T-Duality Invariant Higher-Derivative Corrections for Cosmology and $D = 2$ Black Holes

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A mi familia.

Abstract

In this thesis we study duality-invariant higher-derivative (α') corrections to string low energy effective theories. We restrict to the universal massless sector, consisting of the graviton, B-field and dilaton, and specialize to backgrounds with d abelian isometries, which enjoy $O(d,d|\mathbb{R})$-invariance and contain phenomenologically relevant scenarios such as cosmology and black holes as particular cases. The $O(d,d|\mathbb{R})$-symmetry is expected to be preserved to arbitrary orders in derivatives, a fact that motivated Hohm and Zwiebach to arrive at the cosmological classification. Such construction parameterizes a large class of purely-time dependent duality-invariant theories to all orders in α' in terms of a countable infinite number of coefficients. String theories represent single points in this theory space, determined by specific configurations of these coefficients. We compute the first coefficients for bosonic, heterotic and type II string theory by compactifying dilaton-gravity theories up to order α'^3, and bringing them to canonical form. This target-space approach requires one to know the parent theory prior to compactification, which involves complicated beta-function computations from the string sigma model. With the aim of making this process simpler, we propose to begin from a worldsheet action already in cosmological backgrounds. We compute the beta functions for the bosonic string at one and two loops, show how they can be brought to a manifestly $O(d,d|\mathbb{R})$-covariant formulation, and derive the corresponding target-space theory at order α'. We then study a duality-invariant theory due to Hohm, Siegel, and Zwiebach (HSZ), which encodes the massless string modes plus novel massive fields. Integrating out the latter, infinitely many higher-derivative corrections arise for the massless fields, which we bring to the form of the classification and determine the order α'^4 coefficient. If one instead keeps the extra massive fields, HSZ serves as a toy model for an α'-complete theory of strings. We find a two-derivative reformulation of the theory in cosmological backgrounds and determine α'-exact Friedmann equations. We explore the tensionless limit $\alpha' \to \infty$, which features string frame de Sitter vacua, and we set up perturbation theory in $\frac{1}{\alpha'}$. Coming back to generic duality-invariant theories, we revisit the classification for Bianchi Type I cosmologies with q scale factors, and show that only $q-1$ of them have non-trivial α' corrections. In particular, for FRW backgrounds all α' corrections are trivial. We also extend the classification to two-dimensional backgrounds with time-like isometry. To this end, we show that in non-critical string theory the effects of higher-derivative terms are a priori of the same order as two-derivative terms, so that the usual perturbation theory does not apply. We circumvent such obstacle by assuming the α' expansion comes with coefficients that fall off sufficiently fast, and classify the most general higher-derivative terms. This duality-invariant theory space admits black-hole solutions, and we provide perturbative and non-perturbative tools to explore them. For the latter, we prove that the dual of a solution with a regular horizon must have a curvature singularity, and use a parametrization introduced by Gasperini and Veneziano to find α'-deformed black holes with a regular horizon and a singularity. Furthermore, we find subregions in this theory space, probably not containing string theory, in which the black hole geometry exhibits a horizon leading to an interior that, having no singularity in the metric, curvature, or dilaton, is a regular cosmology.

Zusammenfassung

In dieser Arbeit untersuchen wir dualitätsinvariante höher-abgeleitete (α') Korrekturen an effektiven Stringtheorien mit niedriger Energie. Wir beschränken uns auf den universellen masselosen Sektor, bestehend aus Graviton, B-Feld und Dilaton, und spezialisieren uns auf Hintergründe mit d abelschen Isometrien, die $O(d, d|\mathbb{R})$-Invarianz genießen und phänomenologisch relevante Szenarien wie Kosmologie und Schwarze Löcher als Sonderfälle enthalten. Es wird erwartet, dass die $O(d, d|\mathbb{R})$-Symmetrie bis zu beliebigen Ablei-tungsordnungen erhalten bleibt, eine Tatsache, die Hohm und Zwiebach zu der kosmologischen Klassifikation motivierte. Eine solche Konstruktion parametrisiert eine große Klasse von rein zeitabhängigen dualitätsinvarianten Theorien für alle Ordnungen in α' in Form einer abzählbar unendlichen Anzahl von Koeffizienten. Stringtheorien stellen einzelne Punkte in diesem Theorieraum dar, die durch spezifische Konfigurationen dieser Koeffizienten bestimmt werden. Wir berechnen die ersten Koeffizienten für bosonische, heterotische und Typ-II-Stringtheorien, indem wir Dilatongravitationstheorien bis zur Ordnung α'^3 kompaktieren und sie in kanonische Form bringen. Dieser Zielraum-Ansatz setzt voraus, dass man die übergeordnete Theorie vor der Verdichtung kennt, was komplizierte Beta-Funktionsberechnungen aus dem String-Sigma-Modell erfordert. Um diesen Prozess zu vereinfachen, schlagen wir vor, von einer Worldsheet-Aktion auszugehen, die bereits im kosmologischen Hintergrund vorhanden ist. Wir berechnen die Betafunktionen für den bosonischen String bei einer und zwei Schleifen, zeigen, wie sie in eine offensichtlich $O(d, d|\mathbb{R})$-kovariante Formulierung gebracht werden können, und leiten die entsprechende Zielraumtheorie der Ordnung α' ab. Anschließend untersuchen wir eine dualitätsinvariante Theorie von Hohm, Siegel und Zwiebach (HSZ), die die masselosen Stringmoden und neuartige massive Felder kodiert. Integriert man letztere heraus, ergeben sich für die masselosen Felder unendlich viele Korrekturen höherer Ableitung, die wir in die Form der Klassifikation bringen und den Koeffizienten der Ordnung α'^4 bestimmen. Behält man stattdessen die extra massiven Felder bei, so dient die HSZ als Spielzeugmodell für eine α'-komplette Theorie der Strings. Wir finden eine zweifach abgeleitete Neuformulierung der Theorie für kosmologische Hintergründe und bestimmen α'-genaue Friedmann-Gleichungen. Wir untersuchen den spannungsfreien Grenzwert $\alpha' \to \infty$, der sich durch Wir untersuchen den spannungslosen Grenzwert $\alpha' \to \infty$, der de Sitter-Vakua mit String-Rahmen aufweist, und stellen die Störungstheorie in $\frac{1}{\alpha'}$ auf. Wir kehren zu generischen dualitätsinvarianten Theorien zurück und überprüfen die Klassifikation für Bianchi Typ I Kosmologien mit q Skalenfaktoren und zeigen, dass nur $q-1$ von ihnen nicht-triviale α'-Korrekturen haben. Insbesondere für FRW-Hintergründe sind alle α'-Korrekturen trivial. Wir erweitern die Klassifikation auch auf zweidimensionale Hintergründe mit zeitähnlicher Isometrie. Zu diesem Zweck zeigen wir, dass in der nicht-kritischen Stringtheorie die Effekte der höher-abgeleiteten Terme a priori der gleichen Ordnung sind wie zweidimensionale Terme, so dass die übliche Störungstheorie nicht anwendbar ist. Wir umgehen dieses Hindernis, indem wir annehmen, dass die α'-Expansion Koeffizienten hat, die ausreichend schnell abfallen, und klassifizieren die allgemeinsten höher-abgeleiteten Terme. Dieser dualitätsinvariante Theorieraum lässt Lösungen für Schwarze Löcher zu, und wir bieten perturbative und nicht-perturbative Werkzeuge an, um sie zu erforschen. Für letztere beweisen wir, dass das Dual einer Lösung mit einem regulären Horizont eine Krümmungssingularität haben muss, und verwenden eine von Gasperini und Veneziano eingeführte Parametrisierung, um α'-deformierte Schwarze Löcher mit einem regulären Horizont und einer Singularität zu finden. Außerdem finden wir Unterregionen in diesem Theorieraum, die wahrscheinlich nicht die Stringtheorie enthalten, in denen die Geometrie des Schwarzen Lochs einen Horizont aufweist, der zu einem Inneren führt, das keine Singularität in der Metrik, der Krümmung oder dem Dilaton aufweist und eine reguläre Kosmologie darstellt.

