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# Nonextensive black hole thermodynamics from generalized Euclidean path integral and Wick's rotation

**F. Barzi**[1,2,a], **H. El Moumni**[1,b], **K. Masmar**[1,3,c]

[1] LPTHE, Physics Department, Faculty of Sciences, Ibnou Zohr University, Agadir, Morocco
[2] Laboratory of High Energy Physics and Condensed Matter HASSAN II University, Faculty of Sciences Ain Chock, Casablanca, Morocco
[3] CRMEF, Regional Center for Education and Training Professions, Marrakesh, Morocco

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**Abstract** This paper extends the Euclidean path integral formalism to account for nonextensive thermodynamics. Concretely, we introduce a generalized Wick's rotation from real time $t$ to imaginary time $\tau$ such that, $t \to -i f_\alpha(\tau)$, where $f_\alpha$ a differentiable function and $\alpha$ is a parameter related to nonextensivity. The standard extensive formalism is recovered in the limit $\alpha \to 0$ and $f_0(\tau) = \tau$. Furthermore, we apply this generalized Euclidean path integral to black hole thermodynamics and derive the generalized Wick's rotations given the nonextensive statistics. The proposed formulation enables the treatment of nonextensive statistics on the same footing as extensive Boltzmann–Gibbs statistics. Moreover, we define a universal measure, $\eta$, for the nonextensivity character of statistics. Lastly, based on the present formalism, we strengthen the equivalence between the AdS-Schwarzschild black hole in Boltzmann–Gibbs statistics and the flat-Schwarzschild black hole within Rényi statistics and suggest a potential reformulation of the $AdS_5/CFT_4$ duality.

## 1 Introduction

The path integral formulation of quantum mechanics, developed in the mid-twentieth century [1,2], stands as a remarkable synthesis of key theoretical physics concepts and serves as a powerful computational tool. This technique has been applied to a wide range of physical systems across diverse contexts, including quantum mechanics [3], quantum field theory [4,5], gauge field theory [6,7], black hole physics [8], quantum gravity [9,10], string theory [11,12], topology [13,14], condensed matter physics [15–17], and optical communications [18], among others [19–21]. Indeed, in statistical physics, path integrals laid the foundation for the first formulation of the renormalization group transformation and are widely used to study systems with random impurity distributions [22]. In particle physics, they have been crucial in understanding and accounting for the presence of instantons [23]. Quantum field theory benefits from path integrals as the natural framework for quantizing gauge fields. In chemical, atomic, and nuclear physics, this approach has been applied to various semiclassical schemes in scattering theory. Moreover, path integrals offer a powerful means to explore classical and quantum fields' topological and geometrical properties, facilitating novel perturbative and non-perturbative analyses of fundamental natural processes [24].

The Boltzmann–Gibbs (BG) statistics has long been the cornerstone for describing a broad class of physical systems, providing over a century of successful applications.[1] It is particularly effective for systems with predominantly chaotic dynamics, such as classical systems exhibiting mixing, ergodicity, and a positive maximal Lyapunov exponent. However, many complex physical systems fall outside the scope of this framework, especially those where the maximal Lyapunov exponent vanishes, indicating a departure from simple chaotic behavior. To better describe the statistical properties of such systems, various generalized forms of statistical mechanics have emerged, including nonadditive entropies, Kappa-distributions [25,26], $q$-Gaussians [27,28], and Superstatistics [29–32]. These generalized approaches have demonstrated a wide range of applications both within and beyond physics.

[a] e-mail: faical.barzi@edu.uiz.ac.ma
[b] e-mail: h.elmoumni@uiz.ac.ma (corresponding author)
[c] e-mail: karima.masmar@gmail.com

[1] This year's Nobel Prize in Physics honors groundbreaking research on the application of Boltzmann statistics to neural networks and machine learning.

For the building and discussion of relativistic quantum field theoretical models, the concept of *passage to imaginary time* has proven to be an invaluable technique. This approach was initially introduced by Dyson [33] and later formalized by Wick [34], giving rise to the well-known *Wick rotation*. Building on this foundation, Schlingemann [35] provided a more rigorous framework that connected Euclidean and Lorentzian quantum field theories, leveraging the Osterwalder–Schrader theorem [36,37]. This theorem outlines the necessary and sufficient conditions for a consistent transition between these frameworks.

Applying a Wick rotation to the Lorentzian path integral reveals its close resemblance to the partition function in statistical mechanics. Specifically, the Euclidean path integral sums over all possible paths, with each path weighted by an effective energy-like term derived from the action in imaginary time-mirroring how the partition function sums over all states of a system, weighted by their respective energies. This formalism in Quantum Mechanics ($QM$) stems from the *time-evolution operator* $\mathcal{U}(t)$, which satisfies the following differential equation

$$i\hbar\frac{\partial \mathcal{U}}{\partial t} = H\mathcal{U}. \tag{1}$$

Where $H$ is the Hamiltonian operator. The solution of the above equation for a time-independent Hamiltonian reads as

$$\mathcal{U}(t) = \exp\left(\frac{-iHt}{\hbar}\right). \tag{2}$$

Probability amplitudes are given by matrix elements of $\mathcal{U}(t)$. Wick's rotation permits a connection with the partition function of statistical mechanics such that by analytical continuation to imaginary time $t \rightarrow -i\tau$ we get,

$$\mathcal{U}(\tau) = \exp\left(\frac{-H\tau}{\hbar}\right). \tag{3}$$

Which is a solution to the diffusion-like differential equation,

$$\frac{\partial \mathcal{U}}{\partial \tau} = -\frac{H\mathcal{U}}{\hbar} \tag{4}$$

Then, the partition function $Z[\beta]$ is given by,

$$Z[\beta] = tr\left[\exp\left(-\beta H\right)\right] = \oint Dg\ \exp\left(-\mathcal{I}_E[g]\right), \tag{5}$$

where $tr[.]$ is the trace operator, $\beta = \frac{\tau}{\hbar}$, and $\mathcal{I}_E$ is the Euclidean action. The integration is performed over all closed trajectories in phase space. We readily get the thermodynamic free energy from the partition function as,

$$F = U - TS = -\frac{1}{\beta}\ln\left(Z[\beta]\right). \tag{6}$$

Likewise, all other relevant thermodynamic quantities can be computed from $Z[\beta]$.

The setup demonstrates that the exponential form of the time-evolution operator in Eq. (2) aligns with the Boltzmann exponential probability factor in Eq. (3) after applying a Wick rotation. This connection suggests that the extensive nature of Boltzmann–Gibbs statistics in the thermodynamic limit is fundamentally tied to the linearity of the Schrödinger equation-without which, the exponential form of the time-evolution operator would not be feasible. In contrast, nonextensive statistical mechanics modifies the partition function away from its exponential form, typically by introducing new functional forms and parameters to capture nonextensive behavior. However, the linearity of the Schrödinger equation remains unaltered in such approaches, creating a fundamental inconsistency. This inconsistency arises because, while a statistical mechanics framework attempts to generalize Boltzmann–Gibbs statistics, the quantum mechanical side lacks any modification to justify this shift. This discrepancy presents a theoretical gap that is unsatisfactory for a coherent understanding of the relationship between quantum mechanics and nonextensive statistical mechanics. Thus, there is a clear need for a consistent theoretical foundation that links the path integral formulation of quantum mechanics with nonextensive statistical approaches. In this paper, we aim to addr]ess this imbalance and propose a more cohesive framework.

This paper is structured as follows: In Sect. 2 we provide a clear and concise definition of nonextensivity. Section 3, introduces the generalized Euclidean path integral formalism alongside the extended version of Wick's rotation. Section 4 demonstrates the application of this new formalism to derive nonextensive black hole thermodynamics, achieving a comparable status to Boltzmannian statistics. Additionally, we introduce a measure of nonextensivity to quantify the statistical character. Section 5 explores the proposed *Rényi/AdS* equivalence and suggests a potential reformulation of the $AdS_5/CFT_4$ duality. A general discussion and concluding remarks are provided in Sect. 6.

## 2 Nonextensivity and black hole thermodynamics

Before proceeding further, it is essential to clarify the concept of nonextensivity in the context of black hole thermodynamics. It's commonly known that nonextensivity refers to a property of physical systems where the total properties, such as energy, volume, or entropy, do not scale linearly with the system's size or the number of its components. In extensive systems, these properties are directly proportional to system size. However, in nonextensive systems, scaling is non-linear, often due to long-range interactions, such as in gravitational systems, fractal structures, or systems with strong internal correlations. In this sense, nonextensivity signifies a deviation from standard statistical mechanics, specifically the Boltzmann–Gibbs (BG) framework, which assumes additive entropy. This deviation motivates the introduction of generalized, non-additive entropy forms, as explored in Tsallis statistics [38,39]. Whereas entropy scales with volume in typical systems, for black holes, it is proportional to the horizon area. This fundamental distinction underscores the importance of studying black hole thermodynamics through the framework of nonextensive entropy, differing from the traditional Boltzmann–Gibbs approach.

