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Entanglement and Generalized Berry Geometrical Phases in Quantum Gravity

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Abstract: A new formalism is introduced that makes it possible to elucidate the physical and geometric content of quantum space–time. It is based on the Minimum Group Representation Principle (MGRP). Within this framework, new results for entanglement and geometrical/topological phases are found and implemented in cosmological and black hole space–times. Our main results here are as follows: (i) We find the Berry phases for inflation and for the cosmological perturbations and express them in terms of the observables, such as the spectral scalar and tensor indices, n_S and n_T, and the tensor-to-scalar ratio r. The Berry phase for de Sitter inflation is imaginary with the sign describing the exponential acceleration. (ii) The pure entangled states in the minimum group (metaplectic) $Mp(n)$ representation for quantum de Sitter space–time and black holes are found. (iii) For entanglement, the relation between the Schmidt type representation and the physical states of the $Mp(n)$ group is found: This is a *new non-diagonal* coherent state representation complementary to the known Sudarshan diagonal one. (iv) Mean value generators of $Mp(2)$ are related to the adiabatic invariant and topological charge of the space–time, (matrix element of the transition $-\infty < t < \infty$). (v) The basic *even* and *odd* n-sectors of the Hilbert space are intrinsic to the quantum space–time and its discrete levels (in particular, continuum for $n \to \infty$), they do not require any extrinsic generation process such as the standard Schrodinger cat states, and are *entangled*. (vi) The gravity or cosmological domains on one side and another of the Planck scale are *entangled*. Examples: The quantum primordial trans-Planckian de Sitter vacuum and the classical late de Sitter vacuum today; the central quantum gravity region and the external classical gravity region of black holes. The classical and quantum dual gravity regions of the space–time are entangled. (vii) The general classical-quantum gravity duality is associated with the Metaplectic $Mp(n)$ group symmetry which provides the complete full covering of the phase space and of the quantum space–time mapped from it.

Keywords: quantum physics; quantum information; quantum gravity; symmetry groups; trans-planckian physics

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1. Introduction and Results

In recent years, a deep interest has been manifested, not only in the search for a consistent theory of quantum gravity, but also in a fundamental description of the dynamics in the quantum domain.

Recently, in refs. [1,2] we have carried out a construction of the space–time as a generalized coherent state of the $Mp(n)$ group (complying with the principle of the minimum group representation). We have shown that physical states are mappings of the group generators in a vector representation, and that these are expressed in the so-called Sudarshan's diagonal-representation and in a *new nondiagonal* one that leads, as an important consequence, with the *physical states* with spin content $(1/2,\ 1,\ 3/2,\ 2)$.

We have implemented this framework both in black holes and de Sitter space–time, showing the quantum fundamental dynamics. This approach goes well in the direction of refs [3,4] for quantum space–time and its discrete levels, explicit examples and fundamental quantum principles: the classical-quantum duality of Nature including gravity (classical-quantum gravity duality), refs. [5–7].

Now, in this paper, let us take another step in this novel and consistent conceptual description of the universe considering entanglement mechanisms and their relationship with geometrical and topological phases (namely, generalized Berry phases). The groups involved here are non compact.

Our formalism is fundamental to advance towards a true information theory of quantum gravity given that the entire theory is self-consistent and has important points in favor. The basic states that respond to the principle of least group representation satisfy a *new equation of positive energy* conceptually similar to those proposed by Majorana [8,9] and Dirac [10]; they were discussed by Bogomolny in ref. [11] and related to the possible description of dark matter. Precisely, our *generalized coherent states* here generate a map that relates the metric g_{ab}, solution of the positive energy wave equation to the specific subspace of the full Hilbert space where these coherent states live.

Taking advantage of our formalism, we find in this paper the Berry phases for inflation and for the cosmological perturbations and express them in terms of the observables, as the spectral scalar and tensor indices, n_S an n_T, and the tensor to scalar ratio r. The Berry phase for de Sitter inflation is imaginary with the sign describing the exponential acceleration. For entanglement, we find the pure entangled states for de Sitter space–time as well as for black holes.

The complete covering of the Hilbert space with the complete covering of the quantum space–time is realized by the Metaplectic group symmetry which equivalently provides the complete global CPT symmetric states.

We show here that:

The Classical-Quantum Duality of space–time is also realized in the $Mp(n)$ symmetry; because of the complete covering, the global complete space–time (and full phase space completion) is needed to make the classical-quantum space–time duality manifest (and the phase space mapped from it).

- It is worth mentioning that similar discrete levels can be obtained from the global (complete) classical-quantum duality including gravity [3,5–7], namely classical-quantum gravity duality: The two *even* and *odd* (local) carts or sectors here and their global $(\pm)$ sum of states, reflect a relation between the $Mp(n)$ symmetry and the classical-quantum gravity duality.
- The two $\sqrt{(2n+1)}$ and $\sqrt{2n}$, *even* and *odd* sets separately are local coverings and they are *entangled*. The symmetric or antisymmetric sum of these states are global covering states, and they are necessary to cover the *whole* manifold.
- Moreover, the corresponding $(\pm)$ global states are complete, CPT symmetric and unitary, the levels $n = 0, 1, 2, ...$, cover the whole Hilbert space $\mathcal{H} = \mathcal{H}_{(+)} \oplus \mathcal{H}_{(-)}$ and all the space–time regimes. In the Metaplectic group representation $Mp(n)$, this corresponds to the state sectors $\mathcal{H}_{(1/4)}$ (even) and $\mathcal{H}_{(3/4)}$ (odd), and as we find here these states *are entangled.*
- The total n states range over *all* scales from the lowest excited levels to the highest excited ones covering the two dual branches $(+)$ and $(-)$ or Hilbert space sectors and corresponding space–time coverings. The two $(+)$ and $(-)$ dual sectors are entangled.

The consequences of these results are interesting because the classical-quantum gravity duality allows signals or states in the quantum gravity (trans-Planckian) domaine, or semiquantum gravity (inflationary) domaine, do appear as low energy effects in the semiclassical/classical universe today.

From the results of this paper, we see that the gravity domains from one side and the other of the Planck scale *are entangled*, for instance, the black hole quantum interior and the classical exterior gravity domains. Similarly, for the cosmological domains: The quantum trans-Planckian primordial vacuum (constant curvature de Sitter phase, followed by a quasi-de Sitter (inflationary) phase), and the late classical dual de Sitter Universe today are *entangled*.

We have also performed an interpretation of the quadratic quantum inflationary fluctuations in terms of the well-developed theory of time-dependent oscillators, their coherent, squeezed and cat states. The consequences of such states for the cosmological evolution and the detection of CMB fluctuations remain unclear, but hopefully such results will emerge from these investigations.

Other new entanglement results of this paper are as follows:

- The precise relation between the Schmidt-type representation in the density matrix context and the physical state fulfilling the Minimal Group Representation Principle (MGRP), which is bilinear in the basic states of the $Mp(n)$ group, is found.
- The mapping for the physical state refers to a *new non-diagonal* coherent state representation complementary to that of the known Sudarshan diagonal representation.
- The basic states in the Minimal Group Representation sense: $|1/4>$ and $|3/4>$ (belonging to the even and odd sectors of the Hilbert space, respectively) are intrinsic fundamental parts of the very structure of the space–time itself and do not require an additional extrinsic generation process as in the standard Schrodinger cat states and their entanglement.

This paper is organized as follows:

In Section 2 we describe the Berry geometrical phases for non-compact groups and the coherent state quantum evolution. In Section 3 we apply this framework to find the Berry phases for de Sitter inflation and the inflationary (tensor and scalar) perturbations. Section 4 is devoted to entanglement, its density matrix description and our generalized coherent states which allow us to map the space–time metric into the Hilbert space. Section 5 deals with the $Mp(n)$ coherent states describing the quantum evolution of space–time, the associated adiabatic invariants and the topological structure. In Sections 6 and 7 we find the entanglement with semi-coherent states and generalized Schrodinger cat states in the quantum space–time structure. In Section 8 we apply these results to find the entanglement in de Sitter and black hole space–times. Section 9 is devoted to Remarks and Conclusions.

2. Geometrical Phases and Noncompact Groups

2.1. SU(1,1) Coherent States and the Berry Phase

Our starting point is the coset coherent state definition according to Perelomov–Klauder [12–14] via the action of a displacement operator $D(\xi)$ belonging to the coset and generally unitary, as follows:

$$| \psi \rangle = U(\xi) | \psi_0 \rangle \rightarrow | k, \xi \rangle = D(\xi) | k, 0 \rangle$$

$$\begin{aligned} | k, \zeta \rangle &= \exp(\xi K_+ - \xi^* K_-) | k, 0 \rangle \\ &= \left(1 - |\zeta|^2\right)^k \exp(\zeta K_+) | k, 0 \rangle \end{aligned}$$

where

$$\xi = -(\chi/2)\, e^{-i\varphi}, \quad \zeta = -\tanh(\chi/2)\, e^{-i\varphi} = \frac{\xi}{|\xi|} \tanh |\xi|$$

The parameter ζ is restricted by $|\zeta| < 1$ (*disk*), and the number k is known as the Bargmann index which separates different irreducible representations.

ζ is taken to be a slow function of time, as usual.

Notice that the displacement operator D does not contain K_0. We will give a realization of the generators K_+, K_- and K_0 are in the next section.

The state $\mid k,\ 0\ \rangle$ meets the conditions: $K_- \mid k,\ 0\ \rangle\ =\ 0$ and $K_0 \mid k,\ 0\ \rangle\ =\ k| \ k,\ 0\ \rangle$.

If one assumes the diagonal operator K_0 like the Hamiltonian we have that $|k,\ \xi\rangle$ is the eigenstate of the following Hamiltonian:

$$\begin{aligned}\widetilde{H}\ &=\ D\,(\,\xi\,)\,K_0\ D^{\dagger}\ (\,\xi\,)\\ &=\ K_0\ \cosh\ \chi\ +\ \frac{1}{2}\left(\,e^{-i\varphi}\ K_+\ +\ e^{i\varphi}\ K_-\,\right)\ \sinh\ \chi\end{aligned}$$

Taking into account ζ as a slowly varying function of the evolution parameter of the system (e.g., time) and

$$\langle\ k,\ \zeta\ |\ k,\ \zeta\ \rangle\ =\ 1$$

$$\langle\ k,\ \zeta\ |K_+|\ k,\zeta\ \rangle\ =\ \frac{2\,k\,\zeta^*}{\left(\,1\ -\ |\ \zeta\ |^2\,\right)}\ =\ -\ k\ \sinh\chi\,e^{i\varphi}$$

then

$$\langle\ k,\ \zeta\ |\ \frac{d}{dt}\ |\ k,\ \zeta\ \rangle\ =\ k\ \frac{\zeta^*\ \frac{d\zeta}{dt}\ -\ \zeta\ \frac{d\zeta^*}{dt}}{\left(\,1\ -\ |\ \zeta\ |^2\,\right)}$$

From the definitions for the Berry phase, with $R(t)$ the set of parameters (generally of geometrical origin) we have as follows:

$$\begin{aligned}\gamma(\,C\,)\ &=\ i\ \int_0^T dt\ \langle\ n,\ R\ |\ \frac{d}{dt}\ |\ n,\ R\ \rangle\\ &=\ i\ \int_C dR\ \langle\ n,\ R\ |\ \nabla_R\ |\ n,\ R\ \rangle\end{aligned}$$

and the Berry phase expresses as

$$\gamma_k(\,C\,)\ \ =\ -i\,k\ \int_C\ \frac{\zeta^*\ d\zeta\ -\ \zeta\ d\zeta^*}{\left(1\ -\ \ |\ \zeta\ |^2\,\right)}$$