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List of Publications

This thesis is based on the following published research articles and preprints:

[1] *"String Dualities at Order α'^3"*
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[2] *"Beta functions for the duality-invariant sigma model"*
R. Bonezzi, T. Codina and O. Hohm
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[3] *"General string cosmologies at order α'^3"*
T. Codina, O. Hohm and D. Marques
Phys. Rev. D **104** (2021) no.10, 106007
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[4] *"Duality invariant string beta functions at two loops"*
R. Bonezzi, T. Codina and O. Hohm
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[5] *"An α'-complete theory of cosmology and its tensionless limit"*
T. Codina, O. Hohm and D. Marques
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[6] *"2D black holes, Bianchi I cosmologies, and α'"*
T. Codina, O. Hohm and B. Zwiebach
Phys. Rev. D **108** (2023) no.2, 026014
DOI: 10.1103/PhysRevD.108.026014

[7] *"On black hole singularity resolution in $D = 2$ via duality-invariant α' corrections"*
T. Codina, O. Hohm and B. Zwiebach
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Ich erkläre, dass ich die Dissertation selbständig und nur unter Verwendung der von mir gemäß §7 Abs. 3 der Promotionsordnung der Mathematisch-Naturwissens-chaftlichen Fakultät, veröffentlicht im Amtlichen Mitteilungsblatt der Humboldt-Universität zu Berlin Nr. 42/2018 am 11.07.2018 angegebenen Hilfsmittel angefertigt habe

Ort, Datum	Tomás Codina

I declare that I have completed the thesis independently using only the aids and tools specified. I have not applied for a doctor's degree in the doctoral subject elsewhere and do not hold a corresponding doctor's degree. I have taken due note of the Faculty of Mathematics and Natural Sciences PhD Regulations, published in the Official Gazette of Humboldt-Universität zu Berlin no. 42/2018 on 11/07/2018.

Place, date	Tomás Codina

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Chapter 1

Introduction

1.1 Motivation

In the realm of theoretical physics, the quest for a unified theory that reconciles the pillars of modern physics, general relativity for large scales, and quantum mechanics for subatomic particles, remains an enduring challenge. This search is driven by the need to understand the behavior of matter and energy at the most fundamental level, especially in extreme conditions. Typical examples of such scenarios are the earliest moments of the universe or inside black holes: massive objects characterized by an event horizon that forms, for instance, when the size of a collapsing star falls below a limiting radius. Both examples exhibit a singularity, a point in spacetime at which curvature becomes infinite. In cosmology, this corresponds to the big-bang, the initial point in time where the entire universe was in an infinitely compact state. Black hole singularities, on the other hand, can be generated during the formation of such massive objects upon gravitational collapse [8]. It is exactly close to these singular points where Einstein's classical theory of gravity breaks down, exhibiting the urgent need for quantum gravity theories.

One of the most promising candidates in this direction is string theory [9, 10], a framework that originated in the late 1960s in the context of nuclear physics as a model for the strong force. After being abandoned in favor of quantum chromodynamics (QCD), it was realized that string theory was in fact a theory of gravity, not a theory of hadrons. The classical theory describes one-dimensional objects, the strings, propagating in a D-dimensional spacetime, also called target space. The dynamics of such extended objects is obtained from the so-called worldsheet action, that parameterizes the area covered by the string while moving in space and time. This two-dimensional surface can have different topologies, characterized by the number of "holes" on it. This leads to a genus expansion weighted by the string coupling constant g_s, in which the leading order has no holes and represents the classical limit of the theory. g_s corrections to the tree-level action correspond to quantum string loop effects.

