The Schwinger-Keldysh Formalism from an Effective Field Theory Perspective

by

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Abstract

The coset construction is a tool for systematically building low energy effective actions for Nambu-Goldstone modes. This technique is typically used to compute time-ordered correlators appropriate for S-matrix computations for systems in their ground state. In this paper, we extend this technique to the Schwinger-Keldysh formalism, which enables one to calculate a wider variety of correlators and applies also to systems in a mixed state. We focus our attention on internal symmetries and demonstrate that, after identifying the appropriate symmetry breaking pattern, Schwinger-Keldysh effective actions for Nambu-Goldstone modes can be constructed using the standard rules of the coset construction. Particular emphasis is placed on the thermal state and ensuring that correlators satisfy the KMS relation. We also discuss explicitly the power counting scheme underlying our effective actions. We comment on the similarities and differences between our approach and others that have previously appeared in the literature. In particular, our prescription does not require the introduction of additional "diffusive" symmetries and retains the full non-linear structure generated by the coset construction. We demonstrate our approach with a series of explicit examples, including a computation of the finite-temperature two-point functions of conserved spin currents in non-relativistic paramagnets, antiferromagnets, and ferromagnets. Along the way, we also clarify the discrete symmetries that set antiferromagnets apart from ferromagnets, and point out that the dynamical KMS symmetry must be implemented in different ways in these two systems. Lastly, we introduce the concept of "ajar systems" as an intermediate case between closed and open systems, where the timescale for charge exchange with the environment is parametrically larger than all other characteristic time scales. The Schwinger-Keldysh effective action for such systems exhibits weak explicit symmetry breaking, which we systematically describe using spurion techniques. We then focus on an example with an ajar system with $U(1)$ symmetry, calculating leading-order corrections to correlation functions in both diffusive and spontaneously broken phases.

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Chapter 1

Introduction

Non-equilibrium physics is concerned with the study of systems that are not in thermodynamic equilibrium, i.e. they exhibit dynamic behavior due to the ongoing exchange of energy, matter, or information with their surroundings. Unlike equilibrium systems, which usually admit relatively simple descriptions of their properties, non-equilibrium systems are often far more complex, displaying phenomena such as dissipation, large fluctuations, self-organization, and phase change. These systems can be found in a wide range of contexts, from biological and chemical processes to physical systems such as hydrodynamics [1], systems driven by an external force [2], dynamics of open systems [3] and even quantum information [4] in the context of error correction. Understanding non-equilibrium physics requires new theoretical frameworks, as traditional thermodynamics and statistical mechanics, which primarily focus on equilibrium states, are often insufficient to describe the rich dynamics at play in these systems.

While non-equilibrium classical systems with dissipation and fluctuations, like viscous fluid dynamics and kinetic theory have been investigated since much earlier [5], the systematic study of physics out of equilibrium stretches back to the start of the 20th century. An important one among them is Onsager's work on deriving reciprocal relations of forces and fluxes [6], valid for a system that is in local equilibrium. By this we mean that the system shows a substantial variation of a relevant thermodynamic quantity across its extent, but when zoomed in down to scales much smaller than the typical length scale of these variations we observe a state of quasi-equilibrium. This led to the development of general linear response theory, aspects of which include the Green-Kubo relations [7,8] and generalization of the fluctuation-dissipation theorem [9]. Subsequent advances include extension to nonlinear response via Mori-Zwanzig formalism [10], the connection between dissipation and stability [11] and the use of quantum master equations [12, 13] equations to study dynamics of out of equilibrium systems. The method of Thermofield Dynamics, originally developed in the thermal equilibrium context, [14] has also been extended to non-equilibrium systems [15].

While the work cited above was based in the operator formalism, non-equilibrium physics was also studied from the path integral and field theory perspective. Feynman and Vernon [16] used the path integral approach to model linear dissipative systems, with external effects parametrized through the "influence functional" in the Lagrangian. Schwinger [17] and Keldysh [18] developed a general path integral construction that relied on the doubling of the degrees of freedom and a closed time path for the generating functional. Caldeira and Leggett [19] considered a simple quantum mechanical system coupled to a bath of harmonic oscillators to model quantum dissipation.

Another avenue of research has involved using the approaches above to derive effective theories for out of equilibrium systems at finite temperature. The fluctuating hydrodynamics approach was developed to include noise and external effects [1] in classical and quantum systems in the hydrodynamic regime. The Schwinger-Keldysh formalism was also extended to describe a wide variety of emergent or effective systems, including quantum dissipation [20–22], hydrodynamics [23–26], gravity [27], cosmology [28, 29] holography [30] condensed matter systems [2] and systems with broken symmetries [31–33].

In this work, we extend the existing EFT methods developed for diffusion and hydrodynamics at finite temperature [23, 34] to systems that exhibit Spontaneous Symmetry Breaking (SSB). To do this, we generalize the Callan-Coleman-Wess-Zumino coset construction [35] to the Schwinger-Keldysh formalism (also called the in-in formalism), and derive the building blocks necessary to build effective actions. We consider specifically closed systems with internal symmetries, either global or gauged; and also investigate the case of an open system in the limit of weak interaction with the environment.

The outline of the thesis is as follows:

- Chapter 2 discusses Spontaneously Broken Symmetries (SSB) and their EFTs. We will especially focus on the method of Coset Construction, which allows for the systematic construction of EFTs by utilizing the symmetry breaking pattern as an input.

- Chapter 3 introduces the Schwinger-Keldysh formalism, also known as the in-in formalism, which can be used to do field theory calculations in a non-equilibrium setting and at finite temperature. After a brief introduction to the formalism and a discussion of its systematics, we will compare it with other finite temperature formalisms found in the literature.

- Chapter 4 focuses on how the Schwinger-Keldysh formalism can be used to build effective actions that describe non-equilibrium systems. We will consider the relevant degrees of freedom, the additional symmetry structure that comes from doubling of the fields, and what expansion parameters are relevant when considering a system at finite temperature. Special focus will be on systems

that are initially in a thermal state, and how this results in the dynamical Kubo-Martin-Schwinger symmetry.

- Chapter 5 is on the extension of coset construction methods to the Schwinger-Keldysh formalism. The focus of this section is on systems described by thermal states that exhibit spontaneously broken internal symmetries. The chapter concludes with the application of the in-in coset machinery to the calculation of current-current correlators of paramagnets, anti-ferromagnets and ferromagnets.

- Chapter 6 extends the Schwinger-Keldysh coset construction to systems interacting weakly with the environment through spurions. The spurion method is briefly explained, and then employed in the case of a thermal system with global $U(1)$ symmetry weakly coupled to a bath.

- Chapter 7 provides a summary of our results and considers possible extensions and avenues of investigation.

Chapter 2

Effective Field Theories of Spontaneously Broken Symmetries and Coset Construction

2.1 Effective Field Theories

Natural laws admit a hierarchy of scales, whereby it is possible to ignore microscopic (or macroscopic) physical phenomena below (above) a certain range of scales; length, time, energy, momentum etc., that an experiment can access. This means we do not have to track all position and momenta of gas molecules, numbering around $\sim 10^{23}$ to design an internal combustion engine; or factor in the accelerating rate of the expansion of the universe to predict when the next solar eclipse will happen. We can go further and derive effective theories that to first order neglect the effects of scales far beyond the relevant range, and then systematically include such relatively unimportant effects perturbatively.

This philosophy of building effective theories is also the framework followed by Effective Field Theories (EFTs), where effective actions for field theories are constructed by assigning to each term a ratio E/Λ for $E \ll \Lambda$ where E is the relevant scale (not necessarily energy) and Λ is the cutoff scale above which the theory has no predictive power. Interactions assigned higher powers of this ratio are more irrelevant, and the EFT is able to make predictions in a controlled manner. For energies $E \sim \Lambda$, all interactions become relevant and one then needs to use a different EFT to describe physics at this new scale, with a different cutoff $\Lambda' \gg \Lambda$.

In this way the understanding of EFTs have evolved from phenomenological models created to match experiments (for example Fermi's theory of weak interactions), to the modern framework of treating every Quantum Field Theory (QFT), including the Standard Model, as an EFT with a range of validity and organized in a power counting scheme of relevant scales.

The EFT methods have seen a wide range of application in recent years. Among

these include chiral perturbation theory [36], binary systems and gravitational radiation [37, 38], material science [39], cosmology [40], collider physics [41], condensed matter systems [42], active matter [43], hydrodynamics [44] and EFT of the Standard Model [45]. For an overview of the current state of the art we refer to [46, 47]. Textbook treatment of the subject can be found in [48, 49], see also [50–61] for reviews focusing on various fields of application.

When constructing EFTs, there are three main principles to consider [60]: degrees of freedom, symmetries, and expansion parameters.

Degrees of Freedom

When building an EFT, we need to choose what the degrees of freedom should be that describes the system. There is an element of freedom in such a choice, but what is important is to pick a choice that is relevant, in the sense that it captures the essential characteristics of the low energy dynamics, but is as economical as possible in terms of added structure.

While it may be possible (although not necessarily practical) in some cases to use the same degrees of freedom that describes a system at high energy also for low energies (e.g. QED), this is generically not the case. This is especially true in the cases where low energy dynamics involve bound states, which cannot be derived perturbatively from the high energy degrees of freedom. A prime example of such a theory is QCD, where the high energy dynamics can be captured perturbatively using quark degrees of freedom (due to asymptotic freedom), but at low energies the quarks become confined leading to bound states such as hadrons and mesons, and it is not possible to talk about asymptotic in and out quark states. Thus, the relevant degrees of freedom governing low energy dynamics must be chosen to fit observations and constraints that come from the high energy theory, such as symmetries and resulting conservation laws.

Symmetries

An important component of EFTs is their symmetry content. Any term written down in an EFT must obey the observed symmetries of the system that we want to describe. The presence or lack of a symmetry, or how badly it is broken, are all inputs in the construction of an EFT and the dynamics of the relevant degrees of freedom mentioned above. Some examples of such symmetries are:

- Space-time: Lorentz, Galilean invariance
- Global: Flavor Symmetry in Quantum Chromodynamics (QCD)
- Local (Gauged): Electrodynamics (Classical and Quantum), Standard Model
- Accidental: $SO(4)$ symmetry of the Hydrogen atom, Baryon number conservation in the Standard Model

- Spontaneously Broken: Ferromagnetism, BCS Superconductivity
- Approximate: Chiral Symmetry in QCD
- Anomalous: Axial $U_A(1)$ symmetry in Quantum Electrodynamics with massless fermions

Symmetries dictate what terms can or cannot be included. Any term that obeys the prescribed symmetries of the EFT can and should be included in the action. This means that in principle the action consists of an infinite number of terms. To manage this we need an organizing principle, which brings us to our other guiding principle.

Expansion Parameters

In order to make sense of the generically infinite possible terms in the EFT, they must be organized in a way that quantifies how strongly the inclusion of each term affects the physics. Each EFT therefore comes equipped with one or more expansion parameters, usually as a ratio of scales. For example, this ratio can be E/Λ as in particle physics, where E is the relevant energy scale and Λ is the energy scale where UV effects start to contribute. Other examples include v/c where v is the velocity and c is the speed of light, which comes up when adding relativistic corrections to Newtonian physics; E/k_bT in effective descriptions of statistical systems in the high temperature limit; and d/r in the multipole expansion of the electric potential of a dipole with separation d. Then, each term in the theory is assigned a power of this parameter such that terms that have less dominant effects come with a higher power of the parameter. The range of validity of an effective theory is also governed by the expansion parameter in that when it is close to one, all terms start to contribute equally, and the perturbative expansion breaks down. In this context of critical phenomena, this can also be interpreted as when a phase transition takes place and all scales of the system start to contribute.

All three principles outlined above will be important when building EFTs in the Schwinger-Keldysh formalism that exhibit spontaneous symmetry breaking (SSB). In the following sections, we will first focus on the physics of systems featuring SSB and then describe how their EFTs can be systematically constructed using coset construction.

2.2 Spontaneous Symmetry Breaking and the Goldstone Theorem

The fundamental symmetries of the universe are not always easy to discern. For instance, our immediate surroundings often obscure the underlying principle that there is no preferred location.[1] Even when a symmetry is hidden by circumstances,

[1] Despite the claims of many New Yorkers.

i.e. it is *spontaneously broken*, it still constrains the dynamics of a system, and it leads to observable consequences. A few of examples of systems exhibiting spontaneous symmetry breaking (SSB) are: the Higgs mechanism and chiral symmetry breaking in particle physics; (anti)-ferromagnetism and BCS superconductivity in condensed matter systems; phase transitions in cosmology [62].

The existence of SSB is intimately related to the existence of an order parameter, defined in terms of the relevant degrees of freedom of the system, that acquires a non-zero expectation value for a subset of the parameter space of the theory.

In Quantum Field Theories (QFTs), the order parameter usually corresponds to the expectation value of a field. This can be either a fundamental degree of freedom of the theory like the Higgs field; or a composite field, in the case of chiral symmetry breaking the condensate value of a quark-antiquark pair. This implies that the ground state of the theory corresponds to a non-trivial vacuum state.

More concretely for the case of continuous symmetries, given an operator $\hat{\phi}(t, \vec{x})$, that corresponds to the order parameter, we can define

$$\delta_g \hat{\phi}(t, \vec{x}) \equiv g\hat{\phi}(t, \vec{x})g^{-1} - \hat{\phi}(t, \vec{x}) \tag{2.1}$$

which quantifies the change due to the action of the group element g belonging to the symmetry group G. In the case of SSB, we expect the ground state to be symmetric only under a subset $H \subset G$ of the full symmetry group. This implies that the expectation value of the above will vanish for $h \in H$, since h leaves the SSB vacuum invariant (up to a phase). Next we express a general group element g in terms of the operators $\hat{Q}_i$ that generate the symmetries[2]. Using $\hat{Q}_i$, we can construct a general group element $g = e^{\alpha^i \hat{Q}_i}$. Finally, we insert this into (2.1) and take the expectation value to find

$$\langle 0|[\hat{Q}_i, \hat{\phi}(x, t)]|0\rangle \neq 0 \tag{2.3}$$

for all $\hat{Q}_i$ that correspond to a broken direction of the symmetry. This expression is well-defined in the infinite volume limit (assuming the operator $\hat{\phi}(t, \vec{x})$ is localized to a finite domain of spacetime [61]), which allows us to express $\hat{Q}_i$ inside the commutator in terms of the charge density operator J_i^0 (cf. footnote 2):

$$\langle 0|[\hat{Q}_i, \hat{\phi}(x, t)]|0\rangle = \int d^3x \, \langle 0|[\hat{J}^0(\vec{x}), \hat{\phi}(t, \vec{x})]|0\rangle \tag{2.4}$$

[2]The usual definition of Q_i are given as the conserved charge operators:

$$\hat{Q}_i \equiv \int d^3x \hat{J}_i^0(t, \vec{x}), \tag{2.2}$$

where J^μ is the Noether current associated with G. This definition becomes problematic for SSB, since it is not well-defined in the infinite volume limit [61, 63].

inserting complete sets of states:

$$\begin{aligned}\langle 0|[\hat{Q}_i,\hat{\phi}(x,t)]|0\rangle &= \int d^3x' \sum_n \int \frac{d^3k}{(2\pi)^3} \langle 0|J^0(x')|n_{\vec{k}}\rangle \langle n_{\vec{k}}|\hat{\phi}(x,t)|0\rangle \\ &\qquad - \langle 0|\hat{\phi}(x,t)|n_{-\vec{k}}\rangle \langle n_{-\vec{k}}|J^0(x')|0\rangle \\ &= \int d^3x' \sum_n \int \frac{d^3k}{(2\pi)^3} e^{-ikx'} \Big[\langle 0|J^0(0)|n_{\vec{k}}\rangle \langle n_{\vec{k}}|\hat{\phi}(x,t)|0\rangle \\ &\qquad - \langle 0|\hat{\phi}(x,t)|n_{-\vec{k}}\rangle \langle n_{-\vec{k}}|J^0(0)|0\rangle \Big] \\ &= \sum_n \int d^3k e^{iE_n(\vec{k})t} \delta(\vec{k}) \Big[\langle 0|J^0(0)|n_{\vec{k}}\rangle \langle n_{\vec{k}}|\hat{\phi}(x,t)|0\rangle \\ &\qquad - \langle 0|\hat{\phi}(x,t)|n_{-\vec{k}}\rangle \langle n_{-\vec{k}}|J^0(0)|0\rangle \Big] \end{aligned} \tag{2.5}$$

where, assuming $[\hat{Q}_i,\hat{P}_\mu]=0$, i.e. Q_i do not generate spacetime symmetries; we used $\hat{J}^0(x) = e^{i\hat{P}_\mu x^\mu}\hat{J}(0)e^{-i\hat{P}_\mu x^\mu}$. Since $d\hat{Q}_i/dt = 0$, the time dependence should only be due to $\hat{\phi}(t,\vec{x})$; implying that

$$E_n(\vec{k}) \xrightarrow{\vec{k}\to 0} 0 \qquad \text{as} \qquad \vec{k}\to\vec{0}. \tag{2.6}$$

From this follows the celebrated Goldstone Theorem: for each broken generator, there is an independent gapless mode. These modes often dominate the behavior of the system at large enough length scales and will be important in writing low energy effective actions for systems with SSB. We emphasize that the results in this section apply to SSB of *internal* symmetries. We will briefly discuss SSB of spacetime symmetries and the resulting modes in Sec. 2.6 and 2.7.

2.3 Coset Construction of Internal Symmetries

Spontaneously broken symmetries act non-linearly on the Goldstone fields, and for this reason it is often non-trivial to identify all possible invariant operators that should be included in the low-energy effective action. Fortunately, there exists a systematic procedure to achieve this, which goes under the name of *coset construction* [35, 64–66]. In the simplest scenarios, this method takes as its only input the symmetry breaking pattern $G \to H$, where G is the fundamental symmetry group, and H is the subgroup of symmetries that remain manifest and are realized linearly. The output of this construction is a series of building blocks that can be easily combined to write down the most general local effective action $S_{\rm EFT}[\vec{\pi}]$ describing the Goldstone modes $\vec{\pi}$. Zero-temperature, time-ordered correlators and the corresponding S-matrix

elements can then be calculated using the standard path-integral representation of the generating functional:

$$Z[J] \equiv \int \mathcal{D}\vec{\pi}\, e^{iS_{\rm EFT}[\vec{\pi}]+i\int \mathrm{d}^{d+1}x\, \vec{J}(x)\cdot\vec{\pi}(x)}. \tag{2.7}$$

As previously discussed, one of the tenets of modern EFTs is the principle that all the terms compatible with the symmetries of a physical system should appear in its corresponding effective action. In the presence of spontaneous symmetry breaking the identification of all such operators can be technically challenging, as the symmetries are non-linearly realized on the Goldstone modes of the system, and the standard framework for organizing the classification is the *coset construction* [35, 65, 66]. Essential elements of the technique are reviewed below in the particular case of spontaneously broken *internal* symmetries; a more detailed discussion can be found for instance in [60].

2.4 The Maurer-Cartan Form

The first step is to distinguish those symmetries that are spontaneously broken from those that are not. Letting the breaking pattern be $G \to H$ and denoting the generators of spontaneously broken symmetries by X_i, the coset construction analysis starts by forming the *coset parametrization* $\Omega \in G/H$ which can be canonically written as

$$\Omega = e^{i\pi^j(x)X_j}\ . \tag{2.8}$$

Every broken generator[3] appears with an associated Goldstone field $\pi^i(x)$. The transformation properties of the Goldstone fields under the action of a generic symmetry transformation g are defined by the relation

$$g\,\Omega(\vec{\pi}) = \Omega(\vec{\pi}\,')\,h(g,\vec{\pi}), \tag{2.9}$$

where h is some element of the unbroken subgroup which could in principle depend on g and the fields $\pi^i(x)$.

The transformation laws for π^i encapsulated by (2.9) are in general non-linear and contain infinitely many terms when expanding in fields. Therefore, the operators which respect the requisite symmetries are severely constrained. The central building block for constructing such operators is the *Maurer-Cartan 1-form*, $\boldsymbol{\omega} = \Omega^{-1}\mathrm{d}\Omega$, which is a Lie-algebra valued 1-form that can be conveniently written as

$$\boldsymbol{\omega} = \Omega^{-1}\partial_\mu\Omega\,\mathrm{d}x^\mu \equiv i\,\left(D_\mu\pi^i X_i + \mathcal{A}^B_\mu T_B\right)\mathrm{d}x^\mu\ , \tag{2.10}$$

[3]The broken generators are of course only specified up to the addition of unbroken generators—and in fact this ambiguity can often be leveraged to simplify explicit calculations. One should include in the coset parametrization only those broken generators that are not equivalent up to an unbroken transformation.

where we have denoted unbroken generators with T_B.

The Maurer-Cartan form is useful because the above coefficients enjoy relatively simple transformation properties under the relevant G-symmetries [66]:

- The $D_\mu\pi^i$ factors are non-linear combinations of the Goldstone fields of the form $D_\mu\pi^i = \partial_\mu\pi^i + \mathcal{O}(\pi^2)$ which transform linearly under (2.9):

$$D_\mu\pi^i \quad \longrightarrow \quad h(g,\vec{\pi})^i{}_j D_\mu\pi^j \ , \tag{2.11}$$

  where $h(g,\vec{\pi})^i{}_j$ is a (possibly reducible) representation of the unbroken transformation $h(g,\vec{\pi})$. Because of $D_\mu\pi^i$'s linear transformation property (2.11), it is conventional to refer to this building block as a "covariant derivative", but we wish to emphasize that this terminology is somewhat misleading: because of $\pi^i(x)$'s highly non-linear transformation laws (2.9) one cannot interpret $D_\mu\pi^i(x)$ as a combination of a partial derivative and a connection acting on $\vec{\pi}(x)$ in any standard manner.

- The $\mathcal{A}_\mu^B$ coefficients transform like connections under $h(g,\vec{\pi})$ and can be used to define the following covariant derivative:

$$\nabla_\mu = (\partial_\mu + i\mathcal{A}_\mu^B T_B) \ . \tag{2.12}$$

  We use the ∇_μ to symbol to represent proper covariant derivatives, which distinguish them from the D_μ notation used for Goldstone "covariant derivative" defined above. This derivative can act on any operator $\mathcal{O}^I$ in a linear representation of $h(g,\vec{\pi})$—be that a Goldstone covariant derivative, some matter field, or a combination thereof—and it yields a quantity that once again transforms linearly:

$$\nabla_\mu\mathcal{O}^I \quad \longrightarrow \quad h(g,\vec{\pi})^J{}_I \nabla_\mu\mathcal{O}^I. \tag{2.13}$$

The most general G-symmetric in-out effective action describing Goldstone modes and their interactions with matter fields Ψ^I is then of the functional form

$$S = \int \mathrm{d}^{d+1}x\, \mathcal{L}(D_\mu\pi^i, \Psi^I, \nabla_\mu), \tag{2.14}$$

where all indices are contracted using the appropriate invariant tensors associated with the unbroken group H.

The effective action as constructed above is exactly invariant under all symmetries. However, we must also include Wess-Zumino terms; terms which are invariant only up to a total derivative and can be systematically constructed using the Maurer-Cartan form defined above [67]. To describe the process very briefly, these terms can be obtained by starting with the wedge product of $d+2$ copies of the Maurer-Cartan form and contracting them in an invariant way. If the resulting $d+2$ form α is exact, i.e. there exists a $d+1$ form such that $\alpha = d\beta$, and furthermore if β shifts by a total derivative under G-transformations, then β is a Wess-Zumino term that should be included in the effective action.

2.5 Gauge Symmetries

One can also extend the previous construction to the case in which some or all the symmetries are gauged. This requires the introduction of a gauge field $A_\mu = \vec{A}_\mu \cdot \vec{Q}$, where the generators $\vec{Q}$ stand for some or all the T_A's and X_i's. The gauged Maurer-Cartan form then reads:

$$\boldsymbol{\omega} = \Omega^{-1} \left(\partial_\mu + iA_\mu\right) \Omega \, \mathrm{d}x^\mu \ , \tag{2.15}$$

where $A_\mu \longrightarrow g(x) \left(A_\mu - i\partial_\mu\right) g^{-1}(x)$ under a gauge transformation, as usual. The various components of $\boldsymbol{\omega}$ defined in Eq. (2.10) transform as before, except that now the building blocks $\mathcal{D}_\alpha \pi^i$ and $\mathcal{A}_\mu^B$ also depend on the gauge field A_μ.

The gauged coset construction is, of course, natural to consider in the context of spontaneously broken gauge theories, in which case A_μ is a dynamical field. Even in theories with spontaneously broken global symmetries, though, it can be fruitful to introduce non-dynamical gauge fields to aid in the path-integral description of such systems. More precisely, treating the A_μ as an external field, one can define a generating functional for correlators of the conserved currents $\mathcal{J}^\mu$ associated with the symmetries that have been gauged, as in:

$$Z[A] = \int \mathcal{D}\vec{\pi} \, e^{iS_{\text{gauged}}} \ , \qquad i^n \frac{\delta^n Z[A]}{\delta A_{\mu_1}(x_1) \ldots \delta A_{\mu_n}(x_n)}\bigg|_{A=0} \equiv \langle T \mathcal{J}^{\mu_1}(x_1) \ldots \mathcal{J}^{\mu_n}(x_n) \rangle . \tag{2.16}$$

It is straightforward to show that $Z[A]$ is gauge-invariant (in the absence of anomalies), from which the conservation of the corresponding currents $\mathcal{J}^\mu$ (and Ward-identities, more generally) follows. See e.g. [36,68,69] for reviews of this construction.

2.6 Coset Construction of Spacetime Symmetries

While this thesis will focus on *internal* symmetries that are spontaneously broken, for completeness we briefly mention the case of spontaneously broken spacetime symmetries [65, 66, 70–72]. For such systems the Goldstone theorem does not hold, but the coset construction method can still be used. Since we want the unbroken symmetry group H to be realized linearly, we need to treat spacetime translations P_μ as broken and include them in the coset element even if they are not physically broken. If translations are actually broken, they are included with a dynamical field as the parameter, if not, the generators are accompanied by spacetime coordinates.

Another caveat is that if the commutator of one broken generator with unbroken spacetime translations leads to another broken generator, then one can replace the dynamical field associated to the first in terms of derivatives of the second. This is dubbed the Inverse Higgs mechanism [73, 74]. Finally, the component of the Maurer-Cartan form along the P_μ-direction leads to the vielbeins $e_\alpha{}^\mu$ that connect the group

structure to the spacetime coordinates. Their utility is in expressing the Maurer-Cartan form of a broken spacetime generator $\mathcal{D}_\mu \pi^\nu$, that has one spacetime and one group index as an object $e_\alpha{}^\mu \mathcal{D}_\mu \pi^\nu$ that acts covariantly under the "internal" group transformation.

We will briefly demonstrate the methods described above with the example of a 4D brane moving in 5D Minkowski space [75, 76]. The symmetry group of the latter is $G = ISO(4,1)$ with generators (P_A, J_{AB}), with indices A, B ranging from 0 to 5, and it is spontaneously broken down to $H' = ISO(3,1)$ with the generators $(P_\mu, J_{\mu\nu})$. Since we want H to be linearly realized, we will choose $H = SO(3,1)$ with $(J_{\mu\nu}$ instead and take the coset element to be

$$\Omega = e^{x^\mu P_\mu} e^{\pi P_5} e^{\xi^a J_{\alpha 5}} \tag{2.17}$$

where the physically broken generators are P_5, the translations along the 5-direction, and $J_{\alpha 5}$ corresponding to boosts and rotations involving the 5th dimension. Using the commutation relations for the generators of G,

$$\begin{aligned} [P_A, P_B] &= 0, \\ [J_{AB}, P_C] &= \eta_{AC} P_B - \eta_{BC} P_A, \\ [J_{AB}, J_{CD}] &= \eta_{AC} J_{BD} - \eta_{AD} J_{BC} + \eta_{BC} J_{AD} - \eta_{BD} J_{AC} \end{aligned} \tag{2.18}$$

the components of the Maurer-Cartan form can now be calculated [75]:

$$\begin{aligned} \omega_P^\alpha &= dx^\alpha - \frac{1}{2} \frac{\psi^\alpha \psi_\nu}{1 + \frac{\psi^2}{4}} dx^\nu + \frac{\psi^\alpha}{1 + \frac{\psi^2}{4}} \partial_\mu \pi dx^\mu, \\ \omega_{P_5} &= \frac{1 - \frac{\psi^2}{4}}{1 + \frac{\psi^2}{4}} \partial_\mu \pi dx^\mu - \frac{\psi_\mu}{1 + \frac{\psi^2}{4}} dx^\mu, \\ \omega_J &= \frac{\partial_\mu \psi}{1 + \frac{\psi^2}{4}} dx^\mu \end{aligned} \tag{2.19}$$

where

$$\psi_\mu \equiv \xi_\mu \frac{\tanh \sqrt{-\frac{\xi^2}{4}}}{\sqrt{-\frac{\xi^2}{4}}}. \tag{2.20}$$

To implement the inverse Higgs mechanism, we look at the following commutator

$$[J_{\nu 5}, P_\mu] = i \eta_{\mu\nu} P_5, \tag{2.21}$$

which implies that solving $\omega_{P_5} = 0$ we can replace ψ^μ in terms of $\partial^\mu \pi$. The solution to the constraint is

$$\psi_\mu = \frac{2 \partial_\mu \pi}{1 + \sqrt{1 + (\partial \pi)^2}}. \tag{2.22}$$

Using the constraint again, we can simplify ω_P^α:

$$\omega_P^\alpha = \left(\delta_\mu^\alpha + \frac{1}{2}\frac{\psi_\mu \psi^\alpha}{1-\frac{\psi^2}{4}}\right) dx^\mu = e_\mu{}^\alpha dx^\mu. \tag{2.23}$$

We can now use ω_P and ω_J to build the effective action using H-invariant contractions using the invariants $\eta^{\mu\nu}$ and $\epsilon^{\mu\nu\rho\sigma}$. This leads to the following leading order term

$$\mathcal{L} = -\frac{1}{4!}\int \epsilon_{\mu\nu\rho\sigma}\omega_P^\mu \wedge \omega_P^\nu \wedge \omega_P^\rho \wedge \omega_P^\sigma = \int d^4x\sqrt{1+(\partial\pi)^2} \tag{2.24}$$

which is the Dirac-Born-Infeld action.

2.7 Classification of Goldstone Modes

The Goldstone theorem specifies the number and dispersion relations for low energy modes in spontaneously broken systems, but it only holds for cases where Lorentz invariance is a symmetry. However, when Lorentz invariance is explicitly broken, the low energy spectrum may have fewer degrees of freedom showing non-identical dispersion relations. A few examples of such systems are ferromagnets [77], Bose-Einstein condensation of cold atomic systems [78,79] and Kaon condensation [80–82]. This has led to work in generalizing the Goldstone Theorem to systems where Lorentz invariance is explicitly broken, and the classification of the types of Goldstone modes that appear in them [83–87]. In this section we briefly summarize these results.