The appropriateness of standard Boltzmann–Gibbs statistics has been widely debated, particularly in the context of black hole thermodynamics. In this framework, the limitations of conventional stability analyses are often attributed to the nonadditivity of entropy, as seen in *the nonadditive and nonextensive* Bekenstein–Hawking entropy. These challenges suggest that using Boltzmann–Gibbs statistics in self-gravitating systems may yield unreliable results, as extensive quantities, such as mass, may not be locally well-defined.

In the thermodynamic limit, BG statistics implies extensivity in the sense of scaling with the system's number of constituents. However, the Bekenstein–Hawking entropy is inherently nonextensive. Therefore, it seems inconsistent to apply the BG thermodynamics along with a nonextensive entropy.

By using a weaker additivity rule specifically, Abe's rule-a novel strategy emerges [40].

$$H_\xi(S_{ab}) = H_\xi(S_a) + H_\xi(S_b) + \xi H_\xi(S_a) H_\xi(S_b). \tag{7}$$

Here, $H_\xi$ denotes a differentiable function of entropy $S$, $\xi$ is a real parameter, and $(a)$ and $(b)$ are two independent systems. The Boltzmann–Gibbs–Shannon entropy is given by,

$$S_{BG} = -\sum_{i=1}^{\Omega} p_i \ln p_i, \tag{8}$$

with $\Omega \in \mathbb{N}$ as the total number of configurations and $p_i$ as their probabilities. One can easily verify that for independent systems $(a)$ and $(b)$ we have an additive rule,

$$S_{BG}(a \cup b) = S_{BG}(a) + S_{BG}(b). \tag{9}$$

Which realizes Eq. (7) for $\xi = 0$ and $H_\xi(S) = S_{BG}$. A well-known of nonextensive entropy is the Tsallis entropy [41], expressed as follows:

$$S_q = \frac{1 - \sum_{i=1}^{\Omega} p_i^q}{q - 1} \equiv S_T, \quad q \in \mathbb{R}. \tag{10}$$

The standard BG entropy $S_{BG}$ is recovered by setting $q \to 1$. Here, defining $H_\xi(S) = S_T$, the pseudo-additivity rule Eq. (7), reads as

$$S_T(a \cup b) = S_T(a) + S_T(b) + (1 - q) S_T(a) S_T(b). \tag{11}$$

Biró and Van [42] introduced the formal logarithm of the Tsallis entropy, corresponding to the Rényi entropy [43]

$$L(S_T) = \frac{1}{\lambda} \ln\left[1 + \lambda S_T\right] \equiv S_R. \tag{12}$$

Where $\lambda = 1 - q$. Remarkably, this definition imparts a formal additive composition law to the Rényi entropy, such as,

$$\begin{aligned} S_{\mathrm{R}}(a \cup b) &= \frac{1}{\lambda} \ln\left[1 + \lambda\left(S_{\mathrm{T}}(a) + S_{\mathrm{T}}(a) + \lambda S_{\mathrm{T}}(a) S_{\mathrm{T}}(a)\right)\right] \\ &= \frac{1}{\lambda} \ln\left[\left(1 + \lambda S_{\mathrm{T}}(a)\right)\left(1 + \lambda S_{\mathrm{T}}(a)\right)\right] \\ &= \frac{1}{\lambda} \ln\left(1 + \lambda S_{\mathrm{T}}(a)\right) + \frac{1}{\lambda} \ln\left(1 + \lambda S_{\mathrm{T}}(a)\right) \\ &= S_{\mathrm{R}}(a) + S_{\mathrm{R}}(a). \end{aligned} \tag{13}$$

Thus, Rényi entropy provides an effective approach for addressing the nonextensive properties of black holes while preserving entropy additivity. Additionally, we observe that the pseudo-additivity rule that the Tsallis entropy obeys allows the Rényi entropy, which is the formal logarithm of the Tsallis entropy, to capture the non-additive nature of black hole entropy.

Since Boltzmann–Gibbs (BG) entropy is additive, as shown in Eq. (9), and extensive in the thermodynamic limit, it cannot be associated with *the nonadditive and nonextensive Bekenstein–Hawking entropy*. Therefore, black hole thermodynamics should not be viewed as a limiting case of BG statistical mechanics but rather as a limit of a nonextensive statistical mechanics, where the entropy, in the thermodynamic limit, does not scale with the system's size.

Finally, it is crucial to clearly distinguish between the concepts of extensivity and additivity in entropy. For over a century, physicists have primarily studied locally interacting systems, where Boltzmann–Gibbs (BG) entropy has served as the standard entropic form that satisfies the thermodynamic requirement of extensivity. However, this focus has led to a common misconception in various physical contexts and textbooks, where additivity and extensivity are incorrectly treated as synonymous. This misinterpretation can result in misunderstandings and unintended consequences. The essential difference lies in the fact that, unlike additivity, extensivity depends not only on the functional form of the entropy but also on the nature of the correlations within the system components, particularly whether these correlations are local or nonlocal. In this context, BG entropy is inherently additive but would be extensive (nonextensive) if the system's elements are locally (nonlocally) correlated. However, in black hole thermodynamics, entropy is fundamentally nonadditive and should not be connected to BG entropy. This raises an interesting question about terminology-why call it *nonextensive black hole thermodynamics* when black hole thermodynamics is inherently nonextensive? Indeed, black hole entropy is both nonextensive and nonadditive, though traditionally it has been discussed in relation to the extensive framework of BG thermodynamics. Our choice to emphasize the nature of the thermodynamics in the name reflects this departure. Another possible term could be *non-Boltzmannian black hole thermodynamics*.

## 3 Generalized Euclidean path integral

We address this discrepancy by extending the existing procedure that connects the two frameworks: *Wick's rotation*. Specifically, we introduce a *generalized Wick's rotation*, offering a more comprehensive approach to bridge the gap between the path integral formulation of quantum mechanics and nonextensive statistical mechanics as

$$t \longrightarrow -i f_\alpha(\tau). \tag{14}$$

Where $\alpha$ is a real parameter related to nonextensivity such that, as $\alpha \to 0$, we recover the BG Wick's rotation $t \longrightarrow -i\tau$, that is $\lim_{\alpha\to 0} f_\alpha(\tau) = \tau$. We observe that the relationship between nonextensivity and the parameter $\alpha$, representing the generalized Wick's rotation, stems from the finding that each deviation from BG statistical mechanics introduces a new set of parameters to quantify nonextensivity. As with the parameter $\alpha$, we recover the BG statistics after these parameters are assumed to vanish.

Applying the generalized rotation, Eq. (14), to Eq. (1), we get

$$\frac{\partial \mathcal{U}_\alpha}{\partial \tau} = -\frac{H\mathcal{U}_\alpha}{\hbar} f_\alpha^{'}(\tau). \tag{15}$$

Where $f_\alpha^{'}(\tau)$ is the first derivative of $f_\alpha(\tau)$. The generalized rotated operator $\mathcal{U}_\alpha$ is then given by

$$\mathcal{U}_\alpha(\tau) = \exp\left(-\frac{H}{\hbar}\int_0^\tau d\tilde{\tau}\; f_\alpha^{'}(\tilde{\tau})\right) = \exp\left(-H\int_0^\beta d\tilde{\beta}\, k_\alpha(\tilde{\beta})\right). \tag{16}$$

Here we put $k_\alpha(\beta) \equiv f_\alpha^{'}(\tau)$. The partition function is therefore generalized as

$$Z_\alpha[\beta] = tr\,[\mathcal{U}_\alpha] = \oint Dg\, \exp\left(-\mathcal{I}_E^\alpha[g]\right). \tag{17}$$

Herein, $\mathcal{I}_E^\alpha$ is the nonextensive Euclidean action derived from the Lorentzian action $\mathcal{I}_L$ through the transformation Eq. (14) such that

$$i\mathcal{I}_L = i\int dt\, L(t) = -\int d\tau f_\alpha^{'}(\tau)\,(-L(\tau)) = -\int d\tau\,(-L^\alpha(\tau)) = -\mathcal{I}_E^\alpha. \tag{18}$$