Strings can have bosonic or fermionic degrees of freedom, they can be closed or open, oriented or unoriented. Each of these choices leads to a different theory, but they all share several common features. One of these features is the presence of a critical dimension for the target space, required for consistency of the theory. This is $D = 26$ for bosonic string or $D = 10$ for superstring theory. The latter incorporates fermionic degrees of freedom through supersymmetry [11], a symmetry that exhchanges bosons with fermions. Strings in non-critical dimensions are also possible and will be explained in detail later on.

From a first quantization of the free classical theory one can determine its spectrum, consisting of infinitely many string excitations. From the point of view of the target space, each of these excitations corresponds to a different field associated with a different point-particle. The spectrum contains a massless sector as well as an infinite tower of massive fields, whose masses are inversely proportional to the length of the string $\ell_s \sim 10^{-32}$ cm. The latter is usually expressed in terms of $\alpha' \sim \ell_s^2$, which together with g_s are the only free parameters of the theory! While the spectrum depends on each string theory, they all share a common universal massless sector composed of three states: the graviton, B-field, and dilaton. From the point of view of the target space, the graviton is a symmetric spin-two particle that plays the role of the spacetime metric, the B-field is a two-form, and the dilaton is a scalar field, whose vacuum expectation value ϕ_0 is related to the string coupling via $g_s = e^{\phi_0}$.

The connection to the standard field-theory picture can be obtained from two different yet equivalent approaches. One is via string scattering amplitudes, and the other one through a renormalization group beta function computation. The latter consists on demanding quantum consistency of the worldsheet theory in the following way [12–16]: the fields associated with the string's spectrum can be used as sources of curvature for the same spacetime where the string propagates. The resulting theory enjoys conformal invariance at the classical level but this is not preserved quantum-mechanically unless certain constraints are imposed on the background fields [17–19]. These constraints, being perturbative in α', are nothing but the fields' equations of motion, that are associated with target-space actions. These are the so-called string low energy effective theories. They are standard D-dimensional field theories incorporating novel string effects, two of the most important ones being the presence of the so-called duality groups and of infinitely many higher-derivative terms.

Duality groups can be explained by recalling the higher dimensional nature of the target space, a fact that makes the connection to phenomenology not immediate. Such a connection can be achieved by means of a mechanism called dimensional reduction or compactification. Here, a theory in D dimensions, dubbed parent theory, is related to an effective one in $n < D$. The remaining $d = D - n$ "extra" dimensions form the so-called internal space. The physics of the effective theory, heavily depends on how extra dimensions are curled-up in the internal space. In particular, when the theory possesses d abelian isometries, meaning that effective fields do not depend on the internal coordinates, a global $O(d, d|\mathbb{R})$ symmetry group emerges [20–24]. This is a target-space manifestation of a string phenomenon called T-duality [25], related to the intrinsic one-dimensional nature of strings. This symmetry can be used to constrain the couplings of the effective action, a feature that motivated the development of Double Field Theory (DFT) [26–28], an $O(D, D|\mathbb{R})$-invariant reformulation of the effective theory.

Higher-derivative terms, on the other hand, are a consequence of an infinite perturbative expansion in α', where the order α'^n of the power series contains couplings with $2(n+1)$ derivatives. These α' (string) corrections to the two-derivative theory represent a significant departure from traditional field theories. By sending α' to zero, we are in the low energy limit where string effects are suppressed. In this limit, the associated equations of motion for the massless spin-two field coincide with Einstein's equations and so general relativity emerges as a prediction of string theory from quantum considerations! String low energy effective theories are therefore theories of gravity, where the other fields of the string spectrum play the role of the matter content. In particular, effective theories coming from the superstring are supersymmetric extensions of gravity, known as supergravities, discovered independently to string theory [29, 30]. By turning on α' we get an infinite tower of higher-derivative corrections to standard (super)gravities. For the purely metric sector, for instance, these include Riemann square terms at order α'. These corrections are not unique, they are defined up to α'-perturbative field redefinitions. While theories in different field bases are physically equivalent, depending on the problem, computations can be easier in one frame or another. In practice, these field redefinitions are used to remove ambiguities or redundancies of the theory.

The complexity of the processes to get the string effective theories when going to higher orders in perturbation theory grows considerably fast. As a consequence, the string corrections have been determined only to a few orders in α' [29, 31–38], and a computation to all orders is out of reach at the moment. This represents a huge obstacle in understanding the scope of string theory as a quantum theory of gravity. Indeed, incorporating more orders in the series expansion allows us to probe string theory at smaller scales, going beyond classical gravity and its limitations. Theoretically, by including the infinite tower of corrections one would probe the spacetime at the string scale. It becomes imperative to have a full control of α' corrections in order to understand the implications of string theory at the extreme regimes where general relativity breaks down.

A promising direction for getting an α'-complete formulation is to use T-duality to constrain the possible higher-derivative couplings of the effective theories. This relies on a result due to Sen, that T-duality should be preserved to all orders in α' [23]. We currently know two successful examples exploiting such technique: the Hohm-Siegel-Zwiebach (HSZ) theory [26] and the cosmological classification of [39, 40]. While these are very exciting news, it is worth emphasizing that those models provide us with a whole class of string-inspired duality-invariant theories. We know string theory should be a single point in this theory space, although determining its exact location is an open problem.

HSZ is a spacetime theory based on double field theory, encoding the dynamics of the universal massless sector of string theory, as well as novel massive modes. Its construction was based on a non-standard chiral Conformal Field Theory (CFT) and so it does not correspond to any conventional string theory. Nevertheless, HSZ theory shares crucial features of any string theory, such as duality invariance under $O(d,d|\mathbb{R})$ for backgrounds with d abelian isometries, and the presence of infinitely many higher-derivative corrections for the massless fields, obtained upon integrating out the massive modes. The resulting effective theory coincides with string low energy effective theories only at zero order in α'. If instead of integrating out the massive fields one keeps them, HSZ carries only a finite number of derivatives and it serves as an α'-exact toy model for an effective theory with string massive modes. Such construction becomes very useful in a context where incorporating genuine string massive modes into a target-space description is an open problem.