For Lorentz invariant EFTs, the lowest order term we can write for the Goldstone modes is given by[4]

$$\mathcal{L} = f^2\omega_\mu^i\omega_i^\mu = D_\mu\pi^i D^\mu\pi_i + \dots \tag{2.25}$$

where f is an overall scale. This leads to an identical dispersion relation $\omega \sim c|\mathbf{k}| + \dots$ for each Goldstone mode.

For systems without Lorentz invariance at zero density, there is less restriction on the form of the EFT. To proceed, it is easier to switch to an equivalent form for the EFT, by considering the π^i's to be the coordinates parametrizing the manifold of ground states, i.e. the coset space G/H [68]. Assuming rotational invariance, the lowest order effective Lagrangian can be written as [82]

$$\mathcal{L} = -c_a\omega_t^a + \frac{1}{2}\bar{g}_{ab}\omega_t^a\omega_t^b - \frac{1}{2}g_{ab}\omega_i^a\omega_i^b + \dots \tag{2.26}$$

[4]For simplicity, we assume the π^i's form an irreducible representation of H. If they instead form a reducible representation, then the kinetic term for each irreducible component should in principle have a different coefficient up front.

where ω_t^a, ω_i^a are the time and space components of the Maurer-Cartan form, and the constants are the lowest order in the expansion of the metric and einbein of the internal manifold which obey the following constraints: $f_{ab}{}^d c_d = 0$, $f_{ab}{}^d g_{dc} + f_{ac}{}^d g_{bd} = 0$, and $f_{ab}{}^d \bar{g}_{dc} + f_{ac}{}^d \bar{g}_{bd} = 0$. In fact, $c_a = \langle \hat{q}_a(t, \vec{x}) \rangle$ [68], and the Lorentz invariant case is recovered when $e_a = 0$ and $\bar{g}_{ab} = g_{ab}$. Expanding the Maurer-Cartan components, we find to lowest order,

$$\mathcal{L} = -\langle \hat{J}_a^0(t, \vec{x}) \rangle \, \partial_t \pi^a - \langle \hat{J}_a^0 \rangle \, \frac{1}{2} f_{bc}{}^a \pi^b \partial_t \pi^c + \frac{1}{2} \bar{g}_{ab} \partial_t \pi^a \partial_t \pi^b - \frac{1}{2} g_{ab} \vec{\nabla} \pi^a \vec{\nabla} \pi^b. \tag{2.27}$$

The first term is a total derivative which we drop, and we can re-express the constants in the second term as a matrix:

$$\rho_{ab} \equiv \langle J_a^0 \rangle \, f_{bc}{}^a = -i \, \langle [\hat{Q}_b, J_c^0(t, \vec{x})] \rangle \tag{2.28}$$

and we obtain

$$\mathcal{L} \simeq \frac{1}{2} \rho_{ab} \pi^a \partial_t \pi^b + \frac{1}{2} \bar{g}_{ab} \partial_t \pi^a \partial_t \pi^b - \frac{1}{2} g_{ab} \vec{\nabla} \pi^a \vec{\nabla} \pi^b. \tag{2.29}$$

With the right choice of orthogonal matrix, ρ_{ab} can be diagonalized to have the form

$$\rho_{ab} = \left(\begin{array}{cccc|c} 0 & \lambda_1 & & & \\ -\lambda_1 & 0 & & & \\ & & 0 & \lambda_2 & \\ & & -\lambda_2 & 0 & \\ & & & & \ddots \\ \hline & & & & 0 \end{array} \right). \tag{2.30}$$

This form of ρ_{ab} means we have two different types of Goldstone modes. The Type A modes correspond to the block diagonal elements where all $\rho_{ab} = 0$, and therefore a dispersion relation $\omega = \pm c_s |k|$, where $c_s = g_{ab}/\bar{g}_{ab}$. The Type B modes correspond to non-zero elements of ρ_{ab}, and its form implies that π^{2l} and π^{2l-1} are conjugate variables, with the implication that these modes have a quadratic dispersion relation $\omega = k^2/2m$ where $m \sim \rho_{ab}/g_{ab}$. The general statement for the classification of Goldstone modes can thus be summarized as

$$n_{\text{Broken Gen.}} = n_A + 2n_B, \qquad n_B = \frac{1}{2} \operatorname{rank} \rho, \qquad n_A = n_{\text{Broken Gen.}} - \frac{1}{2} \operatorname{rank} \rho, \tag{2.31}$$

These counting rules have been generalized even further for the cases of finite density and SSB of spacetime symmetries [87], with the classification now containing four types of Goldstone modes:

1. Linear gapless modes corresponding to Type A above.

2. Quadratic gapless modes corresponding to Type B above.

3. Fixed gap modes where the gap is of the order of the chemical potential and dictated by the SSB pattern.

4. Unfixed gap modes that have a gap of the order of the chemical potential but depend on additional free parameters.

In addition to deriving the number of Goldstone modes for the first three categories above, the authors also provided a bound for the fourth one:

$$n_2 \leq n_4 \leq n_2 + n_3. \tag{2.32}$$

In Sec. 5.3, we will encounter Type I and Type II modes when discussing Nambu-Goldstone modes for magnets at finite temperature.

Chapter 3

The Schwinger-Keldysh (In-In) Formalism

3.1 Outline of the Formalism

Generating functionals of the form (2.7) are appropriate for so-called *in-out* calculations, in which the state of the system is specified both at early and late times [88]. Scattering events are the prototypical example of this scenario. There are however many other physical quantities which require instead in-in calculations, in which the (pure or mixed) state of a system is only specified at some initial time, and afterward the system evolves according to its own dynamics. The observables relevant for cosmology, hydrodynamics, and generic dissipative systems, for example, are most readily phrased within this latter framework, and finite-temperature effects can be included in a natural manner. In this setting, calculations are instead often performed using the *Schwinger-Keldysh* generating functional, whose path-integral representation requires a *doubling* of fields, the analog of (2.7) being

$$Z[J_1, J_2] \equiv \int \mathcal{D}\vec{\pi}_1 \mathcal{D}\vec{\pi}_2 \, e^{iS_{\rm EFT}[\vec{\pi}_1,\vec{\pi}_2]+i\int \mathrm{d}^{d+1}x\, \vec{J}_1(x)\cdot\vec{\pi}_1(x)-\vec{J}_2(x)\cdot\vec{\pi}_2(x)} \,. \tag{3.1}$$

In this section, we discuss the Schwinger-Keldysh formalism and elucidate the origins of the doubling of the degrees of freedom. We work in the Heisenberg picture, in which operators evolve in time and the states remain stationary. We start with the time evolution of the density matrix representing the state of the system (we're suppressing space dependence):

$$\rho(t) = U(t, t_i)\rho_0 U^\dagger(t, t_i) \tag{3.2}$$

where ρ_0 is the initial state of the system, and $U(t, t_i)$ is the unitary time evolution operator. Formally it has the solution

$$U(t_f, t_i) = \mathrm{T}\exp\left(-i\int_{t_i}^{t_f} H(t')dt'\right) \tag{3.3}$$

where T is the time-ordering operator. Note that unitary evolution of $\rho(t)$ means that its trace is preserved for all times. Finding the expectation value of any operator $\mathcal{O}$ at time $t_i < t < t_f$ is possible by using

$$\langle \mathcal{O}(t) \rangle = \frac{\mathrm{Tr}(\rho(t)\mathcal{O})}{\mathrm{Tr}(\rho_0)} = \frac{1}{\mathrm{Tr}(\rho_0)} \mathrm{Tr}\big(\rho_0 U^\dagger(t,t_i)\mathcal{O}U(t,t_i)\big) \tag{3.4}$$

Using two properties of time evolution operator, namely $U(t_3,t_2)U(t_2,t_1) = U(t_3,t_1)$ and $U^\dagger(t_2,t_1) = U(t_1,t_2)$, one can rewrite the above as:

$$\langle \mathcal{O}(t) \rangle = \frac{1}{\mathrm{Tr}(\rho_0)} \mathrm{Tr}(U(t_i,t_f)U(t_f,t)\mathcal{O}U(t,t_i)\rho_0) \tag{3.5}$$

This suggests an interpretation of the above as the insertion of an observable $\mathcal{O}$ at time t on a closed time contour that evolves from t_i to t to t_f and then back to t_i. In the following we will take $t_f = \infty$ and $t_i = -\infty$ to make contact with the usual QFT methods. Defining the generating functional

$$Z \equiv \frac{\mathrm{Tr}\big(U(\infty,-\infty)\rho_0 U^\dagger(\infty,-\infty)\big)}{\mathrm{Tr}(\rho_0)} \tag{3.6}$$

and taking $\mathrm{Tr}\,\rho_0 = 1$, we can make contact with path integral methods by inserting complete sets of states.

$$\begin{aligned} Z = \int D\varphi_a D\varphi_b D\varphi_c \, \langle \varphi_a, \infty | U(\infty,-\infty) | \varphi_b, -\infty \rangle \, \langle \varphi_b, -\infty | \rho_0 | \varphi_c, -\infty \rangle \\ \times \langle \varphi_c, -\infty | U^\dagger(\infty,-\infty) | \varphi_a, \infty \rangle \, . \end{aligned} \tag{3.7}$$

It is possible to denote each matrix element in the above expression in terms of path integrals. For example the second element above can be written as

$$\langle \varphi_b, \infty | U(\infty,-\infty) | \varphi_c, \infty \rangle = \int_{\varphi(-\infty)=\varphi_b}^{\varphi(\infty)=\varphi_a} \mathcal{D}\varphi \, e^{iS[\varphi]}, \tag{3.8}$$

and the third term can be acquired from the above by complex conjugation (we are using $\mathcal{D}\varphi$ for the path integral measure and $D\varphi$ for integration over the field configurations at the boundaries). In terms of path integrals, the generating functional now has the form

$$Z = \int D\varphi_a D\varphi_b D\varphi_c \rho_{ac} \int_{\varphi_1(-\infty)=\varphi_b}^{\varphi_1(\infty)=\varphi_a} \mathcal{D}\varphi_1 e^{iS[\varphi_1]} \int_{\varphi_2(\infty)=\varphi_a}^{\varphi_2(-\infty)=\varphi_c} \mathcal{D}\varphi_2 e^{-iS[\varphi_2]}, \tag{3.9}$$

where we have defined $\rho_{ac} \equiv \langle \varphi_b, -\infty | \rho_0 | \varphi_c, -\infty \rangle$. Things to note here, that will be further explored later, are that we have two copies of the fields, with the combination of path integrals tracing a closed-time path contour (see Fig. 3.3 for the case of an

initial thermal state). Furthermore, the boundary conditions for the two fields match at positive infinity, which has important symmetry implications for the generating functional. Finally, the initial state of the system is accounted for by the matrix element ρ_{ac}, and for a system initially in a thermal state this leads to a manifestation of the Kubo-Martin-Schwinger condition.

The formalism can be generalized in the usual way to calculate correlators of observables. To demonstrate this, consider a *closed* quantum system described by some fields collectively denoted by φ and corresponding action $S[\varphi]$, and that is in a state specified by the density matrix ρ.[1] The most general observables of the system are phrased in terms of correlation functions of the form

$$\langle \mathcal{O}^{(n)}(x_n) \dots \mathcal{O}^{(1)}(x_1) \rangle \equiv \mathrm{Tr}\, \rho\, \mathcal{O}^{(n)}(x_n) \dots \mathcal{O}^{(1)}(x_1) \ , \tag{3.10}$$

where the $\mathcal{O}^{(i)}$'s are local operators built out of the fields φ and their derivatives evaluated at points x_i that are not in any particular order. It is possible to choose generic orderings of operators that are not time-ordered [89], but in many applications one is interested in (linear combinations of) the restricted set of observables where the operator product is of the following factorized form:

$$\langle \bar{T} \left[\mathcal{O}^{(n)}(x_n) \dots \mathcal{O}^{(m+1)}(x_{m+1}) \right] T \left[\mathcal{O}^{(m)}(x_m) \dots \mathcal{O}^{(1)}(x_1) \right] \rangle \ , \tag{3.11}$$

where T and $\bar{T}$ denote time-ordered and anti-time-ordered products, respectively. Specializing further to the case of a single operator $\mathcal{O}(x)$ for clarity of presentation, correlators of this form can be obtained systematically by differentiating a generating functional that depends on two external currents:

$$Z[J_1, J_2] \equiv \mathrm{Tr} \left[\rho\, \bar{T} e^{-i \int J_2 \mathcal{O}}\, T e^{i \int J_1 \mathcal{O}} \right] . \tag{3.12}$$

Generators for correlators of more than one operator or with more general time-orderings can be similarly constructed, but the one in Eq. (3.12) will be sufficient for our purposes.

Inserting resolutions of the identity $\mathbf{1} = \int D\varphi\ |\varphi, \pm\infty\rangle \langle \varphi, \pm\infty|$ between the various factors inside the trace, we can rewrite the generating functional as

$$\begin{aligned} Z[J_1, J_2] = \int D\varphi_a D\varphi_b D\varphi_c\ \langle \varphi_a, -\infty | \rho | \varphi_c, -\infty \rangle \langle \varphi_c, -\infty | \bar{T} e^{-i \int J_2 \mathcal{O}} | \varphi_b, +\infty \rangle \\ \times \langle \varphi_b, +\infty | T e^{i \int J_1 \mathcal{O}} | \varphi_a, -\infty \rangle , \end{aligned} \tag{3.13}$$

where the last factor admits a path integral representation of the form

$$\langle \varphi_b, +\infty | T e^{i \int J_1 \mathcal{O}} | \varphi_a, -\infty \rangle = \int_{\varphi(-\infty)=\varphi_a}^{\varphi(+\infty)=\varphi_b} \mathcal{D}\varphi\, e^{iS[\varphi] + i \int \mathrm{d}^{d+1}x\, J_1 \mathcal{O}} \ , \tag{3.14}$$

[1] We are working in the Heisenberg picture, where operators and their eigenstates evolve in time, while the state of the system does not.

and a similar, conjugated expression holds for the second factor.

It is customary for the preceding construction to be summarized in the compact form

$$Z[J_1, J_2] = \int_\rho \mathcal{D}\varphi_1 \mathcal{D}\varphi_2 \, \exp\left[iS[\varphi_1] - iS[\varphi_2] + i \int \mathrm{d}^{d+1}x \, J_1 \mathcal{O}(\varphi_1) - J_2 \mathcal{O}(\varphi_2) \right] . \tag{3.15}$$

We see therefore that calculating correlation functions of the form (3.11) requires a doubling of the field content: this is a direct consequence of the fact that expectation values, as opposed to transition amplitudes, require time-evolving *both* the corresponding bras and kets, with one set of fields performing each such evolution—see [88] for a related discussion.

In order to calculate correlators, we can extend the logic of perturbative in-out calculations to the in-in path integral (3.15) above. This involves evaluating the two path integrals at the quadratic level, and then adding the two copies of the interactions in a perturbative manner. For simplicity, we choose $\mathcal{O} = \varphi$ in what follows.

One of the key differences here is that unlike the in-out boundary conditions, where we assume adiabatic evolution from vacuum, here we have the matrix element ρ_{ac} at initial time. While the initial state can be taken to be arbitrary [90], two common choices found in the literature are the vacuum, and the thermal state $\rho = e^{-\beta \mathcal{H}}$. The choice of vacuum leads to the path integrals being decoupled at the boundary conditions, and tree level interactions do not mix between the two copies, although loop level contributions may involve a combination of both.

The thermal initial state $\rho = e^{\beta \mathcal{H}}$ can be interpreted as time evolution along imaginary time by $t = -i/T = -i\beta$. This means we can think of the matrix element ρ_{ac} as a Euclidean path integral:

$$\langle \varphi_a, -\infty | e^{-\beta \mathcal{H}} | \varphi_c, -\infty \rangle = \int_{\varphi(-\infty - i\beta)}^{\varphi(-\infty) = \varphi_c} \mathcal{D}\varphi e^{-iS[\varphi]} \tag{3.16}$$

(when evaluating path integrals we take a finite time t_i as the initial time and send $t_i \to -\infty$ at the very end). We also assume that interactions and sources J_i are turned off at $t = -\infty$, and adiabatically switched on (off) along the top (bottom) contour. The path integrals for the free actions can be evaluated using

$$\int_{\varphi_f(t_f)}^{\varphi_i(t_i)} \mathcal{D}\varphi e^{iS_{free}[\varphi, J]} = e^{iS[\varphi_{cl}, J]} \tag{3.17}$$

where φ_{cl} is the classical solution of the equations of motion with the boundary conditions $\varphi_{cl}(t_i) = \varphi_i$, $\varphi_{cl}(t_f) = \varphi_f$ to evaluate each matrix element, and integrate over the fields $\varphi_a, \varphi_b, \varphi_c$ in (3.9), to obtain

$$-i \ln Z[J_1, J_2] \equiv W[J_1, J_2] = \int d^4x \begin{pmatrix} J_1(x) & J_2(x) \end{pmatrix} \begin{pmatrix} G_{11}(x, x') & G_{12}(x, x') \\ G_{21}(x, x') & G_{22}(x; x') \end{pmatrix} \begin{pmatrix} J_1(x') \\ J_2(x') \end{pmatrix} \tag{3.18}$$

where the Green's functions are

$$\begin{aligned} iG_{11}(x,x') &= \langle T\varphi_1(x)\varphi_1(x')\rangle\,, \qquad & iG_{12}(x,x') &= \langle \varphi_2(x')\varphi_1(x)\rangle \\ iG_{21}(x,x') &= \langle \varphi_2(x)\varphi_1(x')\rangle\,, & iG_{22}(x,x') &= \langle \bar{T}\varphi_2(x)\varphi_2(x')\rangle \end{aligned} \tag{3.19}$$

where T ($\bar{T}$) denotes (anti-)time ordering operator. This generating functional now contains a 2×2 matrix, and we have 4 Green's functions instead of the 1 for the in-out case. We can identify $G_{11} = G_F$ as the Feynman propagator in the first leg, and $G_{22} = \tilde{G}_F$ the anti-time ordered counterpart on the second leg of the contour. Also note that due to path ordering along the contour, mixed Green's functions always have φ_1 to the right of φ_2.

We can now use the generating functional in (3.18) and the two copies of the interactions to derive Feynman rules and calculate correlation functions. However, the physical interpretation of these correlators built out of 1- and 2-fields is not immediately obvious.

To make connection with physical observables, we first start with the fact that the four Green's functions defined in (3.19) are not completely independent. To see this, we note that the following combinations of Green's functions is zero:

$$\begin{aligned} i\,(G_{11}(x,x') &+ G_{22}(x,x') - G_{12}(x,x') - G_{21}(x,x')) \\ &= \langle T\varphi_1(x)\varphi_1(x')\rangle + \langle \bar{T}\varphi_2(x)\varphi_2(x')\rangle - \langle \varphi_2(x')\varphi_1(x)\rangle - \langle \varphi_1(x)\varphi_2(x')\rangle \\ &= \theta(t-t')\,\langle\varphi(x)\varphi(x')\rangle + \theta(t'-t)\,\langle\varphi(x')\varphi(x)\rangle \\ &+ \theta(t'-t)\,\langle\varphi(x)\varphi(x')\rangle + \theta(t-t')\,\langle\varphi(x)\varphi(x')\rangle \\ &- \langle\varphi(x')\varphi(x)\rangle - \langle\varphi(x)\varphi(x')\rangle \\ &= [\theta(t-t') + \theta(t'-t) - 1]\,[\langle\varphi(x)\varphi(x')\rangle + \langle\varphi(x')\varphi(x)\rangle] = 0 \end{aligned} \tag{3.20}$$

where $\theta(t-t')$ is the step function. In the third line we used the fact that the Wightman functions of the various combinations of the two copies all evaluate to the same functions, and in the last line we used the identity $\theta(t)+\theta(-t)-1=0$. We can make use of this redundancy by switching to the Keldysh basis [18]:

$$\begin{aligned} \varphi_r &= \frac{1}{2}\,(\varphi_1+\varphi_2)\,, \qquad & \varphi_a &= \varphi_1 - \varphi_2 \\ J_r &= \frac{1}{2}\,(J_1+J_2)\,, & J_a &= J_1 - J_2. \end{aligned} \tag{3.21}$$

The Green's functions in the Keldysh basis become

$$\begin{aligned} iG_{rr} &= \langle\varphi_r\varphi_r\rangle = \frac{1}{4}\,(G_{11}+G_{22}+G_{12}+G_{21}) \\ iG_{ra} &= \langle\varphi_r\varphi_a\rangle = \frac{1}{2}\,(G_{11}-G_{22}-G_{12}+G_{21}) \\ iG_{ar} &= \langle\varphi_a\varphi_r\rangle = \frac{1}{2}\,(G_{11}-G_{22}+G_{12}-G_{21}) \\ iG_{aa} &= \langle\varphi_a\varphi_a\rangle = 0. \end{aligned} \tag{3.22}$$

Apart from the advantage of reducing the number of propagators in the Feynman rules, the Keldysh basis also has a nice physical interpretation in that the r-fields correspond to the physical degrees of freedom in the $\hbar \to 0$ limit, and the a-fields regulate classical and quantum effects. We will expand more on this in Sec. 4.5, and also discuss the various properties of the Keldysh basis correlators in Sec. 4.7.

3.2 Comparison with Other Finite Temperature Formalisms

The Schwinger-Keldysh (SK) formalism is not the only finite temperature used in the literature. In this section we briefly mention other methods and their applications. A detailed comparison of the three finite temperature methods discussed above and in this section is given in [91], for a textbook discussion, see also [92].

3.2.1 Imaginary Time Formalism

The imaginary time formalism, also called the Matsubara formalism [93], utilizes the fact that the Hamiltonian H generates time-translations in quantum mechanics through the unitary operator $U(t) = e^{-it\mathcal{H}}$. This allows for the interpretation that the thermal state given by

$$\rho = e^{-\beta\mathcal{H}}$$

can be taken as generating a time-translation along the imaginary time axis by $-i\beta$, where the inverse temperature β is defined to be $\beta = (k_b T)^{-1}$. The partition function of the thermal system is given by

$$Z(\beta) = \operatorname{Tr}\rho = \int dx \, \langle x|e^{-\beta\mathcal{H}}|x\rangle \, . \tag{3.23}$$

We can rewrite the partition function as a Euclidean path integral by taking the paths to be defined on imaginary time $\tau = -it$, with start and end times $\tau_i = 0$ and $\tau_f = \beta$. Apart from the imaginary time, the second main difference compared to zero temperature real time quantum mechanics is that we now have a periodicity condition due to the trace operation. This means when calculating the path integral, we need to impose the condition $x_i(\tau_i = 0) = x_f(\tau_f = \beta)$. Defining the Euclidean action as

$$S_E \equiv \int_0^\beta d\tau \mathcal{L}_E, \tag{3.24}$$

the partition function is given by

$$Z(\beta) = \int \mathcal{D}x e^{-S_E[x]}. \tag{3.25}$$

The path integral derivation above can be generalized to quantum field theory at finite temperature. For example, for a scalar field $\phi(t, x)$ and external current J, the partition function can be written as:

$$Z[J, \beta] = \int \mathcal{D}\phi e^{-S_E[\phi] - \int d^4x J\phi} \tag{3.26}$$

again with the periodic boundary condition $\phi(0, \mathbf{x}) = \phi(\beta, \mathbf{x})$. For fermions, the degrees of freedom are taken to be Grassmann variables $\psi, \bar{\psi}$; and the boundary condition is anti-periodic $\psi(0, \mathbf{x}) = -\psi(\beta, \mathbf{x})$.

A crucial difference arising from the finite time interval is that Fourier transforming to frequency space leads to discrete values for frequencies:

$$\omega_n = \begin{cases} \dfrac{2n\pi}{\beta} & \text{Bosons,} \\ \dfrac{(2n+1)\pi}{\beta} & \text{Fermions} \end{cases} \tag{3.27}$$

where $n = 0, \pm 1, \pm 2, \cdots$.

As an example, we look at the propagator of a theory with a real scalar field. The free two-point function is

$$iG(\omega_n, \vec{k}) = \frac{1}{\omega_n^2 + \vec{k}^2 + m^2} = \frac{1}{\left(\frac{2\pi n}{\beta}\right)^2 + \vec{k}^2 + m^2}. \tag{3.28}$$

We can then proceed using Feynman rules to do perturbative calculations, with the difference that when integrating over loops the time integral now becomes the sum $\sum_{n=-\infty}^{\infty}$.

To conclude, the imaginary time formalism allows perturbative field theory calculations for systems at equilibrium. The main advantage of this formalism is that unlike the real-time formalisms that we will explore below, one does not need to double the degrees of freedom and therefore the diagrammatics are considerably simpler. The downside however is that it is not suited for non-equilibrium systems.

3.2.2 Thermofield Double

The Thermofield Double or Thermofield Dynamics (TFD) is another real time finite temperature formalism. The starting point is how ensemble averages of an observable $\mathcal{O}$ with respect to a thermal system are calculated in quantum mechanics:

$$\langle O \rangle_\beta = \frac{1}{Z(\beta)} \operatorname{Tr}\left(e^{-\beta H}\mathcal{O}\right) = \frac{1}{Z(\beta)} \sum_n e^{-\beta E_n} \langle n|O|n\rangle \tag{3.29}$$

where for now we assume the Hamiltonian operator H has discrete eigenvalues and orthonormal eigenstates ($\langle n|m\rangle = \delta_{nm}$) and the normalization factor is $Z(\beta) \equiv \operatorname{Tr} e^{-\beta H}$.

Our eventual goal is to calculate time-ordered products of observables with asymptotic in and out states that correspond to the thermal state $\rho = e^{-\beta H}$, allowing us to use QFT methods at finite temperature. To begin with, we need to construct a thermal vacuum state which we will call $|0,\beta\rangle$. We want the following relationship to hold for the ensemble average:

$$\langle 0,\beta|\mathcal{O}|0,\beta\rangle = \frac{1}{Z(\beta)}\sum_n e^{-\beta E_n}\langle n|\mathcal{O}|n\rangle\,. \tag{3.30}$$

Let's first assume that we can express this thermal vacuum state in terms of the Hilbert space spanned by H. We find

$$\begin{aligned}\langle 0,\beta|\mathcal{O}|0,\beta\rangle &= \sum_n\sum_m \langle 0,\beta|n\rangle\langle n|\mathcal{O}|m\rangle\langle m|0,\beta\rangle \\ &\equiv \sum_n\sum_m f_n^*(\beta)f_m(\beta)\langle n|\mathcal{O}|m\rangle\end{aligned} \tag{3.31}$$

where we have defined the overlap $\langle m|0,\beta\rangle \equiv f_m(\beta)$, a function of β. For the above to be equal to the right-hand side of (3.30), we need

$$f_n^*(\beta)f_m(\beta) = \frac{1}{Z(\beta)}e^{-\beta E_n}\delta_{mn}. \tag{3.32}$$

Notice the right-hand side implies the requirement that f_i's behave like orthonormal vectors, yet the left-hand side is just a product of two complex numbers, which is not in general zero for $m \neq n$. So we cannot define a state $|0,\beta\rangle$ with just the original Hilbert space. A way out is to "double" it by introducing a second copy of the system, such that the full Hilbert space is spanned by the tensor product of the two sets of eigenstates: $|n,\tilde{m}\rangle \equiv |n\rangle\otimes|\tilde{m}\rangle$. We then define

$$|0,\beta\rangle \equiv \sum_n f_n(\beta)\,|n,\tilde{n}\rangle \tag{3.33}$$

and calculate the expectation value (observables are taken to be in the original Hilbert space):

$$\begin{aligned}\langle 0,\beta|\mathcal{O}|0,\beta\rangle &= \sum_n\sum_m f_n^*(\beta)f_m(\beta)\langle n,\tilde{n}|\mathcal{O}|m,\tilde{m}\rangle \\ &= \sum_n\sum_m f_n^*(\beta)f_m(\beta)\langle n|\mathcal{O}|m\rangle\langle\tilde{n}|\tilde{m}\rangle \\ &= \sum_n f_n^*(\beta)f_n(\beta)\langle n|\mathcal{O}|n\rangle\,.\end{aligned} \tag{3.34}$$

The above result will equal (3.30) if we choose

$$f_n(\beta) = f_n^*(\beta) = \frac{1}{Z^{1/2}(\beta)}e^{-\beta E_n/2}. \tag{3.35}$$

We see that the cost of defining a thermal vacuum was to double the Hilbert space, which was also the case in the Schwinger-Keldysh formalism.

As an example of the generalization of the above to field theory, we consider the case of the free real scalar field. To be able to use the QFT machinery developed for in-out states, we first need to connect the thermal vacuum defined above to the true vacuum $|0,\tilde{0}\rangle$ of the doubled Fock space. We first introduce a (formally) unitary operator made up of creation and annihilation operators $a_k^\dagger$, a_k for each momentum mode k:

$$U(\beta) = e^{-iG(\beta)}\,, \qquad G(\beta) \equiv -i\sum_k \theta_{\vec{k}}(\beta)\left(\tilde{a}_{\vec{k}}\otimes a_{\vec{k}} - \tilde{a}_{\vec{k}}^\dagger \otimes a_{\vec{k}}^\dagger\right) \tag{3.36}$$

where $\theta_{\vec{k}}$ is chosen such that

$$\cosh\theta_{\vec{k}}(\beta) = \frac{1}{\sqrt{1-e^{-\beta\omega_k}}}\,, \qquad \sinh\theta_{\vec{k}}(\beta) = \frac{e^{-\beta\omega_k/2}}{\sqrt{1-e^{-\beta\omega_k}}}, \tag{3.37}$$

and we will assume the fields are relativistic: $\omega_k = \sqrt{\vec{k}^2+m^2}$. With this definition, the thermal vacuum can be obtained through[2]

$$|0,\beta\rangle = U(\beta)\,|0,\tilde{0}\rangle\,. \tag{3.38}$$

Using the transformation

$$a(\beta) = U(\beta)aU^\dagger(\beta), \tag{3.39}$$

the same operator induces a Bogoliubov transformation, that leads to the following thermal creation and annihilation operators:

$$\begin{aligned} a_{\vec{k}}(\beta) &= \cosh\theta_{\vec{k}}a_{\vec{k}} - \sinh\theta_{\vec{k}}a_{\vec{k}}^\dagger \\ \tilde{a}_{\vec{k}}(\beta) &= \cosh\theta_{\vec{k}}\tilde{a}_{\vec{k}} - \sinh\theta_{\vec{k}}\tilde{a}_{\vec{k}}^\dagger. \end{aligned} \tag{3.40}$$

These new creation and annihilation operators create or destroy thermal modes when acting on the thermal vacuum. With the above definitions, we can now use the usual QFT techniques to calculate time-ordered correlation functions and to demonstrate we look at the propagator of the free real scalar theory.

