Moduli spaces of curves and enumerative geometry via topological recursion

Lewański, D.

Publication date
2017

Document Version
Final published version

License
Other

Link to publication

Citation for published version (APA):
Lewański, D. (2017). *Moduli spaces of curves and enumerative geometry via topological recursion*. [Thesis, fully internal, Universiteit van Amsterdam].

Moduli spaces of curves and enumerative geometry via topological recursion

Danilo Lewański

Moduli spaces of curves and enumerative geometry via topological recursion

Academisch Proefschrift

ter verkrijging van de graad van doctor
aan de Universiteit van Amsterdam
op gezag van de Rector Magnificus
prof. dr. ir. K. I. J. Maex
ten overstaan van een door het College voor Promoties ingestelde
commissie, in het openbaar te verdedigen in
de Agnietenkapel
op woensdag op 5 juli 2017, te 10:00 uur

door

Danilo Lewański

geboren te Montecchio Emilia, Italië.

Promotiecommissie

Promotor:

Prof. dr. S. V. Shadrin
Korteweg-de Vries Instituut voor Wiskunde
Universiteit van Amsterdam

Overige leden:

Dr. R. R. J. Bocklandt
Korteweg-de Vries Instituut voor Wiskunde
Universiteit van Amsterdam

Prof. dr. A. Chiodo
Institút de Mathématiques de Jussíeu
UPMC, Paris

Prof. dr. C. F. Faber
Mathematisch Instituut
Universiteit Utrecht

Prof. dr. J. P. Harnad
Centre de Recherches Mathématiques
Université de Montréal

Dr. H. B. Posthuma
Korteweg-de Vries Instituut voor Wiskunde
Universiteit van Amsterdam

Prof. dr. J. V. Stokman
Korteweg-de Vries Instituut voor Wiskunde
Universiteit van Amsterdam

Faculteit der Natuurwetenschappen, Wiskunde en Informatica

FSC www.fsc.org
MIX
Paper from responsible sources
FSC® C128610

This research was funded by the Netherlands Organization for Scientific Research.

Ai miei genitori

Typesetting in LaTeX.
Cover design by Malte Taylor.

Contents

1 Introduction

The thesis considers several enumerative geometric problems concerning the topology of the moduli space of curves and their combinatorics. These enumerative geometric problems are analysed from different intertwined points of view and using different mathematical tools, including Hurwitz theory, Givental theory, cohomological field theories, integrable hierarchies, Fock spaces, quantum curves, and a relatively new powerful technique introduced by Chekhov, Eynard and Orantin known as topological recursion. These subjects lie in the interplay between enumerative algebraic geometry, differential geometry and mathematical physics.

The introduction is organised as follows. Section 1.1 contains a brief motivation on the interplay of these different points of view, Sections 1.2–1.14 are devoted to the mathematical and physical background necessary to state the results of the thesis, Section 1.15 outlines the content of thesis and describes the division in chapters, and finally Sections 1.15.1–1.15.5 describe in detail the results, chapter by chapter. Every chapter is self-contained and can be read independently. At the end of the thesis summaries in English and in Dutch are provided.

1.1 Motivation

An important conjecture formulated by Witten in 1991, and proved by Kontsevich in 1992, was one of the main motivations to investigate the connection between integrable hierarchies and enumerative problems in algebraic geometry. It roughly says that the generating series encoding numbers with a certain geometric interpretation obeys a certain integrable hierarchy; that is, it is a solution of a certain infinite list of partial differential equations (PDEs) involving infinitely many variables. This hierarchy of PDEs is known as KdV integrable hierarchy, since the first equation of the list is the Korteweg - de Vries equation, which models 1 dimensional waves on shallow water. These PDEs describe recursive relations among the aforementioned numbers, and since the initial data comes from algebraic geometry, these numbers are identified uniquely. The geometric meaning of the numbers lies in the intersection theory of certain classes defined on the moduli spaces of curves, which is one of the main objects of study in algebraic geometry.

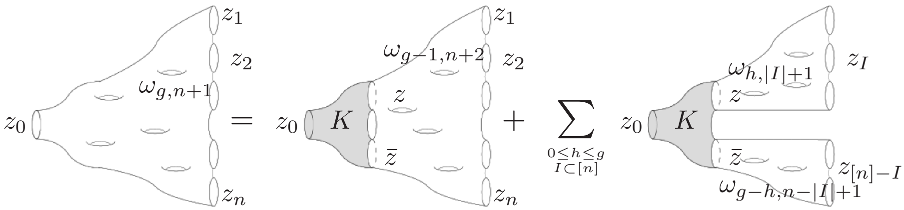
Witten's conjecture has important implications in the context of two dimensional gravity, in which one is interested in certain integrals over the space of all gravitational fields (Riemannian metrics) over the surface formed by space and time. To compute such integrals, two seemingly different approaches have been developed. The first approach uses sums over triangulations of surfaces, the second involves the integration over spaces of conformally equivalent metrics. The latter corresponds to the generating series collecting the numbers mentioned above. In proving the conjecture, Kontsevich thus proved that these two approaches are in fact equivalent.

The Witten-Kontsevich result turned out to be just the tip of the iceberg. Many more generating series are solutions to more involved integrable hierarchies. Often the geometric meaning of the numbers still lies in the intersection theory of certain cohomological classes, but for more involved classes. The appropriate mathematical tools that capture the behaviour of these numbers are called cohomological field theories, which generalise Gromov-Witten theories. Gromov-Witten theory counts algebraic curves in a fixed target variety with fixed cohomology class, and provides a formal mathematical counterpart of topological string theory in physics, in which one is interested in counting the embeddings of closed strings' world-sheets in space-time. Roughly speaking, cohomological field theories axiomatise and generalise Gromov-Witten theories, substituting the cohomology of the target variety with an arbitrary vector space. They were introduced by Manin and Kontsevich. The Witten-Kontsevich generating series plays a universal role, since it corresponds to the trivial cohomological field theory, or, in terms of Gromov-Witten, it encodes the Gromov-Witten theory with a single point as target variety. Semi-simple cohomological field theories are described by the Givental-Teleman classification. More explicitly, CohFT's with fixed semi-simple topological field theory admit the transitive action of a matrix group, called the Givental group, that takes values in the endomorphisms of the vector space of the theory, so that each semi-simple cohomological field theory can be obtained by the action of an element of this group on a topological field theory.

On the other hand, the list of PDE's Witten-Kontsevich generating series satisfies can be reformulated in terms of differential operators of the Virasoro type. Virasoro constraints encode relations between invariants of various nature, and they are ubiquitous in the fields of random matrix models, classical integrable systems, statistical physics and string theory. They are equivalent to loop equations in random matrix models, and their solutions are constructed by a universal procedure. It is formulated in terms of differential geometry on a Riemann surface, and it carries the information of the underlying cohomological field theory - this is the Chekhov-Eynard-Orantin topological recursion.

The Chekhov-Eynard-Orantin (CEO) topological recursion is a recent and powerful method motivated by random matrix theory and statistical mechanics to compute invariants recursively through the topology of moduli spaces of curves. Specialisations of these invariants recover many known invariants, including Weil-Petersson volumes, Gromov-Witten invariants, Hurwitz numbers, Tutte's enumeration of maps, knot polynomials or asymptotics of random matrices expectation values. CEO topological recursion has been chosen to be the topic of 2016 AMS Symposium — where it celebrated ten years since the formulation — for its fast development and its capacity to attract researchers from many different areas of mathematics and physics.

Topological recursion takes as input a spectral curve — often an algebraic curve with some additional structure — and produces a family of differential forms defined on the product of several copies of the curve. These differentials $\omega_{g,n}$, indexed by two non-negative integers g and n, are defined by a universal recursion on $2g-2+n$ based on the glueing of surfaces. Often the numbers of interests are the coefficients of these differentials $\omega_{g,n}$.



$$\text{Spectral Curve} \xrightarrow{\text{Topological recursion}} \text{Invariants } \omega_{g,n}.$$

Let us name two simple examples. Consider the spectral curves described by pairs of functions $(y(z), x(z))$ on the Riemann sphere, given by the Airy curve (z, z^2) and by the curve $(\frac{sin(2\pi z)}{4\pi}, z^2)$. Let us run the topological recursion and look at the coefficients of the output differentials $\omega_{g,n}$. The former gives the number of the Witten-Kontsevich generating series, the latter gives the volumes of moduli spaces of hyperbolic spaces computed by Mirzakhani's recursion. More complicated examples involve the Lambert curve and its generalisation for Hurwitz theory, Hitchin systems, equations of mirror symmetric theory of a Calaby-Yau threefold, to name a few.

One of the most important results in the intersection theory of the moduli spaces of curves is the celebrated ELSV formula, due to Ekedahl, Lando, Shapiro, and Vainshtein [21]. In particular, the ELSV formula expresses connected Hurwitz numbers in terms of Hodge integrals on the moduli spaces of curves, and plays a central role in many of the alternative proofs of Witten's conjecture that appeared after the first proof by Kontsevich. On the other side, the topological recursion and ELSV-type formulae are very much related. In the first place, Eynard and Orantin [25] proved that the asymptotic behaviour of the correlation differentials $\omega_{g,n}$ near a regular zero of the spectral curve is described by the Airy curve, and therefore the generated numbers are fundamentally related to the moduli space of curves. Here, again, the Witten-Kontsevich generating series shows its universality. Secondly, Eynard [23, 22] showed that the generated numbers are an explicit combination of Hodge integrals on the moduli space of curves and the moduli space of a-coloured stable curves. This form already presents many similarities with the structure of the ELSV formula. Thirdly, Dunin-Barkowski, Orantin, Shadrin, and Spitz [16] identify CEO topological recursion and Givental theory. This identification explicitly connects the generated numbers with cohomological field theories (and Gromov-Witten theories), and it can been used to explicitly express the correlation differentials in terms of intersection theory of the moduli space of curves, given a spectral curve. In particular, it can be used to provide new proofs of the ELSV formula and its generalisations.

1.2 Semi-infinite wedge formalism

In this section we recall the semi-infinite wedge formalism, tailored for our future use. It is nowadays a standard tool in Hurwitz theory. We refer the reader, for instance, to [34] and references therein for a complete exposition.

Let V be an infinite-dimensional complex vector space with a basis labeled by half-integers. Denote the basis vector labeled by $m/2$ by $\underline{m/2}$, so $V = \bigoplus_{i\in\mathbb{Z}+\frac{1}{2}} \mathbb{C}\underline{i}$.

Definition 1.2.1. The semi-infinite wedge space $\bigwedge^{\frac{\infty}{2}}(V) = \mathcal{V}$ is defined to be the span of all of the semi-infinite wedge products of the form

$$\underline{i_1} \wedge \underline{i_2} \wedge \cdots$$

for any decreasing sequence of half-integers (i_k) such that there is an integer c with $i_k + k - \frac{1}{2} = c$ for k sufficiently large. The constant c is called the *charge*. We give $\mathcal{V}$ an inner product $(\cdot,\cdot)$ declaring its basis elements to be orthonormal.

Remark 1.2.2. By definition 1.2.1 the charge-zero subspace $\mathcal{V}_0$ of $\mathcal{V}$ is spanned by semi-infinite wedge products of the form

$$\underline{\lambda_1 - \frac{1}{2}} \wedge \underline{\lambda_2 - \frac{3}{2}} \wedge \cdots$$

for some integer partition λ. Hence we can identify integer partitions with the basis of this space:

$$\mathcal{V}_0 = \bigoplus_{n\in\mathbb{N}} \bigoplus_{\lambda\vdash n} \mathbb{C}v_\lambda$$

The empty partition $\emptyset$ plays a special role. We call

$$v_\emptyset = \underline{-\frac{1}{2}} \wedge \underline{-\frac{3}{2}} \wedge \cdots$$

the vacuum vector and we denote it by $|0\rangle$. Similarly we call the covacuum vector its dual with respect to the scalar product $(\cdot,\cdot)$ and we denote it by $\langle 0|$.

The *vacuum expectation value* or *disconnected correlator* $\langle\mathcal{P}\rangle^\bullet$ of an operator $\mathcal{P}$ acting on $\mathcal{V}_0$ is defined to be:

$$\langle\mathcal{P}\rangle^\bullet := (|0\rangle, \mathcal{P}|0\rangle) =: \langle 0|\mathcal{P}|0\rangle$$

We also define the functions

$$\zeta(z) = e^{z/2} - e^{-z/2} = 2\sinh(z/2)$$

and

$$\mathcal{S}(z) = \frac{\zeta(z)}{z} = \frac{\sinh(z/2)}{z/2}.$$

Definition 1.2.3. The following list contains the useful operators for the purpose of the thesis.

i) For k half-integer the operator $\psi_k \colon (\underline{i_1} \wedge \underline{i_2} \wedge \cdots) \mapsto (\underline{k} \wedge \underline{i_1} \wedge \underline{i_2} \wedge \cdots)$ increases the charge by 1. Its adjoint operator ψ_k^* with respect to $(\cdot,\cdot)$ decreases the charge by 1.

ii) The normally ordered products of ψ-operators

$$E_{i,j} := \begin{cases} \psi_i\psi_j^*, & \text{if } j > 0 \\ -\psi_j^*\psi_i & \text{if } j < 0 \end{cases}.$$

preserve the charge and hence can be restricted to $\mathcal{V}_0$ with the following action. For $i \neq j$, $E_{i,j}$ checks if v_λ contains $\underline{j}$ as a wedge factor and if so replaces it by $\underline{i}$. Otherwise it yields 0. In the case $i = j > 0$, we have $E_{i,j}(v_\lambda) = v_\lambda$ if v_λ contains $\underline{j}$ and 0 if it does not; in the case $i = j < 0$, we have $E_{i,j}(v_\lambda) = -v_\lambda$ if v_λ does not contain $\underline{j}$ and 0 if it does. This gives a projective representation of $\mathcal{A}_\infty$, the Lie algebra of complex $\mathbb{Z}\times\mathbb{Z}$ matrices with only finitely many non-zero diagonals [34].

iii) The diagonal operators are assembled into the operators

$$\mathcal{F}_n := \sum_{k\in\mathbb{Z}+\frac{1}{2}} \frac{k^n}{n!} E_{k,k}.$$

The operator $C := \mathcal{F}_0$ is called the *charge operator*, while the operator $E := \mathcal{F}_1$ is called *energy operator*. Note that $\mathcal{F}_0$ vanishes identically on $\mathcal{V}_0$. We say that an operator $\mathcal{P}$ on $\mathcal{V}_0$ is of energy $c \in \mathbb{Z}$ if $[\mathcal{P}, E] = c\,\mathcal{P}$.

In other words, if $\mathcal{P}$ is an operator of energy c, then it maps a basis element of energy k into a combination of basis elements that all have energies $k - c$. The operator $E_{i,j}$ has energy $j - i$, hence all the $\mathcal{F}_n$'s have zero energy. It will be important to us that operators with positive energy annihilate the vacuum while negative energy operators are annihilated by the covacuum, explicitly: let $\mathcal{M}$ be any operator, let $\mathcal{P}$ have positive energy and $\mathcal{N}$ have negative energy, then $\langle \mathcal{M}\mathcal{P}\rangle^\bullet = 0$ and $\langle \mathcal{N}\mathcal{M}\rangle^\bullet = 0$.

iv) For n any integer and z a formal variable one has the energy n operators:

$$\mathcal{E}_n(z) = \sum_{k\in\mathbb{Z}+\frac{1}{2}} e^{z(k-\frac{n}{2})} E_{k-n,k} + \frac{\delta_{n,0}}{\zeta(z)}.$$

v) For n any nonzero integer one has the energy n operators:

$$\alpha_n = \mathcal{E}_n(0) = \sum_{k\in\mathbb{Z}+\frac{1}{2}} E_{k-n,k}.$$

The operator α_n is adjoint to α_{-n} with respect to the scalar product $(\cdot,\cdot)$ described above.

The commutation formula for $\mathcal{E}$ operators reads:

$$[\mathcal{E}_a(z), \mathcal{E}_b(w)] = \zeta\left(\det\begin{bmatrix} a & z \\ b & w\end{bmatrix}\right) \mathcal{E}_{a+b}(z+w)$$

and in particular $[\alpha_k, \alpha_l] = k\delta_{k+l,0}$.

Note that $\mathcal{E}_k(z)\big|0\big\rangle = 0$ if $k > 0$, while $\mathcal{E}_0(z)\big|0\big\rangle = \zeta(z)^{-1}\big|0\big\rangle$. We will also use the $\mathcal{E}$ operator without the correction in energy zero, i.e.

$$\tilde{\mathcal{E}}_0(z) = \sum_{k\in\mathbb{Z}+\frac{1}{2}} e^{zk} E_{k,k} = \sum_{n=0}^{\infty} \mathcal{F}_n z^n = C + Ez + \mathcal{F}_2 z^2 + \ldots$$

which annihilates the vacuum and obeys the same commutation rule as $\mathcal{E}_0$. It is known that the operators $\mathcal{F}_r$ have eigenvectors v_λ with eigenvalues

$$\frac{1}{r!}\sum_{i=1}^{\ell(\lambda)} (\lambda_i - i + \frac{1}{2})^r - (-i + \frac{1}{2})^r.$$

1.3 Hurwitz numbers and their variations

Hurwitz theory studies coverings of Riemann surfaces with prescribed ramifications. For a complete introduction to the topic we refer, e.g., to [6]. In the following we introduce the standard Hurwitz numbers and the variations that we study in the thesis.

The number of possibly disconnected Hurwitz coverings $h^{\bullet}_{\mu,\nu,B}$ of degree d over the 2-sphere with ramifications described by the partitions μ over zero, and ν over infinity, with $|\mu| = |\nu| = d$, and other non specified ramifications codified by the element B in the group algebra $\mathbb{Q}(\mathfrak{S}_d)$ can be defined by considering the coefficient of $C_{(1^d)}$ in the product $C_\mu C_\nu B$:

$$h^{\bullet}_{\mu,\nu,B} := \frac{1}{d!}[C_{(1^d)}]C_\mu C_\nu B$$

For every disconnected Hurwitz number, it is possible to define the *connected* counterpart by requiring the additional condition that the corresponding permutations act transitively on the set $\{1, \dots, d\}$. The geometric meaning of this condition indeed corresponds to the count of the connected coverings. We use the common convention in literature of indicating by $h^\bullet$ the disconnected numbers and by h° the connected ones. If $B \in \mathcal{Z}(\mathbb{Q}(\mathfrak{S}_d))$, then its action in the left regular representation is given by the diagonal matrix $\mathsf{egv}(B)$, whose action in the irreducible representation λ is the multiplication by an eigenvalue $\mathsf{egv}_\lambda(B)$. The elements C_α lie in the center and their eigenvalues are given by $\mathsf{egv}_\lambda(C_\alpha) := |C_\alpha|\chi_\lambda(\alpha)/\dim\lambda$, where $\dim\lambda$ and χ_λ are the dimension and the character of the representation λ, and $|C_\alpha|$ is the number of permutations of the cycle type α. This implies that

$$\begin{aligned} h^{\bullet}_{\mu,\nu,B} &= \frac{\mathrm{Tr}\left(\mathsf{egv}(C_\mu)\,\mathsf{egv}(C_\nu)\,\mathsf{egv}(B)\right)}{(d!)^2} \\ &= \sum_{\lambda\vdash n}\left(\frac{\dim\lambda}{d!}\right)^2 \mathsf{egv}_\lambda(C_\mu)\,\mathsf{egv}_\lambda(C_\nu)\,\mathsf{egv}_\lambda(B) \\ &= \frac{1}{Z_\mu Z_\nu}\sum_{\lambda\vdash d}\chi_\lambda(\mu)\chi_\lambda(\nu)\,\mathsf{egv}_\lambda(B), \end{aligned} \tag{1.1}$$

where $Z_\mu = \prod \mu_i \prod_{i=1}^d (j_i)!$ for $\mu = (1^{j_1}2^{j_2}\cdots d^{j_d}) = (\mu_1 \geq \cdots \geq \mu_{\ell(\mu)})$.

Let us discuss some examples. It is well-known that

$$\mathsf{egv}_\lambda(C_2) = \frac{1}{2}\sum_{i=1}^{\ell(\lambda)}(\lambda_i - i + \frac{1}{2})^2 - (-i + \frac{1}{2})^2.$$

The Hurwitz number $h^{\bullet}_{\mu,\nu,B}$ for $B = C_2^{2g-2+\ell(\mu)+\ell(\nu)}$ is the standard double Hurwitz number for possibly disconnected surfaces of genus g [46]. Consider an element $\overline{C}_r$ such that

$$\mathsf{egv}_\lambda(\overline{C}_r) = \frac{1}{r!}\sum_{i=1}^{\ell(\lambda)}(\lambda_i - i + \frac{1}{2})^r - (-i + \frac{1}{2})^r.$$

It is the so-called completed r-cycle [47] (in some normalization), and the Hurwitz number $h^{\bullet}_{\mu,\nu,B}$ for $B = \overline{C}_r^b$, $b(r-1) = 2g - 2 + \ell(\mu) + \ell(\nu)$, is the double Hurwitz number with completed r-cycles for possibly disconnected surfaces of genus g [51].

In some cases, one can consider the enumeration of coverings up to automorphisms that fix the preimages of two special points (say, 0 and ∞ in $\mathbb{C}\mathrm{P}^1$) pointwise. In this case, we use the following formula instead of the one given by Equation (1.1):

$$\frac{1}{\prod_{i=1}^{\ell(\mu)} \mu_i \prod_{i=1}^{\ell(\nu)} \nu_i} \sum_{\lambda \vdash d} \chi_\lambda(\mu)\chi_\lambda(\nu)\,\mathsf{egv}_\lambda(B).$$

We consider the Jucys-Murphy elements $\mathcal{J}_k \in \mathbb{Q}(\mathfrak{S}_d)$, $k = 2, \dots, d$, defined as

$$\mathcal{J}_k := (1\ k) + (2\ k) + \cdots + (k-1\ k).$$

They generate a maximal commutative subalgebra of $\mathbb{Q}(\mathfrak{S}_d)$ called Gelfand-Tsetlin algebra. It is an important result of Jucys that *symmetric polynomials in the Jucys-Murphy elements* generate the center of the group algebra. It is hence possible to define several variation of the Hurwitz numbers using the standard bases of symmetric polynomials, evaluated at the Jucys-Murphy elements.

Let ρ be a standard Young tableau of a Young diagram $\lambda \vdash d$. We denote by i_k and j_k the column and the row indices of the box labeled by k. By

$$\mathsf{cr}^\rho := (i_1 - j_1, i_2 - j_2, \dots, i_d - j_d)$$

we denote the content vector of the tableau. Jucys [36] proves that

$$\mathsf{egv}_\lambda(B(\mathcal{J}_2, \dots, \mathcal{J}_d)) = B(\mathsf{cr}^\rho_2, \dots, \mathsf{cr}^\rho_d)$$

for any symmetric polynomial B in $d-1$ variables and any choice of ρ. Since it does not depend on ρ, we can always use some arbitrary choice of the Young tableau, for instance, filling the diagram from left to right, and denote by cr^λ the content vector for this choice. This implies the following:

Lemma 1.3.1. *If $B = B(\mathcal{J}_2, \dots, \mathcal{J}_d)$ is a symmetric polynomial in the Jucys elements, then*

$$h^\bullet_{\mu,\nu,B} = \frac{1}{\prod_{i=1}^{\ell(\mu)} \mu_i \prod_{i=1}^{\ell(\nu)} \nu_i} \sum_{\lambda \vdash d} \chi_\lambda(\mu)\chi_\lambda(\nu) B(\mathsf{cr}^\lambda_2, \dots, \mathsf{cr}^\lambda_d).$$

Let us consider several bases of the symmetric polynomials, in particular, we denote by σ_b the elementary symmetric polynomials, with h_b the homogeneous complete, and with p_b the power sums. We define the following blocks of ramifications B, which are arguably most important for applications:

$$B^<_b := \sigma_b(\mathcal{J}_2, \dots, \mathcal{J}_d);\ \ B^\leq_b := h_b(\mathcal{J}_2, \dots, \mathcal{J}_d);\ \ B^\times_b := p_b(\mathcal{J}_2, \dots, \mathcal{J}_d);$$

$$B^{|}_b := \sum_{\substack{\alpha \in (\mathfrak{S}_d/\sim) \\ \ell(\alpha) = d-b}} C_\alpha;\ \ B^{||}_b := \sum_{k=1}^{b} (-1)^{k+b} \sum_{\substack{\vec{\alpha} \in (\mathfrak{S}_d/\sim)^k \\ \sum \ell(\alpha_i) = kd-b}} \prod_{i=1}^{k} C_{\alpha_i}$$

Definition 1.3.2. Let us call the double Hurwitz numbers corresponding to the blocks of ramifications above *strictly monotone*, *monotone* or *weakly monotone*, *atlantes*, *free single* and *free group*, respectively. In case $\nu = (r, r, \dots, r)$ for a positive integer $r \geq 2$, we add the adjective *orbifold* to the definition; if $r = 1$, we remove the adjective *double*.

1.4 Hurwitz numbers expressed in terms of semi-infinite wedge formalism

Double Hurwitz numbers take a very convenient expression in terms of semi-infinite wedge formalism, provided one can find an operator $\mathcal{F}_B$ such that $\mathcal{F}_B.v_\lambda = \mathsf{egv}(B)v_\lambda$. Explicitly, the translation in terms of the semi-infinite wedge formalism is accomplished by the use of the following (see, e.g. [34]). For a partition μ we have

$$\prod_{i=1}^{\ell(\mu)} \alpha_{-\mu_i}|0\rangle = \sum_{\lambda:|\lambda|=|\mu|} \chi_\lambda(\mu) v_\lambda, \qquad \text{and} \qquad \langle 0| \prod_{i=1}^{\ell(\mu)} \alpha_{\mu_i} v_\lambda = \chi_\lambda(\mu).$$

For example, the r-completed cycles double Hurwitz numbers read

$$h^\bullet_{\mu,\nu,\overline{C}^b_r} = \left\langle \prod_{i=1}^{\ell(\mu)} \frac{\alpha_{\mu_i}}{\mu_i} \mathcal{F}^b_r \prod_{i=1}^{\ell(\nu)} \frac{\alpha_{-\nu_i}}{\nu_i} \right\rangle.$$

In general, it is useful to pack Hurwitz numbers in formal generating series summing over the genus of the coverings, where an auxiliary variable u takes care of the Riemann-Hurwitz count b, considered as a function of the genus. The generating series in this case reads

$$\sum_{g=0} \frac{h^\bullet_{\mu,\nu,\overline{C}^b_r}}{b!} u^b = \left\langle \prod_{i=1}^{\ell(\mu)} \frac{\alpha_{\mu_i}}{\mu_i} e^{u\mathcal{F}_r} \prod_{i=1}^{\ell(\nu)} \frac{\alpha_{-\nu_i}}{\nu_i} \right\rangle.$$

The specialisation for $r = 2$ recovers the generating series for the usual double Hurwitz numbers. In general, the connected counterpart of any Hurwitz number has the same expression in the semi-infinite wedge formalism as the disconnected one, but in terms of connected correlators, indicated with $\langle \dots \rangle^\circ$, which are defined from the disconnected ones by use of the inclusion-exclusion formula, (for a precise definition see [18]). From now on, we will refer as $h^{r-spin}_{g,\mu}$ and as $h^{[r]}_{g,\mu}$ to the Hurwitz numbers with r-completed cycles (or r-spin Hurwitz numbers) and r-orbifold Hurwitz numbers, respectively. The numbers $h_{g,\mu}$ will indicate $h^{[1]}_{g,\mu}$.

1.5 Moduli spaces of curves and tautological rings

In this section we introduce the needed tautological classes on the moduli space of curves and a list of conjectures due to Faber concerning a natural part of its cohomology. For an introduction on the moduli spaces of curves see, e.g., [54].

For $2g-2+n > 0$, let $\mathcal{M}_{g,n}$ be the moduli spaces of smooth, compact, connected, algebraic curves of genus g with n distinct marked labelled points $p_1, \dots, p_n$. This space admits a Deligne-Mumford smooth compactification $\overline{\mathcal{M}}_{g,n}$ of complex dimension $3g-3+n$ that is a complex orbifold as analytic space. The space $\overline{\mathcal{M}}_{g,n}$ parametrises algebraic curves of genus g with n marked points that are smooth away from finitely many singularities analytically isomorphic to $\{xy = 0\}$ in $\mathbb{C}^2$ (nodes), with finite groups of automophisms. The latter condition is the stability condition of the curves, and is equivalent to the requirement that every irreducible component of the curve has negative Euler characteristic. Concretely, this means that every rational component has at least 3 special points and every torus component has at least 1 special point, with special points indicating both nodes and the marked points $p_1, \dots, p_n$. Let us consider the system of spaces $\{\overline{\mathcal{M}}_{g,n}\}_{2g-2+n>0}$ alltogether. Let us define three natural maps between these spaces, which are called *tautological*.

i). Let $\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ be the morphism that forgets the last marked point p_{n+1}. We refer to $\pi^{(m)} : \overline{\mathcal{M}}_{g,n+m} \to \overline{\mathcal{M}}_{g,n}$ as the morphism forgetting the last m marked points.

ii). Let $\mathrm{gl}^{(1)} : \overline{\mathcal{M}}_{g,n+2} \to \overline{\mathcal{M}}_{g+1,n}$ the morphism that glues the last two marked points together producting a curve with genus raised by one.

iii). Let $\mathrm{gl}^{(2)} : \overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} \to \overline{\mathcal{M}}_{g_1+g_2,n_1+n_2}$ the morphism that glues the last marked point of the first curve with the last marked point of the second curve.

The image of the curves may not be stable, but it can be stabilised by contracting the rational components with two marked points.

The cohomology of $\overline{\mathcal{M}}_{g,n}$, considered here with rational coefficients, has several natural classes, called tautological classes. Let $(C; p_1, \dots, p_n)$ be a point in $\overline{\mathcal{M}}_{g,n}$, and consider the cotangent line to the curve C at the i-th marked point. These cotangent lines give raise to a line bundle $\mathcal{L}_i$ over $\overline{\mathcal{M}}_{g,n}$. Denote the first Chern class $c_1(\mathcal{L}_i)$ by $\psi_i \in H^2(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$, by $\kappa_j \in H^{2j}(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$ the pushforward $\pi_*(\psi_{n+1}^{j+1})$, and by $\kappa_{k_1,k_2,\dots,k_m}$ the multi-index κ-classes

$$\kappa_{k_1,k_2,\dots,k_m} = (\pi^{(m)})_* \left(\psi_{n+1}^{k_1+1} \psi_{n+2}^{k_2+1} \dots \psi_{n+m}^{k_m+1} \right) \in H^{2\sum_{i=1}^m k_i}(\overline{\mathcal{M}}_{g,n}; \mathbb{Q}).$$

Multi-index κ-classes can be expressed as polynomials in κ-classes with one index. It is possible to construct a rank g vector bundle over $\overline{\mathcal{M}}_{g,n}$ whose generic fiber over the smooth point $(C; p_1, \dots, p_n)$ is constituted by the abelian differentials on the curve C. This vector bundle is called the Hodge bundle Λ_g and by λ_j one denotes the j-th Chern class $c_j(\Lambda_g) \in H^{2j}(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$.

The system of cohomological tautological rings $\{R^*(\overline{\mathcal{M}}_{g,n})\}_{2g-2+n>0}$ is defined to be the smallest system of $\mathbb{Q}$-subalgebras $R^*(\overline{\mathcal{M}}_{g,n}) \subset H^*(\overline{\mathcal{M}}_{g,n})$ closed under pushforwards of the tautological maps defined above. As a result of the definition, the tautological rings are also closed under pullbacks. The tautological ring of $\mathcal{M}_{g,n}$ is defined by restriction. The d-th tautological ring $R^d(\overline{\mathcal{M}}_{g,n})$ is defined as $R^*(\overline{\mathcal{M}}_{g,n}) \cap H^{2d}(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$.

1.5.1 Faber's conjectures

The rank of the tautological ring is in general smaller than that of the full cohomology ring of $\mathcal{M}_{g,n}$. Nevertheless, it is actually not easy to construct examples of non-tautological cohomology classes, and most of the geometrically defined cohomology classes happen to be tautological. The structure of the tautological ring $R^*(\mathcal{M}_g)$, for $g \geq 2$, is the object of Faber conjectures [27]. Faber's conjectures for the open moduli spaces, if true, completely determine the structure of the tautological ring $R^*(\mathcal{M}_g)$.

i). **Socle**. $R^{>g-2}(\mathcal{M}_g) = 0$ and $R^{g-2}(\mathcal{M}_g) \cong \mathbb{Q}$.

ii). **Top intersection**. For k_i non-negative integers such that $k_1 + \cdots + k_m = g - 2$:

$$\kappa_{k_1,\dots,k_m} = \frac{(2g-3+m)!(2g-3)!!}{(2g-2)! \prod_{i=1}^m (2k_i+1)!!} \kappa_{g-2}.$$

iii). **Perfect pairing**. For any $0 \leq i \leq g-2$, the cup product defines a non-degenerate pairing

$$R^i(\mathcal{M}_g) \times R^{g-2-i}(\mathcal{M}_g) \to R^{g-2}(\mathcal{M}_g) = \mathbb{Q}.$$

For $g \geq 2$ and $n \geq 1$, Faber's conjectures have been generalised in [5] to the tautological ring $R^*(\mathcal{M}_{g,n})$. Again, if true, these conjectures completely determine the structure of $R^*(\mathcal{M}_{g,n})$.

i). **Socle**. $R^{>g-1}(\mathcal{M}_{g,n}) = 0$ and $R^{g-1}(\mathcal{M}_{g,n}) \cong \mathbb{Q}^n$. The classes ψ_i^{g-1}, with $i = 1, 2, \ldots, n$, form a basis in $R^{g-1}(\mathcal{M}_{g,n})$.

ii). **Top intersection**. Suppose that $d_1 + \cdots + d_n + k_1 + \ldots + k_m = g-1$ and that $d_i, k_j \geq 0$. Then we have the following equation in $R^{g-1}(\mathcal{M}_{g,n})$:

$$\prod_{i=1}^n \psi_i^{d_i} \cdot \kappa_{k_1,k_2,\ldots,k_m} = \frac{(2g-1)!!}{\prod_{i=1}^n (2d_i+1)!! \prod_{j=1}^m (2k_j+1)!!} \frac{(2g-3+n+m)!}{(2g-2+n)!} \times$$
$$\times \sum_{i=1}^n \frac{(2g-2+n)d_i + \sum k_j}{g-1} \psi_i^{g-1}.$$

iii). **Perfect pairing**. A polynomial in $\psi_1, \ldots, \psi_n, \kappa_1, \ldots, \kappa_{g-1}$ vanishes if and only if its products with all classes of complementary dimension vanish in $R^{g-1}(\mathcal{M}_{g,n})$.

1.6 Cohomological field theories

Cohomological field theories were introduced by Kontsevich and Manin in [38] and generalise Gromov-Witten theories by substituting the cohomology of the target space with an arbitrary vector space. Let V be a finite dimensional vector space over $\mathbb{Q}$ with a basis $\{e_1, \ldots, e_N\}$, let η be a non-degenerate bilinear-form on V, and let $\mathbb{1}$ be a distinguished element of V. Denote by $\eta_{j,k}$ the evaluation $\eta(e_j, e_k)$ and by $\eta^{j,k}$ the inverse matrix. A *cohomological field theory with unit* (CohFT) is a system of elements

$$\left\{\alpha_{g,n} \in H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}) \otimes (V^*)^{\otimes n}\right\}_{2g-2+n>0}$$

compatible with the tautological maps in the following sense:

i). Each element $\alpha_{g,n}$ is invariant with respect to the action of the symmetric group $\mathfrak{S}_n$ acting simultaneously on the marked points of the curves in $\overline{\mathcal{M}}_{g,n}$ and on the copies of V^*.

ii). $(\mathrm{gl}^{(1)})^* \left(\alpha_{g,n}(e_{i_1} \otimes \cdots \otimes e_{i_n})\right) = \sum_{j,k=1}^N \eta^{j,k} \alpha_{g-1,n+2}(e_{i_1} \otimes \cdots \otimes e_{i_n} \otimes e_j \otimes e_k)$.

iii). $(\mathrm{gl}^{(2)})^* \left(\alpha_{g,n}(e_{i_1} \otimes \cdots \otimes e_{i_n})\right) =$
$\sum_{j,k=1}^N \eta^{j,k} \alpha_{g_1,n_1+1}(e_{i_1} \otimes \cdots \otimes e_{i_n} \otimes e_j) \cdot \alpha_{g_2,n_2+1}(e_{i_{n_1+2}} \otimes \cdots \otimes e_{i_{n_1+n_2+1}} \otimes e_k)$.

iv). $\pi^* \alpha_{g,n}(e_{i_1} \otimes \cdots \otimes e_{i_n}) = \alpha_{g,n+1}(e_{i_1} \otimes \cdots \otimes e_{i_n}, \mathbb{1})$.

v). $\alpha_{0,3}(e_{i_1} \otimes e_{i_2} \otimes \mathbb{1}) = \eta_{i_1,i_2}$.

If the properties $i) - iii)$ hold, and at least one among properties $iv)$ and $v)$ does not hold, we refer to the system of $\alpha_{g,n}$ as a cohomological field theory without unit (CohFT/1). A CohFT composed of only degree zero classes is called a topological field theory. CohFTs turn V into an algebra with respect to the associative quantum product $\bullet$ defined by $\eta(v_1 \bullet v_2, v_3) = \alpha_{0,3}(v_1, v_2, v_3)$ for which the element $\mathbb{1}$ plays the role of unit element. A CohFT is called semisimple if the algebra $(V, \bullet)$ is semisimple. Semisimple CohFTs carry the transitive action of a huge group introduced by Givental, whose elements are called Givental R-matrices. A Givental R-matrix is an $\mathrm{End}(V)$-valued power series

$$R(\zeta) = 1 + \sum_{l=1} R_l \zeta^l = \exp\left(\sum_{l=1} r_l \zeta^l\right), \quad R_l, r_l \in \mathrm{End}(V)$$

satisfying the symplectic condition

$$R(\zeta)\, R^*(-\zeta) = 1 \in \mathrm{End}(V)$$

where R^* is the adjoint of R with respect to η. By the Givental - Teleman classification [28, 52], every semi-simple CohFT is obtained by the action of a Givental R-matrix on a topological field theory. The Givental action has a concrete expression in terms of an action over the sum of stable dual graphs. The general definitions can be found in many sources, see, e.g., [16, 28]. In the following two sections, instead, we show how the computations can be applied to a specific case that is used later on in the thesis, working out in details the example of Chiodo classes. This result appears in [40], which corresponds to Chapter 3 of the thesis, but the computations are there omitted.

1.7 An explicit example: Chiodo classes

In this section we recall Chiodo classes and we show their Givental decomposition; for more details we refer the reader to [9, 10, 7, 32, 51]. For $2g - 2 + n > 0$, consider a nonsingular curve with distinct markings $[C, p_1, \dots, p_n] \in \mathcal{M}_{g,n}$, and let $\omega_{\log} = \omega_C(\sum p_i)$ be its log canonical bundle. Let $r \geq 1$, $1 \leq a_1, \dots, a_n \leq r$ and $0 \leq s \leq r$ be integers satisfying the condition

$$(2g - 2 + n)s - \sum_{i=1}^{n} a_i = 0 \qquad \mod\ r \tag{1.2}$$

This condition guarantees the existence of rth tensor roots L of the line bundle

$$\omega_{\log}^{\otimes s}\left(-\sum a_i p_i\right)$$

on C. For the moduli space of such rth tensor roots a natural compactification $\overline{\mathcal{M}}^{r,s}_{g;a_1,\dots,a_n}$ was constructed in [8, 33].

Let $\pi : \mathcal{C}^{r,s}_{g;a_1,\dots,a_n} \to \overline{\mathcal{M}}^{r,s}_{g;a_1,\dots,a_n}$ be the universal curve, let $\mathcal{L} \to \mathcal{C}^{r,s}_{g;a_1,\dots,a_n}$ be the universal rth root, and let $\epsilon : \overline{\mathcal{M}}^{r,s}_{g;a_1,\dots,a_n} \to \overline{\mathcal{M}}_{g,n}$ be the forgetful map (in order for ϵ to be unramified in the orbifold sense, the target $\overline{\mathcal{M}}_{g,n}$ is changed into the moduli space of r-stable curves, meaning that for each stable curve there is an extra $\mathbb{Z}_r$ stabilizer at each node, see [8]). Recall the generating series for the Bernoulli polynomials

$$\sum_{l=0} B_l(x) \frac{t^l}{l!} = \frac{te^{xt}}{e^t - 1}$$

where the usual Bernoulli numbers are $B_l(0) = B_l$. We are interested in the Chiodo classes [9]

$$\mathcal{C}_{g,n}(r,s;a_1,\dots,a_n) := \epsilon_* c\big(-R^*\pi_*\mathcal{L}\big) =$$
$$\epsilon_* \exp\left(\sum_{l=1}^{\infty}(-1)^l(l-1)!\mathrm{ch}_l(r,s;a_1,\dots,a_n)\right) \in H^{even}(\overline{\mathcal{M}}_{g,n}),$$

where the Chiodo formula for the Chern characters reads

$$\mathrm{ch}_l(r,s;a_1,\dots,a_n) = \frac{B_{l+1}(\frac{s}{r})}{(l+1)!}\kappa_l - \sum_{i=1}^{n}\frac{B_{l+1}(\frac{a_i}{r})}{(l+1)!}\psi_i^l$$
$$+\frac{r}{2}\sum_{a=1}^{r}\frac{B_{l+1}(\frac{a}{r})}{(l+1)!}(j_a)_*\frac{(\psi')^l+(-1)^{l-1}(\psi'')^l}{\psi'+\psi''}.$$

Here j_a is the boundary map that represents the boundary divisor with remainder a at one of the two half edges, and ψ', ψ'' are the ψ-classes at the two branches of the node. For $r = s = 1$, and moreover $a_i = 1$, for $i = 1,\dots,n$, the map ϵ is the identity map and the specialisation recovers Mumford's formula [45] for the total Chern class of the dual of the Hodge bundle $c(\Lambda_g^\vee)$:

$$\mathcal{C}_{g,n}(1,1;1,\dots,1) = \exp\Bigg(-\Bigg[\sum_{l=1}^{\infty}\frac{B_{l+1}}{l(l+1)}\kappa_l - \sum_{i=1}^{n}\frac{B_{l+1}}{l(l+1)}\psi_i^l$$
$$+\frac{1}{2}\frac{B_{l+1}}{l(l+1)}j_*\frac{(\psi')^l+(-1)^{l-1}(\psi'')^l}{\psi'+\psi''}\Bigg]\Bigg)$$
$$= c(\Lambda_g^\vee) = 1-\lambda_1+\lambda_2-\dots+(-1)^g\lambda_g,$$

where the identity $B_l(1) = (-1)^l B_l$ is used. The formula in [45] is slightly different due to a different Bernoulli number convention and a missprint in the κ term.

1.7.1 Expression in terms of stable graphs

Let us recall the expression of the Chiodo class in terms of the sum of products of contributions decorating stable graphs, in order to compare it with the Givental action, for more details see [32]. The strata of the moduli space of curves correspond to stable graphs

$$\Gamma = (\mathrm{V},\mathrm{E},\mathrm{H},\mathrm{L},g,n:\mathrm{V}\to\mathbb{Z}_{\geq 0},v:\mathrm{H}\to\mathrm{V},\iota:\mathrm{H}\to\mathrm{H})$$

where $\mathrm{V}(\Gamma)$, $\mathrm{E}(\Gamma)$, $\mathrm{H}(\Gamma)$ and $\mathrm{L}(\Gamma)$ respectively denote the sets of vertices, edges, half-edges and leaves of Γ; self-edges are permitted. A half-edge indicates either a leaf or an edge together with a choice of one of the two vertices it is attached to. The function v associates to each half-edge its vertex assignment, while ι is the involution that swaps the two half-edges of the same edge, or leaves the half-edge invariant if it is a leaf. The function $n(v)$ denotes the valence of Γ at v, including both half-edges and legs, and $g(v)$ denotes the genus function. Every vertex v is required to satisfy the stability condition $2g(v)-2+n(v) > 0$, and the genus of a stable graph Γ is defined by $g(\Gamma) := \sum_{v\in V} g(v) + h^1(\Gamma)$. Let $\mathrm{Aut}(\Gamma)$ denote the group of automorphisms of the sets V and H which leave the structures L, g, v, and ι invariant. Let $\mathsf{G}_{g,n}$ be the finite set of isomorphism classes of stable graphs of genus g with n legs. Let moreover $\mathsf{W}_{\Gamma,r,s,\vec{a}}$ be the set of *weightings* $w : \mathrm{H}(\Gamma) \to \{0,\dots,r-1\}$ satisfying the following three properties:

(i) The i-th leaf l_i has weight $w(l_i) = a_i \mod r$, for $i \in \{1, \dots, n\}$.

(ii) For any two half-edges h' and h'' corresponding to the same edge, we have $w(h') + w(h'') = 0 \mod r$.

(iii) The condition in Equation (1.2) is satysfied locally on each component: for any vertex v the sum of the weights associated to the half-edges incident to v is $\sum_{v(h)=v} w(h) = s\big(2g(v) - 2 + n(v)\big) \mod r$.

Proposition 1.7.1 ([32])**.** *The Chiodo class $\mathcal{C}_{g,n}(r, s; a_1, \dots, a_n) \in R^*(\overline{\mathcal{M}}_{g,n})$ is equal to*

$$\sum_{\Gamma \in \mathsf{G}_{g,n}} \sum_{w \in \mathsf{W}_{\Gamma,r,s,\vec{a}}} \frac{r^{|E(\Gamma)| + \sum_{v \in V(\Gamma)} 2g(v) - 1}}{|\operatorname{Aut}(\Gamma)|} \xi_{\Gamma *} \Bigg[\prod_{v \in \mathrm{V}(\Gamma)} e^{-\sum\limits_{l \geq 1} (-1)^{l-1} \frac{B_{l+1}(s/r)}{l(l+1)} \kappa_l(v)} \cdot$$
$$\prod_{i=1}^{n} e^{\sum\limits_{l \geq 1} (-1)^{l-1} \frac{B_{l+1}(a_i/r)}{l(l+1)} \psi_{h_i}^l} \cdot \prod_{\substack{e \in \mathrm{E}(\Gamma) \\ e = (h', h'')}} \frac{1 - e^{\sum\limits_{l \geq 1} (-1)^{l-1} \frac{B_{l+1}(w(h)/r)}{l(l+1)} [(\psi_{h'})^l - (-\psi_{h''})^l]}}{\psi_{h'} + \psi_{h''}} \Bigg]. \tag{1.3}$$

where ξ_Γ is the canonical morphism $\xi_\Gamma : \prod_{v \in \mathrm{V}(\Gamma)} \overline{\mathcal{M}}_{g(v),n(v)} \to \overline{\mathcal{M}}_{g,n}$ of the boundary stratum corresponding to Γ.

1.8 Givental decompositions for Chiodo classes

In this section we show that the action of the R-matrix

$$R^{-1}(\zeta) := \exp\left(- \sum_{l=1}^{\infty} \frac{\operatorname{diag}_{a=1}^{r} B_{l+1}\left(\frac{a}{r}\right)}{l(l+1)} (-\zeta)^l \right)$$

defined as power series valued in the endomorphisms for the vector space

$$V = \langle v_1, \dots, v_r \rangle$$

with

$$\eta(v_a, v_b) = \frac{1}{r} \delta_{a+b \mod r},$$

acting on the topological field theory

$$\alpha_{g,n}^{top}(v_{a_1} \otimes \cdots \otimes v_{a_n}) = r^{2g-1} \delta_{a_1 + \cdots + a_n - s(2g-2+n) \mod r},$$

produces the Chiodo classes. Therefore Chiodo classes determine a semisimple CohFT with a known Givental decomposition. The action of the Givental R-matrix is defined as the sum over stable graphs Γ weighted by $|\operatorname{Aut}(\Gamma)|^{-1}$, with contributions on the leaves, on the edges, on special leaves called *dilaton* leaves, and the topological field theory contributes on the verteces. Chiodo classes are already expressed as a sum over stable graphs in Equation (1.3) with a very similar structure. Let us match the Givental contributions one by one:

Ordinary leaf contributions.

The contribution of the i-th leaf reads

$$\exp\left(-\sum_{l=1}\frac{B_{l+1}(a_i/r)}{l(l+1)}(-\psi_{h_i})^l\right)=\sum_{j=1}^{r}(R^{-1})^j_{a_i}(\psi_{h_i})$$

Dilaton leaf contributions.

Recall that the kappa classes are defined as $\kappa_l=\pi_*(\psi_{n+1}^{l+1})$ under the map that forgets the last marked point $\pi:\overline{\mathcal{M}}_{g,n+1}\to\overline{\mathcal{M}}_{g,n}$. The contributions on the dilaton leaves correspond to the contributions on the vertices in Equation (1.3) before forgetting the corresponding marked point. For the dilaton leaf marked with label $n+i$, for some positive integer i, the contribution reads:

$$\exp\left(-\sum_{l=1}\frac{B_{l+1}(s/r)}{l(l+1)}(-\psi_{n+i})^l(-\psi_{n+i})\right)$$

We check that v_s is the neutral element $\mathbb{1}$ for the quantum product $\bullet$ in flat basis:

$$\begin{aligned}\eta(v_s\bullet v_a,v_b)&=\alpha^{top}_{0,3}(v_s\otimes v_a\otimes v_b)\\&=r^{-1}\delta_{s+a+b-s\mod r}=r^{-1}\delta_{a+b\mod r}\\&=\eta(v_a,v_b)\end{aligned}$$

Hence the contribution of the dilaton leaf $n+i$ is

$$\psi_{n+i}\Big[\mathrm{Id}-\sum_{j=1}^{r}(R^{-1})^j_{\mathbb{1}}(\psi_{n+i})\Big]$$

Edge contributions.

The edge contribution in Equation (1.3), multiplied by the factor $(\psi_{h'}+\psi_{h''})$ and after applying the property of Bernoulli numbers $(-1)^{p+1}B_{p+1}\left(\frac{w(h')}{r}\right)=B_{p+1}\left(\frac{r-w(h')}{r}\right)$, reads

$$1-\exp\left(-\sum_{l=1}\frac{B_{l+1}(\frac{w(h')}{r})}{l(l+1)}(-\psi_{h'})^l\right)\exp\left(-\sum_{p=1}\frac{B_{p+1}(\frac{r-w(h'')}{r})}{p(p+1)}(-\psi_{h'})^p\right).$$

Note that the condition on the weightings $w(h')+w(h'')=0\mod r$ can be taken care of by the scalar product η. Hence we can write the Givental contribution on the edges as

$$\sum_{j_1,j_2}\frac{\eta^{j_1,j_2}-(R^{-1})^{j_1}_{w(h')}(\psi_{h'})\eta^{w(h'),w(h'')}(R^{-1})^{j_2}_{w(h'')}(\psi_{h''})}{\psi_{h'}+\psi_{h''}}$$

Weightings.

Out of the three conditions on the weightings, condition (i) becomes $w(l_i)=a_i$, condition (ii) on the edges is taken care by the bilinear form η, condition (iii) can be substituted by the topological field theory condition.

Powers of r.

Every stable graph contributes with $|E(\Gamma)|+\sum_{v\in V(\Gamma)} 2g(v)-1$ powers of r. Indeed the topological field theory in the vertex v provides $2g(v)-1$ powers of r, and the inverse of η provides one power of r for each edge of Γ.

The expression of the Givental action

Let us indicate with $\{l_1,\dots,l_n,l_{n+1},\dots,l_{n+k}\} = L(\Gamma)$ the set of legs, corresponding to marked points of the curves in $\overline{\mathcal{M}}_{g,n+k}$, and let

$$\xi_\Gamma^{(k)} : \prod_{v\in V(\Gamma)} \overline{\mathcal{M}}_{g(v),n(v)} \to \overline{\mathcal{M}}_{g,n}$$

be the canonical morphism of the boundary stratum corresponding to Γ that forgets the last k marked points. Let us consider functions $w^\vee : H(\Gamma)\to\mathbb{Z}_{\geq 0}$ without *any* further condition. We use here the notation $w^\vee$, instead of w, to remark that the weightings $w^\vee$ decorates the half-edges *after* the application of the endomorphisms R_l^{-1}. Collecting the contributions and the considerations above, we have:

$$\begin{aligned}\mathcal{C}_{g,n}(r,s;a_1,\dots,a_n) = \sum_{k=0} \sum_{\substack{\Gamma\in G_{g,n+k}\\ w^\vee : H(\Gamma)\to\mathbb{Z}_{\geq 0}}} \frac{1}{|\operatorname{Aut}(\Gamma)|}\left(\xi_\Gamma^{(k)}\right)_* \Bigg[\\ \prod_{v\in V(\Gamma)} \alpha^{top}_{g(v),n(v)}\Bigg(\bigotimes_{\substack{h\in H(\Gamma):\\ v(h)=v}} v_{w^\vee(h)}\Bigg) \\ \prod_{i=1}^n (R^{-1})_{a_i}^{w^\vee(l_i)}(\psi_i)\prod_{i=1}^k \psi_{n+i}\Big[\operatorname{Id} - (R^{-1})_{\mathbb{1}}^{w^\vee(l_{n+i})}(\psi_{n+i})\Big] \\ \prod_{e=(h',h'')\in V(\Gamma)} \frac{\eta^{w^\vee(h'),w^\vee(h'')} - \sum_{k_1,k_2}(R^{-1})_{k_1}^{w^\vee(h')}(\psi_{h'})\eta^{k_1,k_2}(R^{-1})_{k_2}^{w^\vee(h'')}(\psi_{h''})}{\psi_{h'}+\psi_{h''}}\Bigg]\end{aligned}$$

The expression above is equivalent to $(R.\alpha^{top})_{g,n}(v_{a_1}\otimes\cdots\otimes v_{a_n})$, i.e. the Givental action of the matrix R on α^{top} correspondent of genus g and n marked points, evaluated in the element $v_{a_1}\otimes\cdots\otimes v_{a_n}$, (see [28, 16, 49]).
Consider then $\mathcal{C}_{g,n}(r,s;a_1,\dots,a_n)$ as the evaluation of a map

$$\mathcal{C}_{g,n}(r,s)\colon V^{\otimes n} \to H^{even}(\overline{\mathcal{M}}_{g,n}),$$

where $V = \langle v_1,\dots,v_r\rangle$, and

$$\mathcal{C}_{g,n}(r,s)\colon v_{a_1}\otimes\cdots\otimes v_{a_n} \mapsto \mathcal{C}_{g,n}(r,s;a_1,\dots,a_n).$$

The previous calculation anticipates a result of Chapter 3

Proposition 1.8.1 ([40]). *For $0 \le s \le r$ the collection of maps $\{\mathcal{C}_{g,n}(r,s)\}$ defined by the Chiodo classes form a semi-simple cohomological field theory with flat unit, obtained by the action of the Givental matrix R on the topological field theory $\alpha^{top}_{g,n}$:*

$$\left(R.\alpha^{top}\right)_{g,n} = \mathcal{C}_{g,n}(r,s).$$

1.9 Local topological recursion

The Chekhov, Eynard and Orantin topological recursion was formalised in 2007 and plays a central role in the thesis. For a complete exposition see, e.g., [24]. In this section, we define a local version of the topological recursion and write the corresponding invariants as a sum over graphs, which allows us to compare it to the Givental action in the next section.

The local version of the CEO topological recursion takes as input the following set of data $\mathcal{S} = (\Sigma, x, y, B)$:

I). A local spectral curve $\Sigma = \sqcup_{i=1}^N U_i$, given by the disjoint union of N open disks with the center points p_i, $i = 1, \dots, N$.

II). A holomorphic function $x\colon \Sigma \to \mathbb{C}$ such that the zeros of its differential dx are $p_1, \dots, p_r$. We will assume the zeros of dx to be simple. Let $x^i = (C_i w_i)^2 + x_i$ be the local normal form of x on the disk U_i for local coordinates w_i and some constants C_i.

III). A holomorphic function $y\colon \Sigma \to \mathbb{C}$ which does not vanish at the zeros of dx. Denote by y^i the restriction of y to the disk U_i.

IV). A symmetric bidifferential B defined on $\Sigma \times \Sigma$ with a double pole on the diagonal with residue 1. If $z = w_i$ and $z' = w_j$ are respectively coordinates on the disks i and j the expansion of B reads
$$B^{i,j}(z,z') = \delta_{i,j} \frac{dz \otimes dz'}{(z-z')^2} + \sum_{k,l=0}^{\infty} B^{i,j}_{k,l} z^k z'^l dz \otimes dz'.$$

The output of the topological recursion procedure consists of a collection of symmetric differentials $\omega^{\mathcal{S}}_{g,n}$ defined on the topological product of the curve $\Sigma^{\times n}$, recursively on $2g-2+n$. Explicitly, for $2g-2+n > 0$, define

$$\omega^{i_0,i_1,\dots,i_n}_{g,n+1}(z_0, z_1, \dots, z_n) := \sum_{j=1}^{N} \operatorname*{Res}_{z\to 0} \frac{\int_{-z}^{z} B^{i_0,j}(z_0,\cdot)}{2\left(y^j(z) - y^j(-z)\right) \mathrm{d}x^j(z)} \left(\omega^{j,j,i_1,\dots,i_n}_{g-1,n+2}(z,-z,z_1,\dots,z_n) + \sum_{A \cup B = \{1,\dots,n\}} \sum_{h=0}^{g} \omega^{j,\mathbf{i}_A}_{h,|A|+1}(z,\mathbf{z}_A)\, \omega^{j,\mathbf{i}_B}_{g-h,|B|+1}(-z,\mathbf{z}_B) \right),$$

with base cases $\omega^i_{0,1}(z) := 0$; and $\omega^{i,j}_{0,2}(z,z') := B^{i,j}(z,z')$.

1.10 ELSV formulae

ELSV-type formulae relate connected Hurwitz numbers to the intersection theory of certain classes on the moduli spaces of curves. Both Hurwitz theory and the theory of moduli spaces of curves benefit from them, since ELSV formulae provide a bridge through which calculations and results can be transferred from one to the other. The original ELSV formula [21] relates simply connected Hurwitz numbers and Hodge integrals. It plays a central role in many of the alternative proofs of Witten's conjecture that appeared after the first proof by Kontsevich (for more details see [41]). The celebrated ELSV formula expresses these numbers in terms of the intersection theory of moduli spaces of curves:

$$\frac{h^{\circ}_{g;\vec{\mu}}}{b!} = \prod_{i=1}^{\ell(\vec{\mu})} \frac{\mu_i^{\mu_i}}{\mu_i!} \int_{\overline{\mathcal{M}}_{g,\ell(\vec{\mu})}} \left(\sum_{l=0}^{g} (-1)^l \lambda_l \right) \prod_{j=1}^{\ell(\vec{\mu})} \sum_{d_j=0} \mu_j^{d_j} \psi_j^{d_j}$$

In the case of r-orbifold Hurwitz numbers, they are known to satisfy the Johnson-Pandharipande-Tseng (JPT) ELSV-type formula [35] (specialised here to the case $G = \mathbb{Z}/r\mathbb{Z}$, U equal to the representation that sends 1 to $e^{\frac{2\pi i}{r}}$, and empty γ):

$$\frac{h^{\circ,[r]}_{g;\vec{\mu}}}{b!} = r^b \prod_{i=1}^{n} \frac{\left(\frac{\mu_i}{r}\right)^{[\mu_i]}}{[\mu_i]!} \int_{\overline{\mathcal{M}}_{g,\ell(\vec{\mu})}} \frac{p_* \sum_{i\geq 0} (-1)^i \lambda_i}{\prod_{j=1}^{\ell(\vec{\mu})} (1 - \frac{\mu_j}{r}\psi_j)},$$

with $b = 2g - 2 + l(\mu) + d/r$ the count of simple ramification points given by Riemann-Hurwitz formula, and the euclidean division by r written as $x = [x]r + \langle x \rangle$, with $0 \geq \langle x \rangle \geq r - 1$. The powers of r are here slightly rearranged, and the products in the denominator are to be understood as geometric power series as in the ELSV formula above. The class $p_* \sum_{i\geq 0} (-r)^i \lambda_i$ is described in [35] via admissible covers.

In the case of r-spin Hurwitz numbers D. Zvonkine conjectured the following formula [53]:

$$\frac{h^{\circ, r-spin}_{g;\vec{\mu}}}{b!} = r^{b-2g-2+\ell(\vec{\mu})} \prod_{i} \frac{\left(\frac{\mu_i}{r}\right)^{[\mu_i]}}{[\mu_i]!} \int_{\overline{\mathcal{M}}_{g,\ell(\vec{\mu})}} \frac{\mathcal{C}_{g,n}(r,1,r-\langle\vec{\mu}\rangle)}{\prod_{j=1}^{\ell(\vec{\mu})} (1 - \frac{\mu_j}{r}\psi_j)},$$

with $rb = 2g - 2 + \ell(\vec{\mu}) + d$ and the same conventions as above.

Every example above expresses numbers enumerating connected Hurwitz covers of a certain kind, depending on a genus parameter and a partition, in terms of some *non - polynomial* factor in the entries of a partition $\vec{\mu}$ and an integral over moduli spaces of curves of a certain class intersected with ψ class. This integral is clearly a *polynomial* of degree $3g - 3 + \ell(\vec{\mu})$ in the μ_i. Conceptually:

$$h^{\circ, condition}_{g,\vec{\mu}} = \mathrm{NonPoly}(\vec{\mu}) \int_{\overline{\mathcal{M}}_{g,\ell(\vec{\mu})}} (\mathrm{Class}) \prod_{j=1}^{\ell(\vec{\mu})} \sum_{d_j=0} c_{d_j}(\mu_j) \psi_j^{d_j}$$

where $c_{d_j}(\mu_j)$ is a polynomial of degree d_j in μ_j, possibly depending on the parameters $\langle \mu_j \rangle$.

1.11 Relation between ELSV formulae and topological recursion: the ELSV-TR equivalence statements

The CEO topological recursion procedure associates to a spectral curve

$$\mathcal{S} = (\Sigma, x(z), y(z), B(z_1, z_2))$$

(see, e.g., [24]) a collection of symmetric correlation differentials $\omega_{g,n}$ defined on the product of the curve $\Sigma^{\times n}$ through a universal recursion on $2g-2+n$. The expansion of these differentials near particular points can unveil interesting invariants, or solutions to enumerative geometric problems.

We say that certain numbers satisfy the topological recursion if there exists a spectral curve such that the expansion of the correlation differentials near *some* point has those numbers as coefficients. The expansion of the correlation differentials takes the form

$$\omega_{g,n}^{\mathcal{S}} = \mathrm{d}_1 \otimes \cdots \otimes \mathrm{d}_n \sum_{\mu_1,\ldots,\mu_n} N_{g,\vec{\mu}}^{\mathcal{S}} \prod_{i=1}^{n} \tilde{x}_i^{\mu_i}$$

for some coefficients $N_{g,\vec{\mu}}$, where $\tilde{x}$ is a function of x that depend on the point of the expansion. Both the simple Hurwitz and the monotone Hurwitz numbers satisfy the topological recursion (see [2, 3, 14, 26, 44]), and their spectral curves are respectively

$$\left(\mathbb{CP}^1, -z + \log(z), z, \frac{\mathrm{d}z_1 \mathrm{d}z_2}{(z_1 - z_2)^2}\right), \left(\mathbb{CP}^1, \frac{z-1}{z^2}, -z, \frac{\mathrm{d}z_1 \mathrm{d}z_2}{(z_1 - z_2)^2}\right) \tag{1.4}$$

In the simple Hurwitz case $\tilde{x} = e^x$, whereas in the monotone case $\tilde{x} = x$. On the other side, it was proved that the expansion of the correlation differentials *have the same structure as the right-hand side of ELSV-type formulae* described above (see Theorem 1.12.1), depending on ingredients that are functions of the spectral curve.

At this point comes the key observation: if one can compute these ingredients explicitly for a given spectral curve $\mathcal{S}$, one proves that

$$N_{g,\vec{\mu}}^{\mathcal{S}} = \mathrm{NonPoly}^{\mathcal{S}}(\vec{\mu}) \int_{\overline{\mathcal{M}}_{g,\ell(\vec{\mu})}} (\mathrm{Class}^{\mathcal{S}}) \prod_{j=1}^{\ell(\vec{\mu})} \sum_{d_j=0} c_{d_j}^{\mathcal{S}}(\mu_j) \psi_j^{d_j},$$

where the non-polynomial part $\mathrm{NonPoly}^{\mathcal{S}}$, the class $Class^{\mathcal{S}}$, and the $c_{d_j}^{\mathcal{S}}(\mu_j)$ are explicit. This allows to formulate equivalence statements in the following sense.

Definition 1.11.1. A *TR-ELSV equivalence statement* for a Hurwitz problem $h_{g,\vec{\mu}}^{\circ,condition}$ and a spectral curve $\mathcal{S}$ asserts the equivalence between the following two propositions:

i) The numbers $h_{g,\vec{\mu}}^{\circ,condition}$ satisfy the topological recursion with input spectral curve $\mathcal{S}$ (i.e. $h_{g,\vec{\mu}}^{\circ,condition} = N_{g,\vec{\mu}}^{\mathcal{S}}$, up to normalisation factors).

ii) $h_{g,\vec{\mu}}^{\circ,condition} = \mathrm{NonPoly}^{\mathcal{S}}(\vec{\mu}) \int_{\overline{\mathcal{M}}_{g,\ell(\vec{\mu})}} (\mathrm{Class}^{\mathcal{S}}) \prod_{j=1}^{\ell(\vec{\mu})} \sum_{d_j=0} c_{d_j}^{\mathcal{S}}(\mu_j) \psi_j^{d_j}$.

An equivalence statement for certain Hurwitz numbers $h_{g,\vec{\mu}}^{\circ,condition}$ and a certain spectral curve $\mathcal{S}$ is useful if for at least one of the two propositions there exists some evidence or a proof.

Thus, if one establishes i) independently of ii), then ii) follows immediately (and vice versa), and hence this equivalence relationship has received much attention in the literature. For example, for the case of simple Hurwitz numbers and the first curve in Equation (1.4), proposition i) was conjectured to hold by Bouchard and Mariño [3], while ii) is the original ELSV formula. Proposition i) was proved in [2, 26, 44], and the equivalence statement was proved in [22], see also [51]. The equivalence statement immediately provides a new proof of i) from ii). The proofs [26, 44] of i) though, make use of a polynomiality property that is extracted from ELSV, hence the equivalence cannot be used in the other direction without falling into a circular argument, unless this polynomiality property can be proved without using ELSV formula. This was done in [17], see also [18, 39], and thus ii) follows from i) by the equivalence statement.

In the case of r-spin Hurwitz numbers, proposition i) is known as r-Bouchard-Mariño conjecture [3], proposition ii) is the r-ELSV formula conjectured by Zvonkine in [53], see also [51]. The equivalence of i) and ii) was established in [51], but since neither of the two has been proven, both remain conjectural.

1.12 DOSS identification between topological recursion and Givental theory

In this section we recall the identification procedure between the CEO topological recursion and Givental theory introduced in [16].

It is known that the correlation differentials take have a very particular internal structure. In particular, this structure is related to the moduli spaces of curves and cohomological field theories in the following way.

Theorem 1.12.1 ([23, 16]). *The correlation differentials $\omega^{\mathcal{S}}_{g,n}$ produced via the topological recursion procedure from the specral curve $\mathcal{S} = (\Sigma, x, y, B)$ are equal to*

$$C^{2g-2+n} \sum_{\substack{i_1,\dots,i_n \\ d_1,\dots,d_n}} \int_{\overline{\mathcal{M}}_{g,n}} \left(\mathrm{Class}^{\mathcal{S}}\right)_{g,n} (e_{i_1} \otimes \cdots \otimes e_{i_n}) \prod_{j=1}^{n} \psi_j^{d_j} \mathrm{d}\left(\left(-\frac{1}{w_j}\frac{\mathrm{d}}{\mathrm{d}w_j}\right)^{d_j} \xi^{\mathcal{S}}_{i_j}\right).$$

Remark 1.12.2. The $\mathrm{Class}^{\mathcal{S}}$ defines a semi-simple CohFT, possibly with a non-flat unit. In this paper we will restrict the attention to CohFT with flat unit and we will write

$$\left(\mathrm{Class}^{\mathcal{S}}\right)_{g,n} = \left(R^{\mathcal{S}}.\alpha^{\mathcal{S},top}\right)_{g,n}$$

to indicate its Givental decomposition (for CohFT with non-flat unit see [49] or [40], Section 2.3).

Let us describe the ingredients in the formula above in terms of the data of the specral curve, following [16]. The only difference with the usual representation is that we incorporate a torus action on cohomological field theories, fixing a point $(C, C_1, \dots, C_r) \in (\mathbb{C}^*)^{r+1}$. This formula doesn't depend on these parameters, though all its ingredients do.

i). The local coordinates w_i on U_i, $i = 1, \dots, r$, are chosen such that $w_i(p_i) = 0$ and $x = (C_i w_i)^2 + x_i$

ii). The underlying topological field theory is given in the idempotent basis by

$$\eta(e_i, e_j) = \delta_{ij},$$
$$\alpha_{g,n}^{\mathcal{S},top}(e_{i_1} \otimes \cdots \otimes e_{i_n}) = \delta_{i_1 \ldots i_n} \left(-2C_i^2 C \frac{\mathrm{d}y}{\mathrm{d}w_i}(0)\right)^{-2g+2-n}.$$

iii). The Givental matrix $R^{\mathcal{S}}(\zeta)$ is given by

$$-\frac{1}{\zeta}(R^{\mathcal{S}})^{-1}(\zeta)_i^j = \frac{1}{\sqrt{2\pi\zeta}} \int_{-\infty}^{\infty} \frac{B(w_i, w_j)}{\mathrm{d}w_i}\bigg|_{w_i=0} \cdot e^{-\frac{w_j^2}{2\zeta}}.$$

iv). The auxiliary functions $\xi_i^{\mathcal{S}} \colon \Sigma \to \mathbb{C}$ are given by

$$\xi_i^{\mathcal{S}}(x) := \int^x \frac{B(w_i, w)}{\mathrm{d}w_i}\bigg|_{w_i=0}$$

v). *DOSS Test*: The following condition for the function y is necessary and sufficient in order for the unit of the cohomological field theory $R^{\mathcal{S}}.\alpha^{\mathcal{S}}$ to be flat.

$$\frac{2C_i^2 C}{\sqrt{2\pi\zeta}} \int_{-\infty}^{\infty} \mathrm{d}y \cdot e^{-\frac{w_i^2}{2\zeta}} = \sum_{k=1}^{r} (R^{-1})_k^i \left(2C_k^2 C \frac{\mathrm{d}y}{\mathrm{d}w_k}(0)\right)$$

This test was formalised in the form above in [19], Section 4. Its name refers to the initials of the authors of [16].

1.13 Quantum curves

In this section we briefly introduce the notion of quantum curve, tailored for the specific context of Hurwitz theory.

For numbers $N_{g;\mu}$ (connected) solutions of some enumerative geometric problem depending on a partition μ and a genus parameter g, define the free energies

$$F_{g,n}(p_1, p_2, \ldots) := \sum_{\mu \colon \ell(\mu)=n} N_{g;\mu}\, p_{\mu_1} \cdots p_{\mu_n}.$$

Consider the full partition function

$$\mathcal{Z}(p_1, p_2, \ldots; \hbar) := \exp\left(\sum_{g,n} F_{g,n}(p_1, p_2, \ldots)\hbar^{2g-2+n}\right)$$

and its principal specialisation $\mathcal{Z}(x, \hbar) = \mathcal{Z}(p_1, p_2, \ldots; \hbar)\Big|_{p_d \mapsto x^d}$. The Schrödinger equation for the principal specialisation of the partition function reads

$$D\left(x, \frac{x}{dx}, \hbar\right).\mathcal{Z}(\hbar, x) = 0$$

for a differential operator D. A quantisation procedure here is meant to be the data of the new variables $\hat{x} = \hat{x}(x, \frac{d}{dx}, \hbar)$ and $\hat{y} = \hat{y}(x, \frac{d}{dx}, \hbar)$. We call *quantum curve* the operator $D(\hat{x}, \hat{y})$. Taking the semi-classical limit gives an analytic complex curve $D(x, y) = 0$, which, conjecturally, gives the topological recursion spectral curve $\Sigma(D)$ that generates the same enumerative geometric problem we started with. Explicitly,

$$\omega_{g,n}^{\Sigma(D)}(x_1, \dots, x_n) = d_1 \dots d_n F_{g,n}(x_1, \dots, x_n).$$

More precisely, Gukov and Sulkowski [29] conjectured that whenever the solutions of an enumerative geometric problem are generated by a spectral curve via CEO topological recursion, there exists a way to *quantise* the spectral curve into the operator $D(\hat{x}, \hat{y})$.

For example, in Hurwitz problems the most used quantisations are

$$\hat{x} = x; \qquad \hat{y} = \hbar x \frac{x}{dx} \quad \text{or} \quad \hat{y} = \hbar \frac{x}{dx}$$

The different choices of quantisation reflect the ambient space in which the spectral curve is defined. The quantum curves for many Hurwitz problems are known, for many other are still to be computed, below we recall the quantum curves computed in [43].

Hurwitz numbers	Spectral curve	Quantum curve
q-orbifold	$x = y^{1/q} e^{-y}$	$\hat{y} - \left(e^{\frac{q-1}{2}\hat{y}} \hat{x} e^{-\frac{q-1}{2}\hat{y}}\right)^q e^{q\hat{y}}$
r-spin	$x = ye^{-y^r}$	$\hat{y} - \hat{x}^{\frac{3}{2}} \exp\left(\frac{\sum_{i=0}^{r} \hat{x}^{-1}\hat{y}^i \hat{x} \hat{y}^{r-i}}{r+1}\right) \hat{x}^{-\frac{1}{2}}$
q-r-mixed	$x = y^{1/q} e^{-y^r}$	$\hat{y} - \hat{x}^{\frac{q+1}{2}} \exp\left(\frac{q}{r+1} \sum_{i=0}^{r} \hat{x}^{-q}\hat{y}^i \hat{x}^q \hat{y}^{r-1}\right) \hat{x}^{-\frac{1}{2}}$

1.14 Methods of derivation of quantum curves

In this section we give a brief recollection of some of the basic concepts of KP integrability and methods of derivation of quantum curves. For more details see, e.g., [50, 42, 1] and references therein.

The KP hierarchy can be described by the bilinear identity satisfied by the tau-function $\tau(\mathbf{t})$, namely

$$\oint_{\infty} e^{\xi(\mathbf{t}-\mathbf{t}', z)} \tau(\mathbf{t} - [z^{-1}]) \, \tau(\mathbf{t}' + [z^{-1}]) dz = 0,$$

where $\xi(\mathbf{t}, z) = \sum_{k=1}^{\infty} t_k z^k$ and we use the standard notation

$$\mathbf{t} \pm \left[z^{-1}\right] = \left\{t_1 \pm \frac{1}{z}, t_2 \pm \frac{1}{2z^2}, t_3 \pm \frac{1}{3z^3}, \dots\right\}.$$

In particular, in Hurwitz-type problems, it is convenient to work in the coordinates $\mathbf{p}$ instead of $\mathbf{t}$, where $p_k = kt_k$, $k \geq 1$.

Let $\langle \Phi \rangle = \langle \Phi_1, \Phi_2, \Phi_3, \dots \rangle$ be the point of the Sato Grassmannian. The infinitesimal symmetries of the KP hierarchy can be conveniently described by working with *Kac–Schwarz* (KS) operators a in the algebra $w_{1+\infty}$ of differential operators in one variable that describes infinitesimal diffeomorphisms of the circle. A Kac-Schwarz operator a for the tau-function τ leaves invariant the corresponding point of the Sato Grassmannian, i.e., $a \langle \Phi \rangle \subset \langle \Phi \rangle$.

For the trivial tau-function $\tau_\emptyset := 1$ with the basis vectors $\Phi_k^\emptyset = x^{1-k}$, $k \geq 1$, we have two obvious KS operators

$$a_\emptyset = -x\frac{\partial}{\partial x}, \qquad b_\emptyset = x^{-1},$$

satisfying the commutation relation $[a_\emptyset, b_\emptyset] = b_\emptyset$. Note that they act on the basis vectors as

$$a_\emptyset \, \Phi_k^\emptyset(x) = (k-1)\Phi_k^\emptyset(x), \qquad b_\emptyset \, \Phi_k^\emptyset(x) = \Phi_{k+1}^\emptyset(x).$$

A generic KP tau-function $\tau_\bullet$ has a expression on the fermionic Fock space of semi-infinite wedge formalism, from which one can find a vector $|\Phi\rangle = g.|0\rangle$ representing the point of the Sato Grassmanian, obtained by an operator g acting on the vacuum. g determines an operator G that generates an admissible basis $\Phi_k^\bullet(x) = G(k)\Phi_k^\emptyset(x)$ for $\Phi^\bullet$. At this point, the operators obtained by conjugation by G

$$a_\bullet := G \, a_\emptyset \, G^{-1}, \qquad b_\bullet := G \, b_\emptyset \, G^{-1}$$

are Kac-Schwarz operators for the tau-function $\tau_\bullet$ satisfying the same commutation relation as $a_\emptyset$ and $b_\emptyset$, and acting in the same way as above with respect to the basis $\Phi_k^\bullet(x)$. Under a certain specialisation, the quantum curve is recovered by the operator $a_\bullet$ acting on the first element of the basis $\Phi_1^\bullet(x)$.

1.15 Outline of contents

The thesis is based on the following works:

1. R. Kramer, F. Labib, D. Lewanski, S. Shadrin, *The tautological ring of $\mathcal{M}_{g,n}$ via Pandharipande-Pixton-Zvonkine r-spin relations.*, arXiv:1703.00681, 2017.
2. A. Popolitov, D. Lewanski, S. Shadrin, D. Zvonkine, *Chiodo formulas for the r-th roots and topological recursion.* Letters in Mathematical Physics, pp.1-19, 2016.
3. A. Alexandrov, D. Lewanski, S. Shadrin, *Ramifications of Hurwitz theory, KP integrability and quantum curves.* Journal of High Energy Physics 2016(5), 2015.
4. P. Dunin-Barkowski, D. Lewanski, A. Popolitov, S. Shadrin, *Polynomiality of orbifold Hurwitz numbers, spectral curve, and a new proof of the Johnson-Pandharipande-Tseng formula.* Journal of the London Mathematical Society 92(3), 2015.
5. R. Kramer, D. Lewanski, S. Shadrin, *Polynomiality of monotone orbifold Hurwitz numbers and Grothendieck's dessins d'enfants.* arXiv:1610.08376, 2016.
6. R. Kramer, D. Lewanski, A. Popolitov, S. Shadrin, *Towards an orbifold generalization of Zvonkine's r-ELSV formula.* arXiv:1703.06725, 2017.

Other works related to these subjects by the same author:

7. D. Lewanski, *Correspondences between ELSV-type formulae and Spectral Curves.* Submitted to Proceedings of the 2016 von Neumann Symposium on Topological Recursion and its Influence in Analysis, Geometry, and Topology.

8. T. Milanov, appendix by D. Lewanski, *$\mathcal{W}$-algebra constraints and topological recursion for A_N-singularity.* International Journal of Mathematics 27 (13), 114-135.

The chapters are structured in the following way. Each chapter of the thesis is based on one of the six papers cited above, in the same order as they appear in the list, with the exception of Chapter 6, which is based on both papers 5 and 6. This choice is due to the fact that the results appearing in those two works are of similar nature and they are proved with similar methods, besides the technical difficulties taking different forms. The notations can differ slightly from chapter to chapter. Part of the material appearing in paper 7 is used in several sections of this introduction. Every chapter is self-contained and can be read independently. In the papers $1-6$, all the authors contributed equally.

1.15.1 Chapter 2: The tautological ring of $\mathcal{M}_{g,n}$ via Pandharipande-Pixton-Zvonkine r-spin relations

In [48], Pandharipande, Pixton, and Zvonkine derive the Givental decomposition for the shifted semi-simple cohomological field theory whose top degree coincides with the r-spin Witten class. This decomposition gives explicit relations in the tautological ring of the moduli spaces $\overline{\mathcal{M}}_{g,n}$. We rearrange and specialise these relations to obtain some restrictions on the dimensions of the tautological rings of the open moduli spaces $\mathcal{M}_{g,n}$. In particular, we give a new proof for the result of Looijenga (for $n = 1$) and Buryak et al. (for $n \geq 2$) that $\dim R^{g-1}(\mathcal{M}_{g,n}) \leq n$. We also give a new proof of the result of Looijenga (for $n = 1$) and Ionel (for arbitrary $n \geq 1$) that $R^i(\mathcal{M}_{g,n}) = 0$ for $i \geq g$ and give some estimates for the dimension of $R^i(\mathcal{M}_{g,n})$ for $i \leq g-2$. Explicitly, we prove the following results. First, we show that every monomial in ψ and κ classes of degree $g-1$ can be expressed in terms of pure monomials in ψ classes.

Proposition 1.15.1. *Let $g \geq 2$ and $n \geq 1$. The ring $R^{g-1}(\mathcal{M}_{g,n})$ is spanned by the monomials $\psi_1^{d_1} \cdots \psi_n^{d_n}$ for $d_1, \ldots, d_n \geq 0$, $\sum_{i=1}^n d_i = g-1$.*

Then, we show that every monomial in ψ classes of degree $g-1$ can be expressed in terms of n generators.

Proposition 1.15.2. *For $n \geq 2$ and $g \geq 2$, every monomial of degree $g-1$ in ψ classes and at most one κ_1-class can be expressed as linear combinations of the following n classes*

$$\psi_1^{g-1},\ \psi_1^{g-2}\psi_2,\ \ldots,\ \psi_1^{g-2}\psi_n,$$

with rational coefficients.

Together, the two propositions give a new proof of

Theorem 1.15.3 ([5])**.** *For $n \geq 2$ and $g \geq 2$*

$$\dim_{\mathbb{Q}} R^{g-1}(\mathcal{M}_{g,n}) \leq n.$$

We moreover prove the vanishing of higher degrees of the tautological ring.

Theorem 1.15.4 ([20, 31])**.** *The tautological ring of $\mathcal{M}_{g,n}$ vanishes in degrees g and higher, that is $R^{\geq g}(\mathcal{M}_{g,n}) = 0$.*

These two theorems consistute the generalized Faber socle conjecture stated above, as the bound $\dim R^{g-1}(\mathcal{M}_{g,n}) \geq n$ can be proved in a relatively simple way, see e.g. [5].