Mp(2), SU(1,1) and Sp(2):

All the groups Mp(2), Sp(2,R), and SU(1,1) are three dimensional. It is possible to parameterize them in several ways that make the homomorphic relations particularly simple. We use two of such parameterizations, both of which are described as:

$$Mp\,(2)\ \rightarrow\ e^{-\,i\,\alpha_1 T_1},\ e^{-\,i\,\alpha_2 T_2},\ e^{-\,i\,\alpha_3 T_3}$$

$$Sp\,(2R)\ \rightarrow\ \begin{pmatrix} e^{\frac{1}{2}\alpha_1} & 0\\ 0 & e^{-\frac{1}{2}\alpha_1}\end{pmatrix},\begin{pmatrix}\cosh\frac{1}{2}\alpha_2 & \sinh\frac{1}{2}\alpha_2\\ \sinh\frac{1}{2}\alpha_2 & \cosh\frac{1}{2}\alpha_2\end{pmatrix},\begin{pmatrix}\cos\frac{1}{2}\alpha_3 & -\sin\frac{1}{2}\alpha_3\\ \sin\frac{1}{2}\alpha_3 & \cos\frac{1}{2}\alpha_3\end{pmatrix}$$

$$SU(1,1)\rightarrow\begin{pmatrix}\cosh\frac{1}{2}\alpha_1 & \sinh\frac{1}{2}\alpha_1\\ \sinh\frac{1}{2}\alpha_1 & \cosh\frac{1}{2}\alpha_1\end{pmatrix},\begin{pmatrix}\cosh\frac{1}{2}\alpha_2 & i\sinh\frac{1}{2}\alpha_2\\ -i\sinh\frac{1}{2}\alpha_2 & \cosh\frac{1}{2}\alpha_2\end{pmatrix},\begin{pmatrix} e^{\frac{i}{2}\alpha_3} & 0\\ 0 & e^{-\frac{i}{2}\alpha_3}\end{pmatrix}$$

where the angle α_3 has the range $(-4\pi, 4\pi]$ for $Mp(2)$, and the range $(-2\pi, 2\pi]$ for $Sp\,(2, R)$ and $SU(1,1)$.

Let us consider the brief description of the relevant symmetry group to perform the realization of the Hamiltonian operator of the problem. This group specifically is the Metaplectic $Mp(2)$ as well as the groups that are topologically covered by it. The generators of $Mp(2)$ are the following:

$$
\begin{aligned}
T_1 &= \frac{1}{4}\,(q\,p\,+\,p\,q) \;=\; \frac{i}{4}\left(a^{+2} - a^2\right), \\
T_2 &= \frac{1}{4}\left(p^2 - q^2\right) \;=\; -\frac{1}{4}\left(a^{+2} + a^2\right), \\
T_3 &= -\frac{1}{4}\left(p^2 + q^2\right) \;=\; -\frac{1}{4}\left(a^+ a \,+\, a\,a^+\right)
\end{aligned} \tag{1}
$$

with the commutation relations,

$$[T_3,\ T_1] \,=\, i\,T_2, \quad [T_3,\ T_2] \,=\, -\,i\,T_1, \quad [T_1,\ T_2] \,=\, -\,i\,T_3$$

being $(q,\ p)$, (alternatively $(a,\ a^+)$) the variables of the standard harmonic oscillator, as usual.

If we rewrite the commutation relations as:

$$[T_3\ ,\ T_1\ \pm\ i\ T_2] \,=\, \pm\,(\ T_1\ \pm\ i\ T_2\ ), \quad [T_1\ +\ i;T_2\ ,\ T_1\ -\ i\ T_2] \,=\, -\,2\ T_3$$

We see that the states $\mid n\ \rangle$ are eigenstates of T_3:

$$T_3 \mid n\ \rangle = -\,\frac{1}{2}\left(n + \frac{1}{2}\right) \mid n\ \rangle$$

and it is easy to see that:

$$T_1\ +\ i\,T_2 \,=\, -\,\frac{i}{2}\,a^2, \quad T_1 -\ i\ T_2 \,=\, \frac{i}{2}\,a^{+2}.$$

2.2. Quadratic Hamiltonians, the Parametric Oscillator and the Group Structure

The problem we are going to face: we know that the general parametric oscillator has a quadratic structure of the known form

$$\widehat{H} \,=\, \frac{1}{2}\left[\,Z(t)\,\widehat{p}^2 \,+\, Y(t)\,(\,\widehat{p}\,\widehat{q} \,+\, \widehat{q}\,\widehat{p}\,) \,+\, X(t)\,\widehat{q}^2\,\right] \tag{2}$$

Therefore, using the group manifold, precisely the $Mp(2)$ group, the association is direct:

$$\widehat{H} \,=\, -i\left[Z(t)\widehat{K}_+ \,+\, 2\,Y(t)\,\widehat{K}_0 \,+\, X(t)\widehat{K}_-\right] \tag{3}$$

since, for Equation (1):

$$\widehat{K}_0 \,=\, i\,T_1, \quad \widehat{K}_+ = -\,i\,(T_2 + T_3), \quad \widehat{K}_- \,=\, i\,(T_2\,-\,T_3)\,,$$

consequently:

$$\left[\widehat{K}_+,\ \widehat{K}_-\right] \,=\, -\,2\,\widehat{K}_0, \quad \left[\widehat{K}_0,\ \widehat{K}_\pm\right] \,=\, \pm\,\widehat{K}_\pm \tag{4}$$

being then one of the possible representations, preserving the commutation relations at the algebra level, the following:

$$\widehat{K}_0 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},\ \widehat{K}_+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\ \widehat{K}_- = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \rightarrow \widehat{H} = -\,i\begin{pmatrix} -Y(t) & X(t) \\ Z(t) & Y(t) \end{pmatrix}. \tag{5}$$

Let us note that formally and for practical purposes we draw on a particular parameterization of $SO(1,2)$, strictly according to the chain for any element A_α

$$A_\alpha :\ \in Mp(2)\ \supset\ Sp(2\mathbb{R})\ \sim\ SU(1,1)\ \supset\ SO(1,2)\ \approx\ L(3) \tag{6}$$

This is for the obvious reason that $Mp(2)$ does not have finite realizations, being the double covering of $SU(1,1)$, $Sp(2R)$, and the quadruple covering of $SO(1,2)$, properties to take into account when signs, phases and connectivity become very important, such as when determining the spectrum of physical states of a particular system.

Consequently, we remark here that the representation Equation (5) is quite useful mainly at the level of making a more precise comparison at the time to pick a point of contact with other methods such as the one based on the Wei–Norman theorem [15]. Strictly speaking (we will return to this point at the end of this paper) one must work independently of the representation as far as non-compact groups are concerned (e.g., infinite dimensional unitary representations).

2.3. Generalized Parametric Oscillator and the Berry Phase

We take again as the starting point the following Hamiltonian:

$$H(t) = Z(t)\frac{p^2}{2m} + \frac{\omega_0}{2}Y(t)\,(\,pq + qp\,) + \epsilon\,X(t)\frac{m(\omega_0\,q\,)^2}{2}, \quad (\,\epsilon = \pm 1\,) \tag{7}$$

Introducing, as usual, annihilation-creation operators in Equation (7)

$$a = \frac{p - i\,m\,\omega_0\,q}{\sqrt{2\,m\,\omega_0\,\hbar}}, \qquad a^+ = \frac{p + i\,m\,\omega_0\,q}{\sqrt{2\,m\,\omega_0\,\hbar}} \tag{8}$$

we obtain

$$H(t) = \frac{1}{2}\Big[-(\epsilon X(t) - Z(t))\Big(a^{+2} + a^2\Big) + iY(t)\Big(a^{+2} - a^2\Big) + (\epsilon X(t) + Z(t))\big(\,aa^+ + a^+a\,\big)\Big] \tag{9}$$

where $(\epsilon = \pm 1)$. We note that using a Bogoliubov-type transformation like

$$\begin{pmatrix} b \\ b^+ \end{pmatrix} = \begin{pmatrix} M_{+-} & M_{-+} \\ M_{--} & M_{++} \end{pmatrix}\begin{pmatrix} a \\ a^+ \end{pmatrix} \tag{10}$$

where

$$M_{\pm\pm} = \frac{1}{2\sqrt{\kappa Z}}\,[\,(\,Z \pm \kappa\,) \pm i\,Y\,] \tag{11}$$

and

$$\kappa = \Big(\epsilon XZ - Y^2\Big)^{1/2} \tag{12}$$

the Hamiltonian takes the form

$$H = \hbar\,\underbrace{\omega(t)}_{\kappa(t)\omega_0}\left(b^+b + \frac{1}{2}\right) \tag{13}$$

due to the canonical commutation relations being preserved, e.g., $[\,b, b^+] = [\,a, a^+] = 1$, and where we have used the relation $(b^+\,b + b\,b^+)$ which follows from Equations (9) and (10):

$$b^+b + b\,b^+ = \frac{1}{4\kappa}\Big[2(Z(t) - \epsilon X(t))\Big(a^{+2} + a^2\Big) + 2(\epsilon X(t) + Z(t))\big(a\,a^+ + a^+a\big) + 2\,iY(t)\Big(a^2 - a^{+2}\Big)\Big] \tag{14}$$

The Heisenberg equations of motion of the system are as follows:

$$\begin{aligned}\frac{d\,b}{d\,t} &= -\frac{i}{\hbar}\,[\,b,\,H\,] + \frac{\partial\,b}{\partial\,t} \\ &= -\,i\,\omega\,b + \frac{i\,Z}{2\,\kappa}\left[\frac{d}{dt}\left(\frac{Y}{Z}\,b - \frac{Y - i\,\kappa}{Z}\,b^+\right)\right],\end{aligned}$$

$$\begin{aligned} \frac{db^+}{dt} &= -\frac{i}{\hbar}\,[\,b^+,\,H\,] \;+\; \frac{\partial\, b^+}{\partial\, t} \\ &= i\,\omega\, b^+ \;-\; \frac{i\,Z}{2\,\kappa}\left[\,\frac{d}{dt}\left(\frac{Y}{Z}\,b^+ \;+\; \frac{Y\,+\,i\,\kappa}{Z}\;b\right)\right], \end{aligned}$$

Notice that for the purpose of finding the Berry phase, the terms of interaction between the creation and annihilation operators can be disregarded as they are beyond the second order of adiabaticity. Consequently, the approximate solution is given by

$$b(t)\;\to\;b(0)\;\exp\left[-\,i\int\left[\omega\,(t)\;-\;\frac{Z}{2\,\kappa}\,\frac{d}{dt}\left(\frac{Y}{Z}\right)\right]dt\right] \tag{15}$$

The second phase factor will be related to the Berry phase of the physical system, e.g.,

$$B_{ph}\;\to\;\int\frac{Z}{2\kappa}\,\frac{d}{dt}\left(\frac{Y}{Z}\right)dt.$$

For the sake of completeness, we have now computed the second order in adiabaticity for the Berry phase and its full expression has the following interesting structure: From the solution of the operators $b(t)$ up to second order in adiabaticity, the Berry phase takes the form

$$B_{ph}\to\underbrace{\int\frac{Z}{2\,\kappa}\,\frac{d}{dt}\left(\frac{Y}{Z}\right)dt}_{1st\ order}\underbrace{-\int\frac{\hbar}{\kappa\,\omega_0}\left[\frac{1}{2}\,\frac{d}{dt}\left(\frac{1}{Z}\frac{dZ}{dt}\right)\;-\;\frac{1}{4}\left(\frac{1}{Z}\,\frac{dZ}{dt}\right)^2\right]}_{2nd\ order}$$

The second term is the second order correction in $1/\hbar$ and as we can see, the dependence is only on $Z(t)$, through $d/dt\,[\,lnZ(t)\,]$ and its quadratic power, while the first order contains the truly parametric oscillator variable $Y(t)$. Clearly, neglecting the second order is fully justified.