Where $L^\alpha(\tau) = f_\alpha^{'}(\tau)\,L(\tau)$ is the generalized Euclidean Lagrangian. In this procedure, the Lorentzian Lagrangian $L(t)$ is rotated to the Euclidean one $L^\alpha(\tau)$. With the above-extended formalism, we reconcile the linearity of *QM* through the Schrödinger equation with the inherent non-linearity of the nonextensive statistical mechanics as proposed by Tsallis and al. [41,44–47]. More precisely, upon performing the standard Wick rotation to imaginary time, $t \to -i\tau$, on the evolution operator defined by the Schrödinger equation, the result is the Boltzmann–Gibbs (BG) statistical partition function, which is extensive in the thermodynamic limit. This leaves no natural framework for introducing a nonextensive formalism that would align with the inherently nonextensive nature of black hole thermodynamics, except by modifying the partition function after Wick rotation. However, such modifications are inconsistent with the standard path integral formalism and compromise the uniqueness of Wick's rotation, as they would require corresponding adjustments to the quantum mechanical time evolution operator and, consequently, the Schrödinger equation itself. Additionally, this approach severs the link between nonextensive thermodynamics and the quantum mechanical evolution operator. Quantum mechanics does not, in fact, prescribe a specific statistical or entropic framework for a system to follow. Consequently, it is overly restrictive to impose an extensive Boltzmann–Gibbs (BG) thermodynamics simply because the standard Wick rotation has been applied to the time evolution operator. Nor does it seem justified to rely on ad hoc modifications to the partition function to enforce a particular entropy form. The proposed approach seeks to unify the relationship between extensive and nonextensive thermodynamics and quantum mechanics within a single framework. This would allow the choice of Wick rotation to reflect the nature of the system's thermodynamics-whether extensive or nonextensive-based on the intrinsic characteristics of the system.

In Fig. 1, we present a schematic diagram of the effect of standard and generalized Wick's rotations. The standard Wick's rotation of the time evolution operator $t \to -i\tau$, results in the BG statistics and extensive thermodynamics, while the generalized rotation produces nonextensive statistics and thermodynamics.

The free energy for the nonextensive statistics is computed as

$$F_\alpha = U_\alpha - \frac{S_\alpha}{\beta} = -\frac{1}{\beta}\ln\left(Z_\alpha\right). \tag{19}$$

Here, $S_\alpha$ represents the nonextensive entropy, and $\beta = \frac{1}{T} = \frac{\partial S_\alpha}{\partial U_\alpha}$ denotes the inverse temperature in the framework of nonextensive thermodynamics. The choice of the nonextensive entropy, $S_\alpha$, determines the functional form of $f_\alpha(\tau)$ through Eq. (19), as will be demonstrated below. Using the

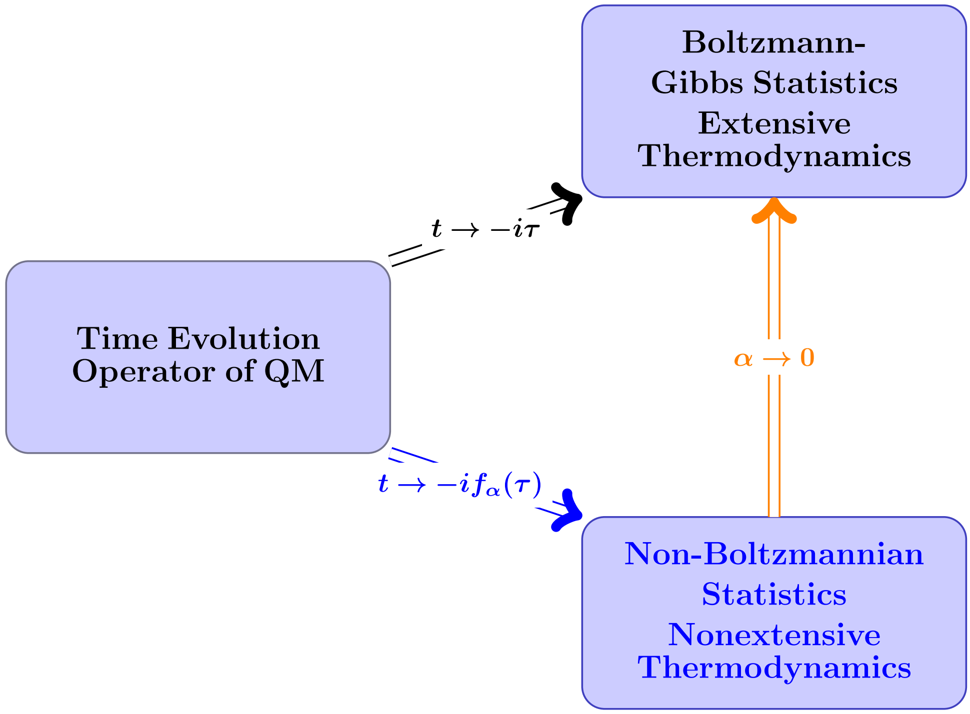


**Fig. 1** Schematic diagram of the standard and generalized Wick's rotations. As the parameter $\alpha$ goes to zero we recover the Boltzmann–Gibbs statistics

well-known saddle point semiclassical approximation, the partition function $Z_\alpha$ takes the following form

$$Z_\alpha \approx \exp\left(-\mathcal{I}_E^\alpha[g_{cl}]\right), \tag{20}$$

where $g_{cl}$ is the classical solution of the system's Euler-Lagrange equations. One then has

$$F_\alpha = U_\alpha - \frac{S_\alpha}{\beta} \approx \frac{\mathcal{I}_E^\alpha[g_{cl}]}{\beta}. \tag{21}$$

$$\implies \beta U_\alpha - S_\alpha = \mathcal{I}_E^\alpha[g_{cl}] \equiv \mathcal{I}_{cl}^\alpha. \tag{22}$$

Equation (22) is the condition to determine $f_\alpha(\tau)$ and the required Wick's rotation. Explicitly, we write for a given spacetime metric $g$,

$$M\frac{\partial S_\alpha}{\partial M} - S_\alpha = -\int_0^\tau d\tilde{\tau}\, L^\alpha[g_{cl}] = \mathcal{I}_{cl}^\alpha \tag{23}$$

Where $L^\alpha[g_{cl}] \equiv L_{cl}^\alpha$ is the Euclidean Lagrangian computed at the saddle point $g_{cl}$ and we have defined $M \equiv U_\alpha$.[2]

It is worth noting that the choice of $f_\alpha(\tau)$ could affect the Euclidean-invariance of the rotated Lagrangian, $L_{cl}^\alpha$, and thus of its Euclidean action, which may supplement further restrictions on the set of possible functions $f_\alpha$, if such invariance is to be preserved. Nonetheless, the Lorentzian invariance is the physical invariance and it is still conserved. Additionally, since Wick's rotation function is derived from the generalized entropies, it is evident that they should inherit the consequences of some or all of the Shannon-Khinchin axioms [48,49]. It is seen from Eq. (16) that the simplest choice which is Euclidean-invariant, corresponds to the obvious BG case, $f_0^{'}(\tau) = k_0(\beta) = 1$. A close inspection of Eq. (23) gives a formal expression for $k_\alpha(\beta) \equiv f_\alpha^{'}(\tau)$ defined in Eq. (16), in the general case,

$$\int_0^\beta d\tilde{\beta}\, k_\alpha(\tilde{\beta})\, L_{cl}(\tilde{\beta}) = S_\alpha - M\frac{\partial S_\alpha}{\partial M} \tag{24}$$

$$\implies k_\alpha(\beta)\, L_{cl}(\beta) = \frac{\partial}{\partial\beta}\left(S_\alpha - M\frac{\partial S_\alpha}{\partial M}\right). \tag{25}$$

That is,

$$k_\alpha(\beta) = \frac{1}{L_{cl}(\beta)}\frac{\partial}{\partial\beta}\left(S_\alpha - M\frac{\partial S_\alpha}{\partial M}\right) \tag{26}$$

$$= \frac{1}{L_{cl}(\beta)}\left(\frac{\partial S_\alpha}{\partial\beta} - \frac{\partial(M\beta)}{\partial\beta}\right) \tag{27}$$

$$= \frac{1}{L_{cl}(\beta)}\left(\frac{\partial S_\alpha}{\partial\beta} - \beta\frac{\partial M}{\partial\beta} - M\right) \tag{28}$$

When a generalized Wick's rotation for a given nonextensive statistics is calculated using Eq. (28), it is placed on equal footing as the Boltzmann–Gibbs statistics which also obeys in the present generalization,

$$k_0(\beta) = 1 = \frac{1}{L_{cl}^0(\beta)}\left(\frac{\partial S_0}{\partial\beta} - \beta\frac{\partial M}{\partial\beta} - M\right), \tag{29}$$

where $S_0 \equiv S_{BG}$, is the Boltzmann–Gibbs entropy. Thus,

$$L_{cl}^0(\beta) = \frac{\partial S_{BG}}{\partial\beta} - \beta\frac{\partial M}{\partial\beta} - M. \tag{30}$$

[2] In subsequent sections, The parameter $M$ will be identified with the mass of black holes.