The cosmological classification is very different in nature to HSZ. It does not correspond to a single theory, but it parameterizes a hole class of models. The idea of this proposal is to use field redefinitions in a systematic way together with T-duality to constrain possible terms of the universal massless sector in cosmological backgrounds. The latter are D-dimensional target spaces in which fields depend on a single time-like coordinate (see [41] for a review on string cosmology). The presence of $O(d,d|\mathbb{R})$-invariance in such backgrounds was corroborated to zeroth order in α' by Meissner and Veneziano in [21, 22], generalizing the scale-factor duality previously found in [20]. Duality invariance was latter checked also at first order in α' in [42], a work that served as motivation for the systematic approach developed in [39] and refined in [40]. The final outcome of those works was a classification to all orders in α', where the action and equations of motion are presented in a fully controllable way, parameterized by a set of countable infinite number of coefficients. Each choice of coefficients corresponds to a different theory in this larger class of duality-invariant cosmological backgrounds, in which string theory represents a single point. Interestingly, the massless sector of HSZ in cosmological backgrounds can also be parameterized by these coefficients. Although the success of the classification relies heavily on the simplicity of cosmological backgrounds, the latter are highly interesting from a phenomenological perspective. They encode Bianchi Type I (BI) universes, described by a homogeneous yet anisotropic metric, of which Friedmann-Robertson-Walker (FRW) backgrounds are particular cases.

These two examples point towards a new program to study quantum theories of gravity. Instead of restricting to string theory, we can change gears and work directly with a bigger space of duality-invariant theories as a theoretical framework to go beyond classical gravity. String theory would represent a particular case of these string-inspired models. With this setup, several research directions are possible. Some of them are introduced in the next section, forming the main research goals of the current work. Some others will be mentioned in Chapter 6 as proposals for follow-up projects.

1.2 Research Goals and Main Results

The main goal of the thesis can be stated in a broad sense as follows: *we want to get closer to a better understanding of string corrections to classical* *gravity and their consequences at scales where general relativity is insufficient.* Obviously, this is too broad for any practical purpose. Since string theory is so vast, however, in order to formulate more concrete research goals we need to restrict the study in several directions. These restrictions will hold all over the thesis but we expect a successful program would require many of them to be relaxed in the future.

- ⊙ We restrict to classical string α' corrections at tree-level in the genus (g_s) expansion. In particular, string quantum effects are not taken into account.
- ⊙ We only deal with bosonic degrees of freedom and we focus on the string universal massless sector containing the graviton, B-field and dilaton.
- ⊙ We are mainly interested in D-dimensional spacetimes with $D-1$ abelian isometries. These backgrounds are the ones in which a classification is currently understood and, despite their simplicity, they contain many phenomenologically relevant scenarios such as cosmological backgrounds [20, 21, 43] and black holes [44–46].

Within this constrained framework, the broad objective stated above can be split into more concrete sub-projects:

- ⊙ Compute the exact coefficients of the cosmological classification coming from string theories.
- ⊙ Solve the α'-complete equations of motions of the larger class of duality-invariant theories and study the implications of these corrected solutions in spacetime regions that are inaccessible in standard gravity ($\alpha' \to 0$ limit).
- ⊙ Extend the program of the cosmological classification to other backgrounds using the same duality-invariant principle.

We now expose the main results of the current work, that allowed us to make progress in all of the above mentioned research directions:

- ⊙ We compute the first coefficients of the classification coming from string theories from two different perspectives.
  - One approach begins from the D-dimensional target-space actions of bo-sonic, heterotic and type II strings for dilaton-gravity backgrounds up to order α'^3. Such string corrected theories were already known in the literature, obtained from beta-function and/or S-matrix calculations. We compactify those theories to cosmological backgrounds and follow the systematic procedure developed in [40] to corroborate they are compatible with an $O(d,d|\mathbb{R})$ duality group and determine the coefficients up to order α'^3 [1, 3]. Following these lines, we take advantage that HSZ is known to all orders in α' and so we perform a cosmological reduction, show consistency of the truncation, and integrate out the massive fields using an iterative procedure to read the coefficients up to order α'^4 [5].

– This target-space approach to determine the coefficients requires one to know the parent theory up to the desired order before compactifying, which in general involves a very complicated worldsheet computation. With the aim of reducing the complexity of the problem, and anticipating that we are ultimately interested in the cosmological classification, we propose to work directly with a worldsheet theory in cosmological backgrounds and perform a beta-function computation from there [4]. Although not $O(d,d|\mathbb{R})$ invariant, this procedure allows one to efficiently determine the $O(d,d|\mathbb{R})$-invariant beta functions. Unfortunately, such procedure turned out not to be simple enough to extend the state-of-the-art results. We managed to perform a two-loop computation for the bosonic string, obtaining the coefficients up to order α', which were in agreement to the ones computed previously in [3, 39, 42].

- ⊙ As a toy model for an α'-exact string effective theory incorporating massless and massive modes, we explore HSZ in cosmological backgrounds keeping the massive fields [5]. We give a two-derivative reformulation of the theory and arrive at a set of α'-exact Friedmann equations, which are then ordinary second order differential equations. We explore the so-called tensionless limit $\alpha' \to \infty$, a limit that can only be taken after bringing the theory to a two-derivative formulation, having no analog in any other target-space description of string theory. We find string frame de Sitter vacua, and we set up perturbation theory in $\frac{1}{\alpha'}$.

- ⊙ We revisit the systematic approach of [40] for Bianchi Type I universes, a particular class of cosmological backgrounds. Surprisingly, we find [6] that when the anisotropic metric is described by q different scale factors, field redefinitions can be used in a systematic way to arrive at an equivalent theory in which only $q-1$ of them are present. As a corollary, when all scale factors are identical, corresponding to a homogeneous FRW background, all α' corrections are trivial. It should be emphasized, however, that the removal of higher-derivative terms is strictly perturbative, so that there may be non-perturbative solutions (such as those found in [40, 47]) that are not accessible for classifications of very restrictive backgrounds. More generally, perturbations or fluctuations away from a background may not preserve any conditions, as in cosmological perturbation theory, where the fluctuations depend on all coordinates.