Similarly to [48, theorem 6], our method also gives a bound for the dimension of the lower degree tautological classes.

Proposition 1.15.5. *Let $p(n)$ be the number of partitions of n, and $p(n,k)$ the number of partitions of n of length at most k.*

$$\dim R^d(\mathcal{M}_{g,n}) \leq \sum_{k=0}^{d} \binom{n+k-1}{k} p(d-k, g-1-d)$$

Remark 1.15.6. If we use the natural interpretation of $\binom{k-1}{k}$ as $\delta_{k,0}$, this does indeed recover [48, theorem 6] in the case $n = 0$.

The proof exhibits an explicit spanning set of this cardinality, consisting of monomials in ψ-classes multiplied with a multi-index κ-class.

1.15.2 Chapter 3: Chiodo formulas for the r-th roots and topological recursion

In this chapter we apply the formula in Theorem 1.12.1 to the spectral curve

$$\mathcal{S}_{r,s} := \left(\mathbb{CP}^1, x(z) = -z^r + \log z, y(z) = z^s, B(z,z') = \frac{\mathrm{d}z\,\mathrm{d}z'}{(z-z')^2}\right)$$

for a global coordinate z on the Riemann sphere and r, s integer parameters. The result of the DOSS identification for this particular spectral curve is summarised in the following. It is important to notice that the Givental R-matrix and the topological field theory coincide with the ones in the Givental decomposition of the Chiodo classes.

Proposition 1.15.7. *The following properties hold:*

i). Choose the constants $C_i = 1/\sqrt{-2r}$ for $i = 1, \dots, r$, and $C = r^{1+s/r}/s$. With this choice the local coordinates w_i on U_i, $i = 1, \dots, r$ satisfy $x = -\frac{w_i^2}{2r} + x(p_i)$.

ii). The underlying topological field theory is given by

$$\eta(v_a, v_b) = \frac{1}{r}\delta_{a+b \mod r}$$
$$\alpha_{g,n}^{\mathcal{S}_{r,s},top}(v_{a_1} \otimes \cdots \otimes v_{a_n}) = r^{2g-1}\delta_{a_1+\cdots+a_n-s(2g-2+n) \mod r}.$$

where the flat coordinates v_a are defined in terms of the idempotents by

$$v_a := \sum_{i=0}^{r-1} \frac{J^{ai}}{r} e_i, \qquad a = 1, \dots, r.$$

iii). The Givental matrix $R^{\mathcal{S}_{r,s}}(\zeta)$ *is given by*

$$R^{\mathcal{S}_{r,s}}(\zeta) = \exp\left(- \sum_{k=1}^{\infty} \frac{\mathrm{diag}_{a=1}^{r} B_{k+1}\left(\frac{a}{r}\right)}{k(k+1)} \zeta^k \right).$$

iv). The auxiliary functions $\xi_a^{\mathcal{S}_{r,s}} : \Sigma \to \mathbb{C}$ *are given by*

$$\xi_a^{\mathcal{S}_{r,s}} = r^{\frac{r-a}{r}} \sum_{n=0}^{\infty} \frac{(nr+r-a)^n}{n!} e^{(nr+r-a)x}$$

v). The DOSS Test is satisfied.

Substituting these ingredients in Theorem 1.12.1 leads to the main theorem of the chapter.

Theorem 1.15.8. *The correlation differentials* $\omega_{g,n}^{\mathcal{S}_{r,s}}$ *are equal to*

$$\mathrm{d}_1 \otimes \cdots \otimes \mathrm{d}_n \frac{r^{2g-2+n+b}}{s^{2g-2+n}} \prod_{j=1}^{n} \frac{\left(\frac{\mu_j}{r}\right)^{[\mu_j]}}{[\mu_j]!} \sum_{\mu_1,\ldots,\mu_n=1}^{\infty} \int_{\overline{\mathcal{M}}_{g,n}} \frac{\mathcal{C}_{g,n}\left(r,s;r-\langle\vec{\mu}\rangle\right)}{\prod_{j=1}^{n}(1-\frac{\mu_i}{r}\psi_i)} \; e^{\sum \mu_j x_j}$$

where $b(r,s) = \left((2g-2+n)s + \sum_{j=1}^{n} \mu_j\right)/r$.

Remark 1.15.9. Note that the case $s = 1$ reproduces Theorem 1.7 in [51], which is the ELSV-TR equivalence statement for r-spin Hurwitz numbers.

Expanding the correlation differentials as

$$\omega_{g,n}^{\mathcal{S}_{r,s}} = \mathrm{d}_1 \otimes \cdots \otimes \mathrm{d}_n \sum_{\mu_1,\ldots,\mu_n=1}^{\infty} \frac{N_{g,\vec{\mu}}^{\mathcal{S}_{r,s}}}{b(r,s)!} \; e^{\sum_{j=1}^{n} \mu_j x_j},$$

we find:

Corollary 1.15.10 ([40])**.**

$$N_{g,\vec{\mu}}^{\mathcal{S}_{r,s}} = b(r,s)! \frac{r^{b(r,s)+2g-2+n}}{s^{2g-2+n}} \prod_{i=1}^{n} \frac{\left(\frac{\mu_i}{r}\right)^{[\mu_i]}}{[\mu_i]!} \int_{\overline{\mathcal{M}}_{g,n}} \frac{C_{g,n}(r,s;r-\langle\vec{\mu}\rangle)}{\prod_{j=1}^{n}(1-\frac{\mu_j}{r}\psi_j)}.$$

Specialising the corollary for $s = r$ and plugging the $r-$orbifold Hurwitz numbers and the curve $\mathcal{S}_{r,r}$ into the TR-ELSV equivalence statement in Definition 1.11.1, we get:

Corollary 1.15.11. *The two propositions are equivalent:*

i) $\quad h_{g,\vec{\mu}}^{\circ,[r]} = N_{g,\vec{\mu}}^{\mathcal{S}_{r,r}}$

ii) $\quad h_{g,\vec{\mu}}^{\circ,[r]} = b(r,r)! r^{b(r,r)} \prod_{i=1}^{n} \frac{\left(\frac{\mu_i}{r}\right)^{[\mu_i]}}{[\mu_i]!} \int_{\overline{\mathcal{M}}_{g,n}} \frac{C_{g,n}(r,r;r-\langle\vec{\mu}\rangle)}{\prod_{j=1}^{n}(1-\frac{\mu_j}{r}\psi_j)}.$

Note that the statement *ii)* is very similar to JPT formula, with the difference that the class $p_* \sum_{i \geq 0} (-r)^i \lambda_i$ is described in [35] via admissible covers, while Chiodo's classes rely on the moduli space of r-th tensor roots. We prove that two approaches are in fact equivalent:

Proposition 1.15.12. $p_* \sum_{i \geq 0} (-r)^i \lambda_i = \mathcal{C}_{g,n}(r, r; r - \langle \vec{\mu} \rangle)$.

Hence we can re-state the corollary as:

Corollary 1.15.13. *The two statements are equivalent:*

i) The r-orbifold Hurwitz numbers satisfy the topological recursion with input the spectral curve $\mathcal{S}_{r,r}$.

ii) The JPT formula holds.

At this point, since JPT formula is proved independently, the equivalence statement implies a new proof of the topological recursion for orbifold Hurwitz numbers. One could legitimately ask the opposite question: can we obtain a new proof of such a result in algebraic geometry just using topological recursion? Several proofs of the topological recursion existed in this case [4, 13], but all of them relied on the JPT Formula in the following way: they extracted a certain polynomiality property from JPT, and proved topological recursion by combining this polynomiality property with a combinatorial formula obtained by counting graphs over surfaces. In order to achieve the converse implication, therefore, an independent proof of the polynomiality property is required. This polynomiality property is proved in Chapter 5 with no use of JPT. Therefore these results together, and via Corollary 1.15.13, provide a new proof of Johnson-Pandharipande-Tseng formula.

1.15.3 Chapter 4: Ramifications of Hurwitz theory, KP integrability and quantum curves

In this chapter we consider several variations of Hurwitz numbers. New examples of quantum curves and vertex operators on the Fock space are derived. Moreover, a systematic organisation of these combinatorial problems provides new and simpler proofs of known results. In particular, we use various versions of these numbers to discuss methods of derivation of quantum spectral curves from the point of view of KP integrability and derive new examples of quantum curves for the families of double Hurwitz numbers.

The logic works as follows. If the generating function of some combinatorial problem is a solution of an integrable hierarchy of type KP, its specialization can be seen as a vector of the corresponding point in the Sato Grassmannian, and this reduces the problem to finding a Kac-Schwarz operator that would annihilate it. Once a quantum curve is known, one can formulate a precise conjecture: the semi-classical limit of the quantum curve is the spectral curve for the CEO topological recursion whose coefficients of the differentials $\omega_{g,n}$ are precisely the numbers of the combinatorial problem we started with. Once the spectral curve for the topological recursion is known, one can conclude that the combinatorial problem can be solved in terms of some intersection numbers on the moduli space of curves and, therefore, can be expressed using Givental theory and admits a formula of type ELSV.

Firstly, we consider many of the Hurwitz numbers described above, in particular the strictly monotone, the weakly monotone, the atlantes, the free single and the free group Hurwitz numbers, and we show that the Jucys correspondence and its extension on the homogenous complete polynomials provides a direct identification between some of these numbers.

Proposition 1.15.14. *The strictly monotone and the free single Hurwitz numbers coincide. The weakly monotone and the free group Hurwitz numbers coincide.*

Explicitly, we show that the blocks of ramifications defining these numbers are interchangable. On the one hand, this provides a short alternative proof of the Harnad-Orlov correspondence [30]. On the other hand, it provides a new interpretation of the enumerative geometric problem, known as enumeration of hypermaps or a particular kind of Grothendieck dessins d'enfant, as the enumeration of strictly monotone orbifold numbers.

Secondly, we provide an expression in the semi-infinite wedge formalism of strictly monotone, weakly monotone and atlantes Hurwitz numbers.

Definition 1.15.15. Define the following operators on the semi-infinite wedge as formal series in z:

$$\mathcal{D}^{(p)}(z) = \frac{\tilde{\mathcal{E}}_0(z)}{\zeta(z)} - E,$$

$$\mathcal{D}^{(h)}(z) = z^{\frac{\tilde{\mathcal{E}}_0\left(z^2\frac{d}{dz}\right)}{\zeta\left(z^2\frac{d}{dz}\right)} - E} := \exp\left(\left[\frac{\tilde{\mathcal{E}}_0\left(z^2\frac{d}{dz}\right)}{\zeta\left(z^2\frac{d}{dz}\right)} - E\right]\log z\right),$$

$$\mathcal{D}^{(\sigma)}(z) = z^{-\frac{\tilde{\mathcal{E}}_0\left(-z^2\frac{d}{dz}\right)}{\zeta\left(-z^2\frac{d}{dz}\right)} + E} := \exp\left(-\left[\frac{\tilde{\mathcal{E}}_0\left(-z^2\frac{d}{dz}\right)}{\zeta\left(-z^2\frac{d}{dz}\right)} - E\right]\log z\right).$$

These operators respectively provide the generating series for the different bases of symmetric polynomials as eigenvalues, with eigenvectors the basis of the charge zero sector:

Proposition 1.15.16.

$$\mathcal{D}^{(p)}(z)v_\lambda = \sum_{k=1}\frac{z^k}{k!}p_k(\mathbf{cr}^\lambda)v_\lambda, \qquad \mathcal{D}^{(h)}(z)v_\lambda = \sum_{k=0} z^k h_k(\mathbf{cr}^\lambda)v_\lambda,$$

$$\mathcal{D}^{(\sigma)}(z)v_\lambda = \sum_{k=0} z^k \sigma_k(\mathbf{cr}^\lambda)v_\lambda.$$

Hence these operators can be used to express generating series or partition functions of Hurwitz problems in which the corresponding blocks of ramification appear, in semi-infinite wedge formalism. This allows us to perform explicit calculations at the level of operators and their commutators, to understand the polynomiality behaviour of the numbers. This has been investigated in Chapter 6.

Thirdly, we consider the relation between topological recursion and ELSV formulae described above, for the case of weakly monotone Hurwitz numbers. In this case the spectral curve is known [12]. So, we derive the following ELSV formula.

Proposition 1.15.17. *We have:*

$$h^{\circ,\leq}_{g,\mu} = \prod_{i=1}^{\ell(\mu)}\binom{2\mu_i}{\mu_i}\int_{\overline{\mathcal{M}}_{g,\ell(\mu)}} e^{\sum_{l=1} A_l\kappa_l}\prod_{j=1}^{\ell(\mu)}\sum_{d_j\geq 0}\psi_j^{d_j}\frac{(2(\mu_j+d_j)-1)!!}{(2\mu_j-1)!!}.$$

Here the coefficients A_i, $i = 1, 2, \dots$, satisfy the following equation:

$$\exp\left(-\sum_{l=1}^{\infty} A_l U^l\right) = \sum_{k=0}^{\infty} (2k+1)!!U^k.$$

Fourthly, we study several examples of KP tau functions arising from Hurwitz theory and we derive quantum curves by the method of Kac-Schwarz operators.

Proposition 1.15.18. *The quantum curve for the monotone r-orbifold Hurwitz numbers is equal to*

$$\hat{x}\left(\hat{x}^{r-1} + \prod_{j=1}^{r} \left(1 + \hat{x}\hat{y} + \hbar(j-1)\right)\hat{y}\right).$$

In particular, for $r = 1$, it reduces to $\hat{x}(\hat{x}\hat{y}^2 + \hat{y} + 1)$, recovering the quantum curves obtained in [11, 14]. With the same methods, we derive an infinite series of linear equations for the tau-function of double monotone Hurwitz numbers

$$\tau_{mm}\left(\mathbf{t}, \tilde{\mathbf{t}}\right) = \langle 0|\exp\left(\sum_{i=1}^{\infty} t_i\alpha_i\right)\mathcal{D}^{(h)}(\hbar)\exp\left(\sum_{i=1}^{\infty} \tilde{t}_i\alpha_{-i}\right)|0\rangle,$$

and the quantum curve for double Hurwitz numbers, which reads

Proposition 1.15.19.

$$\sum_{k=1}^{\infty} k\tilde{t}_k\left(\hat{x}e^{\hat{y}}\right)^k - \hat{y}\Psi(e^x, \hbar) = 0,$$

where $\hat{x} = x\cdot$ and $\hat{y} = \hbar\frac{x}{dx}$ and $\Psi(x, \hbar)$ is the wave function of the partition function

$$\tau_{HH}(\mathbf{t}, \tilde{\mathbf{t}}) = \langle 0|\exp\left(\sum_{i=1}^{\infty} t_i\alpha_i\right)\exp\left(\hbar\mathcal{F}_2\right)\exp\left(\sum_{i=1}^{\infty} \tilde{t}_i\alpha_{-i}\right)|0\rangle.$$

The quantum curve for strictly monotone orbifold Hurwitz numbers is then derived, and we show that this quantum curve indeed coincides with the one earlier obtained in [12] using combinatorics of hypermaps and in [15] using the loop equations for hypermaps, confirming the prediction of Jucys correspondence. Moreover, the interpretation as strictly monotone orbifold Hurwitz numbers provides a short and easy proof. The quantum curve for blocks of atlantes of fixed type is also derived. In this case, the quantum curve reads $\hat{y} - \hat{x}e^{\hat{y}^r}$. This case is very interesting since it provides a counterexample of the usual behaviour of the quantum curves as quantisation of topological recursion spectral curves. Indeed, the dequantization of this quantum curve concides with the dequantization of the quantum curve for the r-spin Hurwitz number

$$\hat{y} - \hat{x}^{3/2}\exp\left(\frac{1}{r+1}\sum_{i=0}^{r} \hat{x}^{-1}\hat{y}^i\hat{x}\hat{y}^{r-i}\right)$$

proved in [43]. Even though the spectral curve and the corresponding r-ELSV formula for the r-spin Hurwitz numbers are still conjectural, there is very strong evidence for these conjectures to be true [51]. From these conjectures we can conclude that the dequantization of $\hat{y} - \hat{x}\exp(\hat{y}^r)$ can not be the spectral curve for the atlantes Hurwitz numbers, suitable for the construction of the topological recursion. Finally, we discuss a one-parameter deformation of single Hurwitz numbers by the co-length (indicated with the formal variable c) of the ramification over infinity. The resulting quantum curve is

$$\left(1 - \left(e^{-\hat{y}}\hat{x}^{-1} - 2c + c^2\hat{x}e^{\hat{y}}\right)\hat{y}\right)\Psi(x,\hbar) = 0$$

which recovers for $c = 0$ is the wave function of the single Hurwitz numbers.

1.15.4 Chapter 5: Quasi-polynomiality of orbifold Hurwitz numbers, spectral curve, and a new proof of the Johnson-Pandharipande-Tseng formula

In this Chapter we address the polynomiality property of orbifold Hurwitz numbers. In particular, we prove the following result:

Theorem 1.15.20. *For $2g - 2 + l(\vec{\mu}) \geq 0$, we have*

$$h^{\circ,[r]}_{g;\vec{\mu}} = \prod_{i=1}^{l(\vec{\mu})} \frac{\mu_i^{[\mu_i]}}{[\mu]!} P^{\langle\vec{\mu}\rangle}_{[r]}(\mu_1, \ldots, \mu_{l(\vec{\mu})})$$

where $P^{\langle\vec{\mu}\rangle}_{[r]}$ is a polynomial of degree $3g - 3 + l(\vec{\mu})$ depending on the parameters $\langle\mu_1\rangle, \ldots, \langle\mu_{l(\vec{\mu})}\rangle$ and $\mu = r[\mu] + \langle\mu\rangle$ denotes the euclidean division by r.

Because of the work described above, this allows to achieve via topological recursion a new proof of the Johnson-Pandharipande-Tseng ELSV Formula, a result that has been previously proved only by pure algebraic geometric methods. The proof follows the methods developed in [17] for simple Hurwitz numbers, applying them to the more general case for orbifold ones.

Step 1. Express the generating function for the disconnected numbers

$$H^{\bullet,[r]}(\vec{\mu}, u) = \sum_{g=0}^{\infty} \frac{h^{\bullet}_{g;\mu}}{b!} u^b$$

in terms of the semi-infinite wedge formalism as

$$H^{\bullet,[r]}(\vec{\mu}, u) = \frac{1}{\prod \mu_i} \left\langle e^{\frac{\alpha_r}{r}} e^{u\mathcal{F}_2} \prod_{i=1}^{l(\vec{\mu})} \alpha_{-\mu_i} \right\rangle^{\bullet}.$$

Step 2. By energy considerations, the expression can be re-arranged in a more treatable form:

$$H^{\bullet,[r]}(\vec{\mu}, u) = \frac{1}{\prod \mu_i} \left\langle \prod_{i=1}^{l(\vec{\mu})} e^{\frac{\alpha_r}{r}} e^{u\mathcal{F}_2} \alpha_{-\mu_i} e^{-u\mathcal{F}_2} e^{-\frac{\alpha_r}{r}} \right\rangle^{\bullet}$$

Step 3. Following the logic used by Okounkov and Pandharipande, the two nested conjugations of the operator $\alpha_{-\mu_i}$ by the exponential operators can be computed explicitly, leading to the expression in the so-called $\mathcal{A}$ operators:

$$H^{\bullet,[r]}(\vec{\mu},u) = r^{\sum\langle\mu_i\rangle} \prod_{i=1}^{l(\vec{\mu})} \frac{u^{\frac{\mu_i}{r}}\mu_i^{[\mu_i]}}{[\mu_i]!} \frac{1}{\prod \mu_i} \left\langle \prod_{i=1}^{l(\vec{\mu})} \mathcal{A}^{[r]}_{\langle\mu_i\rangle}(\mu_i,u) \right\rangle^{\bullet}$$

The precise formula of the r-orbifold $\mathcal{A}$ operators can be found in Chapter 5.

Step 4. The same formula holds for the generating function for the connected numbers

$$H^{\circ,[r]}(\vec{\mu},u) = r^{\sum\langle\mu_i\rangle} \prod_{i=1}^{l(\vec{\mu})} \frac{u^{\frac{\mu_i}{r}}\mu_i^{[\mu_i]}}{[\mu_i]!} \frac{1}{\prod \mu_i} \left\langle \prod_{i=1}^{l(\vec{\mu})} \mathcal{A}^{[r]}_{\langle\mu_i\rangle}(\mu_i,u) \right\rangle^{\circ}$$

where the connected correlators are defined from the disconnected ones by means of the inclusion-exclusion formula.

Step 5. Following the proof of Okounkov and Pandharipande for the correlators associated to the simple Hurwitz numbers, Johnson showed that the disconnected correlators of $\mathcal{A}$ operators for r-orbifold Hurwitz numbers converge to an analytic function for complex variables $(z_1,\dots,z_n)$ in a certain region Ω for sufficiently small u. The evaluation at $(z_1,\dots,z_n) = (\mu_1,\dots,\mu_n)$ recovers the r-orbifold Hurwitz numbers. This allows to consider the Laurent expansion of the correlators near the origin of $\mathbb{C}^n$. Using this expansion, the *unstable* connected correlators, or correlators corresponding to the two pairs $(g,\ell(\mu)) = (0,1),(0,2)$, are computed explicitly.

Step 6. The disconnected correlators are expressed in terms of the connected ones in terms of sums of products of connected ones, by means of the inclusion-exclusion formula. Among these summands, some of them contain, as a factor, unstable connected correlators, and therefore should be excluded from the expression. By using two recursive formulae, we decompose the correlators and show that dropping these unstable summands is equivalent to considering the disconnected correlator of the regular part of the $\mathcal{A}$ operators in $z_1,\dots,z_n$ in the expansion mentioned before.

Step 7. The stable disconnected correlator of the regular part of the $\mathcal{A}$ operators, divided by the product of the μ_i, is shown to be a symmetric polynomial. This implies, by using the inclusion-exclusion formula to perform induction on $\ell(\mu)$, that the same holds for the connected correlators of the $\mathcal{A}$ operators, including regular and principal part.

Step 8. This is not enough, though, to conclude the statement. Indeed, this proves that the result holds only for those partitions μ of d inside the region Ω.

Step 9. For fixed $\eta_1,\mu_2,\dots,\mu_n$, we show rationality in the first variable of the disconnected correlator with bounded degrees of both numerator and denominator for each fixed power of u. Hence the correlator, as a function of the first variable, can be extended as rational function, except at most at the poles corresponding to negative integer values. The evaluation of this rational function and the polynomial function coincide in a Zariski-dense set. This implies the polynomiality of the connected correlator in the first variable. Now, a symmetric complex function in several variables that is polynomial in one of the variables is a polynomial in all the variables. This concludes the proof of the statement.

1.15.5 Chapter 6: Quasi-polynomiality of Grothendieck's dessins d'enfants, orbifold, monotone orbifold, spin, and spin orbifold Hurwitz numbers

This chapter collects the quasi-polynomiality statements for five different Hurwitz problems: orbifold weakly monotone, orbifold strictly monotone, r-orbifold, r-spin, and q-orbifold r-spin Hurwitz numbers.

Theorem 1.15.21. *For $2g-2+l(\vec{\mu}) \geq 0$, we have*

$$i). \quad h^{\circ,r,\leq}_{g;\vec{\mu}} = \prod_{i=1}^{l(\vec{\mu})} \binom{\mu_i + [\mu_i]}{\mu_i} P^{\langle\vec{\mu}\rangle}_{\leq}(\mu_1, \dots, \mu_{l(\vec{\mu})})$$

$$ii). \quad h^{\circ,r,<}_{g;\vec{\mu}} = \prod_{i=1}^{l(\vec{\mu})} \binom{\mu_i - 1}{[\mu_i]} P^{\langle\vec{\mu}\rangle}_{<}(\mu_1, \dots, \mu_{l(\vec{\mu})})$$

$$iii). \quad h^{\circ,[r]}_{g;\vec{\mu}} = \prod_{i=1}^{l(\vec{\mu})} \frac{\mu_i^{[\mu_i]}}{[\mu]!} P^{\langle\vec{\mu}\rangle}_{[r]}(\mu_1, \dots, \mu_{l(\vec{\mu})})$$

$$iv). \quad h^{\circ,r-spin}_{g;\vec{\mu}} = \prod_{i=1}^{l(\vec{\mu})} \frac{\mu_i^{[\mu_i]}}{[\mu]!} P^{\langle\vec{\mu}\rangle}_{r-spin}(\mu_1, \dots, \mu_{l(\vec{\mu})})$$

$$v). \quad h^{\circ,[q],r-spin}_{g;\vec{\mu}} = \prod_{i=1}^{l(\vec{\mu})} \frac{\mu_i^{[\mu_i]_{qr}}}{[\mu]_{qr}!} P^{\langle\vec{\mu}\rangle_{qr}}_{[q],r-spin}(\mu_1, \dots, \mu_{l(\vec{\mu})})$$

where $P^{\langle\vec{\mu}\rangle}_{<}$, $P^{\langle\vec{\mu}\rangle}_{\leq}$, $P^{\langle\vec{\mu}\rangle}_{[r]}$, $P^{\langle\vec{\mu}\rangle}_{r-spin}$ and $P^{\langle\vec{\mu}\rangle_{qr}}_{[q],r-spin}$ are polynomials whose coefficients depend on the parameters in the upper indices, $\mu = r[\mu] + \langle\mu\rangle$ denotes the euclidean division by r and $\mu = qr[\mu]_{qr} + \langle\mu\rangle_{qr}$ denotes the euclidean division by qr.

Let us analyse the result.

i). The first statements confirms Conjecture 23 in [11]. Note that the small difference in the convention of the conjecture does not affect quasi-polynomiality since the polynomials $P_{\leq}$ depend on the parameters $\langle\mu\rangle$.

ii). The second statement confirms Conjecture 12 in [12]. The conjecture is stated in terms of Grothendieck dessin d'enfants, which correspond to strictly monotone orbifold Hurwitz numbers by the Jucys correspondence as shown in Chapter 4.

iii). The third statement, combined with the results in Chapter 3, provides a new proof of the Johnson-Pandharipande-Tseng ELSV formula, as explained in Chapter 3. This statement is already proved in Chapter 5. In this Chapter, though, a new easier proof is provided, by using the residue methods described below. These methods can be seen in some sense as more powerful, since they allow the proof of the fourth statement. The proof of the first two statements could probably have been performed with the methods of Chapter 5, although the Fock space operators for those Hurwitz problems were not yet known.

iv). The fourth statement is a key step towards the proof of Zvonkine conjecture [53] or, by the equivalence statement in [51] and reproved in Chapter 3, towards the proof of the r-Bouchard-Mariño conjecture. Although the non-polynomial part in the third and the fourth statement are the same, these non-polynomialities arise in very different ways from operators encoding radically different geometric meaning, and I am not aware of any reason they should be related.

v). The fifth statement yields an orbifold generalization of Zvonkine's conjecture.

Let us outline the logic of the proof, which is the same in all five cases. In the following we refer to the euclidean division by r. The logic of the proof for the fifth case follows by replacing r by qr.

Step 1.-3. Same as for the proof in Chapter 5.

Step 4. Because the correlator is a symmetric expression in the μ_i, it is enough to show polynomiality of the connected correlators for $[\mu_1]$.

Step 5. Rationality in $[\mu_1]$ is shown, for fixed $[\mu_2], \dots, [\mu_n], \langle\mu_1\rangle, \dots, \langle\mu_n\rangle$ and fixed power of u, using the vanishing near the covacuum and imposing zero total energy.

Step 6. This allows the correlator as a rational function in $[\mu_1]$ to be extended everywhere except possibly at finitely many poles at negative integers and compute the residues.

Step 7. The effect of the residue operator on the $\mathcal{A}$ operators reads

$$\operatorname*{Res}_{[k]=-l} \mathcal{A}_{\langle\mu_1\rangle}(u, [k]r + \langle k\rangle) = c\mathcal{A}_{-\langle k\rangle}(u, lr - \langle k\rangle)^{\flat}$$

for some constant c. Here the operator $\mathcal{A}^{\flat}$ is obtained from $\mathcal{A}$ by replacing the inner operator α_n by its adjoint α_{-n} and inverting the combinatorial prefactors. Each of the five cases clearly presents different $\mathcal{A}$ operators, but this behaviour is common to all of them (although we prove it separately for each case), with possibly different constants c.

Step 8. This allows a very good control of the possible decompositions of the correlator into summands with unstable correlators as factors; more precisely, it implies the vanishing of the residues in all connected correlators, obtained from the disconnected ones by the inclusion-exclusion formula, except in the two cases $(g, \ell(\mu)) = (0,1), (0,2)$ corresponding to unstable correlators.

Step 9. Once the polynomiality of the connected correlator divided by the product of the μ_i is proved, the degree of the polynomial in μ_1 can be checked to be independent of the choice of the fixed parameters. This concludes the proof of the statements.

In the second part of the chapter we show for the first three cases that the property of quasi-polynomiality is equivalent to the property that the n-point generating function has a natural representation on the n-th cartesian power of a certain algebraic curve, proving necessary conditions for the CEO topological recursion. Moreover we prove that the unstable correlation differentials for the conjectural (or proved) CEO topological recursion coincide with the expression derived from the $\mathcal{A}$-operators. These computations are performed in the case of monotone orbifold, spin, and spin orbifold Hurwitz numbers for the cases $(g, n) = (0,1)$ and $(g, n) = (0,2)$, and for strictly monotone orbifold Hurwitz numbers for the case $(g, n) = (0,1)$. In all cases the computation of the $(0,1)$-numbers was done before. The $(0,2)$-calculation for the spin and orbifold spin Hurwitz numbers is a new result. The $(0,2)$-calculation for the monotone Hurwitz numbers is also a new result, but we learned after completing our calculation that Karev obtained the same formula independently [37]. We show these computations to test the $\mathcal{A}$-operator formula and to demonstrate its power. The computations of the generating function for the $(0,2)$ monotone orbifold Hurwitz numbers, for $(0,2)$ spin Hurwitz numbers and for $(0,2)$ spin orbifold Hurwitz numbers are necessary for the conjectures on topological recursion in [11], [51] and the generalisation of Zvonkine's conjecture proposed in Section 6.11, respectively.

[1] A. Alexandrov. Enumerative Geometry, Tau-Functions and Heisenberg Virasoro Algebra. *Communications in Mathematical Physics*, 338(1):195–249, 2015.

[2] G. Borot, B. Eynard, M. Mulase, and B. Safnuk. A matrix model for simple Hurwitz numbers, and topological recursion. *Journal of Geometry and Physics*, 61(2):522–540, 2011.

[3] V. Bouchard and M. Mariño. Hurwitz numbers, matrix models and enumerative geometry. *arXiv:0709.1458*, page 21, 2007.

[4] V. Bouchard, D. Hernandez Serrano, X. Liu, and M. Mulase. Mirror symmetry for orbifold Hurwitz numbers. *arXiv:1301.4871*, 98:39, 2013.

[5] A. Buryak, S. Shadrin, and D. Zvonkine. Top tautological group of $\mathcal{M}_{g,n}$. *Journal of the European Mathematical Society*, 18(12):2925–2951, 2016.

[6] R. Cavalieri and E. Miles. *Riemann Surfaces and Algebraic Curves : A First Course in Hurwitz Theory.*

[7] A. Chiodo and D. Zvonkine. Twisted r-spin potential and Givental's quantization. *Advances in Theoretical and Mathematical Physics*, 13(5):1335–1369, 2009.

[8] A. Chiodo. Stable twisted curves and their r-spin structures. *Ann. Inst. Fourier (Grenoble)*, 58:1635–1689, 2007.

[9] A. Chiodo. Towards an enumerative geometry of the moduli space of twisted curves and r-th roots. *Compositio Mathematica*, 144(6):1461–1496, 2008.

[10] A. Chiodo and Y. Ruan. LG/CY correspondence: The state space isomorphism. *Advances in Mathematics*, 227(6):2157–2188, 2011.

[11] N. Do and M. Karev. Monotone orbifold Hurwitz numbers. arXiv:1505.06503, 2015.

[12] N. Do and D. Manescu. Quantum curves for the enumeration of ribbon graphs and hypermaps. *Communications in Number Theory and Physics*, 8(4):677–701, 2014.

[13] N. Do, O. Leigh, and P. Norbury. Orbifold Hurwitz numbers and Eynard-Orantin invariants. *arXiv:1212.6850*, page 28, 2012.

[14] N. Do, A. Dyer, and D. Mathews. Topological recursion and a quantum curve for monotone Hurwitz numbers. arXiv:1408.3992, 2014.

[15] P. Dunin-Barkowski, N. Orantin, A. Popolitov, and S. Shadrin. Combinatorics of loop equations for branched covers of sphere. arXiv:1412.1698, 2014

[16] P. Dunin-Barkowski, N. Orantin, S. Shadrin, and L. Spitz. Identification of the Givental Formula with the Spectral Curve Topological Recursion Procedure. *Communications in Mathematical Physics*, 328(2):669–700, 2014.

[17] P. Dunin-Barkowski, M. Kazarian, N. Orantin, S. Shadrin, and L. Spitz. Polynomiality of Hurwitz numbers, Bouchard-Mariño conjecture, and a new proof of the ELSV formula. *Advances in Mathematics*, 279:67–103, 2015.

[18] P. Dunin-Barkowski, D. Lewanski, A. Popolitov, and S. Shadrin. Polynomiality of orbifold Hurwitz numbers, spectral curve, and a new proof of the Johnson-Pandharipande-Tseng formula. *Journal of the London Mathematical Society*, 92(3):547–565, 2015. ISSN 14697750. doi: 10.1112/jlms/jdv047.

[19] P. Dunin-Barkowski, P. Norbury, N. Orantin, A. Popolitov, and S. Shadrin. Dubrovin's superpotential as a global spectral curve. arXiv:1509.06954, 2015.

[20] E. Looijenga. On the tautological ring of $\mathcal{M}_g$. *Inventiones mathematicae*, 121(2):page 411–420, 1995.

[21] T. Ekedahl, S. Lando, M. Shapiro, and A. Vainshtein. Hurwitz numbers and intersections on moduli spaces of curves. *Inventiones Mathematicae*, 146(2):297–327, 2001.

[22] B. Eynard. Intersection numbers of spectral curves. arXiv:1104.0176, 2011.

[23] B. Eynard. Invariants of spectral curves and intersection theory of moduli spaces of complex curves. *Communications in Number Theory and Physics*, 8(3):541–588, 2014.

[24] B. Eynard and N. Orantin. Invariants of algebraic curves and topological expansion. *Communications in Number Theory and Physics*, 1(2):347–452, 2007.

[25] B. Eynard and N. Orantin. Topological recursion in enumerative geometry and random matrices. *J. Phys. A: Math. Theor. 42* , 29(2):2009.

[26] B. Eynard, M. Mulase, and B. Safnuk. The Laplace transform of the cut-and-join equation and the Bouchard-Mariño conjecture on Hurwitz numbers. *Publications of the Research Institute for Mathematical Sciences*, 47 (2):629–670, 2011.

[27] C. Faber. A Conjectural Description of the Tautological Ring of the Moduli Space of Curves. pages 109–129. Vieweg+Teubner Verlag, 1999.

[28] A. Givental. Gromov-Witten invariants and quantization of quadratic hamiltonians. *Mosc. Math. J.*, 1(4): 1–17, 2001.

[29] S. Gukov and P. Sułkowski. A-polynomial, B-model, and quantization. *Journal of High Energy Physics*, 2012 (2), 2012.

[30] J. Harnad and A. Orlov. Hypergeometric τ-functions, Hurwitz numbers and enumeration of paths. *Communications in Mathematical Physics*, 338:267–284, 2015.

[31] E. Ionel. Topological recursive relations in $H^{2g}(\mathcal{M}_{g,n})$. *Inventiones mathematicae*, 148(3):627–658, 2002.

[32] F. Janda, R. Pandharipande, A. Pixton, and D. Zvonkine. Double ramification cycles on the moduli spaces of curves. arXiv:1602.04705, 2016.

[33] T Jarvis. Geometry of the moduli of higher spin curves. Arxiv:9809138, 1998.

[34] P. Johnson. Double Hurwitz numbers via the infinite wedge. *Transactions of the American Mathematical Society*, 367(9):6415–6440, 2015.

[35] P. Johnson, R. Pandharipande, and H. Tseng. Abelian Hurwitz-Hodge integrals. *The Michigan Mathematical Journal*, 60(1):171–198, 2011.

[36] A. Jucys. Symmetric polynomials and the center of the symmetric group ring. *Reports on Mathematical Physics*, 5(1):107–112, 1974.

[37] M. Karev. Personal Communications. 2016.

[38] M. Kontsevich and Y. Manin. Gromov-Witten classes, quantum cohomology, and enumerative geometry. *Communications in Mathematical Physics*, 164(3):525–562, 1994.

[39] R. Kramer, D. Lewanski, and S. Shadrin. Quasi-polynomiality of monotone orbifold Hurwitz numbers and Grothendieck's dessins d'enfants. arXiv:1610.08376, 2016.

[40] D. Lewanski, A. Popolitov, S. Shadrin, and D. Zvonkine. Chiodo formulas for the r-th roots and topological recursion. *Letters in Mathematical Physics*, pages 1–19, 2016.

[41] C.-C. Liu. Lectures on the ELSV formula. arXiv:1004.0853, 2010.

[42] T. Miwa, M. Jimbo, and E. Date. *Solitons : differential equations, symmetries and infinite dimensional algebras*. Cambridge University Press, 2011.

[43] M. Mulase, S. Shadrin, and L. Spitz. The spectral curve and the Schrödinger equation of double Hurwitz numbers and higher spin structures. *Communications in Number Theory and Physics*, 7(1):125–143, 2013.

[44] M. Mulase and N. Zhang. Polynomial recursion formula for linear hodge integrals. *Communications in Number Theory and Physics*, 4(2):267–294, 2010.

[45] D. Mumford. Towards an enumerative geometry of the moduli space of curves. *Arithmetic and geometry, Vol. II*, 36:271–328, 1983.

[46] A. Okounkov. Toda Equations for Hurwitz Numbers. *Mathematical Research Letters*, 453(7):447–453, 2000.

[47] G. Olshanski and S. Kerov. Polynomial functions on the set of Young diagrams. *C. R. Acad. Sci. Paris S er. I Math. 319*, pages 121–126, 1994.

[48] R. Pandharipande, A. Pixton, and D. Zvonkine. Tautological relations via r-spin structures. arXiv:1607.00978, 2016.

[49] R. Pandharipande, A. Pixton, and D. Zvonkine. Relations on $\overline{\mathcal{M}}_{g,n}$ via 3-spin structures. *Amer. Math. Soc.*, (28):279–309, 2015.

[50] M. Sato and Y. Sato. Soliton Equations as Dynamical Systems on Infinite Dimensional Grassmann Manifold. *North-Holland Mathematics Studies*, 81(C):259–271, 1983.

[51] S. Shadrin, L. Spitz, and D. Zvonkine. Equivalence of ELSV and Bouchard-Mariño conjectures for r-spin Hurwitz numbers. *Mathematische Annalen*, 361(3-4):611–645, 2015.

[52] C. Teleman. The structure of 2D semi-simple field theories. *Inventiones mathematicae*, 188(3):525–588, jun 2012.

[53] D. Zvonkine. A preliminary text on the r-ELSV formula. 2006.

[54] D. Zvonkine. An introduction to moduli spaces of curves and their intersection theory. 2011.

2

The tautological ring of $\mathcal{M}_{g,n}$ via Pandharipande-Pixton-Zvonkine r-spin relations

In this chapter we use relations in the tautological ring of the moduli spaces $\overline{\mathcal{M}}_{g,n}$ derived by Pandharipande, Pixton, and Zvonkine from the Givental formula for the r-spin Witten class in order to obtain some restrictions on the dimensions of the tautological rings of the open moduli spaces $\mathcal{M}_{g,n}$. In particular, we give a new proof for the result of Looijenga (for $n = 1$) and Buryak et al. (for $n \geq 2$) that $\dim R^{g-1}(\mathcal{M}_{g,n}) \leq n$. We also give a new proof of the result of Looijenga (for $n = 1$) and Ionel (for arbitrary $n \geq 1$) that $R^i(\mathcal{M}_{g,n}) = 0$ for $i \geq g$ and give some estimates for the dimension of $R^i(\mathcal{M}_{g,n})$ for $i \leq g - 2$.

2.1 Introduction

The study of the tautological ring R^* of the moduli spaces of curves goes back to the classical papers of Mumford and Faber [20, 8], see also [29, 21, 31, 27]. The tautological ring of the moduli space of curves $\overline{\mathcal{M}}_{g,n}$ is additively generated by the so-called dual graphs decorated by ψ- and κ-classes. A dual graph determines a natural stratum in $\overline{\mathcal{M}}_{g,n}$, whose vertices correspond to irreducible components of a generic point in the stratum, the leaves correspond to the marked points, and the edges correspond to the nodes. We decorate each vertex with a non-negative integer equal to the geometric genus of the corresponding irreducible component. Each vertex is also equipped with a multi-index κ-class, and each half-edge, including the leaves, is equipped with a power of the ψ-class of the cotangent line bundle at the corresponding marked point or the corresponding branch of the node. There are many linear relations between these generators called tautological relations.

We can restrict all these classes to the open moduli space $\mathcal{M}_{g,n}$. Then only the graphs with no edges can contribute non-trivially. These graphs just correspond to the classes $\prod_{i=1}^{n} \psi_i^{d_i} \kappa_{e_1,\dots,e_k}$, $d_i \geq 0$, $e_i \geq 1$. There are still many relations among these classes that can be proved, in particular, that $R^i(\mathcal{M}_{g,n}) = 0$ for $i \geq g$, see [19, 15] and also a recent new proof in [3]. In the case $i = g-1$ one can prove that $\dim R^{g-1}(\mathcal{M}_{g,n}) \leq n$, see [19, 2] for the cases $n = 1$ and $n \geq 2$ respectively. In this chapter we give new proofs of all these results as well as some restrictions on the dimensions of the tautological rings for $i \leq g - 2$. Note that, by the non-degeneracy of some matrix of intersection numbers, one can in fact show that $\dim R^{g-1}(\mathcal{M}_{g,n}) = n$, we refer to [2] for that.

We use the tautological relations of Pandharipande-Pixton-Zvonkine [24]. Givental-Teleman theory [13, 28] provides a formula for a homogeneous semi-simple cohomological field theory as a sum over decorated dual graphs as above, see [12, 4, 5, 23]. These formulae can be explained as a result of a certain group action on non-homogeneous cohomological field theories applied to the rescaled Gromov-Witten theory of a finite number of points (also known as topological field theory or degree 0 cohomological field theory), see [9, 28, 26, 23].

In some cases we can obtain this way a graphical formula for a cohomological field theory whose properties we know independently. In particular, the graphical formula might contain classes (linear combinations of decorated dual graphs) that are of dimension higher than the homogeneity property allows for a cohomological field theory. Then theses classes must be equal to zero and give us tautological relations. Alternatively, we might consider the graphical formula as a function of some parameter ϕ parametrizing a path on the underlying Frobenius manifold with $\phi = 0$ lying on the discriminant. If we know independently that the cohomological field theory is defined for any value of ϕ, including $\phi = 0$, then all negative terms of the Laurent series expansion in ϕ near $\phi = 0$ also give tautological relations. See [17, 22] for some expositions. Once we have a relation for the decorated dual graphs in $\overline{\mathcal{M}}_{g,n+m}$, $m \geq 0$, we can multiply it by an arbitrary tautological class, push it forward to $\overline{\mathcal{M}}_{g,n}$, and then restrict it to $\mathcal{M}_{g,n}$. This gives a relation among the classes $\prod_{i=1}^{n} \psi_i^{d_i} \kappa_{e_1,\dots,e_k}$, $d_i \geq 0$, $e_i \geq 1$, in $R^*(\mathcal{M}_{g,n})$.

In the case of the Witten r-spin class [30, 25] the graphical formula and its ingredients are discussed in detail in [11, 9, 6, 24].

Both approaches mentioned above produce the same systems of tautological relations on $\overline{\mathcal{M}}_{g,n}$. Two particular paths on the underlying Frobenius manifold are worked out in detail in [24], and we are using one of them in this chapter. Note that the results of Janda [17, 16, 18] guarantee that these relations work in the Chow ring, see a discussion in [24].

2.1.1 Organization of the chapter

In section 2.2 we recall the relations of Pandharipande-Pixton-Zvonkine. In section 2.3, we use them to give a new proof of the dimension of $R^{g-1}(\mathcal{M}_{g,n})$, up to one lemma whose proof takes up section 2.4. In section 2.5, we extend this proof scheme to show the vanishing of the tautological ring in all higher degrees. Finally, in section 2.6, we give some bounds for the dimensions of the tautological rings in lower degrees.

2.2 Pandharipande-Pixton-Zvonkine relations

In this section we recall the relations in the tautological ring of $\overline{\mathcal{M}}_{g,n}$ from [24] and put them in a convenient form for our further analysis.

2.2.1 Definition

Fix $r \geq 3$. Fix n primary fields $0 \leq a_1, \dots, a_n \leq r-2$. All constructions below depend on an auxiliary variable ϕ and we fix its exponent $d < 0$. A tautological relation $T(g, n, r, a_1, \dots, a_n, d) = 0$ depends on these choices, and it is obtained as $T = r^{g-1} \sum_{k=0}^{\infty} \pi_*^{(k)} T_k / k!$, where T_k is the coefficient of ϕ^d in the expression in the decorated dual graphs of $\overline{\mathcal{M}}_{g,n+k}$ described below, and $\pi^{(k)} \colon \overline{\mathcal{M}}_{g,n+k} \to \overline{\mathcal{M}}_{g,n}$ is the natural projection.

Consider the vector space of primary fields with basis $\{e_0, \dots, e_{r-2}\}$. In the basis $\tilde{e}_i := \phi^{-i/(r-1)} e_i$ we define the scalar product $\eta_{ij} = \langle \tilde{e}_i, \tilde{e}_j \rangle := \phi^{-(r-2)/(r-1)} \delta_{i+j,r-2}$. Equip each vertex of genus h of valency v in a decorated dual graph with a tensor

$$\tilde{e}_{a_1} \otimes \cdots \otimes \tilde{e}_{a_v} \mapsto \phi^{(h-1)(r-2)/(r-1)}(r-1)^h \delta_{(r-1)|h-1-\sum_{i=1}^{v} a_i}.$$

Define matrices $(R_m^{-1})_a^b$, $m \geq 0$, $a, b = 0, \ldots, r-2$, in the basis $\tilde{e}_0, \ldots, \tilde{e}_{r-2}$. We set $(R_m^{-1})_a^b = 0$ if $b \not\equiv a - m \mod r-1$. If $b \equiv a - m \mod r-1$, then $(R_m^{-1})_a^b = (r(r-1)\phi^{r/(r-1)})^{-m} P_m(r, a)$, where $P_m(r, a)$, $m \geq 0$, are the polynomials of degree $2m$ in r, a uniquely determined by the following conditions:

$$\begin{aligned} P_0(r, a) &= 1; \\ P_m(r, a) - P_m(r, a-1) &= ((m - \tfrac{1}{2})r - a)P_{m-1}(r, a-1); \\ P_m(r, 0) &= P_m(r, r-1). \end{aligned}$$

Equip the first n leaves with $\sum_{m=0}^{\infty} (R_m^{-1})_{a_i}^b \psi^m e_b$, $i = 1, \ldots, n$. Equip the k extra leaves (the dilaton leaves) with $-\sum_{m=1}^{\infty} (R_m^{-1})_0^b \psi^{m+1} e_b$, $i = 1, \ldots, n$. Equip each edge, where we denote by ψ' and ψ'' the ψ-classes on the two branches of the corresponding node, with

$$\frac{\eta^{i'i''} - \sum_{m',m''=0}^{\infty} (R_{m'}^{-1})_{j'}^{i'} \eta^{j'j''} (R_{m''}^{-1})_{j''}^{i''} (\psi')^{m'} (\psi'')^{m''}}{\psi' + \psi''} \tilde{e}_{i'} \otimes \tilde{e}_{i''}$$

Then T_k is defined as the sum over all decorated dual graphs obtained by the contraction of all tensors assigned to their vertices, leaves, and edges, further divided by the order of the automorphism group of the graph.

2.2.2 Analysis of relations

There are several observations about the formula introduced in the previous subsection.

1. We obtain a decorated dual graph in $R^D(\overline{\mathcal{M}}_{g,n})$ if and only if the sum of the indices of the matrices R_m^{-1} used in its construction is equal to D.

2. According to [24, Theorem 7], $T(g, n, r, a_1, \ldots, a_n, d)$ is a sum of decorated dual graphs whose coefficients are polynomials in r.

3. Let $A = \sum_{i=1}^{n} a_i$. Then $A \equiv g-1+D \mod r-1$. We can assume that $A = g-1+D+x(r-1)$, $x \geq 0$, since D is bounded by $\dim \overline{\mathcal{M}}_{g,n} = 3g-3+n$, whereas the relations hold for r arbitrarily big. Collecting the powers of ϕ from the contributions above, we obtain $d(r-1) = A + (g-1)(r-2) - rD$. Substituting the expression for A, we have that $d < 0$ if and only if $D \geq g + x$. The relevant cases in this chapter are the cases $x = 0$ and $x = 1$.

These relations, valid for particular $r \geq 3$ and $0 \leq a_1, \ldots, a_n \leq r-2$ are difficult to apply since we have almost no control on the κ-classes coming from the dilaton leaves. We solve this problem in the following way.

Let $x = 0$, consider the degree $D = g$. We have relations with polynomial coefficients for all r much greater than g and $A = 2g-1$. More precisely, for all integers $0 \leq a_1, \ldots, a_n \leq 2g-1$, $\sum_{i=1}^{n} a_i = 2g-1$, we have a relation whose coefficients are polynomials of degree $2g$ in r. In other words, we have a polynomial in r whose coefficients are linear combinations of decorated dual graphs in degree g, and we can substitute any r sufficiently large. Possible integer values of

r determine this polynomial completely, so its evaluation at any other complex value of r is again a relation.

Let $x = 1$, consider the degree $D = g+1$. We have relations with polynomial coefficients for all r much bigger than g and $A = 2g-1+r$. More precisely, for all integers $0 \leq a_1, \dots, a_n \leq r-2$, $\sum_{i=1}^n a_i = 2g-1+r$, we have a relation whose coefficients are polynomials of degree $2g+2$ in r.

Note that in both cases we do not, in general, have polynomiality in $a_1, \dots, a_n$, but we have it for some special decorated dual graphs, under some extra conditions.

We argue below that a good choice of r in both cases is $r = \frac{1}{2}$ (note that we still have to explain what we mean in the case $x = 1$, since the sum A depends on r). In particular, this choice kills all dilaton leaves, and the only non-trivial term that contributes to the sum over k in the definition of $T(g, n, r, a_1, \dots, a_n, d)$ in these cases is T_0.

2.2.3 P-polynomials at $r = \frac{1}{2}$

Recall the $P_m(r, a)$-polynomials of [24] introduced above, and define

$$Q_m(a) \coloneqq \frac{(-1)^m}{2^m m!} \prod_{k=1}^{2m} \Big(a + 1 - \frac{k}{2}\Big)$$

Lemma 2.2.1. *We have $P_m\big(\frac{1}{2}, a\big) = Q_m(a)$.*

Proof. We will use [24, lemma 4.3]. It is clear that $Q_0(a) = 1$ and are $Q_m(0) = Q_m\big(-\frac{1}{2}\big) = \delta_{m,0}$. Furthermore

$$\begin{aligned} Q_m(a) - Q_m(a-1) &= \frac{(-1)^m}{2^m m!}\bigg(\prod_{k=1}^{2m}\Big(a+1-\frac{k}{2}\Big) - \prod_{k=1}^{2m}\Big(a - \frac{k}{2}\Big)\bigg) \\ &= \frac{(-1)^m}{2^m m!}\Big(\big(a + \frac{1}{2}\big)a - \big(a - m + \frac{1}{2}\big)(a-m)\Big)\prod_{k=1}^{2m-2}\Big(a - \frac{k}{2}\Big) \\ &= \frac{1}{2m}\big(-2am + m^2 - \frac{1}{2}m\big)Q_{m-1}(a-1) \\ &= \frac{1}{2}\big(m - \frac{1}{2} - 2a\big)Q_{m-1}(a-1), \end{aligned}$$

so the equations in the lemma are satisfied.

This does not allow us to conclude yet that our $Q_m(a)$ are equal to the $P_m(\frac{1}{2}, a)$, as the lemma only states uniqueness for the $P_m(r, a)$ as polynomials in a and r. However, we can prove equality by induction on m. The cases for $m = 0, 1$ are given explicitly in [24], and can be checked easily.

Now assume $m > 1$ and $P_{m-1}\big(\frac{1}{2}, a\big) = Q_{m-1}(a)$. Then

$$Q_m(a) - Q_m(a-1) = \frac{1}{2}\big(m - \frac{1}{2} - 2a\big)Q_{m-1}(a-1),$$

with the same relation for $P_m\big(\frac{1}{2}, a\big)$. Hence, $P_m\big(\frac{1}{2}, a\big) = Q_m(a) + c$. Using the same relation for $m+1$, we get that

$$\Delta_{m+1}(a) \coloneqq P_{m+1}\big(\frac{1}{2}, a\big) - Q_{m+1}(a) = -\frac{c}{2}a^2 + \frac{2m-3}{4}ac + d$$

We then have that

$$0 = \Delta_{m+1}\big(-\frac{1}{2}\big) - \Delta_{m+1}(0) = -\frac{c}{8} - \frac{2m-3}{8}c = \frac{1-m}{4}c$$

Because $m > 1$ by assumption, this proves $c = 0$, so $P_m\big(\frac{1}{2}, a\big) = Q_m(a)$. □

2.2.4 Simplified relations I

In this subsection we discuss the relations that we can obtain from the substitution $r = \frac{1}{2}$ for the case of $x = 0$ in subsection 2.2.2.

The polynomials $Q_m(a)$, $m = 0, 1, 2, \dots$, discussed in the previous subsection, have degree $2m$ and roots $-\frac{1}{2}, 0, \frac{1}{2}, 1, \dots, m - \frac{3}{2}, m - 1$. Note that on the dilaton leaves in the relation of [24] we always have a coefficient $(R_m^{-1})_0^i$ for some $m \geq 1$. Since for $r = \frac{1}{2}$ we have $(R_m^{-1})_0^i = (-\frac{1}{4}\phi^{-1})^{-m} Q_m(0) = 0$, $m \geq 1$, the graphs with dilaton leaves do not contribute to the tautological relations.

In order to obtain a relation on $\mathcal{M}_{g,n}$ we first consider a relation in $\overline{\mathcal{M}}_{g,n+m}$ that we push forward to $\overline{\mathcal{M}}_{g,n}$ and then restrict to the open moduli space $\mathcal{M}_{g,n}$. Note that only graphs that correspond to a partial compactification of $\mathcal{M}_{g,n+m}$ can contribute non-trivially. Namely, it is a special case of the rational tails partial compactification, where we require in addition that at most one among the first n marked points can lie on each rational tail. We denote this compactification by $\mathcal{M}_{g,n+m}^{\mathrm{rt}[n]}$.

For instance, the dual graphs that can contribute non-trivially to a relation on $\mathcal{M}_{g,n+1}^{\mathrm{rt}[n]}$ are either the graph with one vertex and no edges or the graphs with two vertices of genus g and 0 and one edge connecting them, with leaves labeled by i and $n+1$ attached to the genus 0 vertex and all other leaves attached to the genus g vertex, $i = 1, \dots, n$. These graphs correspond to the divisors in $\mathcal{M}_{g,n+1}^{\mathrm{rt}[n]}$ that we denote by $D_{i,n+1}$.

More generally, we denote by D_I, $I \subset \{1, \dots, n+m\}$, the divisor in $\overline{\mathcal{M}}_{g,n+m}$ whose generic point is represented by a two-component curve, with components of genus g and 0 connected through a node, such that all the points with labels in I lie on the component of genus 0, and all other points lie on the component of genus g. Then the divisors that belong to $\mathcal{M}_{g,n+m}^{\mathrm{rt}[n]}$ are those in which I contains at most one point with a label $1 \leq l \leq n$, and all dual graphs that we have to consider are the dual graphs of the generic points of the strata obtained by the intersection of these divisors.

We denote the relations on $\overline{\mathcal{M}}_{g,n}$ corresponding to the choice of the primary fields $a_1, \dots, a_n$, by $\Omega_{g,n}^D(a_1, \dots, a_n) = 0$, where D is the degree of the class. In this definition we adjust the coefficient, namely, from now on we ignore the pre-factor r^{g-1} in the definition of the relations, as well as the factor $(-\frac{1}{4}\phi^{-1})^{-D}$ coming from the formula for the R-matrices in terms of the polynomials Q. Hence, $\Omega_{g,n}^D(\vec{a})$ is proportional to $T(g, n, \frac{1}{2}, \vec{a}, d(D))$. We will also often write Ω for its restriction to various open parts of the moduli space, such as $\mathcal{M}_{g,n+m}^{\mathrm{rt}[n]}$.

Note that, as we discussed above, there is a condition on the possible degree of the class and the possible choices of the primary fields implied by the requirement that the degree of the auxiliary parameter ϕ must be negative.

We use the following relations in the rest of the chapter: $\Omega_{g,n+m}^D(a_1, \dots, a_{n+m})$, where $D \geq g$, $m \geq 0$, and $\sum_{i=1}^{n+m} a_i = g - 1 + D$ and all primary fields must be non-negative integers.

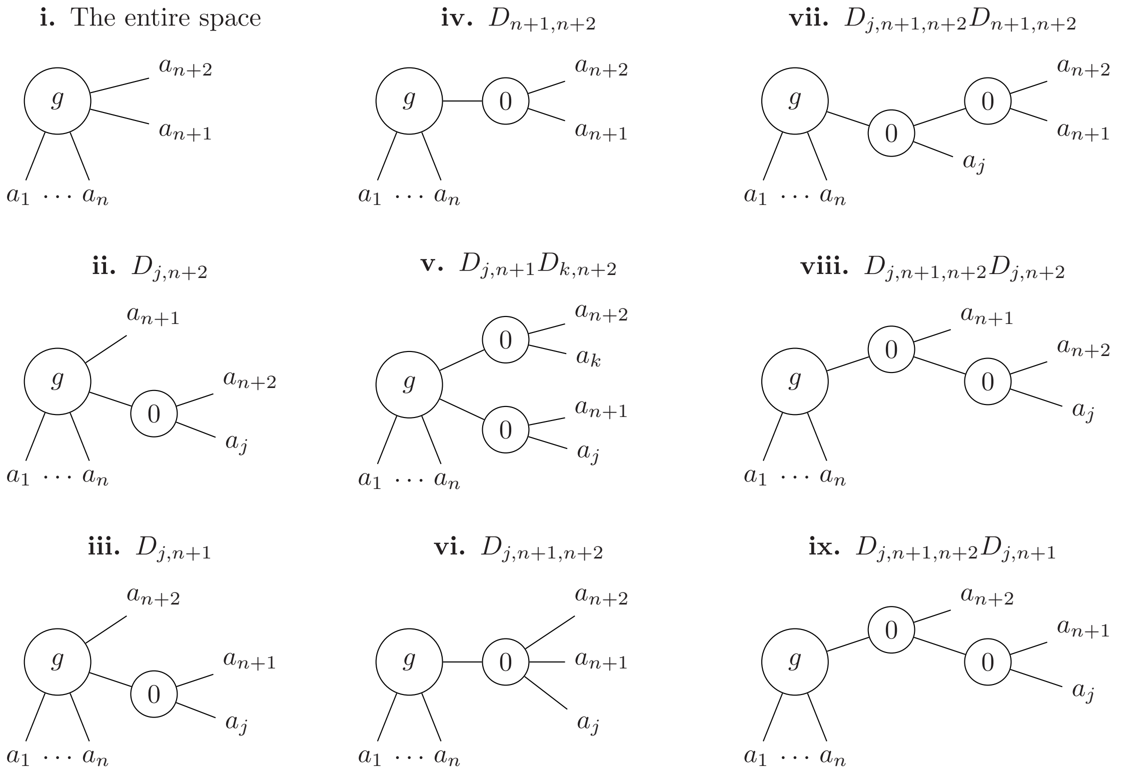


Figure 2.1. Strata in $\mathcal{M}^{\mathrm{rt}[2]}_{g,n+2}$

We sometimes first multiply these relations by extra monomials of ψ-classes before we apply the pushforward to $\overline{\mathcal{M}}_{g,n}$ and/or restriction to $\mathcal{M}_{g,n}$.

2.2.5 Simplified relations II

In this subsection we discuss the relations that we can obtain from the substitution $r = \frac{1}{2}$ for the case of $x = 1$ in subsection 2.2.2.

Let us first list all the dual graphs representing the strata in $\mathcal{M}^{\mathrm{rt}[n]}_{g,n+2}$, see figure 2.1. Note that under an extra condition on the primary fields $a_1, \ldots, a_{n+2}$, namely, that $1 \le a_i \le r - 3 - a_{n+1} - a_{n+2}$ for any $1 \le i \le n$, the coefficients of all these graphs in $T(g, n+2, r, \vec{a}, -1)$, equipped in an arbitrary way with ψ- and κ-classes, are manifestly polynomial in $a_1, \ldots, a_{n+2}, r$. Indeed, this extra inequality guarantees that we can uniquely determine the primary fields on the edges in the Givental formula for all these nine graphs.

Thus, we have a sequence of tautological relations $T(g, n+2, r, \vec{a}, -1)$ in dimension $g+1$ defined for a big enough r, and arbitrary non-negative integers $a_1, \ldots, a_{n+2}$ satisfying $a_1 + \cdots + a_{n+2} = 2g + r - 1$ and $1 \le a_i \le r - 3 - a_{n+1} - a_{n+2}$ for any $1 \le i \le n$. This gives us enough evaluations of the polynomial coefficients of the decorated dual graphs in $\mathcal{M}^{\mathrm{rt}[n]}_{g,n+2}$ to determine these polynomials completely. Thus, we can represent the values of these polynomial coefficients at an arbitrary point $(\tilde{a}_1, \ldots, \tilde{a}_{n+2}, \tilde{r}) \in \mathbb{C}^{n+3}$ as a linear combination of the Pandharipande-Pixton-Zvonkine relations. This representation is non-unique, since we have too many admissible points $(a_1, \ldots, a_{n+2}, r) \in \mathbb{Z}^{n+3}$ satisfying the conditions above. This non-uniqueness is not important for the coefficients of the decorated dual graphs in $\mathcal{M}^{\mathrm{rt}[n]}_{g,n+2}$, since we always get the values of their polynomial coefficients at the prescribed points, but the extension of different linear combinations of the relations to the full compactification $\overline{\mathcal{M}}_{g,n+2}$ can be different. Indeed, the coefficients of the graphs not listed in figure 2.1 can be non-polynomial in $a_1, \ldots, a_{n+2}$ (but they are still polynomial in r).

We can choose one possible extension to the full compactification $\overline{\mathcal{M}}_{g,n+2}$ for each point $(\tilde{a}_1, \ldots, \tilde{a}_{n+2}, \tilde{r}) \in \mathbb{C}^{n+3}$. In particular, we always specialize $r = \frac{1}{2}$, $a_{n+1} = \frac{3}{2}$, $a_{n+2} = -\frac{1}{2}$. The choice $r = \frac{1}{2}$ guarantees that we have no non-trivial dilaton leaves, that is, we have no κ-classes in the decorations of our graphs. We also divide the whole relation by the factor $(\frac{1}{2})^{g-1}(-\frac{1}{4}\phi^{-1})^{1-g}$, as in the previous subsection.

Abusing the notation, we denote these relations by $\Omega^{g+1}_{g,n+2}(a_1, \ldots, a_n, \frac{3}{2}, -\frac{1}{2})$. These relations are defined for arbitrary complex numbers $a_1, \ldots, a_n$ satisfying $\sum_{i=1}^n a_i = 2g - \frac{3}{2}$. Of course, it is reasonable to use half-integer or integer primary fields $a_1, \ldots, a_n$ that would be the roots of the polynomials Q, since this gives us a very good control on the possible degrees of the ψ-classes on the leaves and the edges of the dual graphs.

Let us stress once again that restriction of $\Omega^{g+1}_{g,n+2}(a_1, \ldots, a_n, \frac{3}{2}, -\frac{1}{2})$ to $\mathcal{M}^{\mathrm{rt}[n]}_{g,n+2}$ is well-defined and can be obtained by the specialization of the polynomial coefficients of the dual graphs in figure 2.1 to the point $(a_1, \ldots, a_n, a_{n+1} = \frac{3}{2}, a_{n+2} = -\frac{1}{2}, r = \frac{1}{2})$. We analyze this polynomial coefficients in the next two sections. In the meanwhile, the extension of $\Omega^{g+1}_{g,n+2}(a_1, \ldots, a_n, \frac{3}{2}, -\frac{1}{2})$ from $\mathcal{M}^{\mathrm{rt}[n]}_{g,n+2}$ to $\overline{\mathcal{M}}_{g,n+2}$ is, in principle, not unique, and we only use that it exists.

2.3 The dimension of $R^{g-1}(\mathcal{M}_{g,n})$

In this section we give a new proof of a result in [2] that $\dim R^{g-1}(\mathcal{M}_{g,n}) \leq n$.

2.3.1 Reduction to monomials in ψ-classes

In this subsection we show that any monomial $\psi_1^{d_1} \cdots \psi_n^{d_n} \kappa_{e_1,\ldots,e_m}$ of degree $g-1$ can be expressed as a linear combination of monomials of degree $g-1$ which have only ψ-classes. We prove this fact by considering the relations $\Omega^{g-1+m}_{g,n+m}(a_1, \ldots, a_{n+m})$ for some appropriate choices of the primary fields.

Proposition 2.3.1. *Let $g \geq 2$ and $n \geq 1$. The ring $R^{g-1}(\mathcal{M}_{g,n})$ is spanned by the monomials $\psi_1^{d_1} \cdots \psi_n^{d_n}$ for $d_1, \ldots, d_n \geq 0$, $\sum_{i=1}^n d_i = g - 1$.*

Proof. The tautological ring of the open moduli space is generated by ψ- and κ-classes. Hence, a spanning set for the ring $R^{g-1}(\mathcal{M}_{g,n})$ is

$$\left\{\psi_1^{d_1} \cdots \psi_n^{d_n} \kappa_{e_1,\ldots,e_m} \mid m \geq 0, d_i \geq 0, e_j \geq 1, \sum_{i=1}^n d_i + \sum_{j=1}^m e_j = g - 1\right\}$$

Let $V \subset R^{g-1}(\mathcal{M}_{g,n})$ be the subspace spanned by the monomials

$$\Big\{\psi_1^{d_1}\cdots\psi_n^{d_n} \mid \sum_{i=1}^n d_i = g-1\Big\}.$$

We want to show that $R^{g-1}(\mathcal{M}_{g,n})/V = 0$. We do this by induction on the number m of indices of the κ-class.

Let us start with the case $m=1$. Consider a relation $\Omega^g_{g,n+1}(a_1,\dots,a_{n+1})$ for some admissible choice of the primary fields. In this case we have contributions by the open stratum of smooth curves and by the divisors $D_{n+1,\ell}$, $\ell = 1,\dots,n$. The open stratum gives us the following classes:

$$\sum_{\substack{d_1+\cdots+d_{n+1}=g\\ 0\le d_i\le a_i}} \prod_{i=1}^{n+1} Q_{d_i}(a_i)\prod_{i=1}^{n+1}\psi_i^{d_i}$$

The condition $d_i \le a_i$ follows from the fact that $Q_d(a) = 0$ for $d > a$. The contribution of $D_{n+1,\ell}$ is given by

$$\sum_{\substack{d_1+\cdots+d_n=g-1\\ 0\le d_i\le a_i+\delta_{i\ell}(a_{n+1}-1)}} \prod_{i=1}^{n} Q_{d_i+\delta_{i\ell}}(a_i+\delta_{i\ell}a_{n+1}) \prod_{i\ne \ell,n+1} \psi_i^{d_i} D_{i,\ell}\pi^*(\psi_\ell^{d_\ell})$$

Here $\pi\colon \mathcal{M}^{\mathrm{rt}[n]}_{g,n+1}\to\mathcal{M}_{g,n}$ is the natural projection. The sum of the pushforwards of these classes to $\mathcal{M}_{g,n}$ is equal to

$$0 = \sum_{\substack{d_1+\cdots+d_n+e=g-1\\ d_i\ge 0, e\ge 1}} \prod_{i=1}^n Q_{d_i}(a_i)Q_{e+1}(a_{n+1})\prod_{i=1}^n \psi_i^{d_i}\kappa_e \tag{2.1}$$

in $R^{g-1}(\mathcal{M}_{g,n})/V$. Thus we have equation (2.1) in $R^{g-1}(\mathcal{M}_{g,n})/V$ for each choice of $a_1,\dots,a_{n+1}$ such that $\sum_{i=1}^{n+1} a_i = 2g-1$.

If we choose the lexicographic order on the monomials $\psi_1^{d_1}\cdots\psi_n^{d_n}\kappa_e$, we can then choose the values of the a_i in such a way that the matrix of relations becomes lower triangular, in the following manner. For every monomial $\psi_1^{d_1}\cdots\psi_n^{d_n}\kappa_e$, we choose the relation with primary fields $a_i = d_i$ for $i = 2,\dots,n$, $a_{n+1} = e+1$, and $a_1 = d_1+g-1$. Equation (2.1) allows to express this monomial in terms of similar monomials with the strictly larger exponent of ψ_1, so this set of relations does indeed give a lower-triangular matrix. This matrix is invertible, hence all monomials of the form $\psi_1^{d_1}\cdots\psi_n^{d_n}\kappa_e$ are equal to 0 in $R^{g-1}(\mathcal{M}_{g,n})/V$.

Now assume that all the monomials which have a κ-class with $m-1$ indices or fewer are equal to 0 in $R^{g-1}(\mathcal{M}_{g,n})/V$. Consider a relation $\Omega^{g-1+m}_{g,n+1}(a_1,\dots,a_n,b_1,\dots,b_m)$. This relation, after the push-forward to $\mathcal{M}_{g,n}$, gives many terms with no κ-classes and also with κ-classes with $\le m-1$ indices, and also some terms with κ-classes with m indices. The latter terms are therefore equal to 0 in $R^{g-1}(\mathcal{M}_{g,n})/V$, namely, we have:

$$0 = \sum_{\substack{0\le d_i\le a_i\\ 1\le e_j\le b_j-1}} \Big(\prod_{i=1}^n Q_{d_i}(a_i)\psi_i^{d_i}\Big)\Big(\prod_{j=1}^m Q_{e_j+1}(b_j)\Big)\kappa_{e_1,\dots,e_m} \tag{2.2}$$

for $\sum_{i=1}^n d_i + \sum_{j=1}^m e_j = g-1$. Equation (2.2) is valid for each choice of the primary fields a_i, b_j such that $\sum_{i=1}^n a_i + \sum_{j=1}^m b_j = 2g-2+m$.

Choosing a monomial $\psi_1^{d_1} \cdots \psi_n^{d_n} \kappa_{e_1,\dots,e_m}$, we can choose the primary fields to be $a_i = d_i$ for $i = 2,\dots,n$, $b_j = e_j + 1$ for $j = 1,\dots,m$, and $a_1 = d_1 + g - 1$. Again, this relation expresses our monomial as a linear combination of similar monomials with strictly higher exponent of ψ_1. By downward induction on this exponent, all monomials with m κ-indices vanish in $R^{g-1}(\mathcal{M}_{g,n})/V$ as well.

Thus $R^{g-1}(\mathcal{M}_{g,n})/V = 0$. In other words, any monomial which has a κ-class as a factor can be expressed as a linear combination of monomials in ψ-classes. □

An immediate consequence of this proposition for $n = 1$ is the result of Looijenga.

Corollary 2.3.2 ([19]). *For all $g \geq 2$, $R^{g-1}(\mathcal{M}_{g,1}) = \mathbb{Q}\psi_1^{g-1}$.*

2.3.2 Reduction to n generators

In this subsection we prove the following proposition.

Proposition 2.3.3. *For $n \geq 2$ and $g \geq 2$, every monomial of degree $g-1$ in ψ classes and at most one κ_1-class can be expressed as linear combinations of the following n classes*

$$\psi_1^{g-1},\ \psi_1^{g-2}\psi_2,\ \dots,\ \psi_1^{g-2}\psi_n,$$

with rational coefficients.

Together with the previous subsection this gives a new proof of

Theorem 2.3.4 ([2]). *For $n \geq 2$ and $g \geq 2$*

$$\dim_{\mathbb{Q}} R^{g-1}(\mathcal{M}_{g,n}) \leq n.$$

Remark 2.3.5. Note that the possible κ_1-class is added in proposition 2.3.3 for a technical reason; it seems to be completely unnecessary in the light of proposition 2.3.1. In fact, when we include κ_1, we consider systems of generators approximately twice as large, but this allows us to obtain a much larger system of tautological relations. We do not know of any argument that would allow us to obtain the sufficient number of relations if we consider only monomials of ψ-classes as generators.

We reduce the number of generators by pushing forward enough relations via the map

$$\pi_*^{(2)} \colon R^{g+1}(\overline{\mathcal{M}}_{g,n+2}) \to R^{g-1}(\overline{\mathcal{M}}_{g,n}),$$

where $\pi^{(2)}$ is the forgetful morphism for the last two marked points (we abuse notation a little bit here, restricting the map $\pi^{(2)}$ to $\mathcal{M}_{g,n+2}^{\mathrm{rt}[n]} \to \mathcal{M}_{g,n}$). For $n \geq 2$, let us consider the following vector of primary fields:

$$\vec{a} := \left(a_1 = 2g - \tfrac{3}{2} - A, a_2, \dots, a_n, a_{n+1} = \tfrac{3}{2}, a_{n+2} = -\tfrac{1}{2}\right), \tag{2.3}$$

where $a_i \in \mathbb{Z}_{\geq 0}$, $i = 2,\dots,n$, $A = \sum_{i=2}^n a_i \leq g-2$. We consider the following monomials in $R^{g-1}(\mathcal{M}_{g,n})$:

$$y := \psi_1^{g-2-A} \prod_{i=2}^n \psi_i^{a_i} \kappa_1,$$

$$x_\ell := \psi_1^{g-2-A} \prod_{i=2}^n \psi_i^{a_i+\delta_{i\ell}}, \quad \ell = 2,\dots,n.$$

Lemma 2.3.6. *The tautological relation $\pi_*^{(2)}\Omega_{g,n+2}^{g+1}(\vec{a})$, where $\vec{a}$ is defined in equation (2.3), has the following form:*

$$\begin{aligned} & y \cdot \prod_{i=2}^n Q_{a_i}(a_i) Q_2(\tfrac{3}{2}) \left(Q_{g-1-A}(2g-\tfrac{3}{2}-A) - Q_{g-1-A}(2g-2-A)\right) \qquad (2.4)\\ & - \sum_{\ell=2}^n x_\ell \cdot \prod_{i=2}^n Q_{a_i+2\delta_{i\ell}}(a_i + \tfrac{3}{2}\delta_{i\ell}) \left(Q_{g-1-A}(2g-\tfrac{3}{2}-A) - Q_{g-1-A}(2g-2-A)\right) \\ & = \textit{ terms divisible by } \psi_1^{g-1-A}. \end{aligned}$$

Proof. In order to prove this lemma we have to analyze all strata in $\mathcal{M}_{g,n+2}^{\mathrm{rt}[n]}$. The list of strata is given in figure 2.1. Each stratum should be decorated in all possible ways by the R-matrices with ψ-classes as described in section 2.2.

There are several useful observations that simplify the computation. The leaf labeled by a_i, $i = 2,\dots,n$, is equipped by $\psi_i^{d_i} Q_{d_i}(a_i)$. This implies that $d_i \leq a_i$. Since $Q_{>2}(\frac{3}{2}) = 0$ (respectively, $Q_{>0}(-\frac{1}{2}) = 0$), we conclude that the exponent of ψ_{n+1} is ≤ 2 (respectively, the exponent of ψ_{n+2} is equal to 0). Note that we can obtain a monomial with κ_1-class in the push-forward only if we have ψ_{n+1}^2 in the original decorated graph.

Similar observations are also valid for the exponents of the ψ-classes at the nodes. Note that there are no ψ-classes on the genus 0 components in any strata except for the case of the dual graph **vi**, where we must have a ψ-class at one of the four points (three marked points and the node) of the genus 0 component, otherwise the pushforward is equal to 0. So, for instance, we have ψ^d at the genus g branch of the node on the dual graph **ii** with coefficient $-Q_{d+1}(a_j - \frac{1}{2})$, so in this case $d \leq a_j - 1$. If we have ψ^d at the genus g branch of the node on the dual graph **viii**, then the product of the coefficients that we have on the edges of this graph is equal to $Q_1(a_j - \frac{1}{2}) Q_{d+1}(a_j - \frac{1}{2} + \frac{3}{2} - 1)$, so in this case $d \leq a_j - 1$. And so on; one more example of a detailed analysis of the graphs **vi-ix** is given in lemma 2.4.3 in the next section.

We see that we have severe restrictions on the possible powers of ψ-classes at all points but the one labeled by 1, where the exponent is bounded from below, also after the pushforward. Then it is easy to see by the analysis of the graph contributions as above that the exponent of ψ_1 is $\geq g-2-A$. Let us list all the terms whose pushforwards to $\mathcal{M}_{g,n}$ contain the terms with ψ_1^{g-2-A}.

- One of the classes in $\mathcal{M}_{g,n+2}^{\mathrm{rt}[n]}$ corresponding to graph **i** is $\psi_1^{g-1-A} \prod_{i=2}^n \psi_i^{a_i} \psi_{n+1}^2$ with coefficient $\prod_{i=2}^n Q_{a_i}(a_i) Q_2(\frac{3}{2}) Q_{g-1-A}(2g - \frac{3}{2} - A)$. Its pushforward contains the monomial y and the terms divisible by ψ_1^{g-1-A}.

- Consider graph **ii** for $j=1$. Let $\pi_{n+2}\colon \overline{\mathcal{M}}_{g,n+2} \to \overline{\mathcal{M}}_{g,n+1}$ be the forgetful morphism for the $(n+2)$-nd point. One of the classes corresponding to this graph is
$$\prod_{i=2}^{n} \psi_i^{a_i} \psi_{n+1}^2 D_{1,n+2} (\pi_{n+2})^* (\psi_1^{g-2-A})$$
with coefficient $(-1)\prod_{i=2}^n Q_{a_i}(a_i) Q_2(\frac{3}{2}) \cdot Q_{g-1-A}(2g-2-A)$. Its pushforward is equal to the monomial y.

- Let $\pi_{n+1}\colon \overline{\mathcal{M}}_{g,n+2} \to \overline{\mathcal{M}}_{g,n+1}$ be the forgetful morphism for the $(n+1)$-st point. One of the classes corresponding to graph **iii** for $j=\ell$ is $\prod_{i\neq 1,\ell} \psi_i^{a_i} \psi_1^{g-1-A} D_{\ell,n+1} \cdot (\pi_{n+1})^*(\psi_\ell^{a_\ell+1})$ with coefficient $(-1)\prod_{i\neq 1,\ell} Q_{a_i}(a_i) Q_{a_\ell+2}(a_\ell+\frac{3}{2}) Q_{g-1-A}(2g-\frac{3}{2}-A)$. The pushforward of this class contains the monomial x_ℓ and the terms divisible by ψ_1^{g-1-A}.

- Consider graph **v** for $j=\ell$ and $k=1$. One of the classes corresponding to this graph is $\prod_{i\neq 1,\ell} \psi_i^{a_i} D_{\ell,n+1} (\pi_{n+1})^*(\psi_\ell^{a_\ell+1}) D_{1,n+2} (\pi_{n+2})^*(\psi_1^{g-2-A})$ with coefficient
$$\prod_{i\neq 1,\ell} Q_{a_i}(a_i) Q_{a_\ell+2}(a_\ell+\frac{3}{2}) Q_{g-1-A}(2g-2-A).$$
Its pushforward is equal to the monomial x_ℓ.

Collecting all these terms together, we obtain the left hand side of equation (2.4). Then it is easy to verify case by case that all other graphs and all other possible decorations on these four graphs produce under the push-forward only monomials divisible by ψ_1^{g-1-A}. ☐

Let $a_j > 0$ for $j=2,\dots,n$. Consider a vector of primary fields $\vec{a}^{(j)}$ obtained from $\vec{a}$ by adding $\frac{1}{2}$ to a_1 and subtracting $\frac{1}{2}$ from a_j, that is,
$$\vec{a}^{(j)} := \left(2g-1-A, a_2, \dots, a_{j-1}, a_j - \tfrac{1}{2}, a_{j+1}, \dots, a_n, \tfrac{3}{2}, -\tfrac{1}{2}\right),$$

Lemma 2.3.7. *The tautological relation $\pi_*^{(2)} \Omega_{g,n+2}^{g+1}(\vec{a}^{(j)})$ has the following form:*
$$\begin{aligned} & y \cdot \prod_{i=2}^{n} Q_{a_i}(a_i - \tfrac{1}{2}\delta_{ij}) Q_2(\tfrac{3}{2}) \left(Q_{g-1-A}(2g-1-A) - Q_{g-1-A}(2g-\tfrac{3}{2}-A)\right) \\ & - \sum_{\substack{\ell=2 \\ \ell\neq j}}^{n} x_\ell \cdot \prod_{i=2}^{n} Q_{a_i+2\delta_{i\ell}}(a_i + \tfrac{3}{2}\delta_{i\ell} - \tfrac{1}{2}\delta_{ij}) \left(Q_{g-1-A}(2g-1-A) - Q_{g-1-A}(2g-\tfrac{3}{2}-A)\right) \\ & = \textit{ terms divisible by } \psi_1^{g-1-A}. \end{aligned}$$

Proof. The proof of this lemma repeats the proof of lemma 2.3.6. It is only important to note that the terms that could produce the monomial x_j contribute trivially since they have a factor of $Q_{a_j+2}(a_j - \frac{1}{2} + \frac{3}{2}) = 0$ in their coefficients. ☐

Remark 2.3.8. Note that we have the condition $a_j \geq 0$. Indeed, if $a_j = 0$ we can still try to use $\vec{a}^{(j)}$ as a possible vector of primary fields. But in this case it can contain monomials with lower powers of ψ_1, and hence those relations cannot be used for our induction argument in increasing powers of ψ_1. To see this, consider graph **ii**. The coefficient that we have in this case for the degree d of the ψ-class on the genus g branch of the node is equal to $Q_{d+1}(-\frac{1}{2}-\frac{1}{2})$. Since -1 is not a zero of any polynomial $Q_{\geq 0}$, the degree d can be arbitrarily high, and therefore there is no restriction from below on the degree of ψ_1.

