))$ and the corresponding post-measurement state $\mathrm{tr}(\mathcal{I}_{\mathcal{E}}(\rho))^{-1}\mathcal{I}_{\mathcal{E}}(\rho)$ (for $\mathrm{tr}(\mathcal{I}_{\mathcal{E}}(\rho)) \neq 0$).

In particular, with $S = G$ (i.e., in the case where the sample space is a l.c.s.c. group) and $\mathcal{A} = \mathscr{B}(G)$, for every probe state $\upsilon \in \mathcal{D}(\mathcal{H})$, the mapping

$$\mathscr{B}(G) \ni \mathcal{E} \mapsto \int_{\mathcal{E}} \mathrm{d}\mu_G(g)\ (\mathsf{S}_U(g)\upsilon) =: \mathsf{E}_\upsilon(\mathcal{E}) \quad \text{(where } U\colon G \to \mathcal{U}(\mathcal{H}) \text{ is square integrable)} \tag{42}$$

is a *U-covariant quantum observable*; i.e., a POVM covariant wrt the irreducible representation U:

$$\mathsf{E}_\upsilon(g\mathcal{E}) = U(g)\,\mathsf{E}_\upsilon(\mathcal{E})\,U(g)^*, \quad \forall\, g \in G,\ \forall\, \mathcal{E} \in \mathscr{B}(G). \tag{43}$$

Formula (42) gives the *general expression* of such a POVM [5]. For every $\rho \in \mathcal{D}(\mathcal{H})$, the *positive function*

$$G \ni g \mapsto \mathsf{p}_{\rho,\upsilon}(g) = \mathrm{tr}(\rho\,(\mathsf{S}_U(g)\upsilon)), \tag{44}$$

previously used in the construction of the quantum convolution, is the *probability density,* on the sample space G, of the quantum observable E_υ, when measuring ρ. Namely, it is the Radon-Nikodym derivative of the probability measure $\nu_{\rho,\upsilon}\colon \mathscr{B}(G)\ni\mathcal{E}\mapsto\int_{\mathcal{E}} \mathrm{d}\mu_G(g)\, \mathrm{tr}\big(\rho\,(\mathsf{S}_U(g)\upsilon)\big) = \mathrm{tr}(\rho\,\mathsf{E}_\upsilon(\mathcal{E}))$ wrt μ_G. The mapping

$$\mathscr{B}(G)\ni\mathcal{E}\mapsto\left(\mathcal{T}(\mathcal{H})\ni A\mapsto\int_{\mathcal{E}} \mathrm{d}\mu_G(g)\,\mathrm{tr}\big(A\,(\mathsf{S}_U(g)\upsilon)\big)\,(\mathsf{S}_U(g)\sigma) =: \mathcal{I}_{\mathcal{E}}^{\upsilon,\sigma}(A)\right) \tag{45}$$

is a *quantum instrument* (here, the linear map $\mathcal{I}_{\mathcal{E}}^{\upsilon,\sigma}\colon \mathcal{T}(\mathcal{H})\to\mathcal{T}(\mathcal{H})$ is a *quantum operation*; in particular, a *quantum channel* [20] for $\mathcal{E}=G$). Specifically, it is a U-*covariant* quantum instrument based on G:

$$\mathcal{I}_{g\mathcal{E}}^{\upsilon,\sigma}(\mathsf{S}_U(g)A) = \mathsf{S}_U(g)\big(\mathcal{I}_{\mathcal{E}}^{\upsilon,\sigma}A\big), \quad \forall\, g\in G,\ \forall\,\mathcal{E}\in\mathscr{B}(G),\ \forall\, A\in\mathcal{T}(\mathcal{H}). \tag{46}$$

Clearly, the U-covariant POVM and instrument are related by the relation $\mathrm{tr}(A\,\mathsf{E}_\upsilon(\mathcal{E})) = \mathrm{tr}(\mathcal{I}_{\mathcal{E}}^{\upsilon,\sigma}(A))$.