It is clear that adiabaticity applies in many space–times which are smooth, nonsingular, namely without sudden shock waves for instance. The space–times we are considering in the paper, de Sitter space–time and the regular black holes (with the quantum space–time removed singularity), are widely justified here.

2.4. Coherent State Quantum Evolution

As we have seen so far, from the point of view of the dynamics the relevant symmetries are dominated by the Metaplectic group and the groups covered by it, which define the symplectic and projective characteristics of the quantum phase space. Consequently, to illustrate the motion in the projective Hilbert space it is appropriate to start the corresponding coherent state given by

$$|\,\xi\,\rangle\;=\;e^{-\,|\,\xi\,|^2}\sum_{n=0}^{\infty}|\,n\,\rangle\;=\;e^{\,(\,\xi\,b^{+2}\;-\;\xi^{*}b^{2}\,)}\,|\,0\,\rangle \tag{16}$$

The coherent state Equation (16) comes from the operators $b^+\,b$ corresponding to the diagonal Hamiltonian of Equation (13) is necessary for the analysis of the dynamics of the system.

The Mp(2) Squeezed Vacuum and Physical States:

The displacement operator in the case of the vacuum squeezed state is an element of the Mp(2) group written in the respective variables of the canonical annihilation and creation operators.

$$S(\xi) = \exp \frac{1}{2}\left(\xi^* b^2 - \xi b^{+2} \right) \in Mp(2) \tag{17}$$

Seeing Equations (10)–(18) the relationship is shown directly:

$$\begin{pmatrix} b \\ b^+ \end{pmatrix} \to S(\xi)\begin{pmatrix} a \\ a^+ \end{pmatrix} S^{-1}(\xi) = \begin{pmatrix} \lambda & \mu \\ \mu^* & \lambda^* \end{pmatrix}\begin{pmatrix} a \\ a^+ \end{pmatrix} \tag{18}$$

From Equations (17) and (18), we see that the dynamics of these "square root" fields of Φ_γ in the particular representation that we are interested in are determined by considering these fields as coherent states in the sense that they are eigenstates of a^2 via the action of the Mp(2) group that is of the type:

$$|\,\Psi_{1/4}\,(0,\xi,q)\,\rangle = \sum_{k=0}^{+\infty} f_{2k}\,(0,\xi)\,\frac{(a^\dagger)^{2k}}{\sqrt{(2k)\,!}}\,|\,0\,\rangle \tag{19}$$

$$|\,\Psi_{3/4}\,(0,\xi,q)\,\rangle = \sum_{k=0}^{+\infty} f_{2k+1}\,(0,\xi)\,\frac{(a^\dagger)^{2k+1}}{\sqrt{(2k+1)\,!}}\,|\,0\,\rangle \tag{20}$$

For simplicity, we will take all normalization and fermionic dependence or possible fermionic realization, into the functions $f(\xi)$. Explicitly, at $t = 0$, the states are as follows:

$$\begin{aligned} |\Psi_{1/4}\,(0,\xi,q)\rangle &= f(\xi)\,|\,\alpha_+\rangle \\ |\Psi_{3/4}\,(0,\xi,q)\rangle &= f(\xi)\,|\,\alpha_-\rangle \end{aligned} \tag{21}$$

where $|\alpha_\pm\rangle$ are the coherent basic states in the subspaces $\lambda = \frac{1}{4}$ and $\lambda = \frac{3}{4}$ of the full Hilbert space. In other words, the action of an element of $Mp(2)$ keeps them invariant (coherent), ensuring the irreducibility of such subspace, e.g.,

$$\mathcal{H} \sim \begin{pmatrix} \mathcal{H}_{1/4} & \\ & \mathcal{H}_{3/4} \end{pmatrix} \tag{22}$$

Consequently, the two symmetric and antisymmetric combinations $(\pm)$ of the two sets of states $(1/4,\ 3/4)$ will span *all* the Hilbert space: $\mathcal{H}$:

$$|\Psi_\pm\rangle = |\,\Psi_{1/4}\,\rangle \pm |\,\Psi_{3/4}\,\rangle\,, \qquad |\,\pm\,\rangle = |\,+\,\rangle \pm |\,-\,\rangle \tag{23}$$

The time development is obtained by applying Equation (15) to the combined state (sweeping the entire $\mathcal{H}$):

$$|\xi\rangle_t = e^{-i\Gamma(t,R)}\left|\xi e^{+i\Gamma(t,R)}\right\rangle$$

or, in the respective irreducible subspaces of $\mathcal{H}$ from the point of view of $Mp(n)$:

$$\begin{aligned} |\,\Psi_{1/4}\,\rangle_t &= e^{-i\Gamma(t,R)}\left|\,\Psi_{1/4}e^{+i\Gamma(t,R)}\right\rangle \\ |\,\Psi_{3/4}\,\rangle_t &= e^{-i\Gamma(t,R)}\left|\,\Psi_{3/4}e^{+i\Gamma(t,R)}\right\rangle \end{aligned}$$

where the total phase (dynamical plus Berry phase) is

$$\Gamma(\,t,\,R\,(t\,)\,) = \underbrace{\int \omega(t)\,dt}_{dyn.phase} \underbrace{- \int \frac{Z}{2\kappa}\,\frac{d}{dt}\left(\frac{Y}{Z}\right)dt}_{Berry\ phase}$$

3. Relevant Applications in Cosmology

New illustrative applications of this formalism appear in the context of cosmology and black holes. These situations correspond to the cases $Z(t) = X(t) = 1$.

In general, the relevant term for the emergence of the Berry phase is the linear part in the temporal differential equation of the parametric oscillator, or the $(qp + pq)$ term in the Hamiltonian Equation (9) and thus the non-vanishing coefficient $Y(t)$.

The diagonalization can be always performed through the Bogoliubov transformation Equations (10)–(14) to the quadratic Hamiltonian. Therefore, for these cases, we have as follows:

$$H(t) \;=\; \frac{1}{2}\Big[(1-\epsilon)\Big(a^{+2}+a^2\Big) + iY(t)\Big(a^{+2}-a^2\Big) + (1+\epsilon)\big(aa^+ + a^+a\big)\Big] \tag{24}$$

$$M_{\pm\pm} \;=\; \frac{1}{2\sqrt{\kappa}}[\,1 \,\pm\, \kappa \,\pm\, iY\,] \quad \epsilon \;=\; \pm\,1 \tag{25}$$

($\epsilon \;=\; -1$ corresponding to the inverted, e.g., with imaginary frequency, oscillator).

And the Hamiltonian becomes

$$H \;=\; \hbar\,\omega(t)\Big(b^+b \;+\; \frac{1}{2}\Big) \tag{26}$$

$$\kappa^2 = \Big(\epsilon - Y^2\Big), \qquad \hbar\,\omega(t) = \kappa(t)\;\omega_0 \tag{27}$$

$$b^+b + b\,b^+ = \frac{1}{2\kappa}\;[\;(1-\epsilon)\Big(a^2+a^{+2}\Big) + iY(t)\Big(a^2 - a^{+2}\Big) + (1+\epsilon) + \frac{1}{2}\big(aa^+ + a^+a\big)\;] \tag{28}$$

The Berry phase becoming

$$B_{phase} = \int \frac{dt}{2\sqrt{\epsilon - Y^2}}\;\frac{d}{dt}\,Y \tag{29}$$

3.1. Berry Phase of de Sitter Inflation

Interestingly enough, de Sitter space–time can be described as an *inverted* harmonic oscillator, e.g., with imaginary frequency, both classically and at the quantum level, the Hamiltonian taking the same form of Equation (26) with

$$|\kappa|^2 = (1+Y^2).$$

The description of de Sitter inflation is precisely that of a parametric oscillator with

$$Y(t) = H(t) = \sqrt{\Lambda(t)\,/\,3}\,,$$

the oscillator length being $l_{osc} \;=\; 1/H$. The Berry phase of de Sitter inflation is in fact imaginary, and with the sign describing the increasing exponential factor acceleration

$$B_{\,phase} \;=-\,i\,\int_{t_{in}}^{t_{end}} \frac{dt}{2\,\sqrt{1\;+\;H^2}}\;\;d_t\,H \tag{30}$$

This integral is explicitly solved as a function of H with the limits:

$$H(t_{end}) \equiv H_e, \quad H(t_{in}) \equiv H_i:$$

$$B_{phase} \;=\; \frac{i}{2}\;\ln\left(\frac{H_i \;+\; \sqrt{1\;+\;H_i^2}}{H_e \;+\; \sqrt{1\;+\;H_e^2}}\right) \tag{31}$$

The finite interval $[\,t_{in}\,,\;t_{end}\,]$ indicates the initial and the final time of inflation.

In the early universe inflationary phase, the Hubble values are both very high: $(H_i, H_e) >> 1$, typically of the Grand Unification scale, and therefore, the expression for the Berry phase Equation (31) yields in the early universe:

$$B_{phase} = \frac{i}{2} \ln \left(\frac{H_i}{H_e} \right) + \frac{i}{2} \left(\frac{1}{H_i^2} - \frac{1}{H_e^2} \right) + O \left(\frac{1}{H_i^2 \, H_e^2} \right) \tag{32}$$

Interestingly, the Berry phase for de Sitter inflation expresses as the logarithm of the quotient between the initial and final values of the Hubble constant during inflation. The imaginary value of the Berry phase here appropriately corresponds to the exponentially expanding nature of the de Sitter background.

3.2. Berry Phase of the Inflationary Perturbations

The linear quantum perturbations of Inflation, both the scalar (S) curvature fluctuations and the tensor (T) gravitational k-modes both satisfy a second order Schröedinger type equation in time:

$$Q''(\eta)_{(S,T)} + [\, k^2 - W^2(\eta)_{(S,T)} \,]\, Q(\eta)_{(S,T)} = 0 \tag{33}$$

where η is the conformal time related to the cosmic time t by $dt = a(\eta)\, d\eta$, primes $('')$ denote the second derivative with respect to η, and the potential $W^2(\eta)_{(S,T)}$ felt by the fluctuations is as follows:

$$W^2(\eta)_{(S)} = a''(\eta) / a(\eta) \tag{34}$$

$$W^2(\eta)_{(T)} = z''(\eta) / z(\eta) \tag{35}$$

The variable z appropriately combines the inflaton field $\phi(t)$ and the accelerated expansion background described by the scale factor $a(t)$ driven by $\phi(t)$, with the Hubble parameter being $H = a^{\cdot}(t)/a(t)$:

$$z = a(t) \; \frac{\phi^{\cdot}(t)}{H} \tag{36}$$

$$\frac{d^2 z}{d\eta^2} = a^2 \, (z^{\cdot\cdot} + H\, z^{\cdot}) \tag{37}$$

Therefore, the scalar and tensor gravitational inflationary fluctuations both satisfy similar parametric harmonic oscillator equations with the Hamiltonian $\mathcal{H}$:

$$\mathcal{H} = \frac{1}{2} [\, P^2 + \Omega^2(\eta)_{(S,T)}\, Q^2 \,] \tag{38}$$

$$\Omega^2(\eta)_{(S,T)} \equiv k^2 - W^2(\eta)_{(S,T)}$$

The two typical situations do appear:

$$\Omega(\eta)_{(S,T)} \quad \text{Real} \; : \quad k^2 > W^2(\eta)_{(S,T)}, \quad \text{(sub horizon k-modes)}$$

$$\Omega(\eta)_{(S,T)} \quad \text{Imaginary} : \quad k^2 < W^2(\eta)_{(S,T)}, \quad \text{(super horizon k-modes)}$$

Generically, the field oscillations for which the wavelengths $\lambda = 1/k$ are inside the Hubble radius $1/H$ are named sub horizon modes, therefore, $k > H$ for them. The super horizon modes are those for which the wavelengths are larger than the Hubble horizon: $\lambda > 1/H$, namely $k < H$. Therefore, the above two regimes on the k-modes of the inflationary fluctuations (in conformal time) precisely correspond to these two typical situations.