We start with the mode expansion of the two copies of the scalar field

$$\begin{aligned} \phi(x) &= \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_k}}\left(a_{\vec{k}}e^{ikx} + a_{\vec{k}}^\dagger e^{-ikx}\right) \\ \tilde{\phi}(x) &= \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_k}}\left(\tilde{a}_{\vec{k}}e^{ikx} + \tilde{a}_{\vec{k}}^\dagger e^{-ikx}\right) \end{aligned} \tag{3.41}$$

[2]While $U(\beta)$ is formally unitary, in the continuum limit the resultant thermal vacuum is unitarily inequivalent to $|0,\tilde{0}\rangle$ [94]

where $kx = -\omega_k t + \vec{k}\cdot\vec{x}$. Defining a vector of the two copies

$$\Phi \equiv \begin{pmatrix} \phi(x) \\ \tilde{\phi}(x) \end{pmatrix}, \tag{3.42}$$

we can express the vacuum to vacuum propagator as a two-by-two matrix:

$$iG(x,x') \equiv \langle 0,\tilde{0}|T\Phi(x)\Phi(x')|0,\tilde{0}\rangle = \begin{pmatrix} \langle 0,\tilde{0}|T\phi(x)\phi(x')|0,\tilde{0}\rangle & \langle 0,\tilde{0}|T\phi(x)\tilde{\phi}(x')|0,\tilde{0}\rangle \\ \langle 0,\tilde{0}|T\tilde{\phi}(x)\phi(x')|0,\tilde{0}\rangle & \langle 0,\tilde{0}|T\tilde{\phi}(x)\tilde{\phi}(x')|0,\tilde{0}\rangle \end{pmatrix} \tag{3.43}$$

where time ordering is defined as usual:

$$\langle T\phi(x)\phi(x')\rangle = \theta(t-t')\,\langle\phi(x)\phi(x')\rangle - \theta(t'-t)\,\langle T\phi(x')\phi(x)\rangle \tag{3.44}$$

Using the mode expansion and the commutation relations $[a_{\vec{k}}, a^\dagger_{\vec{k}'}] = \delta(\vec{k}-\vec{k}')$, $[a_{\vec{k}}, a_{\vec{k}'}] = 0$; we arrive at

$$iG(k) = \begin{pmatrix} \dfrac{i}{k^2+m^2} & 0 \\ 0 & -\dfrac{i}{k^2+m^2} \end{pmatrix} \tag{3.45}$$

which agrees with the Schwinger-Keldysh result in the case of an initial vacuum state. Note also that the propagator for the tilde-fields has an overall minus sign, also similar to the in-in case. Turning to the thermal propagator, we make use of $U(\beta)$:

$$G_\beta(x,x') = \langle 0,\beta|T\Phi(x),\Phi(x')\rangle = \langle 0,\tilde{0}|U^\dagger(\beta)T\left(\Phi(x)\Phi(x')\right)U(\beta)|0,\tilde{0}\rangle \tag{3.46}$$

and the Bogoliubov transformation (3.40) induced by it to find

$$iG_\beta(k) = \begin{pmatrix} \frac{i}{k^2+m^2-i\epsilon} & 0 \\ 0 & -\frac{i}{k^2+m^2+i\epsilon} \end{pmatrix} - 2\pi i n_B(\omega_k)\delta(k^2+m^2)\begin{pmatrix} 1 & e^{\beta|\omega_k|/2} \\ e^{-\beta|\omega_k|/2} & 1 \end{pmatrix} \tag{3.47}$$

where $n_B = 1/(e^{\beta|\omega_k|}-1)$ is the Bose-Einstein distribution. With a linear transformation, we can re-express the propagator in a familiar basis:

$$G_\beta(x,x') = \langle 0,\beta|T\Phi(x),\Phi(x')\rangle = \langle 0,\tilde{0}|U^\dagger(\beta)T\left(\Phi(x)\Phi(x')\right)U(\beta)|0,\tilde{0}\rangle \tag{3.48}$$

and the Bogoliubov transformation (3.40) induced by it to find

$$iG_\beta(k) = \begin{pmatrix} \frac{i}{k^2+m^2-i\epsilon} & 0 \\ 0 & -\frac{i}{k^2+m^2+i\epsilon} \end{pmatrix} - 2\pi i n_B(\omega_k)\delta(k^2+m^2)\begin{pmatrix} 1 & e^{\beta|\omega_k|/2} \\ e^{-\beta|\omega_k|/2} & 1 \end{pmatrix} \tag{3.49}$$

The propagator has a temperature independent, diagonal part, and a temperature dependent part that encodes the thermal nature of the system through n_B.

3.2.3 Contours of the Three Formalisms

We end this chapter with a comparison of the time contours in the three finite temperature formalisms [95]. In the imaginary time formalism, the only time interval is along the imaginary axis in the range $[0, \beta)$, so the contour is simply

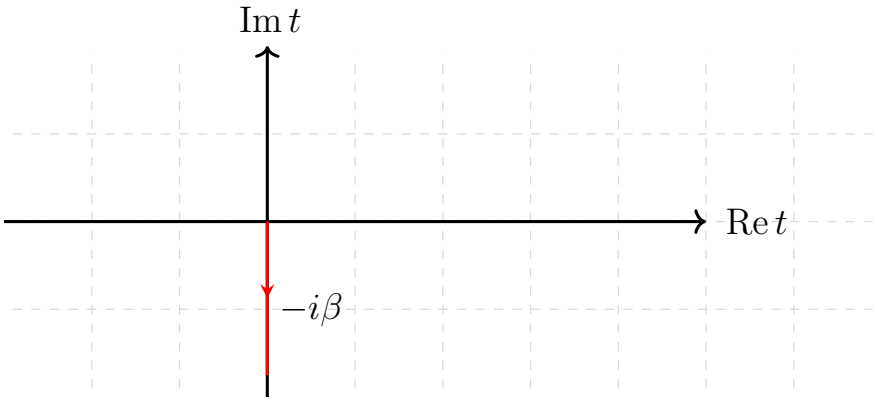

Figure 3.1: The contour used for the imaginary time formalism

To see what the closed time contour of the thermofield formalism looks like, we go back to the ensemble average of an observable $\mathcal{O}$ taken at time t (assuming $[\mathcal{O}, H] = 0$):

$$\begin{aligned}\frac{1}{\operatorname{Tr}\rho}\operatorname{Tr}(\rho(t)\mathcal{O}) &= \frac{1}{Z(\beta)}\operatorname{Tr}\big(e^{-\beta H/2}e^{-\beta H/2}\mathcal{O}(t)\big)\\ &= \frac{1}{Z(\beta)}\sum_n \langle n,-\infty|e^{-\beta H/2}e^{-\beta H/2}U(-\infty,\infty)U(\infty,t)\mathcal{O}U(t,-\infty)|n,-\infty\rangle\\ &= \frac{1}{Z(\beta)}\sum_n \langle n,-\infty|e^{-\beta H/2}U(-\infty,\infty)e^{-\beta H/2}U(\infty,t)\mathcal{O}U(t,-\infty)|n,-\infty\rangle\end{aligned} \tag{3.50}$$

where we used $U^\dagger(t,t') = U(t',t)$ and the semigroup properties of the time evolution operator[3]. We can connect the above to the thermofield ensemble by observing that

$$\begin{aligned}\langle 0,\beta|\mathcal{O}(t)|0,\beta\rangle &= \frac{1}{Z(\beta)}\sum_{n,m}\langle n,\tilde{n}|e^{-\beta E_n/2}\mathcal{O}(t)e^{-\beta E_m/2}|m,\tilde{m}\rangle\\ &= \frac{1}{Z(\beta)}\sum_n\langle n|e^{-\beta H/2}\mathcal{O}e^{-\beta H/2}|n\rangle\\ &= \frac{1}{Z(\beta)}\operatorname{Tr}\big(e^{-\beta H/2}\mathcal{O}(t)e^{-\beta H/2}\big) = \frac{1}{\operatorname{Tr}\rho}\operatorname{Tr}(\rho(t)\mathcal{O}).\end{aligned} \tag{3.51}$$

[3]Note that the above derivation assumes that H is independent of time, but the thermofield formalism has been generalized to the non-equilibrium setting [15]

The closed time path contour for thermofield dynamics described by the last line of (3.50) is given in Fig. 3.2.

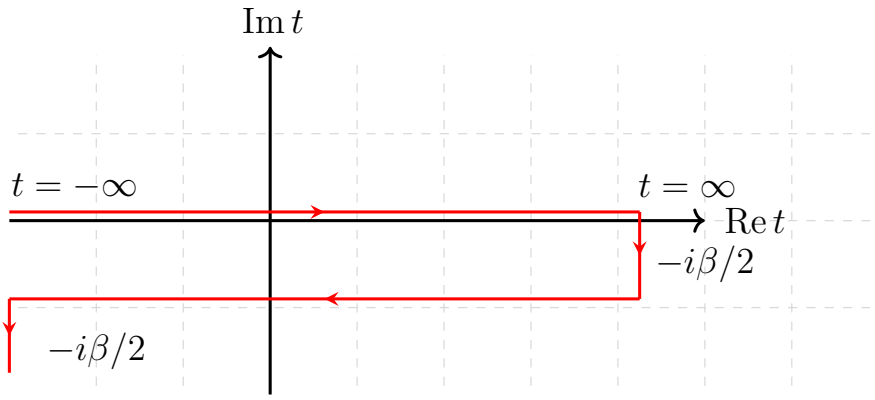

Figure 3.2: The contour used for the thermofield formalism

The contour for the Schwinger-Keldysh formalism is similar, with only a single imaginary time translation at $t = -\infty$, as can be seen from (3.13). A sketch of the resulting contour is given in Fig. 3.3.

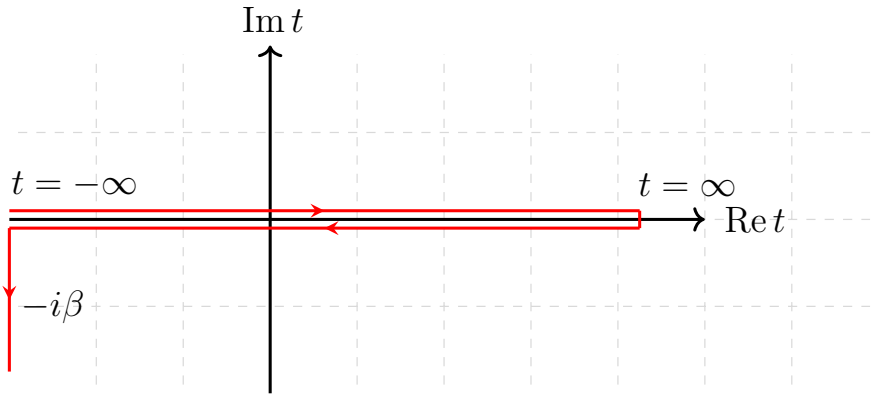

Figure 3.3: The contour used for the Schwinger-Keldysh formalism

Chapter 4

The Schwinger-Keldysh Formalism from an Effective Field Theory Perspective

In this section, we review the Schwinger-Keldysh (or in-in) formalism from an EFT perspective. We will discuss the main features of this formalism (doubling of the field content, implementation of symmetries, power counting, etc.) and strive to keep our remarks as general as possible, so that they equally apply to systems with or without SSB.

4.1 Effective Action

The generating functional given in Eq. (3.15) leaves the dependence on the density matrix ρ and boundary conditions at $t = +\infty$ highly implicit. These are nonetheless important features of the path integral formulation that will play an important role in what follows.

The generator (3.15) corresponds to a path integral over a contour $\mathcal{C}$ which extends from $t = -\infty$ to $t = +\infty$ and back, with appropriate boundary conditions placed in the asymptotic regions; see for instance Fig. 4.1 in appendix 4.6 for a depiction of this contour in the case where ρ is thermal. Consequently, $Z[J_1, J_2]$ generates correlators which are *path-ordered* along $\mathcal{C}$, for the usual reasons, with operators labeled with a 2 always coming after those labeled with a 1. For example, when $\mathcal{O}(\varphi) = \varphi$, the various permutations of the two-point function are explicitly given by

$$\langle \mathcal{P}\varphi_i(x)\varphi_j(x')\rangle = \begin{cases} \langle T\varphi(x)\varphi(x')\rangle & i = j = 1 \\ \langle \bar{T}\varphi(x)\varphi(x')\rangle & i = j = 2 \\ \langle \varphi(x)\varphi(x')\rangle & i = 2, j = 1 \\ \langle \varphi(x')\varphi(x)\rangle & i = 1, j = 2 \end{cases} , \tag{4.1}$$

where $\mathcal{P}$ denotes path-ordering. For brevity, we will omit explicit $\mathcal{P}$'s in subsequent expressions and path-ordering is always implied unless noted otherwise.

The shorthand expression (3.15) is cryptic, at best. A particular shortcoming is that the correlator (4.1) does not arise from simply inverting the kinetic term of $S[\varphi_1]-S[\varphi_2]$, as the notation might suggest. Were this true, all mixed correlators such as $\langle\varphi_1(x)\varphi_2(x')\rangle$ would necessarily be vanishing. Instead, the boundary conditions implicit in (3.15) generate the necessary, non-trivial cross-couplings—see App. 4.6 for an explicit example.

Nuisances such as this one motivate the use of an alternative description based on an *effective action* $S_{\rm EFT}[\varphi_1,\varphi_2]$ in which the boundary conditions implicit in (3.15) are made explicit and encoded into $S_{\rm EFT}$ itself. These effective actions are of the functional form

$$S_{\rm EFT}[\varphi_1,\varphi_2] = S[\varphi_1] - S[\varphi_2] + \Delta S[\varphi_1,\varphi_2] \ , \tag{4.2}$$

where the information about the state ρ of the closed system as well as the boundary conditions are encoded in the cross terms $\Delta S[\varphi_1,\varphi_2]$ [34, 96].[1] We elaborate upon this point later in this section and will also discuss the circumstances under which we expect $S_{\rm EFT}$ to be local. Correlators are computed using $S_{\rm EFT}$ as in

$$\langle\mathcal{O}^{(n)}(x_n)\dots\mathcal{O}^{(1)}(x_1)\rangle = \int \mathcal{D}\varphi_1\mathcal{D}\varphi_2\, e^{iS_{\rm EFT}[\varphi_1,\varphi_2]}\mathcal{O}^{(n)}(x_n)\dots\mathcal{O}^{(1)}(x_1) \ , \tag{4.3}$$

where perturbative computations now proceed in the naive manner in which free propagators are determined by the quadratic terms in $S_{\rm EFT}$ and non-linearities are handled order by order.

A term of the form $\Delta S[\varphi_1,\varphi_2]$ arises also in *open* systems, albeit in a different way [2, 16, 20, 97, 98]. To see this, imagine decomposing our closed system in a subsystem of interest, with degrees of freedom ϕ, and an environment, with degrees of freedom Φ. If we are only interested in correlation functions of the subsystem, we can work with a generating functional of the form

$$Z[J_1,J_2] = \int_\rho \mathcal{D}\phi_1\mathcal{D}\phi_2\mathcal{D}\Phi_1\mathcal{D}\Phi_2\, e^{iS[\phi_1,\Phi_1]-iS[\phi_2,\Phi_2]+i\int \mathrm{d}^{d+1}x\, J_1\mathcal{O}(\phi_1)-J_2\mathcal{O}(\phi_2)} \ , \tag{4.4}$$

where $S[\phi,\Phi] = S[\phi]+S_{\rm int}[\phi,\Phi]$, and $S_{\rm int}[\phi,\Phi]$ captures the dynamics of the environment as well as its interactions with the subsystem. Assuming for simplicity that the state of the full system is factorized, i.e. $\rho=\rho_\phi\otimes\rho_\Phi$, we can integrate out the fields Φ to obtain a correction to the action for the subsystem given by

$$e^{i\Delta S[\phi_1,\phi_2]} = \int_{\rho_\Phi} \mathcal{D}\Phi_1\mathcal{D}\Phi_2\, e^{iS_{\rm int}[\phi_1,\Phi_1]-iS_{\rm int}[\phi_2,\Phi_2]} . \tag{4.5}$$

[1] Note that the fields φ that appear in the effective action (4.2) generically do not coincide with the fields φ in Eq. (3.15)—a standard observation in more traditional in-out effective field theories. Similarly, the functional $S[\varphi_i]$ in Eq. (4.2) will generally differ from the functional $S[\varphi_i]$ in Eq. (3.15); we use the same symbol in both contexts to avoid unnecessary additional notation.

The term $\Delta S[\phi_1, \phi_2]$ that arises by integrating out the environment is known as the *influence functional*, and it generically includes interactions between ϕ_1 and ϕ_2. The origin of these interactions can be traced back to the "off-diagonal" correlation functions of the environment degrees of freedom, e.g. $\langle \Phi_i \Phi_j \rangle$ with $i \neq j$. Alternatively, one could also derive the influence functional using the fact that, for open systems, the evolution between subsequent time slices is not unitary and is performed by the Lindblad operator—see e.g. [2].

Eq. (4.5) shows explicitly that, in the case of an open system, the term $\Delta S[\phi_1, \phi_2]$ contains information about the state of the environment ρ_Φ and its interactions with the subsystem. This is conceptually different from the effective action of a closed system in (4.2), which instead depends on the state of the system under consideration. Of course, one could amend the influence functional so that it also captures the state ρ_ϕ of the subsystem, following the same strategy we outlined for a closed system. By doing so, $\Delta S[\phi_1, \phi_2]$ would now encode once again the state of a closed system, $\rho = \rho_\phi \otimes \rho_\Phi$. This procedure would be equivalent to first introducing the effective action for the total closed system, and then integrating out the environment in the naive manner:

$$\begin{aligned}\int_\rho \mathcal{D}\phi_1 \mathcal{D}\phi_2 \mathcal{D}\Phi_1 \mathcal{D}\Phi_2 e^{iS[\phi_1,\Phi_1]-iS[\phi_2,\Phi_2]} &= \int \mathcal{D}\phi_1 \mathcal{D}\phi_2 \mathcal{D}\Phi_1 \mathcal{D}\Phi_2 e^{iS[\phi_1,\Phi_1]-iS[\phi_2,\Phi_2]+i\Delta S[\phi_1,\phi_2,\Phi_1,\Phi_2]} \\ &= \int \mathcal{D}\phi_1 \mathcal{D}\phi_2 \, e^{iS[\phi_1]-iS[\phi_2]+i\Delta S[\phi_1,\phi_2]}. \end{aligned} \tag{4.6}$$

From this perspective, the difference between an open and a closed system is whether the degrees of freedom appearing in the path integral define a basis for the Hilbert space on which the state ρ is defined.

4.2 Locality and Expansion Parameters

Let's consider an EFT described by an "in-out" effective action $S_{\rm EFT}[\varphi]$ with energy cutoff Λ.[2] This means that any operator in the in-out effective action can be assigned a definite scaling in the ratio E/Λ, with E the typical scale of interest [50, 60]. Any state ρ that is not the vacuum of the EFT will generically introduce additional scales into the problem. We will collectively denote these scales with M; some examples are: the temperature T of a thermal state, the chemical potential μ of a finite density state, the characteristic length scale ℓ of a semi-classical field profile (converted to an energy scale using some characteristic speed), etc. Only states such that $M \ll \Lambda$ can be reliably described within the in-out EFT.

In the approach described in Sec. 4.1, these additional scales M appear explicitly in the Schwinger-Keldysh effective action $S_{\rm EFT}[\varphi_1, \varphi_2]$. While it is perhaps plausible

[2] In non-relativistic EFTs one should distinguish between the cutoffs for energy and momentum—as we will see in the examples discussed in Sec. 5.3. In this section we focus our attention on the energy cutoff to simplify the discussion.

that such an action might be able to reproduce all correlators of the form (3.11), there is certainly no expectation that this EFT should be local for all scales $E \ll \Lambda$—and indeed, it generically won't be. In order to work with a local Schwinger-Keldysh effective action, we need to restrict ourselves to the regime $E \ll M \ll \Lambda$, which will be the focus of this paper. Below the scales M, all information about the state ρ is encoded in the effective action by an infinite tower of irrelevant local operators—the usual way in which UV physics manifests itself at low energies. Thus, the Schwinger-Keldysh effective action now contains a new expansion parameter, E/M.

This point is potentially confusing for thermal states: because EFTs at zero temperature are usually described using a local action, one might expect that the $T \ll E$ regime should also admit a local description by continuity. However, thermality is generically encoded by $\sim e^{-E/T}$ factors [31, 91, 99] which only admit an expansion in powers of E for $T \gg E$, the regime in which being at finite temperature appears as a UV effect.[3]

At scales $E \ll M$, the relevant degrees of freedom are often the Goldstone modes associated with symmetries that are spontaneously broken by the state ρ. In the following sections, we will uncover the principles that one should follow to write down the most general Schwinger-Keldysh effective action for such Goldstone modes.

Irrespective of the state ρ of the system, Schwinger-Keldysh effective actions are also naturally endowed with another expansion parameter, namely $\hbar$, that controls the semi-classical expansion. In the absence of Goldstone modes, fields usually transform linearly under all the symmetries, and a systematic $\hbar$ expansion of the effective action is straightforward to implement [34] (see also Sec. 4.5 for more details). In the presence of Goldstone bosons, however, an expansion of the effective action in powers of $\hbar$ requires some extra care, since a naive implementation would break some of the symmetries that are realized non-linearly, as will become clear in subsequent sections. For this reason, we will not be implementing this expansion in what follows.

4.3 Symmetries

EFTs are specified not only by their field content and expansion parameter(s), but also by their symmetries. It is therefore important to discuss which symmetries one should impose when writing down the most general Schwinger-Keldysh effective action $S_{\text{EFT}}[\varphi_1, \varphi_2]$. The symmetry considerations that must inform the construction of the effective action for closed systems are:

[3] We are being schematic here. More precisely, the scale M is proportional to T but does not necessarily coincide with it. For weakly coupled UV completions, M is more accurately of order $g^4 T \ln g^{-2}$, a scale associated with large-angle scattering events. It is below this scale that the relevant degrees of freedom become the hydrodynamic modes [100]. This situation is a manifestation of the usual fact that, for weakly coupled UV completions, the cutoff of the low-energy EFT can be parametrically smaller than the strong coupling scale (in our case, T).

- *Gauge symmetries*: When the single field action $S[\varphi]$ enjoys a *gauge* symmetry, the action $S[\varphi_1, \varphi_2] = S[\varphi_1] - S[\varphi_2]$ in Eq. (3.15) is invariant under independent gauge transformations of the φ_i's that coincide at $t = \pm\infty$ [34]. As a result, the Schwinger-Keldysh effective action $S_{\rm EFT}[\varphi_1, \varphi_2]$ must be invariant under two copies of the gauge group. This is consistent with the statement in the previous section that one should double all the degrees of freedom—including gauge fields.

- *Continuous global symmetries*: The fact that the two gauge transformations must coincide at $t = \pm\infty$ implies that, in the global limit, the action $S[\varphi_1, \varphi_2] = S[\varphi_1] - S[\varphi_2]$ can only be invariant under the diagonal symmetry group G which transforms the φ_i's simultaneously. This fact is not manifest at the level of the action, since φ_1 and φ_2 would appear to be decoupled; it follows instead from the boundary conditions, and in this sense it can be viewed as a non-local constraint on the symmetries of the system. These boundary conditions, together with the state of the system, are built directly into the Schwinger-Keldysh effective action $S_{\rm EFT}[\varphi_1, \varphi_2]$. Furthermore, gauge symmetries must become physically indistinguishable from global symmetries in the limit of vanishingly small gauge coupling.[4] In the regime where the Schwinger-Keldysh effective action is local, this can only be achieved if $S_{\rm EFT}[\varphi_1, \varphi_2]$ is invariant under two copies of all global symmetries, $G_1 \times G_2$.[5] This enhancement of global symmetries is a direct consequence of the requirement that the Schwinger-Keldysh effective action be local and leads to the existence of two separate Ward identities, which are required to reproduce the same information of a single Ward identity defined on the two segments of the Schwinger-Keldysh contour (see Fig. 4.1 in App 4.6). The symmetry properties of the matrix elements of ρ in the past infinity determine to what extent $G_1 \times G_2$ is spontaneously broken down to a subgroup—we will discuss this more in depth in Sec. 5.2.

- *Discrete symmetries:* The difference between past and future boundary conditions breaks explicitly time reversal, which therefore is not a symmetry of the effective action $S_{\rm EFT}[\varphi_1, \varphi_2]$ even when it is a symmetry of the single-field action $S[\varphi]$. All other discrete symmetries of the single-field action are realized twice in the Schwinger-Keldysh effective action, as is the case for continuous

[4] See e.g. discussion in Sec. 21.4 of [101].

[5] In the ground state, $G_1 \times G_2$ is explicitly broken down to its diagonal subgroup by terms of $\mathcal{O}(\varepsilon)$ when the "$i\varepsilon$" prescription is implemented at the level of the action [96]. This is another manifestation of the fact that the state at $t = -\infty$ is only invariant under the diagonal symmetry group. The factors of $i\varepsilon$ are crucial to reproduce the correct n-point functions, but should not be taken into account when discussing the symmetries of the Schwinger-Keldysh effective action. This is standard EFT practice: for example, the in-out effective action for a $U(1)$ Goldstone is considered to be shift-invariant even though implementing the "$i\varepsilon$" prescription in the action would break the shift symmetry explicitly.

global symmetries. This is required for consistency, since discrete subgroups of continuous symmetries are always realized twice.

- *Emergent symmetries:* The state ρ not only determines whether some of the symmetries are spontaneously broken, but can also give rise to additional "emergent" symmetries in the EFT. For instance, the homogeneity and isotropy of a state would be encoded by an emergent internal $ISO(d)$ symmetry [42, 102]. A more generic example is provided by the thermal state, which endows the effective theory with an additional discrete symmetry—the *dynamical KMS symmetry* (DKMS) [34, 103, 104]—that, in the simplest case,[6] can be implemented on the two copies of the fields as follows:

$$\varphi_1'(t,\vec{x}) = \varphi_1(-t+i\beta/2,\vec{x}) = \varphi_1(-t,\vec{x}) - \frac{i\beta}{2}\partial_t\varphi_1(-t,\vec{x}) + \mathcal{O}(\beta^2), \tag{4.7a}$$

$$\varphi_2'(t,\vec{x}) = \varphi_2(-t-i\beta/2,\vec{x}) = \varphi_2(-t,\vec{x}) + \frac{i\beta}{2}\partial_t\varphi_2(-t,\vec{x}) + \mathcal{O}(\beta^2), \tag{4.7b}$$

where $\beta = 1/T$ (In this paper we are aiming for an effective action up to leading order in an expansion in E/T, which is why we expanded the DKMS symmetry in powers of ∂_t/T).[7] This symmetry ensures that correlation functions of the system satisfy the KMS condition [8, 105], which e.g. for the 2-point correlator of any operator $\mathcal{O}$ reads

$$\langle \mathcal{O}(t,\vec{x})\mathcal{O}(t',\vec{x}')\rangle = \langle \mathcal{O}(t'-i\beta/2,\vec{x}')\mathcal{O}(t+i\beta/2,\vec{x})\rangle. \tag{4.8}$$

Furthermore, when combined with the unitarity conditions discussed below in Sec. 4.4, the DKMS symmetry leads to the existence of an entropy current with non-negative divergence [106]. It is worth mentioning that Eq. (4.7) is not the only possible way of implementing the KMS condition as a symmetry of the effective action—see for instance [107, 108], and [109] for a detailed comparison with the approach put forward in [34] and adopted in this paper.

[6] The existence of the symmetry (4.7) relies on the invariance of the underlying *dynamics* under time reversal. More in general, the DKMS symmetry can be implemented by combining the transformations (4.7) with additional discrete symmetries, e.g. parity and charge conjugation [34]. The important point is that the *ground state* of the system must be invariant under the combined action of these discrete symmetries and time reversal. As we will see in Sec. 5.3.3, this requirement plays an important role for ferromagnets.

[7] We are assuming here that all continuous global symmetries act linearly on the φ_i fields, i.e. that there is no spontaneous symmetry breaking. Furthermore, we have made an additional imaginary time-translation to bring (4.7) to a convenient form—see e.g. [34] for a more general form of these transformation rules.

4.4 Unitarity

In the standard in-out path integral, unitarity implies that the single field effective action $S_{\rm EFT}[\varphi]$ must be real. Similarly, unitarity also restricts the form of the Schwinger-Keldysh effective action $S_{\rm EFT}[\varphi_1, \varphi_2]$, and requires that the following conditions be satisfied [34]:

1. The action must vanish when $\varphi_1 = \varphi_2$: $S_{\rm EFT}[\varphi, \varphi] = 0$.

2. Unlike in the usual in-out path integral, the effective action that appears in the Schwinger-Keldysh generating functional is allowed to be complex. However, under conjugation we must have $S_{\rm EFT}[\varphi_1, \varphi_2]^* = -S_{\rm EFT}[\varphi_2, \varphi_1]$.

3. Furthermore, the imaginary part of the action must be non-negative: $\mathrm{Im}\, S_{\rm EFT}[\varphi_1, \varphi_2] \geq 0$.

In terms of the decomposition (4.2) $S_{\rm EFT}[\varphi_1, \varphi_2] = S[\varphi_1] - S[\varphi_2] + \Delta S[\varphi_1, \varphi_2]$, the first unitarity constraint implies that $\Delta S[\varphi, \varphi] = 0$; the second condition requires $S[\varphi]$ to be purely real and the mixing term to obey $\Delta S[\varphi_1, \varphi_2]^* = -\Delta S[\varphi_2, \varphi_1]$; and the third unitarity condition further imposes $\mathrm{Im}\, \Delta S[\varphi_1, \varphi_2] \geq 0$.

4.5 A Convenient Field Redefinition: Keldysh Rotation

In order to simplify calculations and make the causality properties of various quantities manifest, it is convenient to perform a field redefinition and switch to the so-called *Keldysh basis*. Assuming for now that all symmetries are linearly realized on our fields, this is done by performing a *Keldysh rotation* to introduce the following new degrees of freedom:[8]

$$\varphi_a(x) \equiv \varphi_1(x) - \varphi_2(x) \ , \qquad \varphi_r(x) \equiv \tfrac{1}{2}\left[\varphi_1(x) + \varphi_2(x)\right] . \tag{4.9}$$

The Keldysh basis has several advantages. First, even though the 2-point functions of φ_1 and φ_2 are all generically non-zero, as shown in Eq. (4.1), they are actually not all independent of each other. Writing out the time-orderings, one can easily show that the following identity holds [96]:

$$\langle \varphi_1(x)\varphi_1(x')\rangle + \langle \varphi_2(x)\varphi_2(x')\rangle - \langle \varphi_1(x)\varphi_2(x')\rangle - \langle \varphi_2(x)\varphi_1(x')\rangle = 0 \ . \tag{4.10}$$

[8] See [110] for a detailed discussion of subtleties associated with the Keldysh basis.