A connection with the quantum convolution product associated with the triple $(U,\upsilon,\delta\equiv\delta_e)$ is established by setting $\mathcal{E}=G$; i.e.,

$$\rho\overset{\upsilon}{\odot}\sigma \equiv \rho\underset{\delta}{\overset{\upsilon}{\odot}}\sigma = \mathcal{I}_G^{\upsilon,\sigma}\rho. \tag{47}$$

From relation (46), since $gG=G$, we recover the *left-covariance* property (30) of the quantum convolution product; moreover, the *associativity* of this product translates into the following relation involving the *composition* of the quantum channels $\mathcal{I}_G^{\upsilon,\rho}$ and $\mathcal{I}_G^{\upsilon,\sigma}$:

$$\mathcal{I}_G^{\upsilon,\sigma}\circ\mathcal{I}_G^{\upsilon,\rho} = \mathcal{I}_G^{\upsilon,\tau}, \quad \text{where } \tau=\rho\overset{\upsilon}{\odot}\sigma. \tag{48}$$

For the sake of simplicity, here we have considered the case of the Dirac measure $\varpi=\delta\equiv\delta_e$. What happens in the case where $\varpi\neq\delta$? The quantum observable E_υ is replaced with the *smeared observable* $\mathsf{E}_{\upsilon|\varpi} := \mathsf{E}_\upsilon\circledast\varpi$, namely,

$$\mathsf{E}_{\upsilon|\varpi}(\mathcal{E}) = \int_G \mathrm{d}\varpi(h)\;\mathsf{E}_\upsilon(\mathcal{E}h^{-1}) = \int_G \mathrm{d}\varpi(h)\int_{\mathcal{E}}\mathrm{d}\mu_G(g)\;(\mathsf{S}_U(gh^{-1})\upsilon). \tag{49}$$

Accordingly, the probability density $\mathsf{p}_{\rho,\upsilon}(g) = \mathrm{tr}(\rho\,(\mathsf{S}_U(g)\upsilon))$ is replaced with the convolution of the function $\mathsf{p}_{\rho,\upsilon}$ with the measure ϖ, i.e., with the probability distribution

$$\mathsf{p}_{\rho,\upsilon|\varpi}(g) = \big(\mathsf{p}_{\rho,\upsilon}\circledast\varpi\big)(g) = \int_G \mathrm{d}\varpi(h)\;\mathsf{p}_{\rho,\upsilon}(gh^{-1}). \tag{50}$$

An analogous smearing occurs to any covariant quantum instrument inducing the covariant POVM E_υ [8].

6 Quantum convolutions for every Hilbert space dimension

By means of examples, we will now argue that the quantum convolution can be defined in every Hilbert space dimension. To this end, suppose, at first, that the group G is *compact* and $U\colon G\to\mathcal{U}(\mathcal{H})$ is any *irreducible unitary representation*; thus, in this case, $\mu_G(G)<\infty$, $\mathsf{N}=\dim(\mathcal{H})<\infty$, $\gamma\equiv 1$ and U is *automatically square integrable*. Assuming that μ_G is a *probability measure* — i.e., $\mu_G(G)=1$ — by the *Peter-Weyl theorem* [9], $\mathsf{c}_U=\dim(\mathcal{H})^{-1}=\mathsf{N}^{-1}$ in (19); hence:

$$\rho\underset{\varpi}{\overset{\upsilon}{\odot}}\sigma = \mathsf{N}\int_G \mathrm{d}\mu_G(g)\int_G \mathrm{d}\varpi(h)\;\mathrm{tr}\big(\rho\,(\mathsf{S}_U(g)\upsilon)\big)\,(\mathsf{S}_U(gh)\sigma). \tag{51}$$