Let us distinguish now between zero and non-zero primary fields. Up to relabeling the marked points, we can assume that

$$a_2 = a_3 = \cdots = a_s = 0, \quad \text{and} \quad a_i \geq 1, \quad i = s+1, \ldots, n.$$

Note that, by the definition of the Q-polynomials, the coefficient of y is not zero in all relations in lemmata 2.3.6 and 2.3.7. Dividing these relations by the coefficient of y, we obtain the $n-s$ linearly independent relations:

$$\mathrm{Rel}_0: \quad y - \sum_{l=2}^{n} \frac{Q_{a_l+2}(a_l+3/2)}{Q_{a_l}(a_l)Q_2(3/2)} x_l = \text{terms divisible by } \psi_1^{g-1-A}$$

$$\mathrm{Rel}_j: \quad y - \sum_{l=2}^{n} \frac{Q_{a_l+2}(a_l+3/2)}{Q_{a_l}(a_l)Q_2(3/2)} (1-\delta_{j,l}) x_l = \text{terms divisible by } \psi_1^{g-1-A},$$

for $j = s+1, \ldots, n$. Rescaling the generators by rational non-zero coefficients

$$\tilde{x}_l := -\frac{Q_{a_l+2}(a_l+3/2)}{Q_{a_l}(a_l)Q_2(3/2)} x_l, \quad l = 2, \ldots, n$$

we can represent the relations in the following matrix:

$$M := \begin{array}{c|cccccccccc} & y & \tilde{x}_2 & \cdots & \tilde{x}_s & \tilde{x}_{s+1} & \tilde{x}_{s+2} & \tilde{x}_{s+3} & \cdots & \tilde{x}_n \\ \hline \mathrm{Rel}_0 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 \\ \mathrm{Rel}_{s+1} & 1 & 1 & \cdots & 1 & 0 & 1 & 1 & \cdots & 1 \\ \mathrm{Rel}_{s+2} & 1 & 1 & \cdots & 1 & 1 & 0 & 1 & \cdots & 1 \\ \mathrm{Rel}_{s+3} & 1 & 1 & \cdots & 1 & 1 & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathrm{Rel}_n & 1 & 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 0 \end{array}$$

Let us take linear combinations of the above relations: $\tilde{\mathrm{Rel}}_j := \mathrm{Rel}_0 - \mathrm{Rel}_j$ for $j = s+1, \ldots, n$, and $\tilde{\mathrm{Rel}}_0 := \mathrm{Rel}_0 - \sum_{j=s+1}^{n} \tilde{\mathrm{Rel}}_j$. We obtain:

$$\begin{array}{c|cccccccccc} & y & \tilde{x}_2 & \cdots & \tilde{x}_s & \tilde{x}_{s+1} & \tilde{x}_{s+2} & \tilde{x}_{s+3} & \cdots & \tilde{x}_n \\ \hline \tilde{\mathrm{Rel}}_0 & 1 & 1 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \\ \tilde{\mathrm{Rel}}_{s+1} & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\ \tilde{\mathrm{Rel}}_{s+2} & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 \\ \tilde{\mathrm{Rel}}_{s+3} & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \tilde{\mathrm{Rel}}_n & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1 \end{array}$$

The relation $\tilde{\mathrm{Rel}}_j$ expresses the monomial $\psi_1^{g-2-A}\prod_{i=2}^n \psi_i^{a_i+\delta_{ij}}$ as a linear combination of the generators with higher powers of ψ_1. The relation $\tilde{\mathrm{Rel}}_0$ expresses the monomial $\psi_1^{g-2-A}\prod_{i=2}^n \psi_i^{a_i}\kappa_1$ as linear combination of the monomials $\psi_1^{g-2-A}\prod_{i=2}^n \psi_i^{a_i+\delta_{ij}}$, for $j = 2,\dots,s$ and generators with higher powers of ψ_1. In case no primary field a_i is equal to zero (i. e. $s = 1$), any of the monomials $y, x_2, \dots, x_n$ can be expressed in terms of the generators with strictly bigger power of ψ_1.

Reduction algorithm

Consider a monomial $\psi_1^{g-1-\sum d_i}\psi_2^{d_2}\dots\psi_n^{d_n}$. Let d_M be the maximal element in the list of the d_i's with the lowest index. If $d_M \geq 2$, compute the relations $\tilde{\mathrm{Rel}}_j$ for the following vector of primary fields

$$\left(2g - \tfrac{3}{2} - \textstyle\sum_{i=2}^n d_i, d_2, \dots, d_{M-1}, d_M - 1, d_{M+1}, \dots, d_n,\, d_{n+1} = \tfrac{3}{2},\, d_{n+2} = -\tfrac{1}{2}\right).$$

Since $d_M - 1 \geq 1$, we can use the relation $\tilde{\mathrm{Rel}}_M$ to express the monomial $\psi_1^{g-1-\sum d_i}\cdot\psi_2^{d_2}\cdots\psi_n^{d_n}$ as a linear combination of monomials with higher powers of ψ_1.

We are left to treat the vectors $\vec{d}$ with $d_i = 0$ or 1, $i = 2,\dots,n$. They correspond to the vertices of a unitary $(n-1)$-hypercube with non-negative coordinates. Let s be the number of d_i's equal to zero, so the remaining $(n-1-s)$ d_i's are equal to one, $s = 0,\dots,n-1$. Let us distinguish between the different cases in s.

$s = n-1$: In this case we have ψ_1^{g-1}, a generator.

$s = n-2$: In this case we have the remaining $n-1$ generators $\psi_1^{g-2}\psi_i$ for $i = 2,\dots,n$.

$1 \leq s \leq n-3$: This case can be treated as the case $s = 0$ for some smaller n discussed below. Let us argue by induction on n. For $n \leq 3$, the case $1 \leq s \leq n-3$ does not appear. Let us assume $n \geq 4$. We have at least one zero, so let us assume that $d_j = 0$. Let $\pi_j^{(1)}$ be the morphism that forgets the j-th marked point. If the monomial $\psi_1^{g-n+s}\psi_2^{d_2}\dots\hat{\psi}_j\dots\psi_n^{d_n}$ is expressed as linear combination of generators in $R^{g-1}(\mathcal{M}_{g,n-1})$ (the space where the point with the label j is forgotten), then the pull-back of this relation via $\pi_j^{(1)}$ expresses $\psi_1^{g-n+s}\psi_2^{d_2}\dots\hat{\psi}_j\dots\psi_n^{d_n}$ as a linear combination of the pull-backs of the the $n-1$ generators of $R^{g-1}(\mathcal{M}_{g,n-1})$, ψ_1^{g-1} and $\psi_1^{g-2}\psi_i$, $i \neq 1, j$. To conclude we observe that $(\pi_j^{(1)})^*\psi_1^{g-1} = \psi_1^{g-1}$ and $(\pi_j^{(1)})^*\psi_1^{g-2}\psi_i = \psi_1^{g-2}\psi_i$, $i \neq 1, j$ on the open moduli spaces. Note that the same reasoning does not work in the case $s = n-2$ since the argument for $s = 0$ below uses the assumption $n \geq 3$.

The case $s = 0$

For $n \geq 3$, we show that the monomial $\psi_1^{g-n}\prod_{i=2}^n \psi_i^1$ can be expressed in terms of the generators ψ_1^{g-1}, $\psi_1^{g-2}\psi_i$, $i = 2,\dots,n$, concluding this way the proof of proposition 2.3.3.

Let now $\vec{v}_k$ be the vector of primary fields

$$\vec{v}_k := \Big(a_1 = 2g - \frac{n+2+k}{2}, \underbrace{1,\dots,1}_{k}, \underbrace{\frac{1}{2},\dots\frac{1}{2}}_{n-1-k}, a_{n+1} = \frac{3}{2}, a_{n+2} = -\frac{1}{2}\Big).$$

Similarly as before, let

$$y := \psi_1^{g-n-1} \prod_{i=2}^{n} \psi_i^1 \kappa_1$$
$$\tilde{x}_\ell := -\frac{Q_3(5/2)}{Q_1(1)Q_2(3/2)} \psi_2^1 \cdots \psi_\ell^2 \cdots \psi_n^1 \psi_1^{g-n-1}, \quad \ell = 2, \ldots, n.$$

Consider the monomials

$$\psi_1^{g-n} \prod_{i=2}^{n} \psi_i \quad \text{and} \quad \psi_1^{g-n} \prod_{i=2}^{n} \psi_i^{1-\delta_{i\ell}} \kappa_1, \qquad \ell = 2, \ldots, n.$$

The relations we used in the cases $s \geq 1$ imply that the difference of any two of these monomials is equal to a linear combination of the generators ψ_1^{g-1}, $\psi_1^{g-2}\psi_i$, $i = 2, \ldots, n$. Let c_0 (respectively, c_1, c_2) be the sum of the coefficients of these monomials in the push-forwards of the relations $\Omega_{g,n+2}^{g+1}(\vec{v}_0)$ (respectively, $\Omega_{g,n+2}^{g+1}(\vec{v}_1)$, $\Omega_{g,n+2}^{g+1}(\vec{v}_2)$), and let $\hat{c}_i$ be the normalized coefficients that we get when we divide the relations by the coefficient of y.

Now we can expand, in this special case, the system of linear relations collected in the matrix M above. We have a new linear variable, z defined as $\psi_1^{g-n} \prod_{i=2}^n \psi_i = \psi_1^{g-n} \prod_{i=2}^n \psi_i^{1-\delta_{i\ell}} \kappa_1$, $\ell = 2, \ldots, n$, and an extra linear relation Rel_* corresponding to the vector of primary fields $\vec{v}_2$. Since in this special case in these relations all the terms with the exponent of ψ_1 equal to $g-1-A$, $A = n-1$, are now identified with each other and collected in the variable z, these relations express $z, y, x_2, \ldots, x_n$ in terms of the monomials proportional to ψ_1^{g-A}. The matrix of this system of relations reads:

	z	y	$\tilde{x}_2$	$\tilde{x}_3$	$\tilde{x}_4$	$\ldots$	$\tilde{x}_n$
Rel_0	$\hat{c}_0$	1	1	1	1	$\ldots$	1
Rel_2	$\hat{c}_1$	1	0	1	1	$\ldots$	1
Rel_3	$\hat{c}_1$	1	1	0	1	$\ldots$	1
Rel_4	$\hat{c}_1$	1	1	1	0	$\ldots$	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
Rel_n	$\hat{c}_1$	1	1	1	1	$\ldots$	0
Rel_*	$\hat{c}_2$	1	0	0	1	$\ldots$	1

This matrix is non-degenerate if and only if $\hat{c}_2 - 2\hat{c}_1 + \hat{c}_0 \neq 0$. We prove this non-degeneracy in proposition 2.4.1 in the next section. This completes the proof of proposition 2.3.3 and, as a corollary, theorem 2.3.4.

2.4 Non-degeneracy of the matrix

In this section we compute the sum of the coefficients of the monomials

$$\psi_1^{g-n} \prod_{i=2}^{n} \psi_i \quad \text{and} \quad \psi_1^{g-n} \left(\prod_{i=2}^{n} \psi_i^{1-\delta_{i\ell}} \right) \kappa_1, \quad \ell = 2, \ldots, n,$$

for the three particular sequences of the primary fields. Let us recall the notation. We denote these sums of coefficients by

$$
\begin{aligned}
&c_0 \quad \text{for the primary fields} \quad g-1-n, \frac{1}{2}, \dots, \frac{1}{2};\\
&c_1 \quad \text{for the primary fields} \quad g-\frac{3}{2}-n, 1, \frac{1}{2}, \dots, \frac{1}{2};\\
&c_2 \quad \text{for the primary fields} \quad g-2-n, 1, 1, \frac{1}{2}, \dots, \frac{1}{2}.
\end{aligned}
$$

We denote the sequence of the primary fields by $a_1, \dots, a_n$. The primary fields at the two points that we forget are as usual $a_{n+1} = \frac{3}{2}$ and $a_{n+2} = -\frac{1}{2}$. For each c_i, $i = 0, 1, 2$, we denote by $\hat{c}_i$ the normalized coefficient, namely,

$$
\hat{c}_i := c_i \cdot \left((Q_{g+1-n}(a_1) - Q_{g+1-n}(a_1 - \tfrac{1}{2})) \prod_{i=2}^{n} Q_1(a_i) Q_2(\tfrac{3}{2})\right)^{-1}, \qquad i = 0, 1, 2,
$$

where the sequence of primary field is exactly the one used for the definition of the corresponding c_i, $i = 0, 1, 2$.

The goal is to prove the following non-degeneracy statement:

Proposition 2.4.1. *For any g and n satisfying $3 \leq n \leq g-1$ we have $\hat{c}_0 - 2\hat{c}_1 + \hat{c}_2 \neq 0$.*

We prove this proposition below, in subsection 2.4.3, after we compute the coefficients c_0, c_1, and c_2 explicitly.

2.4.1 A general formula

First, we prove a general formula for any set of primary fields $a_2, \dots, a_n \in \{\frac{1}{2}, 1\}$.

Lemma 2.4.2. *Let all a_i, $i = 2, \dots, n$ be either $\frac{1}{2}$ or 1. We have $a_1 = 2g - \frac{3}{2} - \sum_{i=2}^{n} a_i$. A general formula for the sum of the coefficients of the classes $\psi_1^{g-n} \prod_{i=2}^{n} \psi_i$ and $\psi_1^{g-n} \prod_{i=2}^{n} \psi_i^{1-\delta_{i\ell}} \kappa_1$, $\ell = 2, \dots, n$, in the pushforward to $\mathcal{M}_{g,n}$ is given by*

$$
\begin{aligned}
\prod_{i=2}^{n} Q_1(a_i) \cdot \Bigg[&(2g-2+n) Q_2\left(\tfrac{3}{2}\right) Q_{g-n}(a_1) \\
&+ (2g-2+n) Q_1\left(\tfrac{3}{2}\right) \left(Q_{g+1-n}(a_1) - Q_{g+1-n}(a_1 - \tfrac{1}{2})\right) \\
&+ Q_{g+2-n}(a_1) - Q_{g+2-n}(a_1 - \tfrac{1}{2}) \\
&+ Q_{g+2-n}(a_1 + 1) - Q_{g+2-n}(a_1 + \tfrac{3}{2}) \\
&+ \left(Q_1\left(\tfrac{3}{2}\right) - Q_1(1)\right) Q_{g+1-n}(a_1) \\
+ \sum_{\ell=2}^{n} &\frac{\left(Q_2\left(\tfrac{3}{2}\right)\left(Q_1(a_\ell) - Q_1(a_\ell - \tfrac{1}{2})\right)\right) Q_{g-1}(a_1)}{Q_1(a_\ell)}
\end{aligned}
\tag{2.5}
$$

$$+\sum_{\ell=2}^{n}\frac{\left(Q_3(a_\ell+1)-Q_3(a_\ell+\frac{3}{2})\right)Q_{g-n}(a_1)}{Q_1(a_\ell)}$$
$$\left.+\sum_{\ell=2}^{n}\frac{\left(Q_2\left(\frac{3}{2}\right)Q_0(a_\ell)-Q_2(a_\ell+\frac{3}{2})\right)\left(Q_{g+1-n}(a_1)-Q_{g+1-n}(a_1-\frac{1}{2})\right)}{Q_1(a_\ell)}\right].$$

Proof. The proof is based on the analysis of all possible strata in $\overline{\mathcal{M}}_{g,n+2}$ equipped with all possible monomials of ψ-classes that can potentially contribute non-trivially to $\psi_1^{g-n}\prod_{i=2}^n\psi_i$ and $\psi_1^{g-n}\prod_{i=2}^n\psi_i^{1-\delta_{i\ell}}\kappa_1$, $\ell=1,\dots,n$, under the pushforward. Note that we do not have to consider κ-classes on the strata in $\overline{\mathcal{M}}_{g,n+2}$ since the choice $r=\frac{1}{2}$ guarantees that there are no terms with κ-classes in the Pandharipande-Pixton-Zvonkine relations.

Recall that we denote by D_I, $I\subset\{1,\dots,n+2\}$, the divisor in $\overline{\mathcal{M}}_{g,n+2}$ whose generic point is represented by a two-component curve, with components of genus g and 0 connected through a node, such that all points with labels in I lie on the component of genus 0, and all other points lie on the component of genus g. In this case we denote by ψ_0 the ψ-class corresponding to the node on the genus 0 component.

We denote by $\pi'\colon\overline{\mathcal{M}}_{g,n+2}\to\overline{\mathcal{M}}_{g,n+1}$ the map forgetting the marked point labeled by $n+2$, by $\pi''\colon\overline{\mathcal{M}}_{g,n+1}\to\overline{\mathcal{M}}_{g,n}$ the map forgetting the marked point labeled by $n+1$, and by π their composition $\pi=\pi''\circ\pi'$. Note that $\pi'_*(\prod_{i=1}^{n+1}\psi_i^{d_i})=\sum_{j:d_j>0}\prod_{i=1}^{n+1}\psi_i^{d_i-\delta_{ij}}$, so, since in order to compute π_* we always first apply π'_*, we typically mention below the degree of which ψ-class is reduced. The same we do also for π''_* in the relevant cases.

Let us now go through the full list of possible non-trivial contributions.

- The pushforward of the class $\psi_1^{g-n}\prod_{i=2}^n\psi_i^1\psi_{n+1}^2$ contains $\psi_1^{g-n}\prod_{i=2}^n\psi_i^1$ with coefficient $(2g-2+n)\prod_{i=2}^n Q_1(a_i)Q_{g-n}(a_n)Q_2(\frac{3}{2})$. This explains the first line of equation (2.5). It also contains the terms $\psi_1^{g-n}\prod_{i=2}^n\psi_i^{1-\delta_{i\ell}}\kappa_1$, $\ell=2,\dots,n$, with coefficient $\prod_{i=2}^n Q_1(a_i)Q_{g-n}(a_1)\cdot Q_2(\frac{3}{2})$.

- The pushforward of the class $\psi_1^{g-n}\prod_{i=2}^n\psi_i^{1-\delta_{i\ell}}D_{\ell,n+2}\psi_{n+1}^2$ also gives the term $\psi_1^{g-n}\prod_{i=2}^{n-1}\psi_i^{1-\delta_{i\ell}}\kappa_1$, with coefficient $-\prod_{i=2}^n Q_1(a_i-\frac{\delta_{i\ell}}{2})Q_{g-n}(a_1)Q_2(\frac{3}{2})$. The sum over ℓ of this and the previous coefficient is equal to the sixth line of equation (2.5).

- The pushforward of the class $\psi_1^{g+1-n}\prod_{i=2}^{n+1}\psi_i^1$, where at the first step the map π'_* decreases the degree of ψ_1, gives $\psi_1^{g-n}\prod_{i=1}^n\psi_i^1$ with coefficient $(2g-2+n)\prod_{i=2}^n Q_1(a_i)Q_{g+1-n}(a_1)\cdot Q_1\left(\frac{3}{2}\right)$.

- The push-forward of the class $(\pi')^*(\psi_1^{g-n})\prod_{i=2}^{n+1}\psi_i^1 D_{1,n+2}$ gives $\psi_1^{g-n}\prod_{i=2}^n\psi_i^1$ with coefficient $-(2g-2+n)\prod_{i=2}^n Q_1(a_i)Q_{g+1-n}(a_1-\frac{1}{2})Q_1(\frac{3}{2})$. The sum of this and the previous coefficient is equal to the second line of equation (2.5).

- The pushforward of the class $\psi_1^{g+2-n}\prod_{i=2}^n\psi_i^1$, where both π'_* and π''_* decrease the degree of ψ_1, gives $\psi_1^{g-n}\prod_{i=2}^n\psi_i^1$ with coefficient $\prod_{i=1}^n Q_1(a_i)Q_{g+2-n}(a_1)$.

- The pushforward of the class $(\pi')^*(\psi_1^{g+1-n})\prod_{i=2}^n\psi_i^1 D_{1,n+2}$, where the map π''_* decreases the degree of ψ_1, gives $\psi_1^{g-n}\prod_{i=2}^n\psi_i^1$ with coefficient $-\prod_{i=2}^n Q_1(a_i)Q_{g+2-n}(a_1-\frac{1}{2})$. The sum of this and the previous coefficient is equal to the third line of Equation (2.5).

- The pushforward of the class $(\pi'')^*(\psi_1^{g+1-n})\prod_{i=2}^n \psi_i^1 D_{1,n+1}$, where at the first step the map π'_* decreases the degree of $(\pi'')^*\psi_1$, gives $\psi_1^{g-n}\prod_{i=2}^n \psi_i^1$ with coefficient $-\prod_{i=2}^n Q_1(a_i)$ $Q_{g+2-n}(a_1+\frac{3}{2})$.

- Consider the following seven cases together: $\pi^*(\psi_1^{g-n})\prod_{i=2}^n \psi_i^1 D_{1,n+1,n+2}\cdot(\psi_0+\psi_1+\psi_{n+1}+\psi_{n+2})$ and $\pi^*(\psi_1^{g-n})\prod_{i=2}^n \psi_i^1 D_{1,n+1,n+2}(D_{1,n+1}+D_{1,n+2}+D_{n+1,n+2})$. By Lemma 2.4.3 below, the total sum of their pushforwards is equal to $\psi_1^{g-n}\prod_{i=2}^n \psi_i$ with coefficient $\prod_{i=2}^n Q_1(a_i)\cdot$ $Q_{g+2-n}(a_1-\frac{1}{2}+\frac{3}{2})$. The sum of this and the previous coefficient is equal to the fourth line in Equation (2.5).

- The pushforward of the class $\psi_1^{g-n}\prod_{i=2}^n \psi_i^{1-\delta_{i\ell}} D_{\ell,n+1}(\pi'')^*\psi_\ell^1$, where at the first step the map π'_* decreases the degree of $(\pi'')^*\psi_\ell$, gives $\psi_1^{g-n}\prod_{i=2}^n \psi_i^1$ with coefficient $-\prod_{i=2}^n Q_{1+2\delta_{i\ell}}(a_i+\frac{3\delta_{i\ell}}{2})Q_{g-n}(a_1)$.

- Consider the following seven cases together: $\psi_1^{g-n}\prod_{i=2}^n \psi_i^{1-\delta_{i\ell}} D_{\ell,n+1,n+2}\pi^*\psi_\ell^1\cdot(\psi_0+\psi_\ell+\psi_{n+1}+\psi_{n+2})$ and $\psi_1^{g-n}\prod_{i=2}^n \psi_i^{1-\delta_{i\ell}} D_{\ell,n+1,n+2}\cdot(D_{1,n+1}+D_{1,n+2}+D_{n+1,n+2})\cdot\pi^*\psi_\ell^1$. By lemma 2.4.3 below, the total sum of their pushforwards is equal to $\psi_1^{g-n}\prod_{i=1}^n \psi_i$ with coefficient $\prod_{i=1}^n Q_{1+2\delta_{i\ell}}(a_i+\delta_{i\ell})Q_{g-n}(a_1)$. Note that this coefficient is always equal to zero, since $Q_3(2)=Q_3(\frac{3}{2})=0$, but we included this term here and in equation (2.5) in any case in order to make the whole formula more transparent and homogeneous. The sum of this and the previous coefficient is equal to the seventh line in equation (2.5).

- The pushforward of the class $\psi_1^{g+1-n}\prod_{i=2}^{n-1}\psi_i^1\psi_{n+1}^1$, where, first, the map π'_* decreases the degree of ψ_{n+1}, so it becomes zero, and then the map π''_* decreases the degree of ψ_1, giving $\psi_1^{g-n}\prod_{i=2}^n \psi_i^1$ with coefficient $\prod_{i=2}^n Q_1(a_i)Q_{g+1-n}(a_1)Q_1(\frac{3}{2})$.

- The pushforward of the class $\psi_1^{g+1-n}\prod_{i=2}^{n-1}\psi_i^1 D_{n+1,n+2}$, where the map π''_* decreases the degree of ψ_1, yielding $\psi_1^{g-n}\prod_{i=2}^n \psi_i^1$ with coefficient $-\prod_{i=2}^n Q_1(a_i)\cdot$ $Q_{g+1-n}(a_1)Q_1(\frac{3}{2}-\frac{1}{2})$. The sum over ℓ of this and the previous coefficient is equal to the fifth line of Equation (2.5).

- The push-forward of the class $\psi_1^{g+1-n}\prod_{i=2}^{n-1}\psi_i^{1-\delta_{i\ell}}\psi_{n+1}^2$, where at the first step the map π'_* and decreases the degree of ψ_1, gives $\psi_1^{g-n}\prod_{i=2}^{n-1}\psi_i^{1-\delta_{i\ell}}\kappa_1$ with the coefficient $\prod_{i=2}^n Q_{1-\delta_{i\ell}}(a_i)\cdot$ $Q_{g+1-n}(a_1)Q_2(\frac{3}{2})$.

- The pushforward of the class $(\pi')^*(\psi_1^{g-n})\prod_{i=2}^{n-1}\psi_i^{1-\delta_{i\ell}} D_{1,n+2}\psi_{n+1}^2$ gives the term $\psi_1^{g-n}\prod_{i=2}^{n-1}$ $\psi_i^{1-\delta_{i\ell}}\kappa_1$ with coefficient $-\prod_{i=2}^n Q_{1-\delta_{i\ell}}(a_i)Q_{g+1-n}(a_1-\frac{1}{2})Q_2(\frac{3}{2})$.

- The pushforward of the class $\psi_1^{g+1-n}\prod_{i=2}^{n-1}\psi_i^{1-\delta_{i\ell}} D_{\ell,n+1}(\pi'')^*\psi_\ell^1$, where at the first step the map π'_* decreases the degree of ψ_1, gives $\psi_1^{g-n}\prod_{i=2}^n \psi_i^1$ with coefficient $-\prod_{i=2}^n Q_{1-\delta_{i\ell}}(a_i)\cdot$ $Q_{g+1-n}(a_1)Q_2(a_\ell+\frac{3}{2})$.

- The pushforward of the class $(\pi')^*(\psi_1^{g-n})\prod_{i=2}^{n-1}\psi_i^{1-\delta_{i\ell}} D_{\ell,n+1}(\pi'')^*\psi_\ell^1 D_{1,n+2}$ gives $\psi_1^{g-n}\cdot$ $\prod_{i=2}^n \psi_i^1$ with the coefficient $\prod_{i=2}^n Q_{1-\delta_{i\ell}}(a_i)Q_{g+1-n}(a_1-\frac{1}{2})Q_2(a_\ell+\frac{3}{2})$. The sum over ℓ of this and the previous three coefficients is equal to the eighth line in Equation (2.5).

Thus we have explained how we obtain all terms in Equation (2.5). Note that since $Q_{\geq 1}(-\frac{1}{2})=0$, we can never have a non-trivial degree of ψ_{n+2} in our formulae. For the same reason, the degree of $\psi_2,\dots,\psi_n$ is bounded from above by 1 and the degree of ψ_{n+1} is bounded from above by 2. With this type of reasoning it is easy to see by direct inspection that all other classes of degree $g+1$ do not contain any of the monomials $\psi_1^{g-n}\prod_{i=2}^{n-1}\psi_i$ and $\psi_1^{g-n}\prod_{i=2}^{n-1}\psi_i^{1-\delta_{i\ell}}\kappa_1$, $\ell=2,\dots,n$, with non-trivial coefficients in their push-forwards to $\mathcal{M}_{g,n}$. For instance, for an arbitrary a_ℓ the class $(\pi'')^*(\psi_1^{g-n})\prod_{i=2}^{n}\psi_i^{1-\delta_{i\ell}}D_{\ell,n+2}(\pi')^*\psi_\ell^1 D_{1,n+1}$ gives as result $\psi_1^{g-n}\prod_{i=2}^{n}\psi_i^1$ with the coefficient $\prod_{i=2}^{n}Q_{1-\delta_{i\ell}}(a_i)Q_{g+1-n}(a_1+\frac{3}{2})Q_2(a_\ell-\frac{1}{2})$. But since a_ℓ is either $\frac12$ or 1 and $Q_2(0)=Q_2(\frac12)=0$, this coefficient is equal to zero. □

Lemma 2.4.3. *Let the points* 1, $n+1$, *and* $n+2$ *have arbitrary primary fields* α, β, *and* γ. *Then the pushforward of the part of the class given by*

$$\prod_{i=2}^{n}\psi_i^{d_i}\Big[D_{1,n+1,n+2}\pi^*\psi_1^{d_1}(\psi_0+\psi_1+\psi_{n+1}+\psi_{n+2})$$
$$+D_{1,n+1,n+2}(D_{1,n+1}+D_{1,n+2}+D_{n+1,n+2})\pi^*\psi_1^{d_1}\Big].$$

is equal to $\prod_{i=1}^{n}\psi_i^{d_i}$ *with the coefficient* $\prod_{i=2}^{n}Q_{d_i}(a_i)Q_{d_1+2}(\alpha+\beta+\gamma)$.

Proof. Indeed, the Givental formula for the deformed r-spin class (for a general r) in this case implies that these seven summands have the following coefficients, up to a common factor:

$$\begin{aligned}
\psi_0: &\quad (R^{-1}_{d_1+2})^{\alpha+\beta+\gamma-d_1-2}_{\alpha+\beta+\gamma}-(R^{-1}_{d_1+1})^{\alpha+\beta+\gamma-d_1-2}_{\alpha+\beta+\gamma-1}(R^{-1}_1)^{r-2-(\alpha+\beta+\gamma)}_{r-1-(\alpha+\beta+\gamma)}\\
\psi_1: &\quad -(R^{-1}_{d_1+1})^{\alpha+\beta+\gamma-d_1-2}_{\alpha+\beta+\gamma-1}(R^{-1}_1)^{\alpha-1}_{\alpha}\\
\psi_{n+1}: &\quad -(R^{-1}_{d_1+1})^{\alpha+\beta+\gamma-d_1-2}_{\alpha+\beta+\gamma-1}(R^{-1}_1)^{\beta-1}_{\beta}\\
\psi_{n+2}: &\quad -(R^{-1}_{d_1+1})^{\alpha+\beta+\gamma-d_1-2}_{\alpha+\beta+\gamma-1}(R^{-1}_1)^{\gamma-1}_{\gamma}\\
D_{1,n+1}: &\quad (R^{-1}_{d_1+1})^{\alpha+\beta+\gamma-d_1-2}_{\alpha+\beta+\gamma-1}(R^{-1}_1)^{\alpha+\beta-1}_{\alpha+\beta}\\
D_{1,n+2}: &\quad (R^{-1}_{d_1+1})^{\alpha+\beta+\gamma-d_1-2}_{\alpha+\beta+\gamma-1}(R^{-1}_1)^{\alpha+\gamma-1}_{\alpha+\gamma}\\
D_{n+1,n+2}: &\quad (R^{-1}_{d_1+1})^{\alpha+\beta+\gamma-d_1-2}_{\alpha+\beta+\gamma-1}(R^{-1}_1)^{\gamma+\beta-1}_{\gamma+\beta}
\end{aligned}$$

(In addition to a common factor on the right hand side we also omit the common factor $\pi^*(\psi_1^{d_1})\cdot\prod_{i=2}^{n}\psi_i^{d_i}D_{1,n+1,n+2}$ on the left hand side of this table).

The first term above, $(R^{-1}_{d_n+2})^{\alpha+\beta+\gamma-d_1-2}_{\alpha+\beta+\gamma}$, after the substitution $r=\frac12$ gives us the factor $Q_{d_1+2}(\alpha+\beta+\gamma)$, and times the common factor of $\prod_{i=2}^{n}Q_{d_i}(a_i)$ it is exactly the results we state in the lemma. We have to show that the other seven terms sum up to zero. Indeed, the other seven terms, after substitution $r=\frac12$, are proportional to

$$\begin{aligned}
&Q_1(-\tfrac12-\alpha-\beta-\gamma)+Q_1(\alpha)+Q_1(\beta)+Q_1(\gamma)\\
&-Q_1(\alpha+\beta)-Q_1(\alpha+\gamma)-Q_1(\gamma+\beta)
\end{aligned}$$

Note that $Q_1(-\frac12-x)=Q_1(x)$, so the expression above is proportional to

$$\begin{aligned}
&(\tfrac12+\alpha+\beta+\gamma)(\alpha+\beta+\gamma)+(\tfrac12+\alpha)(\alpha)+(\tfrac12+\beta)(\beta)+(\tfrac12+\gamma)(\gamma)\\
&-(\tfrac12+\alpha+\beta)(\alpha+\beta)-(\tfrac12+\alpha+\gamma)(\alpha+\gamma)-(\tfrac12+\beta+\gamma)(\beta+\gamma)=0.
\end{aligned}$$

□

2.4.2 Special cases of the general formula

In this section we use lemma 2.4.2 in order to derive the formulae for c_0, c_1, and c_2. Since all our expressions are homogeneous (the sum of the indices of the polynomials Q is always equal to $g+1$), we can drop the factor $(-1)^m/2^m$ in the definition of Q_m, $m \geq 0$.

We can substitute the values $Q_1(\frac{3}{2}) = 3$, $Q_1(1) = \frac{3}{2}$, $Q_1(\frac{1}{2}) = \frac{1}{2}$, $Q_1(0) = 0$, $Q_2(\frac{5}{2}) = \frac{45}{4}$, $Q_2(2) = \frac{15}{4}$, $Q_2(\frac{3}{2}) = \frac{3}{4}$, $Q_3(\frac{5}{2}) = \frac{15}{8}$, $Q_3(2) = 0$ in equation (2.5). This gives use the following coefficients of $Q_{g-n}(a_1)$, $Q_{g+1-n}(a_1)$, and $Q_{g+1-n}(a_1 - \frac{1}{2})$:

$$\begin{aligned} \text{in } c_0 : \quad & \left(\tfrac{3g}{2} - \tfrac{9}{4} + \tfrac{3n}{2}\right) Q_{g-n}(a_1) + \left(6g + \tfrac{3}{2} - 3n\right) Q_{g+1-n}(a_1) \\ & + (-6g + 0 + 3n)\, Q_{g+1-1}(a_n - \tfrac{1}{2}) \\ \text{in } c_1 : \quad & \left(\tfrac{3g}{2} - \tfrac{15}{4} + \tfrac{3n}{2}\right) Q_{g-n}(a_1) + \left(6g + \tfrac{1}{2} - 3n\right) Q_{g+1-n}(a_1) \\ & + (-6g + 1 + 3n)\, Q_{g+1-n}(a_1 - \tfrac{1}{2}) \\ \text{in } c_2 : \quad & \left(\tfrac{3g}{2} - \tfrac{21}{4} + \tfrac{3n}{2}\right) Q_{g-n}(a_1) + \left(6g - \tfrac{1}{2} - 3n\right) Q_{g+1-n}(a_1) \\ & + (-6g + 2 + 3n)\, Q_{g+1-n}(a_1 - \tfrac{1}{2}) \end{aligned} \tag{2.6}$$

Note that the primary field a_1 has a different value in these three cases.

Furthermore, we are going to use that

$$Q_{g+2-n}(a_1) - Q_{g+2-n}(a_1 - \tfrac{1}{2}) = \tfrac{(a_1)(a_1 - \frac{1}{2})\cdots(a_1 - g - 1 + n)}{(g+1-n)!} \tag{2.7}$$

$$Q_{g+2-n}(a_1 + 1) - Q_{g+2-n}(a_1 + \tfrac{3}{2}) = \tfrac{-(a_1 + \frac{3}{2})(a_1 + 1)\cdots(a_1 - g + \frac{1}{2} + n)}{(g+1-n)!} \tag{2.8}$$

Let us combine these terms with the terms with Q_{g+1-n} computed above. In the case of c_0 the primary field a_1 is equal to $2g - 1 - \frac{n}{2}$. Then the sum of (2.6), (2.7), and (2.8) is equal to the following expression:

$$\begin{aligned} & \tfrac{(2g-1-\frac{n}{2})\cdots(g-2+\frac{n}{2})}{(g+1-n)!} - (2g - 1 - \tfrac{n}{2}) Q_{g+1-n}(2g - \tfrac{3}{2} - \tfrac{n}{2}) \\ & - (4g + 1 - \tfrac{5n}{2}) Q_{g+1-n}(2g - \tfrac{3}{2} - \tfrac{n}{2}) + (4g + 1 - \tfrac{5n}{2}) Q_{g+1-n}(2g - 1 - \tfrac{n}{2}) \\ & + (2g + \tfrac{1}{2} - \tfrac{n}{2}) Q_{g+1-n}(2g - 1 - \tfrac{n}{2}) - \tfrac{(2g+\frac{1}{2}-\frac{n}{2})\cdots(g-\frac{1}{2}+\frac{n}{2})}{(g+1-n)!} \\ & = -(2g - 1 - \tfrac{n}{2}) \tfrac{(2g-\frac{3}{2}-\frac{n}{2})\cdots(g-\frac{3}{2}+\frac{n}{2})}{(g-n)!} + (4g + 1 - \tfrac{5n}{2}) \tfrac{(2g-1-\frac{n}{2})\cdots(g-1+\frac{n}{2})}{(g-n)!} \\ & - (2g + \tfrac{1}{2} - \tfrac{n}{2}) \tfrac{(2g-\frac{1}{2}-\frac{n}{2})\cdots(g-\frac{1}{2}+\frac{n}{2})}{(g-n)!} \\ & = (3g + \tfrac{5}{2} - 3n) \tfrac{(2g-1-\frac{n}{2})\cdots(g-1+\frac{n}{2})}{(g-n)!} - (2g + \tfrac{1}{2} - \tfrac{n}{2}) \tfrac{(2g-\frac{1}{2}-\frac{n}{2})\cdots(g-\frac{1}{2}+\frac{n}{2})}{(g-n)!} \end{aligned}$$

We can perform the same computation also for c_1 and c_2. Recall also in all three cases the term with Q_{g-n} and the overall coefficients $\prod_{i=1}^{n-1} Q_1(a_i)$ in equation (2.5). We obtain the following expressions:

Corollary 2.4.4. *We have:*

$$c_0 = Q_1(\tfrac{1}{2})^{n-1}\Big[(\tfrac{3g}{2}-\tfrac{9}{4}+\tfrac{3n}{2})\tfrac{(2g-\frac{1}{2}-\frac{n}{2})\cdots(g-0+\frac{n}{2})}{(g-n)!}$$
$$-(2g+\tfrac{1}{2}-\tfrac{n}{2})\tfrac{(2g-\frac{1}{2}-\frac{n}{2})\cdots(g-\frac{1}{2}+\frac{n}{2})}{(g-n)!}+(3g+\tfrac{5}{2}-3n)\tfrac{(2g-1-\frac{n}{2})\cdots(g-1+\frac{n}{2})}{(g-n)!}\Big]$$
$$c_1 = Q_1(\tfrac{1}{2})^{n-2}Q_1(1)\Big[(\tfrac{3g}{2}-\tfrac{15}{4}+\tfrac{3n}{2})\tfrac{(2g-1-\frac{n}{2})\cdots(g-\frac{1}{2}+\frac{n}{2})}{(g-n)!}$$
$$-(2g+0-\tfrac{n}{2})\tfrac{(2g-1-\frac{n}{2})\cdots(g-1+\frac{n}{2})}{(g-n)!}+(3g+\tfrac{5}{2}-3n)\tfrac{(2g-\frac{3}{2}-\frac{n}{2})\cdots(g-\frac{3}{2}+\frac{n}{2})}{(g-n)!}\Big]$$
$$c_2 = Q_1(\tfrac{1}{2})^{n-3}Q_1(1)^2\Big[(\tfrac{3g}{2}-\tfrac{21}{4}+\tfrac{3n}{2})\tfrac{(2g-\frac{3}{2}-\frac{n}{2})\cdots(g-1+\frac{n}{2})}{(g-n)!}$$
$$-(2g-\tfrac{1}{2}-\tfrac{n}{2})\tfrac{(2g-\frac{3}{2}-\frac{n}{2})\cdots(g-\frac{3}{2}+\frac{n}{2})}{(g-n)!}+(3g+\tfrac{5}{2}-3n)\tfrac{(2g-2-\frac{n}{2})\cdots(g-2+\frac{n}{2})}{(g-n)!}\Big]$$

2.4.3 Proof of non-degeneracy

In this subsection we prove proposition 2.4.1. First, observe that the difference of $Q_{g+1-n}(a_1)$ and $Q_{g+1-n}(a_1-\frac{1}{2})$ is equal to $\frac{(a_1)(a_1-\frac{1}{2})\cdots(a_1-g+n)}{(g-n)!}$. We substitute $a_1 = 2g-1+\frac{n}{2}$ for c_0 (respectively, $2g-\frac{3}{2}+\frac{n}{2}$ for c_1 and $2g-2+\frac{n}{2}$ for c_2) and combine the result of corollary 2.4.4 and equation 2.4 in order to obtain the following formulae:

$$\tfrac{3}{4}\hat{c}_0 = (\tfrac{3g}{2}-\tfrac{9}{4}+\tfrac{3n}{2})\tfrac{(2g-\frac{1}{2}-\frac{n}{2})}{(g-\frac{1}{2}+\frac{n}{2})(g-1+\frac{n}{2})}-(2g+\tfrac{1}{2}-\tfrac{n}{2})\tfrac{(2g-\frac{1}{2}-\frac{n}{2})}{(g-1+\frac{n}{2})}+(3g+\tfrac{5}{2}-3n)$$
$$\tfrac{3}{4}\hat{c}_1 = (\tfrac{3g}{2}-\tfrac{15}{4}+\tfrac{3n}{2})\tfrac{(2g-1-\frac{n}{2})}{(g-1+\frac{n}{2})(g-\frac{3}{2}+\frac{n}{2})}-(2g+0-\tfrac{n}{2})\tfrac{(2g-1-\frac{n}{2})}{(g-\frac{3}{2}+\frac{n}{2})}+(3g+\tfrac{5}{2}-3n)$$
$$\tfrac{3}{4}\hat{c}_2 = (\tfrac{3g}{2}-\tfrac{21}{4}+\tfrac{3n}{2})\tfrac{(2g-\frac{3}{2}-\frac{n}{2})}{(g-\frac{3}{2}+\frac{n}{2})(g-2+\frac{n}{2})}-(2g-\tfrac{1}{2}-\tfrac{n}{2})\tfrac{(2g-\frac{3}{2}-\frac{n}{2})}{(g-2+\frac{n}{2})}+(3g+\tfrac{5}{2}-3n)$$

By an explicit computation, we obtain that

$$\tfrac{3}{4}(\hat{c}_0-2\hat{c}_1+\hat{c}_2) = \tfrac{S(g,n)}{(g-\frac{1}{2}+\frac{n}{2})(g-1+\frac{n}{2})(g-\frac{3}{2}+\frac{n}{2})(g-2+\frac{n}{2})},$$

where

$$S(g,n) = -g+\frac{11}{8}n-\frac{9}{4}g^2+\frac{9}{8}gn-\frac{1}{2}g^3+\frac{3}{4}g^2n-\frac{1}{4}n^3$$

We want to prove that this polynomial is never equal to zero in the integer points (g,n) satisfying $3 \le n \le g-1$. We can make a change of variable $n = b+3$, $g = a+b+4$, then we want to prove that $S(a+b+4, b+3)$ never vanishes for any integer $a, b \ge 0$. This is indeed the case since all non-zero coefficients of the polynomial

$$S(a+b+4,b+3) = -\frac{201}{8}-\frac{173}{8}a-\frac{21}{2}b-6a^2-\frac{39}{8}ab-\frac{9}{8}b^2-\frac{1}{2}a^3-\frac{3}{4}a^2b$$

are negative including the constant term. This completes the proof of proposition 2.4.1.

2.5 Vanishing of $R^{\ge g}(\mathcal{M}_{g,n})$

In this section we will give a new proof of the following theorem.

Theorem 2.5.1 ([19, 15]). *The tautological ring of $\mathcal{M}_{g,n}$ vanishes in degrees g and higher, that is $R^{\geq g}(\mathcal{M}_{g,n}) = 0$.*

This theorem and theorem 2.3.4 together consistute the generalized socle conjecture, as the bound $\dim R^{g-1}(\mathcal{M}_{g,n}) \geq n$ can be proved relatively simply, see e.g. [2]. This conjecture is a generalization of one of Faber's three conjectures on the tautological ring of $\mathcal{M}_g$, see [8] for the original conjectures and [2] for the generalization.

The proof consists of three steps: in steps one and two, we show that the pure ψ- and κ-classes vanish, respectively, and in step three we reduce the mixed monomials to the pure cases. The first two steps will be proved in separate lemmata.

Lemma 2.5.2. *Let $g \geq 0$ and $n \geq 1$. Any monomial in ψ-classes of degree at least* $\max(g, 1)$ *vanishes on $\mathcal{M}_{g,n}$.*

Remark 2.5.3. This lemma was originally conjectured by Getzler in [10].

Proof. For $g = 0$, this is well-known, see e.g. [31, proposition 2.13]. So let us assume $g \geq 1$.

We will prove that any monomial in ψ-classes of degree g vanishes. This clearly implies that any monomial of higher degree vanishes as well.

For this, look again at Ω^g, but now on $\overline{\mathcal{M}}_{g,n}$. When restricted to the open part $\mathcal{M}_{g,n}$, the only contributing graph is the one with one vertex of genus g, as the other graphs correspond to boundary divisors by definition. Hence, the equation for the CohFT reduces to

$$\Omega^g_{g,n}(a_1, \dots, a_n)\Big|_{\mathcal{M}_{g,n}} = \begin{cases} -\frac{1}{2}\prod_{i=1}^n \left(\sum_{m_i \geq 0} Q_{m_i}(a_i)\psi_i^{m_i}\right) & \text{if } \sum_{i=1}^n a_i = 2g-1 \\ 0 & \text{else.} \end{cases}$$

We will prove vanishing of all monomials using downward induction on the exponent d_1 of ψ_1, starting with the case of $d_1 = g + 1$. This case trivially gives a zero, as this power cannot occur in a monomial of total degree g.

Now, assuming all monomials with exponent of ψ_1 larger than d_1 vanish, consider the monomial $\psi_1^{d_1} \cdots \psi_n^{d_n}$ for any d_i summing up to g. For the relation, choose $a_i = d_i$ for all $i \neq 1$, and $a_1 = 2g - 1 - \sum_{i=2}^n a_i$. This means $Q_{m_i}(a_i) = 0$ unless $m_i \leq d_i$ or $i = 1$, so the only monomials with non-zero coefficients have exponent of ψ_i at most d_i for $i \neq 1$. Because the total degree is fixed, the only surviving monomial with exponent of ψ_1 equal to d_1 is the one we started with, and this relation expresses it in monomials with strictly larger exponent of ψ_1. By the induction hypothesis, this monomial must be zero. □

Remark 2.5.4. Note that this argument breaks down for degrees lower than g, as the class does not vanish there. Therefore, to get relations in those degrees, one must push forward relations in higher degrees along forgetful maps *on the compactified moduli space*, which contain non-trivial contributions from boundary strata.

Lemma 2.5.5. *Any multi-index κ-class of degree at least g vanishes on $\mathcal{M}_{g,n}$.*

Proof. Fix a degree $d \geq g$, and consider the pure (multi-index) κ-classes in this degree. Without loss of generality, we can assume the amount of indices to be equal to d: this is certainly an upper bound, and adding and extra zero index only multiplies the class by a non-zero factor, using the dilaton equation on the definition of multi-index κ-classes.

We will consider $\Omega^g_{g,n+d}$. In order to get a relation in $R^d(\mathcal{M}_{g,n})$, we should multiply by a class σ of degree $2d-g$, push forward to $\overline{\mathcal{M}}_{g,n}$, and then restrict to $\mathcal{M}_{g,n}$. As we can now assume $d \geq g$, we have $2d-g \geq d$, and we can therefore choose $\sigma = \prod_{j=1}^d \psi_{n+j}^{f_j+1}$, with each $f_j \geq 0$. By choosing such a σ, we ensure that after pushforward and restriction to the open moduli space, none of the contributions from boundary divisors on $\overline{\mathcal{M}}_{g,n+d}$ survive, and only the term with one vertex contributes.

We will use downward induction on the first index of the κ-class. The base case is a first index larger than d, and hence another index being negative, giving a trivial zero.

Now, assume all κ-classes with first index larger than e_1 are zero. Fix a class $\kappa_{e_1,\dots,e_d}$ of degree $d = \sum_{j=1}^d e_j$, choose a set of non-negative integers $\{a_j, f_j \mid 2 \leq j \leq d\}$ such that $a_j + f_j = e_j$, and set $a_1 = 2g-1-\sum_{j=2}^d a_j$ and $f_1 = 0$. We will consider

$$\begin{aligned}
&\pi^d_*\big(\sigma \cdot \Omega^g_{g,n+d}(0,\dots,0,a_1,\dots,a_d)\big)\Big|_{\mathcal{M}_{g,n}} \\
&= \pi^d_*\Big(\prod_{j=1}^d \psi_{n+j}^{f_j+1} \cdot -\frac{1}{2}\prod_{j=1}^d \sum_{m_j\geq 0} Q_{m_j}(a_j)\psi_{n+j}^{m_j}\Big)\Big|_{\mathcal{M}_{g,n}} \\
&= -\frac{1}{2}\sum_{\substack{m_j\geq 0\\ 1\leq j\leq d}} \Big(\prod_{j=1}^d Q_{m_j}(a_j)\Big)\kappa_{f_1+m_1,\dots,f_d+m_d},
\end{aligned}$$

which vanishes. By our choice of a_j, for the product of Q-polynomials to be non-zero, we need $m_j \leq a_j$ for $j \neq 1$. Furthermore, by our choice of f_j, this shows that $f_j + m_j \leq e_j$ for $j \neq 1$. Because we look at a fixed degree d, this means $f_1 + m_1 \geq e_1$, with equality only occuring for $m_j = f_j$, $j \neq 1$, and hence for the κ-class we started with. Hence this relation expresses our chosen class $\kappa_{e_1,\dots,e_d}$ in terms of κ-classes with strictly higher first index, which we already know vanish. □

Remark 2.5.6. Note that we cannot use the vanishing of the ψ-monomials in higher degrees and push these relations forward, as the κ-classes are defined by pushing forward ψ-classes on the compactified moduli space and then restricting to the open part, and not the other way around.

We are now ready to prove the theorem.

Proof of theorem 2.5.1. For general monomial ψ-κ-classes, i.e. classes of the form $\mu = \psi_i^{d_1}\cdots\psi_n^{d_n}\cdot \kappa_{e_1,\dots,e_k}$, we will use induction on the total degree $d = \sum_{i=1}^n d_i + \sum_{j=1}^k e_j$. If all d_i are zero, we are in the case of lemma 2.5.5, so we can assume at least one of them is non-zero, i.e. $\mu = \nu\cdot\psi_i$ for some i.

In degree $d = g$, we get that the degree of ν is $g-1$. By proposition 2.3.1, we know that ν is a polynomial in ψ-classes. Therefore, so is $\mu = \nu\cdot\psi_i$. By lemma 2.5.2, we know μ vanishes.

For the induction step, we know by induction that ν is zero, hence μ is too. This finishes the proof of theorem 2.5.1. □

Because the proof of this theorem only uses the case $x = 0$ from subsection 2.2.2, see also subsection 2.2.4, and only fixed non-negative integer primary fields, all the relations are actually explicit on all of $\overline{\mathcal{M}}_{g,n}$. Hence, we get the following

Proposition 2.5.7. *The Pandharipande-Pixton-Zvonkine relations for $r = \frac{1}{2}$ give an algorithm for computing explicit tautological boundary formulae in the Chow ring for any tautological class on $\overline{\mathcal{M}}_{g,n}$ of codimension at least g. In particular, the intersection numbers of ψ-classes on $\overline{\mathcal{M}}_{g,n}$ can be computed with these relations for any $g \geq 0$ and $n \geq 1$ such that $2g - 2 + n > 0$.*

Remark 2.5.8. The first part of the statement is very similar to [3, Theorem 5], which gave a reduction algorithm based on Pixton's double ramification cycle. It confirms an expectation on [1, Page 7], that "(...)Pixton's relations are expected to uniquely determine the descendent theory, but the implication is not yet proven."

Note that the intersection numbers in ψ- and κ-classes can be expressed as intersection numbers of only ψ-classes by pulling back along forgetful maps, see [31, Corollary 3.23]. By the proposition, all these intersection numbers can then be computed using the PPZ relations.

Proof. The first sentence follows by the comment above the proposition. For the second sentence, we will reduce polynomials in ψ-classes to smaller and smaller boundary strata using our explicit relation. This will be done in the form of an induction on $\dim \overline{\mathcal{M}}_{g,n} = 3g - 3 + n$, the zero-dimensional case $\overline{\mathcal{M}}_{0,3}$ being obvious.

For any $g_1 + g_2 = g$ and $I_1 \sqcup I_2 = \{1, \ldots, n\}$ such that $2g_i + |I_i| - 1 > 0$, write $\rho^{g_1,g_2}_{I_1,I_2} : \overline{\mathcal{M}}_{g_1,|I_1|+1} \times \overline{\mathcal{M}}_{g_2,|I_2|+1} \to \overline{\mathcal{M}}_{g,n}$ for the attaching map, and $D^{g_1,g_2}_{I_1,I_2}$ for the divisor $(\rho^{g_1,g_2}_{I_1,I_2})_*(1)$. Similarly, write $\sigma : \overline{\mathcal{M}}_{g-1,n+2} \to \overline{\mathcal{M}}_{g,n}$ for the glueing map, and δ_{irr} for $\sigma_*(1)$. Then these divisors together form the entire boundary of $\overline{\mathcal{M}}_{g,n}$, and $\rho^*(\psi_i) = \psi_i$ and $\sigma^*(\psi_i) = \psi_i$ for any choice of indices.

Now let g and n be such that $3g - 3 + n > 0$, and choose a polynomial $p(\psi) \in R^{3g-3+n}(\overline{\mathcal{M}}_{g,n})$. Using stability, $3g - 3 + n > g - 1$, so by lemma 2.5.2, this class is zero on $\mathcal{M}_{g,n}$. Since the proof only uses relations without κ-classes, it can be given explicitly as a sum of the boundary divisors given above multiplied with other ψ-polynomials. By the projection formula,

$$\int_{\overline{\mathcal{M}}_{g,n}} \prod_{i=1}^{n} \psi_i^{d_i} D^{g_1,g_2}_{I_1,I_2} (\psi')^{d'} (\psi'')^{d''} = \int_{\overline{\mathcal{M}}_{g_1,|I_1|+1}} \Big(\prod_{i \in I_1} \psi_i^{d_i} \Big) \psi_{n+1}^{d'} \cdot \int_{\overline{\mathcal{M}}_{g_2,|I_2|+1}} \Big(\prod_{i \in I_2} \psi_i^{d_i} \Big) \psi_{n+2}^{d''};$$

$$\int_{\overline{\mathcal{M}}_{g,n}} \prod_{i=1}^{n} \psi_i^{d_i} \delta_{\text{irr}} (\psi')^{d'} (\psi'')^{d''} = \int_{\overline{\mathcal{M}}_{g-1,n+2}} \Big(\prod_{i=1}^{n} \psi_i^{d_i} \Big) \psi_{n+1}^{d'} \psi_{n+2}^{d''},$$

where ψ' and ψ'' are the classes on the half-edges of the unique edge in the dual graphs of the divisors.

All spaces on the right-hand side have a strictly lower dimension, so by induction we can compute those numbers via the PPZ relations. ☐

According to [3, Subsection 3.5], proposition 2.5.7 implies the following theorem.

Corollary 2.5.9 (Theorem ⋆ [14], improved in [7])**.** *Any codimension d tautological class can be expressed in terms of tautological classes supported on curves with at least $d - g + 1$ rational components.*

2.6 Dimensional bound for $R^{\leq g-2}(\mathcal{M}_{g,n})$

Similarly to [24, theorem 6], our method also gives a bound for the dimension of the lower degree tautological classes. For the statement of this proposition, recall that $p(n)$ denotes the number of partitions of n, and $p(n, k)$ denotes the number of partitions of n of length at most k.

Proposition 2.6.1.

$$\dim R^d(\mathcal{M}_{g,n}) \leq \sum_{k=0}^{d} \binom{n+k-1}{k} p(d-k, g-1-d)$$

Remark 2.6.2. If we use the natural interpretation of $\binom{k-1}{k}$ as $\delta_{k,0}$, this does indeed recover [24, theorem 6] in the case $n = 0$.

Proof. We will exhibit an explicit spanning set of this cardinality, consisting of ψ-κ-classes: monomials in ψ-classes multiplied with a multi-index κ-class.

First, a less strict first bound can be obtained as follows: any ψ-κ-class has a definite degree in ψ's, say k. There are $\binom{n+k-1}{k}$ different monomials of degree k in n variables, and furthermore there are as many different multi-index κ-classes of degree $d-k$ as there are partitions of $d-k$, so $p(d-k)$. This gives the first bound

$$\dim R^d(\mathcal{M}_{g,n}) \leq \sum_{k=0}^{d} \binom{n+k-1}{k} p(d-k),$$

which is already close to the statement of the proposition.

To get the actual bound, we will show that any ψ-κ-class with at least $g-d$ κ-indices can be expressed in ψ-κ-classes with strictly fewer κ-indices. Following the logic of the previous paragraph, this proves the bound.

This reduction step is analogous to the proof of lemma 2.5.5. Suppose we have a class $\mu = \psi_1^{d_1} \cdots \psi_n^{d_n} \kappa_{e_1,\cdots e_m}$ with $m \geq g-d$. Choose non-negative integers $\{f_i, a_i\}_{i=1}^{n+m}$ such that the following hold:

$$\begin{aligned} f_1 &= 0; \\ \sum_{i=1}^{n+m} f_i &= d-g+m; \\ a_i + f_i &= d_i, && \text{for } 2 \leq i \leq n; \\ a_{n+j} + f_{n+j} &= e_j + 1, && \text{for } 1 \leq j \leq m; \\ a_1 &= 2g-1-\sum_{j=2}^{m} a_j. \end{aligned}$$

Let $\sigma = \prod_{i=2}^{n+m} \psi_i^{f_i}$, and consider the class

$$\pi_*^m\left(\sigma \cdot \Omega_{g,n+m}^g(a_1, \dots, a_{n+m})\right)\Big|_{\mathcal{M}_{g,n}}.$$

By the second condition on our chosen numbers, which fixes the degree of σ, this expression gives a relation in $R^d(\mathcal{M}_{g,n})$.

There are no ψ-κ-classes with more than m κ-indices in this relation, and the coefficient of any ψ-κ-class with exactly m indices can only come from the open part of $\overline{\mathcal{M}}_{g,n+m}$, as each forgotten point must carry at least two ψ-classes, which would give too high degrees on any rational component.

Therefore, the coefficient of $\psi_1^{p_1}\cdots\psi_n^{p_n}\kappa_{q_1,\dots,q_m}$ must be $\prod_{i=1}^n Q_{p_i-f_i}(a_i)\cdot\prod_{j=1}^m Q_{q_j-f_{n+j}+1}(a_{n+j})$. This is only non-zero if $p_i \leq f_i + a_i = d_i$ for all $i \neq 1$ and $q_j \leq f_{n+j} + a_n + j - 1 = e_j$ for all j. This implies that $p_1 \geq d_1$, with equality only if $p_i = d_i$ and $q_j = e_j$ for all i, j. Hence, this relation expresses the class μ as a linear combination of ψ-κ-classes with less than m κ-indices and ψ-κ-classes with strictly higher exponent of ψ_1. By induction on first the exponent of ψ_1 and then the number of κ-indices, all these classes can be reduced. □

Remark 2.6.3. This argument breaks down for $m < g - d$, as the class σ would have to have a negative degree: our class only vanishes in degree at least g, and to get at most m-index κ-classes, we can only push forward m times, so the lowest degree relation would be in R^{g-m}.

The condition that partitions have length at most $g - 1 - d$ seems dual to Graber and Vakil's Theorem $\star$, corollary 2.5.9, see [14, theorem 1.1].

Bibliography

[1] Alexandr Buryak, Felix Janda, and Rahul Pandharipande. The hypergeometric functions of the Faber-Zagier and Pixton relations. *ArXiv e-prints*, February 2015.

[2] Alexandr Buryak, Sergey Shadrin, and Dimitri Zvonkine. Top tautological group of $\mathcal{M}_{g,n}$. *J. Eur. Math. Soc. (JEMS)*, 18(12):2925–2951, 2016. ISSN 1435-9855. doi: 10.4171/JEMS/657. URL `http://dx.doi.org/10.4171/JEMS/657`.

[3] Emily Clader, Samuel Grushevsky, Felix Janda, and Dimitry Zakharov. Powers of the theta divisor and relations in the tautological ring. *ArXiv e-prints*, May 2016.

[4] Petr Dunin-Barkowski, Sergey Shadrin, and Loek Spitz. Givental graphs and inversion symmetry. *Lett. Math. Phys.*, 103(5):533–557, 2013. ISSN 0377-9017. doi: 10.1007/s11005-013-0606-9. URL `http://dx.doi.org/10.1007/s11005-013-0606-9`.

[5] Petr Dunin-Barkowski, Nicolas Orantin, Sergey Shadrin, and Loek Spitz. Identification of the Givental formula with the spectral curve topological recursion procedure. *Comm. Math. Phys.*, 328(2):669–700, 2014. ISSN 0010-3616. doi: 10.1007/s00220-014-1887-2. URL `http://dx.doi.org/10.1007/s00220-014-1887-2`.

[6] Petr Dunin-Barkowski, Paul Norbury, Nicolas Orantin, Alexandr Popolitov, and Sergey Shadrin. Dubrovin's superpotential as a global spectral curve. *ArXiv e-prints*, September 2015.

[7] C. Faber and R. Pandharipande. Relative maps and tautological classes. *J. Eur. Math. Soc. (JEMS)*, 7(1): 13–49, 2005. ISSN 1435-9855; 1435-9863/e. doi: 10.4171/JEMS/20.

[8] Carel Faber. A conjectural description of the tautological ring of the moduli space of curves. In *Moduli of curves and abelian varieties*, Aspects Math., E33, pages 109–129. Friedr. Vieweg, Braunschweig, 1999.

[9] Carel Faber, Sergey Shadrin, and Dimitri Zvonkine. Tautological relations and the r-spin Witten conjecture. *Ann. Sci. Éc. Norm. Supér. (4)*, 43(4):621–658, 2010. ISSN 0012-9593.

[10] E. Getzler. Topological recursion relations in genus 2. In *Integrable systems and algebraic geometry. Proceedings of the 41st Taniguchi symposium, Kobe, Japan, June 30–July 4, 1997, and in Kyoto, Japan, July 7–11 1997*, pages 73–106. Singapore: World Scientific, 1998. ISBN 981-02-3266-7.

[11] Alexander Givental. A_{n-1} singularities and nKdV hierarchies. *Mosc. Math. J.*, 3(2):475–505, 743, 2003. ISSN 1609-3321. Dedicated to Vladimir I. Arnold on the occasion of his 65th birthday.

[12] Alexander B. Givental. Semisimple Frobenius structures at higher genus. *Internat. Math. Res. Notices*, (23): 1265–1286, 2001. ISSN 1073-7928. doi: 10.1155/S1073792801000605. URL `http://dx.doi.org/10.1155/S1073792801000605`.

[13] Alexander B. Givental. Gromov-Witten invariants and quantization of quadratic hamiltonians. *Mosc. Math. J.*, 1(4):1–17, 2001. URL `http://arxiv.org/abs/math/0108100`.

[14] Tom Graber and Ravi Vakil. Relative virtual localization and vanishing of tautological classes of moduli spaces of curves. *Duke Math. J.*, 130(1):1–37, 2005. ISSN 0012-7094; 1547-7398/e. doi: 10.1215/S0012-7094-05-13011-3.

[15] Eleny-Nicoleta Ionel. Topological recursive relations in $H^{2g}(\mathcal{M}_{g,n})$. *Invent. Math.*, 148(3):627–658, 2002. ISSN 0020-9910. doi: 10.1007/s002220100205. URL `http://dx.doi.org/10.1007/s002220100205`.

[16] Felix Janda. Comparing tautological relations from the equivariant Gromov-Witten theory of projective spaces and spin structures. *ArXiv e-prints*, July 2014.

[17] Felix Janda. Relations in the Tautological Ring and Frobenius Manifolds near the Discriminant. *ArXiv e-prints*, May 2015.

[18] Felix Janda. Relations on $\overline{\mathcal{M}}_{g,n}$ via equivariant Gromov-Witten theory of $\mathbb{P}^1$. *ArXiv e-prints*, September 2015.

[19] Eduard Looijenga. On the tautological ring of $\mathcal{M}_g$. *Invent. Math.*, 121(2):411–419, 1995. ISSN 0020-9910. doi: 10.1007/BF01884306. URL `http://dx.doi.org/10.1007/BF01884306`.

[20] David Mumford. Towards an enumerative geometry of the moduli space of curves. In *Arithmetic and geometry, Vol. II*, volume 36 of *Progr. Math.*, pages 271–328. Birkhäuser Boston, Boston, MA, 1983.

[21] R. Pandharipande. Three questions in Gromov-Witten theory. In *Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002)*, pages 503–512. Higher Ed. Press, Beijing, 2002.

[22] Rahul Pandharipande. A calculus for the moduli space of curves. *ArXiv e-prints*, March 2016.

[23] Rahul Pandharipande, Aaron Pixton, and Dimitri Zvonkine. Relations on $\overline{\mathcal{M}}_{g,n}$ via 3-spin structures. *J. Amer. Math. Soc.*, 28(1):279–309, 2015. ISSN 0894-0347. doi: 10.1090/S0894-0347-2014-00808-0. URL `http://dx.doi.org/10.1090/S0894-0347-2014-00808-0`.

[24] Rahul Pandharipande, Aaron Pixton, and Dimitri Zvonkine. Tautological relations via r-spin structures. *ArXiv e-prints*, July 2016.

[25] Alexander Polishchuk and Arkady Vaintrob. Algebraic construction of Witten's top Chern class. In *Advances in algebraic geometry motivated by physics (Lowell, MA, 2000)*, volume 276 of *Contemp. Math.*, pages 229–249. Amer. Math. Soc., Providence, RI, 2001. doi: 10.1090/conm/276/04523. URL `http://dx.doi.org/10.1090/conm/276/04523`.

[26] Sergey Shadrin. BCOV theory via Givental group action on cohomological fields theories. *Mosc. Math. J.*, 9 (2):411–429, back matter, 2009. ISSN 1609-3321.

[27] Mehdi Tavakol. The moduli space of curves and its invariants. *ArXiv e-prints*, October 2016.

[28] Constantin Teleman. The structure of 2D semi-simple field theories. *Invent. Math.*, 188(3):525–588, 2012. ISSN 0020-9910. doi: 10.1007/s00222-011-0352-5. URL `http://dx.doi.org/10.1007/s00222-011-0352-5`.

[29] Ravi Vakil. The moduli space of curves and its tautological ring. *Notices Amer. Math. Soc.*, 50(6):647–658, 2003. ISSN 0002-9920.

[30] Edward Witten. Algebraic geometry associated with matrix models of two-dimensional gravity. In *Topological methods in modern mathematics (Stony Brook, NY, 1991)*, pages 235–269. Publish or Perish, Houston, TX, 1993.

[31] Dimitri Zvonkine. An introduction to moduli spaces of curves and their intersection theory. In *Handbook of Teichmüller theory. Volume III*, volume 17 of *IRMA Lect. Math. Theor. Phys.*, pages 667–716. Eur. Math. Soc., Zürich, 2012. doi: 10.4171/103-1/12. URL `http://dx.doi.org/10.4171/103-1/12`.

3

Chiodo formulas for the r-th roots and topological recursion

In this chapter we analyze Chiodo's formulas for the Chern classes related to the r-th roots of the suitably twisted integer powers of the canonical class on the moduli space of curves. The intersection numbers of these classes with ψ-classes are reproduced via the Chekhov-Eynard-Orantin topological recursion.

As an application, we prove that the Johnson-Pandharipande-Tseng formula for the orbifold Hurwitz numbers is equivalent to the topological recursion for the orbifold Hurwitz numbers. In particular, this gives a new proof of the topological recursion for the orbifold Hurwitz numbers.

3.1 Introduction

3.1.1 Topological recursion

The topological recursion in the sense of Chekhov, Eynard, and Orantin (see, e.g., [17]) takes as an input a spectral curve (Σ, x, y, B), i.e., the data of a Riemann surface Σ, two functions x and y on Σ with some compatibility condition, and the choice of a bi-differential B on the surface (which is canonical in the case $\Sigma = \mathbb{C}\mathrm{P}^1$, so we will omit it in this case). The recursion produces a collection of symmetric n-differentials $\mathcal{W}_{g,n}$ (called correlation differentials) defined again on the surface whose expansion can generate solutions to enumerative geometric problems.

In particular, under some conditions the expansion of $\mathcal{W}_{g,n}$ are related to the correlators of semi-simple cohomological field theories [11].

3.1.2 Chiodo's formula

In [21], Mumford derived a formula for the Chern character of the Hodge bundle on the moduli space of curves $\overline{\mathcal{M}}_{g,n}$ in terms of the tautological classes and Bernoulli numbers. In [5], Chiodo generalizes Mumford's formula. The moduli stack $\overline{\mathcal{M}}_{g,n}$ is substituted with $\overline{\mathcal{M}}^{r,s}_{g;a_1,\dots,a_n}$, the proper moduli stack of rth roots of the line bundle

$$\omega_{\log}^{\otimes s}\left(-\sum_{i=1}^{n} a_i x_i\right)$$

where $\omega_{\log} = \omega(\sum x_i)$, the integers s, $a_1, \ldots, a_n$ satisfy

$$(2g-2+n)s - \sum_i a_i \in r\mathbb{Z},$$

and the x_i's are the marked points on the curves. Let $\pi\colon \mathcal{C} \to \overline{\mathcal{M}}^r_{g,n}$ be the universal curve and denote by $\mathcal{S} \to \mathcal{C}$ the universal r-th root. Chiodo's formula computes the Chern character $\mathrm{ch}(R^\bullet \pi_* \mathcal{S})$, again in terms of tautological classes and values of Bernoulli polynomials at rational points with denominator r. The push-forward of the corresponding Chern class to the moduli space of curves will be called the *Chiodo class.*

In one particular case we know a relation between the Chiodo classes and the topological recursion. Namely, the coefficients of some expansion of the differentials $\mathcal{W}_{g,n}$ for the spectral curve data $(\Sigma = \mathbb{C}\mathrm{P}^1, x = \log z - z^r, y = z)$ are expressed in terms of the intersection numbers of the Chiodo classes for $s=1$, $r = 1, 2, \ldots$. The main result of this chapter is an extension of this correspondence to arbitrary $s \geq 0$.

3.1.3 Chiodo classes and topological recursion

We consider the spectral curve

$$(\Sigma = \mathbb{C}\mathrm{P}^1, x(z) = -z^r + \log z, y(z) = z^s). \tag{3.1}$$

We prove that (see Theorem 3.4.6)

the expansion of the corresponding correlation differentials in some auxiliary basis of 1-forms is given by the intersection numbers of the corresponding Chiodo class for these particular $r, s \geq 1$.

The case $s = 0$ is exceptional. In this case, the intersection numbers are the same as in the case $s = r$, so we still have to use the spectral curve $(\Sigma = \mathbb{C}\mathrm{P}^1, x(z) = -z^r + \log z, y(z) = z^r)$.

These spectral curves are known in the literature, in some particular cases, in relation to various versions of Hurwitz numbers.

3.1.4 Hurwitz numbers

Hurwitz numbers play an important role in the interaction of combinatorics, representation theory of symmetric groups, integrable systems, tropical geometry, matrix models, and intersection theory on moduli spaces of curves.

There are several kinds of Hurwitz numbers. Simple Hurwitz numbers enumerate finite degree d coverings of the 2-sphere by a genus g connected surface, with a fixed ramification profile $(\mu_1, \ldots, \mu_n)$ over infinity, $\sum_{i=1}^n \mu_i = d$ while the remaining $2g-2+n+d$ ramifications over fixed points are simple.

These Hurwitz numbers are known to be the coefficients of the expansions of the correlation forms of the spectral curve (3.1) for $r = s = 1$. This was conjectured in [3] and proved in several different ways, see, e.g., [16, 9].

Chiodo's formula in this case is reduced to the standard Mumford formula, so the Chiodo class is the Chern class of the dual Hodge bundle on the moduli space of curves. The fact that the same correlation differentials are related, in different expansion, to simple Hurwitz numbers and to the intersection numbers, implies that there is a formula for simple Hurwitz numbers in terms of the intersection numbers. Indeed, it is the celebrated ELSV formula [13]. The equivalence between the topological recursion and the ELSV formula is proved in [15], see also [9, 23].

Another example is r-spin Hurwitz numbers. In this case, the definition is a bit involved; roughly speaking, we still consider the maps of genus g algebraic curves to $\mathbb{C}\mathrm{P}^1$, with a fixed profile over infinity, but the remaining simple ramifications are replaced by more complicated singularities, so-called completed cycles. We refer to [24, 23] for the precise definition.

In this case, the r-spin Hurwitz numbers are conjecturally related by the spectral curve (3.1) for that particular r and $s = 1$, see [20, 23]. The same logic as for the simple Hurwitz numbers implies that this conjecture is equivalent to an ELSV-type formula that expresses the r-spin Hurwitz numbers in terms of intersection numbers [23]. The corresponding ELSV-type formula was conjectured in [25] and is still open.

3.1.5 Orbifold Hurwitz numbers

A case of special interest for us is the r-orbifold Hurwitz numbers. They enumerate finite degree d, $r|d$, coverings of the 2-sphere by a genus g connected surface, with a fixed ramification profile $(\mu_1, \ldots, \mu_n)$ over the infinity, $\sum_{i=1}^{n} \mu_i = d$, the fixed ramification profile $(r, r, \ldots, r)$ over zero, while the remaining $2g - 2 + n + d/r$ ramifications over fixed points are simple.

It is proved in [2, 8] that the r-orbifold Hurwitz numbers satisfy the topological recursion for the spectral curve (3.1) with this particular r and $s = r$. Johnson-Pandharipande-Tseng [19] exhibited an ELSV-type formula that can be restricted to express r-orbifold Hurwitz numbers in terms of intersection numbers. As an application of the general correspondence between the Chiodo formulas and topological recursion, we prove the equivalence of these two statements (see Theorem 5.1).

Since the Johnson-Pandharipande-Tseng formula (the JPT formula, for brevity) is proved independently, our equivalence result implies a proof of the topological recursion of r-orbifold Hurwitz numbers.

It is a new proof of the topological recursion; the existing proofs [2, 8] do use the JPT formula, but only its combinatorial structure, and not the geometry of the classes. The topological recursion is then derived in [2, 8] from an additional recursion relation for r-orbifold Hurwitz numbers called cut-and-join equation.

3.1.6 Further remarks

A natural question is whether we can use the equivalence between the topological recursion and the JPT formula for r-orbifold Hurwitz numbers in order to give a new proof of the JPT formula, as it is done in [9] for the simple Hurwitz numbers. This approach requires a new proof of the topological recursion that wouldn't use the JPT formula. This is done in [10], so we refer there for further details.

Another natural question is whether there is any natural combinatorial and/or geometric problem of Hurwitz type related to the other spectral curves (3.1) for arbitrary r and s. The only indication of a possible relation that we know is that similar spectral curves are used in [20] for the so-called mixed Hurwitz numbers in the context of the quantum spectral curve theory.

3.1.7 Plan of the chapter

In Section 2 we review the semi-simple cohomological field theories, possibly with a non-flat unit, that correspond to Chiodo classes. In Section 3 we recall the general formula of the differentials $\mathcal{W}_{g,n}$ in terms of integrals over moduli spaces of curves as described in [11, 14], while in Section 4 we compute explicitly all the ingredients of that formula and prove our main theorem, Theorem 3.4.6. Finally, in Section 5 we identify the particular Chiodo class with the one used in the JPT formula and prove the equivalence of the JPT formula and the topological recursion for r-orbifold Hurwitz numbers.

3.2 Chiodo classes

In this Section we recall the definition and some simple properties of the Chiodo classes. These classes are defined on the moduli spaces of tensor rth roots of the line bundle $\omega_{\log}^{\otimes s}(-\sum m_i x_i)$, but in this paper we will only need their push-forward to the space of curves $\overline{\mathcal{M}}_{g,n}$. A more detailed discussion of the space of rth roots in the case $s = 0$ is contained in Section 3.5.2. We also refer the reader to [5, 7, 6, 23] for all necessary background and origin of the lemmas in this section.

3.2.1 Definition

Let $r \geq 1$ be an integer and $1 \leq a_1, \ldots, a_n \leq r$, $0 \leq s$ be integers satisfying

$$(2g-2+n)s - \sum_{i=1}^{n} a_i \in r\mathbb{Z} \tag{3.2}$$

Consider the morphisms

$$\overline{\mathcal{C}} \xrightarrow{\pi} \overline{\mathcal{M}}^{r,s}_{g;a_1,\ldots,a_n} \xrightarrow{\epsilon} \overline{\mathcal{M}}_{g,n},$$

where $\overline{\mathcal{M}}^{r,s}_{g;a_1,\ldots,a_n}$ is the space of rth roots $\mathcal{S}^{\otimes r} \xrightarrow{\sim} \omega_{\log}^{\otimes s}(-\sum a_i x_i)$, $\overline{\mathcal{C}}$ is its universal curve, and ϵ is the forgetful morphism to the space of curves. While the boundary strata of $\overline{\mathcal{M}}_{g,n}$ are described by stable graphs, those of $\overline{\mathcal{M}}^{r,s}_{g;a_1,\ldots,a_n}$ are described by stable graphs with a remainder mod r assigned to each half-edge in such a way that the sum of residues on each edge vanishes and that Condition (3.2) is satisfied for each vertex. The boundary divisors correspond to one-edged graphs with two opposite remainders mod r assigned the two half-edges.

The Chern characters of the derived push-forward $R^*\pi_*\mathcal{S}$ are given by Chiodo's formula [5]

$$\begin{aligned}\mathrm{ch}_m(R^*\pi_*\mathcal{S}) &= \frac{B_{m+1}(\frac{s}{r})}{(m+1)!}\kappa_m - \sum_{i=1}^{n} \frac{B_{m+1}(\frac{a_i}{r})}{(m+1)!}\psi_i^m \\ &+ \frac{r}{2}\sum_{a=0}^{r-1} \frac{B_{m+1}(\frac{a}{r})}{(m+1)!}(j_a)_* \frac{(\psi')^m + (-1)^{m-1}(\psi'')^m}{\psi'+\psi''},\end{aligned}$$

where j_a is the boundary map corresponding to the boundary divisor with remainder a at one of the two half-edges and ψ', ψ'' are the ψ-classes at the two branches of the node.

We are interested in the Chiodo classes

$$\begin{aligned}
&\mathrm{C}_{g,n}(r,s;a_1,\ldots,a_n) = \\
&\epsilon_* c(-R^*\pi_*\mathcal{S}) = \\
&\epsilon_* \left[c(R^1\pi_*\mathcal{S})/c(R^0\pi_*\mathcal{S})\right] = \\
&\epsilon_* \exp\left(\sum_{m=1}^{\infty}(-1)^m(m-1)!\mathrm{ch}_m(R^*\pi_*\mathcal{S})\right) \in H^{\mathrm{even}}(\overline{\mathcal{M}}_{g,n}).
\end{aligned} \tag{3.3}$$

An explicit expression of the classes $\mathrm{C}_{g,n}(r,s;a_1,\ldots,a_n)$ in terms of stable graphs, obtained by expanding the exponential in the expression above, is given in [18], Corollary 4.

Consider $\mathrm{C}_{g,n}(r,s;a_1,\ldots,a_n)$ as a coefficient of a map

$$\mathrm{C}_{g,n}(r,s)\colon V^{\otimes n} \to H^{\mathrm{even}}(\overline{\mathcal{M}}_{g,n}),$$

where $V = \langle v_1,\ldots,v_r\rangle$, and

$$\mathrm{C}_{g,n}(r,s)\colon v_{a_1}\otimes\cdots\otimes v_{a_n} \mapsto \mathrm{C}_{g,n}(r,s;a_1,\ldots,a_n).$$

3.2.2 Cohomological field theories

Lemma 3.2.1. *For $0 \le s \le r$ the classes $\{\mathrm{C}_{g,n}(r,s)\}$ form a semi-simple cohomological field theory.*

A semi-simple cohomological field theory (CohFT) is obtained via the action of an element of the upper-triangular Givental group on a topological field theory. In order to determine a topological field theory $\{\omega_{g,n}\}$, we have to fix its scalar product η and $\omega_{0,3}$. An element of the upper-triangular Givental group is determined by a matrix $R(\zeta) \in \mathrm{End}(V)[[\zeta]]$ that should satisfy the symplectic conditions with respect to η.

In the case of $\{\mathrm{C}_{g,n}(r,s)\}$ we have the following description.

Lemma 3.2.2. *For $0 \le s \le r$ the classes $\{\mathrm{C}_{g,n}(r,s)\}$ are given by Givental's action of the R-matrix $R(\zeta)$ on the topological field theory ω with metric η on V, where*

$$\begin{aligned}
&V = \langle v_1,\ldots,v_r\rangle, \\
&R(\zeta) = \exp\left(\sum_{m=1}^{\infty}\frac{\mathrm{diag}_{a=0}^{r-1}B_{m+1}\left(\frac{a}{r}\right)}{m(m+1)}(-\zeta)^m\right), \\
&R^{-1}(\zeta) = \exp\left(-\sum_{m=1}^{\infty}\frac{\mathrm{diag}_{a=0}^{r-1}B_{m+1}\left(\frac{a}{r}\right)}{m(m+1)}(-\zeta)^m\right), \\
&\eta(v_a,v_b) = \frac{1}{r}\delta_{a+b \mod r}, \\
&\omega_{0,3}(v_a\otimes v_b\otimes v_c) = \frac{1}{r}\delta_{a+b+c-s \mod r}, \\
&\omega_{g,n}(v_{a_1}\otimes\cdots\otimes v_{a_n}) = r^{2g-1}\delta_{a_1+\cdots+a_n-s(2g-2+n) \mod r}.
\end{aligned}$$

3.2.3 Cohomological field theories with a non-flat unit

Let us discuss now what happens for $s > r$. We need an extension of the notion of cohomological field theory, namely, we have to consider the cohomological field theories with a non-flat unit, CohFT/1 for brevity.