$\mathcal{H}(P,Q)$ correspond to the (b,b^+) representation, and a canonical (Bogoliubov) transformation as Equation (25): $(P,Q) \rightarrow (q,p)$, [equivalently $(b,b^+) \rightarrow (a,a^+)$], yields $\mathcal{H}$ into the form:

$$\mathcal{H} \;=\; \frac{1}{2}\,[\; p^2 \;+\; Y\,(\,\eta\,)_{\,(S,T)}\,(\,p\,q \;+\; q\,p\,) \;+\; k^2\,q^2\,] \tag{39}$$

$$Y\,(\,\eta\,)_{\,(S)} \;=\; z'\,/\,z, \qquad Y(\,\eta\,)_{\,(T)} \;=\; a'\,/\,a$$

with

$$\Omega^2\,(\,\eta\,)_{\,(S,T)} \;=\; (\,k^2 \;-\; Y^{\,2}_{\,(S,T)}\,) \;-\; Y'_{\,(S,T)} \;=\; \omega^2 \;-\; Y'_{\,(S,T)} \tag{40}$$

$Y'_{(S,T)} \;=\; a''/a \;-\; Y^2$, and recall here that η is the conformal time, $(') \equiv \; d/d\eta$

The Berry phase of the quantum inflationary fluctuations $(S,\ T)$ is then:

$$B_{\,phase\,,k}\;(S,\,T) \;=\; \int_0^{\eta_{\,end}} \frac{d\eta}{2\,\sqrt{\,k^2 \;-\; Y^2_{(S,\,T)}}}\;\; Y'_{(S,\,T)} \tag{41}$$

The η interval in the integral depends on whether it refers to sub-horizon or super-horizon modes, e.g., on whether $k^2 \;>\; Y^2(\,\eta_{\,end})_{\,(S,\,T)}$ or $k^2 \;<\; Y^2(\,\eta_{\,end})_{\,(S,\,T)}$, respectively, $\eta_{\,end}$ being then fixed by:

$$k^2 \;=\; Y^2(\,\eta_{\,end}).$$

Note, that in the case of the cosmological perturbations, the integral is explicitly solved as a function of $Y_{(S,T)}(\eta)$ with limits $Y_{(S,T)}(0) \equiv Y_0$ and $Y_{(S,T)}(\eta_{end}) \equiv Y_e$:

(i) sub horizon modes:

$$B_{phase,\;k}(S,T) \;=\; \frac{1}{2}\;\arctan\left[\;\frac{Y_e - Y_0}{\sqrt{\,(k^2 - Y_e^2\,)(k^2 - Y_0^2\,)} + Y_e Y_0}\;\right]$$

(ii) super horizon modes:

$$B_{phase,\;k}(S,T) \;=\; \frac{i}{2}\;\ln\left(\frac{Y_e \;+\; \sqrt{\,Y_e^2 \;-\; k^2}}{Y_0 \;+\; \sqrt{\,Y_0^2 \;-\; k^2}}\right)$$

We can also express the Berry phase in the inflationary slow roll regime which is well appropriated here because of the Berry phase adiabaticity. $Y(\,\eta)_{\,(S,\,T)}$ can be written in terms of the slow roll parameters $(\epsilon_v,\ \eta_v)$, and therefore, related to the cosmological observables: the scalar and tensor spectral indices n_S and n_T, and the scalar to tensor ratio r:

$$W\,(\,\eta\,)_{\,(S,\,T)} \;=\; \frac{(\,\nu^2_{(S,\,T)} \;-\; 1/4\,)}{\eta^2} \tag{42}$$

$$\begin{aligned} \nu_{\,(S)} \;&=\; 3/2 \;+\; 3\,\epsilon_v \;-\; \eta_v \;=\; 3/2 \;-\; (\,n_s \;-\; 1\,)\,/\,2 \\ \nu_{\,(T)} \;&=\; 3/2 \;+\; \epsilon_v \;=\; 3/2 \;-\; n_T\,/\,2 \\ r \;&=\; 16\,\epsilon_v \;=\; -\,n_T\,/\,8 \end{aligned} \tag{43}$$

These expressions show the explicit operational applications of our framework. The analysis of these findings, their properties and the observable features in terms of real cosmological data exceed the purpose of this paper and deserve future work.

4. Entanglement

Density Matrix Viewpoint

We will develop the entanglement of quantum states in the theoretical context of non-compact groups, in particular, for the Metaplectic group. Besides its several local and global interesting properties, the Metapletic group is strongly (mathematically and physically) supported by the principle of minimum group representation described in detail in ref. [2].

Let us consider the following state

$$|\,\Phi\,\rangle \;=\; |\,\Phi_A\,\rangle|\,\Phi_B\,\rangle$$

The sub-indices A and B indicate the corresponding sub- systems. Usually one could even define (in the context of our previous works) the density matrix for an observable and unobservable sector.

$$|\,\Phi\,\rangle \;=\; |\,\Phi o\,\rangle|\,\Phi d\,\rangle$$

in such a way that the density matrix of the observable system is

$$\rho_o \;=\; Tr_d\,|\,\Phi\,\rangle\,\langle\Phi\,|$$

and the corresponding entropy of the entanglement in this case

$$S_E \;=\; -\,Tr_o\,\rho_o\,\log\rho_o$$

Using the Schmidt decomposition, some pure states can be written as

$$|\,\Phi\,\rangle \;=\; \sum_{i=1}^{n} c_i\,|\,v_{iA}\,\rangle\,\otimes\,|\,u_{iB}\,\rangle \tag{44}$$

where $|\,v_{iA}\,\rangle, |\,u_{iB}\,\rangle$ are orthonormal states in the subsystems A and B, respectively. If we see the structure of the Hilbert space Equation (22) for a state $\in SU(1,1)$ the correspondence between Equation (44) and the results of our previous papers is immediate as we will demonstrate below.

Not all states are separable states (and thus product states). Fix a basis for H_A and a basis for H_B subsystems. The most general state in $H_A \otimes H_B$ is now of the form

$$|\,\Phi\,\rangle \;=\; \sum_{i,j=1} c_{ij}\,|\,v_{iA}\,\rangle \otimes |\,u_{jB}\,\rangle$$

This state is separable if there are vectors such that yielding $c_{ij} = c_{iA}\,c_{jB}/|\,\Phi\,\rangle = \sum_{i=1} c_{iA}\,|\,v_{iA}\,\rangle \otimes \sum_{j=1} c_{jB}\,|\,u_{jB}\,\rangle$. And it is *inseparable* if for any vectors at least for one pair of coordinates $c_{ij} \neq c_{iA}\,c_{jB}$. If a state is *inseparable*, it is called an 'entangled state'. The typical case is one of the Bell states, e.g.,

$$\frac{|\,1_A\,\rangle \otimes |\,0_B\,\rangle - |\,0_A\,\rangle \otimes |\,1_B\,\rangle}{\sqrt{2}}$$

of the four Bell states, which are (maximally) entangled pure states with the basis $\{|\,0_A\,\rangle, |\,1_A\,\rangle\} \in H_A$ and $\{|\,0_B\,\rangle, |\,1_B\,\rangle\} \in H_B$. These are pure states of the $H_A \otimes H_B$ space, but which cannot be separated into pure states of each H_A and H_B space. If the composite system is in such a state, it is impossible to attribute to either system A or system B a definite pure state.

Notice that while the von Neumann entropy of the whole entangled state is zero (as it is for any pure state), the entropy of the subsystems is greater than zero. In this sense, the systems are "entangled".

Recently, in previous works [1,2], we have seen that physical states are mappings of the group generators in a vector representation, and that these are expressed in the so-called Sudarshan's diagonal-representation and in a *new non diagonal* one that leads, as an important consequence, the *physical states* with spin content $\lambda = (1/2,\ 1,\ 3/2,\ 2)$.

Precisely, the *generalized coherent states* generate a map that relates the metric g_{ab}, the solution of the wave equation, to the specific subspace of the full Hilbert space where these coherent states live. Moreover, for operators $\in Mp(2)$ there is an asymmetric kernel leading, for our case, to the following $\lambda = 1$ state:

$$g_{ab}(t, 1, \alpha)|_{HW} = \langle\Psi_{3/4}(t)|L_{ab}|\Psi_{1/4}(t)\rangle = \mathcal{F}\begin{pmatrix}\alpha\\ \alpha^*\end{pmatrix}_{(1)\,ab} \tag{45}$$

where

$$\mathcal{F} = e^{\left[-\left(\frac{m}{\sqrt{2}|\mathbf{a}|}\right)^2[(\alpha+\alpha^*)-B]^2+D\right]}e^{[\xi\varrho(\alpha+\alpha^*)]}|f(\xi)|^2$$

where B and D are given by:

$$B = \left(\frac{|\mathbf{a}|}{m}\right)^2 c_1, \qquad D = \left(\frac{|\mathbf{a}|c_1}{\sqrt{2}m}\right)^2 + c_2 \tag{46}$$

c_1 and c_2 being constants characterizing the solution or its initial conditions; ξ is the fermionic super coordinate of the corresponding group manifold and ϱ is the fermionic part of the superfield solution.

This is so because the non-diagonal projector involved in the reconstruction formula of L_{ab} is formed with the $\Psi_{1/4}$ and $\Psi_{3/4}$ states which span completely the *full* Hilbert space. And this is precisely a particular case of the Schmidt decomposition of an SU(1,1) quantum state, namely

$$\Phi_{ab} \equiv \underbrace{\langle\Psi_{3/4}(t)|L_{ab}|\Psi_{1/4}(t)\rangle}_{Mp(2)representation} \sim \langle\Psi_{3/4}(t)|\begin{pmatrix}a\\ a^+\end{pmatrix}_{ab}|\Psi_{1/4}(t)\rangle \to \underbrace{\Phi_{ab} = \sum_{n,m=0} c_{ab}v_{m1/4} \otimes u^*_{n3/4}}_{Schmidt-repres.} \tag{47}$$

The above correspondence is fully consistent due to the fact that $|\Psi_{1/4}(t)\rangle, |\Psi_{3/4}(t)\rangle$ are pure states (coherent) in each subspace $\in SU(1,1)$ and they are mutually orthogonal.