Exploiting, e.g., the irreducible unitary representations of SU(2), one can construct quantum convolutions for every finite Hilbert space dimension N. For the *maximally mixed state* $\Omega := \mathsf{N}^{-1}I$, we have [5]:

$$\Omega\underset{\varpi}{\overset{\upsilon}{\odot}}\sigma=\Omega, \quad \rho\underset{\varpi}{\overset{\upsilon}{\odot}}\Omega=\Omega, \quad \rho\underset{\varpi}{\overset{\Omega}{\odot}}\sigma=\Omega, \quad \forall\,\rho,\upsilon,\sigma\in\mathcal{D}(\mathcal{H}),\ \forall\,\varpi\in\mathscr{P}(G). \tag{52}$$

According to the first two relations, the quantum channel obtained by fixing one of the two arguments of the quantum convolution is *unital*; hence, by Lemma 2 of [21], this operation is *entropy-nondecreasing*, i.e., the entropy of the product is not smaller of the entropy of the two factors. The third relation shows that choosing Ω as the probe state *trivializes* the product, i.e., the only output state is Ω itself; similarly, setting $\varpi=\mu_G$,

$$\rho\underset{\mu_G}{\overset{\upsilon}{\odot}}\sigma=\Omega, \quad \forall\,\rho,\upsilon,\sigma\in\mathcal{D}(\mathcal{H}). \tag{53}$$

Switching to an infinite-dimensional example, let now G be the *group of translations on phase space* — $G = \mathbb{R}^n \times \mathbb{R}^n$ — $\mathcal{H} = \mathrm{L}^2(\mathbb{R}^n)$ and U the *Weyl system* (with Planck's constant $\hbar \equiv 1$):

$$\big(U(q,p)f\big)(x) = \mathrm{e}^{-\mathrm{i}\,q\cdot p/2}\,\mathrm{e}^{\mathrm{i}\,p\cdot x} f(x-q), \quad (q,p)\in\mathbb{R}^n\times\mathbb{R}^n; \tag{54}$$

namely, $U(q,p) = \mathrm{e}^{-\mathrm{i}\,q\cdot p/2}\,\mathrm{e}^{\mathrm{i}\,p\cdot\hat{q}}\,\mathrm{e}^{-\mathrm{i}\,q\cdot\hat{p}}$ (with $\hat{q}$, $\hat{p}$ denoting *position* and *momentum* operators in $\mathrm{L}^2(\mathbb{R}^n)$), which is an irreducible projective representation, with multiplier $\gamma(q,p\,;\tilde{q},\tilde{p}) = \exp(\mathrm{i}(q\cdot\tilde{p} - p\cdot\tilde{q})/2)$ [13]. The Weyl system is a square integrable representation and, setting $\mathrm{L}^2(G) = \mathrm{L}^2(\mathbb{R}^n\times\mathbb{R}^n, (2\pi)^{-n}\mathrm{d}^n q\,\mathrm{d}^n p;\mathbb{C})$, we have that $\mathsf{c}_U = 1$ in the unimodular orthogonality relations (19) (i.e., $D_U = I$ in (21)). Therefore, in this case, the quantum convolution product associated with the triple (U,υ,ϖ) is of the form

$$\begin{aligned}\tau = \rho \overset{\upsilon}{\underset{\varpi}{\odot}} \sigma = (2\pi)^{-n}\!\int \mathrm{d}^n q\,\mathrm{d}^n p\;\; &\mathrm{tr}\big(\rho\,\big(\mathrm{e}^{\mathrm{i}\,p\cdot\hat{q}}\,\mathrm{e}^{-\mathrm{i}\,q\cdot\hat{p}}\;\upsilon\;\mathrm{e}^{\mathrm{i}\,q\cdot\hat{p}}\,\mathrm{e}^{-\mathrm{i}\,p\cdot\hat{q}}\big)\big)\\ &\times\int \mathrm{d}\varpi(\tilde{q},\tilde{p})\;\big(\mathrm{e}^{\mathrm{i}\,(p+\tilde{p})\cdot\hat{q}}\,\mathrm{e}^{-\mathrm{i}\,(q+\tilde{q})\cdot\hat{p}}\;\sigma\;\mathrm{e}^{\mathrm{i}\,(q+\tilde{q})\cdot\hat{p}}\,\mathrm{e}^{-\mathrm{i}\,(p+\tilde{p})\cdot\hat{q}}\big).\end{aligned} \tag{55}$$