- ⊙ With the aim of extending the classification to other backgrounds, we study the subject of higher-derivative modifications of string theory in non-critical dimensions [6], with a particular focus on the two-derivative black hole solution in $D = 2$[1] [44–46]. We find that the traditional perturbative mindset of critical dimensions does not hold in the non-critical case. All terms in the spacetime action, other than a cosmological-like term, are field redefinition equivalent to terms with arbitrarily many

[1]In string theory usage, the black hole background is generally considered a critical string theory, as one is working directly with a theory of matter central charge 26. The name non-critical strings is reserved for non-conformal field theories coupled to two-dimensional gravity, in which case the Liouville mode of the metric helps restore conformal invariance.

derivatives, with the latter generally of the same order. We circumvent this obstacle by assuming an α' expansion with coefficients that fall off sufficiently fast. Considering field redefinitions consistent with this fall-off structure, we classify the most general duality-invariant higher derivative terms for $D = 2$ theories with one time-like isometry.

⊙ With this most general duality-invariant theory at our disposal, and knowing that the two-derivative truncation leads to a black hole background, we look for α' modifications of black-hole solutions. We analyze these corrections from a perturbative as well as from a non-perturbative perspective. For the former we provide a systematic tool to get corrections to all orders in derivatives [6]. For the non-perturbative approach [7], we use a parametrization recently introduced by Gasperini and Veneziano [48] to find some solutions that are incompatible with a black hole interpretation but also solutions that are α'-deformed black holes where the horizon and singularity are preserved. Furthermore, we find subregions in this duality-invariant theory space, probably not containing string theory, in which the black hole geometry exhibits a horizon leading to an interior that, having no singularity in the metric, curvature, or dilaton, is a regular cosmology. In the process of getting these corrected black hole solutions, we prove that in two-dimensional duality-invariant backgrounds, the dual of a black hole solution with a regular horizon must have a curvature singularity.

1.3 Outline

The reminder of the thesis is organized as follows:

In **Chapter 2** we present an introduction to string low energy effective theories and their most relevant features for the thesis. In here we take the target-space description as starting point, giving no details of how these theories were obtained from the string worldsheet. (The latter discussion is postponed until Chapter 5.) We begin in Section 2.1 with an introduction to the universal bosonic massless sector of supergravity in critical dimensions. After revisiting the two-derivative theory, we study higher-derivative corrections and the important role of field-redefinitions in α'-corrected theories. We summarize the state-of-the-art results for dilaton-gravity theories up to and including α'^3 effects, which then will be used in Chapter 3 to get the full massless sector in cosmological backgrounds. We finish Section 2.1 with a discussion on field redefinitions and higher-derivative corrections for backgrounds in non-critical dimensions. In Section 2.2 we introduce the concepts of compactification and duality focusing on two-derivative theories with d abelian isometries, leading to the celebrated $O(d, d|\mathbb{R})$ duality symmetry. After recovering the well-known Maharana-Schwarz action, we present the cosmological and black-hole backgrounds as particular cases of these toroidal compactifications. Last but not least, Section 2.3 makes use of the duality group $O(d, d|\mathbb{R})$ to introduce DFT and HSZ theory as duality invariant completions of supergravities. In particular, we concentrate on HSZ and compactify the theory to cosmological backgrounds.

In **Chapter 3** we begin with Section 3.1 introducing the systematic of field redefinitions in an abstract fashion, and then we specialize it to the string universal massless sector to arrive at the cosmological classification developed in [40]. Restricting the space of backgrounds even further, we finalize this section revisiting the classification for Bianchi Type I universes with q different scale factors. We show that for one of these scale factors all higher-derivative terms can be removed by field redefinitions, which implies that FRW backgrounds receive no α' corrections. In Section 3.2 we take the α'^3 dilaton-gravity parent theories collected previously in Section 2.1.3, apply a cosmological reduction, and bring the theory to canonical form. We finish the chapter with Section 3.3, where we study HSZ theory in cosmological backgrounds from two different perspectives. We first integrate out the auxiliary fields and then bring the resulting effective action for the massless fields to the minimal basis of the cosmological classification up to and including order α'^4. We then keep the massive modes, bring the theory to a two-derivative reformulation, take the tensionless limit $\alpha' \to \infty$, and find solutions in FRW-like backgrounds upon perturbation theory in $\frac{1}{\alpha'}$.

Chapter 4 deals with two-dimensional black hole backgrounds. We begin in Section 4.1 by reviewing the two-derivative solution. We then revisit the problem of higher-derivative corrections being intrinsically non-perturbative in non-critical dimensions. We sort-out the problem by assuming a fall-off structure of the coefficients next to higher-derivative terms. Under such assumption, we classify the most general duality-invariant theory, in the same spirit of the cosmological classification [40, 47]. In Section 4.2 we build solutions of the α'-exact set of equations we just found. We do it in a perturbative and a non-perturbative fashion. For the latter, we introduce the Gasperini-Veneziano parameterization, from which we recover the standard two-derivative black hole and then build a whole family of α'-corrected singular black hole solutions. We finish in Section 4.3 studying the singularity problem on $D = 2$ black hole backgrounds. We demonstrate that, in duality-invariant theories, a solution with a regular horizon implies a singularity. We finish by building α'-exact regular black hole solutions whose interior regions are regular cosmologies.

In **Chapter 5** we change gears to work at the level of the string worldsheet, and discuss the beta-function approach to derive the corresponding target-space theories. We begin with a review section in 5.1, introducing the bosonic string worldsheet (Polyakov) action and the concepts of Weyl anomaly coefficients, beta functions and background field method, for generic renormalizable two-dimensional sigma models. We then work out a specific example by computing the one-loop beta function for dilaton-gravity worldsheet and obtaining the corresponding two-derivative target-space action. In Section 5.2 we perform a cosmological reduction of the Polyakov action for purely metric backgrounds. After revisiting the Weyl anomaly and the background field method in the context of cosmological backgrounds, we compute the one-loop beta functions and derive the two-derivative cosmological action. In Section 5.3 we extend the previous analysis to two loops. After sorting out various complications that arise at higher-orders, we derive the target-space cosmological action for bosonic string up to order α'.