In the Keldysh basis, this redundancy is made explicit through the fact that $\langle\varphi_a(x)\varphi_a(x')\rangle$ vanishes:[9][10]

$$\langle\varphi_i(x)\varphi_j(x')\rangle = \begin{cases} \frac{1}{2}\langle\{\varphi(x),\varphi(x')\}\rangle & i=j=r \\ 0 & i=j=a \\ \theta(t-t')\langle[\varphi(x),\varphi(x')]\rangle & i=r, j=a \\ \theta(t'-t)\langle[\varphi(x'),\varphi(x)]\rangle & i=a, j=r \end{cases} . \tag{4.11}$$

Another technical advantage to the Keldysh basis also becomes apparent in Eq. (4.11): mixed a-r two-point correlators have manifest causal properties. The retarded Green's function is proportional to $\langle\varphi_r(x)\varphi_a(x')\rangle$, while the advanced one is proportional to $\langle\varphi_a(x)\varphi_r(x')\rangle$. One can think of the mixed correlators as containing information about the system's fundamental dynamics, while r-r correlators encode information about the state of the system.[11]

The effective action $S_{\rm EFT}[\varphi_a,\varphi_r]$ is generally organized as an expansion in φ_a fields which, due to the form of the action in (4.2), can be written as

$$S_{\rm EFT}[\varphi_a,\varphi_r] = \sum_{n\in\{1,3,5,\ldots\}}^{\infty}\left(\prod_{i=1}^{n}\int \mathrm{d}^{d+1}x_i\right)\frac{\delta^n S[\varphi_r]}{\delta\varphi_r(x_1)\ldots\delta\varphi_r(x_n)}\frac{1}{2^{n-1}n!}\varphi_a(x_1)\ldots\varphi_a(x_n) + \Delta S[\varphi_a,\varphi_r] \ , \tag{4.12}$$

where only odd powers of φ_a can arise from the expansion of $S[\varphi_1]-S[\varphi_2]$. In these variables, the unitarity constraints previously discussed in Sec. 4.4 now imply the conditions

1. $\Delta S[\varphi_a=0,\varphi_r]=0$,
2. $\Delta S[\varphi_a,\varphi_r]^* = -\Delta S[-\varphi_a,\varphi_r]$,
3. $\mathrm{Im}\,\Delta S[\varphi_a,\varphi_r]\geq 0$.

The first condition further implies that $\varphi_a=0$ is always a consistent solution of the equations of motion that follow from varying the effective action with respect to φ_r. Variation with respect to φ_a yields instead, at lowest order in φ_a, the classical

[9] This pattern generalizes quite widely: any correlator involving a string of a- and r-operators $\sim\langle\varphi_r^m\varphi_a^n\rangle$ vanishes if among all operators, the one with the largest temporal argument is an a-operator. This non-perturbative result is known as the *largest-time equation* [111], and it implies, in particular, that $\langle\varphi_a^n\rangle$ is vanishing for all n.

[10] Our notation for various correlation functions and some of their properties are summarized in App. 4.7.

[11] For instance, $\langle\varphi_r(\omega,\vec{k})\varphi_r(-\omega,-\vec{k})\rangle \propto \frac{1}{2}+n_{\rm BE}(\beta\omega)$ in a thermal state, where $n_{\rm BE}(\beta\omega)$ is the Bose-Einstein distribution (assuming $\mathcal{O}$ is bosonic)—see App. 4.6 for more details.

equations of motion for φ_r. As a result, the a- and r-fields have also the conceptual advantage of admitting a natural physical interpretation. The φ_r operator can be identified with a classical field in the $\hbar \to 0$ limit, whereas φ_a is responsible for quantum and stochastic effects [96]. In the absence of spontaneous symmetry breaking, this can be made explicit by rescaling $\varphi_a \to \hbar\varphi_a$, so that the effective action admits a straightforward expansion in powers of $\hbar$. This procedure is consistent with semi-classical expansion of in-out effective actions in terms of some dimensionless combination of coupling constants and $\hbar$ [112] in the limit of high occupation number [113]. In the presence of spontaneous symmetry breaking, however, an expansion in powers of $a-$fields becomes more subtle because it obscures (i.e. break explicitly) some of the symmetries realized non-linearly. Understanding how this relates to the semi-classical expansion of in-out effective theories with Goldstones [114] is an open problem that we leave for future study. In the present paper we will not truncate our results at a finite order in $a-$fields.

For future use, we also report here the action of the DKMS symmetry on the fields in the Keldysh basis. Working at lowest order in E/T, we immediately find from the transformation rules (4.7):

$$\varphi_r'(t,\vec{x}) \simeq \varphi_r(-t,\vec{x}) - \tfrac{i}{4}\beta\partial_t\varphi_a(-t,\vec{x}), \tag{4.13a}$$
$$\varphi_a'(t,\vec{x}) \simeq \varphi_a(-t,\vec{x}) - i\beta\partial_t\varphi_r(-t,\vec{x}). \tag{4.13b}$$

Note that, for linearly realized symmetries, the change in φ_r in Eq. (4.13a) is of $\mathcal{O}(\hbar)$, and is therefore often neglected—see e.g. [34].[12] We are keeping this correction here since for our purposes it will be important to distinguish between expansions in powers of E/T and $\hbar$.

In the body of the paper, we focus on constructing low-energy Schwinger-Keldysh effective actions for systems in which one or more symmetries are spontaneously broken. As we will see, working in terms of r- and a-fields becomes especially natural under these circumstances. This is because the Goldstone fields in the Keldysh basis have non-linear transformation properties that resemble those in a more traditional in-out effective theory. This will allow us, in particular, to implement the DKMS symmetry on the Goldstone fields in a way that is consistent with all the non-linearly realized symmetries. Because of the relations (4.9), instead, the Goldstone fields in the $(1,2)$ basis would satisfy much more complicated transformation rules.

[12] Note that, when restoring all factors of $\hbar$, one should replace $\beta \to \hbar\beta$ in order for the temperature to have units of energy.

4.6 Schwinger-Keldysh Path Integral for a Free System at Finite Temperature

In this section, we calculate explicitly the Schwinger-Keldysh generating functional for a free, massive, relativistic $SO(N)$ scalar described by an in-out action.

$$S = -\frac{1}{2}\int d^4x \left(\partial_\mu \Phi^A \partial^\mu \Phi^A + m^2 \Phi^A \Phi^A\right) \ , \qquad A = 1, \ldots, N \ . \tag{4.14}$$

Through this calculation, we show explicitly how the boundary conditions give rise to off-diagonal correlation functions shown e.g. in Eq. (4.1). Second, we show how the $E \ll T$ limit of the 2-point functions can be reproduced using an effective action. This is by now textbook material [96], but we discuss it here for completeness.

Our starting point is the generating functional in Eq. (3.13) with $\varphi = \mathcal{O} = \vec{\Phi}$:

$$\begin{aligned} Z[J_1, J_2] = \int D\vec{\Phi}_a D\vec{\Phi}_b D\vec{\Phi}_c \, \langle \vec{\Phi}_a, -\infty | \, \rho \, | \vec{\Phi}_c, -\infty \rangle \, \langle \vec{\Phi}_c, -\infty | \, \bar{T} e^{-i\int \vec{J}_2 \cdot \vec{\Phi}} \, | \vec{\Phi}_b, +\infty \rangle \\ \times \, \langle \vec{\Phi}_b, +\infty | \, T e^{i\int \vec{J}_1 \cdot \vec{\Phi}} \, | \vec{\Phi}_a, -\infty \rangle \, , \end{aligned} \tag{4.15}$$

When the density matrix is thermal, $\rho \propto e^{-\beta H}$, this functional can be computed from the knowledge of the amplitude

$$\begin{aligned} \langle \vec{\Phi}_f(\mathbf{x}), +t_\star | \, T e^{i\int \mathrm{d}^4x \, \vec{J}\cdot\vec{\Phi}} \, | \vec{\Phi}_i(\mathbf{x}), -t_\star \rangle &\equiv \int_{\vec{\Phi}(-t_\star)=\vec{\Phi}_i(\mathbf{x})}^{\vec{\Phi}(+t_\star)=\vec{\Phi}_f(\mathbf{x})} \mathcal{D}\Phi \, e^{iS[\vec{\Phi}]+i\int \mathrm{d}^4x \, \vec{J}\cdot\vec{\Phi}} \\ &\equiv \langle \vec{\Phi}_f, +t_\star | \vec{\Phi}_i, -t_\star \rangle_{\vec{J}} \, , \end{aligned} \tag{4.16}$$

since each factor on the right-hand side of (4.15) is a special case of the quantity above. In particular, the thermal density matrix factor comes from setting the source $\vec{J}$ to zero and properly Wick rotating. Combining all three factors together, we obtain a single path integral expression for the generating functional, with fields defined along a time-contour $\mathcal{C}$ in the complex plane shown in Fig. 4.1.

Due to the free nature of (4.14), the amplitude in (4.16) can be computed exactly by substituting the classical solution obeying the appropriate boundary conditions in the presence of the source $\vec{J}$ into the action. The explicit form of such a classical

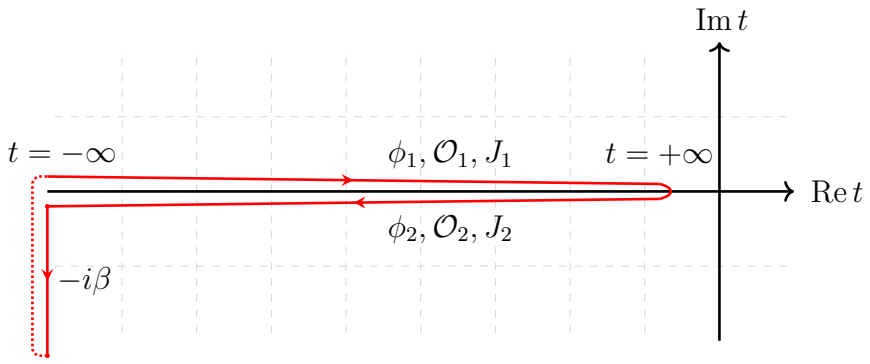

Figure 4.1: The contour $\mathcal{C}$ used for the Schwinger-Keldysh generator (3.13) when $\rho \propto e^{-\beta H}$. The dotted line represents the completion of the trace; there is no evolution along this segment. Other contours for finite-T systems can be found in the literature; see [115] for a review.

solution is[13]

$$\vec{\Phi}_{\rm cl}(t,\mathbf{k}) = K(t,\mathbf{k},t_\star)\cdot\vec{\Phi}_b(\mathbf{k}) + K(-t,\mathbf{k},t_\star)\cdot\vec{\Phi}_a(\mathbf{k}) + \int_{-t_\star}^{t_\star} \mathrm{d}t'\, G(t,t',\mathbf{k},t_\star)\cdot\vec{J}(t',\mathbf{k}) \ , \tag{4.17a}$$

$$K^{AB}(t,\mathbf{k},t_\star) \equiv \delta^{AB}\frac{\sin(\omega_{\mathbf{k}}(t+t_\star))}{\sin(2\omega_{\mathbf{k}}t_\star)} \ , \tag{4.17b}$$

$$G^{AB}(t,t',\mathbf{k},t_\star) \equiv \delta^{AB}\times\Big[\frac{\sin(\omega_{\mathbf{k}}(t-t'))}{2\omega_{\mathbf{k}}}\left[\theta(t-t')-\theta(t'-t)\right] - \frac{2\cos(\omega_{\mathbf{k}}(t+t')) - \cos(\omega_{\mathbf{k}}(2t_\star+t-t')) - \cos(\omega_{\mathbf{k}}(2t_\star+t'-t))}{4\omega_{\mathbf{k}}\sin(2\omega_{\mathbf{k}}t_\star)}\Big] \ , \tag{4.17c}$$

where $\omega_{\mathbf{k}}^2 = \mathbf{k}^2 + m^2$ and where, for now, we are imposing boundary conditions at $\pm t_\star$. The $G^{AB}(t,t',\mathbf{k},t_\star)$ propagator obeys $(\Box_x - m^2)G^{AB}(x,z,t_\star) = -\delta^{AB}\delta^4(x-z)$ in position-space and is symmetric under $t \to t'$ and vanishes at the boundaries where either t' or t equals $\pm t_\star$ while $K^{AB}(x,\mathbf{z},t_\star)$ solves the equation of motion identically, reduces to the identity at the $t = t_\star$ boundary, and vanishes at $t = -t_\star$.

[13]Note that in this appendix we are denoting spatial vectors in boldface to reserve the vector symbol for objects in the fundamental representation of the internal $SO(N)$ group.

Plugging the expression above into the action, the amplitude (4.16) reduces to

$$\begin{aligned}\langle\vec{\Phi}_f,+t_\star|\vec{\Phi}_i,-t_\star\rangle_{\vec{J}} = \exp\Big[&\frac{i}{2}\int\frac{\mathrm{d}^3k}{(2\pi)^3}\,\omega_{\mathbf{k}}\cot(2\omega_{\mathbf{k}}t_\star)\left(\vec{\Phi}_i\cdot\vec{\Phi}_i+\vec{\Phi}_f\cdot\vec{\Phi}_f\right)-2\omega_{\mathbf{k}}\csc(2\omega_{\mathbf{k}}t_\star)\vec{\Phi}_i\cdot\vec{\Phi}_f\\&+i\int_{-t_\star}^{t_\star}\mathrm{d}t\int\frac{\mathrm{d}^3k}{(2\pi)^3}\,\vec{J}(t,\mathbf{k})\cdot K(t,\mathbf{k},t_\star)\cdot\vec{\Phi}_f+\vec{J}(t,\mathbf{k})\cdot K(-t,\mathbf{k},t_\star)\cdot\vec{\Phi}_i\\&+\frac{i}{2}\int_{-t_\star}^{t_\star}\mathrm{d}t\mathrm{d}w\int\frac{\mathrm{d}^3k}{(2\pi)^3}\,\vec{J}(t,-\mathbf{k})\cdot G(t,w,\mathbf{k})\cdot\vec{J}(w,\mathbf{k})\Big]\end{aligned}\tag{4.18}$$

where $\vec{\Phi}_i=\vec{\Phi}_i(\mathbf{k})$ everywhere. Note that this result is manifestly invariant under $SO(N)$. The density matrix components come from setting $t_\star=i\beta/2$ and $\vec{J}\to 0$ in the above.

After calculating the three amplitude factors in (4.15), it is straightforward to stitch them all together by computing the remaining Gaussian path integrals over $\vec{\Phi}_a,\vec{\Phi}_b$, and $\vec{\Phi}_c$. It is this last step that gives rise to the cross terms $\sim J_1\times J_2$. The ultimate expression is written most compactly in terms of the $\vec{J}_a,\vec{J}_r$ Keldysh basis sources:

$$\begin{aligned}&\ln Z[\vec{J}_a,\vec{J}_r]\\&=-\frac{1}{2}\int\frac{\mathrm{d}\omega\mathrm{d}^3k}{(2\pi)^4}\begin{pmatrix}\vec{J}_r(-k) & \vec{J}_a(-k)\end{pmatrix}\cdot\begin{pmatrix}0 & \frac{i}{(\omega+i\epsilon)^2-\omega_{\mathbf{k}}^2}\\ \frac{i}{(\omega-i\epsilon)^2-\omega_{\mathbf{k}}^2} & \left(\frac{1}{2}+\frac{1}{e^{\beta|\omega|}-1}\right)2\pi\delta(\omega^2-\omega_{\mathbf{k}}^2)\end{pmatrix}\cdot\begin{pmatrix}\vec{J}_r(k)\\ \vec{J}_a(k)\end{pmatrix}.\end{aligned}\tag{4.19}$$

In this paper we considered an alternative representation of the generating functional that relies on *effective* fields[14] and for which the usual, naive rules of Gaussian integration can be used. Such a representation allows us to avoid the complicated, multistep process above. From this viewpoint, the generator (4.19) is instead constructed as

$$Z[J_a,J_r]=\int\mathcal{D}\vec{\Phi}_a\mathcal{D}\vec{\Phi}_r\,e^{iS_{\mathrm{EFT}}[\vec{\Phi}_a,\vec{\Phi}_r]+i\int\mathrm{d}^{d+1}x\,\vec{J}_r\cdot\vec{\Phi}_a+\vec{J}_a\cdot\vec{\Phi}_r}\tag{4.20}$$

for some S_{EFT}. Finding the appropriate effective action which reproduces (4.19) is a simple exercise in reverse engineering via standard Gaussian integral formulas, and a convenient form is[15]

$$\begin{aligned}S_{\mathrm{EFT}}[\vec{\Phi}_a,\vec{\Phi}_r]\equiv\int\mathrm{d}^4x\,&\left(-\partial\vec{\Phi}_a\cdot\partial\vec{\Phi}_r-m^2\vec{\Phi}_a\cdot\vec{\Phi}_r-2\varepsilon\vec{\Phi}_a(x)\cdot\partial_t\vec{\Phi}_r(x)\right)\\&+i\varepsilon\int\frac{\mathrm{d}\omega\mathrm{d}^3k}{(2\pi)^4}\,\omega\coth(\beta\omega/2)\vec{\Phi}_a(k)\cdot\vec{\Phi}_a(-k)\ ,\end{aligned}\tag{4.21}$$

[14] In what follows, we denote these effective fields with the same symbol, $\vec{\Phi}$, to streamline the notation.

[15] Equivalence follows from the identity $\lim_{\varepsilon\to0}\frac{\varepsilon}{x^2+\varepsilon^2}=\pi\delta(x)$. The fact that the imaginary terms are $\mathcal{O}(\varepsilon)$ is an artifact of the free limit [96]; they are finite in realistic, interacting systems, see e.g. [116].

where we take $\varepsilon \to 0^+$ at the end of any given computation, as usual.

For emphasis, in contrast to the path integral considered at the start of this section, when using (4.21) there is no need to carefully consider boundary conditions on fields at $t \to \pm\infty$ or the presence of a non-trivial density matrix. Such features are already encoded in $S_{\rm EFT}$ itself. In particular, the explicit factors of β in (4.21) reflect the thermal nature of the system which is also a consequence of the dynamical KMS symmetry [34] of the effective $\vec{\Phi}$ fields. Explicitly, this acts linearly on the fields in frequency space as

$$\vec{\Phi}_a(\omega) \to \cosh(\beta\omega/2)\vec{\Phi}_a(-\omega) - 2\sinh(\beta\omega/2)\vec{\Phi}_r(-\omega) \tag{4.22a}$$

$$\vec{\Phi}_r(\omega) \to \cosh(\beta\omega/2)\vec{\Phi}_r(-\omega) - \frac{1}{2}\sinh(\beta\omega/2)\vec{\Phi}_a(-\omega)\ , \tag{4.22b}$$

and it can be checked that (4.21) is precisely invariant under the above.

For general values of $\beta\omega$, the representation of the generating functional via an effective action (4.21) is not obviously advantageous. Though this construction allows us to more easily use familiar path-integral methods, its convenience is offset by the fact that the terms $\sim \vec{\Phi}_a^2$ are non-local. However, $S_{\rm EFT}$ becomes approximately local in the low-energy limit, $\beta\omega \ll 1$:

$$S_{\rm EFT} \simeq \int \mathrm{d}^4x \left\{ -\partial\vec{\Phi}_a \cdot \partial\vec{\Phi}_r - m^2\vec{\Phi}_a \cdot \vec{\Phi}_r + 2\varepsilon \left(-\vec{\Phi}_a \cdot \partial_t\vec{\Phi}_r + \frac{i}{\beta}\vec{\Phi}_a(x) \cdot \vec{\Phi}_a(x) \right) \right\}\ . \tag{4.23}$$

The structure of the $\mathcal{O}(\varepsilon)$ terms is dictated by the β-expansion of the dynamical KMS symmetry (4.22) in which the symmetry acts on the fields in momentum space as in

$$\vec{\Phi}_a(\omega,\mathbf{k}) \to \vec{\Phi}_a(-\omega,\mathbf{k}) - \beta\omega\vec{\Phi}_r(-\omega,\mathbf{k})\ , \qquad \vec{\Phi}_r(\omega,\mathbf{k}) \to \vec{\Phi}_r(-\omega,\mathbf{k}) - \frac{\beta\omega}{4}\vec{\Phi}_a(-\omega,\mathbf{k})\ , \tag{4.24}$$

under which the $\mathcal{O}(\varepsilon)$ terms in (4.23) are strictly invariant.

Note that the effective action is invariant under two copies of the global $SO(N)$ symmetries provided we disregard the terms proportional to ε, as is customary when discussing the symmetries of effective actions. However, it is easy to show that both copies of the symmetries are realized linearly on our effective fields, meaning that the off-diagonal symmetry is not spontaneously broken in this very simple case. This is an artifact of the free limit, which prevents the existence of a hydrodynamic regime at low energies.

4.7 Correlators

A wide variety of conventions and notations exist for the possible correlators in quantum field theory. Ours are found below, along with various useful relations they satisfy. Given a set of operators $\mathcal{O}^i$ with i some general indices, some of the correlators one might be interested in calculating are

$$\begin{aligned}
\Delta^{ij}(x,x') &\equiv \langle\big[\mathcal{O}^i(x),\mathcal{O}^j(x')\big]\rangle \\
G_W^{ij}(x,x') &\equiv \langle\mathcal{O}^i(x)\mathcal{O}^j(x')\rangle \\
G_R^{ij}(x,x') &\equiv i\theta(t-t')\langle\big[\mathcal{O}^i(x),\mathcal{O}^j(x')\big]\rangle = i\langle\mathcal{O}_r^i(x)\mathcal{O}_a^j(x')\rangle \\
G_A^{ij}(x,x') &\equiv -i\theta(t'-t)\langle\big[\mathcal{O}^i(x),\mathcal{O}^j(x')\big]\rangle = -i\langle\mathcal{O}_a^i(x)\mathcal{O}_r^j(x')\rangle \\
G_S^{ij}(x,x') &\equiv \frac{1}{2}\langle\{\mathcal{O}^i(x),\mathcal{O}^j(x')\}\rangle = \langle\mathcal{O}_r^i(x)\mathcal{O}_r^j(x')\rangle \ ,
\end{aligned} \tag{4.25}$$

where for the last three correlators we have also provided an expression in terms of fields in the Keldysh basis introduced in Sec. 4.5. The momentum-space versions of the above are defined in the usual way, with $G_X^{ij}(\omega,\vec{k})$ determined from a correlator of $\mathcal{O}^i(\omega,\vec{k})$ and $\mathcal{O}^j(-\omega,-\vec{k})$.

The commutator Δ^{ij} and the retarded correlator G_R^{ij} are related in momentum space by

$$G_R^{ij}(\omega,\vec{k}) = \lim_{\varepsilon\to 0^+}\int\frac{\mathrm{d}\omega'}{2\pi}\,\frac{\Delta^{ij}(\omega',\vec{k})}{\omega'-\omega-i\varepsilon} \ , \tag{4.26}$$

as follows from the Fourier-representation of the Heaviside function. The zero-frequency limit of this relation determines the *static susceptibilities* $\chi^{ij}(\vec{k})$,

$$\chi^{ij}(\vec{k}) \equiv \lim_{\omega\to 0} G_R^{ij}(\omega,\vec{k}\neq 0) = \lim_{\varepsilon\to 0^+}\int\frac{\mathrm{d}\omega'}{2\pi}\,\frac{\Delta^{ij}(\omega',\vec{k})}{\omega'-i\varepsilon} \ . \tag{4.27}$$

At finite temperature, the KMS conditions relate various correlators to each other. These can be straightforwardly derived starting from inserting $\mathbf{1} = e^{-\beta H}e^{\beta H}$ judiciously into the Wightman function G_W^{ij} (also called the *dynamical structure factor*) and using the cyclicity of the trace to get (highlighting only the temporal and frequency dependence)

$$G_W^{ij}(t) = G_W^{ji}(-i\beta-t) = e^{i\beta\partial_t}G_W^{ji}(-t) \quad\Longrightarrow\quad G_W^{ij}(\omega) = e^{\beta\omega}G_W^{ji}(-\omega) \ . \tag{4.28}$$

In particular, in the limit $\beta\omega \ll 1$ that is the focus of our paper, this implies a form of the Fluctuation-Dissipation Theorem

$$G_S^{ij}(\omega) \simeq G_W^{ij}(\omega) \approx \frac{\Delta^{ij}(\omega)}{\beta\omega} \ , \tag{4.29}$$

up to corrections of higher order in $\beta\omega$. The first relation above implies that $\mathcal{O}^i$ and $\mathcal{O}^j$ approximately commute, thereby motivating the "classical limit" terminology.

Using the effective actions discussed in this paper it is straightforward to calculate the symmetric G_S^{ij} viewed as an r-r correlator in the language of the Keldysh rotation of Sec. 4.5. The low energy expressions for G_R^{ij}, Δ^{ij}, and G_W^{ij} then follow from G_S^{ij} using (4.26) and (4.29).

4.8 Effective Field Theory of Diffusion

In this section, we will elaborate on methods found in the literature to construct Schwinger-Keldysh EFTs for conserved quantities. The construction followed in this section was first formulated in [23], and used for non-Abelian symmetries [69], and diffusive systems [117–119], see also [34] for a review. When building EFTs, the method used by the above work introduces an additional symmetry called the diffusive symmetry, whose existence distinguishes between the normal phase and the SSB phase. This is not a usual ingredient in the construction of theories with Goldstone modes, and when we generalize coset construction to the Schwinger-Keldysh formalism in Sec. 5.2, we will not require this symmetry. Nevertheless, we think that illustrating its use and the state of the art in Schwinger-Keldysh EFTs through a simple example will be helpful in making comparisons with our SK coset method.

The example we will consider is diffusion in a closed thermal system with $U(1)$ charge. The generating functional for such a system in the SK formalism is (suppressing spacetime indices)

$$e^{iW[A_{1\mu},A_{2\mu}]} = \mathrm{Tr}\left(\rho \bar{T} e^{-i\int A_2^\mu J_{2\mu}} T e^{i\int A_1^\mu J_{1\mu}}\right) \tag{4.30}$$

where ρ is the initial thermal state, $J_{s\mu}$ $(s = 1, 2)$ are the currents associated with the conserved charge and A_s^μ are the external vector fields. The conservation of the current $\partial^\mu J_{i\mu} = 0$ can be imposed by requiring the following transformation rule on the generating functional:

$$W[A_{1\mu}, A_{2\mu}] = W[A_{1\mu} + \partial_\mu \lambda_1, A_{2\mu} + \partial_\mu \lambda_2] \tag{4.31}$$

for arbitrary functions λ_1, λ_2 that vanish at spacetime infinities. To be able to derive an effective action, we need to "integrate in" the modes φ_1, φ_2 that correspond to low energy collective excitations:

$$e^{iW[A_{1\mu},A_{2\mu}]} = \int \mathcal{D}\varphi_1 \mathcal{D}\varphi_2 e^{iS_{EFT}[\varphi_1,\varphi_2;A_{1\mu},A_{2\mu}]}. \tag{4.32}$$

Since we're using φ_s to represent $J_{s\mu}$, their equation of motion should correspond to conservation of $J_{s\mu}$. They should also transform in such a way that (4.31) still holds. These two conditions forces the fields to transform as

$$A_{s\mu} \to A_{s\mu} - \partial_\mu \lambda_s, \qquad \varphi_s \to \varphi_s + \lambda_s \qquad s = 1, 2\,, \tag{4.33}$$

and appear in the combination

$$B_{1\mu} = A_{1\mu} + \partial_\mu \varphi_1 \,, \qquad B_{2\mu} = A_{2\mu} + \partial_\mu \varphi_2 \tag{4.34}$$

such that under the gauge transformation we get:

$$B'_{s\mu} = A_{s\mu} - \partial_\mu \lambda_i + \partial_\mu \varphi_s + \partial_\mu \lambda_s = B_{s\mu}. \tag{4.35}$$

The fact that we have introduced a new degree of freedom to impose the "gauge" symmetry (4.31) means we can interpret φ_s as Stückelberg fields [120].

We can now define currents:

$$\tilde{J}_1^\mu \equiv \frac{1}{i}\frac{\delta S_{EFT}}{\delta A_{1\mu}} \,, \qquad \tilde{J}_2^\mu \equiv -\frac{1}{i}\frac{\delta S_{EFT}}{\delta A_{2\mu}}. \tag{4.36}$$

With these definitions, the equations of motion for φ_s lead to the conservation equations for the currents:

$$\begin{aligned} \frac{\delta S_{EFT}}{\delta \varphi_s(x)} = \int d^4x' \frac{\delta S_{EFT}}{\delta B_{s\mu}(x')}\frac{\delta B_{s\mu}(x')}{\delta \varphi_s(x)} &= \int d^4x' \frac{\delta S_{EFT}}{\delta A_{s\mu}(x')}\partial_\mu \delta(x-x') \\ &= -\partial_\mu \tilde{J}_s^\mu(x) \end{aligned} \tag{4.37}$$

and the correlations of currents $J_{s\mu}$ derived from (4.30) are given by the correlators of $\tilde{J}$ from the EFT:

$$\mathrm{Tr}(\rho J_1^\mu(x) J_2^\nu(y)) = \int \mathcal{D}\varphi_1 \mathcal{D}\varphi_2 \tilde{J}_1^\mu(x)\tilde{J}_2^\mu(y) e^{iS_{EFT}[\varphi_1,\varphi_2;A_{1\mu},A_{2\mu}]}\Big|_{A_1=A_2=0} \tag{4.38}$$

Since the generating functional W results from integrating out the modes φ_s, we generically expect it to be non-local. The effective action S_{EFT} is also generically non-local, since it in turn results from integrating out "fast" modes that govern microscopic physics. But in the hydrodynamic approximation, that is the length scale for the collective excitations λ related to conserved quantities is much bigger than the relaxation length ℓ of microscopic non-conserved processes, then S_{EFT} can be written in a local derivative expansion with powers of $\frac{\ell}{\lambda}\partial_\mu$.