In more detail we have the $Mp(n)$ case from Equation (47):

$$\begin{aligned}\Phi_{ab} &= \sum_{n=0,1,2..}\underbrace{\langle\Psi_{3/4}(t)|2n+1\rangle}_{u^*_{3/4}}\ \underbrace{\langle 2n+1|L_{ab}|2n\rangle}_{c_{ab}}\ \underbrace{\langle 2n|\Psi_{1/4}(t)\rangle}_{v_{1/4}} \\ &= \sum_{n=0,1,2..}\left(1-|\omega|^2\right)\frac{(\omega^*/2)^{2n+1}}{\sqrt{(2n+1)!}}\begin{pmatrix}0\\ \sqrt{2n+1}\end{pmatrix}_{ab}\frac{(\omega/2)^{2n}}{\sqrt{(2n)!}} = \\ &= \sum_{n=0,1,2..}\Xi(n)\begin{pmatrix}0\\ \sqrt{2n+1}\end{pmatrix}_{ab}\omega^*\end{aligned} \tag{48}$$

Here, we have used the discrete basis (series) in the disk of Mp(2) and its coverings.

Meanwhile, the case

$$\begin{aligned}\Psi_{ab} &\equiv \langle\Psi_{1/4}(t)|L_{ab}|\Psi_{3/4}(t)\rangle \neq \langle\Psi_{3/4}(t)|L_{ab}|\Psi_{1/4}(t)\rangle \\ &= \sum_{n=0,1,2..}\underbrace{\langle\Psi_{1/4}(t)|2n\rangle}_{u^*_{3/4}}\ \underbrace{\langle 2n|L_{ab}|2n+1\rangle}_{c_{ab}}\ \underbrace{\langle 2n+1|\Psi_{3/4}(t)\rangle}_{v_{1/4}} \\ &= \sum_{n=0,1,2..}\Xi(n)\begin{pmatrix}\sqrt{2n}\\ 0\end{pmatrix}_{ab}\omega\end{aligned} \tag{49}$$

with the definition

$$\Xi(n) = \left(\frac{|\omega|}{2}\right)^{2n}\frac{\left(1-|\omega|^2\right)}{2\sqrt{(2n)!(2n+1)!}}$$

We can see that the expressions Equations (48) and (49) describe two definite chirality states showing the noncompact analog to the SU(2) case with the respective spinors (discrete series inside the disc with characteristic complex variable ω).

It is also important to remark on the following points:

(i) We have for Mp(2) an ∞-dimensional but unitary representation that maps it to the Lorentz group in 3-dimensional L_3.

(ii) We can see that:

$$\langle \Psi_{3/4}(t)|L_{ab}|\Psi_{1/4}(t)\rangle \neq \langle \Psi_{1/4}(t)|L_{ab}|\Psi_{3/4}(t)\rangle$$

This fact has interesting implications for the realization of the CPT invariance and for the arrow of time. This is so because of the full covering of the Hilbert space by the $|1/4\rangle$ and $|3/4\rangle$ sets of states (*even* and *odd* sets) of the Metaplectic (Mp) group. In particular, the correspondence between the standard Schrodinger cat states and the basic states $|1/4\rangle$ and $|3/4\rangle$ of Mp(2) is discussed in Section 7.

(iii) From the point (ii) it can be seen that there would be a natural transition from the compact spinorial description of SU(2) for example, to the non-compact SU(1,1) description by taking as the basis of this transition the universal covering Mp(2).

Let us also notice that the bilinear state associated with spin 2 in this basis, similar to in Equations (48) and (49), was associated with the space–time metric, providing the space–time discretization condition for small n and the continuous space–time for $n \to \infty$.

Consequently, and due to Equation (45) the entanglement entropy is given by

$$S_E = -Tr \mid \mathcal{F}\,\alpha \mid^2 \log \mid \mathcal{F}\,\alpha \mid^2$$

where $\mathcal{F}$ is due to the expression Equation (45).

5. Quantum Evolution, Adiabatic Invariants and Topological Structure

Now, we will provide the interpretation of the relation between the coherent states in the metaplectic representation and the quantum evolution of the space–time. For this purpose, we will need to start similarly to Equation (18) with the association

$$\begin{pmatrix} b \\ b^+ \end{pmatrix} \to \begin{pmatrix} a_{(+)} \\ a^+_{(+)} \end{pmatrix} \equiv L_{(+)}$$

and

$$\begin{pmatrix} a \\ a^+ \end{pmatrix} \to \begin{pmatrix} a_{(-)} \\ a^+_{(-)} \end{pmatrix} \equiv L_{(-)}$$

where $a_{(\pm)}$ correspond to operators $a(t)$ for $t \to \pm\infty$ (asymptotical values). Consequently

$$a^+_{(-)}a_{(-)}|n(-)\rangle = n|n(-)\rangle, \;\; a^+_{(+)}a_{(+)}|n(+)\rangle = n|n(+)\rangle, \; etc.$$

From Equation (18) we have

$$\begin{aligned} L_{(+)} &= UL_{(-)}U^{-1} = \mathbb{A}L_{(-)}, \\ &\mathbb{A}\in SU(1,1), \end{aligned} \tag{50}$$

but now the element of the group is

$$U \in Mp(2)/U \;=\; e^{-i\gamma\, T_{3(-)}}\, e^{-i\beta\, T_{1(-)}}\, e^{-i\alpha\, T_{3(-)}}$$

We can observe now that

$$M_{mn} = \langle m(+)||n(-)\rangle$$

where as before $a^+_{(+)}a_{(+)}|m(+)\rangle = m|m(+)\rangle$. We can see also that: $a^+_{(+)}a_{(+)} = U\, a^+_{(-)}\, a_{(-)}\, U^{-1}$. From Equation (50):

$$|\, m(+)\,\rangle \;=\; U\,|\, m\,(-)\,\rangle$$

then,

$$\langle m(+)\,||n(-)\rangle \;=\; \langle m(-)|\, U^{-1}\,|n(-)\rangle$$

with our matrix element

$$M_{mn} = \langle\, m|\; e^{i\alpha T_3 -\; i\beta T_1}\, e^{i\gamma T_3}\;|n\rangle$$

where the index $(-)$ was dropped for simplicity. Because $T_3|n\rangle = -\frac{1}{2}\left(n+\frac{1}{2}\right)|n\rangle$:

$$M_{mn} \;=\; e^{-i\alpha\frac{(2m+1)}{4}}\; e^{-i\gamma\frac{(2n+1)}{4}}\,\langle m|\; e^{i\beta T_1}\;|n\rangle$$

We can further consider the following quantity that determines the transition probability of the oscillator from the state $|\, n(-)\,\rangle$ to the state $|\, m(+)\,\rangle$:

$$W_{mn} \;=\; |\; M_{mn}\;|^2 \;=\; \left|\;\langle\; m\;|e^{-\frac{\beta}{4}\left(\,a^{+2}-\,a^2\,\right)}|\; n\;\rangle\;\right|^2$$

In particular,

$$\begin{aligned} W_{nn} &= \left|\;\langle\, n\,|e^{-\frac{\beta}{4}\left(\,a^{+2}-\,a^2\,\right)}|\, n\,\rangle\;\right|^2 \\ &= \left|\;\langle\, n\,|1\;-\;\frac{\beta}{4}\left(a^{+2}-\;a^2\right)\;+\;\frac{\beta}{32}^2\left(a^{+2}-\;a^2\right)^2\;-\;\cdots|n\rangle\;\right|^2 \\ &= 1-\;\frac{\beta}{8}^2\left(\,n^2\;+\;n+1\,\right)^2\;+\;\cdots \end{aligned}$$

By taking $\beta < 1$, β^2 in the above expression is the magnitude determining the accuracy of preservation of the adiabatic invariant of the classical oscillator, generally with time-dependent frequency, (parametric oscillator), as Equation (3) related with the dynamical and geometrical phases.

Let us notice that in the transition probability W_{mn} the relevant generator of the transition between the eigenstates $|\, n\,\rangle$ of T_3 of the respective system is T_1 characteristic of the diabatic-adiabatic evolution of the physical system considered.

6. Entanglement with Semi-Coherent States

Let us briefly analyze in an algebraic description, the origin of the quantum relativistic effects as the prolongated high oscillations effect or so called "Zitterbewegung".

There are two types of states in the "algebro-pseudo-differential" correspondence: the basic (non-observable) states and the observable physical states. In that case, the basic states are coherent states corresponding to the double covering of the $SL(2C)$ group, e.g., the Metaplectic group [1,2]: This is responsible for projecting the symmetries of the six-dimensional $Mp(4)$ group space to the four-dimensional space–time by means of a bilinear combination of the $Mp(4)$ generators. The supermultiplet metric solution for the geometric Lagrangian is

$$g_{ab}(t,\lambda) \;=\; \langle\psi_\lambda(t)\;|\; L_{ab}\;|\;\psi_\lambda(t)\;\rangle$$

As we can see above, the physical state (which appears as a mapping of the non-compact generator of interest and its fundamental coverings) takes precisely the form of a Husimi quasi-probability usually represented as Q. Specifically,

$$g_{ab}\ (0,\lambda)\ =\ \exp\left[\,A\,\right]\ \exp\left[\,\xi\varrho(t)\,\right]\chi_f\ \langle\,\psi_\lambda(\,0\,)\mid\ L_{ab}\mid\psi_\lambda(\,0\,)\,\rangle,$$

$$A\,(\,t\,)\ =\ -\left(\frac{m}{|\gamma|}\right)^2 t^2+c_1\,t+c_2\ ,\qquad (c_1,c_2)\ \in\ C \tag{51}$$

where we have written the corresponding indices for the simplest supermetric state solution, L_{ab} are the corresponding generators $\in Mp(n)$, and χ_f is coming from the odd generators of the big covering group of symmetries of the specific model.

Considering for simplicity the 'square' solution for the three compactified dimensions (spin λ fixed, $\xi\equiv-\left(\overline{\xi}^{\dot{\alpha}}-\xi^\alpha\right)$), the exponential even fermionic part is given by:

$$\varrho(t)\ \equiv\ \overset{\circ}{\phi}_\alpha\left[\ \left(\alpha\,e^{i\omega t/2}\ +\ \beta e^{-i\omega t/2}\right)-\left(\sigma^0\right)_{\dot{\alpha}}^{\alpha}\left(\alpha e^{i\omega t/2}-\beta e^{-i\omega t/2}\right)\right] \tag{52}$$

$$+\frac{2i}{\omega}\left[\left(\sigma^0\right)_{\alpha}^{\ \dot{\beta}}\,\overline{Z}_{\dot{\beta}}+\left(\sigma^0\right)_{\dot{\alpha}}^{\alpha}\,Z_\alpha\right] \tag{53}$$

$\overset{\circ}{\phi}_\alpha$, Z_α, $\overline{Z}_{\dot{\beta}}$ being constant spinors, and α and β are $\mathbb{C}$-numbers (the constant $c_1\in\mathbb{C}$ in Equation (51) due to the obvious physical reasons and the chirality restoration of the superfield solution).