We finish in **Chapter 6** with some conclusions and proposals for follow-up projects.

Chapter 2

String Target-Space Theories

Starting from a worldsheet action one can get the low energy effective theory of strings via string scattering amplitudes or beta-function computations. In this chapter we begin directly at the level of the target-space, postponing its derivation from the beta-function approach until Chapter 5. We introduce the concepts of α' corrections, field redefinitions, dimensional reduction, and T-duality, which will prove essential for the development of the main results of the thesis in following chapters.

While this chapter is mainly dedicated to review known results from the literature, some computations of Section 2.1.3 are taken from [1] and [3], the results of Section 2.1.4 come from [6] and Section 2.3.2 contains results from [5].

2.1 Gravity Theories and α' Corrections

Demanding consistency of the worldsheet action at the quantum level [12–16] imposes constraints on the backgrounds fields [17–19]. These can be interpreted as the equations of motion coming from an effective theory of the D-dimensional target space, whose field content depends on the background to which we coupled the string. In particular, we can consider the universal massless sector, composed of the graviton $G_{\mu\nu}(X)$, B-field $B_{\mu\nu}(X)$ and dilaton $\phi(X)$, which is a subsector of the infinite string spectrum. In here, X^μ with $\mu, \nu = 0, 1, \ldots, D-1$ are the target-space coordinates.

Since the anomaly cancellation mechanism used to arrive at the target-space equations is perturbative in α', this produces an infinite number of higher-derivative corrections in the effective theory where couplings with $2(p+1)$ derivatives sit next to an α'^p factor. In this section we consider dimension D corresponding to the string critical dimension, except for Section 2.1.4 where the non-critical case is addressed.

2.1.1 Two-derivative theory

At lowest (zeroth) order in α' we obtain a two-derivative theory given by

$$\mathcal{I}^{(0)} \equiv \int d^D X \sqrt{-G} e^{-2\phi} \mathcal{L}^{(0)} , \quad \mathcal{L}^{(0)} \equiv \mathcal{R} + 4\, \partial_\mu \phi \partial^\mu \phi - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} , \tag{2.1.1}$$

where $\mathcal{R}$ is the Ricci scalar for the D-dimensional metric $G_{\mu\nu}$, and $H_{\mu\nu\rho} = 3\partial_{[\mu}B_{\nu\rho]}$ is the field-strength of the 2-form. The second and third couplings of the Lagrangian $\mathcal{L}^{(0)}$ correspond to the kinetic terms for the dilaton and B-field respectively. The first term is the Einstein-Hilbert action of general relativity, making explicit a statement of the introduction: Einstein's theory of gravity emerges as a prediction of string theory. In the measure, G stands for the determinant of the metric, and the $e^{-2\phi}$ factor defines the theory in the so-called string-frame. By a suitable field redefinition of the metric, the theory can be rewritten in the more familiar Einstein-frame, where the exponential factor is absent and the Einstein-Hilbert action takes its standard form. The action (2.1.1) corresponds to the bosonic sector of supergravity, also known as $\mathcal{N}=0$ supergravity.

The equations of motion coming from (2.1.1) can be obtained by varying the action with respect to $G_{\mu\nu}, B_{\mu\nu}$ and ϕ

$$\delta\mathcal{I}^{(0)} = \int d^D X \sqrt{-G} e^{-2\phi}\left[E^{(0)}_{G,\mu\nu}\delta G^{\mu\nu} + E^{(0)}_{\phi}\delta\phi + E^{(0)}_{B,\mu\nu}\delta B^{\mu\nu}\right] = 0\,, \tag{2.1.2}$$

with

$$E^{(0)}_{G,\mu\nu} = \mathcal{R}_{\mu\nu} + 2\nabla_\mu\nabla_\nu\phi - \frac{1}{4}H_{\mu\rho\sigma}H_\nu{}^{\rho\sigma} + \frac{1}{4}G_{\mu\nu}E^{(0)}_\phi = 0\,, \tag{2.1.3a}$$

$$E^{(0)}_{\phi} = -2\mathcal{R} + 8\nabla_\mu\phi\nabla^\mu\phi - 8\nabla_\mu\nabla^\mu\phi + \frac{1}{6}H_{\mu\nu\rho}H^{\mu\nu\rho} = 0\,, \tag{2.1.3b}$$

$$E^{(0)}_{B,\mu\nu} = \frac{1}{6}\nabla_\rho H_{\mu\nu}{}^\rho - \frac{1}{3}\nabla_\rho\phi H_{\mu\nu}{}^\rho = 0\,, \tag{2.1.3c}$$

where we use the following convention for the covariant derivative: $\nabla_\mu V^\mu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\rho}V^\rho$. Here $\Gamma^\rho_{\mu\nu}$ are the familiar Christoffel symbols and V^μ a generic spacetime vector. The action (2.1.1) and its equations of motion (2.1.3) are invariant under D-dimensional diffeomorphisms and B-field gauge transformations. Denoting collectively $\Psi \in \{G_{\mu\nu}, B_{\mu\nu}, \phi\}$, these transformations act infinitesimally as

$$\Psi \to \Psi' = \Psi + \delta\Psi\,, \tag{2.1.4}$$

with

$$\delta G_{\mu\nu} = \mathcal{L}_\xi G_{\mu\nu}\,, \quad \delta B_{\mu\nu} = \mathcal{L}_\xi B_{\mu\nu} + 2\partial_{[\mu}\lambda_{\nu]}\,, \quad \delta\phi = \mathcal{L}_\xi\phi\,, \tag{2.1.5}$$

where λ_μ parameterizes B-field gauge transformations, and ξ^μ generates infinitesimal coordinate transformations through the ordinary Lie derivative $\mathcal{L}_\xi$, under which $G_{\mu\nu}$ and $B_{\mu\nu}$ behave as 2-tensors and ϕ is a scalar. On an arbitrary vector V^μ, the Lie derivative acts as follows:

$$\mathcal{L}_\xi V^\mu = \xi^\nu\partial_\nu V^\mu - V^\nu\partial_\nu\xi^\mu\,. \tag{2.1.6}$$