We employ the notation *index* "0" to denote the Boltzmannian statistics, which corresponds to the case where the nonextensive parameter vanishes ($\alpha = 0$). Reciprocally and by construction, the Wick's function $k_0(\beta) \equiv f_0^{'}(\tau)$ defined in Eq. (28), enables to rotate the Lorentzian action to the corresponding Euclidean one, $\mathcal{I}_E^{\alpha}[g_{cl}]$, and to obtain the statistics $S_\alpha$ through the partition function Eq. (20). Concretely, from Eq. (23), one can obtain

$$M\frac{\partial S_\alpha}{\partial M} - S_\alpha = \mathcal{I}_{cl}^{\alpha}[\beta] = \mathcal{I}_{cl}^{\alpha}\left[\frac{\partial S_\alpha}{\partial M}\right], \tag{31}$$

where we used $\beta \equiv \dfrac{\partial S_\alpha}{\partial M}$. Thus, for a given Wick's rotation defined by a function $f_\alpha(\tau)$ which produces a semi-classical Euclidean action $\mathcal{I}_{cl}^{\alpha}[\beta]$, the nonextensive entropy generated by such a Wick's rotation obeys the differential equation

$$M\frac{\partial S_\alpha}{\partial M} - S_\alpha - \mathcal{I}_{cl}^{\alpha}\left[\frac{\partial S_\alpha}{\partial M}\right] = 0. \tag{32}$$

This differential equation is used in the next section[3] to generate statistics from their corresponding Wick's rotations.

## 4 Applications to black hole thermodynamics

One of the main motivations for deriving nonextensive statistics from the generalized Euclidean path integral is the need to introduce nonextensive entropies into black hole thermodynamics in order to account for their nonextensive nature. Through the standard Euclidean path integral formulation, where Wick's rotation function is limited to $f_0(\tau) = \tau$, one can uniquely obtain the Hawking temperature as the thermodynamic temperature of the black hole. A challenge arises when attempting to modify the black hole thermodynamic away from the Boltzmann–Gibbs framework, as altering the entropy inevitably leads to a corresponding change in temperature. This contradicts the traditional understanding of Hawking's temperature, resulting in a tension between the need for nonextensive behavior in black holes and the unique nature of Hawking temperature as derived from standard Wick's rotation. The solution proposed in this study addresses this issue by introducing a generalized Wick's rotation, which produces a temperature that aligns with the chosen statistical framework for the black hole system. In this view, the Hawking temperature is no longer unique; it represents the temperature that aligns specifically with Boltzmannian statistics. Thus, it becomes *equally* possible to apply any statistical model to a given black hole without encountering contradictions. In this approach, we leverage the freedom of Wick's rotations to *encode* nonextensivity, allowing for a unified connection between QM and statistical mechanics, particularly in the application to black hole thermodynamics. Some may question the necessity of such a generalization, citing the already established nonextensivity of the Bekenstein–Hawking entropy [50]. Indeed, the standard Euclidean path integral formalism, obtained through the conventional Wick's rotation, provides a way to derive the Bekenstein–Hawking entropy from the on-shell Euclidean action, as demonstrated by Gibbons and Hawking in their seminal work [9]. However, it is essential to note that the standard Euclidean path integral and its Wick's rotation lead singularly to the BG statistics and, in the thermodynamic limit, to an extensive thermodynamics, which is inadequate for describing the nonextensive nature of black holes.

In the literature, numerous generalizations of the Bekenstein–Hawking entropy have been put forth [51–53]. Away from the shortcomings of the BG thermodynamics for black holes, these nonextensive entropies offer a framework to explain the nonextensivity of black hole thermodynamics. They are typically started in a disputed way [54,55] by directly altering the black hole entropy without taking into account the origin of the black hole temperature, which needs to coincide with the altered entropy. However, only the Hawking temperature is revealed by the standard Euclidean path integral following the well-known Wick's rotation, and assuming another temperature causes a potential conflict with the Euclidean path integral formalism and QM. The present framework offers a new pathway to construct these entropic modifications in accordance with the principles of QM by utilizing the freedom of Wick rotations.

In this section, we assess the applicability of our formalism by analyzing a range of well-known black hole thermodynamic systems, considering both extensive and nonextensive statistical frameworks. Furthermore, we propose a novel universal measure for quantifying the degree of nonextensivity in a given statistical model, derived from the generalized Wick's rotation introduced in the preceding section.

### 4.1 The *4d*-AdS Schwarzschild black hole in Boltzmann–Gibbs statistics

Let's apply Eq. (30) to the case of the four-dimensional asymptotically-AdS Schwarzschild black hole within BG-statistics (4*d* AdS-Sch). We have for the entropy, mass, and horizon radius of this black hole[4]

$$S_{BH} = \pi r_h^2, \quad M = \frac{r_h}{2} - \frac{\Lambda r_h^3}{6} \quad \text{and}$$

[3] See Sect. 4.4.

[4] The horizon radius is given in terms of the cosmological constant $\Lambda < 0$ and the inverse temperature $\beta$, pending the condition that the AdS-Sch black hole phase can appear from the thermal radiation phase. This holds for $\beta^2 < -\dfrac{4\pi^2}{\Lambda} = \beta_{min}^2$ [56]. Similar remarks are true also for subsequent black hole systems.

$$r_h = \frac{\sqrt{\Lambda\beta^2 + 4\pi^2} - 2\pi}{\Lambda\beta}, \tag{33}$$

here $S_{BH}$ is the Bekenstein–Hawking entropy and $\Lambda < 0$ is the cosmological constant associated with the AdS spacetime radius via $\Lambda = \frac{3}{\ell^2}$. Because Bekenstein–Hawking entropy is obtained by the standard Euclidean path integral following the standard Wick's rotation, as shown by Gibbons–Hawking formalism [9], it is therefore associated with the Boltzmann–Gibbs statistics. By substitution in Eq. (30), the Euclidean gravitational Lagrangian $L^0_{cl}(\beta, \Lambda)$ is found to be

$$L^0_{cl}(\beta, \Lambda) = \frac{4\pi^2\left(\beta^2\Lambda - 4\pi\sqrt{\beta^2\Lambda + 4\pi^2} + 8\pi^2\right) - \beta^4\Lambda^2}{3\beta^3\Lambda^2\sqrt{\beta^2\Lambda + 4\pi^2}}, \tag{34}$$

or equivalently in terms of the imaginary time $\tau = \beta$ $(\hbar = 1)$

$$L^0_{cl}(\tau, \Lambda) = \frac{4\pi^2\left(\Lambda\tau^2 - 4\pi\sqrt{\Lambda\tau^2 + 4\pi^2} + 8\pi^2\right) - \Lambda^2\tau^4}{3\Lambda^2\tau^3\sqrt{\Lambda\tau^2 + 4\pi^2}}. \tag{35}$$

Integrating Eq. (35) gives the on-shell gravitational Euclidean action

$$\mathcal{I}^0_{cl}(\Lambda) = \frac{\Lambda\tau^2 + 2\pi\left(\sqrt{\Lambda\tau^2 + 4\pi^2} + 4\pi\right)}{3\Lambda\left(\sqrt{\Lambda\tau^2 + 4\pi^2} + 2\pi\right)} - \frac{\pi}{\Lambda}. \tag{36}$$

Here $\mathcal{I}^0_{cl}(\Lambda)$ is finite since it is the sum of the bulk action which diverges because spacetime has infinite volume, the Gibbon–Hawking action for the boundary contribution and the counterterm action to cancel divergences. For small cosmological constant, $\Lambda \ll 1$, we get

$$L^0_{cl}(\tau, \Lambda) = -\frac{\tau}{8\pi} + \frac{\Lambda\tau^3}{96\pi^3} + O\left(\Lambda^2\right), \tag{37}$$

$$\mathcal{I}^0_{cl}(\Lambda) = \frac{\tau^2}{16\pi} - \frac{\Lambda\tau^4}{384\pi^3} + O\left(\Lambda^2\right). \tag{38}$$

The associated gravitational partition function for small $\Lambda \ll 1$ is given through Eq. (20) such as

$$Z_0(\Lambda) = \exp\left(-\frac{\beta^2}{16\pi} + \frac{\Lambda\beta^4}{384\pi^3}\right). \tag{39}$$

Since $\Lambda$ is negative, the partition function remains bounded. In the limit where the cosmological constant vanishes, $\Lambda \to 0$, we recover the well-known results for the Boltzmann–Gibbs statistics: the 4-dimensional asymptotically flat Schwarzschild classical Lagrangian, $L_{cl}$, the classical action, $\mathcal{I}^0_{cl}$, and the gravitational partition function, $Z_0$, as follows

$$L_{cl}(\tau) = -\frac{\tau}{8\pi}, \quad \mathcal{I}^0_{cl}(\tau) = \frac{\tau^2}{16\pi}, \quad \text{and} \quad Z_0(\beta) = \exp\left(-\frac{\beta^2}{16\pi}\right). \tag{40}$$

We note that Eq. (40) concur with the results attained using various techniques [10,57,58].