The effective action $S_{EFT}[B_{1\mu}, B_{2\mu}]$ must additionally obey the unitarity constraints and DKMS as discussed in Sec. 4.3. The DKMS transformation for $B_{s\mu}$ in the Keldysh basis is:

$$\begin{aligned} B_{r\mu}(t,\vec{x})' &= B_{r\mu}(-t,-\vec{x}) - \tfrac{i}{4}\beta\partial_t B_{a\mu}(-t,-\vec{x}) \\ B_{a\mu}(t,\vec{x})' &= B_{a\mu}(-t,-\vec{x}) - i\beta\partial_t B_{\alpha\mu}(-t,-\vec{x}), \end{aligned} \tag{4.39}$$

where we have assumed $\mathcal{PT}$ invariance for the ground state.

Finally, there is one more symmetry that distinguishes between the normal phase and the SSB phase of the system. In the normal phase S_{EFT} is invariant under the following:

$$B'_{1i} = B_{1i} - \partial_i \lambda(\vec{x}), \qquad B'_{2i} = B_{2i} - \partial_i \lambda(\vec{x}). \tag{4.40}$$

This symmetry is *diagonal*, in the sense that both copies of the field transform under the same space-dependent function $\lambda(\vec{x})$, unlike the two copies of (4.33). In the Keldysh basis this symmetry takes the form

$$B'_{ri} = B_{ri} - \partial\lambda(\vec{x}), \qquad B'_{ai} = B_{ai} \tag{4.41}$$

which implies that in the normal phase, any factor of B_{ri} must always come with at least one time derivative, or with the combination $F_{rij} \equiv \partial_i B_{rj} - \partial_j B_{ri}$. For the SSB case there is no longer such a constraint on B_{ri}.

The diffusive symmetry arises in the normal phase and stems from the fact that if we consider the system as consisting of fluid elements, locally we can consider $B_{s\mu}(x)$ to be external sources for each fluid element labelled by x. For example, with this interpretation a local chemical potential can be defined via

$$B_{s0}(x) = A_{s0}(x) + \partial_0\varphi_s(x). \tag{4.42}$$

Note that the definition includes the collective excitation φ_s, and not just the external source A_{s0}: the fact that fluctuations of collective excitations act as a local source is a manifestation of the local transport of conserved charge. The origin of the diffusive symmetry is then the fact that global $U(1)$ invariance allows for each fluid element, which is independent of one another, to make an independent phase rotation that does not depend on time: $e^{i\lambda(\vec{x})}$. This freedom in the choice of the phase directly results in the symmetry given in (4.40). In the SSB case, the global phase is fixed, meaning there is no freedom to choose an independent phase for each fluid element, and the diffusive symmetry is no longer present.

With the building blocks and symmetries specified, we can build effective actions. We first look at the diffusion case at the quadratic level. The Lagrangian is:

$$\mathcal{L}^{(2)}_{EFT} = \chi B_{at} B_{rt} + \sigma B_{ai}\left(\frac{i}{\beta}B_{ai} - \partial_t B_{ri}\right) + \dots \tag{4.43}$$

where χ and σ are EFT parameters. Due to the condition $\mathrm{Im}\, S_{EFT} \geq 0$, $\sigma \geq 0$ and the form of the last term above comes from DKMS invariance. By taking variational derivatives with respect to $A_{r\mu}, A_{a\mu}$, we can find the currents:

$$\frac{\delta\mathcal{L}_{EFT}}{\delta A_{at}} = \tilde{J}^t_r = \chi B_{rt} \equiv \chi\mu, \tag{4.44}$$

$$\frac{\delta\mathcal{L}_{EFT}}{\delta A_{ai}} = \tilde{J}^i_r = \frac{2i\sigma}{\beta}B_{ai} - \sigma\partial_t B_{ri} \equiv \sigma(E_i - \partial_i\mu) + \frac{2i\sigma}{\beta}B_{ai}, \tag{4.45}$$

$$\frac{\delta\mathcal{L}_{EFT}}{\delta A_{rt}} = \tilde{J}^t_a = \chi B_{at}, \tag{4.46}$$

$$\frac{\delta\mathcal{L}_{EFT}}{\delta A_{ri}} = \tilde{J}^i_a = \sigma\partial_t B_{ri}, \tag{4.47}$$

where in the first line we defined the chemical potential $\mu \equiv B_{rt} = A_{rt} + \partial_t \varphi_r$. In our SK coset construction method, to be discussed in Chapter 5, the chemical potential will correspond to the "matter field" ρ_r. To get the last equality in the second line, we used

$$\begin{aligned} \partial_t B_{ri} = \partial_t A_{ri} + \partial_t \partial_i \varphi_r &= \partial_t A_{ri} - \partial_i A_{rt} + \partial_i \left(A_{rt} + \partial_t \varphi_r \right) \\ &\equiv F_{ti} + \partial_i \mu = -E_i + \partial_i \mu \end{aligned} \tag{4.48}$$

where we have defined the field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and the electric field $E_i = F_{ti}$. For the case $\mu = 0$, (4.45) reduces to Ohm's law. We also observe that since the charge density $n = \tilde{J}^t_r = \chi\mu$, (setting $A_{a\mu} = 0$), the equation of motion for φ_r (obtained by varying w.r.t φ_a) leads to

$$\begin{aligned} 0 = \frac{\delta S_{EFT}}{\delta \varphi_a} = -\chi \partial_t B_{rt} + \sigma \partial_i \partial_t B_{ri} &= -\partial_\mu \tilde{J}^\mu_r \\ &= -\chi \partial_t n - \sigma \partial_i \left(E_i - \frac{1}{\chi} \partial_i n \right), \end{aligned} \tag{4.49}$$

and we recover the diffusion equation in the presence of an external force:

$$\partial_t n - \mathcal{D} \partial_i^2 n = -\sigma \partial_i E_i \tag{4.50}$$

where we defined the diffusion constant $\mathcal{D} \equiv \sigma/\chi$. This allows us to interpret the EFT parameters χ and σ as the charge susceptibility and conductivity. The former also through $n = \chi\mu$ and the latter from the form of $\tilde{J}^i_r$. Furthermore, the dispersion relation at $E_i = 0$ is $\omega = i\mathcal{D}k^2$, as expected from a diffusive mode.

We can equally find an equation of motion for φ_a, which also leads to the conservation of the a-current: $\partial_\mu J^\mu_a = 0$. This indicates that the system is *closed*, unlike in an open system where the current J^μ_a will not be conserved on the equation of motion level.

We also want to mention that the linear relation $n = \chi\mu$ at the quadratic level becomes nonlinear with the inclusion of higher order interactions. This will lead to a nonlinear version of the diffusion equation [118]:

$$\partial_t n(\mu) = \partial_i \left[\sigma(\mu) \partial^i \mu \right] \tag{4.51}$$

where $n(\mu) = \chi \left(\mu + \frac{\chi'}{2} \mu^2 + \dots \right)$ and $\sigma(\mu) = \sigma \left(1 + \frac{\sigma'}{\sigma} \mu + \dots \right)$ where χ', σ' are free EFT parameters that appear at the cubic level. This allows us to handle non-linearites order by order in the EFT; see Section 6.4 for more details.

Finally, we look at the SSB phase of the system. We drop the requirement of diffusive symmetry, which leads to two additional terms in the quadratic action

$$\mathcal{S}^{(2)}_{SSB} = \frac{f^2}{c_s^3} \int d^4x B_{at} B_{rt} - c_s^2 B_{ai} B_{ri} + \sigma B_{ai} \left(\frac{i}{\beta} B_{ai} - \partial_t B_{ri} \right) + \sigma B_{at} \left(\frac{i}{\beta} B_{at} - \partial_t B_{rt} \right), \tag{4.52}$$

where the last term is now included due to the power counting in the SSB phase ($\partial_t \sim \partial_i$). We see that the action leads to the dispersion relation $\omega = c_s|\vec{k}| + \ldots$ for φ_r, which is gapless and linear, as expected from our in-out intuition of SSB of $U(1)$-symmetry.

4.8.1 Effective Field Theory of Hydrodynamics

For completeness, we briefly mention how the above methodology can be used for hydrodynamics, where the conserved charges include energy and momentum as well as any charges due to global symmetries. The current for energy-momentum is the stress-energy tensor $T_{\mu\nu}$, which we assume is conserved. The associated symmetry of W is the diffeomorphism symmetry of the corresponding "source", the metric $g_{s\mu\nu}$, $s = 1, 2$, on the two branches:

$$g^{\xi}_{sAB}(\sigma) = g_{s\mu\nu}\left(\chi_1(\sigma)\right) \frac{\partial \xi^\mu}{\partial \sigma^A} \frac{\partial \xi^\nu}{\partial \sigma^B} \tag{4.53}$$

Following the same reasoning as before, we promote the parameters ξ to dynamical variables $X^\mu_s(\sigma)$, which enforce the above symmetry. To do this we first introduce an auxiliary spacetime σ^A, $A = 0, 1, 2, 3$. The spatial part σ^i labels each fluid element and σ^0 is the internal clock of each such element. The fields X^μ_s then correspond to a map from the fluid spacetime to the physical spacetime and can be viewed as the Lagrangian description of a fluid. The action can now be built using the equivalent of $B_{s\mu}$ from the previous section, which we define as

$$h_{sAB} \equiv g_{s\mu\nu} \frac{\partial X^\mu_s}{\partial \sigma^A} \frac{\partial X^\nu_s}{\partial \sigma^B}, \qquad s = 1, 2, \tag{4.54}$$

and the generating functional can be written as

$$e^{iW[g_{1\mu\nu}, g_{2\mu\nu}]} = \int \mathcal{D}X_1 \mathcal{D}X_2 e^{iS_{EFT}[h_{1AB}, h_{2AB}]}. \tag{4.55}$$

The correlators of $\tilde{T}_{\mu\nu}$ in the EFT description will now correspond to the correlators of $T_{\mu\nu}$ of full theory.

4.9 Connection to Fluctuating Hydrodynamics

In this section we briefly comment on the connection between the previous section's formalism with stochastic hydrodynamics [1], using the $U(1)$ diffusion case discussed above as an example (for a more general treatment refer to [23]).

The quadratic EFT of $U(1)$ diffusion with all sources turned off ($B_{s\mu} \to \partial_\mu \varphi_s$)) is

$$\mathcal{L}_{EFT} = \chi \partial_t \varphi_a \partial_t \varphi_r + \sigma \partial_i \varphi_a \left(\frac{i}{\beta} \partial_i \varphi_a - \partial_t \partial_i \varphi_r \right) \tag{4.56}$$

or in terms of the charge density n:

$$\begin{aligned}\mathcal{L}_{EFT} &= \varphi_a\left(-\partial_t n + \mathcal{D}\partial_i^2 n\right) - i\varphi_a\left(\frac{\sigma}{\beta}\partial_i^2\right)\varphi_a \\ &\equiv \left(-\partial_t n + \mathcal{D}\partial_i^2 n\right)\varphi_a - \frac{i}{2}\varphi_a K_{aa}\varphi_a \end{aligned} \tag{4.57}$$

and the corresponding path integral is

$$Z = \int \mathcal{D}n\mathcal{D}\varphi_a e^{i\int d^4x \mathcal{L}_{EFT}[n,\varphi_a]}. \tag{4.58}$$

Now we employ a Hubbard-Stratonovich transformation using the identity

$$\exp\left\{-\frac{1}{2}\int d^4x \varphi_a K_{aa}\varphi_a\right\} = \int \mathcal{D}\xi \exp\left\{-\int d^4x\, \frac{1}{2}\xi K_{aa}^{-1}\xi - i\xi\varphi_a\right\}. \tag{4.59}$$

Inserting this into the path integral we find

$$Z = \int \mathcal{D}n\mathcal{D}\varphi_a\mathcal{D}\xi \exp\left\{-\int \frac{1}{2}\xi K_{aa}^{-1}\xi\right\}\exp\left\{i\int d^4x\varphi_a\left(-\partial_t n + \mathcal{D}\partial_i^2 n + i\xi\right)\right\}. \tag{4.60}$$

The field φ_a now just a Lagrange multiplier and we evaluate the integral:

$$Z = \int \mathcal{D}n\mathcal{D}\xi \exp\left\{-\int \frac{1}{2}\xi K_{aa}^{-1}\xi\right\}\delta\left(-\partial_t n + \mathcal{D}\partial_i^2 n + \xi\right). \tag{4.61}$$

The above result leads to the stochastic diffusion equation

$$\partial_t n - \mathcal{D}\partial_i^2 n = \xi, \tag{4.62}$$

where ξ is a stochastic force that is Gaussian distributed:

$$\langle\xi\rangle = 0, \qquad \langle\xi(x)\xi(0)\rangle = K_{aa}\delta(x). \tag{4.63}$$

One can make a similar connection with the extension discussed in 4.8.1 to recover Navier-Stokes equations with stochastic fluctuations, making full contact with the formalism of fluctuating hydrodynamics [1]. The main advantage of the EFT formalism discussed above is that we can go beyond and consider nonlinear contributions of higher order interactions between forces and dynamical variables in a systematic way.

Chapter 5

Schwinger-Keldysh Coset Construction

5.1 Introduction

In Sec. 4.8, we discussed current methods used to build Schwinger-Keldysh Effective Field Theories (SK EFTs). These methods introduce the diffusive symmetry, which is not a component of usual tools of constructing Goldstone EFT, including coset construction. Instead, we will investigate in this chapter to what extent the traditional coset construction can be used, without imposing additional symmetries to distinguish normal vs. SSB phases, to write down the doubled-field effective action $S_{\rm EFT}[\vec{\pi}_1, \vec{\pi}_2]$ that defines the Schwinger-Keldysh generating functional (3.1). In the case of Goldstone fields, doubling the field content would naively appear to be in tension with non-linear realization of the spontaneously broken symmetries. We will explicitly address this by providing a systematic prescription for writing down all possible operators that involve two copies of the Goldstone fields and are compatible with all the symmetries. Our results provide a complementary perspective on the modern field-theoretic description of non-equilibrium systems, reviewed recently in, e.g., [34], as the ingredients in both constructions are intimately related. In particular, we study the additional constraints that a finite temperature places on top of the traditional coset-construction rules, as encapsulated by the discrete *dynamical Kubo-Martin-Schwinger* (DKMS) symmetry argued for in [23]. Previous studies of the coset construction within the Schwinger-Keldysh formalism can be found in [32, 121], for instance. See also [122–124] for interesting discussions of Goldstone modes in out-of-equilibrium systems.

5.2 Schwinger-Keldysh Effective Actions from a Coset Construction

We will now discuss how the coset construction reviewed in Sec. 2.3 can also be used to construct Schwinger-Keldysh effective actions. In Sec. 4.3 we argued that, if the single-field effective action is invariant under the global symmetry group G, the corresponding Schwinger-Keldysh effective action for a closed system must enjoy twice as many symmetries—i.e. be invariant under the group $G_1 \times G_2$—in a regime where the action is local. How many of these symmetries are realized linearly will depend on the state of the system ρ. Inspired by the form of the generating functional in Eq. (3.13), we will *postulate* that the symmetry breaking pattern should be determined by acting with a $G_1 \times G_2$ transformation on the state ρ as follows:[1]

$$\rho \;\to\; U_1 \rho\, U_2^{-1}, \tag{5.1}$$

where U_1 (U_2) is an element of G_1 (G_2). Symmetry transformations of this form that do not leave ρ invariant are spontaneously broken. As we will see, this rule of thumb implies different symmetry breaking patterns depending on whether the state ρ is pure or thermal.

5.2.1 Closed Systems in a Pure State

Let us start by considering the case of pure states: $\rho = |\psi\rangle\langle\psi|$. Spontaneous symmetry breaking occurs if there exists a local operator $\mathcal{O}$ whose expectation value on $|\psi\rangle$ is not invariant under some symmetry transformations. More precisely, the symmetry generator X is spontaneously broken if

$$\mathrm{Tr}(\rho[X, \mathcal{O}]) = \langle\psi|[X, \mathcal{O}]|\psi\rangle \neq 0. \tag{5.2}$$

This condition can be satisfied only if the state $|\psi\rangle$ is an eigenstate of some but not all of the generators of the symmetry group G—say, those spanning a subgroup H. As a result, the state ρ remains invariant under a transformation (5.1) only when $U_1 \in H_1$ and $U_2 \in H_2$. In other words, the pure state ρ effectively gives rise to a symmetry breaking pattern $G_1 \times G_2 \to H_1 \times H_2$. Denoting with X_1^i (X_2^i) the broken generators of G_1 (G_2), we introduce for later convenience the linear combinations of

[1] We should stress that Eq. (5.1) is not in contradiction with the familiar statement that symmetry transformations must act on density matrices as $\rho \to U \rho U^{-1}$ in order to preserve the trace. When the state of the system is encoded using a local Schwinger-Keldysh effective action, one may wonder what part of $G_1 \times G_2$ is preserved by the ground state in the Hilbert space of this effective theory. Eq. (5.1) provides a way of answering this question. We will support the validity of this criterion in the hydrodynamic regime by discussing a series of non-trivial examples in Sec. 5.3. Note however that this criterion does not apply to free fields in their ground state—see App. 4.6—for which a hydrodynamic regime does not exist.

generators

$$X_r^i = X_1^i + X_2^i, \qquad X_a^i = X_1^i - X_2^i. \tag{5.3}$$

Then, the coset parametrization in the case of pure states can be chosen to be of the form

$$\Omega = e^{i\pi_r \cdot X_r} e^{i\pi_a \cdot X_a} \qquad \text{(closed system in a pure state).} \tag{5.4}$$

For the purposes of calculating the corresponding Maurer-Cartan form, it is important to keep in mind that, while $[X_1^i, X_2^j] = 0$, the X_r's and X_a's do not commute with each other. Let us now turn our attention to thermal states.

5.2.2 Closed Systems in a Thermal State

It would appear at first sight that the matrix elements of a thermal state $\rho \sim e^{-\mathcal{H}/T}$ should always be invariant under the diagonal subgroup of $G_1 \times G_2$ based on the very definition of symmetry, *i.e.* $U\mathcal{H}\,U^{-1} = \mathcal{H}$. This conclusion however would be too hasty [2, 125, 126]. In order to formulate more precisely the criterion for SSB around a thermal state, we can follow [127] and deform the Hamiltonian by adding an operator $\Delta\mathcal{H}$ that explicitly breaks the global symmetry under consideration: $\mathcal{H} \to \mathcal{H} + \Delta\mathcal{H}$. We will denote the resulting canonical ensemble with $\rho_{\Delta\mathcal{H}}$. Then, SSB occurs whenever there is an order parameter $\mathcal{O}$ whose expectation value is not invariant even in the limit $\Delta H \to 0$. Denoting a symmetry transformation as $U = e^{i\alpha X}$ with X a symmetry generator, this is equivalent to the statement that

$$\lim_{\Delta\mathcal{H}\to 0} \mathrm{tr}(\rho_{\Delta\mathcal{H}}[X, \mathcal{O}]) \neq 0. \tag{5.5}$$

We can now diagonalize the maximum number of generators of G that commute simultaneously with each other and with the modified Hamiltonian $\mathcal{H} + \Delta\mathcal{H}$, and write the thermal state more explicitly as[2]

$$\rho \sim \sum_{E,\vec{Q},\alpha} e^{-E/T} |E, \vec{Q}, \alpha\rangle\langle E, \vec{Q}, \alpha|, \tag{5.6}$$

where E is the eigenvalue of $\mathcal{H} + \Delta\mathcal{H}$, $\vec{Q}$ are the charges associated with the commuting generators, and α is an additional collective index that accounts for all possible degeneracies. The states $|E, \vec{Q}, \alpha\rangle$ form all possible representations of the unbroken group H, and when we act on the thermal state with an $H_1 \times H_2$ transformation as in (5.1), only the elements of the diagonal subgroup H_{diag} will leave ρ invariant.

[2] Working only with eigenstates of those generators that commute with $\mathcal{H} + \Delta\mathcal{H}$ rather than $\mathcal{H}$ alone is formally equivalent to modifying $e^{-\mathcal{H}/T}$ by adding a projection operator [127].

It might be helpful to illustrate this point with a simple example. Consider a system where an internal $SO(4)$ is spontaneously broken down to $SO(3)$. Then, the deformation $\Delta\mathcal{H}$ breaks $SO(4)$ explicitly down to $SO(3)$. We can diagonalize at most one generator of $SO(3)$ together with $\mathcal{H}+\Delta\mathcal{H}$, and as a result the thermal state takes the form

$$\rho \sim \sum_{E,m,j} e^{-E/T}|E,m,j\rangle\langle E,m,j|, \tag{5.7}$$

with m and j the usual $SO(3)$ quantum numbers, playing respectively the role of charge Q and degeneracy parameter α. Let us now act with an $SO(3)_1 \times SO(3)_2$ transformation on Eq. (5.7), and we obtain

$$\begin{aligned}\rho \to & \sum_{E,m,j}\sum_{m',m''} e^{-E/T} D^{(j)}_{m',m}(U_1)|E,m',j\rangle\langle E,m'',j|D^{(j)*}_{m'',m}(U_2) \\ & = \sum_{E,m',m'',j} e^{-E/T} D^{(j)}_{m',m''}(U_1U_2^{-1})|E,m',j\rangle\langle E,m'',j|,\end{aligned} \tag{5.8}$$

where we obtained the second line by using standard properties of the $D^{(j)}_{m',m}$ matrices, namely $D^{(j)}_{m',m}(U^{-1}) = D^{(j)*}_{m,m'}(U)$ and $\sum_m D^{(j)}_{m',m}(U_1)D^{(j)}_{m,m''}(U_2) = D^{(j)}_{m',m''}(U_1U_2)$ [128]. Eq. (5.8) shows that ρ is only invariant under the diagonal subgroup of $SO(3)_1 \times SO(3)_2$, i.e. when $U_1 = U_2$, while the off-diagonal combination of generators is spontaneously broken.

We conclude therefore that a thermal state realizes the symmetry breaking pattern $G_1 \times G_2 \to H_{\rm diag}$, which differs from the pattern of a pure state. Denoting with T_1^A (T_2^A) the generators of H_1 (H_2), we introduce the linear combinations of generators

$$T_r^A = T_1^A + T_2^A, \qquad T_a^A = T_1^A - T_2^A, \tag{5.9}$$

with T_r^A the generators of $H_{\rm diag}$. We are therefore led to the conclusion that the appropriate coset parametrization for a thermal state should be

$$\Omega = e^{i\pi_r\cdot X_r} e^{i\pi_a\cdot X_a} e^{i\varphi_a\cdot T_a}, \qquad \text{(closed system in a thermal state)}, \tag{5.10}$$

where the specific ordering of the various factors has been chosen for later convenience. However, the field content in Eq. (5.10) is by itself incompatible with the principle that Schwinger-Keldysh effective actions should contain twice the number of degrees of freedom as regular in-out effective actions. In other words, the fields φ_a are missing their "r" partners which, among other things, are necessary to implement the DKMS symmetry.

To remedy this situation, we will add to our field content a set of "matter fields" ρ_r^A in the adjoint representation of the unbroken group $H_{\rm diag}$. In accordance with

the standard rules of the coset construction [101], these matter fields will transform under a generic transformation $g \in G_1 \times G_2$ as

$$\rho_r^A \to h(g, \pi_r, \pi_a, \varphi_a, x)^A{}_B \, \rho_r^B , \tag{5.11}$$

with h some element of H_{diag}. As we will see, these fields admit a simple physical interpretation: in the classical limit, they are related to the densities of unbroken charges.

5.2.3 Implementing the dynamical KMS symmetry

When a system is in a thermal state, correlators must satisfy the KMS condition. This, in turn, imposes some restrictions on the form of the Schwinger-Keldysh effective action. When all the symmetries are linearly realized, these restrictions are enforced by invariance under a DKMS symmetry transformation of the form (4.13). We will now discuss how this symmetry should be implemented on the Goldstone fields and the matter fields ρ_r.

To this end, it is helpful to discuss how the KMS condition would affect, say, the 2-point correlation functions of the conserved Noether currents in the effective theory:

$$\langle \mathcal{J}_\mu(t, \vec{x}) \mathcal{J}_\nu(t', \vec{x}') \rangle = \langle \mathcal{J}_\nu(t' - i\beta/2, \vec{x}') \mathcal{J}_\mu(t + i\beta/2, \vec{x}) \rangle, \tag{5.12}$$

where we have suppressed the index labeling the corresponding symmetry generators to streamline the notation. We can introduce a generating functional similar to the one in Eq. (3.12) that allows us to systematically calculate such correlators. In this case, the external sources for the $\mathcal{J}_\mu$'s are gauge fields $A_{1,\mu}$ and $A_{2,\mu}$, and the KMS condition (5.12) implies that the generating functional must satisfy the following property:[3]

$$Z[A_1(t), A_2(t)] = Z[A_1(-t + i\beta/2), A_2(-t - i\beta/2)] \ . \tag{5.13}$$

Switching to the (a, r) basis of fields, and expanding in powers of E/T, this means that Z should be invariant under the following transformation of sources:

$$A_r(t) \to A_r(-t) - \frac{i\beta}{4} \partial_t A_a(-t) + \mathcal{O}(E^2/T^2) \ , \tag{5.14a}$$

$$A_a(t) \to A_a(-t) - i\beta \partial_t A_r(-t) + \mathcal{O}(E^2/T^2). \tag{5.14b}$$

Within the context of the coset construction, external gauge fields can be introduced by gauging the Maurer-Cartan form as discussed in Sec. 2.5:

$$\begin{aligned} &\Omega^{-1} \left(\partial_\mu + i A_\mu^r \cdot X_r + i A_\mu^a \cdot X_a + i A_\mu^a \cdot T_a + i A_\mu^r \cdot T_r \right) \Omega \\ &\qquad \equiv i \left(D_\mu \pi_r \cdot X_r + D_\mu \pi_a \cdot X_a + D_\mu \varphi_a \cdot T_a + \mathcal{A}_\mu \cdot T_r \right) , \end{aligned} \tag{5.15}$$

[3] We are implicitly assuming here that DKMS transformations are implemented without resorting to additional discrete symmetries—see comment in footnote 6. Eq. (5.13) could be generalized by allowing such discrete symmetries to act on the gauge fields on the right-hand side. It is easy to work out how the following results would need to be modified.

where we have denoted with a dot the contraction of indices labeling the various generators. The coset covariant derivatives and connection now depend both on the Goldstone fields and the external gauge fields.

Our goal is to derive how the DKMS symmetry should be implemented on the Goldstone fields and the matter fields ρ_r to ensure that the generating functional is invariant under (5.14). To this end, we will start by considering a situation where all the generators are spontaneously broken. This is not the physical symmetry breaking pattern we are actually interested in: it is just a convenient trick we will use to figure out how our Goldstone and matter fields should transform. If all the symmetries were spontaneously broken, the coset parametrization would read

$$\tilde{\Omega} = e^{i\pi_r \cdot X_r} e^{i\pi_a \cdot X_a} e^{i\varphi_a \cdot T_a} e^{i\varphi_r \cdot T_r} \ , \tag{5.16}$$

and the components of the associated Maurer-Cartan form would be

$$\begin{aligned} &\tilde{\Omega}^{-1} \left(\partial_\mu + iA^r_\mu \cdot X_r + iA^a_\mu \cdot X_a + iA^a_\mu \cdot T_a + iA^r_\mu \cdot T_r\right) \tilde{\Omega} \\ &\quad \equiv i \left(\tilde{D}_\mu \pi_r \cdot X_r + \tilde{D}_\mu \pi_a \cdot X_a + \tilde{D}_\mu \varphi_a \cdot T_a + \tilde{D}_\mu \varphi_r \cdot T_r\right) \\ &\quad = i\, e^{-i\varphi_r \cdot T_r} \left[D_\mu \pi_r \cdot X_r + D_\mu \pi_a \cdot X_a + D_\mu \varphi_a \cdot T_a + \left(ie^{i\varphi_r \cdot T_r} \partial_\mu e^{-i\varphi_r \cdot T_r} + \mathcal{A}_\mu\right) \cdot T_r\right] e^{i\varphi_r \cdot T_r} \ . \end{aligned} \tag{5.17}$$

In the second line, we are showing explicitly how the building blocks of our new coset are related to the "physical" ones defined in Eq. (5.15).

The advantage of considering an auxiliary symmetry breaking pattern where all the symmetries are spontaneously broken is twofold: first, all symmetries are now treated on equal footing and, in particular, all components of our external gauge fields can be obtained from a covariant derivative by turning off the Goldstone fields:

$$\tilde{D}_\mu \pi^i_r \to A^{ri}_\mu, \qquad \tilde{D}_\mu \pi^i_a \to A^{ai}_\mu, \qquad \tilde{D}_\mu \varphi^B_a \to A^{aB}_\mu, \qquad \tilde{D}_\mu \varphi^B_r \to A^{rB}_\mu \ . \tag{5.18}$$

Therefore, if this was the symmetry breaking pattern that we were interested in, we could ensure that the generating functional is invariant under the transformations (5.14) by demanding that the effective action be symmetric under[4]

$$\tilde{D}_\mu \pi_a(t) \to \tilde{D}_\mu \pi_a(-t) - i\beta \partial_t \tilde{D}_\mu \pi_r(-t) + \mathcal{O}(E^2/T^2) \ , \tag{5.19a}$$

$$\tilde{D}_\mu \pi_r(t) \to \tilde{D}_\mu \pi_r(-t) - \frac{i\beta}{4} \partial_t \tilde{D}_\mu \pi_a(-t) + \mathcal{O}(E^2/T^2) \ , \tag{5.19b}$$

$$\tilde{D}_\mu \varphi_a(t) \to \tilde{D}_\mu \varphi_a(-t) - i\beta \partial_t \tilde{D}_\mu \varphi_r(-t) + \mathcal{O}(E^2/T^2) \ , \tag{5.19c}$$

$$\tilde{D}_\mu \varphi_r(t) \to \tilde{D}_\mu \varphi_r(-t) - \frac{i\beta}{4} \partial_t \tilde{D}_\mu \varphi_a(-t) + \mathcal{O}(E^2/T^2) \ . \tag{5.19d}$$

The second advantage is that the coset connection is now trivial, and therefore higher covariant derivatives of the Goldstone fields can be obtained by acting with regular

[4] An important comment on our notation: when $\mu = t$, equations (5.19) reduce to $\tilde{D}_t \pi_r(t) \to \tilde{D}_t \pi_r(-t) + ... = -\tilde{D}_{-t} \pi_r(-t) + ...$, and so on. The same goes for Eqs. (5.21).

partial derivatives on the Maurer-Cartan components; hence, the symmetry transformations (5.19) are covariant under all the non-linearly realized symmetries.