2.1.2 α' corrections and field redefinitions

From the point of view of string low energy effective actions, (2.1.1) is just the leading term in an infinite higher-derivative expansion. The generic structure for different string theories is the same, and is given by

$$\mathcal{I} \equiv \sum_{p\geq 0} \alpha'^p \mathcal{I}^{(p)} \,, \quad \mathcal{I}^{(p)} \equiv \int d^D X \sqrt{-G} e^{-2\phi} \mathcal{L}^{(p)} \,, \tag{2.1.7}$$

where $\mathcal{L}^{(p)}$ contains gauge-invariant terms with $2(p+1)$ derivatives. While (2.1.1) is common to all strings, the specific form of $\mathcal{L}^{(p)}$ for $p > 0$ do depend on the theory under consideration and currently they are just partially known. Only the first few orders were obtained via string scattering or beta-function techniques and, for some theories, they are only known for a subsector of the massless fields. Moreover, as opposed to the zeroth order action, these higher-derivative corrections are not unique, but they are defined up to α'-perturbative field redefinitions. Therefore, two Lagrangians that look very different, having different couplings and numerical coefficients, can in fact describe the same physical system. There is no preferred field basis in general but, depending on the problem at hand, computations can be easier in one or another.

Since field redefinitions play a crucial role for this thesis, it is worth dedicating some time here to understand how they work. To this end, consider (2.1.7) up to order α' only, namely

$$\mathcal{I} = \mathcal{I}^{(0)} + \alpha' \mathcal{I}^{(1)} + \mathcal{O}(\alpha'^2) \,. \tag{2.1.8}$$

Here, the zeroth order is given by (2.1.1) and $\mathcal{I}^{(1)}$ is totally generic, which means that $\mathcal{L}^{(1)}$ contains all possible gauge-invariant four-derivative couplings, each parameterized by a different arbitrary coefficient. Specifically,

$$\mathcal{L}^{(1)} = a_1 \mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma} + a_2 \mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu} + a_3 \mathcal{R}^2 + a_4 H_{\mu\nu\rho}H^{\mu\nu\rho}\nabla_\sigma\nabla^\sigma\phi + \cdots \,, \tag{2.1.9}$$

where a_i are arbitrary constant coefficients, and we displayed just four examples explicitly and hide all other couplings in the ellipsis. By using integration by parts and Bianchi identities of the Riemann tensor, one can show that there are only 20 inequivalent gauge-invariant four-derivative couplings one can write [36]. Obviously, not every choice of a_i leads to a consistent string theory, the latter is attained by specific values of the coefficients, determined by beta-function or S-matrix calculations. We keep (2.1.9) generic because we do not need specific values for the parameters in order to explore how field redefinitions transform the action.

These transformations cannot change (2.1.9) arbitrarily because there are certain invariant quantities that are preserved under field redefinitions. To formalize this statement, we introduce the notion of ambiguous and unambiguous coefficients. The former corresponds to coefficients that can be changed by field redefinitions and the latter to the ones that cannot. Let's consider an example by applying the following pure-graviton field redefinition

$$G_{\mu\nu} \to G'_{\mu\nu} = G_{\mu\nu} + \alpha' \delta G_{\mu\nu} \,, \tag{2.1.10}$$

after which the α'-corrected theory changes as

$$\begin{aligned} \mathcal{I} \to \mathcal{I}' &= \mathcal{I}^{(0)'} + \alpha' \mathcal{I}^{(1)'} + \mathcal{O}(\alpha'^2) \\ &= \mathcal{I}^{(0)} + \alpha' \int d^D X \sqrt{-G} e^{-2\phi} \left[\mathcal{L}^{(1)} + E^{(0)}_{G,\mu\nu} \delta G^{\mu\nu} \right] + \mathcal{O}(\alpha'^2) \,, \end{aligned} \tag{2.1.11}$$

where we are neglecting higher orders in α' and so the effect of (2.1.10) is to shift $\mathcal{L}^{(1)}$ in (2.1.9) by a factor $E^{(0)}_{G,\mu\nu}\delta G^{\mu\nu}$, where $E^{(0)}_{G,\mu\nu}$ is the equation of motion defined in (2.1.3a). Making the choice

$$\delta G^{\mu\nu} = -a_2\mathcal{R}^{\mu\nu}\,, \tag{2.1.12}$$

the shift takes the form

$$\begin{aligned} E^{(0)}_{G,\mu\nu}\delta G^{\mu\nu} = -a_2\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu} - a_2\Big[2\nabla_\mu\nabla_\nu\phi\mathcal{R}^{\mu\nu} - \frac{1}{4}H_{\mu\rho\sigma}H_\mu{}^{\rho\sigma}\mathcal{R}^{\mu\nu} - \frac{1}{2}\mathcal{R}^2 \\ + 2\nabla_\mu\phi\nabla^\mu\phi\mathcal{R} - 2\nabla_\mu\nabla^\mu\phi\mathcal{R} + \frac{1}{24}H_{\mu\nu\rho}H^{\mu\nu\rho}\mathcal{R}\Big]\,. \end{aligned} \tag{2.1.13}$$

It is then easy to see that (2.1.13) allows us to remove the $a_2\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu}$ coupling in (2.1.9), and trade it for terms with other tensor structures. By applying the same trick with specific combinations of $\delta G_{\mu\nu}$ and $\delta\phi$, one can remove any Ricci-tensor-dependent coupling. Such procedure is a bit more tedious, however, requiring integration by parts and the use of Bianchi identities for the Riemann tensor.