This stochastic product — the *phase-space quantum convolution product* — is *commutative*, since it stems from a representation of an *abelian* group. For $\varpi = \delta \equiv \delta_e$, we get the *standard quantum convolution*:

$$\tau = \rho \overset{\upsilon}{\odot} \sigma = (2\pi)^{-n}\!\int\mathrm{d}^n q\,\mathrm{d}^n p\;\;\mathrm{tr}\big(\rho\,\big(\mathrm{e}^{\mathrm{i}\,p\cdot\hat{q}}\,\mathrm{e}^{-\mathrm{i}\,q\cdot\hat{p}}\,\upsilon\,\mathrm{e}^{\mathrm{i}\,q\cdot\hat{p}}\,\mathrm{e}^{-\mathrm{i}\,p\cdot\hat{q}}\big)\big)\,\big(\mathrm{e}^{\mathrm{i}\,p\cdot\hat{q}}\,\mathrm{e}^{-\mathrm{i}\,q\cdot\hat{p}}\,\sigma\,\mathrm{e}^{\mathrm{i}\,q\cdot\hat{p}}\,\mathrm{e}^{-\mathrm{i}\,p\cdot\hat{q}}\big). \tag{56}$$

To justify this term and highlight the commutativity of this stochastic product, in the next subsection we will show how this operation looks like when expressed in terms of the *Wigner distributions* [22–24] $\mathcal{W}_\rho$, $\mathcal{W}_\upsilon$, $\mathcal{W}_\sigma$ and $\mathcal{W}_\tau$ of the states ρ, υ, σ and τ.

7 The quantum convolution and group-covariant symbols of operators

Let $\mathcal{S}(\mathcal{H})$ be the Hilbert space of *Hilbert-Schmidt operators* on $\mathcal{H}$, and let $U\colon G\to\mathcal{U}(\mathcal{H})$ be a *square integrable* projective representation. We can define a *linear isometry* $\mathfrak{D}\colon\mathcal{S}(\mathcal{H})\to\mathrm{L}^2(G)\equiv\mathrm{L}^2(G,\mu_G;\mathbb{C})$ — that may be thought of as a *dequantization map* [19,23,24] — by putting

$$\breve{A}(g) \equiv (\mathfrak{D}\,A)(g) = \mathrm{tr}(U(g)^* A), \quad \forall\, A\in\mathcal{T}(\mathcal{H})\subset\mathcal{S}(\mathcal{H}). \tag{57}$$

We are assuming that G is *unimodular* — otherwise this definition would involve the Duflo-Moore operator too — and using the fact that $\mathcal{T}(\mathcal{H})$ is a $\|\cdot\|_{\mathrm{HS}}$-*dense* subspace of $\mathcal{S}(\mathcal{H})$. The function $\breve{A}\in\mathrm{L}^2(G)$ is called the *generalized Wigner transform* or *group-covariant symbol* of the operator A, which can be easily reconstructed back from its symbol $\breve{A}$ [24]. In the case where G is the group of *phase-space translations* and U is the *Weyl system*, and for $A\equiv\rho\in\mathcal{D}(\mathcal{H})$, $\breve{A}\equiv\breve{\rho}$ is also called the *quantum characteristic function* of ρ, since $\breve{\rho}\in\mathrm{L}^2(\mathbb{R}^n\times\mathbb{R}^n)$ is the *Fourier-Plancherel transform* of the Wigner distribution $\mathcal{W}_\rho$ of ρ [24].