The CohFT/1s are obtained by an extension of the Givental group by translations, which allows one to use the dilaton leaves (in the terminology of [12, 11]) or κ-legs (in the terminology of [22]) with arbitrary coefficients. We refer to the exposition in [22] for further details.

One of the possible descriptions of a CohFT/1 is in terms of stable graphs without any κ-legs. The vertices, leaves, and edges of these graphs are decorated in exactly the same way as in the case of a usual CohFT, but in addition every vertex is also decorated by $\exp(\sum_{m=1}^{\infty} T_m \kappa_m)$ for some constants T_m, $m = 1, 2, \dots$.

In the case of Chiodo classes (3.3) for $s > r$, we have the following:

Lemma 3.2.3. *For $s > r$ the classes $\{\mathrm{C}_{g,n}(r,s)\}$ form a CohFT/1. The corresponding element of the extended Givental group coincides with the one described in Lemma 3.2.2, but instead of the dilaton shift, we decorate each vertex by*

$$\exp\left(\sum_{m=1}^{\infty} (-1)^m \frac{B_{m+1}(\frac{s}{r})}{m(m+1)} \kappa_m\right).$$

3.3 Topological recursion and Givental group

In this Section we revisit the main result of [11, 14]. We present a version a bit refined of it, in order to make precise relation that incorporates a torus action on cohomological field theories.

3.3.1 General background

The input of the local topological recursion consists of a local spectral curve $\Sigma = \sqcup_{i=1}^r U_i$, which is a disjoint union of open disks with the center points p_i, $i = 1, \dots, r$, holomorphic function $x\colon \Sigma \to \mathbb{C}$ such that the zeros of its differential dx are $p_1, \dots, p_r$, holomorphic function $y\colon \Sigma \to \mathbb{C}$, and a symmetric bidifferential B defined on $\Sigma \times \Sigma$ with a double pole on the diagonal with residue 1.

The output is a set of symmetric differentials $\mathcal{W}_{g,n}$ on Σ^n. This set of differentials is canonically associated to the input data via the topological recursion procedure. Under some conditions (for example, when Σ is an open submanifold of a Riemann surface, where dx is a globally defined meromorphic differential, see [14], and we should assume some relation between y and B, see [11] and below), we can represent this set of differentials in terms of the correlators of a CohFT multiplied by some auxiliary differentials. This representation is not canonical, the choice of it is controlled by the action of the group $(\mathbb{C}^*)^r$.

Our goal is to make this action on all ingredients of the formula (that is, the matrix R of a CohFT, its underlying TFT, and the auxiliary differentials) precise.

3.3.2 The formula

We fix a point $(C_1, \dots, C_r) \in (\mathbb{C}^*)^r$. We also fix some additional constant $C \in \mathbb{C}^*$. All constructions in this Section depend on these choices.

We choose a local coordinate w_i on U_i, $i = 1, \dots, r$, such that $w_i(p_i) = 0$ and

$$x = (C_i w_i)^2 + x_i.$$

In this case, the underlying TFT is given by

$$\begin{aligned} \eta(e_i, e_j) &= \delta_{ij}, \\ \alpha^{Top}_{g,n}(e_{i_1} \otimes \cdots \otimes e_{i_n}) &= \delta_{i_1 \dots i_n} \left(-2C_i^2 C \frac{\mathrm{d}y}{\mathrm{d}w_i}(0) \right)^{-2g+2-n}. \end{aligned} \tag{3.4}$$

In particular, the unit vector is equal to $\sum_{i=1}^r \left(-2C_i^2 C \frac{\mathrm{d}y}{\mathrm{d}w_i}(0) \right) e_i$.

The matrix $R(\zeta)$ is given by

$$-\frac{1}{\zeta} R^{-1}(\zeta)^j_i = \frac{1}{\sqrt{2\pi\zeta}} \int_{-\infty}^{\infty} \left. \frac{B(w_i, w_j)}{\mathrm{d}w_i} \right|_{w_i=0} \cdot e^{-\frac{w_j^2}{2\zeta}}. \tag{3.5}$$

We have to check that the function y satisfies the condition

$$\frac{2C_i^2 C}{\sqrt{2\pi\zeta}} \int_{-\infty}^{\infty} \mathrm{d}y \cdot e^{-\frac{w_i^2}{2\zeta}} = \sum_{k=1}^r (R^{-1})^i_k \left(2C_k^2 C \frac{\mathrm{d}y}{\mathrm{d}w_k}(0) \right) \tag{3.6}$$

Finally, the auxiliary functions $\xi_i \colon \Sigma \to \mathbb{C}$ are given by

$$\xi_i(x) := \int^x \left. \frac{B(w_i, w)}{\mathrm{d}w_i} \right|_{w_i=0} \tag{3.7}$$

Using Formulas (3.4) and (3.5) we define a CohFT, whose classes we denote by $\alpha^{Coh}_{g,n}(e_{i_1} \otimes \cdots \otimes e_{i_n})$.

Theorem 3.3.1. *[14, 11] The differentials $\mathcal{W}_{g,n}$ produced by the topological recursion from the input (Σ, x, y, B) are equal to*

$$\begin{aligned} \mathcal{W}_{g,n} = C^{2g-2+n} \sum_{\substack{i_1,\dots,i_n \\ d_1,\dots,d_n}} \int_{\overline{\mathcal{M}}_{g,n}} \alpha^{Coh}_{g,n}(e_{i_1} \otimes \cdots \otimes e_{i_n}) \\ \prod_{j=1}^n \psi_j^{d_j} \mathrm{d} \left(\left(-\frac{1}{w_j} \frac{\mathrm{d}}{\mathrm{d}w_j} \right)^{d_j} \xi_{i_j} \right). \end{aligned} \tag{3.8}$$

In particular, this formula doesn't depend on the choice of $(C_1, \dots, C_r) \in (\mathbb{C}^*)^r$ and $C \in \mathbb{C}^*$, though all its ingredients do.

The proof of this theorem is given by exactly the same argument as in [14, 11], with a different choice of local coordinates near the points p_i, so we omit it here.

Remark 3.3.2. Let us discuss what happens if the condition (3.6) is not satisfied. Still, under the same conditions a version of Theorem 3.3.1 holds. Namely, we can represent the correlation differentials as

$$\mathcal{W}_{g,n} = C^{2g-2+n} \sum_{\substack{i_1,\dots,i_n \\ d_1,\dots,d_n}} \int_{\overline{\mathcal{M}}_{g,n}} \alpha^{Coh/1}_{g,n}(e_{i_1} \otimes \cdots \otimes e_{i_n}) \prod_{j=1}^{n} \psi_j^{d_j} \mathrm{d}\left(\left(-\frac{1}{w_j}\frac{\mathrm{d}}{\mathrm{d}w_j}\right)^{d_j} \xi_{i_j}\right),$$

where the classes $\alpha^{Coh/1}_{g,n}$ are described, in terms of the graphical formalism recalled in Section 3.2.3, via the same TFT and R-matrix as $\alpha^{Coh}_{g,n}$ in Theorem 3.3.1, but instead of the dilaton leaves, we decorate each vertex labeled by i (that is, the one that is decorated by $\alpha^{Top}_{g,n}(e_i \otimes \cdots \otimes e_i)$) with the κ-class

$$\exp\left(\sum_{k=1}^{\infty} T_{i,k}\kappa_k\right),$$

where the constants $T_{i,k}$ are given by

$$\frac{\mathrm{d}y}{\mathrm{d}w_i}(0) \exp\left(\sum_{k=1}^{\infty} T_{i,k}(-\zeta)^k\right) = \frac{1}{\sqrt{2\pi\zeta}} \int_{-\infty}^{\infty} \mathrm{d}y \cdot e^{-\frac{w_i^2}{2\zeta}}.$$

This is a direct corollary of [15, Theorem 3.2], see also [11, Lemma 3.5].

3.4 Computations with the spectral curve

Consider the following initial data on the spectral curve $\Sigma = \mathbb{C}\mathrm{P}^1$ with a global coordinate z:

$$\begin{aligned} x(z) &= -z^r + \log z; \\ y(z) &= z^s; \\ B(z,z') &= \frac{\mathrm{d}z\,\mathrm{d}z'}{(z-z')^2}. \end{aligned} \tag{3.9}$$

In this section we compute all ingredients of the Formula (3.8) for this initial data with a special choice of the torus point. In particular, for $1 \le s \le r$ we prove that the correlation differentials are controlled by a CohFT, and the corresponding CohFT coincides with the one given by Chiodo classes (3.3) considered in the normalized canonical frame.

3.4.1 Local expansions

As it was computed in [23], we can associate with this curve the following local data.

The critical points are

$$p_i := r^{-1/r}\mathrm{J}^i, \qquad i = 0,\dots,r-1,$$

and the critical values of the function x at these points are

$$x_i := x(p_i) = -\frac{1}{r} + \frac{2\pi i\sqrt{-1}}{r} - \frac{\log r}{r}, \qquad i = 0,\dots,r-1.$$

If we choose a local coordinate w_i near the point p_i such that $w_i(p_i) = 0$ and $-w_i^2/2r + x_i = x$, $i = 0, 1, \dots, r-1$, then there are two possible choices for the expansion of the function z in w_i. We fix it to be

$$z(w_i) = r^{-1/r}\mathrm{J}^i + \left(r^{-1-\frac{1}{r}}\mathrm{J}^i\right) w_i + O(w_i^2),$$

With this choice we also fix the expansion of $y = z^s$, namely,

$$y(w_i) = r^{-s/r}\mathrm{J}^{si} + \left(sr^{-1-\frac{s}{r}}\mathrm{J}^{is}\right) w_i + O(w_i^2).$$

Lemma 3.4.1. *We have:*

$$\frac{1}{\sqrt{2\pi\zeta}}\int_{-\infty}^{\infty} \mathrm{d}y(w_i)\cdot e^{-\frac{w_i^2}{2\zeta}} \sim \left(sr^{-1-\frac{s}{r}}\mathrm{J}^{is}\right)\exp\left(-\sum_{m=1}^{\infty}\frac{B_{m+1}\left(\frac{s}{r}\right)}{m(m+1)}(-\zeta)^m\right). \tag{3.10}$$

Proof. This Lemma is analogous to [23, Lemma 4.3]. Indeed, we introduce a new coordinate $t = rz^r$. In this coordinate we have:

$$\begin{aligned} z &= t^{\frac{1}{r}}r^{-\frac{1}{r}}\mathrm{J}^i; \\ -x_i - z^r + \log z &= \frac{1}{r} - \frac{t}{r} + \frac{\log t}{r}; \\ \mathrm{d}z &= t^{\frac{1-r}{r}}r^{-1-\frac{1}{r}}\mathrm{J}^i\mathrm{d}t. \end{aligned}$$

We can then make a change of variables and use the standard asymptotic expansion of the gamma function, cf. the proof of Lemma 4.3 in [23]:

$$\begin{aligned} &\frac{\sqrt{-2r}}{\sqrt{2\pi\zeta}}\int \mathrm{d}y\cdot e^{2r\cdot\frac{(x-x_i)}{2\zeta}} = \frac{sr^{-\frac{1}{2}-\frac{s}{r}}\mathrm{J}^{si}e^{\frac{1}{\zeta}}}{\sqrt{-\pi\zeta}}\int \mathrm{d}t\cdot t^{\frac{s-r}{r}+\frac{1}{\zeta}}e^{-\frac{t}{\zeta}} \\ &\sim \left(s\sqrt{-2}r^{-\frac{1}{2}-\frac{s}{r}}\mathrm{J}^{si}\right)\exp\left(-\sum_{m=1}^{\infty}\frac{B_{m+1}\left(\frac{s}{r}\right)}{m(m+1)}(-\zeta)^m\right). \end{aligned}$$

□

Lemma 3.4.2. *We have:*

$$\begin{aligned} &\frac{1}{\sqrt{2\pi\zeta}}\int_{-\infty}^{\infty}\left.\frac{B(w_i,w_j)}{\mathrm{d}w_i}\right|_{w_i=0}\cdot e^{-\frac{w_j^2}{2\zeta}} \\ &\sim \sum_{c=0}^{r-1}\frac{\mathrm{J}^{cj-ci}}{r}\frac{\exp\left(-\sum_{m=1}^{\infty}\frac{B_{m+1}\left(\frac{c}{r}\right)}{m(m+1)}(-\zeta)^m\right)}{(-\zeta)}. \end{aligned}$$

Proof. This Lemma is just a refined version of Lemma 4.4 in [23], so the proof is exactly the same as there. □

Note that this Lemma means that we have to consider the Givental group action defined by the matrix $R(\zeta)$, where

$$R^{-1}(\zeta)_i^j := \sum_{c=0}^{r-1}\frac{\mathrm{J}^{cj-ci}}{r}\exp\left(-\sum_{m=1}^{\infty}\frac{B_{m+1}\left(\frac{c}{r}\right)}{m(m+1)}(-\zeta)^m\right).$$

We choose the constants $C_1 = \cdots = C_r := 1/\sqrt{-2r}$ and $C := r^{1+s/r}/s$. In particular, with this choice the structure constants of the underlying TFT are given by

$$-2C_i^2 C \frac{\mathrm{d}y}{\mathrm{d}w_i}(0) = \frac{\mathrm{J}^{is}}{r} \tag{3.11}$$

Lemma 3.4.3. *For $1 \le s \le r$ the condition (3.6) is satisfied.*

Proof. This is a direct computation. We have:

$$\begin{aligned}
\frac{2C_i^2 C}{\sqrt{2\pi\zeta}} \int_{-\infty}^{\infty} \mathrm{d}y \cdot e^{-\frac{w_i^2}{2\zeta}} &= -\frac{\mathrm{J}^{is}}{r} \exp\left(-\sum_{m=1}^{\infty} \frac{B_{m+1}\left(\frac{s}{r}\right)}{m(m+1)} (-\zeta)^m \right) \\
&= \sum_{k=1}^{r} \sum_{c=0}^{r-1} \frac{\mathrm{J}^{ci-ck}}{r} \exp\left(-\sum_{m=1}^{\infty} \frac{B_{m+1}\left(\frac{c}{r}\right)}{m(m+1)} (-\zeta)^m \right) \left(-\frac{\mathrm{J}^{ks}}{r} \right) \\
&= \sum_{k=1}^{r} (R^{-1})^i_k \left(2C_k^2 C \frac{\mathrm{d}y}{\mathrm{d}w_k}(0) \right)
\end{aligned}$$

The second equality is true for $0 \le s \le r-1$, and also for $s = r$, since $B_{m+1}(1) = B_{m+1}(0)$ for $m \ge 1$. □

This Lemma implies that we indeed have correlators of a cohomological field theory inside Formula (3.8) in this case.

Finally, Definition (3.7) implies that

$$\xi_i = \frac{r^{-1-\frac{1}{r}} \mathrm{J}^i}{r^{-\frac{1}{r}} \mathrm{J}^i - z},$$

and it is easy to see that

$$-\frac{1}{w} \frac{\mathrm{d}}{\mathrm{d}w} = \frac{1}{r} \frac{\mathrm{d}}{\mathrm{d}x}. \tag{3.12}$$

This completes the description of all the ingredient of the Formula (3.8) for the correlation differentials $\mathcal{W}_{g,n}$.

3.4.2 Correlation differentials in flat basis

In the previous section we described all ingredients of the formula for the correlation differentials (3.8) for the case of the spectral curve data (3.9). In particular, for $1 \le s \le r$ we proved that there are the correlators of a CohFT inside this formula, otherwise we have a CohFT/1. Our goal now is to show that the cohomological field theories obtained in the previous Section is the one given by the same formulas as in Lemmas 3.2.2 and 3.2.3. In order to do that we apply a linear change of variables to the basis $e_0, \ldots, e_{r-1}$ used in the previous Section.

We use the change of basis from $e_0, \ldots, e_{r-1}$ to $v_1, \ldots, v_r$ given by the formula

$$e_i = \sum_{a=1}^{r} \mathrm{J}^{-ai} v_a; \qquad v_a = \sum_{i=0}^{r-1} \frac{\mathrm{J}^{ai}}{r} e_i$$

Lemma 3.4.4. *In the basis $v_1, \dots, v_r$ we have:*

- *The underlying TFT $\alpha^{Top}_{g,n}$ (3.4) with the choice of constants as in Equation (3.11) is given by*

$$\begin{aligned} \eta(v_a, v_b) &= \frac{1}{r}\delta_{a+b \mod r}; \\ \omega_{0,3}(v_a \otimes v_b \otimes v_c) &= \frac{1}{r}\delta_{a+b+c-s \mod r} \\ \omega_{g,n}(v_{a_1} \otimes \cdots \otimes v_{a_n}) &= r^{2g-1}\delta_{a_1+\cdots+a_n-s(2g-2+n) \mod r} \end{aligned} \tag{3.13}$$

- *The R-matrix is given by*

$$R(\zeta) = \exp\left(\sum_{m=1}^{\infty} \frac{\mathrm{diag}^r_{a=1} B_{m+1}\left(\frac{a}{r}\right)}{m(m+1)}(-\zeta)^m\right) \tag{3.14}$$

- *The auxiliary functions ξ_a are given by*

$$\xi_a = r^{\frac{r-a}{r}} \sum_{p=0}^{\infty} \frac{(pr+r-a)^p}{p!} e^{(pr+r-a)x}. \tag{3.15}$$

Proof. The computation of the underlying TFT is fairly simple:

$$\begin{aligned} \eta(v_a, v_b) &= \sum_{i,j=0}^{r-1} \frac{\mathrm{J}^{ai+bj}}{r^2}\eta(e_i, e_j) = \sum_{i=0}^{r-1} \frac{\mathrm{J}^{(a+b)i}}{r^2} = \frac{1}{r}\delta_{a+b \mod r}, \\ \omega_{0,3}(v_a \otimes v_b \otimes v_c) &= \sum_{i=0}^{r-1} \frac{\mathrm{J}^{ai+bi+ci}}{r^3}\omega_{0,3}(e_i \otimes e_i \otimes e_i) \\ &= \sum_{i=0}^{r-1} \frac{\mathrm{J}^{ai+bi+ci-si}}{r^2} = \frac{1}{2}\delta_{a+b+c-s \mod r}, \end{aligned}$$

and the other correlators of the underlying TFT are determined uniquely.

The change of basis for the matrix R^{-1} reads:

$$\begin{aligned} R^{-1}(\zeta)^b_a &= \sum_{i,j=0}^{r-1} \frac{\mathrm{J}^{-jb+ia}}{r} \sum_{c=0}^{r-1} \frac{\mathrm{J}^{cj-ci}}{r} \exp\left(-\sum_{m=1}^{\infty} \frac{B_{m+1}\left(\frac{c}{r}\right)}{m(m+1)}(-\zeta)^m\right) \\ &= \exp\left(-\sum_{m=1}^{\infty} \frac{B_{m+1}\left(\frac{c}{r}\right)}{m(m+1)}(-\zeta)^m\right) \cdot \delta_{c-b \mod r} \cdot \delta_{c-a \mod r} \\ &= \exp\left(-\sum_{m=1}^{\infty} \frac{B_{m+1}\left(\frac{a}{r}\right)}{m(m+1)}(-\zeta)^m\right) \cdot \delta_{a-b}, \end{aligned}$$

which implies Equation (3.14).

Finally, Equation (3.15) follows from Lemma 4.6 in [23]. □

Remark 3.4.5. Observe that Equations (3.13) and (3.14) and Lemma 3.4.3 imply that for $s \leq r$ the cohomological field theory that we have in the flat basis coincides with the one given in Lemma 3.2.2. For $s > r$, where Lemma 3.4.3 does not apply, we have obtained the topological field theory and the R-matrix as in Lemma 3.2.3, but we still have to compare the power series that determines the κ-legs.

Lemma 3.4.4 allows us to rewrite formula (3.8) for the correlation differentials of the spectral curve data (3.9) in the following way.

Theorem 3.4.6. *The correlation differentials of the spectral curve* (3.9) *are equal to*

$$\begin{aligned} \mathcal{W}_{g,n} = & \sum_{\mu_1,\ldots,\mu_n=1}^{\infty} \mathrm{d}_1 \otimes \cdots \otimes \mathrm{d}_n \; e^{\sum_{j=1}^n \mu_j x_j} \\ & \times \int_{\overline{\mathcal{M}}_{g,n}} \frac{\mathrm{C}_{g,n}\left(r,s;r - r\left\langle \frac{\mu_1}{r}\right\rangle, \ldots, r - r\left\langle \frac{\mu_n}{r}\right\rangle\right)}{\prod_{j=1}^n (1 - \frac{\mu_i}{r}\psi_i)} \\ & \times \prod_{j=1}^n \frac{\left(\frac{\mu_j}{r}\right)^{\left\lfloor \frac{\mu_j}{r}\right\rfloor}}{\left\lfloor \frac{\mu_j}{r}\right\rfloor !} \times \frac{r^{2g-2+n+\frac{(2g-2+n)s+\sum_{j=1}^n \mu_j}{r}}}{s^{2g-2+n}}, \end{aligned} \tag{3.16}$$

where $\frac{\mu}{r} = \lfloor \frac{\mu}{r} \rfloor + \langle \frac{\mu}{r} \rangle$ *is the decomposition into the integer and the fractional parts.*

Proof. First, consider the case $s \leq r$. Using Equation (3.8), together with Lemma 3.4.4, Remark 3.4.5, Equation (3.12) and $C = r^{1+s/r}/s$, we have:

$$\begin{aligned} & \mathcal{W}_{g,n}(x_1,\ldots,x_n) \\ & = \sum_{\substack{d_1,\ldots,d_n \geq 0 \\ 1 \leq a_1,\ldots,a_n \leq r}} \frac{r^{2g-2+n+\frac{(2g-2+n)s}{r}}}{s^{2g-2+n}} \int_{\overline{\mathcal{M}}_{g,n}} \mathrm{C}_{g,n}(r,s;a_1,\ldots,a_n) \\ & \times \prod_{j=1}^n \psi_j^{d_j} r^{-d_j} r^{\frac{r-a_j}{r}} \mathrm{d}\left[\left(\frac{\mathrm{d}}{\mathrm{d}x_j}\right)^{d_j} \sum_{p=0}^{\infty} \frac{(pr+r-a_j)^p}{p!} e^{(pr+r-a_j)x_j} \right] \\ & = \mathrm{d}_1 \otimes \cdots \otimes \mathrm{d}_n \sum_{\substack{d_1,\ldots,d_n \geq 0 \\ 1 \leq a_1,\ldots,a_n \leq r}} \int_{\overline{\mathcal{M}}_{g,n}} \mathrm{C}_{g,n}(r,s;a_1,\ldots,a_n) \prod_{j=1}^n \psi_j^{d_j} \\ & \times \frac{r^{2g-2+2n-\sum_{j=1}^n d_j + \frac{(2g-2+n)s - \sum_{j=1}^n a_j}{r}}}{s^{2g-2+n}} \\ & \times \prod_{j=1}^n \sum_{p=0}^{\infty} \frac{(pr+r-a_j)^{p+d_j}}{p!} e^{(pr+r-a_j)x_j}. \end{aligned}$$

Equation (3.16) is just a way to rewrite the last formula using a summation over the parameter $\mu_i = p_i r + r - a_i$ instead of a double summation over p_i and a_i.

In the case $s > r$, we should compute separately the κ-classes. In this case, Remark 3.3.2 and Equation (3.10) imply that the κ-class attached to the vertex of index i (in the basis $e_0, \ldots, e_{r-1}$) is equal to $\exp\left(\sum_{m=1}^{\infty} (-1)^m \frac{B_{m+1}(\frac{s}{r})}{m(m+1)} \kappa_m\right)$. Since it doesn't depend on i, it remains the same in the basis $v_1, \ldots, v_r$, where it coincides with the one given by Lemma 3.2.3. ☐

Remark 3.4.7. Note that in the case $s = 1$ we reproduce Theorem 1.7 in [23].

3.5 Johnson-Pandharipande-Tseng formula and topological recursion

In this Section we consider a special case of the correspondence between the Chiodo formulas and the spectral curve topological recursion. We assume that $s = r$. In this case, the correlation differentials of this spectral curve are known to give the so-called r-orbifold Hurwitz numbers in some expansion.

An r-orbifold Hurwitz number $h_{g;\vec{\mu}}$ is just a double Hurwitz number that enumerates ramified coverings of the sphere by a genus g surface, where one special fiber is arbitrary (given by the partition $\vec{\mu}$ of length n) and one has ramification indices $(r, r, \dots, r)$. Therefore, the degree of the covering $\sum_{i=1}^{n} \mu_i$ is divisible by r and there are $b = 2g - 2 + n + \sum_{i=1}^{n} \mu_i/r$ simple critical points.

The r-orbifold Hurwitz numbers are also known to satisfy the Johnson-Pandharipande-Tseng (JPT) formula that expresses them in terms of the intersection theory of the moduli space of curves. The main goal of this Section is to show that the JPT formula is equivalent to the topological recursion for r-orbifold Hurwitz numbers. In particular, this gives a new proof of the topological recursion for r-orbifold Hurwitz numbers.

3.5.1 The JPT formula

The formula of Johnson, Pandharipande and Tseng is presented in [19] for a general abelian group G, its particular finite representation U and a vector of monodromies γ. Here we consider only the case of $G = \mathbb{Z}/r\mathbb{Z}$, the representation U sends $1 \in \mathbb{Z}/r\mathbb{Z}$ to $e^{\frac{2\pi i}{r}}$, and γ is empty. In this case the JPT formula reads

$$\frac{h_{g;\vec{\mu}}}{b!} = r^{1-g+\sum\langle\frac{\mu_i}{r}\rangle} \prod_{i=1}^{n} \frac{\mu_i^{\lfloor\frac{\mu_i}{r}\rfloor}}{\lfloor\frac{\mu_i}{r}\rfloor!} \int_{\overline{\mathcal{M}}_{g,n}} \frac{\epsilon_* \sum_{i\geq 0}(-r)^i \lambda_i}{\prod_{j=1}^{n}(1-\mu_j\psi_j)}, \tag{3.17}$$

where the class $\epsilon_* \sum_{i\geq 0}(-r)^i\lambda_i$ is described in detail below.

3.5.2 Two descriptions of rth roots

Let $G = \mathbb{Z}/r\mathbb{Z}$ be the abelian group of rth roots of unity. The space $\overline{\mathcal{M}}_{g;a_1,\dots,a_n}(\mathcal{B}G)$ is the space of stable maps to the stack $\mathcal{B}G$ with monodromies $a_i \in \{0, \dots, r-1\}$ at the marked points. This space, and the natural cohomology classes on it, can be constructed in several ways, see, for instance, [1, 4]. Johnson, Pandharipande, and Tseng [19] use the description via admissible covers. Chiodo [5] uses the moduli space of rth roots of the line bundle $\mathcal{O}(-\sum a_i x_i)$. In our work we apply Chiodo's formulas to a result of Johnson, Pandharipande, and Tseng, so we recall and briefly explain the equivalence between the two approaches.

The r-stable curves.

An *r-stable curve* is an orbifold stable curve whose only nontrivial orbifold structure appears at the nodes and at the markings. The neighborhood of a marking is isomorphic to Δ/G, where an

rth root of unity $\rho \in G$ acts on the disc Δ by $z \mapsto \rho z$. The neighborhood of a node in a family of r-stable curves is isomorphic to $(\Delta \times \Delta)/G$, where $\rho \in G$ acts by $(z, w) \mapsto (\rho z, \rho^{-1} w)$.

The moduli space of r-stable curves has the same coarse space as $\overline{\mathcal{M}}_{g,n}$, but an extra factor of G appears in the stabilizer for every node of the curve.

Line bundles over r-stable curves.

A line bundle L over an r-stable curve has a particular structure at the neighborhoods of markings and nodes. At a marking it can be given by the chart $\Delta \times \mathbb{C}$ with the action of an element $\rho \in G$ given by $(z, s) \mapsto (\rho z, \rho^a s)$. Thus the number $a \in \{0, \ldots, r-1\}$ describes the local structure of L at a marking. At a node L can be given a by a chart $(\Delta \times \Delta) \times \mathbb{C}$ with the action of an element $\rho \in G$ given by $(z, w, s) \mapsto (\rho z, \rho^{-1} w, \rho^a s)$. Note, however, that the number a is replaced with $-a$ (mod r) if we exchange z and w. Thus the local structure of L at node is described by assigning to the branches of the node two numbers $a', a'' \in \{0, \ldots, r-1\}$ such that $a' + a'' = 0 \mod r$.

Roots of $\mathcal{O}$.

In [5] an element of $\overline{\mathcal{M}}_{g;a_1,\ldots,a_n}(\mathcal{B}G)$ is an r-stable curve $\mathcal{C}$ with an orbifold line bundle $L \to \mathcal{C}$ endowed with an identification $L^{\otimes r} \xrightarrow{\sim} \mathcal{O}$. The integers $a_i \in \{0, \ldots, r-1\}$ prescribe the structure of L at the markings.

From r-th roots to G-bundles.

To make the connection with the description of $\overline{\mathcal{M}}_{g;a_1,\ldots,a_n}(\mathcal{B}G)$ in [19] we look at the multi-section of L that maps to the section 1 of $\mathcal{O}$ when raised to the power r. This multi-section is a principal G-bundle $\pi : D \to C$ ramified over the markings and the nodes. At a marking with label a the G-bundle has the monodromy given by adding a in $\mathbb{Z}/r\mathbb{Z}$. This can be seen from the G-action $(z, s) \mapsto (\rho z, \rho^a s)$. If we choose $\rho = e^{2\pi i/r}$, a path from z to ρz in the chart corresponds to a loop around the marking in the stable curve and its lifting leads from s to $\rho^a s$ in the fiber of L.

Similarly, at the node the G-bundle has monodromies a' and a'' at the two branches, satisfying $a' + a'' = 0 \mod r$.

Note that, because D is formed by a multi-section of L, the pull-back of L to D has a tautological section. We will denote this section by ϕ_0.

From G-bundles to r-th roots.

In [19] an element of $\overline{\mathcal{M}}_{g;a_1,\ldots,a_n}(\mathcal{B}G)$ is G-cover $\pi : D \to C$ ramified over the markings and the nodes and satisfying the "kissing condition": the monodromies of the G-action over two branches of a node are opposite modulo r. The integers $a_i \in \{0, \ldots, r-1\}$ prescribe the monodromies at the markings. Suppose we are given a principal G-bundle $\pi : D \to C$ like that. Using this data it is easy to construct a line bundle L over the r-stable curve $\mathcal{C}$ corresponding to C. Over any contractible open set $U \subset C$ that does not contain markings and nodes we create a chart $U \times \mathbb{C}$ and identify the r-roots of unity in $\mathbb{C}$ with the sheets of the G-bundle in an arbitrary way that preserves the G-action. At the markings we create the orbi-chart $\Delta \times \mathbb{C}$ endowed with the G-action $(z, s) \mapsto (\rho z, \rho^a s)$ as above and also identify the r-th roots of unity with the sheets of the bundle. The transition maps between the charts are obtained from the matching of the sheets over different charts (every transition map is the multiplication by a locally constant r-th root of unity).

Sections of L and of $K \otimes L^*$.

Let ϕ be a section of L over an open set $U \subset \mathcal{C}$. Then $\pi^*\phi/\phi_0$ is a holomorphic function on $\pi^{-1}(U) \subset D$. Moreover, the G-action on this function has the form $f(\rho z) = \rho^{-1} f(z)$. A global section of L gives rise to a global holomorphic function on D satisfying the above transformation rule. It follows that L has no global sections over $\mathcal{C}$, with the exception of the case where all a_i's vanish, L is the trivial line bundle and $D = C \times G$.

Similarly, let ϕ be a section of $K \otimes L^*$ on an open set $U \subset \mathcal{C}$. Then $\alpha = \pi^*\phi \cdot \phi_0$ is a section of the canonical line bundle K_D over $\pi^{-1}(U)$. Moreover, the G-action on this function has the form $\alpha(\rho z) = \rho\alpha(z)$. In particular, the space of global sections of $K \otimes L^*$ coincides with the space of holomorphic differentials on D satisfying the transformation rule $\alpha(\rho z) = \rho\alpha(z)$.

Two ways of writing R^*p_*L.

Chiodo's formula expresses the Chern character of R^*p_*L, where we denote by $p : \overline{\mathcal{C}}_{g;a_1,\dots,a_n}(\mathcal{B}G) \to \overline{\mathcal{M}}_{g;a_1,\dots,a_n}(\mathcal{B}G)$ the universal curve. Using this formula one can also easily express the total Chern class of $-R^*p_*L$.

According to our remarks above, if there is at least one positive a_i then $R^0p_*L = 0$. In that case R^1p_*L is a vector bundle, and we have $c(-R^*p_*L) = c(R^1p_*L)$.

If all the a_i's vanish, the space $\overline{\mathcal{M}}_{g;a_1,\dots,a_n}(\mathcal{B}G)$ has a special connected component on which the line bundle L is trivial. Over this component $R^0p_*L = \mathbb{C}$. On the other connected components we have, as before, $R^0p_*L = 0$. Therefore the total Chern class of R^0p_*L is equal to 1 and we have, once again, $c(-R^*p_*L) = c(R^1p_*L)$.

Johnson, Pandharipande, and Tseng use the Chern classes λ_i of the vector bundle of equivariant sections of K_D. Our analysis above shows that this vector bundle is the dual of R^1p_*L. In other words, we have

$$c(-R^*p_*L) = \sum(-1)^i\lambda_i, \tag{3.18}$$

which is the equality that we use in our computations.

Remark 3.5.1. In the Johnson-Pandharipande-Tseng formula the monodromies at the markings are given by the remainders modulo r of $-\mu_i$, that is, minus the parts of the ramification profile. Thus if we denote by $a_i = \mu_i \mod r$, we will use Chiodo's formula with remainders $r - a_1, \dots, r - a_n$ at the markings. If an a_i is equal to 0, we can plug either 0 or r in Chiodo's formula. Indeed, we have $B_k(0) = B_k(1)$ for any $k > 1$, thus replacing 0 by r will only affect the Chern character of degree 0, that is not used in the expression for the total Chern class.

In particular, in Equation (3.17) we use the push-forward of $\sum(-1)^i\lambda_i$ to $\overline{\mathcal{M}}_{g,n}$, for monodromies equal to minus the remainders of $\mu_1, \dots, \mu_n$. This class coincides with $\mathrm{C}_{g,n}(r, s; r - a_1, \dots, r - a_n)$ defined by Equation (3.3).

3.5.3 The equivalence

Now we are armed to prove the following

Theorem 3.5.2. *The expansion of the correlation differentials of the spectral curve* (3.9) *for $s = r$ is given by*

$$\mathcal{W}_{g,n} = \sum_{\mu_1,\dots,\mu_n=1}^{\infty} \mathrm{d}_1 \otimes \dots \otimes \mathrm{d}_n \; e^{\sum_{j=1}^n \mu_j x_j} \frac{h_{g;\vec{\mu}}}{b!}, \tag{3.19}$$

if and only if the numbers $h_{g;\vec{\mu}}$ are given by the Johnson-Pandharipande-Tseng formula (3.17).

Proof. The proof is indeed very simple. First, Equation (3.18) allows us to replace Chiodo class in (3.16) with the push-forward of the linear combination of λ-classes. Then we notice the following rescaling of the integral

$$\int_{\overline{\mathcal{M}}_{g,n}} \frac{\pi_* \sum_{i\geq 0}(-r)^i \lambda_i}{\prod_{j=1}^n (1-\mu_i\psi_i)} = r^{3g-3+n} \int_{\overline{\mathcal{M}}_{g,n}} \frac{\pi_* \sum_{i\geq 0}(-1)^i \lambda_i}{\prod_{j=1}^n (1-\frac{\mu_i}{r}\psi_i)}. \tag{3.20}$$

The equivalence then follows from comparison of coefficients in front of particular $\mathrm{d}_1 \otimes \cdots \otimes \mathrm{d}_n \, e^{\sum_{j=1}^n \mu_j x_j}$ in (3.19) and (3.16), which is obvious, modulo the following simple computation of the powers of r. For $s = r$,

$$\prod_{j=1}^n \frac{\left(\frac{\mu_j}{r}\right)^{\lfloor \frac{\mu_j}{r} \rfloor}}{\lfloor \frac{\mu_j}{r} \rfloor !} \frac{r^{2g-2+n+\frac{(2g-2+n)s+\sum_{j=1}^n \mu_j}{r}}}{s^{2g-2+n}} = \prod_{j=1}^n \frac{\mu_j^{\lfloor \frac{\mu_j}{r} \rfloor}}{\lfloor \frac{\mu_j}{r} \rfloor !} r^{2g-2+n+\sum_{j=1}^n \langle \frac{\mu_j}{r} \rangle}$$

is the coefficient in Equation (3.16). This is equal to

$$r^{3g-3+n} r^{1-g+\sum \langle \frac{\mu_i}{r} \rangle} \prod_{i=1}^n \frac{\mu_i^{\lfloor \frac{\mu_i}{r} \rfloor}}{\lfloor \frac{\mu_i}{r} \rfloor !},$$

which is the coefficient of (3.17) after rescaling (3.20). □

Bibliography

[1] D. Abramovich, A. Vistoli. Compactifying the space of stable maps, J. Amer. Math. Soc. **15** (2002), 27–75.

[2] V. Bouchard, D. Hernàndez Serrano, X. Liu, M. Mulase, Mirror symmetry for orbifold Hurwitz numbers, J. Differ. Geom. **98** (2014), 375–423.

[3] V. Bouchard, M. Mariño, Hurwitz numbers, matrix models and enumerative geometry, In: From Hodge theory to integrability and TQFT tt*-geometry, Proc. Sympos. Pure Math. 78, Amer. Math. Soc., 2008, 263–283.

[4] W. Chen and Y. Ruan. Orbifold Gromov-Witten theory, In: Orbifolds in mathematics and physics (Madison, WI, 2001), 2585, Contemp. Math. **310** (2002).

[5] A. Chiodo. Towards an enumerative geometry of the moduli space of twisted curves and r-th roots, Compos. Math. **144** (2008), No. 06, 1461–1496.

[6] A. Chiodo, Y. Ruan. Landau-Ginzburg/Calabi-Yau correspondence for quintic three-folds via symplectic transformations, Invent. Math. **182** (2010), no. 1, 117–165.

[7] A. Chiodo, D. Zvonkine. Twisted Gromov-Witten r-spin potential and Givental's quantization, Adv. Theor. Math. Phys. **13** (2009), no. 5, 1335–1369.

[8] N. Do, O. Leigh, P. Norbury, Orbifold Hurwitz numbers and Eynard-Orantin invariants, arXiv:1212.6850.

[9] P. Dunin-Barkowski, M. Kazarian, N. Orantin, S. Shadrin, L. Spitz, Polynomiality of Hurwitz numbers, Bouchard-Mariño conjecture, and a new proof of the ELSV formula, arXiv: 1307.4729.

[10] P. Dunin-Barkowski, D. Lewanski, A. Popolitov, S. Shadrin, Polynomiality of orbifold Hurwitz numbers, spectral curve, and a new proof of the Johnson-Pandharipandhe-Tseng Formula, preprint 2015.

[11] P. Dunin-Barkowski, N. Orantin, S. Shadrin, L. Spitz, Identification of the Givental formula with the spectral curve topological recursion procedure, Comm. Math. Phys. **328** (2014), no. 2, 669–700.

[12] P. Dunin-Barkowski, S. Shadrin, L. Spitz, Givental graphs and inversion symmetry, Lett. Math. Phys. **103** (2013), no. 5, 533–557.

[13] T. Ekedahl, S. Lando, M. Shapiro, and A. Vainshtein, Hurwitz numbers and intersections on moduli spaces of curves, Invent. Math. 146 (2001), no. 2, 297-327.

[14] B. Eynard, Invariants of spectral curves and intersection theory of moduli spaces of complex curves, arXiv:1110.2949.

[15] B. Eynard, Intersection numbers of spectral curves, arXiv:1104.0176.

[16] B. Eynard, M. Mulase, B. Safnuk, The Laplace transform of the cut-and-join equation and the Bouchard-Mariño conjecture on Hurwitz numbers, arXiv:0907.5224.

[17] B. Eynard, N. Orantin, Invariants of algebraic curves and topological expansion, Commun. Number Theory Phys. 1 (2007), no. 2, 347-452.

[18] F. Janda, R. Pandharipande, A. Pixton, D. Zvonkine, Double ramification cycles on the moduli spaces of curves, arXiv:1602.04705.

[19] P. Johnson, R. Pandharipande, H.-H. Tseng, Abelian Hurwitz-Hodge integrals, Michigan Math. J. **60** (2011), No. 1, 171–198.

[20] M. Mulase, S. Shadrin, L. Spitz, The spectral curve and the Schrödinger equation of double Hurwitz numbers and higher spin structures, arXiv:1301.5580.

[21] D. Mumford, Towards an enumerative geometry of the moduli space of curves, Arithmetic and geometry, Vol. II, 271–328, Progr. Math., 36, Birkhäuser Boston, Boston, MA, 1983.

[22] R. Pandharipande, A. Pixton, D. Zvonkine, Relations on $\overline{\mathcal{M}}_{g,n}$ via 3-spin structures, J. Amer. Math. Soc. 28 (2015), no. 1, 279–309.

[23] S. Shadrin, L. Spitz, D. Zvonkine, Equivalence of ELSV and Bouchard-Mariño conjectures for r-spin Hurwitz numbers, Math. Ann. **361** (2015), no. 3-4, 611–645.

[24] S. Shadrin, L. Spitz, D. Zvonkine, On double Hurwitz numbers with completed cycles, arXiv:1103.3120v1.

[25] D. Zvonkine, A preliminary text on the r-ELSV formula, Preprint 2006.

4

Ramifications of Hurwitz theory, KP integrability and quantum curves.

In this chapter we revisit several recent results on monotone and strictly monotone Hurwitz numbers, providing new proofs. In particular, we use various versions of these numbers to discuss methods of derivation of quantum spectral curves from the point of view of KP integrability and derive new examples of quantum curves for the families of double Hurwitz numbers.

4.1 Introduction

4.1.1 Hurwitz numbers

The purpose of this chapter is to survey some variations of the concept of Hurwitz numbers and their generating functions. Recall that a simple Hurwitz number $h_{g,\mu}$ depends on a genus $g \geq 0$ and a partition $\mu \vdash d$ of length $\ell = \ell(\mu)$, $\mu = (\mu_1 \geq \cdots \geq \mu_\ell)$, $\sum_{i=1}^{\ell} \mu_i = d$. By definition, $h_{g,\mu}$ is the weighted number of ramified coverings of a sphere $\mathbb{C}\mathrm{P}^1$ by a genus g surface, whose degree is d, whose monodromy near $\infty \in \mathbb{C}\mathrm{P}^1$ is a permutation of a cyclic type μ, and these coverings must have simple ramification points over fixed $2g-2+n+d$ points in $\mathbb{C}\mathrm{P}^1 \setminus \{\infty\}$.

These numbers satisfy plenty of interesting properties, and for the thesis the most important ones are

- The generating function of Hurwitz numbers

$$Z(\mathbf{p},\hbar) := \exp\left(\sum_{g,\mu} h_{g,\mu} p_\mu \frac{\hbar^{2g-2+\ell(\mu)+|\mu|}}{(2g-2+\ell(\mu)+|\mu|)!}\right)$$

  is a tau-function of the KP hierarchy [43, 33].

- The principal specialization $\Psi(x,\hbar)$ of the generating function satisfies a differential equation

$$\left(\hat{y} - \hat{x}e^{\hat{y}}\right)\Psi(x,\hbar) = 0, \qquad \hat{x} = x\cdot, \hat{y} = \hbar x \frac{\partial}{\partial x},$$

  called quantum curve [50]. Here by principle specialization we call the substitution $p_\mu = (x/\hbar)^{|\mu|}$ in the formula for Z above.

- Hurwitz numbers for fixed $g \geq 0$ and $\ell \geq 1$ can be arranged into the so-called ℓ-point functions, whose differentials satisfy the topological recursion in the sense of Chekhov–Eynard–Orantin for the spectral curve
$$\log x = \log y - y,$$
see [22].

- There is a formula for $h_{g,\mu}$ in terms of the intersection numbers on the moduli space of curves $\overline{\mathcal{M}}_{g,\ell}$:
$$h_{g.\mu} = \frac{(2g-2+\ell+d)!}{|\mathrm{Aut}(\mu)|} \prod_{i=1}^{\ell(\mu)} \frac{\mu_i^{\mu_i}}{\mu_i!} \int_{\overline{\mathcal{M}}_{g,\ell}} \frac{1-\lambda_1+\cdots\pm\lambda_g}{\prod_{i=1}^{\ell(\mu)}(1-\mu_i\psi_i)}$$
(the ELSV formula) [18, 14].

These results are related to each other, and it is interesting to specify a class of combinatorial problems depending in a natural way on a genus parameter $g \geq 0$ and a partition μ, where the same sequence of results can be derived. Let us explain why we find this sequence of results important.

4.1.2 Outline of the logic

Let us assume that we start with a combinatorial problem depending on a parameter $g \geq 0$ and a partition μ, and its generating function appears to be a KP tau-function. Then we have the following:

Step 1: From KP to quantum curve

In the case when the generating function of some problem of enumerative geometry can be identified with a KP tau-function, the integrable hierarchy often allows us to find a quantum spectral curve. Indeed, the principal specialization of the generating function coincides with the so-called first basis vector of the corresponding point of the Sato Grassmannian, and, as it was observed in [2] (see also [46, 52, 51]), this reduces the problem to a specialization of a suitable Kac–Schwarz operator that would annihilate it.

Step 2: From quantum curve to topological recursion

Once we have a quantum curve, we can formulate a precise conjecture that the differentials of the ℓ-point functions satisfy the topological recursion [21] for the spectral curve obtained by the dequantization of the quantum curve. This relation was made explicit in [30]. Note that the spectral curve should also correspond to the $(g,\ell) = (0,1)$ part of the problem [10], and this property is automatically implied by the quantum curve, see e.g. [41, 11].

Step 3: From topological recursion to intersection numbers

Once we have spectral curve topological recursion, we can immediately conclude that the corresponding combinatorial problem can be solved in terms of some intersection numbers on the moduli space of curves that represent the correlators of a semi-simple cohomological field theory with a possibly non-flat unit [19, 20, 13], and therefore, have expressions in terms of the Givental graphs [16, 13].

Discussion of Steps 1-3

The most important point of this sequence of steps is that Step 1 provides us with a conjectural spectral curve for Step 2 and, therefore, with a conjectural intersection number formula in Step 3. Thus, analysis of the principal specialization in the framework of the KP integrability appears to be a powerful tool that provides very precise conjectural links between combinatorial problems and the intersection theory of the moduli space of curves.

This logic allows one to prove the ELSV-type formulas in some cases, for instance, this way the original ELSV is proved in [14], the Johnson–Pandharipande–Tseng formula for the orbifold Hurwitz numbers is proved in [15], and the conjectural ELSV-type formula for the r-spin Hurwitz numbers is derived, in a new way, in [49]. The corresponding quantum curves (that might be considered as the sources of all these formulas) are derived in [50, 41]. In all these examples, however, the ELSV-type formulas were known before, without any relation to spectral/quantum curves.

4.1.3 Results of the chapter

Rather general models of Hurwitz type are known to be described by the KP/Toda tau-functions [3, 42, 31], thus, the logic that we outline above can be applied to them. In this chapter we focus on the first step for a number of Hurwitz-type theories based on the symmetric functions of the Jucys-Murphy elements in the group algebra of the symmetric group. These theories were considered recently in connection to enumeration of dessins d'enfants [4], expansion of hypegeometric tau-functions [31], study of the HCIZ matrix model [29], and topological recursion [8, 7].

We revisit with new proofs a number of results in [4, 31, 7], namely,

– we establish relations between various geometric interpretations for these Hurwitz-type theories;

– we provide the group operators that generate the corresponding tau-functions;

– we derive the quantum curves from the Kac–Schwarz operators.

Once we have a quantum curve, we can immediately produce an ELSV-type formula. We give a detailed computation for the monotone Hurwitz numbers — this answers a question posed in [29], and, in fact, it is not a conjecture but a theorem since the corresponding Step 2 (a proof of the topological recursion) was derived in [8].

The description of the 2D Toda tau-function for the double monotone Hurwitz numbers in terms of the KP Kac–Schwarz operators allows us to construct the quantum spectral curve for this case. The second set of the Toda times plays the role of linear parameters of the corresponding operator. We use this Kac–Schwarz description in order to derive a system of linear differential operators that annihilate the tau-function for the double monotone Hurwitz numbers and uniquely characterize it.

In addition, we derive a number of new quantum curves for similar Hurwitz theories. In particular, this yields an interesting example for which we can say in advance that the logic outlined above does not apply. Namely, we have an example where the dequantization of the quantum curve doesn't give a spectral curve suitable for the corresponding topological recursion.

4.1.4 Organization of the chapter

In Section 4.2 we briefly recall the necessary facts from the theory of the KP hierarchy. In Section 4.3 we recall the necessary facts from the Jucys theory. In Section 4.4 we define a variety of Hurwitz-type problems that we study in this chapter, and explain the correspondences between them that follow from the Jucys correspondence. In Section 4.5 we embed these Hurwitz-type problems in the framework of the KP formalism. Section 4.6 is devoted to the study of the monotone Hurwitz numbers. We derive in a new way a quantum curve for them, compute the associated ELSV-type formula, and provide the linear constrains for the tau-function of the double monotone Hurwitz numbers. Finally, in Section 4.7 we derive quantum curves for some further examples that are interesting from various points of view (in particular, the one whose classical limit does not give a proper spectral curve).

4.2 KP hierarchy and Kac-Schwarz operators

In this section we give a brief recollection of some of the basic concepts of KP integrability used in this chapter. For more details see, e.g., [45, 47, 24, 40, 2] and references therein.

The KP hierarchy can be described by the bilinear identity satisfied by the tau-function $\tau(\mathbf{t})$, namely

$$\oint_{\infty} e^{\xi(\mathbf{t}-\mathbf{t}',z)}\,\tau(\mathbf{t}-[z^{-1}])\,\tau(\mathbf{t}'+[z^{-1}])dz=0,$$

where $\xi(\mathbf{t},z)=\sum_{k=1}^{\infty}t_kz^k$ and we use the standard notation

$$\mathbf{t}\pm\left[z^{-1}\right]=\left\{t_1\pm\frac{1}{z},t_2\pm\frac{1}{2z^2},t_3\pm\frac{1}{3z^3},\dots\right\}.$$

In Hurwitz-type problems it is often convenient to work in the coordinates $\mathbf{p}$ instead of $\mathbf{t}$, where $p_k=kt_k$, $k=1,2,\dots$.

4.2.1 Semi-infinite wedge space

We consider the vector space $V:=\bigoplus_{c\in\mathbb{Z}}V_c$ spanned by the vectors that are obtained from

$$|0\rangle:=z^0\wedge z^{-1}\wedge z^{-2}\wedge\cdots$$

by applying a finite number of the operators $\psi_i:=z^i\wedge$ and $\psi_i^*:=\frac{\partial}{\partial(z^i)}$, $i\in\mathbb{Z}$. The gradation c is introduced as follows:

$$|0\rangle\in V_0,\ \deg\psi_i=1,\ \deg\psi_i^*=-1,\ i\in\mathbb{Z}.$$

In particular, the vector space V_0 has a basis that consists of the vectors

$$v_\lambda:=z^{\lambda_1-0}\wedge z^{\lambda_2-1}\wedge z^{\lambda_3-2}\wedge\cdots,$$

where $\lambda=(\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_{\ell(\lambda)}\geq 0\geq 0\geq\cdots)$ is a Young diagram. Note that $|0\rangle=v_\emptyset$.

We define the operator $:\psi_i\psi_j^*:$ on V_0 to be $\psi_i\psi_j^*$ if $j>0$ and $-\psi_j^*\psi_i$ if $j\leq 0$. The map $E_{ij}\to\,:\psi_i\psi_j^*:$ gives a projective representation of $\mathfrak{gl}_\infty$ in V_0.

Consider the operators $\alpha_n:=\sum_{i\in\mathbb{Z}}:\psi_{i-n}\psi_i^*:$ defined on V_0. Note that $\alpha_0V_0=0$.

There is a map from V_0 to $\mathbb{C}[[\mathbf{t}]]$ given by

$$V_0 \ni v \mapsto \langle 0| \exp(\sum_{i=1}^{\infty} t_i \alpha_i) v, \tag{4.1}$$

where $\langle 0|$ is the covacuum, that is, the covector that returns the coefficient of $|0\rangle$. For instance, the function that corresponds to v_λ is the Schur function $s_\lambda(\mathbf{t})$.

The description of the tau-functions of the KP hierarchy in this language is the following: the tau-functions correspond to the vectors that belong to the image of the Plücker embedding of the semi-infinite Grassmannian, also called Sato Grassmannian. On the open cell this means that we are looking for the vectors representable as

$$\Phi_1 \wedge \Phi_2 \wedge \Phi_3 \wedge \cdots ,$$

where $\Phi_k(z) = z^{1-k} + \sum_{m=2-k}^{\infty} \Phi_{km} z^m$, $\Phi_{km} \in \mathbb{C}$, are known as basis vectors. This description immediately implies that the group $GL(V_0)$ is the group of symmetries of the KP hierarchy.

The map (4.1) allows to translate the infinitesimal symmetries of the semi-infinite Grassmannian in $\widehat{\mathfrak{gl}}_\infty$ into differential operators that act as infinitesimal symmetries of the KP hierarchy.

There are several examples that are important in this chapter. First of all, we have:

$$\alpha_n \leftrightarrow \widehat{J}_n := \frac{\partial}{\partial t_n}, n > 0; \qquad \alpha_n \leftrightarrow \widehat{J}_n := -n t_{-n}, n < 0,$$

where the operators $\widehat{J}_n$ are defined on $\mathbb{C}[[\mathbf{t}]]$. The energy operator $E \colon V_0 \to V_0$ defined as $E \colon v_\lambda \mapsto |\lambda| v_\lambda$ corresponds to the operator $\widehat{L}_0 \colon \mathbb{C}[[\mathbf{t}]] \to \mathbb{C}[[\mathbf{t}]]$ defined as

$$\widehat{L}_0 := \frac{1}{2} \sum_{i+j=0} {:}\, \widehat{J}_i \widehat{J}_j \,{:},$$

where the normal ordering denoted by ${:} \cdots {:}$ put all operators $\widehat{J}_k$ with positive k to the right of all $\widehat{J}_k$ with negative k. The Casimir operator $\tilde{\mathcal{E}}_0(z) \colon V_0 \to V_0$ acts as follows:

$$\tilde{\mathcal{E}}_0(u) v_\lambda = \sum_{r=0}^{\infty} \frac{u^r}{r!} \sum_{i=1}^{\ell(\lambda)} \left[(\lambda_i - i + \frac{1}{2})^r - (-i + \frac{1}{2})^r \right] v_\lambda.$$

Using the auxiliary functions $\zeta(u) = e^{u/2} - e^{-u/2}$, we can present the corresponding differential operator on $\mathbb{C}[[\mathbf{t}]]$ as

$$\frac{1}{\zeta(u)} \sum_{n=1} \frac{1}{n!} \sum_{\substack{\vec{k} \in (\mathbb{Z}^\times)^n \\ k_1 + \cdots + k_n = 0}} \prod_{i=1}^{n} \frac{\zeta(k_i u)}{k_i} {:}\, \widehat{J}_{k_1} \cdots \widehat{J}_{k_n} \,{:}$$

(see [44, 1, 48]).

4.2.2 Kac-Schwarz operators

A convenient way to describe infinitesimal symmetries of the KP hierarchy is to work with the operators from the algebra $w_{1+\infty}$ (the algebra of differential operators in one variable that describes infinitesimal diffeomorphisms of the circle) acting on the basis vectors Φ_i, $i = 1, 2, \ldots$.

Let us denote by $\langle\Phi\rangle$ the point of the Sato Grassmannian, defined by the set of the basis vectos $\langle\Phi_1,\Phi_2,\Phi_3,\dots\rangle$. We call an operator $a\in w_{1+\infty}$ the Kac–Schwarz (KS) operator for the tau-function τ if for the corresponding point of the Sato Grassmannian we have the stability condition

$$a\,\langle\Phi\rangle \subset \langle\Phi\rangle\,.$$

For the trivial tau-function $\tau_\emptyset := 1$ with the basis vectors $\Phi^\emptyset_k = x^{1-k}$, $k\geq 1$, we have two obvious KS operators

$$\begin{aligned} a_\emptyset &:= -x\frac{\partial}{\partial x}, \\ b_\emptyset &:= x^{-1}. \end{aligned} \tag{4.2}$$

These operators satisfy the commutation relation

$$[a_\emptyset, b_\emptyset] = b_\emptyset. \tag{4.3}$$

The KS operators (4.2) act on the basis vectors as follows:

$$\begin{aligned} a_\emptyset\, \Phi^\emptyset_k(x) &= (k-1)\Phi^\emptyset_k(x), \\ b_\emptyset\, \Phi^\emptyset_k(x) &= \Phi^\emptyset_{k+1}(x). \end{aligned} \tag{4.4}$$

Consider the tau-function

$$\tau_\bullet(\mathbf{t};\tilde{\mathbf{t}}) = e^{\sum_{k=1}^\infty k t_k \tilde t_k} = \sum_\lambda s_\lambda(\mathbf{t}) s_\lambda(\tilde{\mathbf{t}}),$$

where $s_\lambda(t)$ are the Schur functions. From the point of view of the KP hierarchy this tau-function corresponds to the basis vectors

$$\Phi^\bullet_k(x) = e^{\sum_{j=1}^\infty \tilde t_j x^j} x^{1-k}$$

and the KS operators can be obtained from (4.2) by conjugation:

$$\begin{aligned} a_\bullet &:= e^{\sum_{j=1}^\infty \tilde t_j x^j} a_\emptyset\, e^{-\sum_{j=1}^\infty \tilde t_j x^j} = \sum_{k=1}^\infty k\tilde t_k x^k - x\frac{\partial}{\partial x}, \\ b_\bullet &:= e^{\sum_{j=1}^\infty \tilde t_j x^j} b_\emptyset\, e^{-\sum_{j=1}^\infty \tilde t_j x^j} = x^{-1}. \end{aligned}$$

In this case the commutation relation and action of the KS operators on the basis vectors coincide with the ones given by Equations (4.3) and (4.4).

Basis vectors for the points of the Sato Grassmannian, corresponding to the double Hurwitz numbers, can be obtained from (4.2.2) by an action of the operators which are formal series in $x\frac{\partial}{\partial x}$. Corresponding Kac-Schwarz operators (and, in particular, the quantum spectral curve operator) can be obtained from the operators (4.2.2) by a conjugation and also satisfy relations (4.3) and (4.4).

Remark 4.2.1. Let us stress that the algebra of the Kac–Schwarz operators for the trivial tau-function is generated not by the operators $a_\emptyset$ and $b_\emptyset$, but instead by the operators $b_\emptyset^{-1}a_\emptyset$ and $b_\emptyset$. Of course, this is also true for the corresponding Kac–Schwarz operators for all tau-functions, which can be obtained from the trivial one by a simple conjugation, in particular for the generating functions of the Hurwitz numbers (see Remark 4.6.9).

4.3 Symmetric polynomials of Jucys elements

In this section we briefly recall some relations between different bases of the algebra of symmetric polynomials and the Jucys correspondence.

4.3.1 Symmetric polynomials

We consider the elementary symmetric polynomials σ_b, the complete homogeneous polynomials h_b, and the power sums p_b:

$$\sigma_b(x_1,\dots,x_n) := \sum_{1\le i_1<\dots<i_b\le n} x_{i_1}\cdots x_{i_b},$$
$$h_b(x_1,\dots,x_n) := \sum_{1\le \lambda_1\le\dots\le\lambda_b\le n} x_{\lambda_1}\cdots x_{\lambda_b},$$
$$p_b(x_1,\dots,x_n) := \sum_{1\le i\le n} x_i^b.$$

The polynomials σ_b and h_b have the following generating series:

$$\prod_{i=1}^{n}(1+x_i t) = \sum_{b=0}^{\infty}\sigma_b(x_1,\dots,x_n)t^b, \tag{4.5}$$
$$\prod_{i=1}^{n}\frac{1}{(1-x_i t)} = \sum_{b=0}^{\infty}h_b(x_1,\dots,x_n)t^b.$$

The Newton identities describe relations between the power sums p_b and bases σ_b and h_b:

$$\sigma_b = [z^b].\exp\left(-\sum_{i\ge1}\frac{p_i}{i}(-z)^i\right), \tag{4.6}$$
$$h_b = [z^b].\exp\left(\sum_{i\ge1}\frac{p_i}{i}z^i\right).$$

We also have the following relations between σ_b and h_b:

$$h_b = \sum_{k=1}^{b}(-1)^{k+b}\sum_{\substack{\vec\alpha\in(\mathbb{N}^\times)^k\\ |\alpha|=b}}\sigma_{\alpha_1}\cdots\sigma_{\alpha_k}, \tag{4.7}$$
$$\sigma_b = \sum_{k=1}^{b}(-1)^{k+b}\sum_{\substack{\vec\alpha\in(\mathbb{N}^\times)^k\\ |\alpha|=b}}h_{\alpha_1}\cdots h_{\alpha_k}.$$

4.3.2 The Jucys correspondence

Let $\alpha \in \mathfrak{S}_n/\sim$ be a conjugacy class of the symmetric group $\mathfrak{S}_n$ or, equivalently, a partition of n. We denote the number of cycles of α by $\ell(\alpha)$. We denote the formal sum of all permutations with cycle type α as $C_\alpha := \sum_{g\in\alpha} g$. Note that C_α belongs to the center of the group algebra of $\mathfrak{S}_n$, that is, $C_\alpha \in \mathcal{Z}(\mathbb{Q}(\mathfrak{S}_n))$ for any α. The elements C_α span $\mathcal{Z}(\mathbb{Q}(\mathfrak{S}_n))$.

We consider the Jucys-Murphy elements $\mathcal{J}_k \in \mathbb{Q}(\mathfrak{S}_n)$, $k = 2, \dots, n$, defined as

$$\mathcal{J}_k := (1\ k) + (2\ k) + \cdots + (k-1\ k).$$

They generate a maximal commutative subalgebra of $\mathbb{Q}(\mathfrak{S}_n)$ called Gelfand-Tsetlin algebra.

The Jucys-Murphy elements are linked to the center of the group algebra through symmetric polynomials.

Lemma 4.3.1 (Jucys Correspondence [32]). *For $b = 0, \dots, n-1$ we have:*

$$\sigma_b(\mathcal{J}_2, \dots, \mathcal{J}_n) = \sum_{\substack{\alpha\in\mathfrak{S}_n/\sim \\ \ell(\alpha)=n-b}} C_\alpha.$$

This lemma together with the result of Farahat and Higman [23] implies that symmetric polynomials in the Jucys-Murphy elements generate the center of the group algebra.

Using Equation (4.7), we obtain the following expression for the homogeneous complete polynomials of Jucys-Murphy elements:

Lemma 4.3.2. *For $b = 0, \dots, n-1$ we have:*

$$h_b(\mathcal{J}_2, \dots, \mathcal{J}_n) = \sum_{k=1}^{b} (-1)^{k+b} \sum_{\substack{\vec{\alpha}\in(\mathfrak{S}_n/\sim)^k \\ \sum \ell(\alpha_i)=kn-b}} \prod_{i=1}^{k} C_{\alpha_i}$$

We denote $h_b(\mathcal{J}_2, \dots, \mathcal{J}_n)$ by $W_b^{\mathfrak{S}_n}$. Let $\mathscr{C}_m$ be the m-th Catalan number. Let us list the first few examples of $W_b^{\mathfrak{S}_n}$:

$$\begin{aligned}
W_0^{\mathfrak{S}_n} &= 1;\\
W_1^{\mathfrak{S}_n} &= \mathscr{C}_1 C_{(2^1 1^{n-2})} \ (= \sigma_1(\vec{\mathcal{J}}) = p_1(\vec{\mathcal{J}}) = \text{sum of all transpositions});\\
W_2^{\mathfrak{S}_n} &= \mathscr{C}_2 C_{(3^1 1^{n-3})} + \mathscr{C}_1^2 C_{(2^2 1^{n-4})} + \frac{n(n-1)}{2} C_{(1^n)};\\
W_3^{\mathfrak{S}_n} &= \mathscr{C}_3 C_{(4^1 1^{n-4})} + \mathscr{C}_2\mathscr{C}_1 C_{(3^1 2^1 1^{n-5})} + \mathscr{C}_1^3 C_{(2^3 1^{n-6})}\\
&\quad + \left(\frac{(n+1)(n+2)}{2} - 5\right) C_{(2^1 1^{n-2})}.
\end{aligned}$$

Of course, each summand appears if and only if n is big enough to allow the corresponding cycle type.

4.4 Ramifications of Hurwitz theory

In this Section we define the basic objects of study in this chapter — different variations of the classical Hurwitz numbers, whose definition utilizes symmetric functions of Jucys-Murphy elements. We describe a class of problems and their geometric interpretations.

4.4.1 General setup

The general setup is the following. We consider the coefficient of $C_{(1^n)}$ in the product $C_\mu C_\nu B$ for some $B \in \mathbb{Q}(\mathfrak{S}_n)$:

$$h^\bullet_{\mu,\nu,B} := \frac{1}{n!}[C_{(1^n)}]C_\mu C_\nu B$$

If $B \in \mathcal{Z}(\mathbb{Q}(\mathfrak{S}_n))$, then its action in the left regular representation is given by the diagonal matrix $\mathsf{egv}(B)$, whose action in the irreducible representation λ is multiplication by the eigenvalue $\mathsf{egv}_\lambda(B)$. The elements C_α lie in the center and their eigenvalues are given by $\mathsf{egv}_\lambda(C_\alpha) := |C_\alpha|\chi_\lambda(\alpha)/\dim\lambda$, where $\dim\lambda$ and χ_λ are the dimension and the character of the representation λ, and $|C_\alpha|$ is the number of permutations of the cycle type α. This implies that

$$\begin{aligned} h^\bullet_{\mu,\nu,B} &= \frac{\mathrm{Tr}\left(\mathsf{egv}(C_\mu)\,\mathsf{egv}(C_\nu)\,\mathsf{egv}(B)\right)}{(n!)^2} \\ &= \sum_{\lambda\vdash n}\left(\frac{\dim\lambda}{n!}\right)^2 \mathsf{egv}_\lambda(C_\mu)\,\mathsf{egv}_\lambda(C_\nu)\,\mathsf{egv}_\lambda(B) \\ &= \frac{1}{Z_\mu Z_\nu}\sum_{\lambda\vdash n}\chi_\lambda(\mu)\chi_\lambda(\nu)\,\mathsf{egv}_\lambda(B), \end{aligned} \tag{4.8}$$

where $Z_\mu = \prod \mu_i \prod_{i=1}^n (j_i)!$ for $\mu = (1^{j_1}2^{j_2}\cdots n^{j'_n}) = (\mu_1 \geq \cdots \geq \mu_{\ell(\mu)})$.

Let us discuss some examples. One can observe that

$$\mathsf{egv}_\lambda(C_2) = \frac{1}{2}\sum_{i=1}^{\ell(\lambda)}(\lambda_i - i + \frac{1}{2})^2 - (-i+\frac{1}{2})^2.$$

The Hurwitz number $h^\bullet_{\mu,\nu,B}$ for $B = C_2^{2g-2+\ell(\mu)+\ell(\nu)}$ is the standard double Hurwitz number for possibly disconnected surfaces of genus g [43]. Consider an element $\overline{C}_r$ such that

$$\mathsf{egv}_\lambda(\overline{C}_r) = \frac{1}{r!}\sum_{i=1}^{\ell(\lambda)}(\lambda_i - i + \frac{1}{2})^r - (-i+\frac{1}{2})^r.$$

It is the so-called completed r-cycle [36] (in some normalization), and the Hurwitz number $h^\bullet_{\mu,\nu,B}$ for $B = \overline{C}_r^m$, $m(r-1) = 2g-2+\ell(\mu)+\ell(\nu)$, is the double Hurwitz number with completed r-cycles for possibly disconnected surfaces of genus g [48].

In some cases, one can consider the enumeration of coverings up to automorphisms that fix the preimages of two special points (say, 0 and ∞ in $\mathbb{C}\mathrm{P}^1$) pointwise. In this case, we use the following formula instead of the one given by Equation (4.8):

$$\frac{1}{\prod_{i=1}^{\ell(\mu)}\mu_i\prod_{i=1}^{\ell(\nu)}\nu_i}\sum_{\lambda\vdash n}\chi_\lambda(\mu)\chi_\lambda(\nu)\,\mathsf{egv}_\lambda(B).$$

4.4.2 Basic definitions

Let ρ be a standard Young tableau of a Young diagram $\lambda \vdash n$. We denote by i_k and j_k the column and the row indices of the box labeled by k. By

$$\mathsf{cr}^\rho := (i_1 - j_1, i_2 - j_2, \dots, i_n - j_n)$$

we denote the content vector of the tableau. Jucys [32] proves that

$$\mathsf{egv}_\lambda(B(\mathcal{J}_2, \dots, \mathcal{J}_n)) = B(\mathsf{cr}^\rho_2, \dots, \mathsf{cr}^\rho_n) \tag{4.9}$$

for any symmetric polynomial B in $n-1$ variables and any choice of ρ. Since it does not depend on ρ, we can always use some standard choice of the Young tableau, for instance, filling the diagram from left to right, and denote by cr^λ the content vector for this choice. This implies the following:

Lemma 4.4.1. *If $B = B(\mathcal{J}_2, \dots, \mathcal{J}_n)$ is a symmetric polynomial in the Jucys elements, then*

$$h^\bullet_{\mu,\nu,B} = \frac{1}{Z_\mu Z_\nu} \sum_{\lambda \vdash n} \chi_\lambda(\mu)\chi_\lambda(\nu) B(\mathsf{cr}^\lambda_2, \dots, \mathsf{cr}^\lambda_n).$$

Definition 4.4.2. A disconnected double Hurwitz problem is the following set of data: genus g, degree n, two partitions $\mu, \nu \vdash n$, and a vector $\vec{\mathcal{P}} = (\mathcal{P}_1, \dots, \mathcal{P}_m)$, $m \geq 1$, where each $\mathcal{P}_i$ is a central element of $\mathbb{Q}(\mathfrak{S}_n)$. We assign to each $\mathcal{P}_i$ a number b_i and we require the Riemann-Hurwitz equation to hold: $\sum_{i=1}^m b_i = 2g - 2 + \ell(\mu) + \ell(\nu)$. The associated Hurwitz number is then $h^\bullet_{\mu,\nu,B}$ for $B := \prod_{i=1}^m \mathcal{P}_i$, and it can be expressed as

$$h^\bullet_{\mu,\nu,B} = \frac{1}{n!}[C_{(1^n)}] C_\mu C_\nu \prod_{i=1}^m \mathcal{P}_i$$

We call elements $\mathcal{P}_i$ blocks and the vector $\vec{\mathcal{P}}$ the vector of blocks.

Here are some possible blocks (that is, the possible values of $\mathcal{P}_i$, $i = 1, \dots, m$), which are arguably most important for applications:

$$B^<_b := \sigma_b(\mathcal{J}_2, \dots, \mathcal{J}_n);\ B^\leq_b := h_b(\mathcal{J}_2, \dots, \mathcal{J}_n);\ B^\times_b := p_b(\mathcal{J}_2, \dots, \mathcal{J}_n);$$

$$B^{|}_b := \sum_{\substack{\alpha \in (\mathfrak{S}_n/\sim) \\ \ell(\alpha) = n-b}} C_\alpha;\ B^{||}_b := \sum_{k=1}^b (-1)^{k+b} \sum_{\substack{\vec{\alpha} \in (\mathfrak{S}_n/\sim)^k \\ \sum \ell(\alpha_i) = kn-b}} \prod_{i=1}^k C_{\alpha_i}$$

In all these cases $b_i := b$.

In each of this cases we can describe the geometry of the covering that realizes the monodromy of the block. The descriptions follow directly from the definition of the Jucys-Murphy elements $\mathcal{J}_k$, $k = 2, \dots, n$, and the central elements C_λ, $\lambda \vdash n$.

Lemma 4.4.3. *The geometric interpretation of the possible blocks is the following:*

- $B^<_b$ [Strictly Monotone] *We have b simple ramifications, whose monodromies are given by the transpositions $(x_i\ y_i)$, $x_i < y_i$, $i = 1, \dots, b$, with the extra condition $y_i < y_{i+1}$.*

$B_b^{\leq}$ [Monotone] *We have b simple ramifications, whose monodromies are given by the transpositions $(x_i\ y_i)$, $x_i < y_i$, $i = 1, \dots, b$, with the extra condition $y_i \leq y_{i+1}$.*

$B_b^{\times}$ [Atlantes] *We have b simple ramifications, whose monodromies are given by the transpositions $(x_i\ y)$, $i = 1, \dots, b$. Here y is an arbitrary number from 2 to n, which is not fixed in advance, but is the same for all transpositions.*

$B_b^{|}$ [Free Single] *We have one ramification, whose monodromy has no restrictions except for the Euler characteristic of the preimage of the corresponding disk, that is, the monodromy given by a cycle type μ with $\ell(\mu) = n - b$.*

$B_b^{||}$ [Free Group] *We have an arbitrary number k of ramifications, $1 \leq k \leq b$ with no restrictions on the monodromy except for the restriction on the Euler characteristic: the total number of zeros of the differential of the corresponding covering should be equal to b. The coverings are counted with an extra sign $(-1)^{k+b}$.*

The Jucys correspondence given by Lemmas 4.3.1 and 4.3.2 implies the following equalities:

Proposition 4.4.4. *We have $B_b^{<} = B_b^{|}$ and $B_b^{\leq} = B_b^{||}$.*

4.4.3 Examples

Here we survey some examples of disconnected double Hurwitz problems in the sense of Definition 4.4.2 known in the literature.

The Harnad-Orlov correspondence

In [31] Harnad and Orlov prove that a family of 2D Toda tau-functions of hypergeometric type have two different geometric interpretations involving double Hurwitz problems. Their Theorem 2.1 expresses these tau-functions in terms of some Hurwitz numbers of some special type and their Theorem 2.2 deals with enumeration of paths in Cayley graphs. We review these two theorems and show that Jucys correspondence implies their equivalence.

The hypergeometric function $\tau_{(q,w,z)}(\mathbf{t}, \tilde{\mathbf{t}})$ is defined as

$$\tau_{(q,w,z)}(\mathbf{t}, \tilde{\mathbf{t}}) := \sum_{n=0} q^n \sum_{\lambda \vdash n} \prod_{j=1}^{n} \frac{\prod_{a=1}^{l}(1 + \mathsf{cr}_j^{\lambda} w_a)}{\prod_{b=1}^{m}(1 - \mathsf{cr}_j^{\lambda} z_b)} s_\lambda(\mathbf{t}) s_\lambda(\tilde{\mathbf{t}})$$

Here $w = (w_1, \dots, w_l)$ and $z = (z_1, \dots, z_m)$ are the parameters of the tau-function, and their number (l and m respectively) is arbitrary, not necessarily finite. For particular values of these parameters the hypergeometric tau-functions represent all generating functions of the Hurwitz numbers considered below. Using the generating functions of σ_b and h_b (see Equation (4.5)) we rewrite (4.4.3) as

$$\sum_{n=0} q^n \sum_{\lambda \vdash n} \sum_{\substack{c \in \mathbb{N}^l \\ d \in \mathbb{N}^m}} \prod_{a=1}^{l} w_a^{c_a} \sigma_{c_a}(\mathsf{cr}^{\lambda}) \prod_{b=1}^{m} z_b^{d_b} h_{d_b}(\mathsf{cr}^{\lambda}) s_\lambda(\mathbf{t}) s_\lambda(\tilde{\mathbf{t}})$$

Since $s_\lambda(\mathbf{t}) = \sum_{\mu\vdash|\lambda|} \chi_\lambda(\mu)p_\mu(\mathbf{t})/Z_\mu$, the coefficient of

$$q^n \prod_{a=1}^{l}\prod_{b=1}^{m} w_a^{c_a} z_b^{d_b} p_\mu(\mathbf{t})p_\nu(\tilde{\mathbf{t}})$$

in this expression is equal to

$$\sum_{\lambda\vdash n} \frac{\chi_\lambda(\mu)\chi_\lambda(\nu)}{Z_\mu Z_\nu} \prod_{a=1}^{l} \sigma_{c_a}(\mathsf{cr}^\lambda) \prod_{b=1}^{m} h_{d_b}(\mathsf{cr}^\lambda)$$

Lemma 4.4.1, Lemma 4.4.3, and Jucys statement about eigenvalues (4.9) imply that this coefficient is equal to $h^\bullet_{\mu,\nu,B}$ for the vector of blocks given by

$$\vec{\mathcal{P}} := \left(B^<_{c_1}, \ldots, B^<_{c_l}, B^\leq_{d_1}, \ldots, B^\leq_{d_m}\right)$$

This is the way Harnad and Orlov prove [31, Theorem 2.2]. Now Proposition 4.4.4 implies a different interpretation of the same Hurwitz number, namely,

$$\vec{\mathcal{P}} = \left(B^{|}_{c_1}, \ldots, B^{|}_{c_l}, B^{||}_{d_1}, \ldots, B^{||}_{d_m}\right),$$

which proves [31, Theorem 2.1].

Remark 4.4.5. The monotone and strictly monotone blocks are expressed in [31] as counting paths in the Cayley graph of $\mathfrak{S}_n$. For convenience, we express them as a Hurwitz problem here.

Remark 4.4.6. We also adjust a small inconsistency: observe that our weight in each summand of the free group block is $(-1)^{k+b}$, while in [31] it is $(-1)^{n+k+b}$.

Remark 4.4.7. The solutions of the Hurwitz problem in genus zero with $\nu = (1^n)$ and a single block $B^{||}_{b,k}$ (which coincides with $B^{||}_b$ except that the number of groups k is fixed and it is not weighted by sign) is known as Bousquet-Mélou Shaeffer numbers [6], see also [34].

Enumeration of hypermaps

The enumeration of hypermaps, or, more generally, of Grothendieck's dessins d'enfants, is considered in many recent papers in slightly different formulations in relation to the Chekhov–Eynard–Orantin recursion, quantum curves, and KP/Toda integrability. An incomplete list of recent references includes [17, 35, 9, 25, 54, 26, 4, 5].

Enumeration of hypermaps is equivalent to the standard weighted count of the coverings of degree n of a sphere $\mathbb{C}\mathrm{P}^1$ by a surface of genus g (or, rather, a possibly disconnected surface of Euler characteristic $2-2g$) that have three ramification points, 0, 1, and ∞, such that

- The monodromy over 0 has cycle type $\mu \vdash n$, which is a parameter of the enumeration problem.
- The monodromy over ∞ has cycle type $(r^{n/r})$, r is a parameter of the enumeration problem, and we assume that $r|n$.
- The monodromy over 1 is an arbitrary one. Let us denote it by $\kappa \vdash n$. The only restriction that we have here is imposed by the Riemann-Hurwitz formula $2g-2+\ell(\mu)+n/r = n-\ell(\kappa)$.

In our terms, this enumeration problem can be reformulated as a Hurwitz number $h^\bullet_{\mu,\nu,B}$, where $\nu = (r^{|\mu|/r})$, the vector of blocks $\vec{\mathcal{P}} = (\mathcal{P}_1)$ has length 1, and

$$B = \mathcal{P}_1 := B^{|}_{2g-2+\ell(\mu)+|\mu|/r}.$$

Proposition 4.4.4 implies that

$$B = B^{<}_{2g-2+\ell(\mu)+|\mu|/r}.$$

The Hurwitz numbers for the data

$$(g, n, \mu, \nu = (r^{n/r}), \vec{\mathcal{P}} = (B^{\leq}_{2g-2+\ell(\mu)+n/r}))$$

are called monotone orbifold Hurwitz numbers in [7] (orbifold here refers to the type of partition ν), so it is natural to call Hurwitz numbers for the data

$$(g, n, \mu, \nu = (r^{n/r}), \vec{\mathcal{P}} = (B^{<}_{2g-2+\ell(\mu)+n/r}))$$

strictly monotone orbifold Hurwitz numbers. Then the observation above can be reformulated as follows:

Proposition 4.4.8. *The enumeration of hypermaps is equivalent to the strictly monotone orbifold Hurwitz problem.*

Remark 4.4.9. This proposition also implicitly follows from the discussion in [4, Section 1], in a different way.

4.5 Operators for $B_b^<$, $B_b^\leq$, and $B_b^\times$

In this Section we derive the operators that represent the blocks $B_b^<$, $B_b^\leq$, and $B_b^\times$ in the semi-infinite wedge formalism and provide the corresponding differential operators.

4.5.1 Derivation of operators

Recall that the Casimir operator $\tilde{\mathcal{E}}_0(z)$ on V_0 is a $\widehat{\mathfrak{gl}}_\infty$-operator (4.2.1) that generates completed cycles. We would like to construct the same operators for the blocks $B_b^<$, $B_b^\leq$, and $B_b^\times$, that is, we are looking for the operators $\mathcal{D}^{(p)}(z)$, $\mathcal{D}^{(h)}(z)$, and $\mathcal{D}^{(\sigma)}(z)$ defined on V_0 and acting on the basis vectors as follows:

$$\mathcal{D}^{(p)}(z)v_\lambda = \sum_{k=1} \frac{z^k}{k!} p_k(\mathsf{cr}^\lambda)v_\lambda, \qquad \mathcal{D}^{(h)}(z)v_\lambda = \sum_{k=0} z^k h_k(\mathsf{cr}^\lambda)v_\lambda,$$

$$\mathcal{D}^{(\sigma)}(z)v_\lambda = \sum_{k=0} z^k \sigma_k(\mathsf{cr}^\lambda)v_\lambda.$$

Since it is not important how we arrange the generating functions, we do it in the way that is most convenient for the proof below.