Based on the second line of (5.17), they can be expressed equivalently in terms of the physical covariant derivatives $\mathcal{D}_\mu\pi_r, \mathcal{D}_\mu\pi_a, \mathcal{D}_\mu\varphi_a, \nabla_\mu$ and the combination

$$D_\mu\varphi_r \equiv i\, e^{i\varphi_r\cdot T_r}\partial_\mu e^{-i\varphi_r\cdot T_r} + \mathcal{A}_\mu \tag{5.20}$$

as follows:[5]

$$D_\mu\pi_a(t) \to D_\mu\pi_a(-t) - i\beta\nabla_t D_\mu\pi_r(-t) - \beta\,[D_t\varphi_r(-t), D_\mu\pi_r(-t)] + \mathcal{O}(E^2/T^2)\ , \tag{5.21a}$$

$$D_\mu\pi_r(t) \to D_\mu\pi_r(-t) - \frac{i\beta}{4}\nabla_t D_\mu\pi_a(-t) - \frac{\beta}{4}\,[D_t\varphi_r(-t), D_\mu\pi_a(-t)] + \mathcal{O}(E^2/T^2)\ , \tag{5.21b}$$

$$D_\mu\varphi_a(t) \to D_\mu\varphi_a(-t) - i\beta\nabla_t D_\mu\varphi_r(-t) - \beta\,[D_t\varphi_r(-t), D_\mu\varphi_r(-t)] + \mathcal{O}(E^2/T^2)\ , \tag{5.21c}$$

$$D_\mu\varphi_r(t) \to D_\mu\varphi_r(-t) - \frac{i\beta}{4}\nabla_t D_\mu\varphi_a(-t) - \frac{\beta}{4}\,[D_t\varphi_r(-t), D_\mu\varphi_a(-t)] + \mathcal{O}(E^2/T^2)\ . \tag{5.21d}$$

where, on the right-hand side of these equations, $\nabla_t = \partial_t + i\mathcal{A}_t^B(-t)T_B$, and we have streamlined our notation by defining commutators between covariant derivatives, e.g.

$$[D_t\varphi_r(-t), D_\mu\pi_a(-t)]_k \equiv i f_{Ajk} D_t\varphi_r^A(-t) D_\mu\pi_a^j(-t)\ , \tag{5.22}$$

and similarly for the other commutators.[6] Once again, the transformations (5.21) are manifestly covariant under all the non-linearly realized symmetries.

At this point, we notice that the quantity $D_t\varphi_r$ has exactly the same transformation properties as our matter fields ρ_r (we are focusing on internal symmetries, and therefore boosts are irrelevant—i.e. explicitly broken—as far as we are concerned). Therefore, if Eqs. (5.21) involved only $D_t\varphi_r$ and its derivatives, we could simply replace $D_t\varphi_r(t) \to \rho_r(t)$ everywhere and obtain DKMS symmetry transformations involving the Goldstones π_r, π_a, φ_a and the matter fields ρ_r. Unfortunately, Eq. (5.21c) depends on all components of $D_\mu\varphi_r$, but this can be remedied by "commuting" the covariant derivatives of $\nabla_t D_\mu\varphi_r(-t)$: tedious but straightforward manipulations show that

$$\nabla_t D_\mu\varphi_r(-t) - i[D_t\varphi_r(-t), D_\mu\varphi_r(-t)] = \mathcal{F}_{t\mu}(-t) + \nabla_\mu D_t\varphi_r(-t)\ , \tag{5.23}$$

[5] To derive these expressions, we assumed that the structure constants are totally antisymmetric, which is ensured whenever the symmetry group is compact [101].

[6] Note that all four components X_r, X_a, T_a and T_r are in independent representations of T_r and therefore do not mix. This is due to the structure of commutation relations, e.g.the term $e^{-i\varphi_r\cdot T_r} D_\mu\pi_r \cdot X_r e^{i\varphi_r\cdot T_r}$ is a linear combination of X_r generators due to $[X_r, T_r] \sim X_r$. For this reason, (5.19) is related to (5.21) through conjugation with $e^{-i\varphi_r\cdot T_r}$

where $\mathcal{F}_{\mu\nu}$ is the usual field strength tensor associated with $\mathcal{A}_\mu$. Using this identity, and replacing $D_t\varphi_r(t) \to \rho_r(t)$ everywhere,[7] we finally obtain the desired form of the DKMS transformation rules for our Goldstone and matter fields (we only need to consider the $\mu = t$ component of Eq. (5.21d)):

$$D_\mu\pi_a(t) \to D_\mu\pi_a(-t) - i\beta\nabla_t D_\mu\pi_r(-t) + \beta\,[\rho_r(-t), D_\mu\pi_r(-t)] + \mathcal{O}(E^2/T^2)\ , \tag{5.24a}$$

$$D_\mu\pi_r(t) \to D_\mu\pi_r(-t) - \frac{i\beta}{4}\nabla_t D_\mu\pi_a(-t) + \frac{\beta}{4}\,[\rho_r(-t), D_\mu\pi_a(-t)] + \mathcal{O}(E^2/T^2)\ , \tag{5.24b}$$

$$D_\mu\varphi_a(t) \to D_\mu\varphi_a(-t) + i\beta\nabla_\mu\rho_r(-t) - i\beta\mathcal{F}_{t\mu}(-t) + \mathcal{O}(E^2/T^2)\ , \tag{5.24c}$$

$$\rho_r(t) \to -\rho_r(-t) - \frac{i\beta}{4}\nabla_t D_t\varphi_a(-t) + \frac{\beta}{4}\,[\rho_r(-t), D_t\varphi_a(-t)] + \mathcal{O}(E^2/T^2)\ . \tag{5.24d}$$

These transformation rules are the main result of this subsection. They act non-locally at the level of the fields, but to the best of our knowledge there is no fundamental obstruction to having non-local discrete symmetries. Furthermore, because the Goldstone fields enter the effective action only through their covariant derivatives, and these transform locally, it is not hard to impose the DKMS symmetry in practice. In this paper, we will be concerned with the implementation of DKMS relations up to first order in E/T; the systematics of higher order corrections are still an open question that we hope to explore in the near future. We will derive the lowest-order invariant combinations in Sec. (5.2.5).

5.2.4 Relation to other approaches in the literature

We should briefly comment on the relation between our approach and previous results in the literature on out-of-equilibrium effective actions. It was previously proposed that, in the hydrodynamic limit, the Schwinger-Keldysh effective action should contain one Goldstone field for each continuous symmetry, regardless of whether it is spontaneously broken or not (see e.g. [34, 69, 121]). The effective action then must be invariant under an additional local symmetry, which only depends on the spatial coordinates and acts on the fields associated with unbroken generators as follows:

$$e^{i\varphi_r(t,\vec{x})\cdot T_r} \quad \to \quad e^{i\varphi_r(t,\vec{x})\cdot T_r} h_r(\vec{x})\ , \qquad\qquad h_r(\vec{x}) \equiv e^{ic(\vec{x})\cdot T_r}\ . \tag{5.25}$$

In the simplest, abelian case, this symmetry reduces to a local shift, $\varphi_r(t,\vec{x}) \to \varphi_r(t,\vec{x}) + c(\vec{x})$. The symmetry (5.25) is equivalent to the transformation $\tilde{\Omega} \to \tilde{\Omega}\, h_r(\vec{x})$,

[7] This also means replacing $D_t\varphi_r(-t) = -D_{-t}\varphi_r(-t) \to -\rho_r(-t)$.

which in turn implies the following transformation rules for the coefficients of the Maurer-Cartan form:

$$\tilde{D}_\mu \pi_a \cdot X_a \to \tilde{D}_\mu \pi_a \cdot h_r^{-1}(\vec{x}) X_a h_r(\vec{x}) \ , \tag{5.26a}$$

$$\tilde{D}_\mu \pi_r \cdot X_r \to \tilde{D}_\mu \pi_r \cdot h_r^{-1}(\vec{x}) X_r h_r(\vec{x}) \ , \tag{5.26b}$$

$$\tilde{D}_\mu \varphi_a \cdot T_a \to \tilde{D}_\mu \varphi_a \cdot h_r^{-1}(\vec{x}) T_a h_r(\vec{x}) \ , \tag{5.26c}$$

$$\tilde{D}_\mu \varphi_r \cdot T_r \to \tilde{D}_\mu \varphi_r \cdot h_r^{-1}(\vec{x}) T_r h_r(\vec{x}) - i\delta^j_\mu h_r^{-1}(\vec{x}) \partial_j h_r(\vec{x}) \ . \tag{5.26d}$$

This symmetry plays two important roles: (1) it effectively forces us to contract all the indices in a way that is invariant under H_{diag}, even though H_{diag} is formally broken; and (2) it ensures that the fields φ_r appear in the effective action only through $\tilde{D}_t \varphi_r$ and its covariant derivatives—the spatial components $\tilde{D}_i \varphi_r$'s are not allowed building blocks. This, in turn, gives rise to a diffusive behavior for the unbroken currents, and for this reason we'll also refer to invariance under (5.26) as *diffusive symmetry*. This additional symmetry is to be regarded as emergent at low energies, and its physical origin is not particularly clear.[8] Furthermore, its implications have so far been explored mostly in the classical limit, i.e. by working only up to quadratic order in the $a-$type fields. This approach has the advantage that the DKMS symmetry becomes easier to implement [104]—which is why we started this section by considering a similar symmetry breaking pattern. However, in practice one is actually more interested in the properties of the charge density ρ_r rather than the field φ_r, and some authors even resort to an explicit change of variable from the latter to the former—see e.g. [118], which performs precise numerical tests of the EFT approach to diffusion.

This alternative approach yields exactly the same correlation functions for conserved currents as the one developed in this paper. However, given the different number of time derivatives at play, we don't expect these two approaches to be fully equivalent, and ultimately expect the number of propagating degrees of freedom to be different—at least based on our experience with more conventional EFTs. We plan to further investigate this question in the near future, but in the meantime we find the conceptual simplicity of our approach—which relies on the standard rules of the coset construction and doesn't require additional symmetries—particularly compelling.

The symmetry breaking patterns for thermal systems have also been discussed in the literature under the name strong-to-weak spontaneous symmetry breaking [130–133]. Their approach is as follows: For a mixed state ρ with symmetry G and a unitary representation U, ρ is said to be strongly symmetric if

$$U\rho = e^{i\theta}\rho \tag{5.27}$$

for any choice of U and a constant phase θ. The mixed state is instead weakly symmetric if the above isn't true, but

$$U\rho U^{-1} = \rho \tag{5.28}$$

[8] See however [30, 129] for a holographic interpretation of this symmetry.

holds. They then argue that the strongly symmetric system corresponds to the normal phase, whereas the weakly symmetric systems corresponds to the normal phase. This is analogous to what we propose, that is the full symmetry group has the SSB pattern $G_1 \times G_2 \to G_{diag}$ for the diffusive case, and $G_1 \times G_2 \to H_{diag}$ in the SSB phase.

In the context of open systems [131] argues that the symmetries are doubled, and the breaking of strong symmetry down to weak implies that charge is not conserved. This is very similar to the symmetry breaking pattern that we propose in Ch. 6 for weakly coupled open systems. There we distinguish an open system with a symmetry breaking pattern where the off-diagonal symmetry G_a is explicitly broken.

5.2.5 Lowest order DKMS-invariant combinations

Now that we understand how the DKMS symmetry acts on our Goldstone and matter fields up to first order in E/T, we can try to build combinations that are invariant under these symmetries at this order. To this end, it is once again convenient to work at first with our "fictitious" covariant derivatives $\tilde{D}_\mu \pi_r, \dots$ because the DKMS transformation rules (5.19) are simpler than the ones for the physical fields given in (5.24). We will eventually rewrite the combinations we derive purely in terms of the physical fields, but to do so we'll need to impose the additional diffusive symmetry introduced in the previous section.

It is easy to see that the following combination changes by a total derivative up to first order in E/T,

$$\tilde{D}_t\pi_a \cdot \tilde{D}_t\pi_r \quad \to \quad \tilde{D}_t\pi_a \cdot \tilde{D}_t\pi_r - \frac{i\beta}{2}\partial_t\left[\tilde{D}_t\pi_r \cdot \tilde{D}_t\pi_r - \frac{1}{4}\tilde{D}_t\pi_a \cdot \tilde{D}_t\pi_a\right] + \mathcal{O}(E^2/T^2), \tag{5.29}$$

and thus provides an invariant contribution to the effective action. The dot in this equation stands for the most general symmetric contraction that is invariant under the diffusive symmetry. An analogous statement can be made about $\tilde{D}_i\pi_a \cdot \tilde{D}^i\pi_r$ and $\tilde{D}_t\varphi_a \cdot \tilde{D}_t\varphi_r$. Using Eq. (5.17), it is easy to see that these combinations are respectively equal to $D_t\pi_a \cdot D_t\pi_r$, $D_i\pi_a \cdot D^i\pi_r$, and $D_t\varphi_a \cdot D_t\varphi_r \to D_t\varphi_a \cdot \rho_r$, which are therefore invariant under the DKMS symmetry up to a total derivative.

The last combination that would be natural to consider, $\tilde{D}_i\varphi_a \cdot \tilde{D}^i\varphi_r$, would also be invariant under DKMS but, alas, not under the diffusive symmetry. This suggests that we should act with at least one time derivative on $\tilde{D}^i\varphi_r$, but the contraction $\tilde{D}_i\varphi_a \cdot \partial_t\tilde{D}^i\varphi_r$, albeit now invariant under the diffusive symmetry, would now no longer be invariant under DKMS. However, this can be easily remedied by adding a term quadratic in $\tilde{D}_i\varphi_a$ to form the combination

$$\tilde{D}^j\varphi_a \cdot \left(\partial_t\tilde{D}_j\varphi_r - \frac{i}{\beta}\tilde{D}_j\varphi_a\right) \ , \tag{5.30}$$

which is exactly invariant under a DKMS transformation up to $\mathcal{O}(E^2/T^2)$. Performing manipulations analogous to those that took us from Eqs. (5.19) to Eqs. (5.24), we can rewrite this combination in terms of our "physical" fields as follows:

$$D^j\varphi_a \cdot \left(\nabla_j\rho_r + \mathcal{F}_{tj} - \frac{i}{\beta}D_j\varphi_a\right) \ . \tag{5.31}$$

The combinations we have identified above provide the leading kinetic terms for our fields. Of course, the DKMS symmetry can also be imposed at higher orders. Examples of such higher order terms that will play a role in our discussion of antiferromagnets (see Sec. 5.3.2) are

$$D^j\pi_a \cdot \left(-\nabla_t D_j\pi_r + i\left[\rho_r, D_j\pi_r\right] + \frac{i}{\beta}D_j\pi_a\right) \ , \tag{5.32}$$

or an equivalent expression with spatial derivatives replaced by time derivatives.

5.2.6 Power counting rules

As we discussed in Sec. 4.2, in order for our EFT to be well-defined we must be able to assign to each term in the effective action a definite scaling in terms of our expansion parameters. This, in turn, requires us to specify how covariant derivatives and matter fields scale with energy and momentum. The scaling of covariant derivatives is the conventional one—time derivatives scale like energy, spatial derivatives like momentum. The scaling of the Goldstone and matter fields instead, are determined by the form of the quadratic terms in the EFT.
In the next section we will discuss a few concrete examples, and show how these power counting rules can be used in each separate case to estimate the size of various operators in the effective action.

5.3 Examples: Paramagnets, Antiferromagnets, and Ferromagnets

Non-relativistic magnetic systems at finite temperature are endowed with an internal $SO(3)$ symmetry that corresponds to global rotations of all the spins, and thus provide a non-trivial testing ground for our formalism. In the case of paramagnets this $SO(3)$ symmetry remains unbroken, while in (anti-)ferromagnets it is spontaneously broken down to an $SO(2)$ subgroup. The feature that sets ferromagnets apart is that they have a non-zero density of unbroken $SO(2)$ charge—i.e., a non-zero magnetization density. In this section, we will discuss separately these three possibilities, restricting our attention to the internal $SO(3)$ symmetry and neglect the spacetime

symmetries that would also be broken by a finite temperature state.[9] These examples will illustrate how to power count terms in the effective action and how to calculate correlation functions of Noether currents.[10]

5.3.1 Paramagnets

Paramagnets are systems where the internal $SO(3)$ symmetry remains unbroken.[11] As a result, the fields π_r and π_a are absent, the only fields that enter the low energy EFT are the triplet of Goldstone fields $\vec{\varphi}_a$ corresponding to the breaking of $SO(3)_1 \times SO(3)_2$ down to the diagonal subgroup $SO(3)_{\text{diag}}$, and the three associated matter fields $\vec{\rho}_r$. Using the invariant building blocks we have identified in Sec. 5.2.5, we can write the following leading-order effective action for a paramagnet:

$$S_{\text{para}} = \int dt\, d^3x \left[\frac{k_*^3}{E_*}\, \rho_r \cdot D_t \varphi_a - k_*\, D^j \varphi_a \cdot \left(\nabla_j \rho_r + \mathcal{F}^r_{tj} - \frac{i}{\beta} D_j \varphi_a \right) \right] , \quad (5.33)$$

where the dot stands for a contraction of the internal indices with a 3-dimensional Kronecker delta. Note that this action is not invariant under time reversal since this is never a symmetry of Schwinger-Keldysh effective actions, as we have discussed already in Sec. 4.3. Physically, this makes it possible to reproduce a diffusive behavior, as we will see below.

We have chosen to parametrize the two free coefficients in the effective action (5.35) in terms of some microscopic momentum and energy scales, denoted respectively with k_* and E_*, so that our effective action will be organized in powers of k/k_*, E/E_*, and E/T. In fact, there are only two independent expansion parameters, because the first two ratios are related to each other by the free equations of motion for ρ_r, which can be obtained by varying the action with respect to φ_a:[12]

$$\left. \frac{\delta S_{\text{para}}}{\delta \varphi_a} \right|_{\varphi_a = A_a = A_r = 0} = -\frac{k_*^3}{E_*} \partial_t \rho_r + k_* \partial_j \partial^j \rho_r = 0 \ , \qquad \to \qquad \frac{E}{E_*} \sim \frac{k^2}{k_*^2} \ . \quad (5.34)$$

To find the scaling of the fields, we consider the quadratic part of the action with

[9] This amounts to neglecting phonon excitations by working in the incompressible limit and, in particular, treating boosts as if they were explicitly broken. An EFT treatment of the Goldstone modes arising at zero temperature from the simultaneous breaking of $SO(3)$ and spacetime symmetries was recently discussed in [134].

[10] Classic studies of the high-T Noether current correlator for these systems include for instance [127, 135].

[11] See [69] for a more general study of non-Abelian hydrodynamics in the absence of SSB and in the classical limit.

[12] In the simplest case where $E_* = T$, then there is just one independent expansion parameter.

sources turned off:

$$S^{(2)}_{\rm para} = \int dt\, d^3x \left[\frac{k_*^3}{E_*}\, \rho_r \cdot \partial_t \varphi_a - k_*\, \partial^j \varphi_a \cdot \left(\partial_j \rho_r - \frac{i}{\beta} \partial_j \varphi_a \right) \right] . \tag{5.35}$$

We see from the second term that $\rho_r \sim T\varphi_a$. Combining Eq. (5.34) with this relation, and the fact that $d^4x \sim E^{-1}k^{-3}$, we see that the effective action (5.35) schematically scales as

$$S_{para} \sim \frac{T}{E}\frac{k_*}{k} \ . \tag{5.36}$$

Notice also that our expression in (5.35) does not rely on a classical approximation: the covariant derivative $D_t\varphi_a$ is generically a non-linear combination of the fields φ_a, and as such contains terms of all orders in $\hbar$ (see discussion in Sec. 4.5).

Varying instead our action with respect to the gauge fields A^a_μ and A^r_μ yields respectively the conserved currents $\mathcal{J}^\mu_r$ and $\mathcal{J}^\mu_a$ expressed in terms of the external gauge fields, ρ_r, φ_a, and their derivatives. In particular, setting $A^a_\mu = \varphi_a = 0$, the current $\mathcal{J}^\mu_r$ reduces to the *classical* conserved current in the presence of external gauge fields A^r_μ:

$$\mathcal{J}^\mu_r|_{\varphi_a = A_a = 0} = \left(\frac{k_*^3}{E_*}\, \rho_r \ , -k_* \nabla^j \rho_r + k_* F^{tj}_r \right) \ . \tag{5.37}$$

From this, we see that the fields ρ_r are equal to the conserved charge densities up to an overall normalization. In the absence of external gauge fields, we recover the standard constitutive relation $\mathcal{J}^i_r = -\mathcal{D}\partial^i \mathcal{J}^0_r$ with a diffusion coefficient $\mathcal{D} \equiv E_*/k_*^2$. In fact, the equations of motion (5.34) are just a set of diffusion equations for the charge densities $\mathcal{J}^0_r$. When $\rho_r = 0$, instead, the second term in the classical current densities reproduces Ohm's law, $\mathcal{J}^i_r = \sigma E^i_r$, with E^i_r the electric component of the field strength and $\sigma \equiv k_*$ the conductivity.

More in general, we can calculate correlation functions of the currents $\mathcal{J}^\mu_r$ and $\mathcal{J}^\mu_a$ by taking functional derivatives with respect to A^a_μ and A^r_μ of the generating functional

$$Z[A^a_\mu, A^r_\mu] = \int \mathcal{D}\rho_r \mathcal{D}\varphi_a e^{iS_{\rm para}} \ . \tag{5.38}$$

A single derivative with respect to A^a_μ yields the expectation value of (5.37), which coincides with the expectation value of the physical Noether current, $\langle \mathcal{J}^\mu \rangle$. The requirement that paramagnets preserve the $SO(3)$ symmetry in the absence of external fields implies that $\langle \rho_r \rangle = 0$.

Similarly, by taking two functional derivatives we can calculate two point functions:

$$\langle \mathcal{J}^\mu_{rA}(p) \mathcal{J}^\nu_{rB}(k) \rangle = -\frac{\delta^2 Z}{\delta A^A_{\mu a}(-p) \delta A^B_{\nu a}(-k)} \ , \qquad \langle \mathcal{J}^\mu_{rA}(p) \mathcal{J}^\nu_{aB}(k) \rangle = -\frac{\delta^2 Z}{\delta A^A_{\mu a}(-p) \delta A^B_{\nu r}(-k)} \ . \tag{5.39}$$

These current correlators can be inferred from the correlators of ρ_r and φ_a, and correspond to different Green's functions of the physical Noether current $\mathcal{J}^\mu$, along the lines of what we discussed in Sec. 4.5). For example,

$$\langle \mathcal{J}_r^\mu(x)\mathcal{J}_r^\nu(x')\rangle = \tfrac{1}{2}\langle\{\mathcal{J}^\mu(x),\mathcal{J}^\nu(x')\}\rangle \ , \qquad \langle \mathcal{J}_r^\mu(x)\mathcal{J}_a^\nu(x')\rangle = \theta(t-t')\langle[\mathcal{J}^\mu(x),\mathcal{J}^\nu(x')]\rangle \ . \tag{5.40}$$

For simplicity, we will calculate these correlators with vanishing external sources, i.e. setting $A_\mu^r = A_\mu^a = 0$ after taking the appropriate functional derivatives.

At leading order in k/k_* the current correlators can be calculated by approximating the currents $\mathcal{J}_r^\mu$ and $\mathcal{J}_a^\mu$ up to linear order in the fields ρ_r and φ_a; thus, we only need to know the 2-point functions of these fields, which can be obtained simply by inverting the quadratic term in (5.35)—this is in fact one of the main advantages of working with the effective action:

$$\langle \rho_r^A(\omega,\vec{k})\rho_r^B(-\omega,-\vec{k})\rangle' = \frac{2}{\beta}\frac{\mathcal{D}}{\sigma}\frac{\mathcal{D}k^2\delta^{AB}}{\omega^2+\mathcal{D}^2k^4} \tag{5.41a}$$

$$\langle \rho_r^A(\omega,\vec{k})\varphi_a^B(-\omega,-\vec{k})\rangle' = \frac{\mathcal{D}}{\sigma}\frac{\delta^{AB}}{\omega+i\mathcal{D}k^2} \ , \tag{5.41b}$$

$$\langle \varphi_a^A(\omega,\vec{k})\rho_r^B(-\omega,-\vec{k})\rangle' = \frac{\mathcal{D}}{\sigma}\frac{\delta^{AB}}{-\omega+i\mathcal{D}k^2} \ , \tag{5.41c}$$

where the primes on the left-hand side denote the fact that we have dropped the delta functions imposing energy and momentum conservation. Combining these 2-point functions we can easily calculate all the components of the r-r correlator,

$$\langle \mathcal{J}_r^{tA}(\omega,\vec{k})\mathcal{J}_r^{tB}(-\omega,-\vec{k})\rangle' = \frac{2\sigma}{\beta}\frac{k^2\delta^{AB}}{\omega^2+\mathcal{D}^2k^4} \ , \tag{5.42a}$$

$$\langle \mathcal{J}_r^{tA}(\omega,\vec{k})\mathcal{J}_r^{jB}(-\omega,-\vec{k})\rangle' = \frac{2\sigma}{\beta}\frac{\omega k^j\delta^{AB}}{\omega^2+\mathcal{D}^2k^4} \ , \tag{5.42b}$$

$$\langle \mathcal{J}_r^{iA}(\omega,\vec{k})\mathcal{J}_r^{jB}(-\omega,-\vec{k})\rangle' = \frac{2\sigma}{\beta}\left(\delta^{ij}-\frac{\mathcal{D}^2k^2k^ik^j}{\omega^2+\mathcal{D}^2k^4}\right)\delta^{AB} \ , \tag{5.42c}$$

which are precisely of the form needed to ensure current conservation, $k^\mu\langle J_{r\mu}^A J_{r\nu}^B\rangle' = 0$.

The retarded r-a correlator is computed in a similar way, and we have verified that it is also conserved. For completeness, we list here its components:

$$\langle \mathcal{J}_r^{tA}(\omega,\vec{k})\mathcal{J}_a^{tB}(-\omega,-\vec{k})\rangle' = \frac{\sigma k^2\delta^{AB}}{\omega+i\mathcal{D}k^2} \ , \tag{5.43a}$$

$$\langle \mathcal{J}_r^{tA}(\omega,\vec{k})\mathcal{J}_a^{jB}(-\omega,-\vec{k})\rangle' = \langle \mathcal{J}_r^{jA}(\omega,\vec{k})\mathcal{J}_a^{tB}(-\omega,-\vec{k})\rangle' = \frac{\sigma\omega k^j\delta^{AB}}{\omega+i\mathcal{D}k^2} \ , \tag{5.43b}$$

$$\langle \mathcal{J}_r^{iA}(\omega,\vec{k})\mathcal{J}_a^{jB}(-\omega,-\vec{k})\rangle' = \sigma\omega\left(\delta^{ij}-\frac{i\mathcal{D}k^ik^j}{\omega+i\mathcal{D}k^2}\right)\delta^{AB} \ , \tag{5.43c}$$

displaying the usual diffusive pole. All of the above correlators agree with previous well-known results in the literature, see e.g. [127, 135]. Note that the correlators above receive contributions from contact terms—i.e. terms in the effective action (5.35) that are quadratic in the external gauge fields—and these are crucial to ensure current conservation.

5.3.2 Antiferromagnets

Antiferromagnets are systems where the internal $SO(3)$ symmetry is spontaneously broken down to $SO(2)$ by a non-trivial staggered-magnetization order parameter which, without loss of generality, we take to be along the 3-direction. Consequently, the relevant degrees of freedom at low energies are the two doublets of Goldstones π_r and π_a, a single Goldstone φ_a and its associated matter field ρ_r.

The antiferromagnetic ground state differs from the ferromagnetic one in that the magnetization density vanishes. This distinction is often captured by the statement that ferromagnets break time reversal whereas antiferromagnets do not—see e.g. [60, 77]. This is actually only part of the story, because otherwise the Schwinger-Keldysh effective actions for these two systems would be identical given that time reversal is always broken. The staggered magnetization of the antiferromagnetic ground state picks a preferred direction in spin space but not an orientation. Therefore, it breaks rotations by generic angles around the 1- and 2-directions, but is still invariant under a residual discrete subgroup that consist of 180°-rotations around these same axes. These discrete symmetries act on our fields as

$$\rho_r \to -\rho_r, \qquad \varphi_a \to -\varphi_a, \qquad \pi^1_{a,r} \to -\pi^1_{a,r}, \qquad \pi^2_{a,r} \to \pi^2_{a,r}, \tag{5.44}$$

in the case of rotations around the 2-axis; for rotations around the 1-axis, it would be the fields $\pi^1_{a,r}$ that are left invariant.[13] The Schwinger-Keldysh effective action for antiferromagnets is invariant under these discrete transformations, whereas the one for ferromagnets is not. This statement applies equally to zero-temperature effective actions. In that context, antiferromagnets are also separately invariant under time reversal, whereas ferromagnets are only invariant under a combination of discrete rotations and time reversal. This state of affairs is summarized in Table 5.1.

Our Goldstone covariant derivatives transform in a simple way under the discrete symmetry (5.44):

$$D_\mu\varphi_a \to -D_\mu\varphi_a, \qquad D_\mu\pi^1_{a,r} \to -D_\mu\pi^1_{a,r}, \qquad D_\mu\pi^2_{a,r} \to D_\mu\pi^2_{a,r}\ . \tag{5.45}$$

[13] Rotations around the 2-direction flip the sign of the generators of rotations around the 1- and 3-directions. In our notation, this means that $X^1_{a,r} \to -X^1_{a,r}$ and $T_{a,r} \to -T_{a,r}$. The transformation properties of the Goldstone modes are determined as usual by demanding that coset parametrization Ω remains invariant when the generators are transformed this way. The transformation rule for ρ_r should be the same as that of its partner field, φ_a. This is also consistent with that fact that, as we will see, ρ_r is related to the density of spin along the 3-direction.

	AF $T=0$	**AF** $T\neq 0$	**F** $T=0$	**F** $T\neq 0$
T	✓	✗	✗	✗
D	✓	✓	✗	✗
$T+D$	✓	✗	✓	✗

Table 5.1: Discrete symmetries ferromagnets (F) and antiferromagnets (AF) in ordinary ($T=0$) and Schwinger-Keldysh ($T\neq 0$) effective actions. We have denoted with D the residual discrete transformations of the form (5.44). The symbol ✓(✗) indicates unbroken (broken) symmetries.