7.1 The abelian case

We will henceforth suppose that G is an *abelian* l.c.s.c. group, and, as above, denote by $\widehat{G}$ the Pontryagin dual group of G. We say that G is *selfdual* if $G\simeq\widehat{G}$ (isomorphism of topological groups). It is useful, at this point, to introduce the following function $\gamma_\diamond\colon G\times G\to\mathbb{T}$ associated with the *multiplier* γ of the projective representation U:

$$\gamma_\diamond(g,h) := \gamma(g,h)\,\overline{\gamma(h,g)}. \tag{58}$$

We have that

$$\gamma_\diamond(g,h) = \overline{\gamma_\diamond(h,g)}, \quad \gamma_\diamond(g,g^{-1}) = 1. \tag{59}$$

Note that, if the multiplier γ is *trivial* — i.e., of the form

$$\gamma(g,h) = \beta(g)\,\beta(h)\,\overline{\beta(gh)} = \gamma(h,g), \quad (G\text{ being abelian}) \tag{60}$$

for some Borel function $\beta\colon G\to\mathbb{T}$ — then $\gamma_\diamond\equiv 1$, and, accordingly, in general the function $\gamma_\diamond$ depends on the equivalence (or similarity) class [12] of γ only. Let us recall the following fact [25]:

Proposition 6. *The function* $\gamma_\diamond\colon G\times G\to\mathbb{T}$ *is a (continuous) skew-symmetric bicharacter; i.e., in addition to the the first of relations* (59)*, we also have that*

$$\gamma_\diamond(gh,\tilde{g}) = \gamma_\diamond(g,\tilde{g})\,\gamma_\diamond(h,\tilde{g}), \quad \forall\, g,h,\tilde{g}\in G. \tag{61}$$

As a consequence of Proposition 6, we can next define the continuous *homomorphism*

$$\mathsf{h}_\gamma \colon G \ni g \mapsto \gamma_\diamond(\,\cdot\,, g) \in \widehat{G}. \tag{62}$$

We say that the multiplier γ is *nondegenerate* if the map h_γ is *injective*.

Proposition 7 ([25]). *The multiplier γ is nondegenerate iff $\mathsf{h}_\gamma(G)$ is a* dense *subgroup of $\widehat{G}$.*

Definition 2. A nondegenerate multiplier γ is called *regular* in the case where $\mathsf{h}_\gamma(G) = \widehat{G}$ — i.e., if h_γ is an isomorphism of topological groups — so that G is selfdual.

Given a complex Radon measure ν on G — recall that $\mathcal{M}(G)$ is the Banach space of all such measures — we can define its *Fourier-Stieltjes transform* $\widehat{\nu} \colon \widehat{G} \to \mathbb{C}$ (the classical characteristic function of ν), i.e.,

$$\widehat{\nu}(\widehat{g}) := \int_G \mathrm{d}\nu(h)\, \overline{[\![h, \widehat{g}]\!]}, \quad \text{where we have put } [\![h, \widehat{g}]\!] \equiv \widehat{g}(h), \text{ for } h \in G,\ \widehat{g} \in \widehat{G} \text{ (pairing)}. \tag{63}$$

We can then introduce the (classical) *symbol* of $\nu \in \mathcal{M}(G)$ associated with γ as the bounded continuous function $\breve{\nu} \equiv \breve{\nu}_\gamma \colon G \to \mathbb{C}$ defined by

$$\breve{\nu}(g) \equiv \breve{\nu}_\gamma(g) := \widehat{\nu} \circ \mathsf{h}_\gamma(g) = \int_G \mathrm{d}\nu(h)\, \overline{[\![h, \mathsf{h}_\gamma(g)]\!]} = \int_G \mathrm{d}\nu(h)\, \overline{\gamma_\diamond(h, g)}. \tag{64}$$

Proposition 8 ([25]). *The mapping $\mathcal{M}(G) \ni \nu \mapsto \breve{\nu} \equiv \breve{\nu}_\gamma$ is injective if the multiplier γ is nondegenerate.*