By consistency, (and as in the string case), two geometric-physical options are related to the orientability of the superspace trajectory: $\alpha=\pm\beta$. We take without loss of generality $\alpha=+\beta$ then, exactly, there are two possibilities:

(i) The compact case, which is associated with a small mass limit (or $|\gamma|>>1$):

$$\varrho(t)\ =\ \begin{pmatrix}\overset{\circ}{\phi}_\alpha\cos(\omega t/2)\ +\ \frac{2}{\omega}Z_\alpha\\ -\overset{\circ}{\overline{\phi}}_{\dot{\alpha}}\sin(\omega t/2)\ -\ \frac{2}{\omega}\overline{Z}_{\dot{\alpha}}\end{pmatrix} \tag{54}$$

(ii) The non-compact case, which can be associated to the imaginary frequency ($\omega\ \rightarrow\ i\,\omega$: generalized inverted oscillators) case:

$$\varrho(\,t\,)\ =\ \begin{pmatrix}\overset{\circ}{\phi}\cosh\left(\,\omega\,t\,/2\,\right)\ +\ \frac{2}{\omega}Z_\alpha\\ -\overset{\circ}{\overline{\phi}}_{\dot{\alpha}}\ \sinh\left(\,\omega\,t/2\,\right)\ -\ \frac{2}{\omega}\,\overline{Z}_{\dot{\alpha}}\end{pmatrix} \tag{55}$$

Obviously (in both cases), this solution represents a *Majorana fermion* where the $\mathbb{C}$ (or *hypercomplex*) symmetry (wherever the case) is inside the constant spinors.

The spinorial even part of the superfield solution in the exponent becomes:

$$\xi\varrho(t)\ =\ \theta^\alpha\left(\overset{\circ}{\phi}_\alpha\cos(\omega t/2)\ +\ \frac{2}{\omega}Z_\alpha\right)\ -\ \overline{\theta}^{\dot{\alpha}}\left(-\overset{\circ}{\overline{\phi}}_{\dot{\alpha}}\sin(\omega t/2)\ -\ \frac{2}{\omega}\,\overline{Z}_{\dot{\alpha}}\right) \tag{56}$$

We easily see that in the above expression there is a type of continuous oscillation between the chiral and antichiral part of the bispinor $\varrho(t)$, or *Zitterbewegung* as shown qualitatively in Figure 1 for suitable values of the group parameters.

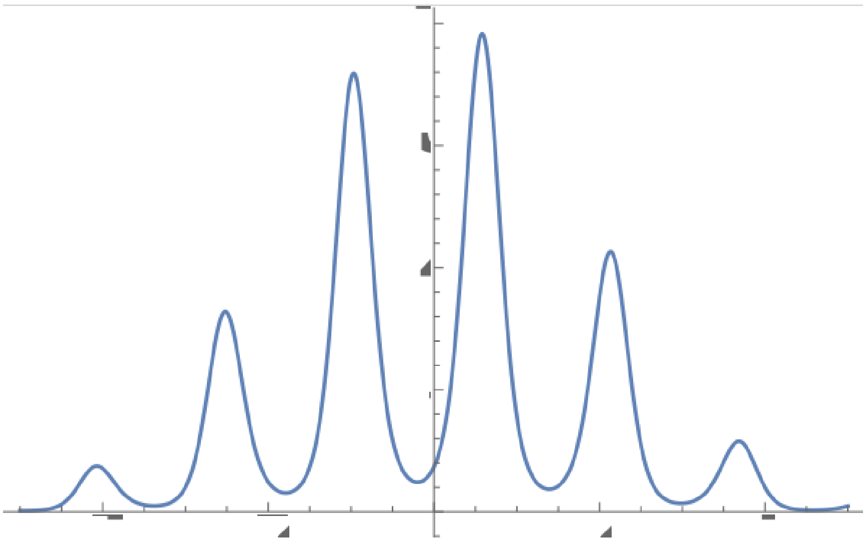

Figure 1. Chiral-antichiral oscillation (zitterbebegung) giving the pattern of cat states from first principles. The asymmetry in the pattern can be seen, marking a preferential temporal evolution.

6.1. Generation and Entanglement of Schrodinger Cat States

Let us remember that, in general, a "cat state" refers to a symmetric and antisymmetric combination of coherent Heisenberg–Weyl (HW) states with the property that they sweep even and odd states of the harmonic oscillations. The example of the definition in ref. [16] in terms of Heisenberg–Weyl displacement operators are

$$\mathcal{D}_{\pm}(\alpha)|\,0\,\rangle \;\equiv\; |\pm\alpha\,\rangle$$

$$\frac{1}{2}\left[\,\mathcal{D}(\alpha)\;\pm\;\mathcal{D}(-\alpha)\,\right]\;=\;\mathcal{D}_{\pm}(\alpha)$$

with the displacement operator standard definition

$$\mathcal{D}(\alpha)\;=\;e^{\left(\alpha\,a^{+}\,-\,\alpha^{*}a\right)}$$

We will now demonstrate, by comparing with the case of ref. [17], that the theoretical construction and physical interpretation presented here and in our previous works [1–7] is relevant from the fundamental point of view as far as the very quantum structure of space–time is concerned.

The starting state in ref. [17] is a cat coherent state as described above, but in our case here it is time-dependent, the evolution operator being a Kerr type (non-linear) Hamiltonian is described in the expression Equation (58). To simplify the problem, the authors of ref. [17] take a fixed interaction time $t=\pi/2\Omega$ where Ω is the coefficient of the anharmonic term:

$$|\,\alpha,\;\pi/2\Omega\,\rangle\;=\;\frac{1}{\sqrt{2}}\left[\,e^{-i\pi/4}\,|\,\beta\,\rangle\;+\;e^{i\pi/4}\,|\,-\beta\,\rangle\,\right]$$

with $\beta\;=\;\alpha\,e^{i\,\omega}$.

Therefore, we observe that:

(i) The photoelectron or heterodyne number operator of ref. [17] is simply

$$\widehat{q}\;=\;\frac{1}{2}\left[\,\cos\left(\,\theta\;+\;\omega\,t\,\right)\left(a+a^{+}\right)\;+\;i\,\sin\left(\theta\;+\;\omega\,t\,\right)\left(a\;-\;a^{+}\,\right)\,\right]\;+\;q_{0} \tag{57}$$

Let us compare with our operator in scalar form

$$\xi\,\varrho(t) \;=\; \xi\,\varrho(t) \;+\; \overline{\xi}\,\overline{\varrho}\,(t)$$
$$=\; \cos(\omega t/2)\left(\theta^{\alpha}\overset{\circ}{\phi}_{\alpha} + \overline{\theta}_{\dot{\alpha}}\overline{\phi}^{\dot{\alpha}}\right) + \sin(\omega t/2)\left(\theta^{\alpha}\overset{\circ}{\phi}_{\alpha} - \overline{\theta}_{\dot{\alpha}}\overline{\phi}^{\dot{\alpha}}\right) + \;\frac{4}{\omega}\left(\theta^{\alpha}Z_{\alpha} - \overline{\theta}_{\dot{\alpha}}\,\overline{Z}^{\dot{\alpha}}\right)$$

Therefore, the identification is immediate between the solution field operators and the photonic creation and annihilation operators in the case of the cited ref. [17].

(ii) The nonlinearity of the Kerr term (nonlinear optical medium) introduces the ingredient $SU(1,1)$ and the covering $Mp(2)$ in the simplest case, namely (remember that Ω is the oscillator non-linearity parameter and takes a specific form when the physical setting is defined)

$$H \;=\; \omega\,\widehat{n} \;+\; \Omega\,\widehat{n}^2 \;=\; (\;\omega \;+\; \Omega\;)a^{+}a \;+\; \Omega\,a^{+2}a^2 \tag{58}$$

The evolution generator in our case contains all the elements of the group $Osp(n)$ (or its covering $OMp(n)$), which is the largest symmetry harmonic oscillator as far as the a variables are concerned, with the odd part given by the generators: $F_{+} = \frac{1}{2}a^{+}$ and $F_{-} = \frac{1}{2}a$. For more detail please see Figure 2:

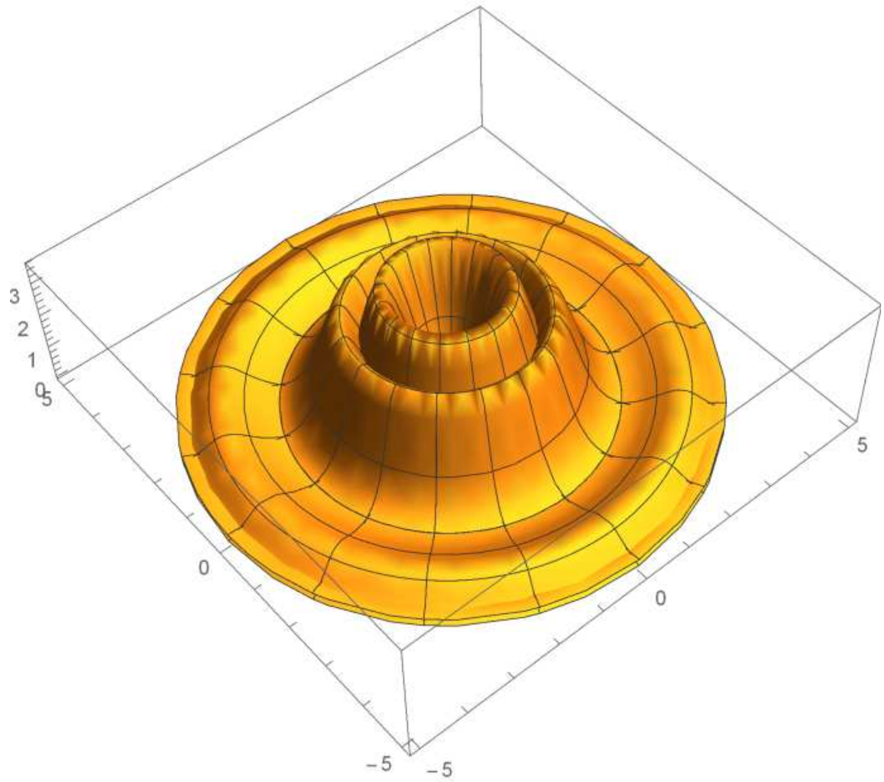

Figure 2. 3D picture of the chiral-antichiral oscillation (cat states pattern).

6.2. *Entanglement of Coherent States and Evolution of Probability*

We consider SU(1,1) states as an example, namely

$$|\,k,\zeta\,\rangle \;=\; e^{\,(\,\xi\,K_{+} - \xi^{*}K_{-})}\,|\,k,\,0\,\rangle$$
$$=\;\left(1 \;-\; |\,\zeta\,|^2\right)^{k} e^{\,\zeta\,K_{+}}\,|\,k,\,0\,\rangle$$

Over the orthonormal basis $|k, m\rangle$ we have

$$|\,k,\zeta\,\rangle \;=\; \Big(1\;-\;|\,\zeta\,|^2\Big)^k \sum_{m=0}^{\infty} \left[\frac{\Gamma(\,m\;+\;2k\,)}{m!\,\Gamma(\,2k\,)}\right] \zeta^m \;|\,k,\,m\,\rangle$$

where

$$\xi \;=\; -\,(\,\chi/2\,)\,e^{-i\varphi}, \quad \zeta \;=\; (\,\xi/\,|\,\xi\,|\,)\,tanh|\,\xi\,| \;=\; -\,tanh\,(\,\chi/2\,)\,e^{-\varphi}$$

Consequently, the parameter ζ is restricted to $|\,\zeta\,| < 1$ (disk) and (χ, φ) is coming from the parametrization of the quotient space is $SU(1,1)/U(1)$ (the upper sheet of the two-sheet hyperboloid), and the standard coherent state is specified by a unit pseudo Euclidean vector:

$$\hat{n} \;=\; (\sinh\chi\cos\varphi,\;\; \sinh\chi\sin\varphi,\;\; \cosh\chi)$$

The overlap

$$\langle\,k,\,\zeta_a\,|\,k,\,\zeta_b\,\rangle \;=\; \Big(1-\,|\,\zeta_a\,|^2\Big)^k\,\Big(1\;-\;|\,\zeta_b\,|^2\Big)^k(\,1\;-\;\zeta_a^*\,\zeta_b\,)^{-2k}$$