## 4.2 The *4d*-flat Schwarzschild black hole in Rényi statistics

We perform the same calculation for the four-dimensional asymptotically flat Schwarzschild black hole within the Rényi nonextensive formalism (4*d* Rényi-Sch) [43,51,59–61]. One obtain,[5]

$$S_R = \frac{1}{\lambda}\ln\left(1 + \lambda\pi r_h^2\right), \quad M = \frac{r_h}{2} \quad \text{and} \quad r_h = \frac{2\sqrt{\pi} - \sqrt{4\pi - \beta^2\lambda}}{\sqrt{\pi}\beta\lambda}. \tag{41}$$

Here, the nonextensivity is measured by the parameter $\alpha = \lambda$ which is assumed to be small, $0 < \lambda \ll 1$, and accounts for the nonlocal and nonextensive nature of black holes. In this limit, the Rényi entropy in Eq. (41) reads

$$S_R = \pi r_h^2 - \frac{\lambda}{2}\left(\pi^2 r_h^4\right) + O\left(\lambda^2\right). \tag{42}$$

By inserting Eq. (41) in Eq. (28), one finds the following expression $(\beta \equiv \tau)$

$$L^\lambda_{cl}(\tau) \equiv L_{cl}(\tau) f^{'}_\lambda(\tau) = \frac{\sqrt{4 - \frac{\lambda\tau^2}{\pi}} - 2}{2\lambda\tau}. \tag{43}$$

Here $L_{cl}(\tau)$ is the Euclidean Lagrangian derived from the asymptotically flat Schwarzschild black hole metric, given by Eq. (40), which is independent of the parameter $\lambda$. The nonextensivity is encoded in the generalized Wick's rotation represented by the function $f_\lambda(\tau)$. A direct calculation of the asymptotically flat Schwarzschild black hole Euclidean action using the Hawking–Gibbon–York method confirms the expression of $L_{cl}(\tau)$ as

$$L_{cl}(\tau) = -\frac{\beta}{8\pi} = -\frac{\tau}{8\pi}. \tag{44}$$

Therefore, the derivative of the Rényi Wick's rotation function $f_\lambda(\tau)$ is found to be

$$f^{'}_\lambda(\tau) = \frac{4\pi}{\lambda\tau^2}\left(2 - \sqrt{4 - \frac{\lambda\tau^2}{\pi}}\right) \tag{45}$$

For small $\lambda \ll 1$, we get for $f_\lambda(\tau)$ and $\mathcal{I}^\lambda_{cl}$

$$f_\lambda(\tau) = \tau + \frac{\lambda\tau^3}{48\pi} + O\left(\lambda^2\right), \tag{46}$$

$$\mathcal{I}^\lambda_{cl} = \frac{\tau^2}{16\pi} + \frac{\lambda\tau^4}{512\pi^2} + O\left(\lambda^2\right). \tag{47}$$

[5] For the Rényi-Sch black hole phase to exist, the condition $\beta < \beta_{min} = \sqrt{\frac{4\pi}{\lambda}}$ must be satisfied [51].

Consequently, the Rényi gravitational partition function, Eq. (20), is given by

$$Z_\lambda = \exp\left(-\frac{\beta^2}{16\pi} - \frac{\lambda\beta^4}{512\pi^2}\right) \tag{48}$$

This Rényi partition function remains finite as long as λ is positive, ensuring that no divergences occur. As $\lambda \to 0$, we recover the Boltzmannian Wick's rotation function, $f_0(\tau) = \tau$. In summary, within this generalized Euclidean path integral formalism and in the limit of small parameter λ, to achieve the nonextensive Rényi black hole thermodynamics, one should perform the Wick rotation of the Lorentzian action such that:

$$t \to -i\left(\tau + \frac{\lambda\tau^3}{48\pi}\right). \tag{49}$$

This is in complete parallel with Wick's rotation to obtain the Boltzmann–Gibbs thermodynamics given by $t \to -i\tau$.

Using Eq. (48), we can verify that we obtain Rényi free energy through Eq. (19)

$$F_\lambda = -\frac{1}{\beta}\ln(Z_\lambda) \tag{50}$$

$$= \frac{\beta^4\lambda}{512\pi^2} + \frac{\beta^2}{16\pi}, \tag{51}$$

$$f_q^{'}(\tau) = \frac{4\sqrt{\pi}\left[(1-q)^2\tau^4 + 8\pi(1-q)\tau^2 + 32\pi^{3/2}\sqrt{4\pi-(1-q)\tau^2} - 64\pi^2\right]}{(1-q)^2\tau^4\sqrt{4\pi-(1-q)\tau^2}}, \tag{57}$$

in the limit of small nonextensivity parameter $\lambda \ll 1$. The Rényi entropy is also recovered

$$S_R = \beta^2\frac{\partial F_\lambda}{\partial\beta} \tag{52}$$

$$= \pi r_h^2 - \frac{\lambda}{2}\left(\pi^2 r_h^4\right) + O\left(\lambda^2\right) \tag{53}$$

$$= S_{BH} - \frac{\lambda}{2}S_{BH}^2 + O\left(\lambda^2\right), \tag{54}$$

where the inverted expression giving $\beta$ as a function of the horizon radius $r_h$

$$\beta = \frac{4\pi r_h}{1+\lambda\pi r_h^2} \tag{55}$$

was used to first order in λ to get Eq. (53) which matches Eq. (42).

Tsallis entropy [47], $S_T$, is connected to Rényi entropy $S_R$, through Eq. (10) by the relation

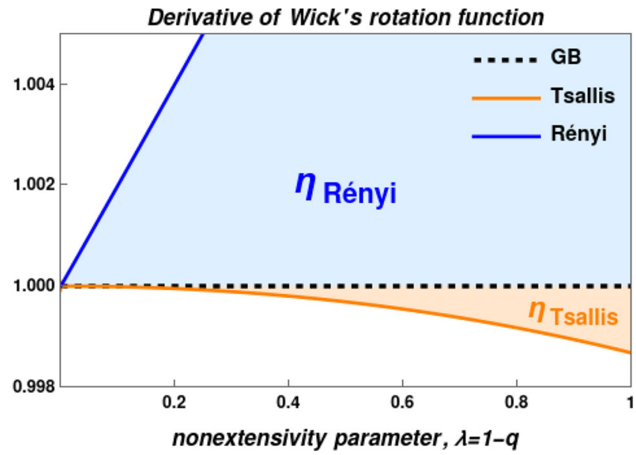


**Fig. 2** Comparing the derivatives of the Wick's rotation function of the Boltzmann–Gibbs, Tsallis, and Rényi statistics. The plot contrasts the nonextensivity nature of Tsallis and Rényi statistics for black hole thermodynamics. The shaded areas are a measure of the nonextensivity of the Rényi statistics $\eta_{\text{Rényi}}$ (Blue area) and the Tsallis statistics $\eta_{\text{Tsallis}}$ (Orange area). We fixed $\tau_0 = 1$

$$S_T = \frac{\exp[(1-q)\,S_R] - 1}{1-q}, \tag{56}$$

where the parameter $q = 1 - \lambda$. It is a simple calculation to compute the derivative of Wick's rotation function, $f_q^{'}(\tau)$, for the Tsallis statistics. One finds through Eq. (28)

which for $q \to 1$ becomes

$$f_q^{'}(\tau) = 1 - \frac{3\tau^4}{256\pi^2}(1-q)^2 + O\left[(1-q)^3\right], \tag{58}$$

$$\Longrightarrow\; f_q(\tau) = \tau - \frac{3\tau^5}{1280\pi^2}(1-q)^2 + O\left[(1-q)^3\right]. \tag{59}$$

Thus, the function, $f_q(\tau)$, differs only by a second order term in the nonextensivity parameter $\lambda = 1 - q$, from the Boltzmann–Gibbs Wick's rotation function $f_0(\tau) = \tau$. As depicted in Fig. 2, comparing with the function $f_\lambda(\tau)$, Eq. (49), it is shown that Rényi (blue line) and Tsallis (orange line) statistics display different nonextensivity characters, with Rényi being more pronounced than Tsallis. Therefore, The present framework offers the possibility to contrast the nonextensive nature of different statistics based on their defining Wick's rotations.