Theorem 3 ([25]). *Let G be* selfdual*, and let the multiplier γ of U be* regular*. For every triple of density operators $\rho, \upsilon, \sigma \in \mathcal{D}(\mathcal{H})$ and every probability measure ϖ on G, the symbol of the quantum convolution product of ρ by σ* (induced by the triple (U, υ, ϖ)) *is of the form*

$$\Big(\rho \underset{\varpi}{\overset{\upsilon}{\odot}} \sigma\Big)^{\smile}(g) = \overline{\gamma(g, g^{-1})}\, \breve{\varpi}(g)\, \breve{\rho}(g)\, \breve{\upsilon}(g^{-1})\, \breve{\sigma}(g) = \breve{\varpi}(g)\, \breve{\rho}(g)\, \overline{\breve{\upsilon}(g)}\, \breve{\sigma}(g). \tag{65}$$

7.2 The phase-space translation group

As a concrete example, consider the vector group $G = \mathbb{R}^n \times \mathbb{R}^n$ and the Weyl system $(q,p) \mapsto U(q,p)$. The multiplier of U is of the form $\gamma(q,p;\tilde{q},\tilde{p}) = \exp(\mathrm{i}(q\cdot\tilde{p} - p\cdot\tilde{q})/2)$ so that $\gamma_\diamond(q,p;\tilde{q},\tilde{p}) = \exp(\mathrm{i}(q\cdot\tilde{p} - p\cdot\tilde{q}))$. Note that γ is a regular multiplier because in this case we can identify G with its dual group $\widehat{G}$ via the *symplectic pairing*

$$G \times \widehat{G} \equiv G \times G \ni (q,p;\tilde{q},\tilde{p}) \mapsto [\![q,p;\tilde{q},\tilde{p}]\!] = \gamma_\diamond(q,p;\tilde{q},\tilde{p}) \in \mathbb{T}. \tag{66}$$

For every probability measure ϖ on G, we can then identify its Fourier-Stieltjes transform $\widehat{\varpi} \colon \widehat{G} \to \mathbb{C}$ — the characteristic function of ϖ — with the function

$$\breve{\varpi}(q,p) := \int_{\mathbb{R}^n\times\mathbb{R}^n} \mathrm{d}\varpi(\tilde{q},\tilde{p})\ \exp(\mathrm{i}(q\cdot\tilde{p} - p\cdot\tilde{q})). \tag{67}$$

By Theorem 3, the quantum convolution — expressed in terms of $\breve{\varpi}$ and of the quantum characteristic functions $\breve{\rho} := \mathrm{tr}(U(q,p)^*\rho)$, $\breve{\upsilon}$, $\breve{\sigma}$ of ρ, υ, σ — has the *explicitly commutative* form of a pointwise product:

$$\Big(\rho \underset{\varpi}{\overset{\upsilon}{\odot}} \sigma\Big)^{\smile}(q,p) = \breve{\varpi}(q,p)\, \breve{\rho}(q,p)\, \overline{\breve{\upsilon}(q,p)}\, \breve{\sigma}(q,p). \quad \text{(the weighted pointwise product of } \breve{\rho} \text{ by } \breve{\sigma}) \tag{68}$$

Let us now express the binary operation (55) in terms of the *Wigner distributions* $\mathcal{W}_\rho$, $\mathcal{W}_\upsilon$, $\mathcal{W}_\sigma$, $\mathcal{W}_\tau$. With $\varpi = \delta$ ($\breve{\varpi} \equiv 1$) — setting $\check{\mathcal{W}}_\upsilon(x) := \mathcal{W}_\upsilon(-x)$, $x \equiv (q,p)$ — we find the following expression [5–8,25]:

$$\mathcal{W}_\tau(z) = \int_{\mathbb{R}^{2n}} \mathrm{d}^{2n}x \left(\int_{\mathbb{R}^{2n}} \mathrm{d}^{2n}y\ \mathcal{W}_\rho(y)\, \check{\mathcal{W}}_\upsilon(x-y)\right) \mathcal{W}_\sigma(z-x). \tag{69}$$