Then, an orthonormal state [18] to $|\,k, \zeta_b\,\rangle$ is as follows:

$$\left|k,\,\widetilde{\zeta_b}\right\rangle \;\equiv\; \Big(1-|\,\zeta_b\,|^2\Big)^k \sum_{m=0}^{\infty} \left[\frac{\Gamma(m+2k)}{m\,!\;\Gamma(2k)}\right] \left[\zeta_b^m + \frac{\Big(1-|\,\zeta_a\,|^2\Big)^k}{(1-\zeta_a^*\,\zeta_b)^{2k}}\;\zeta_a^m\right] |\,k, m\,\rangle$$

where to simplify we denote as semi-coherent state the following:

$$|\,\widetilde{\alpha}\,\rangle \;=\; \frac{|\,\alpha\,\rangle - |\,\beta\,\rangle\langle\,\beta\,|\,\alpha\,\rangle}{\sqrt{1\;-\;|\langle\,\beta\,|\,\alpha\,\rangle|^2}}$$

To make a construction like Bell's for coherent states one orthonormalizes the SU(1,1) states for example to have a basis of the standard type $|1\rangle$ and $|0\rangle$, namely $|\alpha\rangle, |\widetilde{\alpha}\rangle$ (with k fixed) consequently considering $|\alpha\rangle\;|\widetilde{\alpha}\rangle$, ($\alpha$, $\widetilde{\alpha}$ denoting the coherent state eigenvalues). They can be used for computing processes because they are orthonormal:

$$|\,\psi\,\rangle \;=\; \frac{|\,\alpha\,\rangle\,\otimes\,|\,\widetilde{\alpha}\,\rangle\;+\;e^{i\,\varphi}|\,\widetilde{\alpha}\,\rangle\,\otimes\,|\,\alpha\,\rangle}{\sqrt{2}}$$

Let us notice from the expressions of the coherent state as a function of $|k, m\rangle$ that the Bargmann index must be the same.

Figure 3 below shows the dynamics of the probability for $|\widetilde{\alpha}\rangle$ which is a Gaussian-type [19] entangled state where the degree of entanglement varies as a function of time.

By considering that the evolution equation is the Fokker–Planck type due to the fact that the Hamiltonian is of the type of Equation (58), we have:

$$\frac{\partial\,P}{\partial\,t} \;=\; \frac{\epsilon}{2}\sum_i\left[\frac{\partial(w_i\,P)}{\partial\,w_i}\;+\;\frac{1}{2}\,\frac{\partial^2(w_i\,P)}{\partial\,w_i^2}\right]$$

with

$$w_i \;=\; \mathrm{Re}\;\alpha,\;\; \mathrm{Im}\;\alpha,\;\; \mathrm{Re}\;\widetilde{\alpha},\;\; \mathrm{Im}\;\widetilde{\alpha},\;\; (\,i=1,...,4\,),\;\; \widetilde{\alpha} \;=\; \frac{\alpha-\beta\,\langle\,\beta\,|\,\alpha\,\rangle}{\sqrt{1\;-\;|\,\langle\,\beta\,|\,\alpha\,\rangle\,|^2}},$$

and in standard form the probability:

$$P(\,\alpha,\,\alpha^*,\,\widetilde{\alpha},\,\widetilde{\alpha}^*\,,\,t\,) \;=\; |\,\langle\,m, n\,|\,\psi, k\,\rangle\,|^2.$$

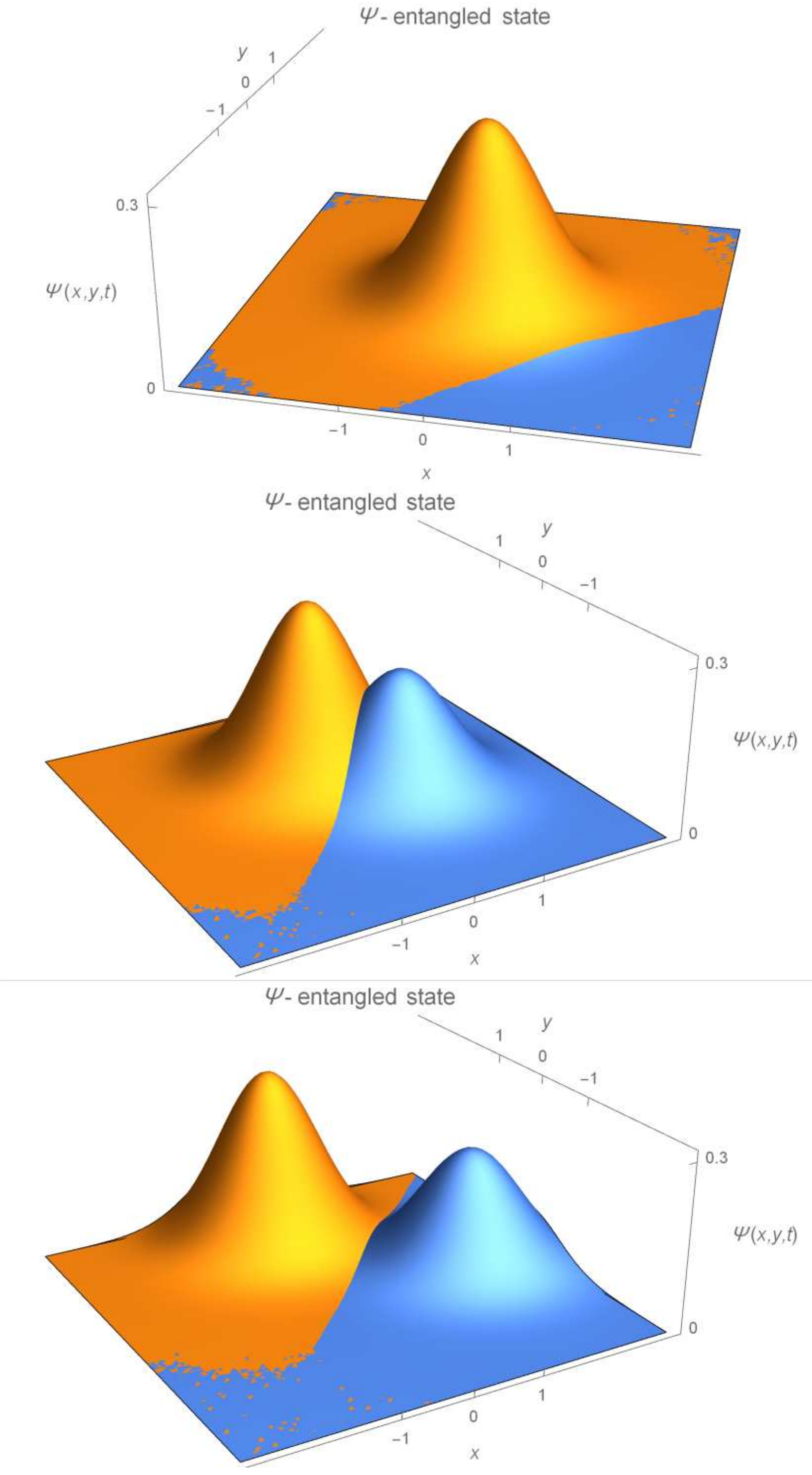

Figure 3. The three images show from top to down the entangled coherent state where the degree of entaglement varies as a function of time from the highest to the lowest degree controlled by the overlap $\langle \alpha \mid \beta \rangle$. Here (x, y) correspond to Re $\widetilde{\alpha}$ and Im $\widetilde{\alpha}$.

7. Schrodinger Cat States and Mp(2): Even and Odd Sectors

From the point of view of the considered Hilbert space, divided into even and odd states we have the following apparent correspondence between standard cat states and the basic states of Mp(2), namely $|1/4\rangle$ and $|3/4\rangle$:

$$\left|\begin{array}{c} Heisenberg--Weyl \\ |\alpha_{+}\rangle \\ |\alpha_{-}\rangle \end{array}\right| \begin{array}{c} \\ \longrightarrow \\ \longrightarrow \end{array} \left|\begin{array}{c} Metaplectic\ Mp(2) \\ |1/4\rangle \\ |3/4\rangle \end{array}\right| \left|\begin{array}{c} \\ \text{Even} \\ \text{Odd} \end{array}\right| \quad (59)$$

where explicitly the standard cat Schrodinger states are

$$|\alpha_{\pm}\rangle = \frac{1}{\sqrt{2 \pm 2\, e^{-2|\alpha|^2}}} [\, |\, \alpha\, \rangle \pm |-\alpha\, \rangle\,]$$

The situation from the point of view of the density matrices is clear in favor of the $Mp(2)$ states that follow the principle of minimum representation. In the case of the standard cat states, the density matrices for the even and odd sectors are the following

$$\rho_{\pm\alpha} = \frac{1}{2\left(1 \pm e^{-2\,|\,\alpha\,|^{2}}\right)} [\, |\, \alpha\, \rangle\langle\, \alpha\, | + |-\alpha\, \rangle\langle -\alpha\, | \pm (\, |-\alpha\, \rangle\langle\, \alpha\, | + |\, \alpha\, \rangle\langle -\alpha\, |\,)] \quad (60)$$

but in the fundamental case of the minimal group representation Mp(2), we evidently have a diagonal representation (as described by Sudarshan) clearly differentiated into the corresponding *even* and *odd* subspaces:

$$\rho_{Mp(2)} = \begin{cases} |\, 1/4\, \rangle\langle\, 1/4\, | & \rightarrow\ even \\ |\, 3/4\, \rangle\langle\, 3/4\, | & \rightarrow\ odd \end{cases} \quad (61)$$

Looking at Equation (59) and comparing it with Equation (58) for the even or odd cases we see that the fundamental substrate of space–time follows the principle of minimum group representation described by the Metaplectic group and not by the given standard Schrodinger cat states: this situation is clearly seen from the tail in the expression for the $\rho_{\pm\alpha}$ Equation (58) while such a tail does not appear in the fundamental expressions of $\rho_{Mp(2)}$ for both $Mp(2)$ even and odd states.

Consequently, in the case of this last description, Equation (61), we see that the geometric substrate of space–time is *quantum in essence* because we are in the minimal representation and it is diagonal (classical and quantum aspects are both represented), and where the power of coherent states in describing *both continuum and discrete space–time* is completely manifest. It must be also noticed:

(i) The states $|1/4\rangle, |3/4\rangle$ are not the same standard cat states as from Ref. [17] but are a fundamental part of the very structure of the space–time itself and do not require an extrinsic generation process.

(ii) The basic states could be forming a generalized state of the type

$$\left|\Psi_{Mp(2)}\right\rangle_{gen} = \frac{1}{\sqrt{|A|^2 \pm |B|^2}} [\, A\, |1/4\rangle \pm B\, |3/4\, \rangle\,]$$

(similar to the standard Schrodinger cat state $\rho_{\pm\alpha}$) giving, in this case, the following density matrix:

$$\rho_{Mp(2)_{gen}} = \frac{1}{|A|^2 \pm |B|^2} \left[|A|^2\, |1/4\rangle\langle 1/4| + |B|^2\, |3/4\rangle\langle 3/4| \pm (B^* A\, |3/4\rangle\langle 1/4| + A^* B |1/4\rangle\langle 3/4|) \right] \quad (62)$$

This expression is comparable in shape to the density matrix $\rho_{\pm\alpha}$ where the parameters A and B may be subject to extrinsic control conditions determined by Majorana Dirac type equations as we established in references [1,2].