Remark 4.5.1. While $\mathcal{D}^{(p)}(z)$ is an element of the $\widehat{\mathfrak{gl}}_\infty$ Lie algebra, operators $\mathcal{D}^{(h)}(z)$ and $\mathcal{D}^{(\sigma)}(z)$ belong to the corresponding group. From the Newton identities it follows that

$$\mathcal{D}^{(h)}(z) = \frac{1}{\mathcal{D}^{(\sigma)}(-z)}.$$

Proposition 4.5.2. *These operators, as the formal series in z, are given by the following formulas:*

$$\mathcal{D}^{(p)}(z) = \frac{\tilde{\mathcal{E}}_0(z)}{\zeta(z)} - E,$$

$$\mathcal{D}^{(h)}(z) = z^{\frac{\tilde{\mathcal{E}}_0\left(z^2\frac{d}{dz}\right)}{\zeta\left(z^2\frac{d}{dz}\right)} - E} := \exp\left(\left[\frac{\tilde{\mathcal{E}}_0\left(z^2\frac{d}{dz}\right)}{\zeta\left(z^2\frac{d}{dz}\right)} - E\right]\log z\right),$$

$$\mathcal{D}^{(\sigma)}(z) = z^{-\frac{\tilde{\mathcal{E}}_0\left(-z^2\frac{d}{dz}\right)}{\zeta\left(-z^2\frac{d}{dz}\right)} + E} := \exp\left(-\left[\frac{\tilde{\mathcal{E}}_0\left(-z^2\frac{d}{dz}\right)}{\zeta\left(-z^2\frac{d}{dz}\right)} - E\right]\log z\right).$$

Proof. The action of the power sums of Jucys elements was computed by Lascoux and Thibon in [37, Proposition 3.3]. The formula for $\mathcal{D}^{(p)}(z)$ is equivalent to their result. Note that the constant term of $\tilde{\mathcal{E}}(z)/\zeta(z)$ is precisely E. The formulas for $\mathcal{D}^{(h)}(z)$ and $\mathcal{D}^{(\sigma)}(z)$ follow from the Newton identities (4.6). □

Remark 4.5.3. Since we know the differential operator (4.2.1) that corresponds to $\tilde{\mathcal{E}}_0$, we immediately obtain the differential operators corresponding to $\mathcal{D}^{(p)}$, $\mathcal{D}^{(h)}$, and $\mathcal{D}^{(\sigma)}$.

Remark 4.5.4. Note that the formula for $\mathcal{D}^{(\sigma)}(z)$ was already observed in [4, Section 3].

Remark 4.5.5. The operators $\mathcal{D}^{(p)}(z_1)$, $\mathcal{D}^{(h)}(z_2)$, and $\mathcal{D}^{(\sigma)}(z_3)$ commute with each other for arbitrary values of z_1, z_2 and z_3.

4.5.2 Some examples

In this Section we list some examples of particular Hurwitz problems whose generating functions are written as vacuum expectations in semi-infinite wedge formalism.

Example 4.5.6. Simple orbifold Hurwitz numbers:

$$\mathcal{Z}(\mathbf{p};\hbar) = \langle 0|\, \exp\left(\sum_{i=1}^{\infty}\frac{p_i\alpha_i}{i}\right)\exp\left(\hbar\mathcal{F}_2\right)\exp\left(\frac{\alpha_{-r}}{\hbar r}\right)\,|0\rangle,$$

where $\mathcal{F}_2 = [z^1]\mathcal{D}^{(\sigma)}(z)$ is the second Casimir. Note that here we could use $\mathcal{D}^{(h)}$ instead of $\mathcal{D}^{(\sigma)}$ since their $[z^1]$ coefficients coincide.

Example 4.5.7. A one-parameter deformation of simple Hurwitz numbers in the tau-function of double Hurwitz numbers:

$$\mathcal{Z}(\mathbf{p},\hbar) = \langle 0|\, \exp\left(\sum_{i=1}^{\infty}\frac{p_i\alpha_i}{i}\right)\exp\left(\hbar\mathcal{F}_2\right)\exp\left(\sum_{i=1}^{\infty}\frac{c^{i-1}\alpha_{-i}}{\hbar}\right)\,|0\rangle$$

Example 4.5.8. Monotone orbifold Hurwitz numbers:

$$\mathcal{Z}(\mathbf{p};\hbar) = \langle 0|\, \exp\left(\sum_{i=1}^{\infty} \frac{p_i \alpha_i}{i}\right) \mathcal{D}^{(h)}(\hbar) \exp\left(\frac{\alpha_{-r}}{\hbar r}\right) |0\rangle$$

Example 4.5.9. Strictly monotone orbifold Hurwitz numbers (or, equivalently, hypermaps, see Proposition 4.4.8):

$$\mathcal{Z}(\mathbf{p};\hbar) = \langle 0|\, \exp\left(\sum_{i=1}^{\infty} \frac{p_i \alpha_i}{i}\right) \mathcal{D}^{(\sigma)}(\hbar) \exp\left(\frac{\alpha_{-r}}{\hbar r}\right) |0\rangle$$

Example 4.5.10. Atlantes orbifold Hurwitz numbers:

$$\mathcal{Z}(\mathbf{p};\hbar) = \langle 0|\, \exp\left(\sum_{i=1}^{\infty} \frac{p_i \alpha_i}{i}\right) \exp(r![z^r]\mathcal{D}^{(p)}(z\hbar)) \exp\left(\frac{\alpha_{-q}}{\hbar q}\right) |0\rangle$$

4.6 Monotone Hurwitz numbers

In this Section we discuss the monotone (orbifold) Hurwitz numbers (see Example 4.5.8 above) from different points of view.

4.6.1 HCIZ matrix integral and basis vectors

According to [29] the generating function of double monotone Hurwitz numbers is described by the Harish-Chandra–Itzykson–Zuber (HCIZ) tau-function. More precisely, let us introduce the tau-function

$$\tau_{HCIZ}\left(\mathbf{t},\tilde{\mathbf{t}},\alpha,N\right) = \sum_{\lambda} \alpha^{|\lambda|} s_\lambda(\mathbf{t}) s_\lambda(\tilde{\mathbf{t}}) \prod_i \frac{\Gamma(N-i+1)}{\Gamma(\lambda_i+N-i+1)},$$

so that the HCIZ matrix integral is given by the Miwa parametrization $t_i = \operatorname{Tr} A^i$, $\tilde{t}_i = \operatorname{Tr} B^i$, $i = 1, 2, \ldots$, of this tau-function

$$\int dU e^{\alpha \operatorname{Tr} UAU^\dagger B} = \tau_{HCIZ}\left(\mathbf{t},\tilde{\mathbf{t}},\alpha,N\right).$$

Here we assume that the $N \times N$ matrices A and B are diagonal, and we normalize the Haar measure on the unitary group $U(N)$ in such a way that $\int dU = 1$. Up to a factor that is not relevant for our computations, HCIZ integral describes a tau-function of the two-dimensional Toda lattice [39, 53].

The generating function of the double monotone Hurwitz numbers is given by

$$\begin{aligned}\tau_{mm}\left(\mathbf{t},\tilde{\mathbf{t}}\right) &= \tau_{HCIZ}\left(\mathbf{t},\tilde{\mathbf{t}},-\hbar^{-1},-\hbar^{-1}\right)\\ &= \sum_{\lambda} s_\lambda(\mathbf{t}) s_\lambda(\tilde{\mathbf{t}}) \prod_{i=1}^{l(\lambda)} \prod_{k=0}^{\lambda_i-1} \frac{1}{1+\hbar(k+i-\lambda_i)},\end{aligned}$$

or, in terms of the semi-infinite wedge product, by

$$\tau_{mm}\left(\mathbf{t},\tilde{\mathbf{t}}\right) = \langle 0|\, \exp\left(\sum_{i=1}^{\infty} t_i\alpha_i\right) \mathcal{D}^{(h)}(\hbar) \exp\left(\sum_{i=1}^{\infty} \tilde{t}_i\alpha_{-i}\right) |0\rangle.$$

Proposition 4.6.1. *We can choose basis vectors of the KP hierarchy with respect to the set of times* $\mathbf{t}$ *in the following way:*

$$\Phi_k^{mm}(x) = G_{mm}(k)\, e^{\sum_{m=1}^{\infty} \tilde{t}_m x^m} x^{1-k},$$

where

$$G_{mm}(k) = \frac{\Gamma(1-k-\hbar^{-1})(-\hbar)^{1-k-D}}{\Gamma(D-\hbar^{-1})}, \quad D = x\frac{\partial}{\partial x}.$$

Remark 4.6.2. To specify the asymptotic we use the operator identity

$$f\left(\frac{\partial}{\partial a}\right) e^a = e^a f\left(\frac{\partial}{\partial a} + 1\right)$$

valid for arbitrary function f, so that for the leading coefficient of the series (4.6.1) we have

$$G_{mm}(k) x^{1-k} = x^{1-k} \frac{\Gamma(1-k-\hbar^{-1})(-\hbar)^{-D}}{\Gamma(1-k+D-\hbar^{-1})} = x^{1-k}.$$

We have

$$\Phi_k^{mm}(x) = \sum_{j=0}^{\infty} \frac{x^{j+1-k}}{\hbar^j j!} s_j(\mathbf{t}) \prod_{l=1}^{j} \frac{1}{1-\hbar(l-k)}.$$

4.6.2 Quantum curve from KS operators

We construct the KS operators by conjugation:

$$\begin{aligned} a_{mm} = G_{mm}\, a_{\bullet}\, G_{mm}^{-1} &= \sum_{k=1}^{\infty} k\tilde{t}_k x^k \frac{(-\hbar)^{-k}\Gamma(D-\hbar^{-1})}{\Gamma(D+k-\hbar^{-1})} - D \\ &= \sum_{k=1}^{\infty} k\tilde{t}_k x^k \prod_{j=0}^{k-1} \frac{1}{1-\hbar(D+j)} - D. \end{aligned}$$

It follows from Equation (4.4) that this operator annihilates the first basis vector. This implies that the wave function given by

$$\begin{aligned} \Psi^{mm}(x,\hbar) &= \Phi_1^{mm}(x)\big|_{\tilde{t}_k \mapsto \tilde{t}_k/\hbar} \\ &= \frac{\Gamma(-\hbar^{-1})(-\hbar)^{-D}}{\Gamma(D-\hbar^{-1})} e^{\frac{1}{\hbar}\sum_{k=1}^{\infty} \tilde{t}_k x^k} \end{aligned}$$

is annihilated by the operator A_{mm}, where

$$A_{mm} := \sum_{k=1}^{\infty} k\tilde{t}_k x^k \prod_{j=0}^{k-1} \frac{1}{1-\hbar(D+j)} - \hbar D. \tag{4.10}$$

We call the operator A_{mm} a general quantum curve.

If $\tilde{t}_k = 0$ for all $k > l$ with some finite l, then the quantum curve can be reduced to a polynomial one:

$$A_{mm} = \sum_{k=1}^{l} k\tilde{t}_k \left(\prod_{j=1}^{k} \frac{1}{1-\hbar(D-j)} \right) x^k - \hbar D$$
$$= \left(\prod_{j=1}^{l} \frac{1}{1-\hbar(D-j)} \right) \tilde{A}_{mm},$$

where

$$\tilde{A}_{mm} := \sum_{k=1}^{l} k\tilde{t}_k x^k \prod_{j=1}^{l-k} (1-\hbar(D-j)) - \hbar D \prod_{j=1}^{l} (1-\hbar(D-j))$$

also annihilates the wave function:

$$\tilde{A}_{mm} \Psi^{mm}(x,\hbar) = 0.$$

Introducing the operators $\hat{x} = x\cdot$, $\hat{y} = -\hbar\frac{\partial}{\partial x}$, we obtain

$$\tilde{A}_{mm} = \sum_{k=1}^{l} k\tilde{t}_k \hat{x}^k \prod_{j=1}^{l-k} (1+\hat{x}\hat{y}+\hbar j) + \hat{x}\hat{y} \prod_{j=1}^{l} (1+\hat{x}\hat{y}+\hbar j).$$

Further specializations of this formula imply the following proposition:

Proposition 4.6.3. *The quantum curve for the monotone r-orbifold Hurwitz numbers is equal to*

$$\hat{x} \left(\hat{x}^{r-1} + \prod_{j=1}^{r} (1+\hat{x}\hat{y}+\hbar(j-1))\,\hat{y} \right).$$

In particular, for $r = 1$, it reduces to

$$\hat{x}(\hat{x}\hat{y}^2 + \hat{y} + 1).$$

Remark 4.6.4. These expressions, up to a factor $\hat{x}$, coincide with the quantum curves obtained in [7, 8].

Proof. Monotone r-orbifold Hurwitz numbers correspond to the specialization

$$\tilde{t}_k = \frac{\delta_{k,r}}{r}.$$

In this case the quantum spectral curve (4.6.2) reduces to

$$A = \hat{x}^r + \hat{x}\hat{y} \prod_{j=1}^{r} (1+\hat{x}\hat{y}+\hbar j)$$
$$= \hat{x} \left(\hat{x}^{r-1} + \prod_{j=1}^{r} (1+\hat{x}\hat{y}+\hbar(j-1))\,\hat{y} \right)$$

□

4.6.3 Linear equations for the tau-function

In this Section we derive some linear equations for the tau-function of double monotone Hurwitz numbers $\tau_{mm}(\mathbf{t},\tilde{\mathbf{t}})$.

Recall that the boson-fermion correspondence allows us to translate the operators in $w_{1+\infty}$ into the differential operators in the variables $\mathbf{t}$ in $\widehat{\mathfrak{gl}}_\infty$. The general formula reads:

$$(xD)^m\, x^k \mapsto \widehat{Y}_{(xD)^m x^k} := \operatorname*{Res}_{x=0}\left(x^{-k} \,{}^*_*\, \frac{(\widehat{J}(x)+\partial_x)^m}{m+1}\widehat{J}(x)\,{}^*_*\right) dx,$$

where the operators $(xD)^m\, x^k$, $m\geq 0$, $k\in\mathbb{Z}$, span $w_{1+\infty}$, $D = x\frac{\partial}{\partial x}$. We refer to [2] for a detailed exposition of this correspondence.

Remark 4.6.5. Note that the operator $\widehat{Y}_a$ is a finite-order differential operator if and only if $a \in w_{1+\infty}$ is a differential operator, that is a polynomial in D.

Proposition 4.6.6. *The tau-function $\tau_{mm}(\mathbf{t},\tilde{\mathbf{t}})$ satisfies the following linear identities:*

$$\widehat{R}_n\tau_{mm}(\mathbf{t},\tilde{\mathbf{t}}) = n\tilde{t}_n\tau_{mm}(\mathbf{t},\tilde{\mathbf{t}}),\quad n=1,2,\dots.$$

where

$$\widehat{R}_n := \sum_{k=0}^{n}(-\hbar)^k \sum_{1\leq i_1<i_2<\cdots<i_k\leq n} \widehat{Y}_{x^{-n}(D-i_1)\cdots(D-i_k)}.$$

Moreover, this system of identities determines the tau-function uniquely up to a constant factor.

Proof. Consider the KS operator

$$b_{mm} := G_{mm}\, b_\bullet\, G_{mm}^{-1} = x^{-1}\frac{(-\hbar)\Gamma(D-\hbar^{-1})}{\Gamma(D-1-\hbar^{-1})} = x^{-1}(1-\hbar(D-1)).$$

Using that $Dx^{-1} = x^{-1}(D-1)$, we have:

$$b_{mm}^n = x^{-n}(1-\hbar(D-1))(1-\hbar(D-2))\cdots(1-\hbar(D-n)).$$

Hence we have that $\widehat{R}_n = \widehat{Y}_{b_{mm}^n}$, $n=1,2,\dots$. Since the operators b_{mm}^n are polynomial in D and preserve $\{\Phi^{mm}\}$, the corresponding differential operators are finite degree operators in $\mathbf{t}$ that satisfy

$$\widehat{Y}_{b_{mm}^n}\tau_{mm} = c_n(\tilde{\mathbf{t}})\tau_{mm}. \tag{4.11}$$

We have to determine the coefficients $c_n(\tilde{\mathbf{t}})$, $n=1,2,\dots$.

Note that Equation (4.11) is obtained by conjugation with $\mathcal{D}^{(\sigma)}$, where $\mathcal{D}^{(\sigma)}$ is now considered also as a differential operator in $\mathbf{t}$, of the following equation for $\tau_\bullet(\mathbf{t},\tilde{\mathbf{t}})$:

$$\widehat{Y}_{(b_\bullet)^n}\tau_\bullet = c_n(\tilde{\mathbf{t}})\tau_\bullet,\quad ,n=1,2,\dots.$$

The last equation can be rewritten as

$$\frac{\partial}{\partial t_n}\tau_\bullet = n\tilde{t}_n\tau_\bullet,\quad ,n=1,2,\dots. \tag{4.12}$$

Thus we see that $c_n(\tilde{\mathbf{t}_n}) = n\tilde{t}_n$, and since Equations (4.12) determine the tau-function $\tau_\bullet$ up to a constant factor, the same is true for Equations (4.11) and the tau-function τ_{mm}. ☐

Remark 4.6.7. By construction, the operators $\widehat{R}_n$, $n \geq 1$, commute.

Example 4.6.8. Let us list the first two operators, $\widehat{R}_1$ and $\widehat{R}_2$. We have:

$$\widehat{R}_1 = \frac{\partial}{\partial t_1} - \hbar \widehat{L}_1,$$
$$\widehat{R}_2 = \frac{\partial}{\partial t_2} - 2\hbar \widehat{L}_2 + \hbar^2 \widehat{M}_2,$$

where

$$\widehat{L}_m = \widehat{Y}_{x^{-m}\left(D - \frac{m+1}{2}\right)} = \frac{1}{2} \sum_{a+b=m} {}^*_* \widehat{J}_a \widehat{J}_b {}^*_*,$$
$$\widehat{M}_m = \widehat{Y}_{x^{-m}\left(D^2 - (m+1)D + \frac{(1+m)(2+m)}{6}\right)} = \frac{1}{3} \sum_{a+b+c=m} {}^*_* \widehat{J}_a \widehat{J}_b \widehat{J}_c {}^*_*$$

are some standard infinitesimal symmetries of KP, see e.g. [2].

Remark 4.6.9. For the tau-function of the double monotone Hurwitz numbers all possible Kac–Schwarz operators that are polynomial in D are given by the polynomials of b_{mm}. However, for particular specializations of the parameters $\tilde{t}_k$, some other polynomial Kac–Schwarz operators can appear. In particular, for the single monotone Hurwitz numbers (Example 4.5.8 with $r = 1$) we have the following Kac–Schwarz operator:

$$c_{mm} := a_{mm}^2 - b_{mm}^{-1} a_{mm} + (1 + \hbar^{-1}) a_{mm} = z - \hbar D + \hbar^2 D(D-1).$$

The corresponding equation for the tau-function is

$$\widehat{Y}_{c_{mm}} \left[\tau_{mm}(\mathbf{t}, \tilde{\mathbf{t}})|_{\tilde{t}_k = \hbar^{-1}\delta_{k,1}, k \geq 1} \right] = 0,$$

where

$$\widehat{Y}_{c_{mm}} = \hbar^2 \widehat{M}_0 - \hbar \widehat{L}_0 + t_1,$$

is equivalent to the cut-and-join equation of [27]. Similar operators can be easily found for higher r.

4.6.4 ELSV-type formula

We denote by $h^{\leq}_{g,\mu}$ the monotone Hurwitz numbers for the connected covering surface of genus g. The generating function for these numbers is the logarithm of the one we have in Example 4.5.8 for $r = 1$. The following quasi-polynomiality property is proved in [27, 28]:

$$h^{\leq}_{g,\mu} = \prod_{i=1}^{n} \binom{2\mu_i}{\mu_i} P^{\leq}_{g,n}(\mu_1, \ldots, \mu_{\ell(\mu)})$$

for some polynomial $P^{\leq}_{g,n}$. Based on this formula the authors conjectured that there should be an ELSV-type formula for these numbers.

The topological recursion for these numbers is proved in [8]. They prove that the expansions of the correlation differentials of the curve $x = -(y+1)/y^2$ are given by

$$\omega_{g,n}^{\leq}(z_1,\dots,z_n) = d_1\cdots d_n \sum_{\vec{\mu}\in(\mathbb{N}^\times)^n} h_{g,\mu}^{\leq} \prod_{i=1}^{\ell(\mu)} x_i^{\mu_i}. \tag{4.13}$$

Remark 4.6.10. Here by $\vec{\mu}$ we denote a vector, that is, we don't assume that $\mu_1 \geq \cdots \geq \mu_n$. We denote by μ the partition of length n whose parts are the ordered components of the vector $\vec{\mu}$.

This is sufficient to prove the following:

Proposition 4.6.11. *We have:*

$$h_{g,\mu}^{\leq} = \prod_{i=1}^{\ell(\mu)} \binom{2\mu_i}{\mu_i} \int_{\overline{\mathcal{M}}_{g,\ell(\mu)}} e^{\sum_{l=1} K_l\kappa_l} \prod_{j=1}^{\ell(\mu)} \sum_{d_j\geq 0} \psi_j^{d_j} \frac{(2(\mu_j+d_j)-1)!!}{(2\mu_j-1)!!}. \tag{4.14}$$

Here the coefficients K_i, $i=1,2,\dots$, satisfy the following equation:

$$\exp\left(-\sum_{l=1}^{\infty} K_l U^l\right) = \sum_{k=0}^{\infty}(2k+1)!!U^k. \tag{4.15}$$

Remark 4.6.12. The cohomological field theory in this formula is given by the class $\exp\left(\sum_{l=1} K_l\kappa_l\right)$. This type of cohomological field theories of rank 1 with a non-flat unit is considered in detail in [38].

Proof. First note that $x(y)$ has a single critical point $y_{cr} = -2$ with the critical value $x_{cr} := x(y_{cr}) = 1/4$. The local coordinate ζ around y_{cr} and its inverse read

$$\zeta := \sqrt{x - x_{crit}} = i\frac{y+2}{2y}; \qquad y = \frac{2i}{2\zeta - i}.$$

We expand y near $\zeta = 0$:

$$y(\zeta) = \sum_{k\geq 0} s_k\zeta^k, \qquad s_k = i^k(-2)^{k+1}, \tag{4.16}$$

in particular for odd coefficients we have $s_{2k+1} = 4i(-4)^k$.

The correlation differentials $\omega_{g,n}$ produced by the topological recursion can be expressed as sums over graphs (see [19, 20, 13]). In the case when the spectral curve has a single branch point Theorem 3.3. in [19] gives an explicit formula for the $\omega_{g,n}$'s. Since the local coordinate ζ is in fact a global coordinate on the sphere, the Bergman kernel is equal to

$$\frac{d\zeta_1 d\zeta_2}{(\zeta_1-\zeta_2)^2}.$$

This means that the Bergman kernel has trivial regular part near the critical point, and the expression in term of stable graphs simplifies sensibly since only stable graphs with a single vertex appear. It can be written as

$$\omega_{g,n}(\vec{\zeta}) = (-2s_1)^{2-2g-n} \sum_{\vec{d}\in\mathbb{N}^n} \prod_{i=1}^{n} (2d_i+1)!! \frac{d\zeta_i}{\zeta_i^{2d_i+2}} \times \tag{4.17}$$
$$\int_{\overline{\mathcal{M}}_{g,n}} \prod_{j=1}^{n} \psi_j^{d_j} \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{\vec{\alpha}\in(\mathbb{N}^\times)^m} \prod_{k=1}^{m} \left(-(2\alpha_k+1)!! \frac{s_{2\alpha_k+1}}{s_1} \right) \kappa_{\alpha_1,\dots,\alpha_m}$$

(cf. [13, Equation (3.53)]), where the coefficients s_k are given by Equation (4.16).

In order to rewrite Equation (4.17) as an expansion in $x_1, \dots, x_n$ near $y = -1$, we observe that

$$\frac{(2a+1)!!d\zeta}{\zeta^{2a+2}} = 2i(-4)^a \, d \sum_{l=0}^{\infty} \binom{2l}{l} x^l \cdot \frac{(2(l+a)-1)!!}{(2l-1)!!}. \tag{4.18}$$

Indeed, this follows from equation

$$\frac{(2a+1)!!d\zeta}{\zeta^{2a+2}} = d\left(-\frac{d}{\zeta d\zeta}\right)^a (-\zeta^{-1}),$$

and expansion

$$-\zeta^{-1} = 2i \sum_{l=0}^{\infty} \binom{2l}{l} x^l.$$

The multi-index kappa classes can be written as exponent of sum of single kappa classes:

$$\sum_{m=0}^{\infty} \frac{1}{m!} \sum_{\vec{\alpha}\in(\mathbb{N}^\times)^m} \prod_{k=1}^{m} f(\alpha_k)\kappa_{\alpha_1,\dots,\alpha_m} = \exp\left(\sum_{l=1}^{\infty} K_l \kappa_l\right),$$

where the coefficients K_l can be computed by the expansion

$$\exp\left(-\sum_{l=1}^{\infty} K_l U^l\right) = 1 - \sum_{k=1}^{\infty} f(k) U^k.$$

This implies that

$$\sum_{m=0}^{\infty} \frac{1}{m!} \sum_{\vec{\alpha}\in(\mathbb{N}^\times)^m} \prod_{k=1}^{m} -(2\alpha_k+1)!!\kappa_{\alpha_1,\dots,\alpha_m} = \exp\left(\sum_{l=1}^{\infty} K_l \kappa^l\right), \tag{4.19}$$

where $\exp\left(-\sum_{l=1}^{\infty} K_l U^l\right) = \sum_{k=0}^{\infty} (2k+1)!! U^k$.

Finally, observe that if $\sum_{k=1}^{n} d_k + \sum_{k=1}^{m} \alpha_m = 3g-3+n$, then

$$(-2s_1)^{2-2g-n} \prod_{k=1}^{n} \left((-4)^{d_k} \cdot (2i)\right) \cdot \prod_{k=1}^{m} \frac{s_{2\alpha_k+1}}{s_1} = 1. \tag{4.20}$$

Now we are ready to complete the proof of the proposition. Note that Equation (4.13) implies that

$$\sum_{\mu\in(\mathbb{N}^\times)^n} h^{\leq}_{g,\mu} x_1^{\mu_1} \cdots x_n^{\mu_n} = \int \cdots \int \omega^{\leq}_{g,n}$$

On the other hand Equations (4.17), (4.18), (4.19), and (4.20) imply that

$$\int \cdots \int \omega^{\leq}_{g,n} =$$
$$\sum_{\vec{\mu} \in (\mathbb{N}^\times)^n} \sum_{\vec{d} \in (\mathbb{N})^n} \int_{\overline{\mathcal{M}}_{g,n}} e^{\sum_{l=1} K_l \kappa_l} \prod_{j=1}^{n} \psi_j^{d_j} \frac{(2(\mu_j + d_j) - 1)!!}{(2\mu_j - 1)!!} \binom{2\mu_j}{\mu_j} x_j^{\mu_j}$$

for K_l given by Equation (4.15), which is equivalent to Equation (4.14). □

Remark 4.6.13. After we shared this formula with colleagues, we learned from N. Do that he and M. Karev derived the same formula independently, using the geometric approach to topological recursion due to M. Kazarian.

4.7 Further examples of quantum curves

4.7.1 Strictly monotone orbifold Hurwitz numbers

By Proposition 4.4.8 strictly monotone orbifold Hurwitz problem is equivalent to the enumeration of hypermaps. Its tau-function is given in Example 4.5.9.

By principal specialization of Schur functions near infinity, the corresponding wave function is equal to

$$\begin{aligned}\Psi(x^{-1}, \hbar) &= \sum_{n=0}^{\infty} \frac{x^{-rn}}{n!\hbar^n r^n} \sum_{k=0}^{\infty} \sigma_k(cr^{(rn,0,\ldots,0)})\hbar^k \qquad (4.21)\\ &= \sum_{n=0}^{\infty} \frac{x^{-rn}}{n!\hbar^n r^n} \prod_{j=1}^{rn-1} (1 + j\hbar).\end{aligned}$$

In order to get a curve, consistent with results of [9, 12], here we consider the wave function as a series in the variable x^{-1} instead of x.

Proposition 4.7.1. *We have:*

$$\left[\hat{x}^{\frac{1}{\hbar}}(\hat{y}^r - \hat{x}\hat{y} + 1)\hat{x}^{-\frac{1}{\hbar}}\right] \Psi(x^{-1}, \hbar) = 0,$$

where $\hat{x} = x\cdot$ *and* $\hat{y} = -\hbar\frac{\partial}{\partial x}$.

Proof. Let a_n be the nth summand in Equation (4.21). We have:

$$\hbar r(n+1)a_{n+1} = x^{-r} \prod_{j=0}^{r-1} [1 + (nr + j)\hbar] a_n.$$

In terms of the operators this can be rewritten as

$$-\hbar x \frac{\partial}{\partial x} a_{n+1} = \left[x^{-1}\left(1 - \hbar x \frac{\partial}{\partial x}\right)\right]^r x^r a_n.$$

Hence we obtain

$$x^{\frac{1}{\hbar}}\left[\left(-\hbar\frac{\partial}{\partial x}\right)^r+\hbar x\frac{\partial}{\partial x}+1\right]x^{-\frac{1}{\hbar}}\,\Psi(x^{-1},\hbar)=0.$$

□

Remark 4.7.2. This quantum curve was earlier obtained in [9] using combinatorics of hypermaps and in [12] using the loop equations for hypermaps. Comparison with this results also forces us to use the variable x^{-1} instead of x.

Remark 4.7.3. Even though we presented here a purely combinatorial derivation of the quantum curve, it is worth mentioning that one can derive it for a more general double strictly monotone Hurwitz problem using the method of Section 4.6.2. In this case the operator given by Equation (4.10) is replaced by

$$\sum_{k=1}^{\infty}k\tilde{t}_k x^k\prod_{j=0}^{k-1}(1+\hbar(D+j))-\hbar D,$$

whose specialization for $\tilde{t}_k=\delta_{k,r}/r$ is equivalent to the operator above after the change of variable $x\mapsto x^{-1}$.

4.7.2 Blocks of atlantes

We consider a Hurwitz theory given by a vector of blocks of atlantes of some fixed type, that is, the vector of blocks is equal to $\vec{\mathcal{P}}=(B_r^\times,\dots,B_r^\times)$ for some fixed $r\geq 1$. We also assume that $\nu=(1^{|\mu|})$, see Example 4.5.10 for $q=1$. The corresponding wave function is equal to

$$\begin{aligned}\Psi(x,\hbar)&=\sum_{n=0}^{\infty}\frac{x^n}{n!\hbar^n}\exp(p_r(cr^{(n,0,\dots,0)})\hbar^r)\\&=\sum_{n=0}^{\infty}\frac{x^n}{n!\hbar^n}\exp\left(\hbar^r\sum_{j=1}^{n-1}j^r\right)\end{aligned}\tag{4.22}$$

Proposition 4.7.4. *We have:*

$$[\hat{y}-\hat{x}e^{\hat{y}^r}]\Psi(x,\hbar)=0,$$

where $\hat{x}=x\cdot$ and $\hat{y}=\hbar x\frac{\partial}{\partial x}$ (it is more convenient to use the exponential coordinate in this case, cf. [41]).

Proof. Let a_n be the nth summand in Equation (4.22). We have:

$$\hbar(n+1)a_{n+1}=xe^{\hbar^r n^r}a_n$$

In terms of the operators this can be rewritten as

$$\hbar x\frac{\partial}{\partial x}a_{n+1}=xe^{(\hbar x\frac{\partial}{\partial x})^r}a_n.$$

Therefore,

$$\left[\hbar x\frac{\partial}{\partial x}-xe^{(\hbar x\frac{\partial}{\partial x})^r}\right]\Psi(x,\hbar)=0.$$

□

Remark 4.7.5. This case is very interesting since we can say in advance that the logic outlined in Section 4.1.2 fails. Indeed, the dequantization of this quantum curve consides with the dequantization of the quantum curve for the r-spin Hurwitz number

$$\hat{y} - \hat{x}^{3/2} \exp\left(\frac{1}{r+1}\sum_{i=0}^{r} \hat{x}^{-1}\hat{y}^{i}\hat{x}\hat{y}^{r-i}\right)$$

proved in [41]. Even though the spectral curve and the corresponding r-ELSV formula for the r-spin Hurwitz numbers are still conjectural, there is very strong evidence for these conjectures to be true [49]. From these conjectures we can conclude that the dequantization of $\hat{y} - \hat{x}\exp(\hat{y}^r)$ can not be the spectral curve for the atlantes Hurwitz numbers, suitable for the construction of the topological recursion.

Indeed, even though in genus zero atlantes Hurwitz numbers coincide with the r-spin Hurwitz numbers (and hence all data of the spectral curve must be the same), in higher genera this is no longer the case.

4.7.3 Double Hurwitz numbers

The partition function of the double Hurwitz numbers is

$$\tau_{HH}(\mathbf{t}, \tilde{\mathbf{t}}) = \langle 0| \exp\left(\sum_{i=1}^{\infty} t_i \alpha_i\right) \exp\left(\hbar \mathcal{F}_2\right) \exp\left(\sum_{i=1}^{\infty} \tilde{t}_i \alpha_{-i}\right) |0\rangle.$$

or

$$\tau_{HH}(\mathbf{t}; \tilde{\mathbf{t}}) = \sum_{\lambda} s_\lambda(\mathbf{t}) s_\lambda(\tilde{\mathbf{t}}) e^{\hbar\, \mathsf{egv}_\lambda(C_2)}.$$

Then, the basis vectors for this tau-function as a KP tau-function with respect to times t_k is

$$\Phi_k^{HH}(x) = e^{\frac{\hbar}{2}\left((D-\frac{1}{2})^2-(k-\frac{1}{2})^2\right)} e^{\sum_{j=1}^{\infty}\tilde{t}_j x^j} x^{1-k}.$$

The wave function is given, as usual, by a rescaling of $\Phi_1^{HH}(x)$:

$$\Psi(x, \hbar) := \Phi_1^{HH}(x)|_{\tilde{t}_k \mapsto \hbar^{-1}\delta_{k,1}, k\geq 1}$$

Proposition 4.7.6. *We have:*

$$\sum_{k=1}^{\infty} k\tilde{t}_k \left(\hat{x}e^{\hat{y}}\right)^k - \hat{y}\Psi(e^x, \hbar) = 0,$$

where $\hat{x} = x\cdot$ and $\hat{y} = \hbar D$.

Proof. To obtain the KS operators for the generating function of double Hurwitz numbers we use the conjugation of the operators (4.2.2):

$$\begin{aligned} a_{HH} &= e^{\frac{\hbar}{2}m_0} a_\bullet\, e^{-\frac{\hbar}{2}m_0} \\ &= \sum_{k=1}^{\infty} k\tilde{t}_k \left(x\exp\left(\hbar D\right)\right)^k - D \\ &= \sum_{k=1}^{\infty} k\tilde{t}_k e^{\frac{\hbar}{2}k(k-1)} x^k \exp\left(\hbar k D\right) - D, \end{aligned} \tag{4.23}$$

where $m_0 := \left(D-\frac{1}{2}\right)^2+\frac{1}{12}$ (the constant $\frac{1}{12}$ is not important for the calculations, but this way we get one of the standard generators of $w_{1+\infty}$, cf. the operator $\widehat{M}_0$ in Example 4.6.8). The KS operators (4.23) act of the basis vectors as follows:

$$a_{HH}\,\Phi_k^{HH}(z)=(k-1)\Phi_k^{HH}(z).$$

The operator a_{HH} annihilates $\Phi_1^{HH}(x)$ and, therefore, describes the quantum spectral curve for this model. Namely, we have

$$A_{HH}\,\Psi(x,\hbar)=0$$

where

$$A_{HH}=\sum_{k=1}^{\infty}k\tilde{t}_k e^{\frac{\hbar}{2}k(k-1)}x^k\exp\left(\hbar kD\right)-\hbar D, \tag{4.24}$$

□

Remark 4.7.7. The wave function in this case is also given by the integral

$$\Psi(x,\hbar)=\frac{e^{-\frac{\hbar}{8}}}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}dy\,\exp\left(-\frac{y^2}{2\hbar}-\frac{y}{2}+\sum_{k=1}^{\infty}\frac{\tilde{t}_k}{\hbar}(xe^y)^k\right).$$

considered as a formal series in $\tilde{t}_k$.

Remark 4.7.8. Particular specifications of $\tilde{\mathbf{t}}_k$ describe interesting examples of this model, in particular usual simple Hurwitz numbers [2], triple Hodge integrals and string amplitude for the resolved conifold [51]. Quantum spectral curves for all these examples are given by specifications of the more general expression (4.24).

A particular example: one-parameter deformation of single Hurwitz numbers

Let us discuss an example of a particular specialization of double Hurwitz numbers given by $\tilde{t}_k = c^{k-1}$, $k=1,2,\dots$. This gives a one-parameter deformation of single Hurwitz numbers considered in Example 4.5.7. Up to a simple combinatorial factor, this is equivalent to the Hurwitz theory for the vector of blocks $(B_2^<,\dots,B_2^<,B_r^<)$ and $\nu=(1^{|\mu|})$ (recall that $B_r^<=B_r^|$ by Proposition 4.4.4).

In this case the wave function is given by

$$\Psi(x,\hbar)=\langle 0|\,\exp\left(\sum_{i=1}^{\infty}\frac{x^i\alpha_i}{i}\right)\exp\left(\hbar\mathcal{F}_2\right)\exp\left(\sum_{i=1}^{\infty}\frac{c^{i-1}\alpha_{-i}}{\hbar}\right)|0\rangle.$$

Equation (4.24) reduces to

$$A_{HH}=\frac{\hat{x}e^{\hat{y}}}{(1-c\,\hat{x}e^{\hat{y}})^2}-\hat{y}.$$

Let us multiply this operator by $\frac{(1-c\,\hat{x}e^{\hat{y}})^2}{\hat{x}e^{\hat{y}}}$. The resulting equation for the wave function

$$\left(1-\left(e^{-\hat{y}}\hat{x}^{-1}-2c+c^2\hat{x}e^{\hat{y}}\right)\hat{y}\right)\Psi(x,\hbar)=0$$

describes the quantum spectral curve for this case.

Remark 4.7.9. The restriction of the wave function $\Psi(x, \hbar)$ to $c = 0$ is the wave function of the single Hurwitz numbers, and in this special case we recover the quantum spectral curve $e^{-\hat{y}}\hat{x}^{-1}\hat{y}-1$, which is equivalent to the one that was proved in this case in [50].

This quantum spectral curve equation suggests that the spectral curve for the one-parameter family of Hurwitz numbers that we consider here should be

$$ye^{-y} - (1 + 2cy)x + c^2 ye^y x^2 = 0,$$

which is a deformation of the Lambert curve.

Bibliography

[1] A. Alexandrov. Matrix models for random partitions. *Nuclear Phys. B*, 851(3):620–650, 2011. ISSN 0550-3213. doi: 10.1016/j.nuclphysb.2011.06.007. URL http://dx.doi.org/10.1016/j.nuclphysb.2011.06.007.

[2] A. Alexandrov. Enumerative geometry, tau-functions and Heisenberg-Virasoro algebra. *Comm. Math. Phys.*, 338(1):195–249, 2015. ISSN 0010-3616. doi: 10.1007/s00220-015-2379-8. URL http://dx.doi.org/10.1007/s00220-015-2379-8.

[3] A. Alexandrov, A. Mironov, A. Morosov, and S. Natanzon. Integrability of Hurwitz partition functions. *J. Phys. A*, 45(4):045209, 10, 2012. ISSN 1751-8113. doi: 10.1088/1751-8113/45/4/045209. URL http://dx.doi.org/10.1088/1751-8113/45/4/045209.

[4] A. Alexandrov, A. Mironov, A. Morozov, and S. Natanzon. On KP-integrable Hurwitz functions. *J. High Energy Phys.*, (11):080, front matter+30, 2014. ISSN 1126-6708.

[5] J. Ambjørn and L. Chekhov. The matrix model for dessins d'enfants. *ArXiv e-prints*, April 2014.

[6] Mireille Bousquet-Mélou and Gilles Schaeffer. Enumeration of planar constellations. *Adv. in Appl. Math.*, 24(4):337–368, 2000. ISSN 0196-8858. doi: 10.1006/aama.1999.0673. URL http://dx.doi.org/10.1006/aama.1999.0673.

[7] N. Do and M. Karev. Monotone orbifold Hurwitz numbers. *ArXiv e-prints*, May 2015.

[8] N. Do, A. Dyer, and D. V. Mathews. Topological recursion and a quantum curve for monotone Hurwitz numbers. *ArXiv e-prints*, August 2014.

[9] Norman Do and David Manescu. Quantum curves for the enumeration of ribbon graphs and hypermaps. *Commun. Number Theory Phys.*, 8(4):677–701, 2014. ISSN 1931-4523. doi: 10.4310/CNTP.2014.v8.n4.a2. URL http://dx.doi.org/10.4310/CNTP.2014.v8.n4.a2.

[10] Olivia Dumitrescu, Motohico Mulase, Brad Safnuk, and Adam Sorkin. The spectral curve of the Eynard-Orantin recursion via the Laplace transform. In *Algebraic and geometric aspects of integrable systems and random matrices*, volume 593 of *Contemp. Math.*, pages 263–315. Amer. Math. Soc., Providence, RI, 2013. doi: 10.1090/conm/593/11867. URL http://dx.doi.org/10.1090/conm/593/11867.

[11] P. Dunin-Barkowski, M. Mulase, P. Norbury, A. Popolitov, and S. Shadrin. Quantum spectral curve for the Gromov-Witten theory of the complex projective line. *ArXiv e-prints*, December 2013.

[12] P. Dunin-Barkowski, N. Orantin, A. Popolitov, and S. Shadrin. Combinatorics of loop equations for branched covers of sphere. *ArXiv e-prints*, December 2014.

[13] P. Dunin-Barkowski, N. Orantin, S. Shadrin, and L. Spitz. Identification of the Givental formula with the spectral curve topological recursion procedure. *Comm. Math. Phys.*, 328 (2):669–700, 2014. ISSN 0010-3616. doi: 10.1007/s00220-014-1887-2. URL http://dx.doi.org/10.1007/s00220-014-1887-2.

[14] P. Dunin-Barkowski, M. Kazarian, N. Orantin, S. Shadrin, and L. Spitz. Polynomiality of Hurwitz numbers, Bouchard-Mariño conjecture, and a new proof of the ELSV formula. *Adv. Math.*, 279:67–103, 2015. ISSN 0001-8708. doi: 10.1016/j.aim.2015.03.016. URL http://dx.doi.org/10.1016/j.aim.2015.03.016.

[15] P. Dunin-Barkowski, D. Lewanski, A. Popolitov, and S. Shadrin. Polynomiality of orbifold Hurwitz numbers, spectral curve, and a new proof of the Johnson-Pandharipande-Tseng formula. *ArXiv e-prints*, April 2015.

[16] Petr Dunin-Barkowski, Sergey Shadrin, and Loek Spitz. Givental graphs and inversion symmetry. *Lett. Math. Phys.*, 103(5):533–557, 2013. ISSN 0377-9017. doi: 10.1007/s11005-013-0606-9. URL http://dx.doi.org/10.1007/s11005-013-0606-9.

[17] Petr Dunin-Barkowski, Nicolas Orantin, Alexandr Popolitov, and Sergey Shadrin. Combinatorics of loop equations for branched covers of sphere. *arXiv preprint arXiv:1412.1698*, 2014.

[18] Torsten Ekedahl, Sergei Lando, Michael Shapiro, and Alek Vainshtein. Hurwitz numbers and intersections on moduli spaces of curves. *Invent. Math.*, 146(2):297–327, 2001. ISSN 0020-9910. doi: 10.1007/s002220100164. URL http://dx.doi.org/10.1007/s002220100164.

[19] B. Eynard. Intersection numbers of spectral curves. *ArXiv e-prints*, April 2011.

[20] B. Eynard. Invariants of spectral curves and intersection theory of moduli spaces of complex curves. *Commun. Number Theory Phys.*, 8(3):541–588, 2014. ISSN 1931-4523. doi: 10.4310/CNTP.2014.v8.n3.a4. URL http://dx.doi.org/10.4310/CNTP.2014.v8.n3.a4.

[21] B. Eynard and N. Orantin. Invariants of algebraic curves and topological expansion. *Commun. Number Theory Phys.*, 1(2):347–452, 2007. ISSN 1931-4523. doi: 10.4310/CNTP.2007.v1.n2.a4. URL http://dx.doi.org/10.4310/CNTP.2007.v1.n2.a4.

[22] Bertrand Eynard, Motohico Mulase, and Bradley Safnuk. The Laplace transform of the cut-and-join equation and the Bouchard-Mariño conjecture on Hurwitz numbers. *Publ. Res. Inst. Math. Sci.*, 47(2):629–670, 2011. ISSN 0034-5318. doi: 10.2977/PRIMS/47. URL http://dx.doi.org/10.2977/PRIMS/47.

[23] H. K. Farahat and G. Higman. The centres of symmetric group rings. *Proc. Roy. Soc. London Ser. A*, 250:212–221, 1959. ISSN 0962-8444.

[24] Masafumi Fukuma, Hikaru Kawai, and Ryuichi Nakayama. Infinite-dimensional Grassmannian structure of two-dimensional quantum gravity. *Comm. Math. Phys.*, 143(2):371–403, 1992. ISSN 0010-3616. URL http://projecteuclid.org/euclid.cmp/1104248961.

[25] Alain Giorgetti and Timothy RS Walsh. Enumeration of hypermaps of a given genus. *arXiv preprint arXiv:1510.09019*, 2015.

[26] I. P. Goulden and D. M. Jackson. The KP hierarchy, branched covers, and triangulations. *Adv. Math.*, 219(3):932–951, 2008. ISSN 0001-8708. doi: 10.1016/j.aim.2008.06.013. URL http://dx.doi.org/10.1016/j.aim.2008.06.013.

[27] I. P. Goulden, M. Guay-Paquet, and J. Novak. Monotone Hurwitz numbers and the HCIZ integral II. *ArXiv e-prints*, July 2011.

[28] I. P. Goulden, Mathieu Guay-Paquet, and Jonathan Novak. Polynomiality of monotone Hurwitz numbers in higher genera. *Adv. Math.*, 238:1–23, 2013. ISSN 0001-8708. doi: 10.1016/j.aim.2013.01.012. URL http://dx.doi.org/10.1016/j.aim.2013.01.012.

[29] I. P. Goulden, Mathieu Guay-Paquet, and Jonathan Novak. Monotone Hurwitz numbers and the HCIZ integral. *Ann. Math. Blaise Pascal*, 21(1):71–89, 2014. ISSN 1259-1734. URL http://ambp.cedram.org/item?id=AMBP_2014__21_1_71_0.

[30] Sergei Gukov and Piotr Sułkowski. A-polynomial, B-model, and quantization. In *Homological mirror symmetry and tropical geometry*, volume 15 of *Lect. Notes Unione Mat. Ital.*, pages 87–151. Springer, Cham, 2014. doi: 10.1007/978-3-319-06514-4_4. URL http://dx.doi.org/10.1007/978-3-319-06514-4_4.

[31] J. Harnad and A. Yu. Orlov. Hypergeometric τ-functions, Hurwitz numbers and enumeration of paths. *Comm. Math. Phys.*, 338(1):267–284, 2015. ISSN 0010-3616. doi: 10.1007/s00220-015-2329-5. URL http://dx.doi.org/10.1007/s00220-015-2329-5.

[32] A.-A. A. Jucys. Symmetric polynomials and the center of the symmetric group ring. *Rep. Mathematical Phys.*, 5(1):107–112, 1974. ISSN 0034-4877.

[33] M. Kazarian. KP hierarchy for Hodge integrals. *Adv. Math.*, 221(1):1–21, 2009. ISSN 0001-8708. doi: 10.1016/j.aim.2008.10.017. URL http://dx.doi.org/10.1016/j.aim.2008.10.017.

[34] M. È. Kazarian and S. K. Lando. Combinatorial solutions to integrable hierarchies. *Uspekhi Mat. Nauk*, 70(3(423)):77–106, 2015. ISSN 0042-1316. doi: 10.4213/rm9661. URL http://dx.doi.org/10.4213/rm9661.

[35] Maxim Kazarian and Peter Zograf. Virasoro constraints and topological recursion for Grothendieck's dessin counting. *Lett. Math. Phys.*, 105(8):1057–1084, 2015. ISSN 0377-9017. doi: 10.1007/s11005-015-0771-0. URL http://dx.doi.org/10.1007/s11005-015-0771-0.

[36] Serguei Kerov and Grigori Olshanski. Polynomial functions on the set of Young diagrams. *C. R. Acad. Sci. Paris Sér. I Math.*, 319(2):121–126, 1994. ISSN 0764-4442.

[37] A. Lascoux and J.-Y. Thibon. Vertex operators and the class algebras of symmetric groups. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)*, 283(Teor. Predst. Din. Sist. Komb. i Algoritm. Metody. 6):156–177, 261, 2001. ISSN 0373-2703. doi: 10.1023/B:JOTH.0000024619.77778.3d. URL http://dx.doi.org/10.1023/B:JOTH.0000024619.77778.3d.

[38] Yuri I. Manin and Peter Zograf. Invertible cohomological field theories and Weil-Petersson volumes. *Ann. Inst. Fourier (Grenoble)*, 50(2):519–535, 2000. ISSN 0373-0956. URL http://www.numdam.org/item?id=AIF_2000__50_2_519_0.

[39] A. Mironov, A. Morozov, and G. W. Semenoff. Unitary matrix integrals in the framework of the generalized Kontsevich model. *Internat. J. Modern Phys. A*, 11(28):5031–5080, 1996. ISSN 0217-751X. doi: 10.1142/S0217751X96002339. URL http://dx.doi.org/10.1142/S0217751X96002339.

[40] T. Miwa, M. Jimbo, and E. Date. *Solitons*, volume 135 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2000. ISBN 0-521-56161-2. Differential equations, symmetries and infinite-dimensional algebras, Translated from the 1993 Japanese original by Miles Reid.

[41] M. Mulase, S. Shadrin, and L. Spitz. The spectral curve and the Schrödinger equation of double Hurwitz numbers and higher spin structures. *Commun. Number Theory Phys.*, 7(1):125–143, 2013. ISSN 1931-4523. doi: 10.4310/CNTP.2013.v7.n1.a4. URL http://dx.doi.org/10.4310/CNTP.2013.v7.n1.a4.

[42] A. Okounkov and R. Pandharipande. Gromov-Witten theory, Hurwitz theory, and completed cycles. *ArXiv Mathematics e-prints*, April 2002.

[43] Andrei Okounkov. Toda equations for Hurwitz numbers. *Math. Res. Lett.*, 7(4):447–453, 2000. ISSN 1073-2780. doi: 10.4310/MRL.2000.v7.n4.a10. URL http://dx.doi.org/10.4310/MRL.2000.v7.n4.a10.

[44] Paolo Rossi. Gromov-Witten invariants of target curves via symplectic field theory. *J. Geom. Phys.*, 58(8):931–941, 2008. ISSN 0393-0440. doi: 10.1016/j.geomphys.2008.02.012. URL http://dx.doi.org/10.1016/j.geomphys.2008.02.012.

[45] Mikio Sato and Yasuko Sato. Soliton equations as dynamical systems on infinite-dimensional Grassmann manifold. In *Nonlinear partial differential equations in applied science (Tokyo, 1982)*, volume 81 of *North-Holland Math. Stud.*, pages 259–271. North-Holland, Amsterdam, 1983.

[46] Albert Schwarz. Quantum curves. *Comm. Math. Phys.*, 338(1):483–500, 2015. ISSN 0010-3616. doi: 10.1007/s00220-015-2287-y. URL http://dx.doi.org/10.1007/s00220-015-2287-y.

[47] Graeme Segal and George Wilson. Loop groups and equations of KdV type. *Inst. Hautes Études Sci. Publ. Math.*, (61):5–65, 1985. ISSN 0073-8301. URL http://www.numdam.org/item?id=PMIHES_1985__61__5_0.

[48] S. Shadrin, L. Spitz, and D. Zvonkine. On double Hurwitz numbers with completed cycles. *J. Lond. Math. Soc. (2)*, 86(2):407–432, 2012. ISSN 0024-6107. doi: 10.1112/jlms/jds010. URL `http://dx.doi.org/10.1112/jlms/jds010`.

[49] S. Shadrin, L. Spitz, and D. Zvonkine. Equivalence of ELSV and Bouchard-Mariño conjectures for r-spin Hurwitz numbers. *Math. Ann.*, 361(3-4):611–645, 2015. ISSN 0025-5831. doi: 10.1007/s00208-014-1082-y. URL `http://dx.doi.org/10.1007/s00208-014-1082-y`.

[50] J. Zhou. Quantum Mirror Curves for $\mathbb{C}^3$ and the Resolved Confiold. *ArXiv e-prints*, July 2012.

[51] J. Zhou. Emergent Geometry of KP Hierarchy. II. *ArXiv e-prints*, December 2015.

[52] J. Zhou. Emergent Geometry of KP Hierarchy. *ArXiv e-prints*, November 2015.

[53] P. Zinn-Justin. HCIZ integral and 2D Toda lattice hierarchy. *Nuclear Phys. B*, 634(3):417–432, 2002. ISSN 0550-3213. doi: 10.1016/S0550-3213(02)00374-7. URL `http://dx.doi.org/10.1016/S0550-3213(02)00374-7`.

[54] P. Zograf. Enumeration of Grothendieck's dessins and KP hierarchy. *ArXiv e-prints*, December 2013.

5

Quasi-polynomiality of orbifold Hurwitz numbers, spectral curve, and a new proof of the Johnson-Pandharipande-Tseng formula

In this chapter we present an example of a derivation of an ELSV-type formula using the methods of topological recursion. Namely, for orbifold Hurwitz numbers we give a new proof of the spectral curve topological recursion, in the sense of Chekhov, Eynard, and Orantin, where the main new step compared to the existing proofs is a direct combinatorial proof of their quasi-polynomiality. Spectral curve topological recursion leads to a formula for the orbifold Hurwitz numbers in terms of the intersection theory of the moduli space of curves, which, in this case, appears to coincide with a special case of the Johnson-Pandharipande-Tseng formula.

5.1 Introduction

5.1.1 Main goal

The main goal of this chapter is to present a new important application of the procedure that allows to relate in a uniform way a class of combinatorial problems to the intersection theory of the moduli space of curves. First, let us describe this procedure. The logic behind it is the following one:

- We start with a combinatorial problem that depends in a natural way on a genus parameter $g \geq 0$ and a vector $\vec{\mu} \in \mathbb{Z}^n_{>0}$.
- We consider the generating functions that solve this combinatorial problem. Quite often we can prove that they can be considered as an expansion of certain symmetric differentials $\omega_{g,n}$ that solve the matrix model topological recursion [10, 9] for a particular spectral curve data.
- Under some mild assumptions, the expansion of the symmetric differentials obtained via the topological recursion can be represented (up to some constants) as

$$\sum_{l(\vec{\mu})=n} \sum_{a_1,\dots,a_n=1}^{r} \left[\int_{\overline{\mathcal{M}}_{g,n}} \frac{S(a_1,\dots,a_n)}{\prod_{j=1}^n (1-\psi_j \frac{d}{dx_j})} \right] \prod_{j=1}^n \xi_{a_j}(x_j).$$

Here r is the number of branching points on the spectral curve, $S(a_1, \dots, a_n)$ is a certain explicitly described tautological class on the moduli space of curves, and $\xi_a(x)$ are some auxiliary functions, $a = 1, \dots, r$, also explicitly described [6, 8].

– This way we solve the original combinatorial problem in terms of the intersection numbers of the tautological classes on the moduli space of curves, and the formula that we get is of ELSV-type [7].

The first instance of this way to derive an ELSV-type formula was presented in [5], where this leads to a new proof of the original ELSV formula for ordinary Hurwitz numbers.

In this chapter we perform this whole procedure for the so-called orbifold Hurwitz numbers [14, 12, 4, 3]. The orbifold Hurwitz numbers are a special case of double Hurwitz numbers [13], where the ramification indices in one special fiber are given by an arbitrary partition μ, and in the other special fiber they are all equal to r. The intersection formula that we obtain via this procedure was previously derived by Johnson, Pandharipande, and Tseng [14], and this way we get a new proof of it.

5.1.2 The known facts about orbifold Hurwitz numbers

Let us collect here the known facts about orbifold Hurwitz numbers so that we can summarize all relevant previous papers about them.

Fact 1: *(JPT Formula)* The orbifold Hurwitz numbers are given by the intersection numbers on the moduli space of curves via the Johnson-Pandharipande-Tseng formula.

Fact 2: *(Quasi-Polynomiality)* The orbifold Hurwitz numbers can be represented, up to a particular combinatorial factor, as the values of a polynomial in n variables $\mu_1, \dots, \mu_n$, whose coefficients depend only on $\vec{\mu} \mod r$.

Fact 3: *(Cut-and-Join)* The orbifold Hurwitz numbers satisfy a simple recursion with a clear topological meaning, which is called the cut-and-join equation [11].

Fact 4: *(Topological Recursion)* The n-point generating functions of orbifold Hurwitz numbers can be represented as expansions of the correlation differentials obtained via the Chekhov-Eynard-Orantin topological recursion procedure.

Let us explain what was known before. First of all, we have the Johnson-Pandharipande-Tseng result itself [14]:

(Definition) ⇒ (JPT formula)

The main results of [3] and [4] can be described as follows:

(JPT formula) ⇒ (Quasi-Polynomiality)

(Quasi-Polynomiality) AND (Cut-and-Join) ⇒ (Topological Recursion).

Here the first implication is obvious; though, until now, there was no other proof of quasi-polynomiality than its derivation from the structure of the Johnson-Pandharipande-Tseng formula. So, we see that the JPT formula is used in a very weak way in these papers; only its general structure appears to be relevant.

In [15] the full power of the JPT formula is employed; as a result it is proved there that

$$\text{(JPT formula)} \Leftrightarrow \text{(Topological Recursion)}$$

In the present chapter, we first give a direct proof of the quasi-polynomiality just from the definition of orbifold Hurwitz numbers. This allows us to use the results of [3, 4] in order to prove the topological recursion. This allows us to use the result of [15] in order to prove, in a new way, the Johnson-Pandharipande-Tseng formula. So, the structure of this chapter can be summarized as follows:

$$\text{(Definition)} \overset{\text{[this chapter]}}{\Longrightarrow} \text{(Quasi-Polynomiality)}$$

$$\text{(Quasi-Polynomiality) AND (Cut-and-Join)} \overset{\text{following [3, 4]}}{\Longrightarrow} \text{(Topological Recursion)}$$

$$\text{(Topological Recursion)} \overset{\text{using [15]}}{\Longrightarrow} \text{(JPT formula)}$$

The first step here is original and it is the main technical result of the present chapter; in the second step we follow [3, 4], though we try to focus more on the main structure of the formulas that represent the abstract loop equations rather than on explicit computations; in the third step we just use the results of [15].

5.1.3 Organization of the chapter

In Section 2 we introduce our basic technical tool — the semi-infinite wedge formalism. In Section 3 we develop further this formalism, in particular, we use it to define the orbifold Hurwitz numbers, and we represent them, in particular, using the so-called $\mathcal{A}$-operators. In Section 4 we analyze further the formula for orbifold Hurwitz numbers in terms of $\mathcal{A}$-operators in order to prove their quasi-polynomiality. In Section 5 we recall the basic setup of the topological recursion. In Section 6 we show how one can use the quasi-polynomiality and the cut-and-join equation for orbifold Hurwitz numbers in order to prove the topological recursion. In Section 7 we use the result of [15] to prove the Johnson-Pandharipande-Tseng formula in a new way.

Throughout this chapter we fix integer $r \geq 1$.

5.2 Semi-infinite wedge formalism

In this section we introduce the semi-infinite wedge formalism. This allows us in the Section 5.3 to express r-orbifold Hurwitz numbers in terms of vacuum expectation of operators acting on the semi-infinite wedge space. For a more complete introduction see e.g. [16, 18, 13].

Let V be an infinite dimensional vector space with a basis labeled by half-integers. Denote the basis vector labeled by $m/2$ by $\underline{m/2}$, so $V = \bigoplus_{i \in \mathbb{Z}+\frac{1}{2}} \underline{i}$.

Definition 5.2.1. The semi-infinite wedge space $\bigwedge^{\frac{\infty}{2}}(V) = \mathcal{V}$ is defined to be the span of all of the semi-infinite wedge products of the form

$$\underline{i_1} \wedge \underline{i_2} \wedge \cdots$$

for any decreasing sequence of half-integers (i_k) such that there is an integer c with $i_k + k - \frac{1}{2} = c$ for k sufficiently large. The constant c is called the *charge*. We give $\mathcal{V}$ an inner product $(\cdot, \cdot)$ declaring its basis elements to be orthonormal.

Remark 5.2.2. By Definition 5.2.1 the charge-zero subspace $\mathcal{V}_0$ of $\mathcal{V}$ is spanned by semi-infinite wedge products of the form

$$\underline{\lambda_1 - \frac{1}{2}} \wedge \underline{\lambda_2 - \frac{3}{2}} \wedge \cdots$$

for some integer partition λ. Hence we can identify integer partitions with the basis of this space:

$$\mathcal{V}_0 = \bigoplus_{n \in \mathbb{N}} \bigoplus_{\lambda \vdash n} v_\lambda$$

The empty partition $\emptyset$ plays a special role. We call

$$v_\emptyset = \underline{-\frac{1}{2}} \wedge \underline{-\frac{3}{2}} \wedge \cdots$$

the vacuum vector and we denote it by $|0\rangle$. Similarly we call the covacuum vector its dual with respect to the scalar product $(\cdot,\cdot)$ and we denote it by $\langle 0|$.

Definition 5.2.3. The *vacuum expectation value* or *disconnected correlator* $\langle \mathcal{P} \rangle^\bullet$ of an operator $\mathcal{P}$ acting on $\mathcal{V}_0$: is defined to be:

$$\langle \mathcal{P} \rangle^\bullet := (|0\rangle, \mathcal{P}|0\rangle) =: \langle 0|\mathcal{P}|0\rangle$$

We also define

$$\zeta(z) = e^{z/2} - e^{-z/2} = 2\sinh(z/2)$$

Definition 5.2.4. This is the list of operators we will use:

i) For k half-integer the operator $\psi_k \colon (\underline{i_1} \wedge \underline{i_2} \wedge \cdots) \mapsto (\underline{k} \wedge \underline{i_1} \wedge \underline{i_2} \wedge \cdots)$ increases the charge by 1. Its adjoint operator ψ_k^* with respect to $(\cdot,\cdot)$ decreases the charge by 1.

ii) The normally ordered products of ψ-operators

$$E_{i,j} := \begin{cases} \psi_i \psi_j^*, & \text{if } j > 0 \\ -\psi_j^* \psi_i & \text{if } j < 0\ . \end{cases}$$

preserve the charge and hence can be restricted to $\mathcal{V}_0$ with the following action. For $i \neq j$ $E_{i,j}$ checks if v_λ contains $\underline{j}$ as a wedge factor and if so replaces it by $\underline{i}$. Otherwise it yields 0. In the case $i = j > 0$, we have $E_{i,j}(v_\lambda) = v_\lambda$ if v_λ contains $\underline{j}$ and 0 if it does not; in the case $i = j < 0$, we have $E_{i,j}(v_\lambda) = -v_\lambda$ if v_λ does not contain $\underline{j}$ and 0 if it does.

iii) The diagonal operators are assembled into the operators

$$\mathcal{F}_n := \sum_{k \in \mathbb{Z} + \frac{1}{2}} \frac{k^n}{n!} E_{k,k}$$

We will be particularly interested in $\mathcal{F}_2$. The operator $\mathcal{F}_0$ is called *charge operator*, while $\mathcal{F}_1$ is called *energy operator*. Note that $\mathcal{F}_0$ identically vanishes on $\mathcal{V}_0$, while $\mathcal{F}_1$ has the basis vectors v_λ as its eigenvectors, with eigenvalues being $|v_\lambda|$ (we refer to $|v_\lambda|$ as the *energy* of basis vector v_λ). We also say that operator $\mathcal{P}$, defined on $\mathcal{V}_0$, is an operator of energy $c \in \mathbb{Z}$ if $-[\mathcal{F}_1, \mathcal{P}]$ is proportional to $\mathcal{P}$ with c being the coefficient of proportionality, i.e. if

$$-[\mathcal{F}_1, \mathcal{P}] = c\,\mathcal{P}$$

In other words, if $\mathcal{P}$ is an operator of energy c, then it maps a basis element of energy k into a combination of basis elements that all have energies $k - c$.

It will be important to us that operators with positive energy annihilate the vacuum while negative energy operators are annihilated by the covacuum, explicitly: let $\mathcal{M}$ be any operator, let $\mathcal{P}$ have positive energy and $\mathcal{N}$ have negative energy, then $\langle \mathcal{M}\mathcal{P}\rangle^\bullet = 0$ and $\langle \mathcal{N}\mathcal{M}\rangle^\bullet = 0$. The operator $E_{i,j}$ has energy $j - i$, hence all the $\mathcal{F}_n$'s have zero energy.

iv) For n any integer and z a formal variable one has the energy n operators:

$$\mathcal{E}_n(z) = \sum_{k \in \mathbb{Z}+1/2} e^{z(k - \frac{n}{2})} E_{k-n,k} + \frac{\delta_{n,0}}{\zeta(z)}.$$

v) For n any nonzero integer one has the energy n operators:

$$\alpha_n = \mathcal{E}_n(0) = \sum_{k \in \mathbb{Z}+1/2} E_{k-n,k}$$

The commutation formula for $\mathcal{E}$ operators is:

$$[\mathcal{E}_a(z), \mathcal{E}_b(w)] = \zeta\left(\det\left[\begin{smallmatrix} a & z \\ b & w \end{smallmatrix}\right]\right) \mathcal{E}_{a+b}(z+w) \tag{5.1}$$

Note that:

$$\mathcal{E}_k(z)\big|0\big\rangle = \begin{cases} \dfrac{1}{\zeta(z)}\big|0\big\rangle, & \text{if } k = 0 \\ 0 & \text{if } k > 0\ . \end{cases}$$

5.3 $\mathcal{A}$ operators

Let r be a positive natural number. The r-orbifold Hurwitz numbers $h^{\bullet,[r]}_{g,\mu}$ enumerate ramified coverings of the 2-sphere by a possibly disconnected genus g surface, where the ramification profile over infinity is given by a partition $\mu = (\mu_1, \dots, \mu_{l(\mu)})$ and the ramification profile over zero is $(r, \dots, r)$, there are simple ramifications over

$$b := 2g - 2 + l(\mu) + \sum_{i=1}^{l(\mu)} \frac{\mu_i}{r}$$

fixed points, and there are no further ramifications. Clearly r should divide the degree $d = |\mu|$ of the covering.

Definition 5.3.1. The genus-generating function of disconnected r-orbifold numbers is the following formal power series:

$$H^{\bullet,[r]}(\vec{\mu}, u) = \sum_{g \geq 0} h^{\bullet,[r]}_{g,\vec{\mu}} \frac{u^b}{b!}$$

The disconnected r-orbifold Hurwitz numbers can be expressed as vacuum expectation in the following way (see [18, 13, 17]) :

$$H^{\bullet,[r]}(\vec{\mu},u) = \sum_{g\geq 0} \left\langle e^{\frac{\alpha_r}{r}} \mathcal{F}_2^b \prod_{i=1}^{l(\mu)} \frac{\alpha_{-\mu_i}}{\mu_i} \right\rangle^{\bullet} \frac{u^b}{b!} = \left\langle e^{\frac{\alpha_r}{r}} e^{u\mathcal{F}_2} \prod_{i=1}^{l(\mu)} \frac{\alpha_{-\mu_i}}{\mu_i} \right\rangle^{\bullet} \tag{5.2}$$

We want to express the vacuum expectation in a more convenient way using the so called $\mathcal{A}$ operators introduced in [18]. We need the notations:

Notation 1. Recall the *Pochhammer symbol*:

$$(x+1)_n = \frac{(x+n)!}{x!} = \begin{cases} (x+1)(x+2)\cdots(x+n) & n \geq 0 \\ (x(x-1)\cdots(x+n+1))^{-1} & n \leq 0 \end{cases} .$$

From the definition, $(x+1)_n$ vanishes for $-n \leq x \leq -1$ an integer, and $1/(x+1)_n$ vanishes for $0 \leq x \leq -(n+1)$ an integer. Let

$$\mathcal{S}(z) = \zeta(z)/z = \frac{\sinh(z/2)}{z/2}$$

Moreover we split rational numbers into integer and fractional parts as follows: for $x \in \mathbb{Q}$ we have

$$x = \lfloor x \rfloor + \langle x \rangle,$$

where $\lfloor x \rfloor \in \mathbb{Z}$ and $0 \leq \langle x \rangle < 1$.

Definition 5.3.2. The following operators will play a central role in the chapter:

$$\mathcal{A}_\eta^{[r]}(z,u) = r^{-\eta/r} (\mathcal{S}(ruz))^{\frac{z-\eta}{r}} \sum_{k\in\mathbb{Z}} \frac{(\mathcal{S}(ruz))^k z^k}{(\frac{z-\eta}{r}+1)_k} \mathcal{E}_{kr-\eta}(uz) \tag{5.3}$$

Define their coefficients in z by $\mathcal{A}_\eta^{[r]}(z,u) = \sum_{k\in\mathbb{Z}} \mathcal{A}_\eta^{[r],(k)} z^k$.

Remark 5.3.3. Our $\mathcal{A}$-operators are at the same time a specialization of Johnson's $\mathcal{A}$-operators in [12] (which we will denote by ${}_J\mathcal{A}$), and a generalization of Okounkov-Pandharipande ones in [18]. Indeed, we will specialize Johnson's formulas and results in [12] using the following assumptions throughout:

$$K = \{e\} \qquad\qquad R = \mathbb{Z}/r\mathbb{Z} \tag{5.4}$$

This implies that every irreducible representation of K is identically one. With these conditions, Equation (5.5) in [12] gives:

$${}_J\mathcal{A}^1_{\frac{a}{r}}(z,u) = \frac{zr^{a/r}}{z+a} \mathcal{S}(ruz)^{\frac{z+a}{r}} \sum_{k\in\mathbb{Z}} \frac{(\mathcal{S}(ruz))^k z^k}{\left(\frac{z+a}{r}+1\right)_k} \mathcal{E}_{kr+a}(uz)$$

The two operators agree in the sense that, for μ positive integers:

$${}_J\mathcal{A}^1_{1-\langle\frac{\mu}{r}\rangle}(\mu,u) = \mathcal{A}^{[r]}_{r\langle\frac{\mu}{r}\rangle}(\mu,u) = r^{-\langle\frac{\mu}{r}\rangle} (\mathcal{S}(ru\mu))^{\lfloor\frac{\mu}{r}\rfloor} \sum_{k\in\mathbb{Z}} \frac{(\mathcal{S}(ru\mu))^k \mu^k}{(\lfloor\frac{\mu}{r}\rfloor+1)_k} \mathcal{E}_{kr-r\langle\frac{\mu}{r}\rangle}(u\mu)$$

Johnson defines his semi-infinite wedge space to be a tensor product between usual semi-infinite wedge space and group K. With K specialized to trivial group, however, his definition reduces to the ordinary semi-infinite wedge space.

Proposition 5.3.4. *The generating function for disconnected orbifold Hurwitz can be expressed in terms of the $\mathcal{A}$ operators by:*

$$H^{\bullet,[r]}(\vec{\mu},u) = r^{\sum_{i=1}^{l(\vec{\mu})}\langle \frac{\mu_i}{r}\rangle} \prod_{i=1}^{l(\vec{\mu})} \frac{u^{\frac{\mu_i}{r}}\mu_i^{\lfloor \frac{\mu_i}{r}\rfloor -1}}{\lfloor \frac{\mu_i}{r}\rfloor !} \left\langle \prod_{i=1}^{l(\vec{\mu})} \mathcal{A}^{[r]}_{r\langle \frac{\mu_i}{r}\rangle}(\mu_i,u) \right\rangle^{\bullet} \tag{5.5}$$

Proof. Both the operators α_r and $\mathcal{F}_2$ annihilate the vacuum, hence we can conjugate each operator $\alpha_{-\mu_i}$ in (5.2) by their exponent getting:

$$H^{\bullet,[r]}(\vec{\mu},u) = \frac{1}{\prod_{i=1}^{l(\vec{\mu})}\mu_i} \left\langle \prod_{i=1}^{l(\vec{\mu})} e^{\frac{\alpha_r}{r}} e^{u\mathcal{F}_2}\alpha_{-\mu_i}e^{-u\mathcal{F}_2}e^{-\frac{\alpha_r}{r}} \right\rangle^{\bullet} \tag{5.6}$$

We recall Equation (2.14) in [18]:

$$e^{u\mathcal{F}_2}\alpha_{-\mu}e^{-u\mathcal{F}_2} = \mathcal{E}_{-\mu}(u\mu)$$

Note that the energy is preserved to be $-\mu$. Commutator rule (5.1) gives:

$$[\alpha_r, \mathcal{E}_{-\mu}(u\mu)] = \zeta(ru\mu)\mathcal{E}_{r-\mu}(u\mu)$$

We expand the last conjugation in nested commutators of the form above obtaining:

$$e^{\frac{\alpha_r}{r}}\mathcal{E}_{-\mu}(u\mu)e^{-\frac{\alpha_r}{r}} = \sum_{k\geq 0}\left(\frac{\zeta(ru\mu)}{r}\right)^k \frac{1}{k!}\mathcal{E}_{kr-\mu}(u\mu) = \sum_{k\geq 0}\frac{u^k\mu^k(\mathcal{S}(ru\mu))^k}{k!}\mathcal{E}_{kr-\mu}(u\mu)$$

Rescaling by $k - \lfloor \frac{\mu}{r}\rfloor \mapsto k$ and using the vanishing properties of the Pochhammer symbol, we can rewrite the last expression as

$$\frac{(u\mu)^{\lfloor \frac{\mu}{r}\rfloor}}{\lfloor \frac{\mu}{r}\rfloor !}(\mathcal{S}(ru\mu))^{\lfloor \frac{\mu}{r}\rfloor}\sum_{k\in\mathbb{Z}}\frac{u^k(\mathcal{S}(ru\mu))^k\mu^k}{(\lfloor \frac{\mu}{r}\rfloor+1)_k}\mathcal{E}_{kr-\langle \frac{\mu}{r}\rangle r}(u\mu)$$

To match the powers of u we conjugate by the exponent of the energy operator $u^{\mathcal{F}_1/r}$. Since $\mathcal{F}_1$ and its adjoint fix the vacuum, this does not affect operator expectations of products of the $\mathcal{A}$-operators. Since $\mathcal{E}_j$ has energy j, the conjugation removes u^k from inside the sum and produces a factor of $u^{\langle \frac{\mu}{r}\rangle}$ outside. Thus we see that the vacuum expectation of the operators in (5.6) can be replaced by the vacuum expectation of the product of

$$\frac{u^{\frac{\mu_i}{r}}\mu_i^{\lfloor \frac{\mu_i}{r}\rfloor}}{\lfloor \frac{\mu_i}{r}\rfloor !}(\mathcal{S}(ru\mu_i))^{\lfloor \frac{\mu_i}{r}\rfloor}\sum_{k\in\mathbb{Z}}\frac{(\mathcal{S}(ru\mu_i))^k\mu_i^k}{(\lfloor \frac{\mu_i}{r}\rfloor+1)_k}\mathcal{E}_{kr-\langle \frac{\mu_i}{r}\rangle r}(u\mu_i)$$

for $i = 1,\ldots,l(\vec{\mu})$. Then, using Equation (5.3) we can rewrite the full formula (5.6) as (5.5). □

Following [18], we define the doubly infinite series:

$$\delta(z,-w) = \frac{1}{w}\sum_{k\in\mathbb{Z}}\left(-\frac{z}{w}\right)^k$$

which is obtained as the difference between the following two expansions:

$$\frac{1}{z+w} = \frac{1}{w} - \frac{z}{w^2} + \frac{z^2}{w^3} - \dots, \qquad |z| < |w|$$

$$\frac{1}{z+w} = \frac{1}{z} - \frac{w}{z^2} + \frac{w^2}{z^3} - \dots, \qquad |z| > |w|$$

The series $\delta(z,-w)$ is a formal δ-function at $z+w=0$ in the sense that:

$$(z+w)\delta(z,-w) = 0$$

We recall the formula for commutators of $\mathcal{A}$, that will be fundamental to prove polynomiality. Below, by $\delta_r(\eta)$ we denote the function of an integer argument that equals to 1 if $\eta \equiv 0 \mod r$ and vanishes otherwise.

Proposition 5.3.5 (Particular case of Lemma V.4. of [12])**.** *Let η_1, η_2 be integer numbers satisfying $0 \le \eta_1, \eta_2 \le r-1$. We have:*

$$[\mathcal{A}^{[r]}_{\eta_1}(z,u), \mathcal{A}^{[r]}_{\eta_2}(w,u)] = \delta_r(\eta_1+\eta_2) zw\delta(z,-w)$$

or equivalently:

$$[\mathcal{A}^{[r],(k)}_{\eta_1}, \mathcal{A}^{[r],(l)}_{\eta_2}] = \delta_r(\eta_1+\eta_2)(-1)^l \delta_{k+l-1}. \tag{5.7}$$

We define $\Omega \subset \mathbb{C}^n$ by

$$\Omega = \left\{ (z_1, \dots, z_n) \in \mathbb{C}^n \middle| \forall k, |z_k| > \sum_{i=1}^{k-1} |z_i| \right\}.$$

Specializing Theorem V.2 of [12] with the convention (5.4) (see also Section 2.4 in [5]) we have the following:

Proposition 5.3.6. *For any integer numbers $\eta_1, \dots, \eta_n$, $0 \le \eta_1, \dots, \eta_n \le r-1$, the Laurent series expansion of*

$$\left\langle \mathcal{A}^{[r]}_{\eta_1}(z_1,u) \cdots \mathcal{A}^{[r]}_{\eta_1}(z_1,u) \right\rangle^{\bullet}$$

in $u, z_1, \dots, z_n$ converges to an analytic function for $(z_1, \dots, z_n) \in \Omega$ and sufficiently small $u \neq 0$.

Notation 2. For brevity in the rest of the chapter we denote $\mathcal{A}^{[r]}_{\eta}(z,u)$ by $\mathcal{A}_\eta(z)$.

5.4 Quasi-polynomiality

In this section we derive quasi-polynomiality of r-orbifold Hurwitz numbers (Theorem 5.4.9). The argument that we use is a suitable generalization of an argument of [5].

5.4.1 Connected vacuum expectations

Proposition 5.3.4 expresses the genus-generating function of disconnected orbifold Hurwitz in terms of vacuum expectation of $\mathcal{A}$-operators. Our first goal is have a similar expression for connected orbifold Hurwitz numbers

Definition 5.4.1. We define the connected correlators $\langle \mathcal{A}_{\eta_1}(z_1) \cdots \mathcal{A}_{\eta_1}(z_1)\rangle^\circ$ in terms of the disconnected correlators $\langle \cdots \rangle^\bullet$ via the inclusion-exclusion formula.

The inverse form of the inclusion-exclusion formula reads (cf. [5]):

$$\langle \mathcal{A}_{\eta_1}(z_1)\dots\mathcal{A}_{\eta_n}(z_n)\rangle_k^\bullet = \sum_{y\in\mathcal{Y}_{n,k}} \prod_{i=1}^{h(y)} \Big\langle \mathcal{A}_{\eta_{c_{i,1}(y)}}(z_{c_{i,1}(y)})\dots\mathcal{A}_{\eta_{c_{i,l_i(y)}(y)}}(z_{c_{i,l_i(y)}(y)})\Big\rangle^\circ_{\lambda_i(y)} \tag{5.8}$$

Here $\mathcal{Y}_{n,k}$ is the finite set of $\{1,\dots,n\}$-Young tableaux y with the following properties:

1. The numbers in the rows should be ascending: $c_{i,j}(y)$ is the number in the i-th row and j-th column, then for any i and for any $j_1 < j_2$ we have $c_{i,j_1}(y) < c_{i,j_2}(y)$. Each row corresponds to an individual connected correlator.

2. For rows of the same length, just for the first column the numbers should be ascending: $l_i(y)$ is length of the i-th row, then if $l_{i_1}(y) = l_{i_2}(y)$ and $i_1 < i_2$, then $c_{i_1,1}(y) < c_{i_2,1}(y)$.

3. $h(y)$ is the number of rows. Rows are labelled by the vector $\{\lambda_i(y)\in\{-1,0,1,\dots\}\}_i$ with $\sum_{i=1}^{h(y)}\lambda_i(y) = k$. The vector $\vec\lambda$ corresponds to the vector of Euler characteristics of correlators with sign exchanged.

Remark 5.4.2. For $n=1$ we have that connected and disconnected correlators coincide, hence we just write $\langle \mathcal{A}_\eta(z)\rangle$.

The connected correlators can be used to express the generating function for connected orbifold Hurwitz numbers:

$$H^{\circ,[r]}(\vec\mu,u) := \sum_{g\geq 0} h^{\circ,[r]}_{g,\vec\mu}\cdot\frac{u^b}{b!}$$

Proposition 5.4.3. *Generating function for connected orbifold Hurwitz numbers equals:*

$$H^{\circ,[r]}(\vec\mu,u) = r^{\sum_{i=0}^{l(\vec\mu)}\langle\frac{\mu_i}{r}\rangle} \prod_{i=1}^{l(\vec\mu)} \frac{u^{\frac{\mu_i}{r}}\mu_i^{\lfloor\frac{\mu_i}{r}\rfloor-1}}{\lfloor\frac{\mu_i}{r}\rfloor!} \Big\langle \prod_{i=1}^{l(\vec\mu)} \mathcal{A}_{r\langle\frac{\mu_i}{r}\rangle}(\mu_i)\Big\rangle^\circ \tag{5.9}$$

Proof. This follows from (5.5) and the observation that taking u^b-coefficient in H° corresponds to the coefficient of $u^{2g-2+l(\vec\mu)}$ in $\langle\prod\mathcal{A}\rangle^\circ$. □

5.4.2 Unstable terms

In this Section we compute explicitly the coefficients of the connected vacuum expectations that correspond to the orbifold Hurwitz number for $g=0$ and $n=1,2$.

First, let us introduce some convenient notations.