Keeping also in mind the requirement of DKMS invariance, the low energy effective action for antiferromagnets up to first subleading order in E/T turns out to be:

$$\begin{aligned} S_{\text{anti}} = \int dt\, d^3x \Bigg[& \frac{\Lambda^2}{c_s^3}\left(D_t\pi_a\cdot D_t\pi_r - c_s^2 D_i\pi_a\cdot D^i\pi_r\right) \\ & + \frac{\Lambda}{c_s^3}\,\sigma^{\mu\nu} D_\mu\pi_a\cdot\left(-\nabla_t D_\nu\pi_r + i\left[\rho_r, D_\nu\pi_r\right] + \frac{i}{\beta}D_\nu\pi_a\right) \\ & + \frac{\sigma}{\mathcal{D}}\,\rho_r D_t\varphi_a - \sigma D^i\varphi_a\left(\partial_i\rho_r + \mathcal{F}^r_{ti} - \frac{i}{\beta}D_i\varphi_a\right)\Bigg], \end{aligned} \tag{5.46}$$

where the dot now stands for a contraction of the internal indices with a 2-dimensional Kronecker delta, $\sigma^{\mu\nu} = \text{diag}(\Sigma_\pi, c_s^2\sigma_\pi, c_s^2\sigma_\pi, c_s^2\sigma_\pi)$ with Σ_π, σ_π both non-negative and dimensionless, and Λ is the energy scale at which spontaneous symmetry breaking occurs. In light of what we learned in the context of paramagnets, we have already parametrized the diffusive sector in terms of the conductivity and the diffusion coefficient. Our action doesn't contain a tadpole for the external field A_0^a, which is consistent with the fact that the expectation value of the $SO(3)$ Noether current must vanish for antiferromagnets. A discussion of the diffusive sector would be very similar to the analysis we carried out for paramagnets, and for this reason we'll mostly focus our attention to the π-sector in what follows.

The power counting rules follow from the quadratic part of the action:

$$\begin{aligned} S^{(2)}_{\text{anti}} = \int dt\, d^3x \Bigg[& \frac{\Lambda^2}{c_s^3}\left(\partial_t\pi_a\cdot\partial_t\pi_r - c_s^2\partial_i\pi_a\cdot\partial^i\pi_r\right) \\ & + \frac{\Lambda}{c_s^3}\,\sigma^{\mu\nu}\partial_\mu\pi_a\cdot\left(-\partial_t\partial_\nu\pi_r + \frac{i}{\beta}\partial_\nu\pi_a\right) \qquad (5.47) \\ & + \frac{\sigma}{\mathcal{D}}\,\rho_r\partial_t\varphi_a - \sigma\partial^i\varphi_a\left(\partial_i\rho_r - \frac{i}{\beta}\partial_i\varphi_a\right)\Bigg] \qquad (5.48) \end{aligned}$$

which leads to $E\sim c_s k$, $\partial_t\pi_r\sim T\pi_a$ for the π-sector and $E\sim k^2$, $\rho_r\sim T\varphi_a$ for the

diffusive sector. Based on these power counting rules, the second line of the action (5.47) is suppressed by one power of E/Λ compared to the first line.[14] More precisely, the first two lines of our action contain terms that schematically scale as follows:

$$S_{\text{anti},\pi} \sim \frac{T}{E}\left(\frac{\Lambda^2}{E^2}+\frac{\Lambda}{E}\right), \tag{5.49}$$

whereas the scaling of the terms in the leading Lagrangian for paramagnets (or the diffusive sector of antiferromagnets) is:

$$S_{\text{para}} \sim \frac{T}{E}\frac{k_*}{k} \ . \tag{5.50}$$

Once again, varying the generating functional with respect to the external gauge fields we can calculate correlators of the currents $\mathcal{J}_r^\mu$ and $\mathcal{J}_a^\mu$. For instance, the 2-point function of $\mathcal{J}_r^t$ along the broken directions are

$$\langle \mathcal{J}_r^{tI}(\omega,\vec{k})\mathcal{J}_r^{tJ}(-\omega,-\vec{k})\rangle' = \frac{2\Lambda}{\beta c_s}\frac{k^2(\sigma_\pi\omega^2+\Sigma_\pi c_s^2k^2)}{(\omega^2-c_s^2k^2)^2}\,\delta^{IJ} \tag{5.51}$$

where $I,J=1,2$ are here the $SO(2)$ subgroup indices, not to be confused with spatial indices. Note that, in deriving Eq. (5.51) we have neglected corrections of $\mathcal{O}(E^2/\Lambda^2)$ that cannot be trusted at the order we are working.

The retarded density-density correlator along the broken directions takes instead the form

$$\langle \mathcal{J}_r^{tI}(\omega,\vec{k})\mathcal{J}_a^{tJ}(-\omega,-\vec{k})\rangle' = \frac{\Lambda^2}{c_s}\frac{k^2\left[1+i(\Sigma_\pi-\sigma_\pi)\omega/\Lambda\right]}{\omega^2-c_s^2k^2+i\left(\Sigma_\pi\omega^2+\sigma_\pi c_s^2k^2\right)\omega/\Lambda}\,\delta^{IJ} \ . \tag{5.52}$$

The poles of this propagator are at

$$\omega \approx \pm c_s k - \frac{ic_s^2k^2}{2\Lambda}(\Sigma_\pi+\sigma_\pi) \ , \tag{5.53}$$

showing that sound modes decay at a rate $\Gamma \sim k^2$. The correlators above are in agreement with classic results in the literature up to the order in E/Λ we are considering [127, 135, 136].

Our treatment of the antiferromagnet is notably different from the standard analyses, which start from the derivative expansion of the equations of motion for the total magnetization $\vec{M}$ and the staggered-magnetization $\vec{N}$ (see e.g. [127]). The effective theory (5.47) was constructed to compute correlators of the conserved currents associated with $\vec{M}$ alone, and the presence of a non-trivial $\vec{N}$ expectation value is encoded in the assumption that $SO(3)$ is spontaneously broken down to $SO(2)$ even though the magnetization density vanishes. This last property is what distinguishes antiferromagnets from ferromagnets, which we will now turn our attention to.

[14] We are relying on the free equation for π_r that follows from the first line of (5.47) to set $E \sim c_s k$.

5.3.3 Ferromagnets

Ferromagnets spontaneously break $SO(3) \to SO(2)$ because the temporal component of the Noether current acquires a non-zero expectation value, which we again assume to be along the 3-direction. As we discussed at the beginning of the previous section, what distinguishes ferromagnets from antiferromagnets is the lack of separate invariance under time reversal and discrete transformations of the form (5.44). Only a combination of these symmetries leaves the ground state invariant, and therefore our DKMS transformations must be amended by acting also with (5.44) on the right-hand side, thus obtaining

$$D_\mu \pi_a^1(t) \to -D_\mu \pi_a^1(-t) + i\beta \nabla_t D_\mu \pi_r^1(-t) + \beta \left[\rho_r(-t), D_\mu \pi_r^1(-t)\right] + \mathcal{O}(E^2/T^2) \ , \tag{5.54a}$$

$$D_\mu \pi_a^2(t) \to D_\mu \pi_a^2(-t) - i\beta \nabla_t D_\mu \pi_r^2(-t) - \beta \left[\rho_r(-t), D_\mu \pi_r^2(-t)\right] + \mathcal{O}(E^2/T^2) \ , \tag{5.54b}$$

$$D_\mu \pi_r^1(t) \to -D_\mu \pi_r^1(-t) + \frac{i\beta}{4} \nabla_t D_\mu \pi_a^1(-t) + \frac{\beta}{4} \left[\rho_r(-t), D_\mu \pi_a^1(-t)\right] + \mathcal{O}(E^2/T^2) \ , \tag{5.54c}$$

$$D_\mu \pi_r^2(t) \to D_\mu \pi_r^2(-t) - \frac{i\beta}{4} \nabla_t D_\mu \pi_a^2(-t) - \frac{\beta}{4} \left[\rho_r(-t), D_\mu \pi_a^2(-t)\right] + \mathcal{O}(E^2/T^2) \ , \tag{5.54d}$$

$$D_\mu \varphi_a(t) \to -D_\mu \varphi_a(-t) - i\beta \nabla_\mu \rho_r(-t) - i\beta \mathcal{F}_{t\mu}(-t) + \mathcal{O}(E^2/T^2) \ , \tag{5.54e}$$

$$\rho_r(t) \to \rho_r(-t) + \frac{i\beta}{4} \nabla_t D_t \varphi_a(-t) + \frac{\beta}{4} \left[\rho_r(-t), D_t \varphi_a(-t)\right] + \mathcal{O}(E^2/T^2) \ . \tag{5.54f}$$

As a result, the part of the action that describes diffusion of the unbroken $SO(2)$ current now admits one additional invariant, i.e. $D_t \varphi_a$:

$$S_{\text{diff}} = \int dt\, d^3x \left[k_\star^3\, D_t \varphi_a + \frac{\sigma}{\mathcal{D}}\, \rho_r D_t \varphi_a - \sigma D^i \varphi_a \left(\partial_i \rho_r + \mathcal{F}_{ti}^r - \frac{i}{\beta} D_i \varphi_a \right) \right]. \tag{5.55}$$

This additional term gives rise to a tadpole term for A_0^a, so that the expectation value of the $SO(3)$ Noether current is now no longer zero:

$$\langle \mathcal{J}_A^0(x) \rangle = k_\star^3\, \delta_A^3 \ . \tag{5.56}$$

This new term also gives rise to a kinetic term for the π's with a single time derivative, since up to quadratic order in the Goldstones we have $D_t \varphi_a = \partial_t \varphi_a +$

$\epsilon_{IJ}\pi_a^I\partial_t\pi_r^J + \ldots$. This changes the dispersion relation of the Goldstone modes, and therefore the power counting scheme. As a result, the effective action for ferromagnets is now $S_{\rm ferro} = S_{\rm diff} + S_\pi$ with[15]

$$S_\pi = \int dt\, d^3x \Bigg[- E_\star k_\star D_i\pi_a \cdot D^i\pi_r + k_\star\, \sigma_\pi D^j\pi_a \cdot \left(-\nabla_t D_j\pi_r + i\,[\rho_r, D_j\pi_r] + \frac{i}{\beta} D_j\pi_a \right) + \mathcal{O}(k_\star/k) \Bigg]. \tag{5.57}$$

This part of the action introduces a characteristic energy scale, $E_\star$, and a dimensionless coupling σ_π. To determine the scaling of the action we again look at the quadratic part involving π's:

$$S_\pi^{(2)} = \int dt d^3x \left[k_\star^3 \epsilon_{IJ}\pi_a^I\partial_t\pi_r^J - E_\star k_\star \partial_i\pi_a \cdot \partial^i\pi_r + k_\star\, \sigma_\pi \partial^j\pi_a \cdot \left(-\partial_t\partial_j\pi_r + \frac{i}{\beta}\partial_j\pi_a \right) \right] \tag{5.58}$$

The leading equations of motion for π_r now imply a quadratic dispersion relation of the form

$$\frac{E}{E_\star} \sim \frac{k^2}{k_\star^2}\,, \tag{5.59}$$

and therefore the size of the terms shown in (5.58) can be estimated to be

$$S_\pi^{(2)} \sim \frac{T}{E}\left(\frac{k_\star^3}{k^3} + \frac{k_\star^3}{k^3} + \frac{k_\star}{k} \right)\,. \tag{5.60}$$

This scaling should be contrasted with the one for the Goldstone sector of antiferromagnets given in Eq (5.49). In both cases the second term is suppressed by a factor of E/scale, however the fact that the two cases correspond to different classes of Nambu-Goldstone modes, namely Type I and Type II, leads to a general difference in how they scale.

Varying the action with respect to the external gauge fields we can first derive the Noether currents in terms of our fields, and then calculate their correlators. For brevity, we are going to report here only the correlation functions of the time com-

[15] A term of the form $\epsilon_{IJ}D_i\pi_a^I D^i\pi_r^J$ is not allowed even in ferromagnets because it is not DKMS-invariant.

ponents of our currents:

$$\langle \mathcal{J}_r^{t3}(\omega,\vec{k})\mathcal{J}_a^{t3}(-\omega,-\vec{k})\rangle' = \frac{\sigma k^2}{\omega + i\mathcal{D}k^2} \ , \tag{5.61a}$$

$$\langle \mathcal{J}_r^{t3}(\omega,\vec{k})\mathcal{J}_r^{t3}(-\omega,-\vec{k})\rangle = k_\star^6 + \frac{2\sigma}{\beta}\frac{k^2}{\omega^2+\mathcal{D}^2k^4} \ , \tag{5.61b}$$

$$\langle \mathcal{J}_r^{tI}(\omega,\vec{k})\mathcal{J}_a^{tJ}(-\omega,-\vec{k})\rangle' = \frac{ik^2k_\star^5(E_\star - i\sigma_\pi\omega)\delta^{IJ} - k_\star^7\omega\epsilon^{IJ}}{\omega^2k_\star^4 - E_\star^2k^4 + 2i\sigma_\pi E_\star\omega k^4} \ , \tag{5.61c}$$

$$\langle \mathcal{J}_r^{tI}(\omega,\vec{k})\mathcal{J}_r^{tJ}(-\omega,-\vec{k})\rangle' = \frac{2\sigma_\pi}{\beta}\frac{k^2k_\star^5\left[\delta^{IJ}\left(E_\star^2k^4+\omega^2k_\star^4\right)+2iE_\star\omega k_\star^2k^2\epsilon^{IJ}\right]}{\left(\omega^2k_\star^4 - E_\star^2k^4\right)^2} \ . \tag{5.61d}$$

In the above expressions, we have only kept the terms in the numerators and denominators that can be trusted given the order in $k/k_\star$ we are working at. Note also that the Eqs. (5.61a) and (5.61b) are consistent with the results we found in the paramagnet section once we take into account that our Noether current now has an expectation value (5.56). The poles in the retarded correlator (5.61c) yield once again the dispersion relations for magnon excitations, which now display a quadratic dispersion relation with a decay rate $\Gamma \sim k^4$:

$$\omega \simeq E_\star\left(\pm\frac{k^2}{k_\star^2} - i\sigma_\pi\frac{k^4}{k_\star^4}\right) \ . \tag{5.62}$$

Chapter 6

Effective Descriptions of Ajar Systems with a $U(1)$ Symmetry

6.1 Approximate Symmetry Breaking and Spurions

In Chapter 2 we considered systems displaying spontaneous symmetry breaking (SSB) and the method of coset construction to write down effective field theories (EFT) for them. With SSB, it is only the ground state that breaks the symmetry, but the dynamics itself (i.e. the action) still exhibits the symmetry, albeit realized nonlinearly in a complicated fashion. However, there are also cases in nature where a certain symmetry is explicitly broken even at the level of the action. If the breaking is small enough, we can still employ the symmetry breaking pattern to construct EFTs, which will now include an additional parameter that is proportional to the explicit symmetry breaking scale. One way to do this systematically is to extend the coset construction method by adding spurions [60, 101], fictitious fields added in a way that respects the full symmetry of the action, including the broken ones, but are then sent to a specifically chosen vacuum expectation value (vev) so that they break the symmetry explicitly. We will demonstrate this method by way of an example that comes up in Quantum Chromodynamics (QCD).

6.1.1 Chiral Perturbation Theory and the Pion Mass

The canonical example for the spurion method is the mass term in the chiral perturbation theory [36, 48, 137]. We first start with the action for QCD with the two lightest quarks u and d:

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}G_{\mu\nu}G^{\mu\nu} - \bar{u}\not{D}u - \bar{d}\not{D}d - m_u\bar{u}u - m_d\bar{d}d \\ &= -\frac{1}{4}G_{\mu\nu}G^{\mu\nu} - u_L^\dagger \not{D}u_L - u_R^\dagger \not{D}u_R - d_L^\dagger \not{D}d_L - d_R^\dagger \not{D}d_R \\ &\quad - m_u u_R^\dagger u_L - m_u u_L^\dagger u_R - m_d d_R^\dagger d_L - m_d d_L^\dagger d_R, \end{aligned} \tag{6.1}$$

where we suppressed the color and spinor indices. We can package the u and d into a flavor doublet:

$$q = \begin{pmatrix} u \\ d \end{pmatrix}, \qquad \bar{q} = \begin{pmatrix} \bar{u} & \bar{d} \end{pmatrix}. \tag{6.2}$$

Rewriting the action in terms of q we find

$$\mathcal{L} = -\frac{1}{4} G_{\mu\nu} G^{\mu\nu} - \bar{q} \not{D} q - m \bar{q} q. \tag{6.3}$$

This action has a global $SU(2)$ flavor (isospin) symmetry that acts on the flavor doublet q:

$$q \to e^{i \vec{\alpha}_I \cdot \vec{T}} q \tag{6.4}$$

where $\vec{T}$ are generators of $\mathfrak{su}(2)$ and $\vec{\alpha}_I$ the group parameter.

Since the masses of u and d quarks are much smaller than the symmetry breaking scale of QCD, $m_u, m_d \ll \Lambda_{QCD}$; we can take the quarks to be massless to first approximation. This leads to the following QCD action:

$$\begin{aligned} \mathcal{L} &= -\frac{1}{4} G_{\mu\nu} G^{\mu\nu} - \bar{q} \not{D} q \\ &= -\frac{1}{4} G_{\mu\nu} G^{\mu\nu} - \bar{u}_L \not{D} u_L - \bar{u}_R \not{D} \bar{u}_R - \bar{u}_L \not{D} u_L - \bar{u}_R \not{D} \bar{u}_R. \end{aligned} \tag{6.5}$$

This Lagrangian again has the above isospin symmetry, but we can now package left- and right-handed spinors into two doublets

$$q_L = \gamma_L \, q = \begin{pmatrix} u_L \\ d_L \end{pmatrix}, \qquad q_R = \gamma_R \, q = \begin{pmatrix} u_R \\ d_R \end{pmatrix} \tag{6.6}$$

where $\gamma_L = \frac{1}{2}(1 - \gamma_5)$ and $\gamma_R = \frac{1}{2}(1 + \gamma_5)$. The Lagrangian is now invariant under the symmetry $SU_L(2) \times SU_R(2)$:

$$\begin{gathered} q_L \to A q_L, \qquad q_R \to B q_R, \qquad \text{or,} \\ q = \begin{pmatrix} u_L \\ u_R \\ d_L \\ d_R \end{pmatrix} \to [A \gamma_L + B \gamma_R] \begin{pmatrix} u_L \\ u_R \\ d_L \\ d_R \end{pmatrix}. \end{gathered} \tag{6.7}$$

Here $A \in SU_L(2)$ and $B \in SU_R(2)$ are unitary matrices. Thus, there is a chiral symmetry, under which the left- and right-handed fermions transform independently under the mixing of the flavors u and d. The origin of this symmetry stems from the fact that in (6.1) mass terms mix left- and right-handed spinors, and so the massless limit decouples them.

The vacuum of QCD spontaneously breaks this symmetry down to the vector (diagonal) group (also called iso-spin) $SU_L(2) \times SU_R(2) \to SU_I(2)$, which is the case where $U = V$. We can thus use coset methods to construct the low-energy EFT of this theory. To parametrize the $G \to H$ symmetry breaking pattern we start with a general element $g \in G$:

$$g = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \begin{pmatrix} e^{i\vec{\alpha}_L \cdot \vec{V}_L} & 0 \\ 0 & e^{i\vec{\alpha}_R \cdot \vec{V}_R} \end{pmatrix} \in G = SU_L(2) \times SU_R(2), \tag{6.8}$$

where $\vec{V}$ are generators of $SU(2)$ for which we choose the Pauli basis: $\vec{V} = \vec{\sigma}/2$. The subscripts on the generators are redundant so we will omit them. The unbroken subgroup is the given by $\vec{\alpha}_L = \vec{\alpha}_R = \vec{\alpha}_I$:

$$h = \begin{pmatrix} e^{i\vec{\alpha}_I \cdot \vec{T}} & 0 \\ 0 & e^{i\vec{\alpha}_I \cdot \vec{T}} \end{pmatrix} \in H = SU_I(2). \tag{6.9}$$

The spontaneously broken off-diagonal components correspond to $\alpha_L = -\alpha_R$, which means we can parametrize the coset element as:

$$\Omega = \begin{pmatrix} e^{i\vec{\pi} \cdot \vec{X}} & 0 \\ 0 & e^{-i\vec{\pi} \cdot \vec{X}} \end{pmatrix}, \tag{6.10}$$

where the $\vec{\pi}$'s are the Goldstone fields. Their transformation under G is given by

$$g(\vec{\alpha}_L, \vec{\alpha}_R)\Omega(\pi) = \Omega(\pi')h(\pi, g(\vec{\alpha}_L, \vec{\alpha}_R)). \tag{6.11}$$

We can find a closed form expressions for $\vec{\pi}'$ for infinitesimal $\vec{\alpha}$:

$$\vec{\pi} \to \vec{\pi} + \vec{\pi} \times \vec{\alpha}_I + \frac{\pi}{2}\left(\tan\frac{\pi}{2} + \cot\frac{\pi}{2}\right)\left[\vec{\alpha}_A - \hat{\pi}(\hat{\pi} \cdot \vec{\alpha}_A)\right] + \hat{\pi}\left(\hat{\pi} \cdot \vec{\alpha}_A\right). \tag{6.12}$$

Here we have identified the combinations $\frac{1}{2}(\vec{\alpha}_L + \vec{\alpha}_R) = \vec{\alpha}_I$ and $\vec{\alpha}_L - \vec{\alpha}_R = \vec{\alpha}_A$. Expanding in powers of $\vec{\pi}$ leads to:

$$\delta\vec{\pi} = \vec{\alpha}_A + \vec{\pi} \times \vec{\alpha}_I + \mathcal{O}(\pi^2), \tag{6.13}$$

where $\pi = |\vec{\pi}|$. To lowest order in $\vec{\pi}$'s, we see the non-linear realization of the broken off-diagonal symmetry through the shift in α_A, and the linear realization of H, under which $\vec{\pi}$ transforms as a triplet.

6.1.2 Coset Construction

With the group element Ω, we construct the Maurer-Cartan Form:

$$\omega_\mu = \Omega^{-1}\partial_\mu\Omega \equiv i\mathcal{D}_\mu\vec{\pi} \cdot X + i\mathcal{A}_\mu \cdot T \tag{6.14}$$

where we have renamed the broken generators $T_L - T_R = T_A = X$ and unbroken generators $T_L + T_R = T_I = V$.

The building blocks have the following expansions:

$$\mathcal{D}_\mu \vec{\pi} = \partial_\mu \vec{\pi} \left(1 - \frac{1}{6}\pi^2\right) + \frac{1}{6}\left(\vec{\pi} \cdot \partial_\mu \vec{\pi}\right) \vec{\pi} + \mathcal{O}(\pi^5); \tag{6.15}$$

$$\mathcal{A}_\mu = -\frac{1}{2}\vec{\pi} \times \partial_\mu \vec{\pi} + \mathcal{O}(\pi^5). \tag{6.16}$$

Lowest order term in derivatives we can write in the Lagrangian is

$$\begin{aligned}
\mathcal{L}_{EFT} &= -\frac{f^2}{2}\mathcal{D}_\mu \vec{\pi} \cdot \mathcal{D}^\mu \vec{\pi} + \mathcal{O}(\partial^4) \\
&= -\frac{f^2}{2}\partial_\mu \vec{\pi} \cdot \partial^\mu \vec{\pi} + \frac{f^2}{6}\vec{\pi}^2 \partial_\mu \vec{\pi} \cdot \partial^\mu \vec{\pi} - \frac{f^2}{6}(\vec{\pi} \cdot \partial_\mu \vec{\pi})^2 + \mathcal{O}(\pi^6, \partial^4) \\
&= -\frac{f^2}{2}\partial_\mu \vec{\pi} \cdot \partial^\mu \vec{\pi} - \frac{f^2}{2}(\vec{\pi} \cdot \partial_\mu \vec{\pi})^2 + \mathcal{O}(\pi^6, \partial^4)
\end{aligned} \tag{6.17}$$

where we integrated by parts going from the second to the third line. Canonically normalizing the Goldstone fields $\vec{\pi}_c \equiv \vec{\pi}/f$ we finally get

$$\mathcal{L}_{EFT} = -\frac{1}{2}\partial_\mu \vec{\pi}_c \cdot \partial^\mu \vec{\pi}_c - \frac{1}{2f^2}(\vec{\pi}_c \cdot \partial_\mu \vec{\pi}_c)^2 + \mathcal{O}(\pi_c^6, \partial^4). \tag{6.18}$$

6.1.3 Mass Term and Explicit Symmetry Breaking

The effective action we derived in the previous section does not have a mass term for the pions. This is because we assumed the quarks to be massless, and the chiral symmetry was only spontaneously broken by the QCD vacuum. Now we want to incorporate the fact that quarks have a small but non-zero mass. The mass term for the quarks in the QCD action can be written using the mass matrix M:

$$\begin{aligned}
\mathcal{L}_m = -\bar{q}Mq &= -\begin{pmatrix}\bar{u} & \bar{d}\end{pmatrix}\begin{pmatrix}m_u & 0 \\ 0 & m_d\end{pmatrix}\begin{pmatrix}u \\ d\end{pmatrix} \\
&= -\bar{q}M\gamma_L q - \bar{q}M\gamma_R q.
\end{aligned} \tag{6.19}$$

First we observe that this term is not invariant under $SU_L(2) \times SU_R(2)$ (it is still invariant under $SU_I(2)$). Instead, it transforms as ($\gamma_L^2 = 1$ and $\gamma_L \gamma_R = 0$)

$$-\bar{q}M\gamma_L q - \bar{q}M\gamma_R q \longrightarrow -\bar{q}B^\dagger M A \gamma_L q - \bar{q}A^\dagger M B \gamma_R q. \tag{6.20}$$

However, the mass term would be invariant if M was instead a field that had the following transformation property:

$$M \to \left[A\gamma_L + B\gamma_R\right] M \left[A\gamma_L + B\gamma_R\right]^\dagger. \tag{6.21}$$

Therefore, if we consider an effective theory where we start out with the mass matrix treated as a field, called a spurion, and send it to its expectation value, i.e. its actual matrix form; then we can capture the effect of the explicit symmetry breaking.

The leading correction involving M will come from a term linear in it. This means we need to write down a term involving Goldstone modes that transform opposite to M. Our coset element Ω transforms in the fundamental representation of G.

$$\Omega \to g\Omega = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} e^{i\vec{\pi}\cdot\vec{X}} & 0 \\ 0 & e^{-i\vec{\pi}\cdot\vec{X}} \end{pmatrix} = \left[A\gamma_L + B\gamma_R\right]\Omega. \tag{6.22}$$

To cancel the transformation of M we need an adjoint representation so that

$$\Xi \to \left[A\gamma_L + B\gamma_R\right]\Xi\left[A\gamma_L + B\gamma_R\right]^\dagger \tag{6.23}$$

In terms of Goldstone fields, we can choose:

$$\Xi \equiv \gamma_L\Omega\gamma_R\Omega^\dagger = e^{2i\vec{\pi}\cdot\vec{X}}, \tag{6.24}$$

so that

$$\begin{aligned} \Xi &\to \gamma_L \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \Omega\gamma_R\Omega^\dagger \begin{pmatrix} A^\dagger & 0 \\ 0 & B^\dagger \end{pmatrix} \\ &= A\left(e^{i\vec{\pi}\cdot\vec{X}}\right)\left(e^{-i\vec{\pi}\cdot\vec{X}}\right)^\dagger B^\dagger = Ae^{2i\vec{\pi}\cdot\vec{X}}B^\dagger, \end{aligned} \tag{6.25}$$

which is the correct transformation property that leaves the combination $\mathrm{tr}(\Xi M)$ invariant under g. The two invariant combinations at linear order are:

$$\mathcal{L}^m_{EFT} = \frac{\Lambda_m^3}{2}\,\mathrm{tr}\left[M(\Xi + \Xi^\dagger)\right] + \ldots \tag{6.26}$$

where we included an overall dimensionful parameter Λ_m. Using Pauli matrix identities and canonically normalizing $\vec{\pi}$ ($\vec{\pi} \to \vec{\pi}_c/f$) we can express the above as:

$$\begin{aligned} \mathcal{L}^m_{EFT} &= (m_u + m_d)\Lambda_m^3 \cos\frac{\pi_c}{f} \\ &= (m_u + m_d)\Lambda_m^3\left[1 - \frac{\vec{\pi}_c\cdot\vec{\pi}_c}{2f^2} + \ldots\right] \\ &= m_\pi\left[f^2 - \frac{1}{2}\vec{\pi}_c\cdot\vec{\pi}_c + \ldots\right], \end{aligned} \tag{6.27}$$

where we identified the pion mass

$$m_\pi = (m_u + m_d)\frac{\Lambda_m^3}{f^2}. \tag{6.28}$$

Notice that we now have two scales, one is the SSB parameter f, related to Λ_{QCD}, and the other is m_π, related to the explicitly breaking scales m_u, m_d, and our EFT has an expansion in both.

6.2 An Example of a UV Model Describing an Open System

In previous sections, we showed that a local Schwinger-Keldysh effective action at low energies ($E \ll T$) for a *closed* system at finite temperature T must possess two copies of all global internal symmetries, forming a group $G_1 \times G_2$, partially realized non-linearly. Conversely, an *open* system, which exchanges conserved charges with an environment, has an effective action invariant only under the diagonal subgroup G_{diag} [2, 32], reflecting the fact that charge conservation is now only satisfied on average in equilibrium, and not as an exact operator statement.

This distinction suggests an intermediate scenario, termed *ajar* systems, where interactions with the environment occur over time scales much longer than internal dynamics. Such systems experience weak explicit symmetry breaking from $G_1 \times G_2$ to G_{diag}, that can be systematically modeled via a spurion technique [60].

Analogous to the in-out example of chiral perturbation theory described above, the effective action for ajar systems can be built by starting with the effective action for the closed version of the system, for which the off-diagonal symmetry G_a, is spontaneously broken. The effects of weak coupling to a bath is then added using spurions, whose vacuum expectation value (vev) corresponds to the bath coupling strength, and we obtain approximate breaking of G_a when spurions are sent to their vevs.