### 4.3 The nonextensivity measure for statistics

In this section, we introduce a method to quantify the nonextensive nature of a given statistical model using the generalized Wick's rotation formalism. For this purpose, we define a measure, denoted as $\eta(S_\alpha)$, corresponding to a specific statistics $S_\alpha$, such that

$$\eta(S_\alpha) = \lim_{\tau\to\tau_0} \left| \int_0^1 \left( f_\alpha^{'}(\tau) - 1 \right) d\alpha \right|, \tag{60}$$

where $f_\alpha^{'}(\tau)$ is the derivative with respect to $\tau$ of the Wick's rotation function of $S_\alpha$ and $\tau_0$ is a suitable nonzero imaginary time taken in the support of $f_\alpha^{'}(\tau)$. Then *a statistics* $S_{\alpha_1}$ is more nonextensive than another statistics $S_{\alpha_2} \iff \eta(S_{\alpha_1}) > \eta(S_{\alpha_2})$. The definition Eq. (60) is stated such that $\eta(S_{BG}) = 0$. In Fig. 2, we illustrated the geometrical meaning of the measure $\eta$ as the shaded areas. Using Eqs. (49) and (58) one finds,[6]

$$\eta_{\,\text{Rényi}} \simeq \frac{1}{32\pi} \quad \text{and} \quad \eta_{\,\text{Tsallis}} \simeq \frac{1}{256\pi^2}, \tag{61}$$

that is, $\eta_{\,\text{Rényi}} > \eta_{\,\text{Tsallis}}$. The measure $\eta$ is independent of the specific nonextensivity parameter(s) on which a given statistical model may rely, providing a universal scale for assessing the degree of nonextensivity. Furthermore, this definition can be extended to accommodate statistics with multiple parameters. A straightforward extension involves replacing the single integral over one parameter with multiple integrals over all relevant parameters in Eq. (60).

A survey of the literature on the numerical quantification of nonextensivity reveals, to our knowledge, a lack of direct proposals in this area. However, Hanel and Thurner [63,64] introduced a two-parameter asymptotic classification of generalized entropies by relaxing the fourth Shannon-Khinchin axiom [48,49], building on earlier work by Tsallis [41,65]. Moreover, numerous studies have established a connection between nonextensivity and complexity [66–70]. In this context, the proposed measure of nonextensivity can also be interpreted as a measure of complexity.

### 4.4 The generation of nonextensive statistics from Wick's rotation

Here, we investigate the reverse approach: given a specific Wick's rotation function and a Lorentzian action, it is possible to derive the corresponding statistical framework. We illustrate this procedure through a straightforward example and then proceed to a more generalized formulation.

[6] A similar calculation for the nonextensive Kaniadakis statistics [53, 62] gives its nonextensivity measure as $\eta_{Kan} \simeq \frac{1}{1536\pi^2}$.

The second simplest generalized Wick's rotation one can postulate, after the Boltzmannian one, is of the form

$$t \to -i(1+\theta)\tau, \tag{62}$$

where $\theta$ is a constant nonextensivity parameter. This rotation generates a nonextensive statistics, $S_\theta$, which can be computed through Eq. (23). Assuming a 4-dimensional asymptotically flat Schwarzschild black hole metric, one writes a differential equation for $S_\theta$. Using Eq. (32) and Eq. (44), we have

$$M\frac{\partial S_\theta}{\partial M} - S_\theta = \mathcal{I}_{cl}^{\theta} = \frac{\beta^2(\theta+1)}{16\pi}. \tag{63}$$

However, since $\beta \equiv \dfrac{\partial S_\theta}{\partial M}$, one finds that $S_\theta$ obeys a differential equation such as,

$$\frac{(\theta+1)}{16\pi}\left(\frac{\partial S_\theta}{\partial M}\right)^2 - M\frac{\partial S_\theta}{\partial M} + S_\theta = 0. \tag{64}$$

Equation (64) has the solution

$$S_\theta = \frac{4\pi M^2}{\theta+1} = \frac{\pi r_h^2}{\theta+1}, \tag{65}$$

in which, we used the relation $r_h = 2M$ for the horizon radius of the asymptotically flat Schwarzschild black hole. It is clear that as the parameter $\theta \to 0$, the entropy becomes that of BG, $S_\theta \to S_{BH}$. This $\theta$-statistics is the simplest generalization of the black hole BG statistics one can contemplate. In the case of *extreme nonextensivity*, $\theta \to \infty$, the black hole entropy $S_\theta$ vanishes and the black hole is in a *perfectly-ordered* thermodynamic state with a unique micro-state.

In a similar manner we generate the Rényi entropy from its Wick's rotation Eq. (46) and Euclidean action Eq. (47)

$$\frac{\lambda}{512\pi^2}\left(\frac{\partial S_R}{\partial M}\right)^4 + \frac{1}{16\pi}\left(\frac{\partial S_R}{\partial M}\right)^2 - M\frac{\partial S_R}{\partial M} + S_R = 0. \tag{66}$$

Which admits as a solution, to the first order in $\lambda$

$$S_R = 4\pi M^2\left(1 - 2\lambda\pi M^2\right) + O(\lambda^2) \tag{67}$$

$$= \pi r_h^2 - \frac{\lambda}{2}\left(\pi^2 r_h^4\right) + O\left(\lambda^2\right). \tag{68}$$

### 4.5 The *4d*-flat Kerr black hole in Rényi statistics

We apply the formalism to the Kerr asymptotically flat black hole in Rényi statistics (4*d* Rényi-Kerr). The expressions of the entropy, mass, and inverse temperature read as

$$S_R = \frac{1}{\lambda}\ln\left[1 + \lambda\pi\left(r_h^2 + a^2\right)\right], \quad M = \frac{r_h^2 + a^2}{2r_h}$$

$$\text{and} \quad \beta = \frac{4\pi r_h \left(r_h^2 + a^2\right)}{\left(r_h^2 - a^2\right)\left[1 + \pi\lambda\left(r_h^2 + a^2\right)\right]} \tag{69}$$

For *small spin parameter* $a \ll r_h$, the horizon radius $r_h$ is expressed in terms of the inverse temperature $\beta$ such as

$$r_h = \frac{\beta}{4\pi} - \frac{8\pi a^2}{\beta} + \frac{\lambda\beta^3}{64\pi^2} + O\left(a^4, \lambda^2\right). \tag{70}$$

An analogous calculation to the preceding sections yields the expression for the Euclidean Lagrangian of Rényi-Kerr black hole as

$$L_{cl}^{\lambda}(\tau, a) = -\frac{\tau}{8\pi} + \lambda\left(-\frac{\tau^3}{128\pi^2} + \frac{\tau a^2}{8}\right) + \frac{2\pi a^2}{\tau}. \tag{71}$$

To retrieve Wick's rotation function, one needs the Euclidean Lagrangian for the asymptotically-flat Kerr black hole which can be found by putting $\lambda = 0$ in Eq. (71)

$$L_{cl}^{0}(\tau, a) = -\frac{\tau}{8\pi} + \frac{2\pi a^2}{\tau}. \tag{72}$$

Thus, applying Eq. (28) and integrating with respect to $\tau$, the Wick's rotation function reads

$$f_\lambda(\tau) = \tau - \pi\lambda a^2\tau + \frac{\lambda\tau^3}{48\pi} + O\left(\lambda^2, a^4\right). \tag{73}$$

We notice that Eq. (73) reduces to the function derived for the 4-dimensional Rényi–Schwarzschild black hole, as given in Eq. (49), in the limit $a \to 0$. Furthermore, Wick's rotation function exhibits dependence on the spin parameter $a$, which is intrinsic to the Lorentzian action of the Kerr metric. As demonstrated in Eq. (69), this dependence arises because, in the case of the Kerr black hole, the Rényi entropy itself becomes a function of $a$.