Notation 3. For any operator $\mathcal{P}(u)$ define

$$\langle \mathcal{P}(u)\rangle_k^\bullet := [u^k]\,\langle \mathcal{P}(u)\rangle^\bullet \qquad \text{(the coefficient of } u^k \text{ in } \langle \mathcal{P}(u)\rangle^\bullet)$$
$$\langle \mathcal{P}(u)\rangle_k^\circ := [u^k]\,\langle \mathcal{P}(u)\rangle^\circ \qquad \text{(the coefficient of } u^k \text{ in } \langle \mathcal{P}(u)\rangle^\circ)$$

Notation 4. We denote by $\mathcal{A}_{\eta,+}(z)$ the positive power part in z of the $\mathcal{A}_\eta(z)$ operator to be:

$$\mathcal{A}_{\eta,+}(z) := \sum_{k\geq 1} \mathcal{A}_\eta^{(k)} z^k \tag{5.10}$$

The terms that we want to compute are

$$\langle \mathcal{A}_{\eta_i}(z_i)\rangle_{-1}^\circ \qquad \text{and} \qquad \langle \mathcal{A}_{\eta_i}(z_i)\mathcal{A}_{\eta_j}(z_j)\rangle_0^\circ$$

Lemma 5.4.4. *Let η, η_1, η_2 be integer number, $0 \leq \eta \leq r-1$. We have:*

$$\langle \mathcal{A}_\eta(z)\rangle_{-1}^\circ = \frac{\delta_{\eta,0}}{z}, \tag{5.11}$$

$$\langle \mathcal{A}_{\eta_1}(z_1)\mathcal{A}_{\eta_2}(z_2)\rangle_0^\circ = \delta_r(\eta_1+\eta_2) z_1 \sum_{k\geq 0}\left(-\frac{z_1}{z_2}\right)^k. \tag{5.12}$$

Proof. In the vacuum expectation of a single operator $\mathcal{A}_\eta(z)$ only zero-energy term can give non-trivial contribution. Since $\mathcal{E}_i$ has energy i, we have:

$$\langle \mathcal{A}_\eta(z)\rangle = \delta_{\eta,0}\frac{\zeta(ruz)^{z/r}}{(ruz)^{z/r}}\frac{1}{\zeta(uz)} = \left[\frac{1}{uz} + \frac{z(rz-1)}{24}u + O(u^2)\right]\delta_{\eta,0}$$

This implies the formula for the genus-zero one-point correlator. The rest of the proof is devoted to the genus-zero two-pointed correlator.

Note that the following formula for the action of $\mathcal{A}_\eta(z)$ on covacuum holds

$$\langle 0|\mathcal{A}_\eta(z) = \frac{\delta_{\eta,0}}{uz}\langle 0| + \langle 0|\mathcal{A}_{\eta,+}(z), \tag{5.13}$$

which follows directly from Equation (5.3) and two observations:

- $\mathcal{E}_{kr-\eta}(uz)$ annihilates the covacuum when $kr-\eta<0$
- Among the terms that do not annihilate the covacuum, only the term with $\mathcal{E}_0(uz)$ is singular in z at $z=0$

Equation (5.8) implies that

$$\begin{aligned}\langle \mathcal{A}_{\eta_1}(z_1)\mathcal{A}_{\eta_2}(z_2)\rangle_0^\circ = \langle \mathcal{A}_{\eta_1}(z_1)\mathcal{A}_{\eta_2}(z_2)\rangle_0^\bullet &- \langle \mathcal{A}_{\eta_1}(z_1)\rangle_{-1}\langle \mathcal{A}_{\eta_2}(z_2)\rangle_1 \\ &- \langle \mathcal{A}_{\eta_1}(z_1)\rangle_1\langle \mathcal{A}_{\eta_2}(z_2)\rangle_{-1}\end{aligned}$$

Applying (5.13) to the first term in the right-hand side we get:

$$\langle \mathcal{A}_{\eta_1}(z_1)\mathcal{A}_{\eta_2}(z_2)\rangle_0^\bullet = \langle \mathcal{A}_{\eta_1,+}(z_1)\mathcal{A}_{\eta_2}(z_2)\rangle_0^\bullet + \langle \mathcal{A}_{\eta_1}(z_1)\rangle_{-1}\langle \mathcal{A}_{\eta_2}(z_2)\rangle_1$$

In the same way, we observe that

$$\langle \mathcal{A}_{\eta_1,+}(z_1)\mathcal{A}_{\eta_2}(z_2)\rangle_0^\bullet = \langle [\mathcal{A}_{\eta_1,+}(z_1), \mathcal{A}_{\eta_2}(z_2)]\rangle_0^\bullet + \langle \mathcal{A}_{\eta_2,+}(z_2)\mathcal{A}_{\eta_1,+}(z_1)\rangle_0^\bullet + \langle \mathcal{A}_{\eta_1}(z_1)\rangle_1 \langle \mathcal{A}_{\eta_2}(z_2)\rangle_{-1}$$

Therefore,

$$\langle \mathcal{A}_{\eta_1}(z_1)\mathcal{A}_{\eta_2}(z_2)\rangle_0^\circ = \langle \mathcal{A}_{\eta_2,+}(z_2)\mathcal{A}_{\eta_1,+}(z_1)\rangle_0^\bullet + \langle [\mathcal{A}_{\eta_1,+}(z_1), \mathcal{A}_{\eta_2}(z_2)]\rangle_0^\bullet$$

The second term here is equal to the right hand side of Equation (5.12) (this follows the commutation rule for coefficients given by Equation (5.7)). In order to complete the proof of the lemma we have to prove that the first term vanishes.

In other words, we consider

$$r^{\frac{\eta_1+\eta_2}{r}}\left\langle \mathcal{A}_{\eta_2}(z_2)\mathcal{A}_{\eta_1}(z_1)\right\rangle^\bullet = (\mathcal{S}(ruz_2))^{\frac{z_2-\eta_2}{r}}(\mathcal{S}(ruz_1))^{\frac{z_1-\eta_1}{r}} \times \\ \times \sum_{k,l\in\mathbb{Z}} \frac{(\mathcal{S}(ruz_2))^k {z_2}^k (\mathcal{S}(ruz_1))^l {z_1}^l}{(\frac{z_2-\eta_2}{r}+1)_k(\frac{z_1-\eta_1}{r}+1)_l}\left\langle \mathcal{E}_{kr-\eta_2}(uz_2)\mathcal{E}_{lr-\eta_1}(uz_1)\right\rangle^\bullet \tag{5.14}$$

We want to show that the coefficient of u^0 in this expression does not contain terms of expansion in z_1, z_2 that have positive degrees in both variables. This implies directly that we have $\langle \mathcal{A}_{\eta_2,+}(z_2)\mathcal{A}_{\eta_1,+}(z_1)\rangle_0^\bullet = 0$.

There are two cases:

- $kr-\eta_2 = kr-\eta_2 = 0$, which implies $k = l = \eta_1 = \eta_2 = 0$. In this case the expression (5.14) is equal to

$$\frac{\mathcal{S}(ruz_2)^{\frac{z_2}{r}}\mathcal{S}(ruz_1)^{\frac{z_1}{r}}}{\zeta(uz_2)\zeta(uz_1)} = \frac{1}{u^2 z_1 z_2} + \frac{1}{24 z_1 z_2}\left(rz_1^3 + rz_2^3 - z_1^2 - z_2^2\right) + O(u^2),$$

  hence all terms in the coefficient of u^0 have negative degree either in z_1 or in z_2.

- $kr-\eta_2 \neq 0$ and $lr-\eta_1 \neq 0$, which implies $kr-\eta_2+lr-\eta_1 = 0$. In this case all factors are formal power series in u, so we can expand all factors in u up to $O(u^1)$. The summand with particular k and l in (5.14) is equal to

$$\frac{z_2^k z_1^l}{(\frac{z_2-\eta_2}{r}+1)_k(\frac{z_1-\eta_1}{r}+1)_l} + O(u)$$

  The condition $kr-\eta_2+lr-\eta_1 = 0$ is satisfied in one of the two possible cases:

  - $\eta_1 = \eta_2 = 0$, $k+l = 0$, $k, l \neq 0$;
  - $\eta_1 + \eta_2 = r$, $k + l = 1$.

  In both cases either k or l is non-positive. Without loss of generality, let's assume that $l \leq 0$ (the other case is symmetric). Then

$$\frac{z_1^l}{(\frac{z_1-\eta_1}{r}+1)_l} = z_1^l\left(\frac{z_1-\eta_1}{r}\cdot\dots\cdot\left(\frac{z_1-\eta_1}{r}+l+1\right)\right)$$

  contains no positive powers of z_1.

□

5.4.3 Vacuum expectations without unstable terms

In this section we give a formula for the disconnected vacuum expectations, where all unstable terms, that is, $\langle \mathcal{A}_\eta(z)\rangle^\circ_{-1}$ and $\langle \mathcal{A}_{\eta_1}(z_1)\mathcal{A}_{\eta_2}(z_2)\rangle^\circ_0$, are dropped. This is a straightforward generalization of the similar formula in [5], which is based on the following simple recursion rules:

Lemma 5.4.5. *We can recursively decompose disconnected correlators as follows:*

$$\langle \mathcal{A}_\eta(z)\prod_i \mathcal{A}_{\eta_i}(z_i)\rangle^\bullet_k = \langle \mathcal{A}_\eta(z)\rangle^\circ_{-1}\langle \prod_i \mathcal{A}_{\eta_i}(z_i)\rangle^\bullet_{k+1} + \langle \mathcal{A}_{\eta,+}(z)\prod_i \mathcal{A}_{\eta_i}(z_i)\rangle^\bullet_k \tag{5.15}$$

$$\begin{aligned}\langle \mathcal{A}_{\eta,+}(z)\mathcal{A}_\sigma(w)\prod_i \mathcal{A}_{\eta_i}(z_i)\rangle^\bullet_k &= \langle \mathcal{A}_\eta(z)\mathcal{A}_\sigma(w)\rangle^\circ_0\langle \prod_i \mathcal{A}_{\eta_i}(z_i)\rangle^\bullet_k \\ &\quad + \langle \mathcal{A}_\sigma(w)\mathcal{A}_{\eta,+}(z)\prod_i \mathcal{A}_{\eta_i}(z_i)\rangle^\bullet_k\end{aligned} \tag{5.16}$$

Proof. Equation (5.13) and the formula for the one-point correlator (5.11) together prove the first equality. The second equality follows from the computation of the two-points correlator (5.12). ☐

This implies the following proposition.

Proposition 5.4.6. *We have:*

$$\begin{aligned}&\langle \mathcal{A}_{\eta_1,+}(z_1)\dots\mathcal{A}_{\eta_k,+}(z_n)\rangle^\bullet_k \\ &= \sum_{y\in\mathcal{Y}^{stab}_{n,k}}\prod_{i=1}^{h(y)}\left\langle \mathcal{A}_{\eta_{c_{i,1}(y)}}(z_{c_{i,1}(y)})\dots\mathcal{A}_{\eta_{c_{i,l_i(y)}(y)}}(z_{c_{i,l_i(y)}(y)})\right\rangle^\circ_{\lambda_i(y)} .\end{aligned} \tag{5.17}$$

where

$$\mathcal{Y}^{stab}_{n,k} = \left\{y\in\mathcal{Y}_{n,k}\,\middle|\, l_i(y)=1\Rightarrow\lambda_i(y)\neq -1,\ l_i(y)=2\Rightarrow\lambda_i(y)\neq 0\right\}$$

In other words, $\langle \mathcal{A}_{\eta_1,+}(z_1)\cdots\mathcal{A}_{\eta_n,+}(z_n\rangle^\bullet_k$ *is equal to* $\langle \mathcal{A}_{\eta_1}(z_1)\cdots\mathcal{A}_{\eta_n}(z_n)\rangle^\bullet_k$ *with all the unstable terms dropped.*

Proof. The proof of this proposition is completely analogous to the proof of Proposition 2.21 in [5]. It is based on the recursion that expresses $\langle \mathcal{A}_{\eta_1}(z_1)\cdots\mathcal{A}_{\eta_n}(z_n)\rangle^\bullet_k$ in terms of the unstable vacuum expectations and $\mathcal{A}_+$-operators using only Equations (5.15) and (5.16). Though the operators here are more general, the recursion rules are still the same, so the same argument can be applied. ☐

5.4.4 Polynomiality

In this section we prove the quasi-polynomiality property for orbifold Hurwitz numbers. First, we show that $\langle \mathcal{A}_{\eta_1,+}(z_1)\dots\mathcal{A}_{\eta_n,+}(z_n)\rangle^\bullet_k/(z_1\cdots z_n)$ is a symmetric polynomial in $z_1,\dots,z_n$ (excluding unstable cases of $k=-1, n=1$, and $k=0, n=2$). This implies that $\langle \mathcal{A}_{\eta_1}(z_1)\dots\mathcal{A}_{\eta_n}(z_n)\rangle^\circ_k/(z_1\cdots z_n)$ is a symmetric polynomial in $z_1,\dots,z_n$ (again, excluding unstable cases). This, in turn, implies quasi-polynomiality of orbifold Hurwitz numbers.

Proposition 5.4.7. *The function*

$$\frac{\langle \mathcal{A}_{\eta_1,+}(z_1)\dots\mathcal{A}_{\eta_n,+}(z_n)\rangle^\bullet_k}{z_1\cdots z_n}$$

is a symmetric polynomial in $z_1,\dots,z_n$ *for* $(n,k)\neq(1,-1),(2,0)$.

Proof. We follow the proof of Proposition 9 in [18]. We have:

i) Boundedness from below: $\langle \mathcal{A}_{\eta_1,+}(z_1)\dots\mathcal{A}_{\eta_n,+}(z_n)\rangle_k^\bullet$ has strictly positive powers in all its variables $z_1,\dots,z_n$, as it follows from the definition of $\mathcal{A}_{\eta,+}(z)$ given by Equation (5.10). So, we can divide by $\prod_{i=1}^n z_i$, and we still have only non-negative powers of $z_1,\dots,z_n$ in the expansion of the quotient.

ii) Symmetry holds because $\mathcal{A}_+$ operators commute which each other, which is a direct consequence of the commutation formula (5.7).

iii) Boundedness from above: Since it is a symmetric function, it is enough to show that the power of z_n is bounded. From the definition of the $\mathcal{A}$-operator we have the following

$$\mathcal{A}_\eta(z)|0\rangle = r^{-\eta/r}\mathcal{S}(ruz)^{(z-\eta)/r}\sum_{k=0}^{\infty}\frac{\mathcal{S}(ruz)^{-k}z^{-k}}{\left(\frac{z-\eta}{r}+1\right)_{-k}}\mathcal{E}_{-kr-\eta}(uz)|0\rangle \tag{5.18}$$

where we used the change of summation index $k\mapsto -k$ since the operators $\mathcal{E}_i$ with positive i annihilate the vacuum.

Since each factor in each summand in (5.18) has at most first order pole in u, it is sufficient to do the following. We expand each factor of each summand in (5.18) in u up to $O(u^{m+1})$, and we show that the degree of z in this expansion is bounded from above. Indeed, in this case, the highest possible power of z in $\mathcal{E}_{-kr-\eta}(uz)$ is m; in $\mathcal{S}(ruz)^{-k}$ it is again m; in $\mathcal{S}(ruz)^{(z-\eta)/r}$ is it equal to $2m$ (one m comes from argument of $\mathcal{S}$, while the other estimates power of z in the binomial coefficient in the expansion of $(1+x)^{(z-\eta)/r}$; finally, the highest possible power of z in $z^{-k}/\left(\frac{z-\eta}{r}+1\right)_{-k}$ is equal to 0.

□

Proposition 5.4.8. *For $(n,k)\neq(1,-1),(2,0)$, the function*

$$\frac{\langle \mathcal{A}_{\eta_1}(z_1)\dots\mathcal{A}_{\eta_n}(z_n)\rangle_k^\circ}{z_1\cdots z_n}$$

is a symmetric polynomial in $z_1,\dots,z_n$.

Proof. We follow the proof of Proposition 2.23 in [5]. We prove the statement by induction on the number of operators in the vacuum expectation. It holds for $n=1$. Suppose it holds for vacuum expectations with any number of operators less than n, and we want to prove it for n operators as well. Let y' be the single-row Young tableau. Consider the partition $\mathcal{Y}_{n,k}^{stab}=\{y'\}\cup\left(\mathcal{Y}_{n,k}^{stab}\setminus\{y'\}\right)$. Then Equation (5.17) implies:

$$\frac{\langle \mathcal{A}_{\eta_1}(z_1)\dots\mathcal{A}_{\eta_n}(z_n)\rangle_k^\circ}{z_1\cdots z_n}=\frac{\langle \mathcal{A}_{\eta_1,+}(z_1)\dots\mathcal{A}_{\eta_n,+}(z_n)\rangle_k^\bullet}{z_1\cdots z_n}$$
$$-\sum_{y\in\mathcal{Y}_{n,k}^{stab}\setminus\{y'\}}\prod_{i=1}^{h(y)}\frac{\left\langle \mathcal{A}_{\eta_{z_{c_{i,1}(y)}}}(z_{c_{i,1}(y)})\dots\mathcal{A}_{\eta_{c_{i,l_i(y)}(y)}}(z_{c_{i,l_i(y)}(y)})\right\rangle_{\lambda_i(y)}^\circ}{z_{c_{i,1}(y)}\cdots z_{c_{i,l_i(y)}(y)}}.$$

The first term on the right hand side is symmetric polynomial in $z_1,\dots,z_n$ by Proposition 5.4.7, the second term is symmetric polynomial by induction hypothesis. □

Now we are ready to prove the quasi-polynomiality of orbifold Hurwitz numbers.

Theorem 5.4.9. *The orbifold Hurwitz numbers* $h^{\circ,[r]}_{g,\mu}$ *for* $(g,n) \neq (0,1),(0,2)$ *can be expressed as follows:*

$$h^{\circ,[r]}_{g;\mu} = (2g-2+l(\mu)+|\mu|/r)! \left(\prod_{i=1}^{n} \frac{\mu_i^{\lfloor \frac{\mu_i}{r} \rfloor}}{\lfloor \frac{\mu_i}{r} \rfloor!} \right) P^{\langle \frac{\vec{\mu}}{r} \rangle}_{g,n}(\mu_1,\dots,\mu_n), \tag{5.19}$$

where $P^{\vec{\epsilon}}_{g,n}(\mu_1,\dots,\mu_n)$ *are some polynomials in* $\mu_1,\dots,\mu_n$*, whose coefficients depend on the parameters* $\vec{\epsilon} = (\epsilon_1,\dots,\epsilon_n)$*,* $0 \le \epsilon_1,\dots,\epsilon_n \le r-1$ *(we have* $\epsilon_i = \langle \frac{\mu_i}{r} \rangle$*).*

Proof. For partitions μ inside the region Ω, this is a direct corollary of Equation (5.9) and Proposition 5.4.8. We show now that the result holds for every partition μ. In order to do so it is enough to show that, for fixed η_1 and $\mu_2,\dots,\mu_n$, the disconnected correlator $\langle \prod_{i=1}^n \mathcal{A}_{\eta_i}(u,\mu_i) \rangle^\bullet$ is a power series in u whose coefficients are rational functions in μ_1 with bounded degrees in both numerator and denominator. Indeed once we show this, this rationality property of the disconnected correlator implies the same rationality property of the connected one by inclusion-exclusion formula. Therefore, for each fixed power of u, the rational function in μ_1 coincides with a polynomial expression in the Zariski dense set given by (a symmetrization of) Ω. This implies that the connected correlator, for each fixed power of u, is a polynomial in the first variable μ_1. A complex symmetric function in several variables that is polynomial in one of the variables is a polynomial in all the variables. This implies the statement.

Let us now prove the rationality property. Setting $\mu_i = \nu_i r + \eta_i$, where $\nu_i = \lfloor \frac{\mu_i}{r} \rfloor$ and $\eta_i = \langle \frac{\mu_i}{r} \rangle$, the operator $\mathcal{A}_{\eta_i}$ reads

$$\mathcal{A}_{\eta_i}(u,\mu_i) = r^{-\frac{\eta_i}{r}} \mathcal{S}(ru\mu_i)^{\nu_i} \sum_{t_i \in \mathbb{Z}} \frac{\mathcal{S}(ru\mu)^{t_i} \mu_i^{t_i-1}}{(\nu_i+1)_{t_i}} \mathcal{E}_{t_i r - \eta_i}(u\mu_i)$$

Let us expand the product of all the t-sums in the disconnected correlator and impose the condition vanishing energy $\sum_{i=1}^{l(\mu)} (t_i r - \eta_i) = 0$. The energy of left-most operator $\mathcal{A}_{\eta_1}$ should be positive, so $t_1 r - \eta_1 \ge 0$, and the Pochhammer symbol vanishes unless $t_i \ge -\nu_i$. Therefore, for each fixed $\eta_1, \nu_2, \eta_2, \dots, \nu_n, \eta_n$, the t_1-sum becomes finite. Since the power of u is fixed, it also gives a bound on the degree in ν_1 both in the numerator and in the denominator. So the coefficient of a particular power of u in the disconnected correlator $\langle \prod_{i=1}^n \mathcal{A}_{\eta_i}(u,\mu_i) \rangle^\bullet$ is a rational function in ν_1. This concludes the proof of the theorem. ☐

5.5 Topological Recursion

In this section we recall the topological recursion of Chekhov, Eynard, and Orantin tailored for our use. For a more detailed introduction we refer to [10, 9].

Definition 5.5.1. A *spectral curve* is a triple (Σ, x, y), where Σ is a Riemann surface (which we assume from now on to be $\mathbb{CP}^1$) and $x, y \colon \Sigma \to \mathbb{C}$ are meromorphic functions, such that the zeroes of dx are disjoint from the zeroes of dy. Moreover the zeros of dx are simple:

In a neighborhood of a point $\alpha \in \Sigma$ such that $dx(\alpha) = 0$ we can define an involution τ_α that preserves function x (deck transformation).

Furthermore, $\Sigma \times \Sigma$ is equipped with a meromorphic symmetric 2-differential with a second order pole on the diagonal, which is called the *Bergman kernel*. In the case of $\mathbb{CP}^1$ the Bergman kernel is unique and in a global coordinate z it reads

$$B = \frac{dz_1 dz_2}{(z_1 - z_2)^2}.$$

Definition 5.5.2. By *topological recursion* we call a recursive procedure that associates to a spectral curve data (Σ, x, y, B) a family of symmetric meromorphic differentials, called *correlation differentials* $\omega_{g,n}(z_1, \ldots, z_n)$ defined on Σ^n, $g \geq 0$, $n \geq 1$.

The first two correlation differentials are given by explicit formulas:

$$\omega_{0,1}(z) = \frac{y\, dx}{x} \qquad\qquad \omega_{0,2}(z_1, z_2) = \frac{dz_1 dz_2}{(z_1 - z_2)^2}$$

The correlation differentials $\omega_{g,n}$, $2g - 2 + n > 0$, are given by:

$$\omega_{g,n}(z_1, z_S) = \sum_{\substack{\alpha\in\Sigma \\ dx(\alpha)=0}} \operatorname*{Res}_{z=\alpha} K(z_1, z)\Bigg[\omega_{g-1,n+1}(z, \tau_\alpha(z), z_S) + \sum_{\substack{g_1+g_2=g \\ I\sqcup J=S}}' \omega_{g_1,|I|+1}(z, z_I)\omega_{g_2,|J|+1}(\tau_\alpha(z), z_J)\Bigg],$$

where $S = \{2, \ldots, n\}$, and in the second sum we exclude the cases when $(g_1, |I|+1)$ or $(g_2, |J|+1)$ is equal to $(0, 1)$. The *recursion kernel* K is defined in the vicinity of each point α, $dx(\alpha) = 0$ by the formula

$$K(z_1, z) := \frac{\int_z^{\tau_\alpha(z)} \omega_{0,2}(\cdot, z_1)}{2(\omega_{0,1}(\tau_\alpha(z)) - \omega_{0,1}(z))}$$

In our case ($\Sigma = \mathbb{CP}^1$, z is a global coordinate), we can use the following formula:

$$K(z_1, z) = \frac{x(z)}{2(y(\tau_\alpha(z)) - y(z))x'(z)} \left(\frac{1}{z - z_1} - \frac{1}{\tau_\alpha(z) - z_1}\right) \frac{dz_1}{dz}$$

In the stable range, $2g - 2 + n > 0$, the correlation differentials $\omega_{g,n}$ have poles only at the zeros of dx. They can be expressed as the sum of their *principle parts*:

$$\omega_{g,n}(z_1, z_S) = \sum_{\substack{\alpha\in\Sigma \\ dx(\alpha)=0}} [\omega_{g,n}(z_1, z_S)]_\alpha \tag{5.20}$$

where by principal part $[\eta(z_1)]_\alpha$ of a 1-form $\eta(z_1)$ we mean the projection defined as a version of Cauchy formula, where we use B instead of the Cauchy kernel:

$$[\eta(z_1)]_\alpha := \operatorname*{Res}_{z=\alpha} \eta(z) \int_\alpha^z B(\cdot, z_1).$$

In fact, there is an equivalent way to reformulate the topological recursion. We say that the symmetric meromorphic differentials satisfy the topological recursion if they satisfy the property (5.20) for $2g-2+n>0$, and also solve the abstract loop equations:

$$\omega_{g,n}(z,z_S)+\omega_{g,n}(\tau_\alpha(z),z_S) \text{ is holomorphic for } z\to\alpha \tag{5.21}$$

$$\omega_{g-1,n+1}(z,\tau_\alpha(z),z_S)+\sum_{\substack{g_1+g_2=g\\ I\sqcup J=S}}\omega_{g_1,1+|I|}(z,z_I)\omega_{g_2,1+|J|}(\tau_\alpha(z),z_J) \tag{5.22}$$

is holomorphic for $z\to\alpha$ with at least double zero at α.

A proof of that can be found in [1, 2].

5.6 The Spectral Curve

In this section we prove the spectral curve for orbifold Hurwitz numbers using the quasi-polynomiality property proved in Section 5.4 and the cut-and-join equation. It is important to stress that we do not use the Johnson-Pandharipande-Tseng [14] formula in this Section.

We consider the n-point function for orbifold Hurwitz numbers for fixed genus g:

$$H^{\circ,[r]}_{g,n}(x_1,\dots,x_n)=\sum_{\vec{\mu}:l(\mu)=n}\frac{h^{\circ,[r]}_{g;\mu}}{b!}\,x_1^{\mu_1}\cdots x_n^{\mu_n}.$$

Theorem 5.6.1. *Consider the correlation differentials $\omega_{g,n}$, $g\ge 0$, $n\ge 1$, for the spectral curve $(\Sigma=\mathbb{CP}^1,z,y)$, where*

$$x(z)=z\exp(-z^r)\qquad\text{and}\qquad y(z)=z^r$$

in some global coordinate z. They have the following analytic expansion near $x_1=x_2=\cdots=x_n=0$:

$$\omega_{g,n}(x_1,\dots,x_n)=\frac{\partial}{\partial x_1}\cdots\frac{\partial}{\partial x_n}H^{\circ,[r]}_{g,n}(x_1,\dots,x_n)dx_1\otimes\cdots\otimes dx_n.$$

for all $(g,n)\neq(0,2)$ For $(g,n)=(0,2)$ we have:

$$\begin{aligned}\omega_{0,2}(x_1,x_2)&=\frac{dz(x_1)\otimes dz(x_2)}{(z(x_1)-z(x_2))^2}\\ &=\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2}H^{\circ,[r]}_{0,2}(x_1,x_2)dx_1\otimes dx_2+\frac{dx_1\otimes dx_2}{(x_1-x_2)^2}.\end{aligned}$$

This theorem is proved in [4] and [3] using the Johnson-Pandharipande-Tseng formula and the cut-and-join equation for the orbifold Hurwitz numbers. We show below that it is enough to use the quasi-polynomiality property given in Theorem 5.4.9 instead of the Johnson-Pandharipande-Tseng formula.

Since, except for the first few steps that have to be adjusted, the arguments of [4] and [3] still work, we refer to these papers for complete computations. Here, after a careful analysis of the consequences of quasi-polynomiality, we just sketch the main big steps of computation in order to give the reader an idea how the abstract loop equations emerge in this context.

Proof of Theorem 5.6.1. First of all, we have to check the formulas for $\omega_{0,1}$ and $\omega_{0,2}$. This can be done by direct inspection, see [3, 4].

We have the following expression for connected numbers where the sum over $\vec{j} \in \mathbb{Z}_+^n$ is finite because of quasi-polynomiality:

$$h^{\circ,[r]}_{g,\mu} = \left(2g - 2 + l(\mu) + \frac{|\mu|}{r}\right)! \cdot \prod_{i=1}^{n} \frac{\mu_i^{\lfloor \frac{\mu_i}{r} \rfloor}}{\lfloor \frac{\mu_i}{r} \rfloor !} \sum_{\vec{j} \in \mathbb{Z}_+^n} c^{\langle \frac{\vec{\mu}}{r} \rangle}_{g,n,\vec{j}} \mu_1^{j_1} \cdots \mu_n^{j_n}.$$

Here $c^{\langle \frac{\vec{\mu}}{r} \rangle}_{g,n,\vec{j}}$ are the coefficients of the polynomial $P^{\langle \frac{\vec{\mu}}{r} \rangle}_{g,n}(\mu_1, \ldots, \mu_n)$ in Theorem 5.4.9. Hence the partition function reads:

$$H^{\circ,[r]}_{g,n}(x_1, \ldots, x_n) = \sum_{\substack{\vec{\mu} \\ l(\vec{\mu})=n}} \sum_{\vec{j} \in \mathbb{Z}_+^n} c^{\langle \frac{\vec{\mu}}{r} \rangle}_{g,n,\vec{j}} \left(\prod_{i=1}^{n} \frac{\mu_i^{\lfloor \frac{\mu_i}{r} \rfloor + j_i}}{\lfloor \frac{\mu_i}{r} \rfloor !} \right) x_1^{\mu_1} \cdots x_n^{\mu_n}$$

Now we apply the Euclidean division to each μ_i with the notations:

$$\mu_i = r\left\lfloor \frac{\mu_i}{r} \right\rfloor + r\left\langle \frac{\mu_i}{r} \right\rangle \qquad \sigma_i = \left\lfloor \frac{\mu_i}{r} \right\rfloor, \qquad \eta_i = r\left\langle \frac{\mu_i}{r} \right\rangle$$

The coefficients $c^{\langle \frac{\vec{\mu}}{r} \rangle}_{g,n,\vec{j}}$ only depends on the residue of the μ_i modulo r. Writing $[r-1]$ for $\{0, \ldots, r-1\}$ we get:

$$H^{\circ,[r]}_{g,n}(x_1, \ldots, x_n) = \sum_{\vec{\beta} \in [r-1]^n} \sum_{\vec{j} \in \mathbb{Z}_+^n} c^{\beta}_{g,n,\vec{j}} \prod_{i=1}^{n} \sum_{r\sigma_i + \eta_i > 0} \frac{(r\sigma_i + \eta_i)^{\sigma_i + j_i}}{\sigma_i !} x_i^{r\sigma_i + \beta_i}$$

Lemma 5.6.2. *The n-point functions $H^{\circ,[r]}_{g,n}(x_1, \ldots, x_n)$ are local expansions around $(x_1, \ldots, x_n) = (0, \ldots, 0)$ of rational functions in $(z_1, \ldots, z_n)$, where*

$$x(z) = z e^{-z^r}.$$

Proof. It is proved in [19, Equation (46)] that

$$\sum_{\sigma=0}^{\infty} \frac{(r\sigma + \eta')^{\sigma}}{\sigma!} x^{r\sigma + \eta'} = \frac{z^{\eta'}}{1 - rz^r}, \quad \eta' = 1, \ldots, r$$

(note that here we use $\eta' = 1, \ldots, r$ instead of $\eta = 0, \ldots, r-1$ in order to take uniformly the sum over $\sigma \geq 0$ rather than $r\sigma + \eta > 0$). This is obviously a rational function in z, as well as

$$\sum_{\sigma=0}^{\infty} \frac{(r\sigma + \eta)^{\sigma + j}}{\sigma!} x^{r\sigma + \eta} = \left(x \frac{d}{dx} \right)^j \frac{z^{\eta}}{1 - rz^r} = \left(\frac{z}{1 - rz^r} \frac{d}{dz} \right)^j \frac{z^{\eta}}{1 - rz^r}. \tag{5.23}$$

So, $H^{\circ,[r]}_{g,n}(x_1, \ldots, x_n)$ is an expansion of a finite linear combination of products of rational functions in $z_1, \ldots, z_n$. ☐

Let us denote by $p_1, \dots, p_r$ the critical points of the function $x(z)$. It is obvious that each function $z^{\eta'}/(1-rz^r)$ is a linear combination with constant coefficients of the functions $1/(z-z(p_i))$, $i = 1, \dots, r$, up to an additive constant (where said additive constant is not of interest to us, since we are dealing with the differentials of these functions). This implies that all functions given by Equation (5.23) are linear combinations of $1/(z - z(p_i))^a$, $i = 1, \dots, r$, $a \geq 1$. So, we have:

Lemma 5.6.3. *The symmetric differentials $\omega_{g,n} := (d_1 \otimes \cdots \otimes d_n) H^{\circ,[r]}_{g,n}(x_1, \dots, x_n)$ are equal to the sum of their principal parts in the coordinate z at the points $p_1, \dots, p_n$.*

This Lemma immediately implies Equation (5.20) for the standard Cauchy kernel in the coordinate z given by $B(z_1, z_2) = dz_1 dz_2/(z_1 - z_2)^2$.

Lemma 5.6.4. *The differentials $\omega_{g,n}(z_1, \dots, z_n)$ satisfy the linear loop equation* (5.21)*, namely, $\omega_{g,n}(z_1, \dots, z_n) + \omega_{g,n}(\tau_i z_1, z_2, \dots, z_n)$ is holomorphic for $z_1 \to p_i$, where by τ_i we denote the deck transformation of function x near the point p_i, $i = 1, \dots, r$.*

Proof. It is sufficient to proof this lemma for the differentials of the functions given by Equation (5.23). Observe that the operator $x\frac{d}{dx}$ preserves this property, namely, if $df(z) + df(\tau_i z)$ is holomorphic for $z \to p_i$, then $d(x\frac{d}{dx})f(z) + d(x\frac{d}{dx})f(\tau_i z)$ is also holomorphic for $z \to p_i$. It is proved in [19, Equation (4.5)] that

$$\frac{z^{\eta'}}{1 - rz^r} = \left(x\frac{d}{dx}\right)\frac{z^{\eta'}}{\eta'}, \quad \eta' = 1, \dots, r.$$

The functions $z^{\eta'}$ are holomorphic, so their differentials satisfy the linear loop equation. Therefore, the differentials of all the functions given by Equation (5.23) satisfy this property as well. ☐

Now we have to explain how we derive the quadratic loop equation (5.22). The cut-and-join equation for double Hurwitz numbers [11] (see also [4]) can be written in the following form:

$$\begin{aligned} 0 = &-(2g-2+n)H^{\circ,[r]}_{g,n}(x_{[n]}) - \sum_{i=1}^{n}(x_i\frac{d}{dx_i})H^{\circ,[r]}_{g,n}(x_{[n]}) \\ &+ \frac{1}{2}\sum_{i\neq j}\left[\frac{x_i}{x_j - x_i}(x_j\frac{d}{dx_j})H^{\circ,[r]}_{g,n-1}(x_{[n]\setminus\{i\}}) + \frac{x_j}{x_i - x_j}(x_i\frac{d}{dx_i})H^{\circ,[r]}_{g,n-1}(x_{[n]\setminus\{j\}})\right] \\ &+ \frac{1}{2}\sum_{i=1}^{n}\left[(x'\frac{d}{dx'})(x''\frac{d}{dx''})H^{\circ,[r]}_{g-1,n+1}(x', x'', x_{[n]\setminus\{i\}})\right]_{x'=x''=x_i} \\ &+ \frac{1}{2}\sum_{i=1}^{n}\sum_{\substack{g_1+g_2=g \\ I\sqcup J=[n]\setminus\{i\}}}\left[(x_i\frac{d}{dx_i})H^{\circ,[r]}_{g_1,|I|+1}(x_i, x_I)\right]\left[(x_i\frac{d}{dx_i})H^{\circ,[r]}_{g_2,|J|+1}(x_i, x_J)\right]. \end{aligned}$$

Consider the symmetrization of this expression in variable x_1 with respect to the deck transformation near the point p_i. Apply further the operator $\prod_{j=2}^{n}(\frac{d}{dx_j})$ to it and cancel the terms that do not contribute to the polar part of this expression at $z(x_1) \to p_i$. The obstruction for the derived expression to be holomorphic at p_i is precisely the quadratic loop equation (5.22).

This computation implicitly contained in [4] and [3] as the first step of the derivation of the topological recursion, see also [5] for a special case of that. We refer here also to [2, Section 2.4], where it is shown how to derive the topological recursion from the abstract loop equations in general situation, where one can easily recognize the general pattern of the argument in [4] for this particular case. □

5.7 Johnson-Pandharipande-Tseng formula

In this section we give a new proof of a special case of the Johnson-Pandharipande-Tseng formula for orbifold Hurwitz numbers. This is a simple corollary of Theorem 5.6.1, and the results obtained in [15].

We consider the space $\overline{\mathcal{M}}_{g,-\vec{\mu}}(B\mathbb{Z}_r)$ of stable maps to the classifying space $B\mathbb{Z}_r$ of the cyclic group of order r, where the vector $-\vec{\mu}$ in the notation corresponds to the prescribing the monodromy data $(-\mu_1 \mod r, \dots, -\mu_n \mod r)$ at the marked points of source curves. One can think about the elements of this space as admissible covers of curves in $\overline{\mathcal{M}}_{g,n}$ with given monodromy at the marked points. Denote by p the forgetful map $\overline{\mathcal{M}}_{g,-\vec{\mu}}(B\mathbb{Z}_r) \to \overline{\mathcal{M}}$.

Consider the action of $\mathbb{Z}_r$ on the $H^0(C, \omega_C)$, where C is the covering curve. Consider its irreducible component that corresponds to the character $U \colon \mathbb{Z}_r \to \mathbb{C}^*$ that send a generator to $\exp(2\pi i/r)$. This component gives us a vector bundle over $\overline{\mathcal{M}}_{g,-\vec{\mu}}(B\mathbb{Z}_r)$, whose Chern classes we denote by λ_i, $i \geq 0$. We denote by $S(\vec{\mu})$ the class

$$S(\langle \vec{\mu}/r \rangle) := r^{1-g+\sum \langle \frac{\mu_i}{r} \rangle} p_* \sum_{i \geq 0} (-r)^i \lambda_i.$$

Theorem 5.7.1. *We have:*

$$\frac{h^{\circ,[r]}_{g;\vec{\mu}}}{b!} = \prod_{i=1}^{l(\vec{\mu})} \frac{\mu_i^{\lfloor \frac{\mu_i}{r} \rfloor}}{\lfloor \frac{\mu_i}{r} \rfloor !} \int_{\overline{\mathcal{M}}_{g,l(\vec{\mu})}} \frac{S(\langle \vec{\mu}/r \rangle)}{\prod_{j=1}^{l(\vec{\mu})} (1 - \mu_j \psi_j)} \tag{5.24}$$

Remark 5.7.2. This is a special case of the Johnson-Pandharipande-Tseng formula proved in [14], see also [12, 4, 3] for further explanation of the class $S(\vec{\mu})$ used in it.

Remark 5.7.3. Note that this formula looks exactly as formula (5.19), but now the coefficients of the polynomial $P^{\langle \frac{\vec{\mu}}{r} \rangle}_{g,n}(\mu_1, \dots, \mu_n)$ are explicitly represented as intersection numbers.

We give here a new proof of Theorem 5.7.1.

Proof. The proof consists of two simple observations. On the one hand, Theorem 5.6.1 says that the expressions

$$d_1 \otimes \cdots \otimes d_n \sum_{l(\vec{\mu})=n} \frac{h^{\circ,[r]}_{g;\vec{\mu}}}{b!} x_1^{\mu_1} \cdots x_n^{\mu_n}$$

are expansions of the symmetric differentials $\omega_{g,n}(z_1, \dots, z_n)$ that satisfy the topological recursion for the spectral curve data $(\mathbb{CP}^1, x = ze^{-z^r}, y = z^r)$. On the other hand, it is proved in [15] that the expansion of the correlation differentials for this spectral curve is given by

$$d_1 \otimes \cdots \otimes d_n \sum_{l(\vec{\mu})=n} \int_{\overline{\mathcal{M}}_{g,n}} \frac{S(\langle \vec{\mu}/r \rangle)}{\prod_{j=1}^{n} (1 - \mu_j \psi_j)} \prod_{i=1}^{n} \frac{\mu_i^{\lfloor \frac{\mu_i}{r} \rfloor}}{\lfloor \frac{\mu_i}{r} \rfloor !} x^{\mu_i}$$

This identifies the left hand side and the right hand side of Equation (5.24). □

Bibliography

[1] G. Borot, B. Eynard, N. Orantin, Abstract loop equations, topological recursion, and applications, arXiv: 1303.5808.

[2] G. Borot, S. Shadrin, Blobbed topological recursion: properties and applications, arXiv: 1502.00981.

[3] V. Bouchard, D. Hernàndez Serrano, X. Liu, M. Mulase, *Mirror symmetry for orbifold Hurwitz numbers*, J. Differ. Geom. 98 (2014), 375–423.

[4] N. Do, O. Leigh, P. Norbury, *Orbifold Hurwitz numbers and Eynard-Orantin invariants*, arXiv: 1212.6850.

[5] P. Dunin-Barkowski, M. Kazarian, N. Orantin, S. Shadrin, L. Spitz, *Polynomiality of Hurwitz numbers, Bouchard-Mariño conjecture, and a new proof of the ELSV formula*, arXiv: 1307.4729.

[6] P. Dunin-Barkowski, N. Orantin, S. Shadrin, L. Spitz, *Identification of the Givental formula with the spectral curve topological recursion procedure*, arXiv: 1211.4021.

[7] T. Ekedahl, S. Lando, M. Shapiro, and A. Vainshtein, *Hurwitz numbers and intersections on moduli spaces of curves*, Invent. Math. 146 (2001), no. 2, 297-327

[8] B. Eynard, *Invariants of spectral curves and intersection theory of moduli spaces of complex curves*, arXiv: 1110.2949.

[9] B. Eynard, *A short overview of the "Topological recursion"*, arXiv: 1412.3286

[10] B. Eynard, N. Orantin, *Invariants of algebraic curves and topological expansion*, Commun. Number Theory Phys. 1 (2007), no. 2, 347-452.

[11] I.P. Goulden, D.M. Jackson, R. Vakil, *Towards the geometry of double Hurwitz numbers*, Adv. Math. 198 (2005), 43–92.

[12] P. Johnson, *Equivariant Gromov-Witten theory of one dimensional stacks*, arXiv: 0903.1068

[13] P. Johnson, *Double Hurwitz numbers via the infinite wedge*, arXiv: 1008.3266.

[14] P. Johnson, R. Pandharipande, H.-H. Tseng *Abelian Hurwitz–Hodge integrals*, Mich. Math. J. **60** (2011), no. 1, 171–198.

[15] D. Lewanski, A. Popolitov, S. Shadrin, D. Zvonkine, *Chiodo formulas for the r-th roots and topological recursion*, preprint 2015.

[16] T. Miwa, M. Jimbo, E. Date, Solitons. Differential equations, symmetries and infinite-dimensional algebras. Translated from the 1993 Japanese original by Miles Reid. Cambridge Tracts in Mathematics, 135. Cambridge University Press, Cambridge, 2000.

[17] A. Okounkov, *Toda equations for Hurwitz numbers.* Math. Res. Lett. **7** (2000), no. 4, 447–453.

[18] A. Okounkov and R. Pandharipande, *The equivariant Gromov-Witten theory of* $\mathbb{P}^1$, arXiv: math/0207233.

[19] S. Shadrin, L. Spitz, D. Zvonkine, *Equivalence of ELSV and Bouchard-Mariño conjectures for r-spin Hurwitz numbers*, Math. Ann. **361** (2015), no. 3-4, 611–645.

6

Quasi-polynomiality of Grothendieck's dessins d'enfants, orbifold, monotone orbifold, spin, and spin orbifold Hurwitz numbers

In this chapter we prove quasi-polynomiality for a number of Hurwitz problems by computing explicitly the residues of certain operators on the Fock space. More precisely, these enumerative problems are monotone orbifold and strictly monotone orbifold Hurwitz numbers; and orbifold, spin, and spin orbifold Hurwitz numbers. The second enumerative problem is also known as enumeration of a special kind of Grothendieck's dessins d'enfants or r-hypermaps. The first two cases answer positively two conjectures proposed by Do-Karev and Do-Manescu. In the third case we obtain a new and easier proof of the quasi-polynomiality proved in Chapter 5. The fourth case provides a key step towards the proof of Zvonkine's conjectural r-ELSV formula that relates Hurwitz numbers with completed $(r+1)$-cycles to the geometry of the moduli spaces of the r-spin structures on curves. The fifth case allow us to propose an orbifold generalization of Zvonkine's conjecture.

In the second part of the chapter we show that the property of quasi-polynomiality is equivalent in the first three cases to the property for the n-point generating function to have a natural representation on the n-th cartesian powers of a certain algebraic curve. These representations are necessary conditions for the Chekhov-Eynard-Orantin topological recursion. In addition to that, we study the $(0,1)$- and $(0,2)$-functions in many of the cases cited above, and we show that these unstable cases are correctly reproduced by the conjectural (or proved) spectral curve initial data.

6.1 Introduction

This chapter is devoted to a combinatorial and analytic study of several kinds of Hurwitz numbers. The five kinds of Hurwitz numbers that we consider in this chapter are the monotone orbifold, the strictly monotone orbifold, the usual orbifold, the spin and the spin orbifold Hurwitz numbers. Note that the theory of the strictly monotone orbifold Hurwitz numbers is equivalent to the enumeration of hypermaps on two-dimensional surfaces, or, in other words, to the enumeration of some special type of Grothendieck's dessins d'enfants.

This type of combinatorial objects is important both for purely combinatorial reasons and also because of the numerous relations that these numbers and their generating functions have to the intersection theory of the moduli spaces of curves, matrix models and topological recursion, and integrable systems. We will not make any attempt to survey this very rich theory, and we refer the interested reader to [1, 2, 3, 7, 8, 9, 10, 11, 15, 20, 21, 22, 23, 24, 25, 30, 31, 35, 40, 41] and references therein.

The Hurwitz numbers of all five types can be efficiently realized as the vacuum expectations in the semi-infinite wedge formalism. These formulae will be the starting point for the chapter, and we use them as the definitions of the corresponding Hurwitz numbers. The equivalence with the usual definitions is established via the character formula, and we refer to [1] for that.

6.1.1 Monotone orbifold, strictly monotone orbifold and usual orbifold Hurwitz numbers

Recall that the first three Hurwitz numbers that we consider, $h^{\circ,r,\leq}_{g;\vec\mu}$, $h^{\circ,r,<}_{g;\vec\mu}$, and $h^{\circ,r}_{g;\vec\mu}$, depend on a genus parameter $g\geq 0$, and a tuple of $n\geq 1$ positive integers $\vec\mu=(\mu_1,\dots,\mu_n)$. It is a natural combinatorial question how these numbers depend on the parameters $\mu_1,\dots,\mu_n$. We prove in this chapter that for $2g-2+n>0$ the dependence on the parameters can be described in a very explicit way. Namely, let us represent any integer a as $r[a]+\langle a\rangle$, $0\leq\langle a\rangle\leq r-1$, and let $\langle\vec\mu\rangle := (\langle\mu_1\rangle,\dots,\langle\mu_n\rangle)$. We will use this notation throughout the article. We prove that there exist polynomials $P^\eta_{\leq}$, $P^\eta_{<}$, and P^η of degree $3g-3+n$ in n variables, whose coefficients depend on η and also on g and r, such that

$$h^{\circ,r,\leq}_{g;\vec\mu} = P^{\langle\vec\mu\rangle}_{\leq}(\mu_1,\dots,\mu_n)\cdot\prod_{i=1}^{n}\binom{\mu_i+[\mu_i]}{\mu_i};$$

$$h^{\circ,r,<}_{g;\vec\mu} = P^{\langle\vec\mu\rangle}_{<}(\mu_1,\dots,\mu_n)\cdot\prod_{i=1}^{n}\binom{\mu_i-1}{[\mu_i]};$$

$$h^{\circ,r}_{g;\vec\mu} = P^{\langle\vec\mu\rangle}(\mu_1,\dots,\mu_n)\cdot\prod_{i=1}^{n}\frac{\mu_i^{[\mu_i]}}{[\mu_i]!}.$$

We call this property quasi-polynomiality. The proof is purely combinatorial and uses some properties of the analogues of the $\mathcal{A}$-operators of Okounkov and Panharipande [37] in the semi-infinite wedge formalism. This statement was known for the usual orbifold Hurwitz numbers [2, 15, 9]. In this case we give a new proof. In the cases of monotone and strictly monotone orbifold Hurwitz numbers, this property was conjectured by Do and Karev in [8] and Do and Manescu in [10], respectively, and no proof was known.

Let us explain why the property of being quasi-polynomial is of crucial importance for these Hurwitz numbers, as well as some further results of this chapter. For that, we recall several connections of the Hurwitz theory to other areas of mathematics.

First of all, there is a connection to the spectral curve topological recursion in the sense of Chekhov-Eynard-Orantin (CEO). This means that the corresponding Hurwitz numbers can be obtained as the coefficients of some particular expansion of the correlation differentials defined on the Cartesian products of some fixed Riemann surface called the spectral curve. These differentials are produced by the CEO topological recursion procedure from a fairly small input data. The input data consists of a curve Σ, a symmetric bi-differential B defined on $\Sigma\times\Sigma$ with a double pole on the diagonal with biresidue 1, and two meromorphic functions, x and y, defined on Σ. This allows us to compute recursively the correlation differentials. We need one more piece of data — the variable in which we want to expand the correlation differentials in order to obtain as coefficients the solutions of the combinatorial problem.

In our cases, the data is the following. The curve Σ is always $\mathbb{C}\mathrm{P}^1$ in all three cases. We denote by z a global coordinate on $\mathbb{C}\mathrm{P}^1$. In the case of $\mathbb{C}\mathrm{P}^1$ the bi-differential $B(z_1, z_2)$ is uniquely determined by its properties and is equal to $dz_1 dz_2/(z_1 - z_2)^2$. The functions x and y are the following:

$$
\begin{aligned}
&x = z(1 - z^r), && y = z^{r-1}/(z^r - 1) && \text{in the monotone case;}\\
&x = z^r + z^{-1}, && y = z && \text{in the strictly monotone case;}\\
&x = \log z - z^r, && y = z^r && \text{in the usual case.}
\end{aligned}
$$

The correlation differentials obtained by the CEO recursion in these cases should be expanded

$$
\begin{aligned}
&\text{in the variable } x && \text{near } x = 0 && \text{in the monotone case;}\\
&\text{in the variable } x^{-1} && \text{near } x = \infty && \text{in the strictly monotone case;}\\
&\text{in the variable } e^x && \text{near } e^x = 0 && \text{in the usual case.}
\end{aligned}
$$

The topological recursion is proved in the case of the usual orbifold Hurwitz numbers in [2, 9], in the case of strictly monotone Hurwitz numbers it was conjectured in [10] and combinatorially proved in [17], based on the original derivation of topological recursion in [5] in the case of the two-matrix model. In the case of monotone orbifold Hurwitz numbers only the case $r = 1$ has been proved in [7], and a general conjecture was made in [8].

The relation between quasi-polynomiality and the topological recursion is the following. We prove in this chapter that a sequence of numbers depending on a tuple $(\mu_1, \dots, \mu_n)$ can be represented as a polynomial in $\mu_1, \dots, \mu_n$ times the non-polynomial factor $\prod_{i=1}^n \binom{\mu_i + [\mu_i]}{\mu_i}$ (respectively, $\prod_{i=1}^n \binom{\mu_i - 1}{[\mu_i]}$, $\prod_{i=1}^n \mu_i^{[\mu_i]}/[\mu_i]!$) if and only if it can be represented as an expansion of a special kind of symmetric n-differential on the curve $x = z(1 - z^r)$ (respectively, $x = z^r + z^{-1}$, $x = \log z - z^r$) in the variable x (respectively, x^{-1}, e^x).

In the case of the usual orbifold Hurwitz numbers it was already known and used in [9, 2, 15], and, in a slightly different situation, in [39]. In the case of monotone and strictly monotone orbifold Hurwitz numbers this equivalence was neither explicitly stated nor proved, though it is implicitly suggested in a conjectural form in [8] for the monotone and in [10] for the strictly monotone cases. Note that since the topological recursion is proved for the strictly monotone Hurwitz numbers independently [5, 17], this equivalence implies the quasi-polynomiality as well.

There are also two unstable cases that have to be studied separately: $(g, n) = (0, 1)$ and $(0, 2)$. In the case $(g, n) = (0, 1)$ (respectively, $(g, n) = (0, 2)$) the topological recursion requires that the generating function of the corresponding Hurwitz numbers is given by the expansion of ydx (respectively, $B(z_1, z_2) - B(x_1, x_2)$). For $(g, n) = (0, 1)$ this property has been proved in all three cases, in [8] for the monotone, in [10] for the strictly monotone and in [9, 2] for the usual orbifold Hurwitz numbers. Basically, such a representation for the $(g, n) = (0, 1)$ generating function is a way to guess a spectral curve for the corresponding combinatorial problem. For $(g, n) = (0, 2)$ this property has been proved for strictly monotone and usual orbifold Hurwitz numbers (indeed, the topological recursion is proved in both cases), but it was not known for the monotone case. We prove this in section 6.9.

Let us remark that this set of properties (namely, representation of the $(0, 1)$ generating function as an expansion of ydx, the $(0, 2)$ generating function as an expansion of $B(z_1, z_2) - B(x_1, x_2)$, and the quasi-polynomiality property for $2g - 2 + n > 0$) is required for the approach to the topological recursion in [12]. Once these properties are established, the topological recursion appears to be a Laplace transform of some much easier recursion property of the corresponding combinatorial problem.

The other important connection for all three Hurwitz theories that we consider here is their relations to the intersection theory of the moduli spaces of curves. It appears that the coefficients of the polynomials in the quasi-polynomial representation of the n-point functions can be represented in terms of some intersection numbers on the moduli spaces of curves. This statement is proved for usual Hurwitz numbers for $r = 1$ in [18] and for any r in [28].

In general, we assume know that being quasi-polynomial is equivalent to being an expansion of a symmetric differential of certain type. Then in this situation there is an equivalence between the topological recursion and representation in terms of the intersection theory of the moduli spaces of curves. The intersection numbers in this case appear to be the correlators of a certain cohomological field theory, possibly with a non-flat unit. This point of view on topological recursion was first suggested by Eynard in [19] and worked out in detail in many examples, see e. g. [14, 13, 39, 34].

In particular, the cohomological field theory for the case of the strictly monotone orbifold Hurwitz numbers is described in [16]. For the monotone orbifold Hurwitz numbers the intersection number formula was derived so far only the case $r = 1$, see [1, 8], and it is based on the proof of the topological recursion in [7].

6.1.2 The r-spin Hurwitz numbers

There is another generalization of Hurwitz numbers [39] natural both from the point of view of the representation theory of the symmetric group [32] and the Gromov-Witten theory of the projective line [38], where the typical singularity has the monodromy type of a completed $(r + 1)$-cycle. These numbers are called the r-spin Hurwitz numbers, and this name is inspired by an ELSV-type formula, called the r-ELSV formula, conjectured by Zvonkine in 2006 [42]. This conjecture relates the r-spin Hurwitz numbers to the intersection numbers on the moduli spaces of r-spin structures, see [42, 39].

In this case the intersection number formula is only conjectural, and no alternative proof of the quasi-polynomiality is known. It is proved in [39] that the conjectural r-ELSV formula is equivalent to the topological recursion on $\mathbb{C}\mathrm{P}^1$ for the following initial data:

$$x(z) = \log z - z^r, \quad y(z) = z, \quad B(z_1, z_2) = dz_1 dz_2/(z_1 - z_2)^2. \tag{6.1}$$

It is also proved in [35] that the differential of the $(0, 1)$-function for r-spin Hurwitz numbers is indeed the expansion of $y\,dx(z)$ in the variable $\exp(x)$ near $\exp(x) = 0$, where x and y are defined in equation (6.1).

The results of this chapter include, as a special case, the proof that the 2-differential obtained from the $(0, 2)$-function of the r-spin Hurwitz numbers is given by the expansion of $B(z_1, z_2)$ in the variables $\exp(x_1), \exp(x_2)$ near the point $\exp(x_1) = \exp(x_2) = 0$, where x and B are defined in equation (6.1), as well as the quasi-polynomiality statement for the (g, n)-functions for $2g - 2 + n > 0$.

6.1.3 The q-orbifold r-spin Hurwitz numbers

It is natural to combine the spin and the orbifold generalizations of the concept of Hurwitz number: the monodromy of one special fiber consists of q-cycles, and the monodromy of the typical singularity is given by the completed $(r+1)$-cycle. This way we get q-orbifold r-spin Hurwitz numbers [35]. There is not much known about this generalization. There is only a quantum curve for this case that is proved in [35]. Note, however, that according to logic outlined in [1], this leads to a guess of the spectral curve for this case, and the spectral curve implies an ELSV-type formula for this type of Hurwitz numbers as well.

The conjectural spectral curve in this case is $\mathbb{C}\mathrm{P}^1$ with the following initial data:

$$x(z) = \log z - z^{qr}, \quad y(z) = z^q, \quad B(z_1, z_2) = dz_1 dz_2/(z_1 - z_2)^2. \tag{6.2}$$

The result of [35] implies that the differential of the $(0,1)$-function is the expansion of $y\,dx(z)$ in $\exp(x)$ near the point $\exp(x) = 0$, where x and y are defined in equation (6.2).

The main result of this section is the quasi-polynomiality statement for the q-orbifold r-spin Hurwitz numbers and the proof that the 2-differential obtained from the $(0,2)$-function of the q-orbifold r-spin Hurwitz numbers is given by the expansion of $B(z_1, z_2)$ in the variables $\exp(x_1), \exp(x_2)$ near the point $\exp(x_1) = \exp(x_2) = 0$, where x and B are defined in equation (6.2). We also prove the statement of [35] about the $(0,1)$-function in a new way.

This allows us to generalize the conjecture of Zvonkine, in the following way. We conjecture that the q-orbifold r-spin Hurwitz numbers satisfy the topological recursion of the initial data given in equation (6.2). By the results of [19, 14] this immediately implies a conjectural ELSV-type formula for these Hurwitz numbers. The particular computation for the initial data (6.2) is performed in [34], where the correlation differentials for this spectral curve are presented in terms of the Chiodo classes [6]. This allows us to obtain a very precise description of the conjectural ELSV-type formula for the q-orbifold r-spin Hurwitz numbers, which reduces in the case $q = 1$ to the original conjecture of Zvonkine.

6.1.4 Organization of the chapter

In section 6.2 we briefly recall the necessary background on semi-infinite wedge formalism. In section 6.3 we review the interplay between symmetric polynomials and Stirling numbers, together with their generating function. In section 6.4 we define the $\mathcal{A}$-operators for the monotone and strictly monotone Hurwitz numbers, and we express the generating series for monotone and strictly monotone Hurwitz numbers in terms of $\mathcal{A}$-operators acting on the Fock space. In section 6.5 we state and prove the first result of the chapter: the quasi-polynomiality for monotone and strictly monotone Hurwitz numbers. In section 6.6 we apply the same methods to the case of orbifold Hurwitz numbers, obtaining a simpler proof of the results in Chapter 5. In section 6.7 we define the q-orbifold r-spin Hurwitz numbers and we present them as the vacuum expectations of $\mathcal{A}$-operators. Applying the same methods, we prove the quasi-polynomiality property. In section 6.8 the polynomiality properties for the first three Hurwitz problems are proved to be equivalent to the analytic properties that are necessary for the Chekhov-Eynard-Orantin topological recursion. In section 6.9 we perform the computations for the unstable $(0,1)$, as an example of the usage of the $\mathcal{A}$-operators, and we prove a formula relating the $(0,2)$-generating function for the monotone orbifold Hurwitz numbers to the expansion of the Bergman kernel. In section 6.10 we consider the unstable correlation differentials for the conjectural spectral curve and reproduce the $(0,1)$- and $(0,2)$-point functions for the q-orbifold r-spin Hurwitz numbers. Finally, in section 6.11 we describe precisely a conjectural ELSV-type formula for the q-orbifold r-spin Hurwitz numbers that generalizes the conjecture of Zvonkine for r-spin Hurwitz numbers.

6.2 Semi-infinite wedge formalism

In this section we briefly recall the semi-infinite wedge formalism. It is nowadays a standard tool in Hurwitz theory, with many good introductions to it. We refer the reader, for instance, to [26] and [1] and references therein for a more complete exposition.

Let V be an infinite-dimensional complex vector space with a basis labeled by half-integers. Denote the basis vector labeled by $m/2$ by $\underline{m/2}$, so $V = \bigoplus_{i \in \mathbb{Z}+\frac{1}{2}} \mathbb{C}\underline{i}$.

Definition 6.2.1. The semi-infinite wedge space $\bigwedge^{\frac{\infty}{2}}(V) = \mathcal{V}$ is defined to be the span of all of the semi-infinite wedge products of the form

$$\underline{i_1} \wedge \underline{i_2} \wedge \cdots$$

for any decreasing sequence of half-integers (i_k) such that there is an integer c with $i_k + k - \frac{1}{2} = c$ for k sufficiently large. The constant c is called the *charge*. We give $\mathcal{V}$ an inner product $(\cdot, \cdot)$ declaring its basis elements to be orthonormal.

Remark 6.2.2. By definition 6.2.1 the charge-zero subspace $\mathcal{V}_0$ of $\mathcal{V}$ is spanned by semi-infinite wedge products of the form

$$\underline{\lambda_1 - \frac{1}{2}} \wedge \underline{\lambda_2 - \frac{3}{2}} \wedge \cdots$$

for some integer partition λ. Hence we can identify integer partitions with the basis of this space:

$$\mathcal{V}_0 = \bigoplus_{n \in \mathbb{N}} \bigoplus_{\lambda \vdash n} \mathbb{C} v_\lambda$$

The empty partition $\emptyset$ plays a special role. We call

$$v_\emptyset = \underline{-\frac{1}{2}} \wedge \underline{-\frac{3}{2}} \wedge \cdots$$

the vacuum vector and we denote it by $|0\rangle$. Similarly we call the covacuum vector its dual with respect to the scalar product $(\cdot, \cdot)$ and we denote it by $\langle 0|$.

Definition 6.2.3. The *vacuum expectation value* or *disconnected correlator* $\langle \mathcal{P} \rangle^\bullet$ of an operator $\mathcal{P}$ acting on $\mathcal{V}_0$ is defined to be:

$$\langle \mathcal{P} \rangle^\bullet := (|0\rangle, \mathcal{P}|0\rangle) =: \langle 0|\mathcal{P}|0\rangle$$

We also define the functions

$$\zeta(z) = e^{z/2} - e^{-z/2} = 2\sinh(z/2)$$

and

$$\mathcal{S}(z) = \frac{\zeta(z)}{z} = \frac{\sinh(z/2)}{z/2}.$$

Definition 6.2.4. This is the list of operators we will use:

i) For k half-integer the operator $\psi_k : (\underline{i_1} \wedge \underline{i_2} \wedge \cdots) \mapsto (\underline{k} \wedge \underline{i_1} \wedge \underline{i_2} \wedge \cdots)$ increases the charge by 1. Its adjoint operator ψ_k^* with respect to $(\cdot, \cdot)$ decreases the charge by 1.

ii) The normally ordered products of ψ-operators

$$E_{i,j} := \begin{cases} \psi_i \psi_j^*, & \text{if } j > 0 \\ -\psi_j^* \psi_i & \text{if } j < 0 \ . \end{cases}$$

preserve the charge and hence can be restricted to $\mathcal{V}_0$ with the following action. For $i \neq j$, $E_{i,j}$ checks if v_λ contains $\underline{j}$ as a wedge factor and if so replaces it by $\underline{i}$. Otherwise it yields 0. In the case $i = j > 0$, we have $E_{i,j}(v_\lambda) = v_\lambda$ if v_λ contains $\underline{j}$ and 0 if it does not; in the case $i = j < 0$, we have $E_{i,j}(v_\lambda) = -v_\lambda$ if v_λ does not contain $\underline{j}$ and 0 if it does. This gives a projective representation of $\mathcal{A}_\infty$, the Lie algebra of complex $\mathbb{Z} \times \mathbb{Z}$ matrices with only finitely many non-zero diagonals [26].

iii) The diagonal operators are assembled into the operators

$$\mathcal{F}_n := \sum_{k \in \mathbb{Z} + \frac{1}{2}} \frac{k^n}{n!} E_{k,k}$$

The operator $C := \mathcal{F}_0$ is called *charge operator*, while the operator $E := \mathcal{F}_1$ is called *energy operator*. Note that $\mathcal{F}_0$ vanishes identically on $\mathcal{V}_0$. We say that an operator $\mathcal{P}$ on $\mathcal{V}_0$ is of energy $c \in \mathbb{Z}$ if

$$[\mathcal{P}, E] = c\,\mathcal{P}$$

The operator $E_{i,j}$ has energy $j - i$, hence all the $\mathcal{F}_n$'s have zero energy. Operators with positive energy annihilate the vacuum while negative energy operators are annihilated by the covacuum.

iv) For n any integer and z a formal variable one has the energy n operators:

$$\mathcal{E}_n(z) = \sum_{k \in \mathbb{Z} + \frac{1}{2}} e^{z(k - \frac{n}{2})} E_{k-n,k} + \frac{\delta_{n,0}}{\zeta(z)}.$$

v) For n any nonzero integer one has the energy n operators:

$$\alpha_n = \mathcal{E}_n(0) = \sum_{k \in \mathbb{Z} + \frac{1}{2}} E_{k-n,k}$$

The commutation relation between basis elements reads

$$[E_{a,b}, E_{c,d}] = \delta_{b,c} E_{a,d} - \delta_{a,d} E_{c,b} + \delta_{b,c}\delta_{a,d}(\delta_{b>0} - \delta_{d>0})\,\mathrm{Id}\,. \tag{6.3}$$

Using this commutation rule, it is useful to compute:

Lemma 6.2.5.

$$\begin{aligned} \left[\sum_{l \in \mathbb{Z}'} g_l E_{l-a,l}, \sum_{k \in \mathbb{Z}'} f_k E_{k-b,k} \right] &= \sum_{l \in \mathbb{Z}'} (g_{l-b} f_l - g_l f_{l-a}) \, E_{l-(a+b),l} \\ &+ \delta_{a+b} \delta_{a>0} (g_{1/2} f_{1/2-a} + \cdots + g_{a-1/2} f_{-1/2}) \\ &+ \delta_{a+b} \delta_{b>0} (g_{1/2-b} f_{1/2} + \cdots + g_{-1/2} f_{b-1/2}). \end{aligned}$$

In particular, with $g_l = f_k = 1$, it is possible to recover the usual commutation formula

$$[\alpha_a, \alpha_b] = a\delta_{a+b}.$$

The commutation formula for $\mathcal{E}$ operators reads:

$$[\mathcal{E}_a(z), \mathcal{E}_b(w)] = \zeta\left(\det\begin{bmatrix} a & z \\ b & w \end{bmatrix}\right)\mathcal{E}_{a+b}(z+w) \tag{6.4}$$

Note that $\mathcal{E}_k(z)\big|0\big\rangle = 0$ if $k > 0$, while $\mathcal{E}_0(z)\big|0\big\rangle = \zeta(z)^{-1}\big|0\big\rangle$. We will also use the $\mathcal{E}$ operator without the correction in energy zero, i.e.

$$\tilde{\mathcal{E}}_0(z) = \sum_{k\in\mathbb{Z}+\frac{1}{2}} e^{zk}E_{k,k} = \sum_{n=0}^{\infty}\mathcal{F}_n z^n = C + Ez + \mathcal{F}_2 z^2 + \dots$$

which annihilates the vacuum and obeys the same commutation rule as $\mathcal{E}_0$.

6.3 Symmetric polynomials and Stirling numbers

In this section we recollect some combinatorial notions used in the rest of the chapter. In particular we recall here some basic facts on homogeneous symmetric polynomials and Stirling numbers, and their interconnection.

6.3.1 Symmetric polynomials

Definition 6.3.1. Let $X = \{x_1, \dots, x_n\}$ be a finite set of variables. The *complete symmetric polynomials* h_k and the *elementary symmetric polynomials* σ_k on X are defined as follows:

$$h_k(X) = \sum_{1\le i_1\le i_2\le\cdots\le i_k\le n} x_{i_1}\cdots x_{i_k}$$
$$\sigma_k(X) = \sum_{1\le i_1< i_2<\cdots< i_k\le n} x_{i_1}\cdots x_{i_k}$$

The properties of these functions are well-documented, see e.g. [36]. We will list some useful properties.

Lemma 6.3.2. *The generating functions of the complete and elementary symmetric polynomials are as follows:*

$$\sum_{k=0}^{\infty} h_k(x_1,\dots,x_n)u^k = \prod_{i=1}^{n}\frac{1}{1-ux_i}$$
$$\sum_{k=0}^{\infty} \sigma_k(x_1,\dots,x_n)u^k = \prod_{i=1}^{n}(1+ux_i)$$

Corollary 6.3.3. *For any finite set of variables X,*

$$\sum_{k=0}^{\infty} h_k(X)u^k \sum_{l=0}^{\infty}\sigma_l(X)(-u)^l = 1 \tag{6.5}$$

The following lemma is an easy consequence of the definitions, and can be proved by induction on the number of arguments.

Lemma 6.3.4. *If the variables in a symmetric polynomial are all offset by the same amount, they can be re-expressed as a linear combination of non-offset symmetric polynomials as follows:*

$$h_k(x_1 + A, \dots, x_n + A) = \sum_{i=0}^{k} \binom{k+n-1}{i} h_{k-i}(x_1, \dots, x_n) A^i \tag{6.6}$$

$$\sigma_k(x_1 + A, \dots, x_n + A) = \sum_{i=0}^{k} \binom{n+i-k}{i} \sigma_{k-i}(x_1, \dots, x_n) A^i$$

6.3.2 Stirling numbers

We now recall some notions on Stirling numbers. A complete treatment of the subject can be found in [4].

Definition 6.3.5. The *(unsigned) Stirling numbers of the first kind* $\left[{i \atop t}\right]$ are defined as coefficients of the following expansion in the formal variable T

$$(T)_i = \sum_{t=0}^{i} \left[{i \atop t}\right] T^t$$

where i, t are nonnegative integers and the subscript indicates the *Pochhammer symbol*:

$$(x+1)_n = \frac{(x+n)!}{x!} = \begin{cases} (x+1)(x+2)\cdots(x+n) & n \geq 0 \\ (x(x-1)\cdots(x+n+1))^{-1} & n \leq 0 \end{cases}.$$

From the definition, $(x+1)_n$ vanishes for integers x satisfying $-n \leq x \leq -1$, and $1/(x+1)_n$ vanishes for integers x satisfying $0 \leq x \leq -(n+1)$.

The *Stirling numbers of the second kind* $\left\{{i \atop t}\right\}$ are defined as coefficients of the following expansion in the formal variable T

$$T^i = \sum_{t=0}^{i} \left\{{i \atop t}\right\} (T-t+1)_t$$

where i, t are nonnegative integers. Note that for $t > i$ we have $\left[{i \atop t}\right] = \left\{{i \atop t}\right\} = 0$.

The complete and elementary polynomials evaluated at integers are linked to the Stirling numbers by the following relation.

$$\sigma_v(1, 2, \dots, t-1) = \left[{t \atop t-v}\right]$$

$$h_v(1, 2, \dots, t) = \left\{{t+v \atop t}\right\} \tag{6.7}$$

The expressions in terms of generating series read

Lemma 6.3.6. *We have:*

$$\left[{j \atop t}\right] = [y^{j-t}].\frac{(j-1)!}{(t-1)!}\mathcal{S}(y)^{-j} e^{yj/2}; \qquad \left\{{j \atop t}\right\} = [y^{j-t}].\frac{j!}{t!}\mathcal{S}(y)^{t} e^{yt/2}.$$

6.4 $\mathcal{A}$-operators for monotone orbifold Hurwitz numbers

In this section we express the generating series for monotone and strictly monotone orbifold Hurwitz numbers in terms of correlators of certain $\mathcal{A}$-operators acting on the Fock space.

6.4.1 Generating series for monotone orbifold Hurwitz numbers

Let us define the genus-generating series for disconnected monotone and strictly monotone orbifold Hurwitz numbers as

$$H^{\bullet,r,\leq}(u,\vec{\mu}) := \sum_{g=0}^{\infty} \left(h^{r,\leq}_{g;\vec{\mu}}\right) u^b, \qquad H^{\bullet,r,<}(u,\vec{\mu}) := \sum_{g=0}^{\infty} \left(h^{r,<}_{g;\vec{\mu}}\right) u^b \tag{6.8}$$

where, by Riemann-Hurwitz, b is the number of simple ramifications

$$b = 2g - 2 + l(\mu) + |\mu|/r.$$

We want to express the generating series through the semi-infinite wedge formalism. In [1] it was proved that the eigenvalue of the operator

$$\mathcal{D}^{(h)}(u) := \exp\left(\left[\frac{\tilde{\mathcal{E}}_0\left(u^2\frac{d}{du}\right)}{\zeta\left(u^2\frac{d}{du}\right)} - E\right].\log u\right)$$

acting on the basis of the charge zero sector of the Fock space is the generating series for the complete symmetric polynomials, in the sense that

$$\mathcal{D}^{(h)}(u).v_\lambda = \sum_{k=0}^{\infty} h_k(\mathsf{cr}^\lambda)u^k v_\lambda,$$

where the set of variables cr^λ is the content of Young tableau λ. Similarly, the operator

$$\mathcal{D}^{(\sigma)}(u) := \exp\left(-\left[\frac{\tilde{\mathcal{E}}_0\left(-u^2\frac{d}{du}\right)}{\zeta\left(-u^2\frac{d}{du}\right)} - E\right].\log u\right)$$

produces as eigenvalue the generating series for elementary symmetric polynomials:

$$\mathcal{D}^{(\sigma)}(u).v_\lambda = \sum_{k=0}^{\infty} \sigma_k(\mathsf{cr}^\lambda)u^k v_\lambda.$$

The generating series in equation (6.8) therefore read respectively

$$H^{\bullet,r,\leq}(u,\vec{\mu}) = \left\langle e^{\frac{\alpha_r}{r}} \mathcal{D}^{(h)}(u) \prod_{i=1}^{n} \frac{\alpha_{-\mu_i}}{\mu_i} \right\rangle^{\bullet} \tag{6.9}$$

and

$$H^{\bullet,r,<}(u,\vec{\mu}) = \left\langle e^{\frac{\alpha_r}{r}} \mathcal{D}^{(\sigma)}(u) \prod_{i=1}^{n} \frac{\alpha_{-\mu_i}}{\mu_i} \right\rangle^{\bullet} \tag{6.10}$$

6.4.2 Conjugations of operators

In this section we prove several lemmata that we will use later.

Lemma 6.4.1. *We have:*

$$\mathcal{O}^h_\mu(u) := \mathcal{D}^{(h)}(u)\alpha_{-\mu}\mathcal{D}^{(h)}(u)^{-1} = \sum_{k\in\mathbb{Z}+\frac{1}{2}}\sum_{v=0}^{\infty} h_v(1+k-1/2,\dots,\mu+k-1/2)u^v E_{k+\mu,k};$$

$$\mathcal{O}^\sigma_\mu(u) := \mathcal{D}^{(\sigma)}(u)\alpha_{-\mu}\mathcal{D}^{(\sigma)}(u)^{-1} = \sum_{k\in\mathbb{Z}+\frac{1}{2}}\sum_{v=0}^{\infty} \sigma_v(1+k-1/2,\dots,\mu+k-1/2)u^v E_{k+\mu,k}.$$

Proof. We prove only the first equation, since the proof for the second is completely analogous. Applying the change of variable $u(z) = -z^{-1}$, we have

$$\mathcal{D}^{(h)}(u(z)) = \exp\left(-\frac{\tilde{\mathcal{E}}_0\left(\frac{d}{dz}\right)}{\zeta\left(\frac{d}{dz}\right)}.\log(-z)\right)(-z)^E =: e^{B(z)}(-z)^E$$

Observe that the operator $B(z)$ has zero energy and hence commutes with $(-z)^E$. On the other hand, the operator $\alpha_{-\mu}$ has energy $-\mu$, hence the conjugation by the operator $(-z)^E$ produces the extra factor $(-z)^\mu$. By the Hadamard lemma we can expand the conjugation as

$$\mathcal{D}^{(h)}(u)\alpha_{-\mu}\mathcal{D}^{(h)}(u)^{-1} = (-z)^\mu\sum_{s=0}^{\infty}\frac{1}{s!}\,\mathrm{ad}^s_{B(z)}(\alpha_{-\mu})$$

It is enough to show that

$$\mathrm{ad}^s_{B(z)}(\alpha_{-\mu}) = \sum_{k\in\mathbb{Z}+\frac{1}{2}}\log\left(\prod_{l=0}^{\mu-1}\frac{1}{(-z-l-k-1/2)}\right)^s E_{k+\mu,k} \tag{6.11}$$

Indeed this would imply

$$\mathcal{D}^{(h)}(u)\alpha_{-\mu}\mathcal{D}^{(h)}(u)^{-1} = \sum_{k\in\mathbb{Z}+\frac{1}{2}}\left(\prod_{l=0}^{\mu-1}\frac{1}{1-(l+k+1/2)(-z^{-1})}\right)E_{k+\mu,k}$$

which proves the lemma by substituting back $u = -z^{-1}$ and expanding in the generating series for complete symmetric polynomials. Let $C(s)$ be the left hand side of equation (6.11). We compute:

$$\begin{aligned}
C(s) &= \left[-\frac{\tilde{\mathcal{E}}_0\left(\frac{d}{dz_s}\right)}{\zeta\left(\frac{d}{dz_s}\right)},\dots\left[-\frac{\tilde{\mathcal{E}}_0\left(\frac{d}{dz_1}\right)}{\zeta\left(\frac{d}{dz_1}\right)},\mathcal{E}_{-\mu}(0)\right]\dots\right].\prod_{i=1}^{s}\log(-z_i)\Big|_{z_i=z}\\
&= (-1)^s\prod_{i=1}^{s}\frac{\zeta\left(\mu\frac{d}{dz_i}\right)}{\zeta\left(\frac{d}{dz_i}\right)}\mathcal{E}_{-\mu}\left(\sum_{i=1}^{s}\frac{d}{dz_i}\right).\prod_{i=1}^{s}\log(-z_i)\Big|_{z_i=z}\\
&= \sum_{k\in\mathbb{Z}+1/2}\prod_{i=1}^{s}\sum_{l=0}^{\infty}-\left(e^{\frac{d}{dz_i}(\mu+k-l-1/2)} - e^{\frac{d}{dz_i}(k-l-1/2)}\right).\log(-z_i)E_{k+\mu,k}\Big|_{z_i=z}
\end{aligned}$$

Observe that the summation over l is the result of the expansion in geometric formal power series of $1/(1-e^{-d/dz_i})$. The expression in the last line equals the right hand side of equation (6.11) since the s operators act independently, and using $e^{a\frac{d}{dz}}f(z) = f(z+a)$. The lemma is proved. □

Let us define the operators:

$$\mathcal{O}^h_\mu(u)^\flat := \mathcal{D}^{(h)}(u)\alpha_\mu\mathcal{D}^{(h)}(u)^{-1} \qquad \mathcal{O}^\sigma_\mu(u)^\flat := \mathcal{D}^{(\sigma)}(u)\alpha_\mu\mathcal{D}^{(\sigma)}(u)^{-1}$$

Lemma 6.4.2. *We have:*

$$\mathcal{O}^h_\mu(u)^\flat = \sum_{k\in\mathbb{Z}+\frac{1}{2}}\sum_{v=0}^{\infty}\sigma_v(1+k-1/2,\dots,\mu+k-1/2)(-u)^v E_{k,k-\mu}$$

$$\mathcal{O}^\sigma_\mu(u)^\flat = \sum_{k\in\mathbb{Z}+\frac{1}{2}}\sum_{v=0}^{\infty}h_v(1+k-1/2,\dots,\mu+k-1/2)(-u)^v E_{k,k-\mu}$$

Proof. This follows from the duality between generating series of complete and elementary symmetric polynomials expressed in equation (6.5), and the form of the $\mathcal{O}$-operators in lemma 6.4.1. □

Corollary 6.4.3. *The different kinds of $\mathcal{O}$-operators can also be written as follows:*

$$\mathcal{O}^h_\mu(u) = \sum_{v=0}^{\infty}\frac{(v+\mu-1)!}{(\mu-1)!}[z^v]\mathcal{S}(uz)^{\mu-1}\mathcal{E}_{-\mu}(uz)$$

$$\mathcal{O}^h_\mu(u)^\flat = \sum_{v=0}^{\mu}\frac{\mu!}{(\mu-v)!}[z^v]\mathcal{S}(uz)^{-\mu-1}\mathcal{E}_{\mu}(-uz)$$

$$\mathcal{O}^\sigma_\mu(u) = \sum_{v=0}^{\mu}\frac{\mu!}{(\mu-v)!}[z^v]\mathcal{S}(uz)^{-\mu-1}\mathcal{E}_{-\mu}(uz)$$

$$\mathcal{O}^\sigma_\mu(u)^\flat = \sum_{v=0}^{\infty}\frac{(v+\mu-1)!}{(\mu-1)!}[z^v]\mathcal{S}(uz)^{\mu-1}\mathcal{E}_{\mu}(-uz)$$

Proof. We will first derive the first equation, starting from lemma 6.4.1. First we use equation (6.6):

$$\begin{aligned}\mathcal{O}^h_\mu(u) &= \sum_{k\in\mathbb{Z}+\frac{1}{2}}\sum_{v=0}^{\infty}h_v(1+k-1/2,\dots,\mu+k-1/2)u^v E_{k+\mu,k}\\ &= \sum_{k\in\mathbb{Z}+\frac{1}{2}}\sum_{v=0}^{\infty}\sum_{i=0}^{v}\binom{v+\mu-1}{i}h_{v-i}(0,\dots,\mu-1)\big(k+\frac{1}{2}\big)^i u^v E_{k+\mu,k}\end{aligned}$$

By equation 6.7 and lemma 6.3.6, we then get:

$$\begin{aligned}\mathcal{O}^h_\mu(u) &= \sum_{k\in\mathbb{Z}+\frac{1}{2}}\sum_{v=0}^{\infty}\sum_{i=0}^{v}\binom{v+\mu-1}{i}[y^{v-i}]\frac{(v+\mu-i-1)!}{(\mu-1)!}\mathcal{S}(y)^{\mu-1}e^{y\frac{\mu-1}{2}}[z^i]i!e^{z(k+\frac{1}{2})}u^v E_{k+\mu,k}\\ &= \sum_{v=0}^{\infty}\frac{(v+\mu-1)!}{(\mu-1)!}[z^v]\mathcal{S}(uz)^{\mu-1}\mathcal{E}_{-\mu}(uz)\end{aligned}$$

For the other equations, the calculation is similar, replacing the equations for the complete symmetric polynomials with their counterparts for the elementary symmetric polynomials where necessary. □

Lemma 6.4.4.

$$e^{\frac{\alpha_r}{r}}\mathcal{O}^h_\mu(u)e^{-\frac{\alpha_r}{r}} = \sum_{t=0}^{\infty}\sum_{v=t}^{\infty}\frac{(v+\mu-1)!}{t!\,(\mu-1)!}u^t[z^{v-t}]\mathcal{S}(uz)^{\mu-1}\mathcal{S}(ruz)^t\mathcal{E}_{tr-\mu}(uz) \tag{6.12}$$

$$e^{\frac{\alpha_r}{r}}\mathcal{O}^h_\mu(u)^\flat e^{-\frac{\alpha_r}{r}} = \sum_{t=0}^{\mu}\sum_{v=t}^{\mu}\frac{\mu!}{t!(\mu-v)!}(-u)^t[z^{v-t}]\mathcal{S}(uz)^{-\mu-1}\mathcal{S}(ruz)^t\mathcal{E}_{tr+\mu}(-uz) \tag{6.13}$$

$$e^{\frac{\alpha_r}{r}}\mathcal{O}^\sigma_\mu(u)e^{-\frac{\alpha_r}{r}} = \sum_{t=0}^{\mu}\sum_{v=t}^{\mu}\frac{\mu!}{t!(\mu-v)!}u^t[z^{v-t}]\mathcal{S}(uz)^{-\mu-1}\mathcal{S}(ruz)^t\mathcal{E}_{tr-\mu}(uz) \tag{6.14}$$

$$e^{\frac{\alpha_r}{r}}\mathcal{O}^\sigma_\mu(u)^\flat e^{-\frac{\alpha_r}{r}} = \sum_{t=0}^{\infty}\sum_{v=t}^{\infty}\frac{(v+\mu-1)!}{t!\,(\mu-1)!}(-u)^t[z^{v-t}]\mathcal{S}(uz)^{\mu-1}\mathcal{S}(ruz)^t\mathcal{E}_{tr+\mu}(-uz) \tag{6.15}$$

Proof. Let us prove equation (6.12). Applying the Hadamard lemma as in lemma 6.4.1 we find

$$\begin{aligned} e^{\frac{\alpha_r}{r}}\mathcal{O}^h_\mu(u)e^{-\frac{\alpha_r}{r}} &= \sum_{t=0}^{\infty}\frac{1}{t!r^t}\operatorname{ad}^t_{\alpha_r}\Big(\sum_{v=0}^{\infty}\frac{(v+\mu-1)!}{(\mu-1)!}[z^v]\mathcal{S}(uz)^{\mu-1}\mathcal{E}_{-\mu}(uz)\Big) \\ &= \sum_{t=0}^{\infty}\sum_{v=0}^{\infty}\frac{(v+\mu-1)!}{t!\,(\mu-1)!r^t}[z^v]\mathcal{S}(uz)^{\mu-1}\operatorname{ad}^t_{\alpha_r}\mathcal{E}_{-\mu}(uz) \end{aligned}$$

By equation (6.4), we know

$$\operatorname{ad}_{\alpha_r}\mathcal{E}_{-\mu}(uz) = \zeta(ruz)\mathcal{E}_{r-\mu}(uz)$$

Using this t times, we get that

$$\begin{aligned} e^{\frac{\alpha_r}{r}}\mathcal{O}^h_\mu(u)e^{-\frac{\alpha_r}{r}} &= \sum_{t=0}^{\infty}\sum_{v=0}^{\infty}\frac{(v+\mu-1)!}{t!\,(\mu-1)!r^t}[z^v]\mathcal{S}(uz)^{\mu-1}\zeta(ruz)^t\mathcal{E}_{tr-\mu}(uz) \\ &= \sum_{t=0}^{\infty}\sum_{v=0}^{\infty}\frac{(v+\mu-1)!}{t!\,(\mu-1)!}u^t[z^{v-t}]\mathcal{S}(uz)^{\mu-1}\mathcal{S}(ruz)^t\mathcal{E}_{tr-\mu}(uz) \end{aligned}$$

For the other equations, the calculation is completely analogous, using that $\mathcal{S}$ is an even function. This finishes the proof of the lemma. □

6.4.3 $\mathcal{A}$-operators

Let us now define the $\mathcal{A}$-operators for the r-orbifold monotone Hurwitz numbers as

$$\mathcal{A}^h_{\langle\mu\rangle}(u,\mu) = \sum_{t\in\mathbb{Z}}\sum_{v=t}^{\infty}\frac{([\mu]+\mu+1)_{v-1}}{([\mu]+1)_t}[z^{v-t}]\mathcal{S}(uz)^{\mu-1}\mathcal{S}(ruz)^{t+[\mu]}\mathcal{E}_{tr-\langle\mu\rangle}(uz) \tag{6.16}$$

$$\mathcal{A}^\sigma_{\langle\mu\rangle}(u,\mu) = \sum_{t=-\infty}^{\mu-[\mu]}\sum_{v=t}^{\mu-[\mu]}\frac{(\mu-[\mu]-v+1)_{v-1}}{([\mu]+1)_t}[z^{v-t}]\mathcal{S}(uz)^{-\mu-1}\mathcal{S}(ruz)^{t+[\mu]}\mathcal{E}_{tr-\langle\mu\rangle}(uz) \tag{6.17}$$

where $\mu = r[\mu]+\langle\mu\rangle$ denotes the euclidean division by r.