Thus, the Wigner distribution $\mathcal{W}_\tau$ of the standard quantum convolution of ρ by σ can be obtained via a *double* convolution of Wigner functions, with the Wigner distribution $\mathcal{W}_\upsilon$ of the probe state υ playing a *key role* (see section 8). The function

$$(q,p) \mapsto \int_{\mathbb{R}^{2n}} \mathrm{d}^n\tilde{q}\, \mathrm{d}^n\tilde{p}\ \ \mathcal{W}_\rho(\tilde{q},\tilde{p})\, \check{\mathcal{W}}_\upsilon(q-\tilde{q}, p-\tilde{p}), \tag{70}$$

which is involved in the iterated convolution, is a *probability distribution* wrt the Lebesgue measure on $\mathbb{R}^n \times \mathbb{R}^n$. E.g., for $n = 1$, and choosing the pure state $|\psi\rangle\langle\psi|$ — with $\psi(q) = (2\pi)^{-1/4}\, \mathrm{e}^{-q^2/4}$ — as the probe state υ, whose Wigner function is of the form

$$\mathcal{W}_\upsilon(q,p) \equiv \mathcal{W}_\psi(q,p) = \pi^{-1}\, \mathrm{e}^{-(q^2+p^2)}, \tag{71}$$

we obtain the following probability distribution (the so-called *Husimi-Kano function* [22]):

$$\mathcal{K}_\rho(q,p) = \pi^{-1} \int_{\mathbb{R}^2} \mathrm{d}\tilde{q}\, \mathrm{d}\tilde{p}\;\; \mathcal{W}_\rho(\tilde{q},\tilde{p})\; \mathrm{e}^{-(q-\tilde{q})^2-(p-\tilde{p})^2}. \tag{72}$$

Hence, in this case, the Wigner function of the quantum convolution of ρ by σ is of the form

$$\mathcal{W}_\tau(q,p) = \int_{\mathbb{R}^2} \mathrm{d}\tilde{q}\, \mathrm{d}\tilde{p}\;\; \mathcal{K}_\rho(\tilde{q},\tilde{p})\, \mathcal{W}_\sigma(q-\tilde{q}, p-\tilde{p}). \tag{73}$$

8 Epilogue: concluding remarks

At the end of the day, we come to the interesting conclusion that the weighted pointwise product (68) may be thought of as a *direct way* for defining a *commutative stochastic product*. Indeed, the pointwise product of two quantum characteristic functions is *not*, in general, a function of the same kind [10, 11]. *Instead*, the pointwise product of a *classical* characteristic function by a *quantum* characteristic function, is a function of the latter type. Moreover, for every pair of states $\rho, \upsilon \in \mathcal{D}(\mathcal{H})$, the pointwise product $\breve{\rho}\,\overline{\breve{\upsilon}}$ is a *classical* characteristic function; namely, the (symplectic) Fourier-Stieltjes transform of the probability measure $\nu_{\rho,\upsilon}$ on $\mathbb{R}^n \times \mathbb{R}^n$, with $\mathrm{d}\nu_{\rho,\upsilon}(q,p) = (2\pi)^{-n}\, \mathrm{tr}\big(\rho\,(\mathsf{S}_U(q,p)\,\upsilon)\big)\, \mathrm{d}^n q\, \mathrm{d}^n p$, where U is the Weyl system. Explicitly, considering the *convolution* $\mathcal{W}_\rho \circledast \check{\mathcal{W}}_\upsilon$ of $\mathcal{W}_\rho$ with $\check{\mathcal{W}}_\upsilon$, where $\check{\mathcal{W}}_\upsilon(q,p) := \mathcal{W}_\upsilon(-q,-p)$, we have:

$$\begin{aligned}\big(\breve{\rho}\,\overline{\breve{\upsilon}}\big)(q,p) &= \int \frac{\mathrm{d}^n\tilde{q}\, \mathrm{d}^n\tilde{p}}{(2\pi)^n}\; \mathrm{tr}\big(\rho\,\big(\mathrm{e}^{\mathrm{i}\tilde{p}\cdot\hat{q}}\, \mathrm{e}^{-\mathrm{i}\tilde{q}\cdot\hat{p}}\, \upsilon\, \mathrm{e}^{\mathrm{i}\tilde{q}\cdot\hat{p}}\, \mathrm{e}^{-\mathrm{i}\tilde{p}\cdot\hat{q}}\big)\big)\, \exp(\mathrm{i}(q\cdot\tilde{p} - p\cdot\tilde{q})) \\ &= \int \mathrm{d}^n\tilde{q}\, \mathrm{d}^n\tilde{p}\;\; \big(\mathcal{W}_\rho \circledast \check{\mathcal{W}}_\upsilon\big)(\tilde{q},\tilde{p})\, \exp(\mathrm{i}(q\cdot\tilde{p} - p\cdot\tilde{q})).\end{aligned} \tag{74}$$

In fact, classical and quantum characteristic functions belong to (the even and odd parts of) a $\mathbb{Z}_2$-*graded algebra*, and the map

$$\Big(\overbrace{\;\breve{\rho}\;}^{\text{quantum}}, \overbrace{\;\breve{\sigma}\;}^{\text{quantum}}\Big) \longmapsto \Big(\overbrace{\;\breve{\varpi}\,\breve{\rho}\,\overline{\breve{\upsilon}}\;}^{\text{classical}}, \overbrace{\;\breve{\sigma}\;}^{\text{quantum}}\Big) \longmapsto \overbrace{\;\breve{\varpi}\,\breve{\rho}\,\overline{\breve{\upsilon}}\,\breve{\sigma}\;}^{\text{quantum}} \tag{75}$$

involves the pointwise product of two *classical* characteristic functions — $\breve{\varpi}$ and $\breve{\rho}\,\overline{\breve{\upsilon}}$ — which is still a function of this type (the Fourier-Stieltjes transform of the convolution of two probability measures), multiplied pointwise by the *quantum* characteristic function $\breve{\sigma}$, that eventually provides a function of the latter kind. The relevant technical point, here, is that the quantum characteristic functions are *functions of quantum positive type* [10, 11]. We say that a function $\mathcal{Q} \in \mathrm{L}^2(\mathbb{R}^n \times \mathbb{R}^n)$ is of quantum positive type if

$$\int_{\mathbb{R}^n\times\mathbb{R}^n} \mathcal{Q}(q,p)\; (f^* \star f)(q,p)\; \mathrm{d}^n q\, \mathrm{d}^n p \geq 0, \quad \text{(the QPTF condition; compare with (7))} \tag{76}$$

for all $f \in \mathrm{L}^2(\mathbb{R}^n\times\mathbb{R}^n)$, where $f \mapsto f^*$ is the *involution* defined by $f^*(q,p) := \overline{f(-q,-p)}$ $(f \in \mathrm{L}^2(\mathbb{R}^n\times\mathbb{R}^n))$, and the binary operation $(\cdot)\star(\cdot)$ is the *twisted convolution*

$$\big(f \star k\big)(q,p) := (2\pi)^{-n} \int_{\mathbb{R}^n\times\mathbb{R}^n} f(\tilde{q},\tilde{p})\, k(q-\tilde{q}, p-\tilde{p})\; \mathrm{e}^{\frac{\mathrm{i}}{2}(q\cdot\tilde{p}-p\cdot\tilde{q})}\, \mathrm{d}^n\tilde{q}\, \mathrm{d}^n\tilde{p}, \tag{77}$$

i.e., the *star product* [24] realizing the operator product in terms of phase-space functions. The quantum characteristic functions are precisely the normalized *continuous* functions of quantum positive type [10].

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