8. Entanglement in Quantum de Sitter Space–Time and Black Hole Space–Times

In this Section, we implement the entanglement results obtained in this paper in two important quantum gravitational examples: de Sitter and Black Hole quantum space–times. The basis of these quantum space–time descriptions can be found in refs. [1,3–5,7].

The de Sitter space–time admits at both the classical and quantum levels an *inverted*, (i.e., with imaginary frequency) harmonic oscillator description, where the *oscillator constant* $\kappa_{osc} = c/l_{osc}$ and oscillator length l_{osc} are given by [3,7]:

$$l_{osc\ dS}^{-2} = \left(\frac{m\omega}{\hbar}\right)_{dS} = H^2 = \frac{8\,\pi\,G\,\Lambda}{3} \tag{63}$$

The *oscillator length* l_{osc} is classically the Hubble radius, the Hubble constant $H = \kappa$ being the surface gravity, as the black hole surface gravity is the inverse of (twice) the black hole radius.

For Anti-de Sitter space–time, the description is similar and, (because the AdS two sheet hyperbolid embedding in Minkowski space–time with respect to the deS one sheet hyperboloid), Anti-de Sitter is associated to the real frequency (non inverted) harmonic oscillator.

For the (Schwarzschild) black hole space–time description, the physical magnitudes as the oscillator constant and the oscillator length are related to the black hole mass M by:

$$l_{osc\ BH}^{-2} = \left(\frac{m\omega}{\hbar}\right)_{BH} = l_P^{-2}\left(\frac{m_P}{M}\right)^2 \tag{64}$$

l_P being the Planck length and m_P the Planck mass:

$$l_P = (\,2\,G\,\hbar\,/\,c^3\,)^{1/2}, \qquad h_P = c\,/\,l_P$$

The discrete states and their spectrum describe the quantum space–time levels.

Interestingly, the Metaplectic group states with its *both* sectors and discrete representations, $|\,2\,n\,\rangle$ and $|\,2\,n+1\,\rangle$, *even* and *odd* states, fully *cover* the *complete* Hilbert space $\mathcal{H}$:

$$\mathcal{H} = \mathcal{H}_{(+)} \oplus \mathcal{H}_{(-)} \tag{65}$$

The $(\pm)$ symmetric and antisymmetric sum of the two (*even* and *odd*) states provide the *complete* covering of the Hilbert space and of the space–time mapped from it:

$$\Psi(n) = \Psi^{(+)}(2n) + \Psi^{(-)}(2n+1) \tag{66}$$

The complete covering of the Hilbert space with the complete covering of the quantum space–time is realized by the Metaplectic group symmetry which equivalently provides the CPT symmetric states and unitarity states. For a different approach in the searching of these properties in black hole states see, e.g., refs. [20–22].

The Classical-Quantum Duality of space–time is also realized in the $Mp(n)$ symmetry, because the complete covering, the global complete space–time (and full phase space completion) is needed to make manifest the classical-quantum duality of the space–time (and its phase space mapped from it).

- It is worth mentioning that similar discrete levels can be obtained from the global (complete) classical-quantum duality including gravity [3,5,6], namely classical-quantum gravity duality: The two *even* and *odd* (local) carts or sectors and their (global) $(\pm)$ sum of states, reflect a relation between the $Mp(n)$ symmetry and the classical-quantum duality.
- The two $\sqrt{(2n+1)}$ and $\sqrt{2n}$, *even* and *odd* sets are local coverings and they are *entangled* one to another. The symmetric or antisymmetric sum of these sectors is *global* and required to cover the *whole* manifold.
- Moreover, the corresponding $(\pm)$ global states are complete, CPT symmetric and unitary, the levels n = 0, 1, 2, ..., cover the whole Hilbert space $\mathcal{H} = \mathcal{H}_{(+)} \oplus \mathcal{H}_{(-)}$ and all the space–time regimes.

- The total n states range over *all* scales from the lowest excited levels to the highest excited ones covering the two dual branches $(+)$ and $(-)$ or Hilbert space sectors and corresponding space–time coverings. The two $(+)$ and $(-)$ dual sectors are entangled.

This is interesting because the classical-quantum gravity duality allows that signals or states in the quantum gravity (trans-Planckian) domaine, or semiquantum gravity (inflationary) domaine, do appear as low energy effects in the semiclassical/classical universe today.

From the results of this paper, we have seen that such gravity or cosmological domains *are entangled*: The quantum trans-Planckian primordial phase is a quantum constant curvature de Sitter phase, followed by a quasi-de Sitter (inflationary) phase, and the late Universe today is a semiclassical and classical gravity de Sitter phase. The most quantum and primordial (trans-Planckian) period and its dual: the most classical and late (today) one are *entangled*.

As is well known, de Sitter space–time is the realistic phase of the accelerated expansion of our universe today as shown from the robust set of observational data, cosmo and astro observations. De Sitter or quasi-de Sitter space–time is also the cosmic inflationary phase for the early universe. In between these two asymptotic stages, the radiation and matter-dominated cosmological eras are described by the well-known nonaccelerated expansions, e.g., classical or semiclassical (nonquantum) space–time. Entanglement between states of several kinds (eg radiation fields, matter fields, semiclassical geometry) can occur in such stages, and with different degrees of intensity and variability. Enhancement and conditions of the entanglement in such stages depend too of other effects (such as the presence of additional thermal features, magnetic fields, or other fields and sources).

Evidently, the stability of the entanglement will depend on the coherence of the states that form it. Consequently, if the symmetries in the generation of coherent states are preserved, the entanglement will survive. The entanglement between the asymptotic in and out de Sitter stages of the universe are well-suited examples of it, de Sitter or quasi de Sitter stages being maximally symmetric space–times, non-singular and of constant curvature.

Semi-classical (or semi-quantum) de Sitter or quasi-de Sitter space–time is the cosmic inflationary phase for the early universe in the modern Standard Model of the Universe that is supported on theoretical grounds by the General Theory of Relativity for gravity and QFT (Quantum Field Theory) for matter and particles, and on observational grounds by a wide set of cosmic and astro concordant observations: CMB (Cosmic Microwave Background), LSS (Large Scale Structure) among other data.

Quantum de Sitter space–time for the early universe does appear as the earlier phase from which cosmic de Sitter inflation is a continuation, it does also appear on the grounds of the classical-quantum gravity duality, its connection to the generalized (quantum) Synge algebra, and in the description of quantum space–time in terms of the generalized coherent states of the Metaplectic group.

9. Remarks and Conclusions

The results of this paper have both: fundamental and practical implications of quantum physics *from a novel perspective*:

(i) For the quantum space–time structure properties from one side, by studying in its description concepts as the Berry phase and more generally geometrical (Berry type) phases which until now have been most purely studied in quantum systems but not in quantum gravity as we do here: This is in the quantum space–time context including its trans-Planckian domain and the role of the *Metaplectic group* which is non-compact in this case.

(ii) Second, the new results on entanglement with the states of the Metaplectic group and its covering, which can be taken into account for the searching of new measurable signals: from black holes, the gravitational wave domain and the high energy domain, or the de Sitter primordial phases (inflation and before inflation), and the late de Sitter cosmological vacuum (today dark energy).

(iii) Some points here considering the coherent state solutions and the entangled case are the following: As we have pointed out in ref. [23], a remarkable property of the simple solution given by the physical state $g_{ab}(x)$ is that it is localized in a particular position of space–time. The supermetric coefficients a and $a*$ play the important role of localizing the fields in the bosonic part of the superspace, in a similar and suggestive form as the well-known "warp factors" of multidimensional gravity for a positive (or negative) tension brane.

However, the essential difference here is due to the c-numbers a and $a*$ coming from the B_0 (even) fermionic part of the superspace under consideration. Therefore, no additional and/or topological structures that break the symmetries of the model (i.e., the reflection Z_2-symmetry) are required in our description: the natural structure of the superspace does produce this effect due to the symmetries of the Metaplectic group.

(iv) The Coherent (Gaussian type) solution is a very well-defined physical state in any Hilbert space from the mathematical point of view, contrarily to the case in the literature, e.g., $u(y) = c\ \exp(-H|y|)$ [24] and references therein. In such a case, it was possible to find a manner to include it in any Hilbert space, but it was strongly needed to take special mathematical and physical assumptions whose meanings are not clear.

(v) In the entanglement case, as we can see in the Figures (Figures 2 and 3), all the above properties are preserved. The locality is subject to the degree of entanglement and the limit when $a \to \infty$ where the Gaussian condition (envelope) is lost. In such a limit, only the odd (fermionic) part of the super-manifold survives.

(vi) There is clear evidence of a *time arrow* coming from the physical states as a bilinear combination of the basic states of the Metaplectic $Mp(n)$ group. This appears in the appreciable asymmetry displayed by Figure 1 "Zitterbebegung" as in the fact (CPT) that:

$$\langle\, \Psi_{3/4}(t)\mid L_{ab}\mid \Psi_{1/4}\,(t)\,\rangle \neq \langle\, \Psi_{1/4}(t)\mid L_{ab}\mid \Psi_{3/4}(t)\,\rangle$$

(vii) The obtained Berry phase applied to the de Sitter inflation case is *imaginary* describing the inflationary exponential factor acceleration, as it must be. We also consider the case of cosmic perturbations in the slow roll regime and relate the Berry phase to the cosmological observables: scalar and tensor spectral indices n_s and n_T and the ratio of tensor to scalar perturbations).

(viii) From the density matrix viewpoint in the entanglement context, the precise relation between the Schmidt type representation and the physical state fulfilling the Minimal Group Representation Principle (MGRP) that is bilinear in the basic states of the $Mp(n)$ group, is found. The mapping for the physical state refers to a *new non-diagonal* coherent state representation complementary to that of the Sudarshan diagonal representation.

(ix) In the number basis $|n>$ the physical state corresponding to $s = 2$ (graviton, related to the space–time metric) provides the discretization of the space–time for a small n, going to the continuum for $n \to \infty$.

(x) The basic states in the Minimal Group Representation sense: $|1/4>$ and $|3/4>$ (belonging to the even and odd sectors of the Hilbert space, respectively) are a fundamental part of the very structure of the space–time itself and do not require an additional extrinsic generation process as in the standard Schrodinger cat states and their entanglement.

(xi) The entanglement results in quantum de Sitter space–time admit at both the classical and quantum levels an inverted, (i.e., with imaginary frequency) harmonic oscillator description, (a real frequency (non inverted) harmonic oscillator in AdS).

(xii) In the entanglement results for the (Schwarzschild) black hole space–time, the physical magnitudes as the oscillator constant and the oscillator length are related to the black hole mass M and the Planck mass m_P : $l_{osc\ BH}^{-2} = (m\omega/\hbar)_{BH} = l_P^{-2}\,(m_P/M)^2$.

The external and internal regions of the black hole are classical-quantum duals of each other and are *entangled*. The entanglement occurs from the continuum of external semiclassical/classical states and the discrete very quantum, Planckian and trans-Planckian states. Discrete states here describe the quantum space–time levels and their spectrum.

(xiii) Gravity or cosmological domains from one side and the other of the Planck scale are entangled: The quantum trans-Planckian primordial de Sitter phase (followed by a quasi-de Sitter (inflationary) phase), and the late Universe today semiclassical and classical gravity de Sitter phase are dual of each other and *are entangled*. This is interesting because the classical-quantum gravity duality allows signals or states in the quantum gravity (trans-Planckian) primordial domain to appear as low energy effects in the semiclassical/classical gravity universe today. Such effects deserve future investigations.

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