In order to motivate the general approach, put forward in the following section, for Schwinger-Keldysh spurion construction for ajar systems, we consider a concrete UV model where a complex scalar field is coupled linearly to a set of complex harmonic oscillators in thermal equilibrium, which we will treat as the bath. This simple system can be thought of as a field theory generalization of the Caldeira-Leggett model [19]. Other examples of UV descriptions of open systems can be found in [21, 138] Our presentation will follow closely that of [96]. The in-out microscopic Lagrangian is $\mathcal{L} = \mathcal{L}_{sys} + \mathcal{L}_{bath} + \mathcal{L}_{int}$

$$\mathcal{L}_{sys} = -\partial_\mu \phi^\dagger \partial^\mu \phi - m^2 \phi^\dagger \phi \tag{6.29a}$$

$$\mathcal{L}_{bath} = \sum_s \partial_t \chi_s^\dagger \partial_t \chi_s - \Gamma_s^2 \chi_s^\dagger \chi_s \tag{6.29b}$$

$$\mathcal{L}_{int} = \sum_s g_s (\chi_s^\dagger \phi + \phi^\dagger \chi_s), \tag{6.29c}$$

where Γ_s and g_s are the frequency and interaction strength of each mode. This action is invariant under a $U(1)$ symmetry acting on ϕ and the χ_s's as follows:

$$\phi' = e^{i\alpha}\phi \ , \qquad \chi_s' = e^{i\alpha}\chi_s \ . \tag{6.30}$$

The corresponding microscopic Schwinger-Keldysh action can be obtained by doubling all degrees of freedom and calculating $\mathcal{L}_{\text{SK}} = \mathcal{L}(\phi_1, \chi_{s,1}) - \mathcal{L}(\phi_2, \chi_{s,2}) \equiv$

$\mathcal{L}^{\rm SK}_{sys} + \mathcal{L}^{\rm SK}_{bath} + \mathcal{L}^{\rm SK}_{int}$. Switching from the $(1,2)$ basis to the Schwinger-Keldysh basis of fields,

$$\phi_r = \tfrac{1}{2}\left(\phi_1 + \phi_2\right) \ , \quad \phi_a = \phi_1 - \phi_2 \ , \qquad \chi_{s,r} = \tfrac{1}{2}\left(\chi_{s,1} + \chi_{s,2}\right) \ , \quad \chi_{s,a} = \chi_{s,1} - \chi_{s,2} \ , \tag{6.31}$$

we can express the various components of the Schwinger-Keldysh Lagrangian as follows:

$$\mathcal{L}^{\rm SK}_{sys} = -\partial_\mu \phi^\dagger_a \partial^\mu \phi_r - m^2 \phi^\dagger_a \phi_r + h.c. \tag{6.32a}$$

$$\mathcal{L}^{\rm SK}_{bath} = \sum_s \partial_t \chi^\dagger_{a,s} \partial_t \chi_{r,s} - \Gamma_s^2 \chi^\dagger_{a,s} \chi_{r,s} + h.c. \tag{6.32b}$$

$$\mathcal{L}^{\rm SK}_{int} = \sum_s g_s \left(\chi^\dagger_{a,s} \phi_r + \chi^\dagger_{r,s} \phi_a + h.c.\right) . \tag{6.32c}$$

Note that this action is invariant under two copies of the $U(1)$ symmetry, acting separately on the "1" and "2" fields. Equivalently, the diagonal $U(1)$ symmetry acts simultaneously on the "r" and "a" degrees of freedom as in Eq. (6.30), while the second one mixes the "r" and "a" fields as follows:

$$\phi'_r = \phi_r \cos\alpha + \tfrac{i}{2}\phi_a \sin\alpha \ , \qquad \phi'_a = \phi_a \cos\alpha + 2i\phi_r \sin\alpha \ , \tag{6.33}$$

with a similar action on $\chi_{s,r}$ and $\chi_{s,a}$, and α the parameter of the $U(1)$ transformation.

Since the Lagrangian (6.32) is quadratic, we can integrate out the bath degrees of freedom exactly, treating the $\phi_{r,a}$ fields as sources during the process. This leads to the following (non-local) open effective Lagrangian for the $\phi_{r,a}$'s [96]:

$$\mathcal{L}^{\rm SK}_{\phi} = -\partial_\mu \phi^\dagger_a \partial^\mu \phi_r - m^2 \phi^\dagger_a \phi_r + h.c. + \phi^\dagger_r D^{-1}_{ra} \phi_a + \phi^\dagger_a D^{-1}_{ar} \phi_r + \phi^\dagger_a D^{-1}_{aa} \phi_a \tag{6.34}$$

Let us now assume that the frequencies of the harmonic oscillators have an Ohmic distribution, such that the spectral density can be approximated as follows:

$$J(\omega) = 2\pi \sum_s (g_s^2/\Gamma_s)\delta(\omega - \Gamma_s) \overset{\rm Ohmic}{\longrightarrow} 4\gamma\omega \ , \tag{6.35}$$

where γ is a constant for small frequencies. The Ohmic assumption ensures that the ϕ's are able to exchange charge with the bath at arbitrarily small energies, so that, after integrating out the χ_s's, we are left with an open system. The retarded (advanced) kernels reduce to

$$D^{-1}_{ra(ar)}(\omega) = \int \frac{d\omega'}{2\pi} \frac{\omega' J(\omega')}{\omega'^2 - (\omega \pm i\varepsilon)^2} \quad \overset{\rm Ohmic}{\longrightarrow} \quad \text{constant} \pm 2i\gamma\omega \quad \overset{\rm F.T.}{\longrightarrow} \quad \mp 2\gamma\delta(t-t')\partial'_t \ , \tag{6.36}$$

where the upper (lower) sign applies to the ra (ar) kernel. Note that, when carrying out the Fourier transform in the last step we dropped the constant part, since it

would simply renormalize the mass term for ϕ. In order for D_{aa}^{-1} to become local, we additionally need to take the high temperature limit. Its full non-local form is

$$D_{aa}^{-1} = \frac{1}{2}\coth\left(\frac{\omega}{2T}\right)\left[D_{ra}^{-1} - D_{ar}^{-1}\right] \overset{\text{Ohmic}}{\longrightarrow} (4i\gamma\omega)\coth\left(\frac{\omega}{2T}\right) \overset{\beta\to 0}{\longrightarrow} (2i\gamma\omega)\left(\frac{2T}{\omega}\right) = 4iT\gamma. \tag{6.37}$$

With these approximations, Eq. (6.34) reduces to the following Lagrangian that is invariant under the DKMS symmetry at lowest order and describes the open dynamics of our complex scalar field:

$$\mathcal{L}_{open}^{\text{SK}} \simeq -\partial_\mu\phi_a^\dagger\partial^\mu\phi_r - m^2\phi_a^\dagger\phi_r + h.c. + \frac{2i\gamma}{\beta}\phi_a^\dagger\left(\phi_a + i\beta\partial_t\phi_r\right) + \frac{2i\gamma}{\beta}\phi_a\left(\phi_a^\dagger + i\beta\partial_t\phi_r^\dagger\right) \ . \tag{6.38}$$

Note that this Lagrangian is invariant under the DKMS transformation

$$\phi_r'(x) = \phi_r^\dagger(-x) - \tfrac{i\beta}{4}\partial_t\phi_a^\dagger(-x) + \mathcal{O}(\beta^2) \ , \qquad \phi_a'(x) = \phi_a^\dagger(-x) - i\beta\partial_t\phi_r^\dagger(-x) + \mathcal{O}(\beta^2) \tag{6.39}$$

but the last two terms in Eq. (6.38) explicitly break $U(1)_1 \times U(1)_2$ down to the diagonal subgroup $U(1)_{\text{diag}}$. The strength of this breaking is $\sim \gamma$. It will be helpful to check this by reverting to the $1-2$ basis and focusing on the non-derivative terms. Up to an overall constant, we have

$$\gamma\phi_a^\dagger\phi_a = \gamma(\phi_1^\dagger - \phi_2^\dagger)(\phi_1 - \phi_2) = \gamma(\phi_1^\dagger\phi_1 + \phi_2^\dagger\phi_2 - \phi_1^\dagger\phi_2 - \phi_2^\dagger\phi_1) \ , \tag{6.40}$$

and we see that the last two terms are only invariant under $U(1)_{\text{diag}}$.

Following the spurion logic, we can make the above term invariant by replacing the symmetry breaking parameter γ with two complex spurion fields O_1 and O_2 that couple to the ϕ_i's as follows:

$$O_1^\dagger O_1\phi_1^\dagger\phi_1 + O_2^\dagger O_2\phi_2^\dagger\phi_2 - O_2^\dagger O_1\phi_1^\dagger\phi_2 - O_1^\dagger O_2\phi_2^\dagger\phi_1 \ . \tag{6.41}$$

To recover the terms in (6.40), the spurion vev should be $\langle O_1\rangle = \langle O_2\rangle = \sqrt{\gamma}$. Note that the spurion fields appear only in the quadratic combinations discussed in the Introduction—there are no linear terms in O_1 or O_2.

Adding a symmetry breaking potential for ϕ to the original Lagrangian (6.29) would not modify the argument that leads to (6.41), since the ϕ_i's were treated as external sources while integrating the bath degrees of freedom. In this case, we can parametrize $\phi_i = (v + \sigma_i)e^{i\pi_i}$, where v is the expectation value of the complex scalar. After integrating out the radial modes σ_i at tree level, the expression (6.41) reduces to

$$v^2\left(O_1^\dagger O_1 + O_2^\dagger O_2 - O_2^\dagger O_1 e^{-i\pi_a} - O_1^\dagger O_2 e^{i\pi_a}\right), \tag{6.42}$$

where we have introduced $\pi_a = \pi_1 - \pi_2$. Informed by the form of the spurion terms above, we will now introduce a general spurion method to build effective actions of ajar systems, and apply our methods for the case of a $U(1)$ system.

6.3 Extension of Schwinger-Keldysh Coset Construction to Open Systems

We start our general construction by considering a spurion field that transforms bilinearly under $G_1 \times G_2$,

$$O \to U_1 O U_2^\dagger. \tag{6.43}$$

Assigning it an expectation value $\langle O \rangle = \gamma$ invariant under G_{diag} will allow us to systematically account symmetry-breaking effects. The parameter γ characterizes the interaction strength with the environment; small γ relative to the EFT cutoff ensures approximate symmetry, yielding predictions for corrections to observables.

Before proceeding, we emphasize the difference between our approach and that of [33]. The latter studied the hydrodynamic behavior of closed systems exhibiting approximate symmetries (see also [139]), introducing two spurion fields O_1 and O_2 transforming as

$$O_1 \to U_1 O_1, \quad O_2 \to U_2 O_2. \tag{6.44}$$

When these fields acquire non-zero expectation values, the full symmetry group $G_1 \times G_2$ is weakly broken. This allows the authors of [33] to consider, for example the in-in version of what was discussed in Section 6.1. In contrast, our setup preserves the diagonal subgroup G_{diag}. Thus, an open system can be viewed as a particular case of this general scenario in which spurions appear only through combinations preserving G_{diag}, specifically $O_1^\dagger O_1$, $O_2^\dagger O_2$, and $O = O_1 O_2^\dagger$ (and its conjugate).[1] The first two combinations, scalars under $G_1 \times G_2$, modify exactly invariant terms by an amount proportional to the spurion expectation value $\langle O \rangle$. As we will show, these invariant contributions are crucial for ensuring unitarity in the underlying microscopic theory. Consequently, our effective action will depend not only on O but also explicitly on $\langle O \rangle$, slightly departing from the usual spurion formalism.

In the remainder of this chapter, we will illustrate our approach in the simplest case of abelian symmetries, $G = U(1)$. In this case, our spurion field will be a complex field that transforms under $U(1)_1 \times U(1)_2$ like

$$O \to e^{i(\alpha_1 - \alpha_2)} O \ , \tag{6.45}$$

ensuring that the spurion expectation value preserves $U(1)_{\text{diag}}$. Gauged $U(1)$ symmetries in open systems—where the off-diagonal symmetry is badly broken—were recently discussed in [22].

[1] We do not need to consider additional structures involving derivatives, because they would drop out once the spurions are replaced with their expectation value.

6.4 $U(1)$ Diffusion

6.4.1 Closed Systems

We start by reviewing the effective field theory (EFT) of $U(1)$ diffusion for closed systems in a thermal state. In this case, the Schwinger-Keldysh effective action must be invariant under $U(1)_1 \times U(1)_2$ *spontaneously* broken down to $U(1)_{\text{diag}}$ [140]. The corresponding Goldstone mode, φ_a, shifts under the "axial" $U(1)$, is a singlet under $U(1)_{\text{diag}}$, and is paired with a "matter field" ρ_r[2]. Thermality implies that all correlation functions satisfy the KMS condition. This can be enforced at the level of the effective action by imposing invariance under the dynamical KMS (DKMS) symmetry [103, 104]:[3]

$$\varphi_a'(x) = -\varphi_a(-x) - i\beta\rho_r(-x) + \mathcal{O}(\beta^2), \tag{6.46a}$$
$$\rho_r'(x) = \rho_r(-x) - \tfrac{i\beta}{4}\partial_t^2\varphi_a(-x) + \mathcal{O}(\beta^2). \tag{6.46b}$$

We have expanded these transformation rules in powers of $\beta = 1/T$, anticipating the fact that the effective action will be organized in powers of E/T. A discussion of these transformations at all orders in β and of the systematics of power counting in E/T can be found in [141].

The Schwinger-Keldysh effective action must also satisfy some constraints that follow from unitarity considerations [34]:

$$S[\varphi_a = 0, \rho_r] = 0 \ , \tag{6.47a}$$
$$S^*[\varphi_a, \rho_r] = -S[-\varphi_a, \rho_r] \ , \tag{6.47b}$$
$$\text{Im}\, S \geq 0 \ . \tag{6.47c}$$

At lowest order in derivatives and quadratic order in fields, the effective action that satisfies these constraints and is invariant under all the symmetries mentioned above is [118][4]

$$S = \int d^4x \left[n(\rho_r)\partial_t\varphi_a + \sigma(\rho_r)\partial_i\varphi_a \left(\frac{i}{\beta}\partial_i\varphi_a - \partial_i\rho_r \right) \right], \tag{6.48}$$

where $d^4x \equiv dtd^3r$ and n and σ are analytic around $\rho_r = 0$, and $\sigma \geqslant 0$ to satisfy (6.47c). Imposing invariance under charge conjugation constrains n (σ) to contain

[2] The a and r subscript are introduced to match standard notation in the literature—see e.g. [34].

[3] The DKMS symmetry relies on a discrete symmetry of the action involving time reversal. While CPT is a natural candidate for Lorentz-invariant actions, C is spontaneously broken at finite charge density. Thus, implementing DKMS via PT, with additional C symmetry imposed through $\varphi_a'(x) = -\varphi_a(x)$ and $\rho_r'(x) = -\rho_r(x)$, becomes more convenient. The transformation rules (6.46) assume invariance under PT, and differ from the ones in [140] which are built using CT.

[4] In the incompressible limit, boosts are explicitly broken by neglecting thermal bath phonons. The DKMS symmetry is realized perturbatively up to corrections of $\mathcal{O}(E/T)$.

only odd (even) powers of ρ_r. Demanding that the terms in (6.48) are all of the same order implies the power counting relations $\rho_r \sim T\varphi_a$ and $\partial_t \sim \partial_i^2$. Interestingly, the first of these relations also implies that the second term on the RHS of (6.46b) is negligible compared to the first one—an approximation usually justified by invoking an expansion in powers of $\hbar$.

The equation of motion obtained by varying S with respect to ρ_r always admits the solution $\varphi_a = 0$, which is the background we will be interested in. On this background, the equation for φ_a reduces to a diffusion equation for $n(\rho_r)$:

$$\partial_t n(\rho_r) = \partial_i \left[\sigma(\rho_r)\partial^i \rho_r\right] , \tag{6.49}$$

In fact, $n(\rho_r)$ is also the Noether charge density associated with invariance under shifts of φ_a. These results suggest identifying ρ_r with fluctuations of the chemical potential around equilibrium.

6.4.2 Leading Corrections in Ajar Systems

We will now supplement the action S for a closed system with the leading terms that depend on the spurion field O. We will need to ensure that these additional terms are invariant under the DKMS symmetry and satisfy the appropriate unitarity constraints. To this end, we will proceed in two steps.

First, it will be convenient to think again of our spurion field as being a composite object of the form $O = O_1 O_2^\dagger$. This is because the transformation properties of the operators $O_{1,2}$ under the DKMS symmetry are quite simple [34]:

$$O'_{1,2}(x) = O^\dagger_{1,2}(-x) + \mathcal{O}(\beta\partial_t O^\dagger_{1,2}) , \tag{6.50}$$

where the explicit form of the second term on the righthand side will not be needed because it will vanish when the spurions are replaced with their expectation value. The first two unitarity constraints can also be easily amended to account for the fields $O_{1,2}$ [34]:

$$S[\varphi_a = 0, \rho_r, O_1, O_2 = O_1] = 0 , \tag{6.51a}$$
$$S^*[\varphi_a, \rho_r, O_1, O_2] = -S[-\varphi_a, \rho_r, O_2, O_1] . \tag{6.51b}$$

Second, following the usual template for coupling Goldstone modes to additional "matter fields" [101], we will find it convenient to work directly with the combination $\tilde{O} = Oe^{-i\varphi_a}$, since it is invariant under $U(1)_1 \times U(1)_2$. By combining our definitions, we can easily derive the transformation rule for $\tilde{O}$ and its complex conjugate under DKMS:

$$\tilde{O}'(x) = \tilde{O}^\dagger(-x) + \beta\rho_r(-x)\tilde{O}^\dagger(-x) + \mathcal{O}(\beta\partial_t) , \tag{6.52a}$$
$$\tilde{O}^{\dagger\prime}(x) = \tilde{O}(-x) - \beta\rho_r(-x)\tilde{O}(-x) + \mathcal{O}(\beta\partial_t) , \tag{6.52b}$$

while the first two unitarity constraints reduce to

$$S[\varphi_a = 0, \rho_r, \tilde{O} \geqslant 0] = 0 \ , \tag{6.53a}$$

$$S^*[\varphi_a, \rho_r, \tilde{O}] = -S[-\varphi_a, \rho_r, \tilde{O}^\dagger] \ . \tag{6.53b}$$

At linear order in O and $\langle O \rangle$, there is only one term that is invariant under the DKMS symmetry, and satisfies *all three* unitarity constraints.

$$\Delta S = \frac{i}{\beta} \int d^4x F(\rho_r) \left[2\langle O \rangle - \left(\tilde{O}^\dagger + \tilde{O} \right) + \frac{\beta}{2} \rho_r \left(\tilde{O}^\dagger - \tilde{O} \right) \right] , \tag{6.54}$$

where $F(\rho_r)$ is analytic around $\rho_r = 0$ and, without loss of generality, such that $F(0) = 1$. Note that the first term proportional to $\langle \mathcal{O} \rangle$ is crucial to ensure that the unitarity condition (6.53a) is satisfied. Replacing $O \to \langle O \rangle = \gamma$, this expression reduces to

$$\Delta S = \int d^4x \, 4\gamma(\rho_r) \sin \tfrac{\varphi_a}{2} \left(\frac{i}{\beta} \sin \tfrac{\varphi_a}{2} - \frac{\rho_r}{2} \cos \tfrac{\varphi_a}{2} \right) , \tag{6.55}$$

where we have defined $\gamma(\rho_r) \equiv \gamma F(\rho_r)$.

It is easy to check that this expression is invariant under the DKMS transformations (6.46) up to corrections of $\mathcal{O}(E/T)$ and satisfies the unitarity constraints (6.47) provided $\gamma(\rho_r) \geqslant 0$. In the presence of charge conjugation symmetry, $\gamma(\rho_r)$ only contains even powers of ρ_r (under C, $O'(x) = O^\dagger(x)$). The nonlinear structure of (6.55) implies that, in the limit of weak interaction with the environment, any n-point function will introduce only one new free parameter at leading order, i.e. successive measurements of each n-point function will fix the coefficient of the $(n-1)$th parameter in the expansion of $\gamma(\rho_r)$. We will now present the corrections to the 2- and 3-point functions for the charge density in ajar systems due to the terms in (6.55).

6.4.3 Density correlation functions

The quadratic action for diffusive ajar systems follows from expanding $S + \Delta S$ to quadratic order in ρ_r and ϕ_a. Inverting these terms yields the 2-point functions for φ_a and ρ_r, or equivalently φ_a and the charge density n:

$$\begin{aligned} \langle n(p) n(-p) \rangle &= \frac{2}{\beta} \frac{(\gamma + \sigma k^2)}{\omega^2 + (\gamma/\chi + \mathcal{D}k^2)^2} , \\ \langle n(p) \varphi_a(-p) \rangle &= \frac{1}{\omega + i\,(\gamma/\chi + \mathcal{D}k^2)} , \end{aligned} \tag{6.56}$$

where $p = (\omega, \vec{k})$, $\chi = dn/d\rho_r|_{\rho_r=0}$, $\sigma = \sigma(\rho_r = 0)$, and $\mathcal{D} = \sigma/\chi$ is the diffusion coefficient, and delta functions enforcing energy-momentum conservation have been

omitted. Notably, environmental interactions gap only the imaginary part of the pole, unlike an approximate $U(1)$ symmetry, where the real part is also gapped.

The leading 3-point function of charge density arises from expanding $S + \Delta S$ to cubic order in ρ_r and φ_a. The closed-system result was recently derived in [118]. Expressing interactions in terms of $n \simeq \chi(\rho_r + \frac{1}{2}\chi'\rho_r^2)$ yields:

$$S_3 + \Delta S_3 = \int d^4x \left[\frac{\sigma\chi'}{\chi^2} n\, \partial_i\varphi_a n + \sigma' n\, \partial_i\varphi_a \partial^i \left(\frac{i}{\beta}\varphi_a - \frac{n}{\chi} \right) + \frac{\gamma\chi'}{2\chi^2} n^2 \varphi_a + \gamma' n\, \varphi_a \left(\frac{i}{\beta}\varphi_a - \frac{n}{\chi} \right) \varphi_a \right], \tag{6.57}$$

where primes denote derivatives with respect to n. The resulting three-point function of n is (with $q_i \equiv \sqrt{\mathcal{D}} k_i$ and $\tilde{\gamma} \equiv \gamma/\chi$):

$$\begin{aligned} &\langle n(p_1) n(p_2) n(p_3) \rangle = \\ &(\chi T)^2 \Bigg\{ 12\tilde{\gamma} \frac{\chi'}{\chi} \frac{(\tilde{\gamma}+q_1^2)\,(\tilde{\gamma}+q_2^2)\,(\tilde{\gamma}+q_3^2)}{\left[\omega_1^2 + (\tilde{\gamma}+q_1^2)^2\right]\left[\omega_2^2 + (\tilde{\gamma}+q_2^2)^2\right]\left[\omega_3^2 + (\tilde{\gamma}+q_3^2)^2\right]} \\ &\quad + \left[\frac{4\chi'}{\chi} - \frac{2\sigma'}{\sigma} \right] \frac{(\tilde{\gamma}+q_1^2)\,(\tilde{\gamma}+q_2^2)\,(\tilde{\gamma}+q_3^2)\,(q_1^2+q_2^2+q_3^2)}{\left[\omega_1^2 + (\tilde{\gamma}+q_1^2)^2\right]\left[\omega_2^2 + (\tilde{\gamma}+q_2^2)^2\right]\left[\omega_3^2 + (\tilde{\gamma}+q_3^2)^2\right]} \\ &\quad + \frac{4\sigma'}{\sigma} \frac{\omega_2\omega_3 (q_2\cdot q_3)(\tilde{\gamma}+q_1^2) + \omega_1\omega_3 (q_1\cdot q_3)(\tilde{\gamma}+q_2^2) + \omega_1\omega_2 (q_1\cdot q_2)(\tilde{\gamma}+q_3^2)}{\left[\omega_1^2 + (\tilde{\gamma}+q_1^2)^2\right]\left[\omega_2^2 + (\tilde{\gamma}+q_2^2)^2\right]\left[\omega_3^2 + (\tilde{\gamma}+q_3^2)^2\right]} \\ &\quad - \frac{4\gamma'}{\chi^2} \frac{3\left[(\tilde{\gamma}+q_1^2)\,(\tilde{\gamma}+q_2^2)\,(\tilde{\gamma}+q_3^2)\right] + \omega_2\omega_3\,(\tilde{\gamma}+q_1^2) + \omega_1\omega_3\,(\tilde{\gamma}+q_2^2) + \omega_1\omega_2\,(\tilde{\gamma}+q_3^2)}{\left[\omega_1^2 + (\tilde{\gamma}+q_1^2)^2\right]\left[\omega_2^2 + (\tilde{\gamma}+q_2^2)^2\right]\left[\omega_3^2 + (\tilde{\gamma}+q_3^2)^2\right]} \Bigg\}. \end{aligned} \tag{6.58}$$

6.5 $U(1)$ Goldstone Mode

6.5.1 Effective action for Ajar Systems

The approach discussed in the previous section applies also when the $U(1)$ symmetry is spontaneously broken. In this case, the corresponding Schwinger-Keldysh effective action for a closed system realizes the entire group $U(1)_1 \times U(1)_2$ nonlinearly. The relevant degrees of freedom are the two Goldstone modes π_r and π_a, which transform under the DKMS symmetry as follows [140]:

$$\pi_a'(x) = -\pi_a(-x) + i\beta\partial_t\pi_r(-x) + \mathcal{O}(\beta^2), \tag{6.59a}$$
$$\pi_r'(x) = -\pi_r(-x) - \tfrac{i\beta}{4}\partial_t\pi_a(-x) + \mathcal{O}(\beta^2). \tag{6.59b}$$

The effective action must be invariant under (6.59) as well as shifts of $\pi_{r,a}$, and must satisfy unitarity constraints of the form (6.47) and (6.53) with $\rho_r \to \partial_t\pi_r, \varphi_a \to \pi_a$.

The shift symmetries imply that our fields must enter the effective Lagrangian with at least one derivative. If our system was closed, the effective action would be [142]

$$S = \frac{f^2}{c_s^3}\int d^4x \left\{ \partial_t\pi_a\partial_t\pi_r - c_s^2\partial_i\pi_a\partial_i\pi_r + \frac{\Sigma_\pi}{f}\,\partial_t\pi_a\left(\frac{i}{\beta}\partial_t\pi_a - \partial_t^2\pi_r\right) \right. \\ \left. + \frac{\sigma_\pi c_s^2}{f}\,\partial_i\pi_a\left(\frac{i}{\beta}\partial_i\pi_a - \partial_t\partial_i\pi_r\right) + \cdots \right\}, \tag{6.60}$$

where we have shown explicitly only the terms quadratic in the fields. From the quadratic action, we deduce the power-counting rules $\pi_a \sim (E/T)\pi_r$ and $\partial_t \sim \partial_i$.

The exchange of charge with an environment explicitly breaks the shift symmetry of π_a. In ajar systems, where this breaking is soft, it can again be modeled by a spurion O, or equivalently $\tilde{O} = Oe^{-i\pi_a}$, that now transforms under DKMS as follows:

$$\tilde{O}'(x) = \tilde{O}^\dagger(-x) - \beta\partial_t\pi_r(-x)\tilde{O}^\dagger(-x) + \mathcal{O}(\beta^2)\ , \tag{6.61a}$$
$$\tilde{O}^{\dagger\prime}(x) = \tilde{O}(-x) + \beta\partial_t\pi_r(-x)\tilde{O}(-x) + \mathcal{O}(\beta^2)\ , \tag{6.61b}$$

At leading order in the spurion field, the effective action receives the following correction

$$\Delta S = \frac{i}{\beta}\int d^4x F(\partial_t\pi_r, \partial_i\pi_r\partial^i\pi_r)\left[2\langle O\rangle - \left(\tilde{O}^\dagger + \tilde{O}\right) + \frac{\beta}{2}\partial_t\pi_r\left(\tilde{O}^\dagger - \tilde{O}\right)\right], \tag{6.62}$$

where, once again, we can assume $F(0,0) = 1$ without loss of generality.

6.5.2 Goldstone correlation functions

The two-point functions that result from inverting the quadratic part of $S + \Delta S$ are

$$\langle\pi_r\pi_r\rangle = \frac{2c_s^3}{\beta f^2}\frac{\tilde{\gamma} + \frac{1}{f}(\Sigma_\pi\omega^2 + \sigma_\pi c_s^2k^2)}{(\omega^2 - c_s^2k^2)^2 + \omega^2\left[\tilde{\gamma} + \frac{1}{f}(\Sigma_\pi\omega^2 + \sigma_\pi c_s^2k^2)\right]^2}, \\ \langle\pi_r\pi_a\rangle = \frac{c_s^3}{f^2}\frac{i}{\omega^2 - c_s^2k^2 + i\tilde{\gamma}\omega + \frac{i}{f}(\Sigma_\pi\omega^3 + \sigma_\pi c_s^2\omega k^2)}, \tag{6.63}$$

where $\tilde{\gamma} \equiv \gamma c_s^3/f^2$. The second correlator is simply the retarded Green's function for the Goldstone mode.

Once again, we see that "openness" adds a finite contribution to the imaginary part of the Goldstone dispersion relation, rather than turning this mode into a pseudo-Goldstone boson by gapping the real part, as would be the case for an approximate symmetry. Physically, it means that the Goldstone mode can only propagate for wavenumbers $k \gtrsim \tilde{\gamma}/c_s$. Higher-point functions for the Goldstone mode can be calculated in a straightforward way by expanding $S + \Delta S$ up to the desired order.

Chapter 7

Conclusions

The coset construction is heralded for its general applicability, ranging from its origins in nuclear physics [35] to more recent applications to systems at finite density [102, 143], conformal field theories [144, 145] and gravity [76, 146], just to name a few. However, so far this technique has largely been applied to the limited case of regular effective actions for the purposes of computing scattering amplitudes or time-ordered correlators around pure states. In the present paper, we have extended the construction to Schwinger-Keldysh effective actions, which can more naturally incorporate the effects of non-trivial density matrices and facilitate the computation of a more diverse set of correlators. We focused on spontaneously broken internal symmetries, with particular emphasis on thermal states. Our main conclusion is that, once the correct symmetry breaking pattern has been properly identified, the standard rules of the coset construction can be brought to bear to write down Schwinger-Keldysh effective actions. We would like to highlight in particular the advantages of the framework developed in this paper:

- We retain the full non-linear structure inherent to the coset construction, thus preserving all the symmetries realized non-linearly. This should be contrasted with the common practice of linearizing Schwinger-Keldysh actions in the a-fields, focusing on the classical regime.

- In previous work on the effective Schwinger-Keldysh field theory of thermal systems, a mysterious, diffusive symmetry was needed to differentiate the normal phase from a spontaneously broken one [34]. In our approach, no such symmetry is needed: the symmetry breaking pattern together with basic principles such as unitarity dictate the relevant degrees of freedom and their transformation properties under all the symmetries.

In order to illustrate our framework we calculated 2-point functions of conserved spin currents for paramagnets, anti-ferromagnets, and ferromagnets in Sec. 5.3. Our analysis generalizes the classic work in [127, 135], reproducing their results in the appropriate limits.

There are various avenues along which to extend the present work. First, it would be interesting to generalize our framework to include spontaneously broken spacetime symmetries. This could be used to extend the coset-based approach to condensed matter systems put forward in [102, 147], and would provide a different viewpoint on recent developments surrounding EFTs for dissipative hydrodynamics [34, 108]. Second, by eschewing the classical limit, our approach could also shed a new light on the quantum properties of perfect fluids [148, 149]. Third, it would be interesting to further explore how to systematically build DKMS-invariants at higher order in E/T. And, finally, we would like to investigate the symmetry breaking pattern associated with finite density non-thermal density matrices, and understand how the properties of such states can be encoded in a Schwinger-Keldysh effective action. We leave all of this for future work.

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