Proposition 6.4.5.

$$H^{\bullet,r,\leq}(u,\vec{\mu}) = u^{\frac{d}{r}} \prod_{i=1}^{l(\vec{\mu})} \binom{\mu_i + [\mu_i]}{\mu_i} \left\langle \prod_{i=1}^{l(\vec{\mu})} \mathcal{A}^h_{\langle \mu_i \rangle}(u,\mu_i) \right\rangle^\bullet \tag{6.18}$$

$$H^{\bullet,r,<}(u,\vec{\mu}) = u^{\frac{d}{r}} \prod_{i=1}^{l(\vec{\mu})} \binom{\mu_i - 1}{[\mu_i]} \left\langle \prod_{i=1}^{l(\vec{\mu})} \mathcal{A}^\sigma_{\langle \mu_i \rangle}(u,\mu_i) \right\rangle^\bullet \tag{6.19}$$

where $\mu = r[\mu] + \langle \mu \rangle$ denotes the euclidean division by r.

Proof. Let us prove equation (6.18). Observe that both the operators $\tilde{\mathcal{E}}$ and α_r annihilate the vacuum. Hence inserting the operators $\mathcal{D}^{(h)}$ and e^{α_r} acting on the vacuum does not change the expression in equation (6.9):

$$H^{\bullet,r,\leq}(u,\vec{\mu}) = \left\langle \prod_{i=1}^{n} \frac{1}{\mu_i} e^{\frac{\alpha_r}{r}} \mathcal{D}^{(h)}(u) \alpha_{-\mu_i} (\mathcal{D}^{(h)}(u))^{-1} e^{\frac{-\alpha_r}{r}} \right\rangle^\bullet$$

The operators in the correlator are given by formula (6.12), divided by μ. For every $i = 1, \dots, n$, rescale the t-sum in formula (6.12) by $t_{\text{new}} := t - [\mu_i]$ and the v-sum by $v_{\text{new}} := v - [\mu_i]$, and conjugate by the operator $u^{\mathcal{F}_1/r}$. The latter operation has the effect of annihilating the factor u^t and of creating a factor $u^{\mu_i/r}$ that can be written outside the sum. Extracting the binomial coefficient in equation (6.18) and extending the t-sum over all integers (since the Pochhammer symbol in the denominator is infinite for $t < -[\mu_i]$) proves equation (6.18).

The proof for equation (6.19) is analogous, starting from the operator given by formula (6.14). After rescaling the t- and v-sums and conjugating with $u^{\mathcal{F}_1/r}$, we extract from the correlator the factor

$$\frac{(\mu-1)!}{[\mu]!(\mu - [\mu] - 1)!}$$

Here, we can also extend the sum to $+\infty$, because the Pochhammer symbol in the numerator is zero for the added terms. Proposition 6.4.5 is proved. □

Define the operators

$$\mathcal{A}^h_{\langle \mu \rangle}(u,\mu)^\flat = u^{\mu/r} \mu \binom{\mu + [\mu]}{\mu} u^{\mathcal{F}_1/r} e^{\frac{\alpha_r}{r}} \mathcal{O}^h_\mu(u)^\flat e^{-\frac{\alpha_r}{r}} u^{-\mathcal{F}_1/r}$$

and

$$\mathcal{A}^\sigma_{\langle \mu \rangle}(u,\mu)^\flat = u^{\mu/r} \mu \binom{\mu - 1}{[\mu]} u^{\mathcal{F}_1/r} e^{\frac{\alpha_r}{r}} \mathcal{O}^\sigma_\mu(u)^\flat e^{-\frac{\alpha_r}{r}} u^{-\mathcal{F}_1/r}.$$

Proposition 6.4.6. *We have the following:*

$$\mathcal{A}^h_{\langle \mu \rangle}(u,\mu)^\flat = \sum_{t=0}^{\mu} \sum_{v=t}^{\mu} \frac{(-1)^t (\mu + [\mu])! \mu}{t!(\mu - v)![\mu]!} [z^{v-t}] \mathcal{S}(uz)^{-\mu-1} \mathcal{S}(ruz)^t \mathcal{E}_{tr+\mu}(-uz) \tag{6.20}$$

$$\mathcal{A}^\sigma_{\langle \mu \rangle}(u,\mu)^\flat = \sum_{t=0}^{\infty} \sum_{v=t}^{\infty} \frac{(-1)^t (v + [\mu] - 1)! \mu}{t!(\mu - [\mu] - 1)![\mu]!} [z^{v-t}] \mathcal{S}(uz)^{\mu-1} \mathcal{S}(ruz)^t \mathcal{E}_{tr+\mu}(-uz) \tag{6.21}$$

Proof. Let us prove equation (6.20). The conjugation of $\mathcal{O}$ by the operator $e^{\alpha_r/r}$ is given by formula (6.13). The conjugation with $u^{\mathcal{F}_1/r}$ annihilates the factor u^t and produces a factor $u^{-\mu/r}$, which simplifies with $u^{\mu/r}$. This proves equation (6.20). Equation (6.21) is proved in the same way using the conjugation given by formula (6.15). The proposition is proved. ☐

6.5 Quasi-polynomiality results for monotone and strictly monotone Hurwitz numbers

In this section we state and prove the quasi-polynomiality property for monotone and strictly monotone orbifold Hurwitz numbers.

Definition 6.5.1. We define the connected operators $\langle \prod_{i=1}^{n} \mathcal{A}_{\eta_i}(u,\mu_i)\rangle^{\circ}$ in terms of the disconnected correlator $\langle \prod_{i=1}^{n} \mathcal{A}_{\eta_i}(u,\mu_i)\rangle^{\bullet}$ by means of the inclusion-exclusion formula, see, e. g., [13, 15].

The monotone Hurwitz numbers are expressed in terms of connected correlators as

$$h^{\circ,r,\leq}_{g;\vec{\mu}} = [u^{2g-2+l(\vec{\mu})}].\prod_{i=1}^{l(\vec{\mu})} \binom{\mu_i+[\mu_i]}{\mu_i} \left\langle \prod_{i=1}^{l(\vec{\mu})} \mathcal{A}^{h}_{\langle\mu_i\rangle}(u,\mu_i) \right\rangle^{\circ}$$

$$h^{\circ,r,<}_{g;\vec{\mu}} = [u^{2g-2+l(\vec{\mu})}].\prod_{i=1}^{l(\vec{\mu})} \binom{\mu_i-1}{[\mu_i]} \left\langle \prod_{i=1}^{l(\vec{\mu})} \mathcal{A}^{\sigma}_{\langle\mu_i\rangle}(u,\mu_i) \right\rangle^{\circ}$$

We are now ready to state and prove the main result of the chapter.

Theorem 6.5.2 (Quasi-polynomiality for monotone and strictly monotone orbifold Hurwitz numbers)**.** *For $2g-2+l(\vec{\mu}) \geq 0$, the monotone and strictly monotone orbifold Hurwitz numbers can be expressed as follows:*

$$h^{\circ,r,\leq}_{g;\vec{\mu}} = \prod_{i=1}^{l(\vec{\mu})} \binom{\mu_i+[\mu_i]}{\mu_i} P^{\langle\vec{\mu}\rangle}_{\leq}(\mu_1,\dots,\mu_{l(\vec{\mu})})$$

$$h^{\circ,r,<}_{g;\vec{\mu}} = \prod_{i=1}^{l(\vec{\mu})} \binom{\mu_i-1}{[\mu_i]} P^{\langle\vec{\mu}\rangle}_{<}(\mu_1,\dots,\mu_{l(\vec{\mu})})$$

where $P^{\langle\vec{\mu}\rangle}_{<}$ and $P^{\langle\vec{\mu}\rangle}_{\leq}$ are polynomials of degree $3g-3+l(\vec{\mu})$ depending on the parameters $\langle\mu_1\rangle,\dots,\langle\mu_{l(\vec{\mu})}\rangle$, and $\mu = r[\mu]+\langle\mu\rangle$ denotes the euclidean division by r.

Remark 6.5.3. The two statements of theorem 6.5.2 confirm respectively conjecture 23 in [8] and conjecture 12 in [10]. Note that the small difference in the conjecture 23 does not affect quasi-polynomiality since the polynomials $P_{\leq}$ depend on the parameters $\langle\mu\rangle$. Conjecture 12 is stated for Grothendieck dessin d'enfants, which indeed correspond to strictly monotone Hurwitz numbers by the Jucys correspondence (see for example [1] for details).

Remark 6.5.4. Note that since we allow the coefficients of the polynomials $P^{\langle\vec{\mu}\rangle}_{\leq}$ and $P^{\langle\vec{\mu}\rangle}_{<}$ to depend on $\langle\vec{\mu}\rangle$, we can equivalently consider them as polynomials in $[\mu_1],\dots,[\mu_n]$, $n := l(\vec{\mu})$. The latter way is more convenient in the proof.

Proof. We will show that, for fixed η_i, the connected correlator $\langle \prod_{i=1}^n \mathcal{A}_{\eta_i}(u,\mu_i)\rangle^\circ$ is a power series in u with polynomial coefficients in all μ_i, for both the operators $\mathcal{A}^h$ and $\mathcal{A}^\sigma$. As these are symmetric functions in the μ_i, it is sufficient to prove polynomiality in μ_1. Indeed, if a symmetric function $P(\mu_1,\dots,\mu_n)$ is polynomial in the first variable, it can be written in the form $P(\mu_1,\dots,\mu_n) = \sum_{k=0}^d a_k(\mu_2,\dots,\mu_n)\mu_1^k$. To check that each coefficient of P is also polynomial in μ_2, we can compute the values of P at the points $\mu_1 = 1,\dots,d+1$ and show that these values are polynomial in μ_2. But the values of P at these particular values of μ_1 can be computed using the symmetry of P as $P(\mu_2,\dots,\mu_n,\mu_1)$, so they are polynomial in μ_2. Proceeding this way, we establish polynomiality of P in all arguments.

We will first consider the disconnected correlator $\langle \prod_{i=1}^n \mathcal{A}_{\eta_i}(u,\mu_i)\rangle^\bullet$ where, setting $\mu_i = \nu_i r + \eta_i$ to stress the independence the parameters $\nu_i = [\mu_i]$ and $\eta_i = \langle \mu_i \rangle$ here, the operator $\mathcal{A}$ is either

$$\mathcal{A}^h_{\eta_i}(u,\mu_i) = \sum_{t_i\in\mathbb{Z}}^{\infty}\sum_{v_i=t_i}^{\infty} \frac{(\nu_i+\mu_i+1)_{v_i-1}}{(\nu_i+1)_{t_i}}[z^{v_i-t_i}]\mathcal{S}(uz)^{\mu_i-1}\mathcal{S}(ruz)^{t_i+\nu_i}\mathcal{E}_{t_i r-\eta_i}(uz)$$

in the monotone case or

$$\mathcal{A}^\sigma_{\eta_i}(u,\mu_i) = \sum_{t_i=-\infty}^{\mu_i}\sum_{v_i=t_i}^{\mu_i} \frac{(\mu_i-\nu_i-(v_i-1))_{v_i-1}}{(\nu_i+1)_{t_i}}[z^{v_i-t_i}]\mathcal{S}(uz)^{-\mu_i-1}\mathcal{S}(ruz)^{t+\nu_i}\mathcal{E}_{tr-\eta_i}(uz)$$

in the strictly monotone case. In both cases, if we expand the product of all the t-sums in the disconnected correlator, we get the condition $\sum_{i=1}^{l(\mu)}(t_i r - \eta_i) = 0$, as the total energy of the operators in a given monomial must be zero. Furthermore, $t_1 r - \eta_1 \geq 0$, since the first $\mathcal{E}$ would get annihilated by the covacuum otherwise, and $t_i \geq -\nu_i$ (otherwise the symbol $1/(\nu_i+1)_{t_i}$ vanishes), so if we fix $\eta_1, \nu_2, \eta_2, \dots, \nu_n, \eta_n$, the t_1-sum becomes finite. Since the power of u is fixed, it also gives a bound on the degree in ν_1. So the coefficient of a particular power of u in the disconnected correlator $\langle \prod_{i=1}^n \mathcal{A}_{\eta_i}(u,\mu_i)\rangle^\bullet$ is a rational function in ν_1.

Because the coefficients are rational functions, we can extend them to the complex plane, and it makes sense to talk about their poles. The only possible poles must come from $\frac{1}{(\nu+1)_t}$ (because we only look at non-negative exponents of u), and all of these poles are simple. Let us calculate the residue at $\nu = -l$, for $l = 1, 2, \dots$

Lemma 6.5.5. *The residue of the $\mathcal{A}$-operators is, up to a linear multiplicative constant, equal to the correspondent operator $\mathcal{A}^\flat$ with a negative argument. More precisely,*

$$\operatorname*{Res}_{\nu=-l} \mathcal{A}^h_\eta(u,\nu r+\eta) = \frac{1}{lr-\eta}\mathcal{A}^h_{-\eta}(u,lr-\eta)^\flat \qquad \textit{if } \eta \neq 0 \tag{6.22}$$

$$\operatorname*{Res}_{\nu=-l} \mathcal{A}^h_0(u,\nu r) = \frac{1}{lr(r+1)}\mathcal{A}^h_0(u,lr)^\flat \qquad \textit{if } \eta = 0 \tag{6.23}$$

$$\operatorname*{Res}_{\nu=-l} \mathcal{A}^\sigma_\eta(u,\nu r+\eta) = \frac{1}{lr-\eta}\mathcal{A}^\sigma_{-\eta}(u,lr-\eta)^\flat \qquad \textit{if } \eta \neq 0 \tag{6.24}$$

$$\operatorname*{Res}_{\nu=-l} \mathcal{A}^\sigma_0(u,\nu r) = \frac{1}{lr(r-1)}\mathcal{A}^\sigma_0(u,lr)^\flat \qquad \textit{if } \eta = 0 \tag{6.25}$$

Proof. Let us prove equations (6.22) and (6.23) together. The only contributing terms have $t \geq l$, so we calculate

$$\begin{aligned}
&\operatorname*{Res}_{\nu=-l} \mathcal{A}^h_\eta(u,\mu) \\
&\quad = \sum_{t=l}^{\infty}\sum_{v=t}^{\infty} \frac{(\nu+\mu+1)_{v-1}(\nu+l)}{(\nu+1)_t}[x^{v-t}]\mathcal{S}(xu)^{\mu-1}\mathcal{S}(rxu)^{t+\nu}\mathcal{E}_{tr-\eta}(xu)\Big|_{\nu=-l} \\
&\quad = \sum_{t=l}^{\infty}\sum_{v=t}^{\infty} \frac{(\mu-l+1)_{v-1}}{(1-l)_{l-1}(t-l)!}(-1)^{v-t}[x^{v-t}]\mathcal{S}(-xu)^{\mu-1}\mathcal{S}(-rxu)^{t-l}\mathcal{E}_{tr-\eta}(-xu) \\
&\quad = \sum_{t=0}^{\infty}\sum_{v=t}^{\infty} \frac{(-1)^{l+v-t-1}(\mu-l+1)_{v+l-1}}{(l-1)!t!}[x^{v-t}]\mathcal{S}(xu)^{\mu-1}\mathcal{S}(rxu)^{t}\mathcal{E}_{tr-\mu}(-xu)
\end{aligned}$$

where we kept writing μ for $-lr+\eta$. As this is negative, however, it makes sense to rename it $\mu=-\lambda$. Substituting and collecting the minus signs from the Pochhammer symbol, we get

$$\begin{aligned}
&\operatorname*{Res}_{\nu=-l} \mathcal{A}^h_\eta(u,\mu) \\
&\quad = \sum_{t=0}^{\lambda}\sum_{v=t}^{\lambda} \frac{(-1)^t(\lambda+1-v)_{v+l-1}}{(l-1)!t!}[x^{v-t}]\mathcal{S}(ux)^{-\lambda-1}\mathcal{S}(rux)^{t}\mathcal{E}_{tr+\lambda}(-ux) \\
&\quad = \sum_{t=0}^{\lambda}\sum_{v=t}^{\lambda} \frac{(-1)^t(\lambda+l-1)!}{(l-1)!t!(\lambda-v)!}[x^{v-t}]\mathcal{S}(ux)^{-\lambda-1}\mathcal{S}(rux)^{t}\mathcal{E}_{tr+\lambda}(-ux)
\end{aligned}$$

Because $\lambda = lr-\eta$, we have $l=[\lambda]+1-\delta_{\eta 0}$ and $\eta=-\langle\lambda\rangle$. Recalling equation (6.20), we obtain the result. Equations (6.24) and (6.25) follow from the analogous computation of the residue and the comparison with equation (6.21). □

In the following we will use the notation $\mathcal{A}$ and $\mathcal{D}$ without specifying the symmetric polynomial chosen, since the argument is valid for both the choices of $(\mathcal{A}^h,\mathcal{D}^h)$ and $(\mathcal{A}^\sigma,\mathcal{D}^\sigma)$. Lemma 6.5.5 implies that we can express the residues in μ_1 of the disconnected correlator as follows:

$$\operatorname*{Res}_{\nu_1=-l}\Big\langle \prod_{i=1}^n \mathcal{A}_{\eta_i}(u,\mu_i)\Big\rangle^\bullet = c(l,\eta_1)\Big\langle \mathcal{A}_{-\eta_1}(u,lr-\eta_1)^\flat \prod_{i=2}^n \mathcal{A}_{\eta_i}(u,\mu_i)\Big\rangle^\bullet.$$

where $c(l,\eta_1)$ is the coefficient in lemma 6.5.5. Recalling equations (6.9) and (6.18) for the monotone case and equations (6.10) and (6.19) for the strictly monotone case and realising that the corresponding operator $\mathcal{A}^\flat$ is given by the same conjugations as the normal $\mathcal{A}$-operator, but starting from α_μ instead of $\alpha_{-\mu}$, we can see that this reduces to

$$\operatorname*{Res}_{\nu_1=-l}\Big\langle \prod_{i=1}^n \mathcal{A}_{\eta_i}(u,\mu_i)\Big\rangle^\bullet = C\Big\langle e^{\frac{\alpha_r}{r}}\mathcal{D}(u)\alpha_{lr-\eta_1}\prod_{i=2}^n \alpha_{-\mu_i}\Big\rangle^\bullet \tag{6.26}$$

for some specific coefficient C that depends only on l and η_1.

Because $[\alpha_k,\alpha_l]=k\delta_{k+l,0}$, and $\alpha_{lr-\eta_1}$ annihilates the vacuum, this residue is zero unless one of the μ_i equals $lr-\eta_1$ for $i\geq 2$.

Now return to the connected correlator. It can be calculated from the disconnected one by the inclusion-exclusion principle, so in particular it is a finite sum of products of disconnected correlators. Hence the connected correlator is also a rational function in ν_1, and all possible poles must be inherited from the disconnected correlators. So let us assume $\mu_i = lr - \eta_1$ for some $i \geq 2$. Then we get a contribution from (6.26), but this is canceled exactly by the term coming from

$$\operatorname*{Res}_{\nu_1=-l}\left\langle \mathcal{A}_{\eta_1}(u,\mu_1)\mathcal{A}_{-\eta_1}(u,lr-\eta_1)\right\rangle^{\bullet}\left\langle \prod_{\substack{2\leq j\leq n\\ j\neq i}} \mathcal{A}_{\eta_j}(u,\mu_j)\right\rangle^{\bullet}$$

$$= C\left\langle e^{\frac{\alpha_r}{r}}\mathcal{D}(u)\alpha_{lr-\eta_1}\alpha_{-(lr-\eta_1)}\right\rangle^{\bullet}\left\langle e^{\frac{\alpha_r}{r}}\mathcal{D}(u)\alpha_{lr-\eta_1}\prod_{\substack{2\leq j\leq n\\ j\neq i}}\alpha_{-\mu_j}\right\rangle^{\bullet}$$

Hence, the connected correlator has no residues, which proves it is polynomial in ν_1. Therefore, it is also a polynomial in μ_1, see remark 6.5.4. This completes the proof of the polynomiality.

Now, once we know that the coefficient of u^{2g-2+n}, $2g-2+n \geq 0$, of a connected correlator $\langle\prod_{i=1}^n \mathcal{A}_{\eta_i}(u,\mu_i)\rangle^\circ$ is a polynomial in $\mu_1,\dots,\mu_n$, or, equivalently, in $\nu_1,\dots,\nu_n$, we can compute its degree. The argument is the same in both cases, monotone and strictly monotone, so let us use the formulas for the $\mathcal{A}^h$-operators. We can compute the degree of the connected correlator considered as a rational function. Once we know that it is a polynomial, we obtain the degree of the polynomial. For the computation of the degree in $\nu_1,\dots,\nu_n$ it is sufficient to observe that $\sum_{i=1}^n (v_i - t_i) = 2g-2+n$, therefore $\prod_{i=1}^n (\nu_i+\mu_i+1)_{v_i-1}/(\nu_i+1)_{t_i}$ has degree $2g-2$. Moreover, the leading term in $\langle\prod_{i=1}^n \mathcal{E}_{t_i r-\eta_i}(uz)\rangle^\circ$ has degree $n-2$ in uz and $n-1$ in $\nu_1,\dots,\nu_n$, and the coefficient of $(uz)^{2g}$ in the product of $\mathcal{S}\cdot\prod_{i=1}^n \mathcal{S}(uz)^{\mu_i-1}\mathcal{S}(ruz)^{t_i+\nu_i}$, where $\mathcal{S}$ without an argument denotes the $\mathcal{S}$-functions coming from the connected correlator $\langle\prod_{i=1}^n \mathcal{E}_{t_i r-\eta_i}(uz)\rangle^\circ$ divided by its leading term, is a polynomial of degree $2g/2 = g$ in $\nu_1,\dots,\nu_n$. So, the total degree in $\nu_1,\dots,\nu_n$ is equal to $2g-2+n-1+g = 3g-3+n$.

This completes the proof of the theorem. □

6.6 Quasi-polynomiality for the usual orbifold Hurwitz numbers

In the case of the usual orbifold Hurwitz numbers, quasi-polynomiality was already known, see [2, 9, 15]. However, all known proofs use either the Johnson-Pandharipande-Tseng formula [28] (the ELSV formula [18] for $r=1$) or very subtle analytic tools due to Johnson [27] (Okounkov-Pandharipande [37] for $r=1$). In the second approach, presented in [13, 15], the analytic continuation to the integral points outside the area of convergence requires an extra discussion, which is so far omitted. So, it would be good to have a more direct combinatorial proof of quasi-polynomiality for usual orbifold Hurwitz numbers, and we will reprove it here using the same technique as for the (strictly) monotone orbifold Hurwitz numbers.

Definition 6.6.1. The usual orbifold $\mathcal{A}$-operators are given by

$$\mathcal{A}_{\langle\mu\rangle}(u,\mu) := r^{-\frac{\langle\mu\rangle}{r}}\mathcal{S}(ru\mu)^{[\mu]}\sum_{t\in\mathbb{Z}}\frac{\mathcal{S}(ru\mu)^t\mu^{t-1}}{([\mu]+1)_t}\mathcal{E}_{tr-\langle\mu\rangle}(u\mu)$$

Remark 6.6.2. Up to slightly different notation and a shift by one in the exponent of μ, these are the $\mathcal{A}$-operators of [15].

The importance of these operators is given in the following proposition:

Proposition 6.6.3. *[15, proposition 3.1] The generating function for disconnected orbifold Hurwitz can be expressed in terms of the $\mathcal{A}$-operators by:*

$$H^\bullet(u,\vec{\mu}) = \sum_{g=0}^{\infty} h^\circ_{g;\vec{\mu}} u^b = r^{\sum_{i=1}^{l(\vec{\mu})} \frac{\langle \mu_i \rangle}{r}} \prod_{i=1}^{l(\vec{\mu})} \frac{u^{\frac{\mu_i}{r}} \mu_i^{[\mu_i]}}{[\mu_i]!} \Big\langle \prod_{i=1}^{l(\vec{\mu})} \mathcal{A}_{\langle \mu_i \rangle}(u,\mu_i) \Big\rangle^\bullet \tag{6.27}$$

The proof of this proposition amounts to the calculation

$$r^{\frac{\langle \mu \rangle}{r}} \frac{u^{\frac{\mu}{r}} \mu^{[\mu]}}{[\mu]!} \mathcal{A}_{\langle \mu \rangle}(u,\mu) = u^{\frac{\mathcal{F}_1}{r}} e^{\frac{\alpha_r}{r}} e^{u\mathcal{F}_2} \alpha_{-\mu} e^{-u\mathcal{F}_2} e^{-\frac{\alpha_r}{r}} u^{-\frac{\mathcal{F}_1}{r}} \tag{6.28}$$

With these data, we can start our scheme of proof.

Lemma 6.6.4. *The operator $\mathcal{A}_{\langle \mu \rangle}(u,\mu)^\flat$ (in the same sense as before) is given by*

$$\mathcal{A}_{\langle \mu \rangle}(u,\mu)^\flat = \frac{r^{\frac{\langle \mu \rangle}{r}}}{[\mu]!} \sum_{t \geq 0} (-1)^t \frac{\mathcal{S}(ru\mu)^t \mu^{t+[\mu]}}{t!} \mathcal{E}_{tr+\mu}(-u\mu)$$

Proof. The proof is very analogous to the proof of [15, proposition 3.1].

We do the same commutation as for the $\mathcal{A}$-operators, but starting from α_μ. First recall [37, equation (2.14)]:

$$e^{u\mathcal{F}_2} \alpha_\mu e^{-u\mathcal{F}_2} = \mathcal{E}_\mu(-u\mu)$$

The second conjugation gives

$$\begin{aligned} e^{\frac{\alpha_r}{r}} e^{u\mathcal{F}_2} \alpha_\mu e^{-u\mathcal{F}_2} e^{-\frac{\alpha_r}{r}} &= e^{\frac{\alpha_r}{r}} \mathcal{E}_\mu(-u\mu) e^{-\frac{\alpha_r}{r}} \\ &= \sum_{t=0}^{\infty} \Big(\frac{\zeta(-ru\mu)}{r} \Big)^t \frac{1}{t!} \mathcal{E}_{tr+\mu}(-u\mu) \\ &= \sum_{t=0}^{\infty} \frac{(-u\mu)^t \mathcal{S}(-ru\mu)^t}{t!} \mathcal{E}_{tr+\mu}(-u\mu) \end{aligned}$$

And the third conjugation finally shifts the exponent of u:

$$u^{\frac{\mathcal{F}_1}{r}} e^{\frac{\alpha_r}{r}} e^{u\mathcal{F}_2} \alpha_\mu e^{-u\mathcal{F}_2} e^{-\frac{\alpha_r}{r}} u^{-\frac{\mathcal{F}_1}{r}} = u^{-\frac{\mu}{r}} \sum_{t=0}^{\infty} \frac{(-\mu)^t \mathcal{S}(-ru\mu)^t}{t!} \mathcal{E}_{tr+\mu}(-u\mu)$$

Comparing this to equation (6.28) and multiplying by the coefficient finishes the proof. □

Theorem 6.6.5 (Quasi-polynomiality for usual orbifold Hurwitz numbers)**.** *For $2g-2+l(\mu) \geq 0$, the usual orbifold Hurwitz numbers can be expressed as follows:*

$$h^{\circ,r}_{g;\vec{\mu}} = r^{\sum_{i-1}^{l(\vec{\mu})} \frac{\langle \mu_i \rangle}{r}} \prod_{i=1}^{l(\vec{\mu})} \frac{u^{\frac{\mu_i}{r}} \mu_i^{[\mu_i]}}{[\mu_i]!} P^{\langle \vec{\mu} \rangle}(\mu_1, \dots, \mu_{l(\vec{\mu})})$$

where $P^{\langle \mu \rangle}$ are polynomials of degree $3g-3+l(\vec{\mu})$ whose coefficients depend on the parameters $\langle \mu_1 \rangle, \dots \langle \mu_{l(\mu)} \rangle$ and $\mu = r[\mu] + \langle \mu \rangle$ denotes the euclidean division by r.

Remark 6.6.6. As stated before, this result is not new. It has been proved in several ways in [2, 9, 15]. We add this new proof for completeness.

Proof. We will show that, for fixed η_i, the connected correlator $\langle \prod_{i=1}^n \mathcal{A}_{\eta_i}(u,\mu_i)\rangle^\circ$, $n = l(\vec{\mu})$, is a power series in u with polynomial coefficients in all μ_i for the operators $\mathcal{A}$. As these are symmetric functions in the μ_i, it is again sufficient to prove polynomiality in μ_1, or, equivalently (see remark 6.5.4) in $\nu_1 := [\mu_1]$.

We will first consider the disconnected correlator $\langle \prod_{i=1}^n \mathcal{A}_{\eta_i}(u,\mu_i)\rangle^\bullet$ where, setting $\mu_i = \nu_i r + \eta_i$, the operator $\mathcal{A}$ is

$$\mathcal{A}_{\eta_i}(u,\mu_i) := r^{-\frac{\eta_i}{r}} \mathcal{S}(ru\mu_i)^{\nu_i} \sum_{t_i\in\mathbb{Z}} \frac{\mathcal{S}(ru\mu_i)^{t_i}\mu_i^{t_i-1}}{(\nu_i+1)_{t_i}} \mathcal{E}_{t_i r-\eta_i}(u\mu_i)$$

If we expand all of the t-sums in the disconnected correlator, we get the condition $\sum_{i=1}^{l(\mu)}(t_i r - \eta_i) = 0$, as the total energy of the operators in a given monomial must be zero. Furthermore, $t_1 r - \eta_1 \geq 0$, since the first $\mathcal{E}$ would get annihilated by the covacuum otherwise, and $t_i \geq -\nu_i$ (otherwise the symbol $1/(\nu_i+1)_{t_1}$ vanishes), so if we fix $\eta_1, \nu_2, \eta_2, \dots, \nu_n, \eta_n$, the t_1-sum becomes finite. Since the power of u is fixed, it also gives a bound on the degree in ν_1. So the coefficient of a particular power of u in the disconnected correlator $\langle \prod_{i=1}^n \mathcal{A}_{\eta_i}(u,\mu_i)\rangle^\bullet$ is a rational function in ν_1.

Again, because the coefficients are rational functions, we can extend them to the complex plane, and it makes sense to talk about poles. The only possible poles must come from $\frac{1}{(\nu+1)_t}$ or $\mu = 0$. These poles are all simple, except possibly for the last case. Let us calculate the residue at $\nu = -l$, for $l = 1, 2, \dots$.

Lemma 6.6.7. *The residue of the $\mathcal{A}$-operators at negative integers is, up to a multiplicative constant, equal to the corresponding $\mathcal{A}^\flat$-operators with a negative argument. More precisely,*

$$\operatorname*{Res}_{\nu=-l} \mathcal{A}_\eta(u,\nu r+\eta) = \mathcal{A}_{-\eta}(u, lr-\eta)^\flat \qquad \text{if } \eta \neq 0$$
$$\operatorname*{Res}_{\nu=-l} \mathcal{A}_0(u,\nu r) = \frac{1}{r}\mathcal{A}_0(u, lr)^\flat \qquad \text{if } \eta = 0$$

Proof. Let us prove both equations together. The only contributing terms have $t \geq l$, so we calculate

$$\begin{aligned}\operatorname*{Res}_{\nu=-l} \mathcal{A}_\eta(u,\mu) &= r^{-\frac{\eta}{r}} \mathcal{S}(ru\mu)^\nu \sum_{t\geq l} \frac{\mathcal{S}(ru\mu)^t \mu^{t-1}(\nu+l)}{(\nu+1)_t}\mathcal{E}_{tr-\eta}(u\mu)\Big|_{\nu=-l} \\ &= r^{-\frac{\eta}{r}} \mathcal{S}(ru\mu)^{-l} \sum_{t\geq l} \frac{\mathcal{S}(ru\mu)^t \mu^{t-1}}{(1-l)_{l-1}(t-l)!}\mathcal{E}_{tr-\eta}(u\mu)\end{aligned}$$

where we kept writing μ for $-lr+\eta$. As this is negative, however, it makes sense to rename it $\mu = -\lambda$. Substituting and collecting the minus signs from the Pochhammer symbol, we get

$$\begin{aligned}\operatorname*{Res}_{\nu=-l} \mathcal{A}_\eta(u,\mu) &= \frac{(-1)^{l-1} r^{-\frac{\eta}{r}}}{(l-1)!} \mathcal{S}(ru\lambda)^{-l} \sum_{t\geq l} (-1)^{t-1} \frac{\mathcal{S}(ru\lambda)^t \lambda^{t-1}}{(t-l)!}\mathcal{E}_{tr-\eta}(-u\lambda) \\ &= \frac{r^{-\frac{\eta}{r}}}{(l-1)!} \sum_{t\geq 0} (-1)^t \frac{\mathcal{S}(ru\lambda)^t \lambda^{t+l-1}}{(t-l)!}\mathcal{E}_{tr+\lambda}(-u\lambda)\end{aligned}$$

Because $\lambda = lr - \eta$, we have $l = [\lambda] + 1 - \delta_{\eta 0}$ and $\eta = -\langle \lambda \rangle$. Recalling equation (6.20), we obtain the result. □

Because of lemma 6.6.7, we can express the residues in μ_1 of the disconnected correlator as follows:

$$\operatorname*{Res}_{\nu_1=-l} \Big\langle \prod_{i=1}^{n} \mathcal{A}_{\eta_i}(u,\mu_i) \Big\rangle^{\bullet} = c(\eta_1) \Big\langle \mathcal{A}_{-\eta_1}(u, lr - \eta_1)^{\flat} \prod_{i=2}^{n} \mathcal{A}_{\eta_i}(u,\mu_i) \Big\rangle^{\bullet}$$

where $c(\eta_1)$ is the coefficient in lemma 6.6.7. Recalling equation (6.27) and realising that the $\mathcal{A}^{\flat}$-operator is given by the same conjugations as the corresponding $\mathcal{A}$-operator, but starting from α_μ in stead of $\alpha_{-\mu}$, we can see that this reduces to

$$\operatorname*{Res}_{\nu_1=-l} \Big\langle \prod_{i=1}^{n} \mathcal{A}_{\eta_i}(u,\mu_i) \Big\rangle^{\bullet} = C \Big\langle e^{\frac{\alpha_r}{r}} \mathcal{D}(u) \alpha_{lr-\eta_1} \prod_{i=2}^{n} \alpha_{-\mu_i} \Big\rangle^{\bullet}$$

for some specific coefficient C that depends only on η_1 and l.

For the pole at zero, we see the only contributing terms must have $t \leq 0$, but we also need $tr - \eta \geq 0$, in order for the $\mathcal{E}$ not to get annihilated by the covacuum. Therefore, we need only consider the case $\eta = 0$ and the term $t = 0$. However, this term in $\Big\langle \prod_{i=1}^{n} \mathcal{A}_{\eta_i}(u,\mu_i) \Big\rangle^{\bullet}$ cancels against the term coming from

$$\Big\langle \mathcal{A}_{\eta_1}(u,\mu_1) \Big\rangle^{\bullet} \Big\langle \prod_{i=2}^{n} \mathcal{A}_{\eta_i}(u,\mu_i) \Big\rangle^{\bullet}$$

as that has exactly the same conditions $\eta = t = 0$ in order for the first correlator not to vanish.

The rest of the proof is completely parallel to that of theorem 6.5.2, only the computation of the degree of the polynomial makes some difference.

The degree of the coefficient of u^{2g-2+n} (where $2g - 2 + n \geq 0$) of a connected correlator $\langle \prod_{i=1}^{n} \mathcal{A}_{\eta_i}(u,\mu_i) \rangle^{\circ}$ can be computed in the following way. The coefficient $\prod_{i=1}^{n} \mu_i^{t_i - 1}/(\nu_i + 1)_{t_i}$ has degree $-n$ in $\nu_1, \dots, \nu_n$ and degree 0 in u. The leading term of the connected correlator $\langle \prod_{i=1}^{n} \mathcal{E}_{t_i r - \eta_i}(u\mu_i) \rangle^{\circ}$ has degree $n - 1 + n - 2 = 2n - 3$ in $\nu_1, \dots, \nu_n$ and degree $n - 2$ in u. The coefficient of u^{2g} in the series $\mathcal{S} \cdot \prod_{i=1}^{n} \mathcal{S}(ru\mu_i)^{\nu_i + t_i}$, where $\mathcal{S}$ without argument denotes the S-functions coming from the connected correlator $\langle \prod_{i=1}^{n} \mathcal{E}_{t_i r - \eta_i}(u\mu_i) \rangle^{\circ}$ divided by its leading term, is a polynomial of degree $(3/2) \cdot 2g = 3g$ in $\nu_1, \dots, \nu_n$. So, the total degree in $\nu_1, \dots, \nu_n$ is equal to $-n + 2n - 3 + 3g = 3g - 3 + n$.

This completes the proof of the theorem. □

6.7 Quasi-polynomiality for spin and orbifold spin Hurwitz numbers

In this section we prove quasi-polynomiality for orbifold spin Hurwitz numbers. We will define the q-orbifold r-spin Hurwitz numbers as vacuum expectations of certain operators. We will then rewrite this expression to isolate the non-poynomial behaviour and get a formula for the supposed polynomial part as a vacuum expectation of a product of $\mathcal{A}$-operators. This line of thought originates from Okounkov and Pandharipande [37] and has also been used in e.g. [27, 15, 33] to prove quasi-polynomiality of several different kinds of Hurwitz numbers.

We will write $\mu = a[\mu]_a + \langle \mu \rangle_a$ for the integral division of an integer μ by a natural number a. If $a = qr$, we may omit the subscript.

Definition 6.7.1. The *disconnected q-orbifold r-spin Hurwitz numbers* are

$$h^{\bullet,q,r}_{g;\vec{\mu}} := \left\langle \left(\frac{\alpha_q}{q}\right)^{\frac{|\mu|}{q}} \frac{1}{\left(\frac{|\mu|}{q}\right)!} \frac{\mathcal{F}^b_{r+1}}{b!(r+1)^b} \prod_{i=1}^{l(\vec{\mu})} \frac{\alpha_{-\mu_i}}{\mu_i} \right\rangle^{\bullet}, \tag{6.29}$$

where, by Riemann-Hurwitz, the number of $(r+1)$-completed cycles is

$$b = \frac{2g-2+l(\mu)+\frac{|\mu|}{q}}{r}. \tag{6.30}$$

The *connected q-orbifold r-spin Hurwitz numbers* $h^{\circ,q,r}_{g;\vec{\mu}}$ are defined via the inclusion-exclusion formula from the disconnected ones.

Remark 6.7.2. This formula can be interpreted as follows: we count covers of $\mathbb{P}^1$, reading from 0 to ∞. At the point 0, we have ramification profile μ, corresponding to the product of α's on the right. The point ∞ has *orbifold* ramification, profile $[q, q, \dots, q]$, corresponding to the α's on the left, divided by the extra symmetry factor $\left(\frac{|\mu|}{q}\right)!$. In the middle, the ramification profiles are *completed* $r+1$*-cycles*, corresponding to the $\mathcal{F}_{r+1}$. These are formal linear combinations of ramification profiles, with 'leading term' (most ramified) $[r+1, 1, \dots, 1]$, see [38].

Definition 6.7.3. The *generating series of q-orbifold r-spin Hurwitz numbers* is defined as

$$H^{\bullet,q,r}(\vec{\mu}, u) := \sum_{g=0}^{\infty} h^{\bullet,q,r}_{g;\vec{\mu}} u^{rb} = \left\langle e^{\frac{\alpha_q}{q}} e^{u^r \frac{\mathcal{F}_{r+1}}{r+1}} \prod_{i=1}^{l(\vec{\mu})} \frac{\alpha_{-\mu_i}}{\mu_i} \right\rangle^{\bullet}.$$

The *free energies* are defined as

$$F^{q,r}_{g,n}(x_1, \dots, x_n) := \sum_{\mu_1,\dots,\mu_n=1}^{\infty} h^{\circ,q,r}_{g;\vec{\mu}} e^{\sum_{i=1}^n \mu_i x_i}$$

We now introduce $\mathcal{A}$-operators to capture the supposed quasi-polynomial behaviour of the q-orbifold r-spin Hurwitz numbers in the Fock space formalism.

Definition 6.7.4 ($\mathcal{A}$-operators)**.**

$$\mathcal{A}^{q,r}_{\langle\mu\rangle}(u,\mu) := \frac{1}{\mu} \sum_{\substack{l\in\mathbb{Z}+1/2\\ s\in\mathbb{Z}}} \frac{(u^r\mu)^s}{([\mu]+1)_s} \Bigg[\sum_{t=0}^{\infty} \frac{\Delta_q^t}{q^t t!} \left(\frac{(l+\mu)^{r+1}-l^{r+1}}{\mu(r+1)}\right)^{s+[\mu]} E_{l+\mu-qt,l}$$
$$+ \delta_{\langle\mu\rangle_q,0} \sum_{j=1}^{q} \frac{\Delta_q^{[\mu]_q-1}}{q^{[\mu]_q}[\mu]_q!} \left(\frac{(l+\mu)^{r+1}-l^{r+1}}{\mu(r+1)}\right)^{s+[\mu]} \Bigg|_{l=1/2-j} \mathrm{Id} \Bigg],$$

where Δ_q is the q-backward difference operator acting on functions of l, i.e. $(\Delta_q f)(l) = f(l) - f(l-q)$.

Remark 6.7.5. In this definition, u is a formal variable, while μ—at this point—is a positive integer. That is, for fixed μ,

$$\mathcal{A}^{q,r}_{\langle\mu\rangle}(u,\mu) \in \mathcal{A}_\infty[\![u]\!].$$

Indeed, for fixed $[\mu]$ and fixed power of u, t is bounded from above by $r(s+[\mu])$, so only finitely many diagonals are non-zero.

These operators do indeed capture the conjectured polynomial behaviour, as is seen in the following proposition.

Proposition 6.7.6.

$$H^{\bullet,q,r}(\vec{\mu},u) = \prod_{i=1}^{l(\vec{\mu})} \frac{(u^r\mu_i)^{[\mu_i]}}{[\mu_i]!} \Big\langle \prod_{i=1}^{l(\vec{\mu})} \mathcal{A}^{q,r}_{\langle\mu_i\rangle}(\mu_i,u) \Big\rangle^{\bullet}. \tag{6.31}$$

Proof. Since both $\mathcal{F}_{r+1}$ and α_q annihilate the vacuum, their exponents act as the identity operator on the vacuum. Hence we can write

$$H^{\bullet,q,r}(\vec{\mu},u) = \Big\langle \prod_{i=1}^{l(\vec{\mu})} e^{\frac{\alpha_q}{q}} e^{\frac{u^r\mathcal{F}_{r+1}}{r+1}} \frac{\alpha_{-\mu_i}}{\mu_i} e^{-\frac{u^r\mathcal{F}_{r+1}}{r+1}} e^{-\frac{\alpha_q}{q}} \Big\rangle^{\bullet}.$$

Lemma 6.7.7. *The conjugation with exponents of $\mathcal{F}$ reads*

$$\mathcal{O}^{[r]}_{\mu}(u) := e^{u^r\frac{\mathcal{F}_{r+1}}{r+1}} \alpha_{-\mu}\, e^{-u^r\frac{\mathcal{F}_{r+1}}{r+1}} = \sum_{l\in\mathbb{Z}+1/2}\sum_{s=0}^{\infty} \frac{(u^r\mu)^s}{s!}\left(\frac{(l+\mu)^{r+1}-l^{r+1}}{\mu(r+1)}\right)^s E_{l+\mu,l}.$$

Proof. As $\mathrm{Ad}(e^X) = e^{\mathrm{ad}\,X}$, we have

$$e^{u^r\frac{\mathcal{F}_{r+1}}{r+1}} \alpha_{-\mu}\, e^{-u^r\frac{\mathcal{F}_{r+1}}{r+1}} = \sum_{s=0}^{\infty}\frac{u^{rs}}{(r+1)^s s!}\mathrm{ad}^s_{\mathcal{F}_{r+1}}\alpha_{-\mu}.$$

Applying lemma 6.2.5 with $a=0$ and $g_l = l^{r+1}$, we see that every application of the operator $\mathrm{ad}_{\mathcal{F}_{r+1}}$ produces an extra factor $\big((l+\mu)^{r+1}-l^{r+1}\big)$. Multiplying and dividing by μ^s yields the result. ☐

Lemma 6.7.8. *The conjugation with exponents of α_q is given as follows:*

$$\begin{aligned}\frac{1}{\mu}e^{\frac{\alpha_q}{q}}\mathcal{O}^{[r]}_{\mu}(u)e^{-\frac{\alpha_q}{q}} = &\sum_{l\in\mathbb{Z}+1/2}\sum_{s=0}^{\infty}\frac{(u^r\mu)^s}{\mu\, s!}\sum_{t=0}\frac{\Delta_q^t}{q^t t!}\left(\frac{(l+\mu)^{r+1}-l^{r+1}}{\mu(r+1)}\right)^s E_{l+\mu-qt,l}\\ &+\delta_{\langle\mu\rangle_q,0}\sum_{s=0}^{\infty}\frac{(u^r\mu)^s}{\mu\, s!}\sum_{j=1}^{q}\frac{\Delta_q^{[\mu]_q-1}}{q^{[\mu]_q}[\mu]_q!}\left(\frac{(l+\mu)^{r+1}-l^{r+1}}{\mu(r+1)}\right)^s\Bigg|_{l=1/2-j}\mathrm{Id}\,.\end{aligned}$$

Proof. Apply $\mathrm{Ad}(e^X) = e^{\mathrm{ad}\,X}$ as before and lemma 6.2.5 with $a=q$. The component of the identity can only occur if the total energy is zero, i.e. if $\mu = qt$. ☐

Re-indexing $s \mapsto s+[\mu]$ we get the equation for the $\mathcal{A}$-operators, where we use that, for $s < -[\mu]$, the Pochhammer symbol vanishes, so we can extend the sum over all integers. ☐

6.7.1 The operators $\mathcal{A}^{\flat}$

Following the schedule of the previous sections, we would like to calculate the $\mathcal{A}^{\flat}$-operators.

Lemma 6.7.9.

$$\mathcal{O}^{[r]}_{\mu}(u)^{\flat} := e^{u^r \frac{\mathcal{F}_{r+1}}{r+1}} \alpha_\mu e^{-u^r \frac{\mathcal{F}_{r+1}}{r+1}} = \sum_{l\in\mathbb{Z}+1/2} \sum_{s=0}^{\infty} \frac{(u^r\mu)^s}{s!} \left(\frac{(l-\mu)^{r+1} - l^{r+1}}{\mu(r+1)} \right)^s E_{l-\mu,l}.$$

Proof. This is completely analogous to the proof of lemma 6.7.7, only changing the sign of μ in appropriate places. ☐

Lemma 6.7.10.

$$\mu e^{\frac{\alpha_q}{q}} \mathcal{O}^{[r]}_{\mu}(u)^{\flat} e^{-\frac{\alpha_q}{q}} = \sum_{l\in\mathbb{Z}+1/2} \sum_{s=0}^{\infty} \frac{\mu(u^r\mu)^s}{s!} \sum_{t=0} \frac{\Delta_q^t}{q^t t!} \left(\frac{(l-\mu)^{r+1} - l^{r+1}}{\mu(r+1)} \right)^s E_{l-\mu-qt,l}$$

Proof. This is completely analogous to the proof of lemma 6.7.8, bearing in mind that the coefficient of the identity is zero, as both operators in the repeated adjunction have positive energy. ☐

In defining the $\mathcal{A}$-operators, we extracted the coefficient

$$\frac{(u^r\mu)^{[\mu]}}{[\mu]!}.$$

Hence, the $\mathcal{A}^\flat$-operators should include this factor. Therefore we get

Lemma 6.7.11.

$$\mathcal{A}^{q,r}_{\langle\mu\rangle}(u,\mu)^{\flat} = \sum_{l\in\mathbb{Z}+1/2} \sum_{s=0}^{\infty} \frac{\mu(u^r\mu)^{s+[\mu]}}{s![\mu]!} \sum_{t=0}^{\infty} \frac{\Delta_q^t}{q^t t!} \left(\frac{(l-\mu)^{r+1} - l^{r+1}}{\mu(r+1)} \right)^s E_{l-\mu-qt,l} \tag{6.32}$$

6.7.2 Quasi-polynomiality

Definition 6.7.12. An expression defined on a subset $S \subset \mathbb{C}$ is *polynomial* if there exists a polynomial p, defined on $\mathbb{C}$, that agrees with this expression on S. We then use p as a definition of this expression at all other $x \in \mathbb{C}$.

The goal of this section is to prove the following statement.

Theorem 6.7.13 (Quasi-polynomiality)**.** *For $2g-2+\ell(\vec{\mu}) > 0$, the q-orbifold r-spin Hurwitz numbers can be expressed in the following way:*

$$h^{\circ,q,r}_{g,\vec{\mu}} = \prod_{i=1}^{l(\vec{\mu})} \frac{\mu_i^{[\mu_i]}}{[\mu_i]!} P_{\langle\vec{\mu}\rangle}(\mu_1,\dots,\mu_{l(\vec{\mu})}),$$

where P are symmetric polynomials in the variables $\mu_1,\dots,\mu_{l(\vec{\mu})}$ whose coefficients depend on the parameters $\langle\mu_1\rangle,\dots,\langle\mu_{l(\vec{\mu})}\rangle$.

Remark 6.7.14. We prove that the degree of P has a bound that does not depend on the entries of the partition $\vec{\mu}$. The actual computation of the degree in this case is difficult, and it is not necessary for the purpose of topological recursion. However, these numbers are expected to satisfy an ELSV-type formula (see conjecture 6.11). The conjecture would imply that the degree is equal to $3g-3+n$.

Remark 6.7.15. Note that since we allow the coefficients of the polynomials $P_{\langle\vec{\mu}\rangle}$ to depend on $\langle\vec{\mu}\rangle$, we can equivalently consider them as polynomials in $[\mu_1], \dots, [\mu_n]$, $n := l(\vec{\mu})$. The latter way is more convenient in the proof.

Comparing the statement of theorem 6.7.13 to equation (6.31), it is clear that the polynomials P must be the connected correlators of the $\mathcal{A}$-operators, defined via inclusion-exclusion from the disconnected versions. To prove this theorem, we will therefore first consider the disconnected correlators, and show that the coefficient of a fixed power of u is a symmetric rational function in the μ_i, with only prescribed simple poles. The residues at these poles are explicitly related to the $\mathcal{A}^\flat$-operators, and cancel in the inclusion-exclusion formula, proving quasi-polynomiality.

First we need some technical lemmata, analysing the dependence on μ of single terms in the sums of the $\mathcal{A}$-operators.

Lemma 6.7.16. *The coefficients of the polynomial in* l, $\frac{\Delta_q^{x+m}}{q^{x+m}(x+m)!}l^{p+x}$, *are themselves polynomial in* x *for any* p *and* m. *More precisely, the coefficient* $c^p_{m,a}(x)$ *of* l^a *has degree* $2p-a-2m$.

Proof. There is a version of the Leibniz rule for the backwards difference operator:

$$\Delta_q(fg)(l) = (\Delta_q f)(l)g(l) + f(l-q)(\Delta_q g)(l).$$

Repeated application of this rule gives the following:

$$\begin{aligned}
\frac{\Delta_q^{x+m}}{q^{x+m}(x+m)!}l^{p+x} &= \sum_{i_0+\cdots+i_{x+m}=p-m} (l-q(x+m))^{i_{x+m}}\cdots(l-q\cdot 0)^{i_0} \\
&= h_{p-m}(l-q(x+m),\cdots,l) \\
&= \sum_{a=0}^{p-m}\binom{p+x}{a} h_{p-m-a}\big(-q,\dots,-q(x+m)\big)l^a \\
&= \sum_{a=0}^{p-m}\binom{p+x}{a}\left\{ {x+p-a \atop x+m} \right\}(-q)^{p-m-a}l^a.
\end{aligned}$$

Here we used 6.3.4. So the coefficient of l^a is given by

$$c^p_{m,a}(x) = (-q)^{p-m-a}\binom{p+x}{a}\left\{ {x+p-a \atop x+m} \right\}.$$

This binomial coefficient can be written as

$$\frac{1}{a!}(x+p)\cdots(x+p-a+1),$$

which is a polynomial in x of degree a.

The Stirling number, on the other hand, requires a more subtle proof. Define $f_t(x) = \left\{ {x+t \atop x} \right\}$. We prove f_t is a polynomial of degree $2t$ inductively on t, starting with $f_0(x) \equiv 1$.

For the induction step, recall the recursion relation for Stirling numbers, which can be written as follows:

$$\left\{ {x+t \atop x} \right\} - \left\{ {x-1+t \atop x-1} \right\} = x\left\{ {x-1+t \atop x} \right\}.$$

In other notation, $(\Delta_1 f_t)(x) = x f_{t-1}(x)$. By induction, $\Delta_1 f_t$ is polynomial of degree $2t-1$, hence f_t itself can be written as a polynomial of degree $2t$. The Stirling number we require is given by $f_{p-a-m}(x+m)$, which is of degree $2(p-a-m)$. Adding degrees yields the result. □

Remark 6.7.17. Note that the equation $\Delta_1 f = 0$ has non-polynomial solutions, e.g. $f(x) = \sin(2\pi x)$. However, we only prove that the functions in question can be represented as polynomials, not that there is no other analytic continuation.

Lemma 6.7.18. *For fixed $r, i, s, \langle\mu\rangle \in \mathbb{Z}_{\geq 0}$ the expression*

$$\frac{\Delta_q^{i+[\mu]_q}}{q^{i+[\mu]_q}(i+[\mu]_q)!}\left(\frac{(l+\mu)^{r+1}-l^{r+1}}{\mu(r+1)}\right)^{s+[\mu]}$$

is polynomial in $[\mu]$ (in the sense of definition 6.7.12), of degree $2rs - 2i - 2\langle[\mu]_q\rangle_r$.

Proof. Expanding explicitly using Newton's binomial formula,

$$Q_\mu^r(l) := \frac{(l+\mu)^{r+1}-l^{r+1}}{\mu(r+1)} = \sum_{i=0}^{r}\binom{r+1}{i+1}\frac{\mu^i l^{r-i}}{(r+1)}.$$

Let us now consider the coefficient in front of $l^{r([\mu]_{qr}+s)-a}$ for some particular values of "offset" a:

$$\left[l^{r([\mu]+s)-0}\right] Q_\mu^r(l)^{[\mu]+s} = 1;$$

$$\left[l^{r([\mu]+s)-1}\right] Q_\mu^r(l)^{[\mu]+s} = \binom{[\mu]+s}{1}\binom{r+1}{2}\frac{\mu}{(r+1)};$$

$$\left[l^{r([\mu]+s)-2}\right] Q_\mu^r(l)^{[\mu]+s} = \binom{[\mu]+s}{1}\binom{r+1}{3}\frac{\mu^2}{(r+1)} + \binom{[\mu]+s}{2}\binom{r+1}{2}^2\frac{\mu^2}{(r+1)^2};$$

$$\vdots$$

$$\left[l^{r([\mu]+s)-a}\right] Q_\mu^r(l)^{[\mu]+s} = \sum_{\lambda\vdash a}\binom{[\mu]+s}{\{\lambda_i^T-\lambda_{i+1}^T\}_{i\geq 1}}\left(\prod_{i=1}^{\ell(\lambda)}\frac{1}{r+1}\binom{r+1}{\lambda_i+1}\right)\mu^a,$$

where the multinomial coefficient is

$$\binom{[\mu]+s}{\{\lambda_i^T-\lambda_{i+1}^T\}_{i\geq 1}} := \frac{([\mu]+s)!}{([\mu]+s-\ell(\lambda))!\prod_{i\geq 1}(\lambda_i^T-\lambda_{i+1}^T)!}.$$

Clearly, this is a polynomial in $[\mu]$ of degree $2a$—one a comes from μ^a and the other from the multinomial coefficient in the summand, corresponding to the partition $[1^a]$.

Furthermore, it has zeroes at $[\mu] \in \mathbb{Z}_{\geq 0}$ for which $r([\mu]+s)-a < 0$ (i.e. when we want to extract a coefficient in front of the negative power of l). This is because the contributions of partitions λ with more than $[\mu]+s$ parts are zero thanks to the multinomial coefficient and partitions with $\ell(\lambda) \leq [\mu]+s$ will have at least one part for which the corresponding binomial coefficient will be zero.

Let us denote

$$\mathrm{Poly}_{a,s,r}([\mu]) = \left[l^{r([\mu]+s)-a}\right] Q_\mu^r(l)^{[\mu]+s}$$

Using lemma 6.7.16), denoting $i' = i + \langle [\mu]_q \rangle_r$ for brevity and noting $[\mu]_q = r[\mu] + \langle [\mu]_q \rangle_r$, we have

$$\begin{aligned}\frac{\Delta_q^{i'+r[\mu]}}{q^{i'+r[\mu]}(i'+r[\mu])!} Q^r_\mu(l)^{s+[\mu]} &= \frac{\Delta_q^{i'+r[\mu]}}{q^{i'+r[\mu]}(i'+r[\mu])!} \sum_{a=0}^{r([\mu]+s)} \mathrm{Poly}_{a,s,r}([\mu]) l^{r([\mu]+s)-a} \\ &= \sum_{a=0}^{r[\mu]+rs} \sum_{k=0}^{rs-i'-a} l^k c^{rs-a}_{i',k}(r[\mu])\, \mathrm{Poly}_{a,s,r}([\mu]) \\ &= \sum_{a=0}^{rs-i'} \sum_{k=0}^{rs-i'-a} l^k c^{rs-a}_{i',k}(r[\mu])\, \mathrm{Poly}_{a,s,r}([\mu]),\end{aligned}$$

where crucially in the last equality, we can choose upper summation limit of the first sum to be independent of $[\mu]$. We can do this, because:

- for $a > rs - i'$ the coefficients $c^{rs-a}_{i',k}(r[\mu])$ are zero;
- for a particular value of $[\mu] \in \mathbb{Z}_{\geq 0}$ it could happen that $r([\mu]+s) < rs - i'$. But we know that for $a > r([\mu]+s)$, $\mathrm{Poly}_{a,s,r}([\mu]) = 0$. So, adding these zero terms does not change the sum.

We see that we have arrived at a manifestly polynomial expression, which completes the proof.

The degree follows as the degree of $\mathrm{Poly}_{a,s,r}([\mu])$ is $2a$ and that of $c^{rs-a}_{i',k}$ is $2(rs-a)-k-2i'$. ☐

These lemmata can be applied to prove the rationality of the disconnected correlators of $\mathcal{A}$-operators.

Proposition 6.7.19. *For fixed power of u and fixed $[\mu_2], \dots, [\mu_n]$, and $\langle \vec{\mu} \rangle$,*

$$\Big\langle \prod_{i=1}^{l(\vec{\mu})} \mathcal{A}^{q,r}_{\langle \mu_i \rangle}(\mu_i, u) \Big\rangle^{\bullet}$$

is a rational function in the variable $[\mu_1]$, with only simple poles at negative integers and at $[\mu_1] = -\langle \mu \rangle$.

Proof. Let us make some observations about the following expression, where we write $\mu = \mu_1$,

$$\Big\langle \sum_{\substack{l \in \mathbb{Z}+1/2 \\ s \in \mathbb{Z}}} \frac{(u^r \mu)^s}{\mu([\mu]+1)_s} \sum_{t=0} \frac{\Delta_q^t}{q^t t!} Q^r_\mu(l)^{s+[\mu]} E_{l+\mu-qt,l} \prod_{j=2}^{l(\vec{\mu})} \mathcal{A}^{q,r}_{\langle \mu_i \rangle}(\mu_i, u) \Big\rangle^{\bullet}$$

First of all, the energy of the operators on the left should be positive, meaning that $\mu - qt < 0$. On the other side, the exponent of the finite difference operator cannot be greater than the degree of the polynomial to which it is applied, implying $t \leq r(s+[\mu])$. Combining these two restrictions, one obtains that $rs + r[\mu] \geq [\mu]_q = r[\mu] + \langle [\mu]_q \rangle_r$. Solving for s gives $s \geq \frac{\langle [\mu]_q \rangle_r}{r} \geq 0$.

Moreover, the correlator is zero unless the sum of the energies is zero, which means

$$(\mu - qt) + \sum_{j=2}^{l(\vec{\mu})} \mu_j - qt_j = 0. \tag{6.33}$$

Since the other μ_j are fixed, it is clear that $-i := [\mu]_q - t$ does not depend on μ. We can rewrite the expression as

$$\left\langle \sum_{\substack{l\in\mathbb{Z}+1/2\\ s\geq 0}} \frac{(u^r\mu)^s}{\mu([\mu]+1)_s} \sum_{i=0}^{N} \frac{\Delta_q^{i+[\mu]_q}}{q^{i+[\mu]_q}(i+[\mu]_q)!} Q^r_\mu(l)^{s+[\mu]} E_{l+\langle\mu\rangle_q - qi,l} \prod_{j=2}^{l(\vec{\mu})} \mathcal{A}_{\langle\mu_i\rangle}(\mu_i,u) \right\rangle^\bullet, \tag{6.34}$$

where N does not depend on μ. Fixing the power of u reduces the s-sum to a finite sum, as for the other $\mathcal{A}$-operators the power of u is bouned from below by $-[\mu_i]$. Now, the first fraction is clearly a rational function in $[\mu]$ while the second is polynomial by lemma 6.7.18. Hence, the entire correlator is a finite sum of rational functions, so it is rational itself.

The only possible poles can come from the Pochhammer symbol in the denominator, or the factor $\frac{1}{\mu}$, and hence are at $-s, 1-s, \dots, -1$ and at $[\mu] = -\frac{\langle\mu\rangle}{qr}$. □

To prove the connected correlator is a polynomial, we should therefore analyse these poles. As they are simple, we need only calculate the residues, which we do in the following proposition.

Lemma 6.7.20. *The residue of the $\mathcal{A}$-operators at negative integers is, up to a linear multiplicative constant, equal to the $\mathcal{A}^\flat$-operators with a negative argument. More precisely,*

$$\operatorname*{Res}_{\nu=-m} \mathcal{A}^{q,r}_\eta(u,\nu qr+\eta) = \frac{u^r}{mqr-\eta}\mathcal{A}^{q,r}_{-\eta}(u,mqr-\eta)^\flat \qquad \text{if } \eta\neq 0; \tag{6.35}$$

$$\operatorname*{Res}_{\nu=-m} \mathcal{A}^{q,r}_0(u,\nu qr) = \frac{1}{mq^2r^2}\mathcal{A}^{q,r}_0(u,mqr)^\flat \qquad \text{if } \eta = 0. \tag{6.36}$$

Here the residue is taken term-wise in the power series in u, and the factor u^{-r} means a shift of terms.

Remark 6.7.21. Note that the first formula is slightly different from the one in 6.6.7 in the case $r=1$. This is because in that section, an extra conjugation with $u^{\frac{\mathcal{F}_1}{r}}$ was performed, resulting in different $\mathcal{A}$-operators.

Proof. Let us prove equations (6.35) and (6.36) together. The only contributing terms have $s\geq m$, so we calculate

$$\begin{aligned}
\operatorname*{Res}_{\nu=-m} \mathcal{A}^{q,r}_\eta(u,\mu) &= \sum_{\substack{l\in\mathbb{Z}+1/2\\ s\geq m}} \frac{(u^r\mu)^s(\nu+m)}{\mu(\nu+1)_s} \sum_{t=0}^{\infty}\frac{\Delta_q^t}{q^t t!}\left(\frac{(l+\mu)^{r+1}-l^{r+1}}{\mu(r+1)}\right)^{s+\nu} E_{l+\mu-qt,l}\Bigg|_{\nu=-m}\\
&= \sum_{\substack{l\in\mathbb{Z}+1/2\\ s\geq m}} \frac{(u^r\mu)^s}{\mu(1-m)_{m-1}(s-m)!} \sum_{t=0}^{\infty}\frac{\Delta_q^t}{q^t t!}\left(\frac{(l+\mu)^{r+1}-l^{r+1}}{\mu(r+1)}\right)^{s-m} E_{l+\mu-qt,l}\\
&= \sum_{\substack{l\in\mathbb{Z}+1/2\\ s\geq m}} \frac{(u^r\mu)^s(-1)^{m-1}}{\mu(m-1)!(s-m)!} \sum_{t=0}^{\infty}\frac{\Delta_q^t}{q^t t!}\left(\frac{(l+\mu)^{r+1}-l^{r+1}}{\mu(r+1)}\right)^{s-m} E_{l+\mu-qt,l},
\end{aligned}$$

where we kept writing μ for $-mr+\eta$. As this is negative, however, it makes sense to rename it $\mu = -\lambda$. Substituting and shifting the s-summation, we get

$$\begin{aligned}\operatorname*{Res}_{\nu=-m} \mathcal{A}^{q,r}_{\eta}(u,\mu) &= \sum_{\substack{l\in\mathbb{Z}+1/2\\ s\geq m}} \frac{(-u^r\lambda)^s(-1)^{m-1}}{-\lambda(m-1)!(s-m)!}\sum_{t=0}^{\infty}\frac{\Delta_q^t}{q^t t!}\left(\frac{(l-\lambda)^{r+1}-l^{r+1}}{-\lambda(r+1)}\right)^{s-m} E_{l-\lambda-qt,l}\\ &= \sum_{\substack{l\in\mathbb{Z}+1/2\\ s\geq m}} \frac{(u^r\lambda)^s}{\lambda(m-1)!(s-m)!}\sum_{t=0}^{\infty}\frac{\Delta_q^t}{q^t t!}\left(\frac{(l-\lambda)^{r+1}-l^{r+1}}{\lambda(r+1)}\right)^{s-m} E_{l-\lambda-qt,l}\\ &= \sum_{\substack{l\in\mathbb{Z}+1/2\\ s\geq 0}} \frac{(u^r\lambda)^{s+m}}{\lambda(m-1)!s!}\sum_{t=0}^{\infty}\frac{\Delta_q^t}{q^t t!}\left(\frac{(l-\lambda)^{r+1}-l^{r+1}}{\lambda(r+1)}\right)^{s} E_{l-\lambda-qt,l}.\end{aligned}$$

Because $\lambda = mr-\eta$, we have $m = [\lambda]+1-\delta_{\eta 0}$ and $\eta = -\langle\lambda\rangle$. Recalling equation (6.32), we obtain the result. □

Proof of theorem 6.7.13. The Hurwitz numbers are symmetric in their arguments, hence the P must be as well. By the same argument as for the previous sections, it suffices to prove polynomiality in the first argument.

Lemma 6.7.20 implies that we can express the residues in μ_1 of the disconnected correlator as follows:

$$\operatorname*{Res}_{\nu_1=-m}\left\langle \prod_{i=1}^{n}\mathcal{A}_{\eta_i}(u,\mu_i)\right\rangle^{\bullet} = c(m,\eta_1)\left\langle \mathcal{A}_{-\eta_1}(u,mqr-\eta_1)^{\flat}\prod_{i=2}^{n}\mathcal{A}_{\eta_i}(u,\mu_i)\right\rangle^{\bullet}.$$

where $c(m,\eta_1)$ is the coefficient in lemma 6.7.20. Recalling equations (6.29) and (6.31) and realizing that the $\mathcal{A}^{\flat}$-operator is given by the same conjugations as the normal $\mathcal{A}$-operator, but starting from α_μ instead of $\alpha_{-\mu}$, we can see that this reduces to

$$\operatorname*{Res}_{\nu_1=-m}\left\langle \prod_{i=1}^{n}\mathcal{A}_{\eta_i}(u,\mu_i)\right\rangle^{\bullet} = C\left\langle e^{\frac{\alpha_q}{q}}e^{u^r\frac{\mathcal{F}_{r+1}}{r+1}}\alpha_{mqr-\eta_1}\prod_{i=2}^{n}\frac{\alpha_{-\mu_i}}{\mu_i}\right\rangle^{\bullet} \tag{6.37}$$

for some specific coefficient C that depends only on m, η_1, and the μ_i.

Because $[\alpha_k,\alpha_l] = k\delta_{k+l,0}$, and $\alpha_{mqr-\eta_1}$ annihilates the vacuum, this residue is zero unless one of the μ_i equals $mqr-\eta_1$ for $i\geq 2$.

Now return to the connected correlator. It can be calculated from the disconnected one by the inclusion-exclusion principle, so in particular it is a finite sum of products of disconnected correlators. Hence the connected correlator is also a rational function in ν_1, and all possible poles must be inherited from the disconnected correlators. So let us assume $\mu_i = mqr-\eta_1$ for some $i\geq 2$. Then we get a contribution from (6.37), but this is canceled exactly by the term coming from

$$\begin{aligned}&\operatorname*{Res}_{\nu_1=-m}\left\langle \mathcal{A}_{\eta_1}(u,\mu_1)\mathcal{A}_{-\eta_1}(u,mqr-\eta_1)\right\rangle^{\bullet}\left\langle \prod_{\substack{2\leq j\leq n\\ j\neq i}}\mathcal{A}_{\eta_j}(u,\mu_j)\right\rangle^{\bullet}\\ &\quad= C\left\langle e^{\frac{\alpha_q}{q}}e^{u^r\frac{\mathcal{F}_{r+1}}{r+1}}\alpha_{mqr-\eta_1}\alpha_{-(mqr-\eta_1)}\right\rangle^{\bullet}\left\langle e^{\frac{\alpha_q}{q}}e^{\frac{u^r\mathcal{F}_{r+1}}{r+1}}\prod_{\substack{2\leq j\leq n\\ j\neq i}}\frac{\alpha_{-\mu_j}}{\mu_j}\right\rangle^{\bullet},\end{aligned}$$

where the *same* C occurs.

For the pole at $[\mu] = -\frac{\langle\mu\rangle}{qr}$, the only contributing term in equation (6.34) has $s = 0$, so we get

$$\Big\langle \sum_{l\in\mathbb{Z}+1/2} \frac{1}{\mu} \sum_{i=0}^{N} \frac{\Delta_q^{i+[\mu]_q}}{q^{i+[\mu]_q}(i+[\mu]_q)!} Q_\mu^r(l)^{[\mu]} E_{l+\langle\mu\rangle_q - qi, l} \prod_{j=2}^{l(\vec{\mu})} \mathcal{A}_{\langle\mu_i\rangle}(\mu_i, u) \Big\rangle^\bullet.$$

From the proof of lemma 6.7.18, we can clearly see that $\mathrm{Poly}_{a,0,r}([\mu])$ is divisible by μ if $a > 0$, so we need $a = 0$ there. This implies we have only

$$c^0_{i',k}(r[\mu]) = (-q)^{-k-i'} \binom{r[\mu]}{k} \begin{Bmatrix} r[\mu]-k \\ r[\mu]+i' \end{Bmatrix},$$

so we clearly need $k = i' = 0$, and thus $i = 0$ and $\langle[\mu]_q\rangle_r = 0$. As the first $\mathcal{A}$-operator acts on the covacuum, we still need $qi - \langle\mu\rangle_q \geq 0$, so $\langle\mu\rangle_q = 0$. As now $\langle\mu\rangle_{qr} = \langle\mu\rangle_q + q\langle[\mu]_q\rangle_r = 0$, we get that this term cancels against the same term from

$$\Big\langle \mathcal{A}_0(u,\mu_1) \Big\rangle^\bullet \Big\langle \prod_{i=2}^{\ell(\vec{\mu})} \mathcal{A}_{\eta_i}(u,\mu_i) \Big\rangle^\bullet.$$

Hence, the connected correlator has no residues, which proves it is polynomial in ν_1. Therefore, it is also a polynomial in μ_1, see remark 6.7.15. This completes the proof of the polynomiality in $[\mu_1]$.

To be able to conclude that the connected correlator is polynomial in all $[\mu_1]\dots[\mu_n]$ we must show that the degree in $[\mu_1]$ of the connected correlator does not depend on $[\mu_2]\dots[\mu_n]$.

Since a connected correlator is a finite sum over products of disconnected correlators, given by the inclusion-exclusion formula, and the number of summands does not depend on $[\mu_2]\dots[\mu_n]$, the estimate on the degree of the connected correlator follows from estimates on degrees of disconnected correlators. The degree of the disconnected correlator, which is a rational function in $[\mu_1]$ by proposition 6.7.19, is defined as the leading exponent in the limit $[\mu_1] \to +\infty$.

Let us consider summands in the disconnected correlator (6.34) corresponding to a particular choice of $s_j \geq -[\mu_j]$, for $2 \leq j \leq n$. The contribution of genus g covers is extracted by taking the coefficient in front of $u^{2g-2+n+\frac{1}{q}\sum_{i=1}^n \langle\mu_i\rangle}$, so we have

$$s = \frac{2g-2+n}{r} + \frac{1}{rq}\sum_{i=1}^n \langle\mu_i\rangle - \sum_{j=2}^n s_j$$

First of all, the factor $\frac{\mu^s}{\mu([\mu]+1)_s}$ contributes -1 to the degree. Then, by lemma 6.7.18 the degree of

$$\frac{\Delta_q^{i+[\mu]_q}}{q^{i+[\mu]_q}(i+[\mu]_q)!} Q_\mu^r(l)^{s+[\mu]} \tag{6.38}$$

is $2rs - 2i - 2\langle[\mu]_q\rangle$. It looks like the sum over i in (6.34) goes from zero, so the highest degree of these polynomials depends on $[\mu_2]\dots[\mu_n]$ (through s and estimates for s_j), but we are to obtain a finer estimate on the lower limit of summation.

We have

$$t_j \leq r(s_j + [\mu_j]) \text{ for } 2 \leq j \leq n,$$

since exponents of difference operators cannot be greater than the exponent of the polynomials to which they are applied. Combined with the condition (6.33) that the sum of the energies should be zero, this gives

$$i \geq \frac{1}{q}\left(\langle\mu\rangle_q + \sum_{j=2}^{n}\langle\mu_j\rangle\right) - r\sum_{j=2}^{n} s_j,$$

which means that the degree of (6.38) is bounded from above by

$$2(2g-2+n) + \frac{2}{q}\sum_{i=1}^{n}\langle\mu_i\rangle - 2\langle[\mu]_q\rangle_r - \frac{2}{q}\left(\langle\mu\rangle_q + \sum_{j=2}^{n}\langle\mu_j\rangle\right) = 2(2g-2+n),$$

which does not depend on $[\mu_2]\dots[\mu_n]$.

Thus, the degree of the disconnected correlator does not depend on $[\mu_2]\dots[\mu_n]$, and hence the degree of the connected correlator does not depend on $[\mu_2]\dots[\mu_n]$ either. □

6.8 Correlation functions on spectral curves

In this section we explain the relation of the polynomiality statements with the fact that the n-point generation functions can be represented via correlation functions defined on the n-th cartesian power of a spectral curve. The results concerning the monotone and strictly monotone Hurwitz numbers in this section are new, while in the case of usual Hurwitz numbers it is well-known and we recall it here for completeness.

The set-up for the problems considered in this chapter is the following: We consider a spectral curve $\mathbb{C}\mathrm{P}^1$ with a global coordinate z, with a function $x = x(z)$ on it. Let $\{p_0, \dots, p_{r-1}\}$ be the set of the z-coordinates of the critical points of x. We consider the n-point generating function of a particular Hurwitz problem, for a fixed genus g, and we want it to be an expansion of a symmetric function on $\left(\mathbb{C}\mathrm{P}^1\right)^{\times n}$ of a particular type:

$$\sum_{0\leq\alpha_1,\dots,\alpha_n\leq r-1} P_{\vec{\alpha}}\left(\frac{d}{dx_1},\dots,\frac{d}{dx_n}\right)\prod_{i=1}^{n}\xi_{\alpha_i}(x_i) \tag{6.39}$$

Here the $P_{\vec{\alpha}}$ are polynomials in n variables of degree $3g-3+n$, and the functions $\xi_\alpha(x)$ are defined as (the expansions of) some functions that form a convenient basis in the space spanned by $1/(p_\alpha - z)$, $\alpha = 0, \dots, r-1$.