Lastly, the natural exponentiation of the Euclidean action, gives the gravitational partition function for the 4$d$ Rényi-Kerr,

$$Z_\lambda[a] = \exp\left[-\frac{\tau^2}{16\pi} + 2\pi a^2\log(\tau) + \lambda\left(\frac{a^2\tau^2}{16} + \frac{\tau^4}{512\pi^2}\right)\right]. \tag{74}$$

In the limit of vanishing spin parameter, $a \to 0$, we have the Rényi-Sch partition function, Eq. (48).

### 4.6 The *4d*-flat Schwarzschild black hole in Barrow statistics

In Barrow statistics [52], the nonextensivity is quantified by the parameter $\Delta$. We have for the Barrow entropy, mass, and horizon radius of the 4$d$ asymptotically flat Schwarzschild black hole (4$d$ Barrow-Sch)

$$S_B = \left(\pi r_h^2\right)^{1+\frac{\Delta}{2}}, \quad M = \frac{r_h}{2} \quad \text{and}$$

$$r_h = \left(\frac{\pi^{-\frac{\Delta-1}{2}}\beta}{2(\Delta+2)}\right)^{\frac{1}{\Delta+1}}. \tag{75}$$

Injecting these expressions in Eq. (28), we obtain the following expression for the Euclidean Lagrangian

$$L_{cl}^{\Delta}(\tau) \equiv L_{cl}(\tau) f_\Delta^{'}(\tau) = -2^{-\frac{\Delta+2}{\Delta+1}}\left(\frac{\tau}{\pi^{\frac{\Delta}{2}+1}(\Delta+2)}\right)^{\frac{1}{\Delta+1}}. \tag{76}$$

Just as before, $L_{cl}(\tau)$, is the Euclidean Lagrangian for the asymptotically flat Schwarzschild black hole metric, which is independent of the parameter $\Delta$ and given in Eq. (44). After integration with respect to imaginary time $\tau$, one gets the exact expression of Wick's rotation function for the Barrow statistics as

$$f_\Delta(\tau) = 2^{\frac{2\Delta+1}{\Delta+1}}\pi(\Delta+1)\left(\frac{\tau}{\pi^{\frac{\Delta}{2}+1}(\Delta+2)}\right)^{\frac{1}{\Delta+1}}. \tag{77}$$

For small Barrow parameter $\Delta \ll 1$

$$f_\Delta(\tau) = \tau + \frac{\tau}{2}\left[1 - \ln\left(\frac{\tau^2}{16\pi}\right)\right]\Delta + O\left(\Delta^2\right). \tag{78}$$

The generalized classical Euclidean action for Barrow statistics and the corresponding gravitational partition function read

$$\mathcal{I}_{cl}^{\Delta} = \frac{\tau^2}{16\pi} - \Delta\frac{\tau^2}{32\pi}\ln\left(\frac{\tau^2}{16\pi}\right) + O\left(\Delta^2\right), \tag{79}$$

$$Z_\Delta = \exp\left[-\frac{\tau^2}{16\pi} + \Delta\frac{\tau^2}{32\pi}\ln\left(\frac{\tau^2}{16\pi}\right)\right]. \tag{80}$$

Once again, the BG statistical mechanics are found in the limit of vanishing parameter $\Delta \to 0$. As a summary, to obtain the nonextensive Barrow statistical mechanics, to first order in $\Delta$, one should Wick rotate the Lorentzian action such as

$$t \longrightarrow -i\left(\tau + \frac{\tau}{2}\left[1 - \ln\left(\frac{\tau^2}{16\pi}\right)\right]\Delta\right). \tag{81}$$

In connection with the nonextensivity measure, applying Eq. (60) to the Barrow statistics reveals its expression to be

$$\eta(S_\Delta) \simeq \frac{4\log\left(2\sqrt[4]{\pi}\right) - 1}{4}. \tag{82}$$

### 4.7 The higher dimensional flat Schwarzschild black hole in Rényi statistics

In this section, we extend section (4.2) to higher dimensions and show how to compute Wick's rotation function in this case. We start with the generalization of Eq. (41) to $d$-dimensional spacetime

$$S_R = \frac{1}{\lambda}\ln\left(1+\frac{\lambda\pi^{\frac{d-1}{2}}r_h^{d-2}}{2\Gamma\left(\frac{d-1}{2}\right)}\right),\quad M = \frac{\pi^{\frac{d}{2}-\frac{3}{2}}r_h^{d-3}(d-2)}{8\Gamma\left(\frac{d}{2}-\frac{1}{2}\right)}$$
$$\text{and } \beta = \frac{4\pi^{\frac{3}{2}}r_h\Gamma\left(\frac{d}{2}-\frac{3}{2}\right)}{\pi^{\frac{d}{2}}\lambda r_h^{d-2}+2\sqrt{\pi}\Gamma\left(\frac{d}{2}-\frac{1}{2}\right)}. \tag{83}$$

Here, we express the inverse temperature $\beta$ in terms of the horizon radius $r_h$, as the direct inversion of this relationship is generally not feasible. The classical Euclidean action is then given by

$$\mathcal{I}_{cl}^{\lambda,d} = \frac{\pi^{d/2}(d-2)r_h^{d=2}}{(d-3)\left[\lambda\pi^{d/2}r_h^{d-2}+2\sqrt{\pi}\Gamma\left(\frac{d-1}{2}\right)\right]}$$
$$-\frac{1}{\lambda}\ln\left(1+\frac{\pi^{\frac{d-1}{2}}\lambda r_h^{d-2}}{2\Gamma\left(\frac{d-1}{2}\right)}\right). \tag{84}$$

The corresponding Euclidean Lagrangian, $L_{cl}^{\lambda,d}$, can be computed as

$$L_{cl}^{\lambda,d} = -\frac{\partial\mathcal{I}_{cl}^{\lambda,d}}{\partial\tau} = -\frac{\partial\mathcal{I}_{cl}^{\lambda,d}}{\partial r_h}\frac{dr_h}{d\tau}, \tag{85}$$

then the Wick's rotation function $f_\lambda^d(\tau)$ is determined by

$$L_{cl}^d\frac{df_\lambda^d}{d\tau} \equiv L_{cl}^{\lambda,d}, \tag{86}$$
$$\Longrightarrow \frac{df_\lambda^d}{d\tau} = \frac{L_{cl}^{\lambda,d}}{L_{cl}^d}, \tag{87}$$
$$\Longrightarrow \frac{df_\lambda^d}{dr_h} = \frac{L_{cl}^{\lambda,d}}{L_{cl}^d}\frac{d\beta}{dr_h},\quad \text{since}\quad \beta\equiv\tau. \tag{88}$$

Here, $L_{cl}^d$, is the Euclidean Lagrangian of the $d$-dimensional asymptotically flat Schwarzschild black hole for $\lambda = 0$. It generalizes Eq. (44) as

$$L_{cl}^d = -\frac{\pi^{\frac{d}{2}-\frac{3}{2}}r_h^{d-3}(d-2)}{8\Gamma\left(\frac{d}{2}-\frac{1}{2}\right)}. \tag{89}$$

From Eq. (88), we obtain

$$\frac{df_\lambda^d}{dr_h} = \frac{2\pi\left(\Gamma\left(\frac{d-3}{2}\right)-\pi^{\frac{d-1}{2}}\lambda r_h^{d-2}\right)}{\Gamma\left(\frac{d-1}{2}\right)+4\pi^{\frac{d-1}{2}}\lambda r_h^{d-2}}. \tag{90}$$

For small parameter $\lambda \ll 1$, a straightforward integration yields the Rényi Wick's rotation function in $d$-dimensions in terms of $r_h$ to be

$$f_\lambda^d(r_h) = \frac{2}{(d-3)\Gamma\left(\frac{d-1}{2}\right)}$$
$$\times\left[2\pi\Gamma\left(\frac{d-1}{2}\right)r_h - \pi^{\frac{d+1}{2}}\lambda r_h^{d-1}\right], \tag{91}$$

and $r_h$ is expressed by Eq. (83) as an implicit function of $\beta$. When $\lambda$ vanishes, one gets from Eqs. (91) and (83)

$$f_0^d(r_h) = \frac{2\pi\Gamma\left(\frac{d-3}{2}\right)}{\Gamma\left(\frac{d-1}{2}\right)}r_h = \frac{4\pi}{d-3}r_h \quad\text{and}$$
$$\beta = \frac{2\pi\Gamma\left(\frac{d}{2}-\frac{3}{2}\right)}{\Gamma\left(\frac{d}{2}-\frac{1}{2}\right)}r_h = \frac{4\pi}{d-3}r_h. \tag{92}$$

That is, $f_0^d(r_h) = \beta = \tau$. Thus, we arrive at the standard Boltzmann–Gibbs Wick's rotation. This can be seen directly by inspection of Eq. (88); For $\lambda = 0$, it reduces to

$$\frac{df_0^d}{dr_h} = \frac{d\beta}{dr_h} \Longrightarrow \frac{df_0^d}{d\beta} = 1 \Longrightarrow f_0^d(\beta) = \beta = \tau. \tag{93}$$

After showing the applicability of the proposed formalism to some black hole/statistics combinations, we explore in the next section the Rényi/AdS equivalence.