The reason we are interested in the particular degree $3g-3+n$, is in short due to this being the dimension of the moduli space of curves $\overline{\mathcal{M}}_{g,n}$. Somewhat more explicitly, we expect an ELSV-type formula to hold, as it does in the usual orbifold case—the ELSV-formula itself for $r = 1$ [18] and the Johnson-Pandharipande-Tseng formula for general r [28], for more explanations and examples we refer to [19, 14, 34, 15, 1]. The topological recursion implies [19] that the correlation differentials are given by the differentials of

$$\sum_{0\leq\alpha_1,\dots,\alpha_n\leq r-1}\left[\int_{\overline{\mathcal{M}}_{g,n}}\frac{C_{\vec{\alpha}}}{\prod_{i=1}^{n}\left(1-\psi_i\frac{d}{dx_i}\right)}\right]\prod_{i=1}^{n}\xi_{\alpha_i}(x_i),$$

where $C_{\vec{\alpha}}$ is some class in the cohomology of $\overline{\mathcal{M}}_{g,n}$. Because the complex cohomological degree of the ψ-classes is one, this implies that we have a polynomial in the derivatives of degree $\dim \overline{\mathcal{M}}_{g,n} = 3g-3+n$.

6.8.1 Monotone orbifold Hurwitz numbers

In the case of the monotone orbifold Hurwitz numbers the conjectural spectral curve is given by $x = z(1-z^r)$ [8]. The conjecture on the topological recursion assumes the expansion of equation (6.39) in $x_1, \dots, x_n$ near $x_1 = \cdots = x_n = 0$, so we have the following expected property of orbifold Hurwitz numbers:

$$\sum_{\vec{\mu} \in (\mathbb{N}^\times)^n} h^{\circ,r,\leq}_{g;\vec{\mu}} \prod_{i=1}^n x_i^{\mu_i} = \sum_{0 \leq \alpha_1, \dots, \alpha_n \leq r-1} P_{\vec{\alpha}} \left(\frac{d}{dx_1}, \dots, \frac{d}{dx_n} \right) \prod_{i=1}^n \xi_{\alpha_i}(x_i). \tag{6.40}$$

In this case the critical points are given by $p_i = \zeta^i (r+1)^{-1/r}$, $i = 0, \dots, r-1$, where ζ is a primitive r-th root of 1. This means that up to some non-zero constant factors that are not important, we have the space of functions spanned by:

$$\xi_i'' = \frac{1}{1 - \zeta^{-i}(r+1)^{1/r} z}, \qquad i = 0, 1, \dots, r-1$$

Consider a non-degenerate change of basis $\xi_k' = \sum_{i=0}^{r-1} \zeta^{ki}/r \cdot \xi_i''$. We have:

$$\xi_k' = \frac{\left((r+1)^{1/r} z\right)^k}{1-(r+1)z^r}, \qquad k = 0, 1, \dots, r-1$$

Observe that $x = z(1-z^r)$ implies

$$\frac{d}{dx} = \frac{1}{1-(r+1)z^r} \frac{d}{dz}$$

Therefore, the functions ξ_k' are given up to non-zero constant factors C_k' by

$$\xi_k' = C_k' \frac{d}{dx} \frac{z^{k+1}}{k+1}, \qquad k = 0, 1, \dots, r-1$$

Thus, the suitable set of basis functions for the representation of the n-point function in the form of equation (6.40) is given by

$$\xi_i := \frac{d}{dx} \left(\frac{z^{i+1}}{i+1} \right), \qquad i = 0, \dots, r-1$$

Lemma 6.8.1. *For $i = 0, \dots, r-1$, we have:*

$$\xi_i(x) = \sum_{\substack{\mu=0 \\ r|\mu-i}}^{\infty} \binom{\mu + [\mu]}{\mu} x^\mu \tag{6.41}$$

Proof. In order to compute the expansion of z^{i+1} in x, we compute the residue:

$$\oint z^{i+1}\frac{dx}{x^{n+1}} = \oint \frac{1-(r+1)z^r}{(1-z^r)^{n+1}}\frac{z^{i+1}dz}{z^{n+1}} = \oint \frac{dz}{z^{n-i}}(1-(r+1)z^r)\sum_{j=0}^{\infty}\binom{n+j}{j}z^{rj}$$

This residue is nontrivial only for $n = kr+i+1$, $k \geq 0$, and in this case it is equal to the coefficient of z^{kr}, that is,

$$\binom{kr+k+i+1}{k} - (r+1)\binom{kr+k+i}{k-1} = \frac{(i+1)\cdot(kr+k+i)!}{k!(kr+i+1)!}$$

Thus

$$\frac{z^{i+1}}{i+1} = \sum_{k=0}^{\infty}\binom{kr+k+i}{k}\frac{x^{kr+i+1}}{kr+i+1}$$

which implies the formula for $\xi_i = (d/dx)\left(z^{i+1}/(i+1)\right)$, $i = 0,\ldots,r-1$, if we set $\mu = kr+i$. $\square$

The explicit formulae for the expansions of functions ξ_i in the variable x given by equation (6.41) imply a particular structure for the coefficients of the expansion given by equation (6.39), that is, for monotone orbifold Hurwitz numbers. In fact we have:

Proposition 6.8.2. *The coefficient of $x_1^{\mu_1}\cdots x_n^{\mu_n}$ of the expansion in $x_1,\ldots,x_n$ near zero of an expression of the form*

$$\sum_{0\leq k_1,\ldots,k_n\leq r-1} P_{k_1,\ldots,k_n}\left(\frac{d}{dx_1},\ldots,\frac{d}{dx_n}\right)\prod_{i=1}^{n}\xi_{k_i} \tag{6.42}$$

where $P_{k_1,\ldots,k_n}$ are polynomials of degree $3g-3+n$ and ξ_k is equal to $\frac{d}{dx}\frac{z^{k+1}}{k+1}$, is represented as

$$\prod_{i=1}^{n}\binom{\mu_i+[\mu_i]}{\mu_i}\cdot Q_{\langle\mu_1\rangle,\ldots,\langle\mu_n\rangle}([\mu_1],\ldots,[\mu_n])$$

where $\mu_i = r[\mu_i]+\langle\mu_i\rangle$, is the euclidean division, and $Q_{\eta_1,\ldots,\eta_n}$ are some polynomials of degree $3g-3+n$ whose coefficients depend on $\eta_1,\ldots,\eta_n \in \{0,\ldots,r-1\}$.

Proof. The coefficient of x^μ in $(d/dx)^p\xi_q$ is non-trivial if and only if $\langle\mu\rangle + p \equiv q \mod r$. In this case, the coefficient of x^μ is equal to

$$\binom{[\mu+p]+\mu+p}{[\mu+p]}(\mu+1)_p = \binom{\mu+[\mu]}{\mu}\cdot\frac{([\mu+p]+\mu+p)![\mu]!}{(\mu+[\mu])![\mu+p]!} \tag{6.43}$$

Represent p as $p = -\langle\mu\rangle + sr + \ell \geq 0$, $0 \leq \ell \leq r-1$. Then the second factor on the right hand side of equation (6.43) can be rewritten as

$$\frac{(([\mu]+s)(r+1)+\ell)!}{([\mu](r+1)+\langle\mu\rangle)!([\mu]+1)_s}$$

Observe that we can cancel the factors $([\mu]+1),([\mu]+2),\ldots,([\mu]+s)$ in the denominator with the factors $([\mu]+1)(r+1),([\mu]+2)(r+1),\ldots,([\mu]+s)(r+1)$ in the numerator. Since $([\mu]+1)(r+1) >$

$[\mu](r+1)+\langle\mu\rangle$, after this cancellation the numerator is still divisible by $([\mu](r+1)+\langle\mu\rangle)!$. So, this factor is a polynomial of degree p in $[\mu]$, with the leading coefficient $(r+1)^{p+s}[\mu]^p$.

Since the only possible nontrivial coefficient of x^μ in $(d/dx)^p\xi_q$ is a common factor $\binom{\mu+[\mu]}{\mu}$ multiplied by a polynomial of degree p in $[\mu]$, the coefficient of $\prod_{i=1}^n x_i^{\mu_i}$ in the whole expression (6.42) is also given by a common factor $\prod_{i=1}^n \binom{\mu_i+[\mu_i]}{\mu_i}$ multiplied by a polynomial in $[\mu_1],\dots,[\mu_n]$ of the same degree as $P_{k_1,\dots,k_n}$. □

Thus the quasi-polynomiality property of monotone orbifold Hurwitz numbers is equivalent to the property that the n-point functions can be represented in a very particular way (given by equation (6.40)) on the corresponding conjectural spectral curve, cf. [8, conjecture 23].

6.8.2 Strictly monotone orbifold Hurwitz numbers

In this case the spectral curve topological recursion follows from the two-matrix model consideration [5], and it was combinatorially proved in [17], see also [10]. From these papers it does follow that the n-point function is represented as an expansion of the following form:

$$\sum_{\vec\mu\in(\mathbb{N}^\times)^n} h^{\circ,r,<}_{g;\vec\mu}\prod_{i=1}^n x_i^{-\mu_i} = \sum_{0\le\alpha_1,\dots,\alpha_n\le r-1} P_{\vec\alpha}\left(\frac{d}{dx_1},\dots,\frac{d}{dx_n}\right)\prod_{i=1}^n \xi_{\alpha_i}(x_i) \tag{6.44}$$

for the curve $x=z^{r-1}+z^{-1}$. The goal of this section is to show the equivalence of this representation to the quasi-polynomiality property of strictly monotone orbifold Hurwitz numbers.

The critical points of x are given by $p_i=\zeta^i(r-1)^{-1/r}$, $i=0,\dots,r-1$, so, repeating the argument for the previous section and using that in this case

$$-\frac{1}{z^2}\frac{d}{dx}=\frac{1}{1-(r-1)z^r}\frac{d}{dz}$$

we see that a good basis of functions ξ_i can be chosen as

$$\xi_i=\frac{1}{z^2}\frac{d}{dx}\left(\frac{z^{i+1}}{i+1}\right),\qquad i=0,\dots,r-1$$

The expansion of these function in x^{-1} near $x=\infty$ is given by the following lemma:

Lemma 6.8.3. *For $i=0,\dots,r-1$, we have:*

$$\xi_i(x)=\sum_{\substack{\mu=1\\ r\mid\mu-i}}^{\infty}\binom{\mu-1}{[\mu]}x^{-\mu}$$

Proof. We compute the coefficient of $x^{-\mu}$ as the residue

$$\oint\frac{1}{z^2}\frac{d}{dx}\left(\frac{z^{i+1}}{i+1}\right)x^{\mu-1}dx=\oint-\frac{z^{i+1}}{i+1}d\left(\frac{(1+z^r)^{\mu-1}}{z^{\mu+1}}\right)$$

We see that his residue can be non-trivial only if $\mu+1\equiv i+1 \mod r$, and in this case it is equal to $\binom{\mu-1}{[\mu]}$. □

The proof of the following statement repeats the proof of proposition 6.8.2.

Proposition 6.8.4. *The coefficient of* $x_1^{-\mu_1}\cdots x_n^{-\mu_n}$ *of the expansion in* $x_1^{-1},\dots,x_n^{-1}$ *near infinity of an expression of the form*

$$\sum_{0\le k_1,\dots,k_n\le r-1} P_{k_1,\dots,k_n}\left(\frac{d}{dx_1},\dots,\frac{d}{dx_n}\right)\prod_{i=1}^n \xi_{k_i}$$

where $P_{k_1,\dots,k_n}$ *are polynomials of degree* $3g-3+n$ *and* ξ_k *is equal to* $\frac{1}{z^2}\frac{d}{dx}\left(\frac{z^{k+1}}{k+1}\right)$*, is represented as*

$$\prod_{i=1}^n \binom{\mu_i-1}{[\mu_i]}\cdot Q_{\langle\mu_1\rangle,\dots,\langle\mu_n\rangle}([\mu_1],\dots,[\mu_n])$$

where $\mu_i = r[\mu_i]+\langle\mu_i\rangle$ *and* $Q_{\eta_1,\dots,\eta_n}$ *are some polynomials of degree* $3g-3+n$ *whose coefficients depend on* $\eta_1,\dots,\eta_n\in\{0,\dots,r-1\}$*.*

Thus the polynomiality property of strictly monotone orbifold Hurwitz numbers is also equivalent to the property that the n-point functions can be represented in a very particular way (given by equation (6.44)) on the corresponding spectral curve, cf. [10, conjecture 12].

Note that [10] has a binomial $\binom{\mu_i-1}{[\mu_i-1]}$, which is equal to ours unless $\langle\mu_i\rangle=0$. In that case it differs by a factor $r-1$, which can be absorbed in the polynomial.

6.8.3 Usual orbifold Hurwitz numbers

The spectral curve topological recursion for the usual orbifold Hurwitz numbers is proved in [9, 2], see also [15, 34]. The corresponding spectral curve is given by the formula $x=\log z - z^r$, and the computations for this curves are also performed in [39] in relation to a different combinatorial problem. From these papers it does follow that the n-point function is represented as an expansion of the following form:

$$\sum_{\vec\mu\in(\mathbb{N}^\times)^n} h^{\circ,r}_{g;\vec\mu}\prod_{i=1}^n e^{\mu_i x^i} = \sum_{0\le\alpha_1,\dots,\alpha_n\le r-1} P_{\vec\alpha}\left(\frac{d}{dx_1},\dots,\frac{d}{dx_n}\right)\prod_{i=1}^n \xi_{\alpha_i}(x_i) \qquad (6.45)$$

It also follows from these papers that the good basis of functions ξ_i is given by

$$\xi_i = \frac{d}{dx}\left(\frac{z^{i+1}}{i+1}\right) = \frac{z^i}{1-rz^r}, \qquad i=0,\dots,r-1$$

and the expansions of these functions in e^x near $e^x=0$ is given by

$$\xi_i(x) = \sum_{\substack{\mu=0\\ r|\mu-i}}^{\infty} \frac{\mu^{[\mu]}}{[\mu]!}e^{\mu x}, \qquad i=0,\dots,r-1$$

For these functions the differentiation with respect to x is the same as the multiplication by the corresponding degree of e^x, so the following statement is obvious:

Proposition 6.8.5. *The coefficient of* $e^{\mu_1 x_1} \cdots e^{\mu_n x_n}$ *of the expansion in* $e^{x_1}, \dots, e^{x_n}$ *near zero of an expression of the form*

$$\sum_{0 \le k_1, \dots, k_n \le r-1} P_{k_1,\dots,k_n}\left(\frac{d}{dx_1}, \dots, \frac{d}{dx_n}\right) \prod_{i=1}^{n} \xi_{k_i}$$

where $P_{k_1,\dots,k_n}$ *are polynomials of degree* $3g-3+n$ *and* ξ_k *is equal to* $\frac{d}{dx}\left(\frac{z^{k+1}}{k+1}\right)$*, is represented as*

$$\prod_{i=1}^{n} \frac{\mu_i^{[\mu_i]}}{[\mu_i]!} \cdot Q_{\langle \mu_i \rangle, \dots, \langle \mu_n \rangle}([\mu_1], \dots, [\mu_n])$$

where $\mu_i = r[\mu_i] + \langle \mu_i \rangle$ *and* $Q_{\eta_1,\dots,\eta_n}$ *are some polynomials of degree* $3g-3+n$ *whose coefficients depend on* $\eta_1, \dots, \eta_n \in \{0, \dots, r-1\}$.

Thus the polynomiality property of usual orbifold Hurwitz numbers is also equivalent to the property that the n-point functions can be represented in a very particular way (given by equation (6.45)) on the corresponding spectral curve.

6.9 Computations for unstable correlation function for monotone Hurwitz numbers

In this section we prove that the unstable correlation differentials for the conjectural (or proved) CEO topological recursion spectral curve coincide with the expression derived from the $\mathcal{A}$-operators. These computations are performed in the case of monotone orbifold Hurwitz numbers for the cases $(g, n) = (0, 1)$ and $(g, n) = (0, 2)$, and for strictly monotone orbifold Hurwitz numbers for the case $(g, n) = (0, 1)$.

Note that in both cases the computation of the $(0, 1)$-numbers was done before, see [8, 10, 5, 17]. The $(0, 2)$-calculation for the monotone Hurwitz numbers is a new result, but we learned after completing our calculation that Karev obtained the same formula independently [29].

We show these computations here to test the $\mathcal{A}$-operator formula and to demonstrate its power. The computation of the generating function for the $(0, 2)$ monotone orbifold Hurwitz numbers is necessary for the conjecture on topological recursion in [8].

6.9.1 The case $(g, n) = (0, 1)$

In this section we check that the spectral curve reproduces the correlation differential for $(g, n) = (0, 1)$ obtained from the $\mathcal{A}$-operators of section 6.4.

The monotone case

Since in the case of $n = 1$ there is no difference between connected and disconnected Hurwitz numbers, the $(0, 1)$-free energy for monotone Hurwitz numbers reads:

$$F_{0,1}^{\le}(x) := \sum_{\mu=1}^{\infty} [u^{-1+d/r}] H^{\bullet, r, \le}(u, \mu) x^{\mu}$$

Of course, in this formula only $\mu = [\mu]r$, $[\mu] \geq 0$, can contribute non-trivially. Let us compute what we get. We have:

$$\begin{aligned} &[u^{-1+d/r}]H^{\bullet,r,\leq}(u,\mu) = \frac{(\mu+[\mu])!}{\mu![\mu]!}[u^{-1}]\left\langle \mathcal{A}^h_{\langle\mu\rangle}(u,\mu)\right\rangle \\ &= \frac{(\mu+[\mu])!}{\mu![\mu]!}\cdot\frac{(\mu+[\mu]+1)_{-2}}{([\mu]+1)_0}\cdot[z^{-1}]\mathcal{S}(z)^{\mu-1}\mathcal{S}(rz)^{0+[\mu]}\left\langle\mathcal{E}_0(z)\right\rangle \\ &= \frac{(\mu+[\mu])!}{\mu![\mu]!}\frac{1}{(\mu+[\mu])(\mu+[\mu]-1)} \end{aligned}$$

(here we used in the second line equation (6.16), where t and v deliberately must be equal to 0 and -1 respectively).

Thus we have (replacing μ by $r[\mu]$ everywhere):

$$F^{\leq}_{0,1} = \sum_{[\mu]=1}^{\infty}\frac{(r[\mu]+[\mu]-2)!}{(r[\mu])![\mu]!}x^{r[\mu]}$$

Theorem 6.9.1. *We have:* $\omega^{\leq}_{0,1} \coloneqq dF^{\leq}_{0,1} = -ydx$.

Proof. The spectral curve gives $y = -z^r/x$. In lemma 6.8.1 we have shown that

$$z^i = \sum_{k=0}^{\infty}\frac{(kr+k+i-1)!}{k!(kr+i)!}ix^{kr+i} = \sum_{k=0}^{\infty}\frac{(kr+k+i-1)!}{(k+1)!(kr+i-1)!}\frac{(ki+i)}{(kr+i)}x^{kr+i} \tag{6.46}$$

So,

$$\begin{aligned} -ydx &= \sum_{j=0}^{\infty}\frac{(kr+k+r-1)!}{(k+1)!(kr+r-1)!}x^{kr+r-1}dx \\ &= \sum_{k+1=1}^{\infty}\frac{((k+1)r+(k+1)-2)!}{(k+1)!((k+1)r-1)!}x^{(k+1)r-1}dx = dF^{\leq}_{0,1} \end{aligned}$$

(for the last equality we just identify $[\mu]$ with $k+1$). □

The strictly monotone case

Similarly, for strictly monotone Hurwitz numbers the $(0,1)$-free energy reads:

$$F^{<}_{0,1}(x) \coloneqq \sum_{\mu=1}^{\infty}[u^{-1+d/r}]H^{\bullet,r,\leq}(u,\mu)x^{-\mu} - \log(x)$$

Again, only $\mu = [\mu]r$, $[\mu] \geq 0$ can contribute non-trivially. We have:

$$\begin{aligned} &[u^{-1+d/r}]H^{\bullet,r,\leq}(u,\mu) = \frac{(\mu-1)!}{(\mu-[\mu]-1)![\mu]!}[u^{-1}]\left\langle\mathcal{A}_{\langle\mu\rangle}(u,\mu)\right\rangle \\ &= \frac{(\mu-1)!}{(\mu-[\mu]-1)![\mu]!}(\mu-[\mu]+2)_{-2} \\ &= \frac{(\mu-1)!}{(\mu-[\mu]+1)![\mu]!} \end{aligned}$$

(here we used in the second line equation (6.17), where t and v deliberately must be equal to 0 and -1 respectively). Thus we have (replacing μ by $r[\mu]$ everywhere):

$$dF_{0,1}^{<} = -\frac{1}{x}\sum_{[\mu]=1}^{\infty}\frac{(r[\mu])!}{([\mu]r-[\mu]+1)![\mu]!}x^{-r[\mu]}dx - \frac{dx}{x} \tag{6.47}$$

Theorem 6.9.2. *We have:* $\omega_{0,1}^{<} := dF_{0,1}^{<} = ydx$.

Proof. The spectral curve reads $x = z^{r-1} + z^{-1}$ and $y = z$. Let us expand $z = \sum_{n=0}^{\infty} a_n x^n$ and compute the coefficients by

$$a_n = \oint z\frac{dx}{x^{n+1}} = -\oint [1-(r-1)z^r]z^n \sum_{j=0}\binom{n+j}{j}(-z^r)^j dz$$

This residue is nontrivial only for $n = -rj-1$, $j \leq 0$, hence we should extract in the two summands the j-th and the $(j-1)$-st term respectively. Therefore, the residue reads

$$\begin{aligned}&(-1)^{j-1}\left[\binom{-rj-1+j}{j} + (r-1)\binom{-rj-1+j-1}{j-1}\right]\\ =&(-1)^j\binom{-rj-1+j}{j}\frac{1}{(-rj+j-1)}\end{aligned}$$

Hence

$$\begin{aligned}ydx = zdx &= \sum_{j=0}^{\infty}(-1)^j\binom{-rj-1+j}{j}\frac{1}{(-rj+j-1)}x^{-jr-1}dx\\ &= -\frac{1}{x}\sum_{j=0}^{\infty}(-1)^j\frac{(-rj)_j}{j!(rj-j+1)}x^{-jr}dx\\ &= -\frac{1}{x}\sum_{j=0}^{\infty}\frac{(rj)!}{j!(rj-j+1)!}x^{-jr}dx = dF_{0,1}^{<}\end{aligned}$$

where, in order to obtain the last line, we collected the minus signs from the Pochhammer symbol. For the last equality we identify $[\mu]$ with j and incorporate the term $[\mu] = 0$ inside the sum in formula (6.47). □

6.9.2 The case $(g, n) = (0, 2)$

In this section we use equation (6.18) in order to check whether the holomorphic part of the expansion of the unique genus zero Bergman kernel gives the differential $d_1d_2F_{0,2}^{\leq}$. More precisely, we prove the following theorem:

Theorem 6.9.3. *We have:*

$$\frac{dz_1dz_2}{(z_1-z_2)^2} = \frac{dx_1dx_2}{(x_1-x_2)^2} + d_1d_2F_{0,2}^{\leq}(x_1,x_2)$$

Proof. It is sufficient to prove that

$$\log(z_1 - z_2) = \log(x_1 - x_2) + F_{0,2}(x_1, x_2) + C_1(x_1) + C_2(x_2) \tag{6.48}$$

where C_1, C_2 are some functions of one variable.

We apply the Euler operator

$$E := x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}$$

to both sides of this formula. Using that $\partial_x = (1-(r+1)z^r)^{-1}\partial_z$, we observe that in the coordinates z_1, z_2 the Euler operator has the form

$$E := \frac{1 - z_1^r}{1-(r+1)z_1^r} \cdot z_1 \frac{\partial}{\partial z_1} + \frac{1 - z_2^r}{1-(r+1)z_2^r} \cdot z_2 \frac{\partial}{\partial z_2}$$

We have:

$$\begin{aligned}
E\log(z_1 - z_2) &= 1 + r \cdot \frac{z_1^r + z_1^{r-1} z_2 + \cdots + z_2^r (r+1) z_1^r z_2^r}{(1-(r+1)z_1^r)(1-(r+1)z_2^r)} \\
&= 1 + r \frac{\partial^2}{\partial x_1 \partial x_2} \left(\frac{z_1^{r+1} z_2}{(r+1)\cdot 1} + \frac{z_1^r z_2^2}{r \cdot 2} + \cdots + \frac{z_1 z_2^{r+1}}{1 \cdot (r+1)} - \frac{z_1^{r+1} z_2^{r+1}}{r+1} \right) \\
&= 1 + \frac{r}{r+1} \frac{\partial^2}{\partial x_1 \partial x_2} (z_1 z_2 - x_1 x_2) + r \frac{\partial^2}{\partial x_1 \partial x_2} \left(\frac{z_1^r z_2^2}{r \cdot 2} + \cdots + \frac{z_1^2 z_2^r}{2 \cdot r} \right)
\end{aligned}$$

Using equation (6.46), we finally obtain the following formula for $E\log(z_1 - z_2)$:

$$r \cdot \sum_{\substack{i_1, i_2 = 1 \\ i_1 + i_2 = r}}^{r-1} \sum_{k_1, k_2 = 0}^{\infty} \frac{(k_1 r + k_1 + i_1)!}{k_1!(k_1 r + i_1)!} \frac{(k_2 r + k_2 + i_2)!}{k_2!(k_2 r + i_2)!} x_1^{k_1 r + i_1} x_2^{k_2 r + i_2} \tag{6.49}$$

for the degrees of x_1, x_2 not divisible by r (Case I), and

$$\begin{aligned}
&\frac{1}{r+1} + \frac{r}{r+1} \sum_{k_1, k_1 = 0}^{\infty} \binom{k_1 r + k_1}{k_1} \binom{k_2 r + k_2}{k_2} x_1^{k_1 r} x_2^{k_2 r} \\
&= 1 + \frac{r}{r+1} \sum_{\substack{k_1, k_1 = 0 \\ (k_1, k_2) \neq (0,0)}}^{\infty} \binom{k_1 r + k_1}{k_1} \binom{k_2 r + k_2}{k_2} x_1^{k_1 r} x_2^{k_2 r}
\end{aligned} \tag{6.50}$$

if one of the exponents, and, therefore, both of them, are divisible by r (Case II).

Now we apply the Euler operator E to the right hand side of equation (6.48). We obtain the following expression:

$$1 + \tilde{C}_1(x_1) + \tilde{C}_2(x_2) + \sum_{\substack{\mu_1, \mu_2 \geq 1 \\ r | (\mu_1 + \mu_2)}} h^{\circ, r, \leq}_{0; (\mu_1, \mu_2)} x_1^{\mu_1} x_2^{\mu_2} (\mu_1 + \mu_2)$$

We have to prove that the sum of equations (6.49) and (6.50) is equal to this expression.

Let us compute $h^{\circ,r,\leq}_{0;(\mu_1,\mu_2)}$. Equation (6.18) implies that

$$h^{\circ,r,\leq}_{0;(\mu_1,\mu_2)} = \binom{\mu_1+[\mu_1]}{\mu_1}\binom{\mu_2+[\mu_2]}{\mu_2}\cdot\left\langle \mathcal{A}_{\langle\mu_1\rangle}(u,\mu_1)\mathcal{A}_{\langle\mu_2\rangle}(u,\mu_2)\right\rangle^\circ$$

Since we have to use connected correlators, it implies that in the $\mathcal{A}_{\langle\mu_1\rangle}$-operator we have to take only the operators $\mathcal{E}$ with the positive indices, and in the $\mathcal{A}_{\langle\mu_2\rangle}$-operator we have to take only the operators $\mathcal{E}$ with the negative indices. Specializing the formula further, and using that $[\zeta_1^0\zeta_2^0]\left\langle\mathcal{E}_v(\zeta_1)\mathcal{E}_{-v}(\zeta_2)\right\rangle^\circ = v$, we have:

$$h^{\circ,r,\leq}_{0;(\mu_1,\mu_2)} = \sum_{t=1}^{[\mu_2]+1}\frac{(\mu_1+[\mu_1]+t-1)!}{\mu_1!([\mu_1]+t)!}(tr-\langle\mu_1\rangle)\frac{(\mu_2+[\mu_2]-t)!}{\mu_2!([\mu_2]+1-t)!}$$

in Case I, and

$$h^{\circ,r,\leq}_{0;(\mu_1,\mu_2)} = \sum_{t=1}^{[\mu_2]}\frac{(\mu_1+[\mu_1]+t-1)!}{\mu_1!([\mu_1]+t)!}\cdot tr\cdot\frac{(\mu_2+[\mu_2]-t-1)!}{\mu_2!([\mu_2]-t)!}$$

in Case II. Note that in Case II, we omit the contributions from the $t=0$ part, as it cancels the strictly diconnected correlator in the inclusion-exclusion formula.

So, in order to complete the proof of the theorem we have to show that

$$\begin{aligned}(\mu_1+\mu_2)&\sum_{t=1}^{[\mu_2]+1}\frac{(\mu_1+[\mu_1]+t-1)!}{\mu_1!([\mu_1]+t)!}(tr-\langle\mu_1\rangle)\frac{(\mu_2+[\mu_2]-t)!}{\mu_2!([\mu_2]+1-t)!}\\ &= r\cdot\binom{\mu_1+[\mu_1]}{\mu_1}\binom{\mu_2+[\mu_2]}{\mu_2}\end{aligned}\tag{6.51}$$

in Case I (cf. equation (6.49)) and

$$\begin{aligned}(\mu_1+\mu_2)&\sum_{t=1}^{[\mu_2]}\frac{(\mu_1+[\mu_1]+t-1)!}{\mu_1!([\mu_1]+t)!}\cdot t\cdot\frac{(\mu_2+[\mu_2]-t-1)!}{\mu_2!([\mu_2]-t)!}\\ &= \frac{1}{r+1}\cdot\binom{\mu_1+[\mu_1]}{\mu_1}\binom{\mu_2+[\mu_2]}{\mu_2}\end{aligned}\tag{6.52}$$

in Case II.

Let us show this for Case I first. Observe that $tr-\langle\mu_1\rangle = ([\mu_1]+t)r-\mu_1$ and $\mu_1+\mu_2 = ([\mu_1]+[\mu_2]+1)r$, so we can rewrite the left hand side of equation (6.51) as

$$\begin{aligned}r\cdot(\mu_1+\mu_2)\cdot&\sum_{t=1}^{[\mu_2]+1}\frac{(\mu_1+[\mu_1]+t-1)!}{\mu_1!([\mu_1]+t-1)!}\frac{(\mu_2+[\mu_2]-t)!}{\mu_2!([\mu_2]+1-t)!}\\ -r\cdot([\mu_1]+[\mu_2]+1)\cdot&\sum_{t=1}^{[\mu_2]+1}\frac{(\mu_1+[\mu_1]+t-1)!}{(\mu_1-1)!([\mu_1]+t)!}\frac{(\mu_2+[\mu_2]-t)!}{\mu_2!([\mu_2]+1-t)!}\end{aligned}$$

Let us omit the factor r since we have it in the right hand side of equation (6.51). Let us multiply the first summand by μ_1 and the second summand by $([\mu_1]+t)$. We get identical sums with the opposite signs. So, this expression divided by r is equal to

$$\begin{aligned} &\sum_{t=1}^{[\mu_2]+1} \frac{(\mu_1+[\mu_1]+t-1)!}{\mu_1!([\mu_1]+t-1)!} \frac{(\mu_2+[\mu_2]-t)!}{(\mu_2-1)!([\mu_2]+1-t)!} \\ &- \sum_{t=1}^{[\mu_2]} \frac{(\mu_1+[\mu_1]+t-1)!}{(\mu_1-1)!([\mu_1]+t)!} \frac{(\mu_2+[\mu_2]-t)!}{\mu_2!([\mu_2]-t)!} \\ &=: \sum_{t=1}^{[\mu_2]+1} A_t - \sum_{t=1}^{[\mu_2]} B_t \end{aligned}$$

We can reshuffle the summands in this expression in the following way:

$$A_{[\mu_2]+1} - B_{[\mu_2]} + A_{[\mu_2]} - B_{[\mu_2]-1} + \cdots + A_2 - B_1 + A_1$$

Now we add up term by term, starting at the left. First we get

$$\begin{aligned} A_{[\mu_2]+1} - B_{[\mu_2]} &= \binom{\mu_1+[\mu_1]+[\mu_2]}{\mu_1}\binom{\mu_2}{\mu_2} - \binom{\mu_1+[\mu_1]+[\mu_2]-1}{\mu_1-1}\binom{\mu_2}{\mu_2} \\ &= \binom{\mu_1+[\mu_1]+[\mu_2]-1}{\mu_1}\binom{\mu_2}{\mu_2} \end{aligned}$$

Iterating this, get get the following sequence of expressions:

$$\begin{aligned} &A_{[\mu_2]+1} - B_{[\mu_2]} + A_{[\mu_2]} \\ &\quad = \binom{\mu_1+[\mu_1]+[\mu_2]-1}{\mu_1}\binom{\mu_2}{\mu_2} + \binom{\mu_1+[\mu_1]+[\mu_2]-1}{\mu_1}\binom{\mu_2}{\mu_2-1} \\ &\quad = \binom{\mu_1+[\mu_1]+[\mu_2]-1}{\mu_1}\binom{\mu_2+1}{\mu_2} \\ &A_{[\mu_2]+1} - B_{[\mu_2]} + A_{[\mu_2]} - B_{[\mu_2]-1} \\ &\quad = \binom{\mu_1+[\mu_1]+[\mu_2]-1}{\mu_1}\binom{\mu_2+1}{\mu_2} - \binom{\mu_1+[\mu_1]+[\mu_2]-2}{\mu_1-1}\binom{\mu_2+1}{\mu_2} \\ &\quad = \binom{\mu_1+[\mu_1]+[\mu_2]-2}{\mu_1}\binom{\mu_2+1}{\mu_2} \end{aligned}$$

eventually ending up at

$$A_{[\mu_2]+1} - B_{[\mu_2]} + \cdots + A_1 = \binom{\mu_1+[\mu_1]}{\mu_1}\binom{\mu_2+[\mu_2]}{\mu_2}$$

which gives us equation (6.51).

In Case II, the computation is similar. Observe that $t = ([\mu_1]+t) - \mu_1/r$ and $(\mu_1+\mu_2)/r = [\mu_1]+[\mu_2]$, so we can rewrite the left hand side of equation (6.52) in the following way:

$$\begin{aligned} &(\mu_1+\mu_2) \cdot \sum_{t=1}^{[\mu_2]} \frac{(\mu_1+[\mu_1]+t-1)!}{\mu_1!([\mu_1]+t-1)!} \frac{(\mu_2+[\mu_2]-t-1)!}{\mu_2!([\mu_2]-t)!} \\ &- ([\mu_1]+[\mu_2]) \cdot \sum_{t=1}^{[\mu_2]} \frac{(\mu_1+[\mu_1]+t-1)!}{(\mu_1-1)!([\mu_1]+t)!} \frac{(\mu_2+[\mu_2]-t-1)!}{\mu_2!([\mu_2]-t)!} \end{aligned}$$

Again, if we multiply the first summand by μ_1 and the second summand by $([\mu_1]+t)$, this yields identical sums with opposite signs. Cancelling these terms, we get that this expression is equal to

$$\begin{aligned}
&\sum_{t=1}^{[\mu_2]} \binom{\mu_1+[\mu_1]+t-1}{\mu_1}\binom{\mu_2+[\mu_2]-t-1}{\mu_2-1} \\
&-\sum_{t=1}^{[\mu_2]-1} \binom{\mu_1+[\mu_1]+t-1}{\mu_1-1}\binom{\mu_2+[\mu_2]-t-1}{\mu_2} \\
&=: \sum_{t=1}^{[\mu_2]} A'_t - \sum_{t=1}^{[\mu_2]-1} B'_t
\end{aligned}$$

Reshuffling the summands in this expression in the same way as for Case I, we would now get

$$A'_{[\mu_2]} - B'_{[\mu_2]-1} + A'_{[\mu_2]-1} - B'_{[\mu_2]-2} + \cdots + A'_2 - B'_1 + A'_1$$

We will calculate this in the same way as before: we start at the right and at the next term one at a time. First we get

$$\begin{aligned}
A'_{[\mu_2]} - B'_{[\mu_2]-1} &= \binom{\mu_1+[\mu_1]+[\mu_2]-1}{\mu_1}\binom{\mu_2}{\mu_2} - \binom{\mu_1+[\mu_1]+[\mu_2]-2}{\mu_1-1}\binom{\mu_2}{\mu_2} \\
&= \binom{\mu_1+[\mu_1]+[\mu_2]-2}{\mu_1}\binom{\mu_2}{\mu_2}
\end{aligned}$$

Iterating this, the next few calculations give us the following result:

$$\begin{aligned}
&A'_{[\mu_2]} - B'_{[\mu_2]-1} + A'_{[\mu_2]-1} \\
&= \binom{\mu_1+[\mu_1]+[\mu_2]-2}{\mu_1}\binom{\mu_2}{\mu_2} + \binom{\mu_1+[\mu_1]+[\mu_2]-2}{\mu_1}\binom{\mu_2}{\mu_2-1} \\
&= \binom{\mu_1+[\mu_1]+[\mu_2]-2}{\mu_1}\binom{\mu_2+1}{\mu_2} \\
&A'_{[\mu_2]} - B'_{[\mu_2]-1} + A'_{[\mu_2]-1} - B'_{[\mu_2]-2} \\
&= \binom{\mu_1+[\mu_1]+[\mu_2]-2}{\mu_1}\binom{\mu_2+1}{\mu_2} - \binom{\mu_1+[\mu_1]+[\mu_2]-3}{\mu_1-1}\binom{\mu_2+1}{\mu_2} \\
&= \binom{\mu_1+[\mu_1]+[\mu_2]-3}{\mu_1}\binom{\mu_2+1}{\mu_2}
\end{aligned}$$

And finally we get the following result:

$$\begin{aligned}
A'_{[\mu_2]} - B'_{[\mu_2]-1} + \cdots + A'_1 &= \binom{\mu_1+[\mu_1]}{\mu_1}\binom{\mu_2+[\mu_2]-1}{\mu_2} \\
&= \frac{1}{r+1}\binom{\mu_1+[\mu_1]}{\mu_1}\binom{\mu_2+[\mu_2]}{\mu_2}
\end{aligned}$$

which gives us equation (6.52).

This way we prove equation (6.48) is satisfied up to the kernel of the Euler operator. Since neither the left hand side nor the right hand side of equation (6.48) contain the terms in the kernel of the Euler operator, we see that equation (6.48) is satisfied, and this completes the proof of the theorem. □

6.10 Computations for unstable correlation functions for orbifold spin Hurwitz numbers

In this section we prove that the unstable correlation differentials for the spectral curve

$$\begin{cases} X = e^x & = ze^{-z^{qr}} \\ y & = z^q \end{cases} \tag{6.53}$$

coincide with the expression derived from the $\mathcal{A}$-operators. The unstable $(0,1)$-energy was already derived in [35] using the semi-infinite wedge formalism, we derive it here again to test our $\mathcal{A}$-operators. The computation for the unstable $(0,2)$-energy is a new result and fixes the ambiguity for the coordinate z on the spectral curve.

6.10.1 The case $(g,n) = (0,1)$

In this section we check that the spectral curve reproduces the correlation differential for $(g,n) = (0,1)$ obtained from the $\mathcal{A}$-operators. Explicitly, we show:

$$dF_{0,1}^{q,r}(x) = y\,dx. \tag{6.54}$$

Clearly, when dealing with a single $\mathcal{A}$-operator inside the correlator, only the coefficient of the identity operator contributes, since $\langle E_{i,j}\rangle = 0$. Hence, by definition 6.7.4 and equation (6.31), we compute, using that connected and disconnected correlators are equal in this case:

$$\begin{aligned}
F_{0,1}^{q,r}(x) &:= \sum_{\mu=1}^{\infty} [u^{-1+\frac{\mu}{q}}].H^{\circ,q,r}(u,\mu)e^{x\mu} \\
&= \sum_{\mu=1}^{\infty} \frac{\mu^{[\mu]}}{[\mu]!}[u^{\frac{\mu}{q}-1}] \sum_{s=0}^{\infty} \frac{\delta_{\langle\mu\rangle_q,0}}{\mu} \frac{u^{r([\mu]+s)}\mu^s}{([\mu]+1)_s} \sum_{j=1}^{q} \frac{\Delta_q^{[\mu]_q-1}}{q^{[\mu]_q}[\mu]_q!} Q_\mu^r(l)^{[\mu]+s}\Big|_{l=\frac{1}{2}-j} e^{x\mu} \\
&= \sum_{m=1}^{\infty}\sum_{s=0}^{\infty}[u^{m-1}] \frac{u^{r([m]_r+s)}(mq)^{s+[m]_r-1}}{([m]_r+s)!} \sum_{j=1}^{q} \frac{\Delta_q^{m-1}}{q^m m!} Q_{mq}^r(l)^{[m]_r+s}\Big|_{l=\frac{1}{2}-j} e^{xmq} \\
&= \sum_{n=0}^{\infty} \frac{\big(q(nr+1)\big)^{n-1}}{n!} \sum_{j=1}^{q} \frac{\Delta_q^{nr}}{q^{nr+1}(rn+1)!} Q_{(nr+1)q}^r(l)^n\Big|_{l=\frac{1}{2}-j} e^{x(nr+1)q} \\
&= \sum_{n=0}^{\infty} \frac{\big(q(nr+1)\big)^{n-1}}{n!} \sum_{j=1}^{q} \frac{1}{q(rn+1)}\Big|_{l=\frac{1}{2}-j} e^{x(nr+1)q} \\
&= q\sum_{n=0}^{\infty} \frac{\big(q(nr+1)\big)^{n-2}}{n!} e^{x(nr+1)q},
\end{aligned}$$

where the third line follows by setting $\mu = mq$, the fourth line by setting $m = nr+1$ and $s = 0$, and the fifth line because $\frac{\Delta^d}{q^d d!}$ on a monic polynomial of degree d gives 1.

As shown in [35], we have:

$$dF_{0,1}^{q,r}(x) = \left(\frac{W(-qre^{xqr})}{-qr}\right)^{1/r} dx,$$

where W is the Lambert curve $W(z) := -\sum_{n=1}^{\infty} \frac{n^{n-1}}{n!}(-z)^n$. The properties of the Lambert curve (see [35] for details) imply that the spectral curve (6.53) does satisfy equation (6.54), which can be shown by explicitly computing $(ze^{-z^{qr}})^{qr} = e^{qrx}$.

6.10.2 The case $(g, n) = (0, 2)$

In this section we prove that the $(0, 2)$-correlation differential coincides with the usual Bergman kernel on the genus zero spectral curve.

Let us first compute the $(0, 2)$-energy from the $\mathcal{A}$-operators.

Lemma 6.10.1.

$$F_{0,2}^{q,r}(X_1, X_2) = \sum_{\substack{\mu_1,\mu_2=1 \\ qr|\mu_1+\mu_2 \\ qr|\mu_1}}^{\infty} \frac{\mu_1^{[\mu_1]}}{[\mu_1]!} \frac{\mu_2^{[\mu_2]}}{[\mu_2]!} \frac{X_1^{\mu_1} X_2^{\mu_2}}{(\mu_1 + \mu_2)} + qr \sum_{\substack{\mu_1,\mu_2=1 \\ qr|\mu_1+\mu_2 \\ qr\nmid\mu_1}}^{\infty} \frac{\mu_1^{[\mu_1]}}{[\mu_1]!} \frac{\mu_2^{[\mu_2]}}{[\mu_2]!} \frac{X_1^{\mu_1} X_2^{\mu_2}}{(\mu_1 + \mu_2)}$$

Proof. Let us write $\mu := \mu_1 + \mu_2$.

By definition 6.7.4, we have that

$$F_{0,2}^{q,r}(X_1, X_2) = \sum_{\mu_1,\mu_2=1}^{\infty} \frac{1}{\mu_1\mu_2} [u^{\frac{\mu}{q}}] \left\langle \tilde{\mathcal{A}}(u, \mu_1)\tilde{\mathcal{A}}(u, \mu_2) \right\rangle^{\circ} X^{\mu_1} X^{\mu_2}, \tag{6.55}$$

where

$$\tilde{\mathcal{A}}(u, \mu_i) = \sum_{l_i \in \mathbb{Z}+1/2} \sum_{s_i=0}^{\infty} \frac{(u^r \mu_i)^{s_i}}{s_i!} \sum_{t_i=0} \frac{\Delta_q^{t_i}}{q^{t_i} t_i!} Q_{\mu_i}^r (l_i)^{s_i} E_{l_i+\mu_i-qt_i, l_i}.$$

Note that the coefficient of the identity operator in $\tilde{\mathcal{A}}$ does not appear — indeed we are now interested in connected correlators and, in the case of 2-points correlators, we have the simple relation $\langle \mathcal{A}_1\mathcal{A}_2\rangle^\circ = \langle \mathcal{A}_1\mathcal{A}_2\rangle^\bullet - \langle \mathcal{A}_1\rangle\langle \mathcal{A}_2\rangle$. The contributions of the identity operators coincide precisely with the last summand.

Let us now make some observation about equation (6.55). Analysing the energy and the coefficient of u, we find

$$\mu = q(t_1 + t_2) = qr(s_1 + s_2) \qquad \text{and} \qquad \mu_2 > qt_2 \geq 0.$$

Moreover, the only term that can contribute in the correlator is the coefficient of the identity operator, produced by the commutation relation of E-operators described by formula (6.3). Hence we compute that $F_{0,2}^{q,r}(X_1, X_2)$ is equal to

$$\sum_{\mu_1,\mu_2=1}^{\infty} \sum_{s_1+s_2=\frac{\mu}{qr}} \sum_{\substack{t_1+t_2=\frac{\mu}{q} \\ 0\leq qt_2<\mu_2}} \sum_{l=1/2}^{\mu_2-qt_2-1/2} \frac{\mu_1^{s_1-1}\mu_2^{s_2-1}}{s_1!s_2!} \frac{\Delta_q^{t_1}}{q^{t_1}t_1!} Q_{\mu_1}^r(l)^{s_1} \frac{\Delta_q^{t_2}}{q^{t_2}t_2!} Q_{\mu_2}^r(l - \mu_2 + t_2)^{s_2} X_1^{\mu_1} X_2^{\mu_2}.$$

Let us now observe that the sum of the degrees of the two difference operators equals the sum of the degrees of the polynomials to which they are applied. By lemma 6.7.16, whenever the power of the difference operator is greater than the degree of the polynomial, the result equals zero. Hence the only nonvanishing terms should satisfy $t_1 = rs_1$ and $t_2 = rs_2$. We proved that $F_{0,2}^{q,r}(X_1, X_2)$ equals

$$\sum_{\mu_1,\mu_2=1} \sum_{\substack{s_1,s_2=0\\ s_1+s_2=\mu/qr}} (\mu_2 - qrs_2)\frac{\mu_1^{s_1-1}\mu_2^{s_2-1}}{s_1!s_2!} X^{\mu_1}X^{\mu_2}\delta_{qrs_2<\mu_2}$$

We distinguish now two cases: the case in which the μ_i are divisible by qr and the case in which the remainders are non-zero.

Case $\mu_1 = qr\nu_1$

In this case $\mu_2 = qr\nu_2$ and the Kronecker delta gives $s_2 = 0, \dots, \nu_2 - 1$, which implies $s_1 = \nu_1 + 1, \dots, \nu_1 + \nu_2$. We split $(\mu_2 - qrs_2)$ in two terms, and remove the summand for $s_1 = \nu_1 + \nu_2$ from the sum. Writing s for s_1, we get that the coefficient of $X_1^{qr\nu_1}X_2^{qr\nu_2}$ is given by

$$(qr)^{\nu_1+\nu_2-1}\left[\sum_{s=\nu_1+1}^{\nu_1+\nu_2-1}\left(\frac{\nu_1^{s-1}\nu_2^{\nu_1+\nu_2-s}}{s!(\nu_1+\nu_2-s)!} - \frac{\nu_1^{s-1}\nu_2^{\nu_1+\nu_2-s-1}}{s!(\nu_1+\nu_2-s-1)!}\right) + \frac{\nu_1^{\nu_1+\nu_2-1}}{(\nu_1+\nu_2)!}\right].$$

Multiplying and dividing by $(\nu_1+\nu_2)!$ and collecting binomial coefficients we get

$$\frac{(qr)^{\nu_1+\nu_2-1}}{(\nu_1+\nu_2)!}\left[\sum_{s=\nu_1+1}^{\nu_1+\nu_2-1}\left(\binom{\nu_1+\nu_2}{s}\nu_1^{s-1}\nu_2^{\nu_1+\nu_2-s} - (\nu_1+\nu_2)\binom{\nu_1+\nu_2-1}{s}\nu_1^{s-1}\nu_2^{\nu_1+\nu_2-(s+1)}\right) + \nu_1^{\nu_1+\nu_2-1}\right].$$

Distributing the factor $(\nu_1 + \nu_2$ and simplifying binomial coefficients, we get

$$\frac{(qr)^{\nu_1+\nu_2-1}}{(\nu_1+\nu_2)!}\left[\sum_{s=\nu_1+1}^{\nu_1+\nu_2-1}\left(\binom{\nu_1+\nu_2-1}{s-1}\nu_1^{s-1}\nu_2^{\nu_1+\nu_2-s} - \binom{\nu_1+\nu_2-1}{s}\nu_1^{s}\nu_2^{\nu_1+\nu_2-s-1}\right) + \nu_1^{\nu_1+\nu_2-1}\right].$$

This is a telescoping sum, of which the only surviving term is

$$\frac{(qr)^{\nu_1+\nu_2}}{qr(\nu_1+\nu_2)}\frac{\nu_1^{\nu_1}}{\nu_1!}\frac{\nu_2^{\nu_2-1}}{(\nu_2-1)!} = \frac{1}{\mu_1+\mu_2}\frac{\mu_1^{[\mu_1]}}{[\mu_1]!}\frac{\mu_2^{[\mu_2]}}{[\mu_2]!}.$$

Case $\mu_1 = qr\nu_1 + i$**, with** $0 < i < qr$**.**

In this case $\mu_2 = qr\nu_2 + (qr - i)$ and the Kronecker delta gives $s_2 = 0, \dots, \nu_2$, which implies $s_1 = \nu_1 + 1, \dots, \nu_1 + \nu_2 + 1$. We split $(\mu_2 - qrs_2)$ in two terms, and remove the summand for $s_1 = \nu_1 + \nu_2 + 1$ from the sum. Writing s for s_1, the coefficient of $X_1^{\mu_1}X_2^{\mu_2}$ equals

$$\sum_{s=\nu_1+1}^{\nu_1+\nu_2}\left[\frac{\mu_1^{s-1}}{s!}\frac{\mu_2^{\nu_1+\nu_2-s+1}}{(\nu_1+\nu_2-s+1)!} - qr\frac{\mu_1^{s-1}}{s!}\frac{\mu_2^{\nu_1+\nu_2-s}}{(\nu_1+\nu_2-s)!}\right] + \frac{\mu_1^{\nu_1+\nu_2}}{(\nu_1+\nu_2+1)!}.$$

The rest of the proof is completely analogous to the first case. The only remaining term is

$$\frac{qr}{\mu_1+\mu_2}\frac{\mu_1^{[\mu_1]}}{[\mu_1]!}\frac{\mu_2^{[\mu_2]}}{[\mu_2]!}.$$

Summing up the first case and the second case for $i = 1, \ldots, qr - 1$ yields the statement. This concludes the proof of the lemma. $\square$

We are now armed to prove the main result of this section.

Theorem 6.10.2.

$$\frac{dz_1 dz_2}{(z_1 - z_2)^2} = \frac{dX_1 dX_2}{(X_1 - X_2)^2} + d_1 d_2 F_{0,2}^{q,r}(X_1, X_2)$$

Proof. It is enough to show that the Euler operator

$$E := X_1 \frac{d}{dX_1} + X_2 \frac{d}{dX_2} = \frac{z_1}{1 - qr z_1^{qr}} \frac{d}{dz_1} + \frac{z_2}{1 - qr z_2^{qr}} \frac{d}{dz_2}$$

applied to both sides of

$$\log(z_1 - z_2) = \log(X_1 - X_2) + F_{0,2}^{q,r}(X_1, X_2)$$

gives equal expressions up to at most functions of a single variable $X_1 C(X_1)$ and $X_2 C(X_2)$. Let us compute the left hand side first:

$$\begin{aligned}
&E \log(z_1 - z_2) = \\
&= \left(\frac{z_1}{1 - qr z_1^{qr}} - \frac{z_2}{1 - qr z_2^{qr}} \right) \frac{1}{z_1 - z_2} \\
&= 1 + \frac{1}{(1 - qr z_1^{qr})(1 - qr z_2^{qr})} \left(qr(z_1^{qr} + z_1^{qr-1} z_2 + \cdots + z_1^{qr}) - (qr)^2 z_1^{qr} z_2^{qr} \right) \\
&= 1 + \frac{d}{dx_1} \frac{d}{dx_2} \left(qr \left(\frac{z_1^{qr} \log(z_2)}{qr} + \frac{z_1^{qr-1} z_2}{qr - 1} + \frac{z_1^{qr-2} z_2^2}{2(qr-2)} + \cdots + \frac{\log(z_1) z_2^{qr}}{qr} \right) - z_1^{qr} z_2^{qr} \right) \\
&= 1 + \frac{d^2}{dx_1 dx_2} \left(z_1^{qr} x_2 + x_1 z_2^{qr} + qr \left(\frac{z_1^{qr-1} z_2}{qr - 1} + \frac{z_1^{qr-2} z_2^2}{2(qr-2)} + \cdots + \frac{z_1 z_2^{qr-1}}{qr - 1} \right) + z_1^{qr} z_2^{qr} \right) \\
&= 1 + \sum_{\substack{k \geq 1 \\ qr | k}} \frac{k^{[k]}}{[k]!} X_1^k + \sum_{\substack{l \geq 1 \\ qr | l}} \frac{l^{[l]}}{[l]!} X_2^l + qr \sum_{\substack{\mu_1, \mu_2 \\ qr | \mu_1 + \mu_2 \\ qr \nmid \mu_1}}^{\infty} \frac{\mu_1^{[\mu_1]}}{[\mu_1]!} \frac{\mu_2^{[\mu_2]}}{[\mu_2]!} X_1^{\mu_1} X_2^{\mu_2} + \sum_{\substack{\mu_1, \mu_2 \\ qr | \mu_1 + \mu_2 \\ qr | \mu_1}}^{\infty} \frac{\mu_1^{[\mu_1]}}{[\mu_1]!} \frac{\mu_2^{[\mu_2]}}{[\mu_2]!} X_1^{\mu_1} X_2^{\mu_2}
\end{aligned}$$

where in the last equality we used the fact

$$\frac{d}{dx} \left(\frac{z^i}{i} \right) = \sum_{\mu : qr | \mu - i}^{\infty} \frac{\mu^{[\mu]}}{[\mu]!} X^\mu \qquad \text{for } i = 1, \ldots, qr - 1,$$

$$\frac{d}{dx} z^{qr} = \sum_{\mu : qr | \mu}^{\infty} \frac{\mu^{[\mu]}}{[\mu]!} X^\mu,$$

which was proved in [39, Lemma 4.6]—substitute qr for r there. By lemma 6.10.1, the right hand side reads:

$$E\Big(\log(X_1 - X_2) + F^{q,r}_{0,2}(X_1,X_2)\Big) = \\ 1 + qr \sum_{\substack{\mu_1,\mu_2 \\ qr|\mu_1+\mu_2 \\ qr\nmid\mu_1 i}}^{\infty} \frac{\mu_1^{[\mu_1]}}{[\mu_1]!}\frac{\mu_2^{[\mu_2]}}{[\mu_2]!} X^{\mu_1}X^{\mu_2} + \sum_{\substack{\mu_1,\mu_2 \\ qr|\mu_1+\mu_2 \\ qr|\mu_1}}^{\infty} \frac{\mu_1^{[\mu_1]}}{[\mu_1]!}\frac{\mu_2^{[\mu_2]}}{[\mu_2]!} X^{\mu_1}X^{\mu_2}$$

This concludes the proof of the theorem. □

6.11 A generalization of Zvonkine's conjecture

In this section we use the result of Chapter 3 in order to give a precise formulation of the orbifold version of Zvonkine's r-ELSV formula.

Conjecture 6.11.1. We propose the following formula for the q-orbifold r-spin Hurwitz numbers:

$$h^{\circ,q,r}_{g,\mu_1,\dots,\mu_n} = \int_{\overline{\mathcal{M}}_{g,n}} \frac{\mathrm{C}_{g,n}\left(rq,q;qr - qr\left\langle\frac{\mu_1}{qr}\right\rangle,\dots,qr-qr\left\langle\frac{\mu_n}{qr}\right\rangle\right)}{\prod_{j=1}^n(1-\frac{\mu_i}{qr}\psi_i)} \\ \times r^{2g-2+n}(qr)^{\frac{(2g-2+n)q+\sum_{j=1}^n\mu_j}{qr}} \times \prod_{j=1}^n \frac{\left(\frac{\mu_j}{qr}\right)^{\left\lfloor\frac{\mu_j}{qr}\right\rfloor}}{\left\lfloor\frac{\mu_j}{qr}\right\rfloor!}.$$

Here the class $\mathrm{C}_{g,n}\left(rq,q;qr-qr\left\langle\frac{\mu_1}{qr}\right\rangle,\dots,qr-qr\left\langle\frac{\mu_n}{qr}\right\rangle\right)$ is the Chiodo class [6]. We use the same notation as in [34], and we recall briefly its definition following the exposition there.

Let $\overline{\mathcal{M}}^{qr,r}_{g,n}$ be the space of qr-th roots $S^{\otimes qr} \cong \omega^{\otimes q}_{\log}\left(\sum_{i=1}^n\left(qr\left\langle\frac{\mu_i}{qr}\right\rangle - qr\right)x_i\right)$, with $\omega_{\log} := \omega(\sum_{i=1}^n x_i)$, on the curves $(C,x_1,\dots,x_n)\in\overline{\mathcal{M}}_{g,n}$. Note that the degree of the sheaf

$$\omega^{\otimes q}_{\log}\left(\sum_{i=1}^n\left(qr\left\langle\frac{\mu_i}{qr}\right\rangle - qr\right)x_i\right)$$

is equal to $q(2g-2+n) + qr\sum_{i=1}^n\left\langle\frac{\mu_n}{qr}\right\rangle - nqr$ and is divisible by qr (this follows from the Riemann-Hurwitz formula, that is, from the fact that b given by equation (6.30) is integer).

We denote by $\pi\colon \mathcal{C}\to\overline{\mathcal{M}}^{qr,r}_{g,n}$ the universal curve over $\overline{\mathcal{M}}^{qr,r}_{g,n}$, with universal qr-th root line bundle $\mathcal{S}\to\mathcal{C}$, and by $\epsilon\colon\overline{\mathcal{M}}^{qr,r}_{g,n}\to\overline{\mathcal{M}}_{g,n}$ the projection to the moduli space of curves. We define

$$\mathrm{C}_{g,n}\left(rq,q;qr-qr\left\langle\frac{\mu_1}{qr}\right\rangle,\dots,qr-qr\left\langle\frac{\mu_n}{qr}\right\rangle\right) := \epsilon_*\left(c\left(R^1\pi_*\mathcal{S}\right)/c\left(R^0\pi_*\mathcal{S}\right)\right).$$

This definition can be made very explicit, namely, there is an expression of the Chiodo classes in tautological classes via the Givental graphs. We refer to Chapter 3 for further explanation on Chiodo classes.

In the special case $q = 1$ this conjecture is reduced to Zvonkine's 2006 conjecture [42]. In the case $r = 1$ it is proved in [34] that this conjecture is equivalent to the Johnson-Pandharipande-Tseng formula first derived in [28]. In the case $q = r = 1$ this conjecture reduces to the ELSV formula first derived in [18].

Bibliography

[1] A. Alexandrov, D. Lewanski, S. Shadrin. Ramifications of Hurwitz theory, KP integrability and quantum curves, *Journal of High Energy Physics*, 10.1007/JHEP05(2016)124

[2] V. Bouchard, D. Hernàndez Serrano, X. Liu, M. Mulase, Mirror symmetry for orbifold Hurwitz numbers, *J. Differ. Geom.* 98 (2014), 375–423.

[3] M. Bousquet-Mélou and G. Schaeffer. Enumeration of planar constellations. *Adv. in Appl. Math.*, 24(4):337–368, 2000.

[4] C. Charalambides, Enumerative Combinatorics, *Chapman and Hall CRC*, 2002.

[5] L. Chekhov, B. Eynard, N. Orantin, Free energy topological expansion for the 2-matrix model, *Journal of High Energy Physics*, JHEP 0612:053, 2006.

[6] A. Chiodo. Towards an enumerative geometry of the moduli space of twisted curves and rth roots. Compos. Math., 144(6):14611496, 2008.

[7] N. Do, A. Dyer, D. Mathews, Topological recursion and a quantum curve for monotone Hurwitz numbers, arXiv: 1408.3992.

[8] N. Do, M. Karev, Monotone orbifold Hurwitz numbers, *Combinatorics and graph theory.* Part V, Zap. Nauchn. Sem. POMI, 446, POMI, St. Petersburg, 2016, 40–69.

[9] N. Do, O. Leigh, P. Norbury, Orbifold Hurwitz numbers and Eynard-Orantin invariants, arXiv: 1212.6850

[10] N. Do, D. Manescu, Quantum curves for the enumeration of ribbon graphs and hypermaps, *Commun. Number Theory Phys.*, 8(4):677–701, 2014.

[11] O. Dumitrescu, M. Mulase, Edge-contraction on dual ribbon graphs, 2D TQFT, and the mirror of orbifold Hurwitz numbers. 2015. arXiv:1508.05922.

[12] O. Dumitrescu, M. Mulase, B. Safnuk, A. Sorkin, The spectral curve of the Eynard-Orantin recursion via the Laplace transform, *Contemp. Math.*, 593 (2013) 263, arXiv: 1202.1159.

[13] P. Dunin-Barkowski, M. Kazarian, N. Orantin, S. Shadrin, L. Spitz, Polynomiality of Hurwitz numbers, Bouchard-Mariño conjecture, and a new proof of the ELSV formula, *Advances in Mathematics*, 279, 67–103. arXiv:1307.4729.

[14] P. Dunin-Barkowski, N. Orantin, S. Shadrin, L. Spitz, Identification of the Givental formula with the spectral curve topological recursion procedure, *Communications in Mathematical Physics*, 328 (2), 669–700. arXiv: 1211.4021.

[15] P. Dunin-Barkowski, D. Lewanski, A. Popolitov, S. Shadrin, Polynomiality of orbifold Hurwitz numbers, spectral curve, and a new proof of the Johnson-Pandharipande-Tseng formula. *Journal of the London Mathematical Society.* Second Series, 92 (3), 547–565, 2015.

[16] P. Dunin-Barkowski, P. Norbury, N. Orantin, A. Popolitov, S. Shadrin, Dubrovin's superpotential as a global spectral curve, arXiv: 1509.06954

[17] P. Dunin-Barkowski, N. Orantin, A. Popolitov, S. Shadrin, Combinatorics of loop equations for branched covers of sphere, arXiv: 1412.1698

[18] T. Ekedahl, S. Lando, M. Shapiro and A. Vainshtein, Hurwitz numbers and intersections on moduli spaces of curves, *Invent. Math.* 146 (2001) 297, math.AG/0004096

[19] B. Eynard, Invariants of spectral curves and intersection theory of moduli spaces of complex curves, *Commun.Num.Theor.Phys.* 08 (2014) 541–588. arXiv:1110.2949.

[20] A. Giorgetti, T. R. S. Walsh, Enumeration of Hypermaps of a Given Genus, arXiv:1510.09019.

[21] I. Goulden, M. Guay-Paquet, and J. Novak. Monotone Hurwitz numbers and the HCIZ integral II. *ArXiv e-prints*, July 2011.

[22] I. Goulden, M. Guay-Paquet, and J. Novak. Polynomiality of monotone Hurwitz numbers in higher genera. *Adv. Math.*, 238:1–23, 2013.

[23] I. Goulden, M. Guay-Paquet, and J. Novak. Monotone Hurwitz numbers and the HCIZ integral. *Ann. Math. Blaise Pascal*, 21(1):71–89, 2014.

[24] J. Harnad. Weighted Hurwitz numbers and hypergeometric τ-functions: an overview. 2015. arXiv:1504.03408.

[25] J. Harnad and A. Orlov. Hypergeometric τ-functions, Hurwitz numbers and enumeration of paths. *Comm. Math. Phys.*, 338(1):267284, 2015.

[26] P. Johnson, Double Hurwitz numbers via the infinite wedge, *Transactions of the American Mathematical Society*, 367 (9), 6415–6440, 2015.

[27] P. Johnson, Equivariant Gromov-Witten theory of one dimensional stacks, arXiv:0903.1068v1.

[28] P. Johnson, R. Pandharipande, H.-H. Tseng Abelian Hurwitz-Hodge integrals, *Mich. Math. J.* 60 (2011), no. 1, 171–198.

[29] M. Karev, *Private communications.*

[30] M. Kazarian and P. Zograf. Rationality in map and hypermap by genus. 2016. arXiv:1609.05493v1.

[31] M. Kazarian and P. Zograf. Virasoro constraints and topological recursion for Grothendiecks dessin counting. *Lett. Math. Phys.*, 105(8):1057–1084, 2015.

[32] S. Kerov and G. Olshanski. Polynomial functions on the set of Young diagrams. C. R. Acad. Sci. Paris Sér. I Math., 319(2):121126, 1994.

[33] R. Kramer, D. Lewanski, and S. Shadrin. Quasi-polynomiality of monotone orbifold Hurwitz numbers and Grothendiecks dessins denfants. arXiv:1610.08376, 2016.

[34] D. Lewanski, A. Popolitov, S. Shadrin, D. Zvonkine, Chiodo formulae for the r-th roots and topological recursion, arXiv: 1504.07439.

[35] M. Mulase, S. Shadrin, L. Spitz, The spectral curve and the Schrdinger equation of double Hurwitz numbers and higher spin structures. 2013. *Communications in Number Theory and Physics 7*, no. 1, 125–143.

[36] I. MacDonald, Symmetric functions and Hall polynomials, *Oxford University Press*, 1998.

[37] A. Okounkov and R. Pandharipande, The equivariant Gromov-Witten theory of $\mathbb{P}^1$, arXiv: math/0207233.

[38] A. Okounkov and R. Pandharipande. Gromov-Witten theory, Hurwitz theory, and completed cycles. Ann. of Math. (2), 163(2):517560, 2006.

[39] S. Shadrin, L. Spitz, D. Zvonkine, Equivalence of ELSV and Bouchard-Mariño conjectures for r-spin Hurwitz numbers, *Math. Ann.* 361 (2015), no. 3-4, 611–645.

[40] P. Zinn-Justin. HCIZ integral and 2D Toda lattice hierarchy. *Nuclear Phys. B*, 634(3):417432, 2002.

[41] P. Zograf. Enumeration of Grothendiecks dessins and KP hierarchy, 2013. arXiv:1312.2538.

[42] . D. Zvonkine. A preliminary text on the r-ELSV formula. Preprint, 2006.

Summary

Summary

How many circles are tangent to three given circles on the plane in generic position? This famous problem formulated more than 2000 years ago is known as one of the very first problems in enumerative algebraic geometry. Classical problems in algebraic geometry include, for example, the count of algebraic curves of a fixed degree satisfying certain tangency conditions, and generally depend on several parameters. The solutions of such problems — the numbers — can grow very fast as functions of these parameters, and more structure is necessary to get a control on them. Enormous progresses on approaching these problems were made in the framework of Gromov-Witten theory. The definition of the theory relies on the concept of moduli space of curves and its generalisations.

The moduli space of curves is a key object of study in algebraic geometry. The structure of its cohomology ring, whose formal study was initiated by Mumford, is rich and mysterious. Geometrically important loci belong to a specific part of the cohomology, called tautological. Faber's three conjectures describe the structure of the tautological ring of the open moduli space of curves without marked points, later generalised for marked points by Buryak, Shadrin, and Zvonkine.

One of the main results of the thesis shows how the tautological relations derived by Pandharipande, Pixton and Zvonkine from the Givental decomposition of Witten's r-spin classes can be used to re-prove with different methods aspects of the tautological ring (vanishing in higher degrees, first generalised Faber conjecture), as well as to prove new results (estimations of the number of generators in lower degrees).

Statistical mechanics is interested in computing quantities defined combinatorially. This study involves the enumeration of combinatorial objects, such as graphs, tessellations, and triangulations or n-angulations of very many different kinds.

A general principle that permeates transversely enumeration problems in algebraic geometry and in statistical physics is the idea that complicated cases can be recovered by the simpler ones through a recursive formula. Knowing the base cases and the recursive formula provides then an algorithm to compute all the numbers. A second general principle suggests that the numbers can be conveniently packaged in generating series. The recursive formulae, hence, take the form of differential operators applied to such generating series.

The Chekhov-Eynard-Orantin (CEO) topological recursion is a new extremely powerful theory to deal with enumerative geometric problems arising from different areas of mathematics and physics, including algebraic geometry and statistical mechanics. It takes a spectral curve as input and, through a recursive procedure based on the pair of pants decomposition of surfaces, recovers an infinite collection of differentials defined on the product of the spectral curve whose coefficients are the numbers of some enumerative geometric problem.

The enumerative geometric problems generated by this recursion are fundamentally connected with the moduli spaces of curves. Two of the first examples of topological recursion that were derived provide new proofs of Witten's conjecture and Mirzakhani's recursion of volumes of moduli spaces of hyperbolic structures.

In order to prove that an enumerative geometric problem satisfies CEO topological recursion, a certain polynomial structure of the problem need to be understood. Roughly speaking, this polynomial structure is equivalent to state that the differentials are correctly defined on the spectral curve of the recursion. This thesis approaches the study of the polynomiality structure and provides an explicit answer for four different problems in Hurwitz theory, first adapting known methods, then providing new methods which turn out to be more powerful then the existing ones. In the case of monotone orbifold Hurwitz numbers and Grothendieck dessins d'enfant, the polynomiality structure confirms two conjectures by Do and Karev and by Do and Manescu. In the third problem, the r-orbifold Hurwitz numbers, the polynomiality structure it is used to provide the first proof of CEO topological recursion that does not dependent on Johnson-Pandharipande-Tseng ELSV formula. CEO topological recursion has been recently identified with Givental theory by Dunin-Barkowski, Orantin, Shadrin, and Spitz. By the use of Chiodo classes, the thesis provides the explicit identification between topological recursion and Givental theory for r-orbifold Hurwitz numbers. As a corollary, this result implies a new proof of Johnson-Pandharipande-Tseng ELSV formula. In the fourth case, the r-spin Hurwitz numbers, the polynomiality structure provides a key step towards the proof of a conjectural ELSV formula for Hurwitz numbers described by completed cycles. Completed cycles play a crucial role in the Okounkov and Pandharipande correspondence between Hurwitz theory and Gromov-Witten theory. This ELSV formula, hence, would provide an identification between certain Gromov-Witten correlators and the intersection theory of the moduli space of r-spin structures. In addition, the thesis provides a new ELSV formula, for the enumerative problem of monotone Hurwitz numbers.

Quantum curves can be thought as differential operators annihilating a certain generating series of the enumerative problem, and hence codify some important relations between the numbers. Surprisingly, quantum curves in many cases store all the informations needed. More explicitly, it is proved for many problems that the semi-classical limit of the quantum curve recovers the spectral curve which generates the entire list of numbers via CEO topological recursion. In this thesis several new quantum curves have been discovered (double Hurwitz numbers and double monotone Hurwitz numbers) or re-proved, with the use of the method of Kac-Schwarz operators. New operators on the Fock space have been constructed. These operators turned out to be key ingredients in the proof of polynomiality structure for Grothendieck dessins d'enfant and monotone Hurwitz numbers described above.

Samenvatting

Hoeveel cirkels zijn er die drie gegeven cirkels in een vlak raken? Deze meer dan tweeduizend jaar oude beroemde vraag is een van de allereerste problemen in de enumeratieve algebraïsche meetkunde. Klassieke problemen in de algebraïsche meetkunde zoals bijvoorbeeld het tellen van algebraïsche krommen van vaste graad die voldoen aan zekere raakvergelijkingen hangen in het algemeen af van een aantal parameters. De oplossingen voor deze problemen kunnen erg snel groeien en er is meer structuur nodig om controle te krijgen over deze getallen. Enorme vooruitgang in het onderzoek naar deze problemen heeft geleid tot gromov-wittentheorie. De definitie van deze theorie gebruikt het concept moduliruimte van krommen en generalisaties van deze ruimte.

De moduliruimte van krommen is een belangrijk object binnen het gebied van algebraïsche meetkunde. De structuur van zijn cohomologiering, als eerste formeel onderzocht door Mumford, is rijk en mysterieus. Belangrijke loci behoren tot het zogenaamde tautologische deel van de cohomologie. De drie vermoedens van Faber (1999) beschrijven de structuur van de tautologische ring van de open moduliruimte van krommen zonder gemarkeerde punten. Dit is later gegeneraliseerd tot krommen met gemarkeerde punten door Buryak, Shadrin en Zvonkine.

Één van de voornaamste resultaten uit deze scriptie laat zien hoe de tautologische relaties die afgeleid zijn van de giventaldecompositie van Wittens r-spinklassen door Pandharipande, Pixton en Zvonkine gebruikt kunnen worden om via een andere weg verschillende eigenschappen van de tautologische ring te bewijzen (verdwijnen in hogere graden, eerste gegeneraliseerde vermoeden van Faber). Ook worden met de tautologische relaties nieuwe resultaten gevonden (schatting van het aantal generatoren in lagere graad).

Het asymptotische gedrag van een aantal combinatorische modellen is een interessant onderwerp binnen statistische mechanica. Zo worden er bijvoorbeeld combinatorische objecten zoals grafen, betegelingen en triangulaties of n-angulaties van veel verschillende soorten geteld.

Een algemeen principe in telproblemen binnen algebraïsche meetkunde en statistische fysica is het idee dat gecompliceerde objecten herleid kunnen worden tot simpelere objecten via een recursieve formule. De basisobjecten en de recursieve formule geven een algoritme om alle oplossingen van het telprobleem te berekenen. Een tweede algemeen principe is dat een getallenrij gemakkelijk samengevat kan worden in een voortbrengende functie. De recursieve formules opereren dan als differentiaaloperatoren op die voortbrengende functies.

De topologische recursie van Chekhov, Eynard en Oratin is een recente en extreem krachtige theorie om meetkundige telproblemen binnen verschillende wis- en natuurkundige disciplines, zoals algebraïsche meetkunde en statistische mechanica, aan te pakken. De recursie neemt als invoer een spectraalkromme en geeft op basis van broekdecomposities een oneindige verzameling van differentialen gedefinieerd op het product van de spectraalkromme, waarvan de coëfficiënten een meetkundig telprobleem oplossen.

De telproblemen die gegenereerd worden door deze recursies zijn fundamenteel verbonden met de moduliruimte van krommen. Twee van de eerste voorbeelden van topologische recursies die zijn afgeleid met deze techniek geven een nieuw bewijs van het vermoeden van Witten en van Mirzakhani's recursie van volumen van moduliruimten van hyperbolische structuren.

Om te bewijzen dat een meetkundig telprobleem aan de topologische recursie voldoet, moet het probleem een zekere polynomiale structuur hebben. Simpel gezegd is deze polynomiale structuur equivalent aan de uitspraak dat alle differentialen correct gedefinieerd zijn op de spectraalkromme van de recursie. In deze scriptie wordt de polymiale structuur benaderd en wordt een expliciete oplossing gegeven voor vier verschillende problemen in hurwitztheorie, allereerst gebruik makende van bekende methoden en vervolgens met nieuwe methoden die krachtiger blijken te zijn dan de bestaande methoden.

In het geval van monotone orbifold-hurwitzgetallen en Grothendiecks dessins d'enfant bevestigt de polynomiale structuur twee vermoedens van Do en Karev en van Do en Manescu.

In het derde probleem, de r-orbifold-hurwitzgetallen, wordt de polynomiale structuur gebruikt om het eerste bewijs van de topologische recursie te geven dat niet gebaseerd is op Johnson-Pandharipande-Tsengs ELSV-formule. De topologische recursie is recentelijk geïdentificeerd met giventaltheorie door Dunin-Barkowski, Orantin, Shadrin en Spitz. Door gebruik te maken van chiodoklassen geeft deze scriptie een expliciete identificatie tussen topologische recursie en giventaltheorie voor r-orbifold-hurwitzgetallen. Als gevolg impliceert dit resultaat een nieuw bewijs voor Johnson-Pandharipande-Tsengs ELSV-formule.

In het vierde probleem, de r-spin-hurwitzgetallen, geeft de polynomiale structuur een sleutelstap tot het bewijzen van een vermoedde ELSV-formule voor hurwitzgetallen beschreven door gecompleteerde cykels. Gecompleteerde cykels spelen een cruciale rol in Okounkov en Pandharipandes correspondentie tussen hurwitztheorie en gromov-wittentheorie. Deze ELSV-formule zou dus een identificatie tussen bepaalde gromov-wittencorrelatiefuncties en de doorsnijdingstheorie van de moduliruimte van r-spin structuren geven. Daarnaast geeft deze scriptie ook een nieuwe ELSV-formule voor telproblemen van monotone hurwitzgetallen.

Kwantumkrommen kunnen worden beschouwd als differentiaaloperatoren die bepaalde voorbrengende functies van telproblemen annihileren en coderen daardoor een aantal belangrijke relaties tussen die getallen. Verrassend genoeg bevatten kwantumkrommen in veel gevallen alle informatie die nodig is. Explicieter betekent dit dat voor veel problemen bewezen is dat de semi-klassieke limiet van de kwantumkromme de spectraalkromme herleidt die op zijn beurt alle oplossingen via de topologische recursie genereert. In deze scriptie wordt een aantal nieuwe kwantumkrommen ontdekt (dubbele hurwitzgetallen en dubbele monotone hurwitzgetallen) of herbewezen met de methode van kac-schwarzoperatoren. Nieuwe operatoren op de fockruimte worden geconstrueerd. Deze operatoren blijken een sleutelrol te hebben in het bewijs van de polynomiale structuur voor Grothendiecks dessins d'enfant en monotone hurwitzgetallen zoals boven beschreven.

Acknowledgements

Sergey, your way of thinking and working had a substantial influence on my way to approach Mathematics. During these years you've been extremely present, energetic and generous to share and discuss your ideas. Moreover, your teamwork style provided us with tons of inputs and thoughts for discussions. I'm deeply grateful for each and every one of these things. Despite your being practical and trying to remind us what is essential in this job, I guess my writing will always remain ”too long and too poetic”, as probably this paragraph could appear.

I would like to thank KdVI, UvA, and NWO for the support, and The Netherlands for heartly welcoming me and many Italian PhD students, and making our life here so incredibly easy.

I would like to thank my collaborators Petya, Sasha, Reinier, Dimitri, Sasha, Todor, and Farrokh. I learned a lot from you, and I hope that we will share many more exciting projects in the future. In particular I thank the first three named coauthors, for the uncountably many discussions we shared, the fun both in Amsterdam and during conferences. I'm moreover grateful to Guido, Hessel and Raf for the discussions, the seminars and the time spent at the canteen.

I thank the scientific community to which the topics of this thesis are related. I am very grateful to the efforts and the energies of everyone who has been promoting this community and organising conferences related to the topological recursion. Moreover, I would like to thank in particular Alessandro, Anupam, Axel, Dima, Elba, Federico, Gaëtan, Marvin, Max, Norm, Paul, Priya, Raphaël, and Sjabbo, for sharing many moments and adventures during conferences, making them very joyful experiences. Thanks to all the members of this community for teaching me a lot of Mathematics and for contributing in the development of the topic.

I would like to thank the people that I met from the department of Amsterdam: Abhishek, Alice, Apo, Arie, Asma, Bart, Brendan, David, Dirk, George, Guusje, Jacobien, Jan, Jeannette, Joe, Kayed, Kirsten, Lance, Lorenzo, Masha, Marcelo, Mariska, Masoumeh, Nicos, Nikos, Peter, Sonja, Wessel, from Utrecht: Boris, Dana, Davide, Ioan, João, Ralph, Roy and from Leiden: Andrea, Chloe, Giulio, Marta, Mima, Rosa, and Yan. Some of you made me feel welcomed when I arrived. Others of you arrived later, carrying new enthusiasm. All of you contributed to make these years very enjoyable ones.

Thanks to Jarno and Jacob. You showed me how life can be easy, how dutch people can be gezellig and attached to their families, how bad dutch music can be. You trained my liver and my legs. You both remain simple ”Caprese” people and mediocre HoTS players. I owe you much!

Thanks to all the amazing people working for AIMS, AMI, and SAMI, promoting mathematical activities in Africa. We had a lot of fun at the Mathcamps and I hope we'll make many more together.

Even outside the departments of Mathematics is possible to meet beautiful people sometimes. Ale, Davide, Kleo, Minerva, and Sarah, thanks for the great times and for reminding me what real ”food” is from time to time. Yes, the majority of you are mathematicians, but I'll keep pretending that I met you for reasons unrelated to Maths. Moreover, i would like to thank Ergin, KaLe for being amazing flatmates, and the mates of BJJ and Judo for giving me strength (I mean before and after the back-pain).

A Lorenzo ”Com'è?” Ruffoni, Camilla ”Gianna” Felisetti, Alberto ”Bidé” Landuzzi, Jack ”Frenoamano” Ferrari, spero che il Coast to Coast to Coast sia il peggiore tra tutti i viaggi che faremo insieme. Siete incredibili compagni di corso e di vita.

Michele Pancera, questa storia dell'apicoltore vegano che ammazza alberi non se l'era bevuta nessuno, diciamocelo. Meno male che sei tornato ad un sano ipocrita vegetarianesimo. Non mi hai permesso di essere con te nei tuoi giorni bui, ma sei stato con me nei miei. La tua amicizia è molto preziosa, e mi ricorda sempre la canzone *"Dormire sotto le tende e sottovoce stare fermi a parlare..."*.

Grazie alla mia fantastica famiglia: Alberto, Anna, Antonella, Chiara, Elisa, Fabio, Federica, Federico, Francesca, Giulio, Leonardo, Marta, Maurizio, Michaela, Riccardo, e Stefano; e in particolare ai nonni Elvira, Lucia, Luciano, Ryszard[1].

Grazie soprattutto alla Mamma Anto, al Babbo Rudy e al Fratello Nico, per il vostro amore e supporto. Quello che mi avete insegnato fa di me ciò che sono. Non è venuto fuori un granché, ma non si può dire che non ci abbiate provato.

Jule, tu sei per me dove io mi sento a casa, e ciò che di più importante Amsterdam mi abbia fatto trovare.

[1] 1918 Varsavia — 1996 Bologna.