## 5 Rényi/AdS equivalence

In the context of the conjectured equivalence between the 4d AdS Schwarzschild black hole in Boltzmann–Gibbs statistics (AdS-Sch) and the 4d nonextensive flat Schwarzschild black hole in Rényi statistics (Rényi-Sch), we propose to equate their partition functions. That is

$$Z_{AdS_4} = Z_\lambda, \tag{94}$$

where $Z_{AdS_4} \equiv Z_0(\Lambda)$. Apart from strong connections revealed by recent studies [71–74], which points to such equivalence, the present framework also provides support by remarking that their Euclidean actions, Eqs. (38) and (47), share the same functional form. To first order in the parameters, $\Lambda$ and $\lambda$, they have the following partition functions

$$Z_{AdS_4} = \exp\left(-\frac{\beta^2}{16\pi}+\frac{\Lambda\beta^4}{384\pi^3}\right), \tag{95}$$

and

$$Z_\lambda = \exp\left(-\frac{\beta^2}{16\pi}-\frac{\lambda\beta^4}{512\pi^2}\right). \tag{96}$$

Thus, applying (94) gives the relation

$$\Lambda = -\frac{3\pi}{4}\lambda + O\left(\lambda^2\right), \tag{97}$$

which aligns with previous findings obtained using the *Hamiltonian approach to thermodynamics* [72]. Thermodynamically, this indicates that a 4$d$ AdS-Schwarzschild black hole is equivalent to a 4$d$ Rényi–Schwarzschild black hole, provided that Eq. (97) holds.

In the context of the $AdS_5/CFT_4$ duality, The GKP-Witten relation [75,76] proposes the equality of the partition functions of string theory on 5-dimensional $AdS_5$ spacetime and the 4-dimensional conformally invariant gauge theory such as,

$$Z_{AdS_5} = Z_{CFT_4}. \tag{98}$$

In the large-$N_c$ limit for the gauge side of Eq. (98), one can use the saddle-point-approximation for the gravity side, which gives $Z_{AdS_5}$ in terms of the Euclidean classical action. Consequently, a parameter dictionary can be derived for this gauge/gravity duality such as

$$N_c^2 = \frac{\pi}{2G_5}\left(-\frac{6}{\Lambda_5}\right)^{\frac{3}{2}} \tag{99}$$

where $N_c$ is the number of charges of the gauge group and $G_5$ is Newton's constant in 5-dimensional spacetime. The gravitational partition functions of the 5$d$ AdS-Sch and Rényi-Sch black holes are calculated as

$$Z_{AdS_5} = \exp\left(-\frac{\beta^3}{32\pi} + \frac{\Lambda_5\beta^5}{256\pi^3}\right) \quad \text{and}$$

$$Z_\lambda^5 = \exp\left(-\frac{\beta^3}{32\pi} - \frac{\lambda_5\beta^6}{512\pi^2}\right) \tag{100}$$

Thus, an equivalence holds between these two thermodynamic systems if,

$$\Lambda_5 = -\frac{\pi}{2}\lambda_5\beta + O\left(\lambda_5^2\right) \tag{101}$$

An interesting interpretation of Eq. (101) comes from considering the extended phase space. It is usual to treat the cosmological constant $\Lambda$ as the thermodynamic pressure $P$ [77,78],

$$P = -\frac{\Lambda_5}{8\pi}. \tag{102}$$

Similarly, one defines the Rényi pressure [71], $P_R$ in 5-dimensional spacetime as,

$$P_R = \frac{3\pi}{16}\lambda_5 r_h. \tag{103}$$

From the expression of $\beta$ in terms of the horizon radius $r_h$, Eq. (69), it is straightforward to obtain an inverted expression in five dimensions and for small $\lambda$ such as,

$$r_h = \frac{\beta}{2\pi} + O\left(\lambda_5\right) \tag{104}$$

By substitution in (103), we have for the Rényi pressure,

$$P_R = \frac{3}{32}\lambda_5\beta + O\left(\lambda_5^2\right). \tag{105}$$

A quick comparison of Eqs. (101) and (105), taking into consideration Eq. (102) gives,

$$\Lambda_5 = -8\pi P = -\frac{16\pi}{3}P_R, \tag{106}$$

$$\implies P = \frac{2}{3}P_R. \tag{107}$$

In this respect, Eq. (99) can be reformulated in terms of pressure as

$$N_c^2 = \frac{\pi}{2G_5}\left(\frac{3}{4\pi P}\right)^{\frac{3}{2}} = \frac{\pi}{2G_5}\left(\frac{9}{8\pi P_R}\right)^{\frac{3}{2}}. \tag{108}$$

Also, in terms of length scales, one has the usual AdS spacetime radius $L = \sqrt{\frac{3}{4\pi P}}$ and the Rényi length scale $L_\lambda = \sqrt{\frac{9}{8\pi P_R}}$, then

$$N_c^2 = \frac{\pi}{2G_5}L^3 = \frac{\pi}{2G_5}L_\lambda^3. \tag{109}$$

In view of the *Rényi* $_5/AdS_5$ equivalence,[7] we see that the expression of the $AdS_5/CFT_4$ correspondence induces the duality *Rényi*$_5/CFT_4$ with the dictionary,

$$N_c^2 = \frac{\pi}{2G_5}L_\lambda^3 \quad \text{and} \quad \lambda_t = \left(\frac{L_\lambda}{l_s}\right)^4. \tag{110}$$

Where $\lambda_t$ is the 't Hooft parameter and $l_s$ is the string length. The two sides of the duality in this new form are both flat theories. A further investigation of this reformulation is warranted.

## 6 Conclusion

In this study, we extend the Euclidean path integral formalism to derive nonextensive thermodynamics from the Lorentzian action via a generalized Wick's rotation. This framework places nonextensive and extensive statistics on equal footing, allowing them to emerge from Wick's rotation of the same underlying action. In essence, this approach decouples the Lorentzian action, which defines the equations of motion for a given system, from the choice of statistical framework, thereby restoring the freedom to apply any desired statistics independently of the action itself. Furthermore, this method enables any statistical form to be systematically implemented as a rotation of the Lorentzian action, provided a suitable Wick's rotation function exists, rather than imposing the statistics artificially at a later stage. By selecting different Wick's rotation functions and examining their deviations from extensivity, new statistical models can be generated. Specifically, constructing a Wick's rotation function that diverges from the standard Boltzmannian form, $f_0(\tau) = \tau$, to varying extents facilitates the creation of weakly or strongly

[7] This equivalency can be easily extended to all $d$-dimensional spacetimes *Rényi*$_d/AdS_d$, $d > 3$. In particular, for $d = 4$ spacetime, $P_R = \frac{3\lambda}{32.}$, consequently one has $P = P_R$.

nonextensive statistics. The principal conceptual advantage of this generalization is that it integrates nonextensivity into the linear framework of quantum mechanics, which has traditionally been restricted to Boltzmann–Gibbs statistics through the standard Wick's rotation.

Furthermore, We proceeded to apply the generalized Euclidean path integral in the saddle point approximation to black hole thermodynamics and computed Wick's rotation functions for a variety of black holes under different statistics. We showed that Rényi statistics is strongly nonextensive in contrast to the Tsallis one by introducing a novel measure $\eta$ to quantify the nonextensive character of any statistics. Thus, it provides an order in the set of all non-extensive statistical mechanics. Furthermore, we examined the *Rényi/AdS* equivalence which we translated to a *Rényi/CFT* correspondence whenever the *AdS/CFT* one holds. Moreover, in this formulation, we have flat theories on both sides of the duality. Also, this suggests a novel type of correspondence, namely, *a gauge/statistics duality*. The investigation of such a theme is one of the main motivations of this study and we intend to deepen our inquiry to reach further insights.

**Data Availability Statement** This manuscript has no associated data. [Authors' comment: Data sharing not applicable to this article as no datasets were generated or analysed during the current study.]

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