The upshot of this procedure is that by a suitable α' field redefinition of the metric, we can exchange any coupling containing $\mathcal{R}_{\mu\nu}$ from $\mathcal{L}^{(1)}$ for couplings which do not depend on the Ricci tensor at that order in α'. This comes at expenses of inducing $\mathcal{O}(\alpha'^2)$ terms, which we can neglect if we are concerned with $\mathcal{O}(\alpha')$ effects only and perturbation theory holds[1]. Connecting to the definitions introduced above, the coefficients next to couplings with Ricci tensors are ambiguous coefficients. It can be shown that the same holds for terms containing Ricci scalars and dilaton couplings. Moreover, 17 of the 20 coefficients in (2.1.9) are ambiguous [36]. On the other hand, there are no two-derivative covariant field redefinitions of the form

$$G'_{\mu\nu} = G_{\mu\nu} + \alpha'\delta G_{\mu\nu}\,,\quad B'_{\mu\nu} = B_{\mu\nu} + \alpha'\delta B_{\mu\nu}\,,\quad \phi' = \phi + \alpha'\delta\phi\,, \tag{2.1.14}$$

that can change the coefficients next to the couplings: $\mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma}$, $\mathcal{R}^{\mu\nu\rho\sigma}H_{\mu\nu\lambda}H_{\rho\sigma}{}^\lambda$ and $H^{\mu\nu\rho}H_{\mu\sigma}{}^\lambda H_{\nu\lambda}{}^\delta H_{\rho\delta}{}^\sigma$. These are the only three unambiguous coefficients in (2.1.9).

From our previous example, we see we can use field redefinitions to "remove" any term containing Ricci tensors. We write remove between quotes because we do not really eliminate the terms, but we trade them for other couplings. However, since the original Lagrangian (2.1.9) was totally generic, these redefinitions just renormalize other already-present coefficients and so the net effect is to set the coefficient next to the Ricci-tensor-dependent term to zero. This same observation holds for any of the 17 ambiguous coefficients: by suitable field redefinitions we can set them to zero at expenses of renormalizing other coefficients in (2.1.9). However, we cannot set all of them to zero simultaneously, because the field redefinition required to remove one, could make others non-vanishing. How many of these 17 ambiguous coefficients can we set to zero simultaneously? To answer this, we introduce the notion of essential coefficients, which are specific combinations of the ambiguous coefficients

[1]This last condition does not hold in the non-critical case, as we will see in Section 2.1.4!

that remain invariant under field redefinitions. Then, by field redefinitions, any action can be brought to a minimal basis containing as many couplings as unambiguous plus essential coefficients. In the case of (2.1.9), it can be shown that there are only 5 essential coefficients and so we can pick a basis such that (2.1.9) takes the minimal form [36]

$$\begin{aligned}\mathcal{L}^{(1)} = \; & a_1 \mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma} + a_2 \mathcal{R}^{\mu\nu\rho\sigma} H_{\mu\nu\lambda} H_{\rho\sigma}{}^{\lambda} + a_3 H^{\mu\nu\rho} H_{\mu\sigma}{}^{\lambda} H_{\nu\lambda}{}^{\delta} H_{\rho\delta}{}^{\sigma} \\ & + a_4 H_{\mu\rho\sigma} H_{\nu}{}^{\rho\sigma} H^{\mu\lambda\delta} H^{\nu}{}_{\lambda\delta} + a_5 H_{\mu\rho\sigma} H_{\nu}{}^{\rho\sigma} \nabla^{\mu}\phi \nabla^{\nu}\phi + a_6 \left(H_{\mu\nu\rho}H^{\mu\nu\rho}\right)^2 \\ & + a_7 H_{\mu\nu\rho} H^{\mu\nu\rho} \nabla_{\sigma}\phi \nabla^{\sigma}\phi + a_8 \left(\nabla_{\mu}\phi\nabla^{\mu}\phi\right)^2 \,,\end{aligned} \tag{2.1.15}$$

where the first line contains the unambiguous coefficients mentioned above.

Given a minimal basis, all the physical information of the theory is encoded in the essential plus unambiguous coefficients. For instance, the coefficients for bosonic, heterotic and type II strings for the whole massless sector are very well known, and the Lagrangian $\mathcal{L}^{(1)}$ can be written in a unified form for all strings as [36, 49, 50]

$$\begin{aligned}\mathcal{L}^{(1)} = \frac{a+b}{8} \Big[& \mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma} - \frac{1}{2}\mathcal{R}^{\mu\nu\rho\sigma} H_{\mu\nu\lambda} H_{\rho\sigma}{}^{\lambda} \\ & + \frac{1}{24} H^{\mu\nu\rho} H_{\mu\sigma}{}^{\lambda} H_{\nu\lambda}{}^{\delta} H_{\rho\delta}{}^{\sigma} - \frac{1}{8} H_{\mu\rho\sigma} H_{\nu}{}^{\rho\sigma} H^{\mu\lambda\delta} H^{\nu}{}_{\lambda\delta}\Big] + \frac{b-a}{4} H^{\mu\nu\rho}\Omega_{\mu\nu\rho} \,.\end{aligned} \tag{2.1.16}$$

Here $\Omega_{\mu\nu\rho}(\omega) = \text{Tr}\left(\omega_{[\mu}\partial_{\nu}\omega_{\rho]} + \frac{2}{3}\omega_{[\mu}\omega_{\nu}\omega_{\rho]}\right)$ is the Lorentz Chern-Simons three-form for the spin connection $\omega_{\mu\alpha}{}^{\beta} = e_{\alpha}{}^{\nu}\nabla_{\mu}e_{\nu}{}^{\beta}$. The latter is defined in terms of the vielbein $e_{\mu}{}^{\alpha}$, such that $G_{\mu\nu} = e_{\mu}{}^{\alpha}\eta_{\alpha\beta}e_{\nu}{}^{\beta}$ with $\eta_{\alpha\beta}$ being the flat Minkowski metric. By choosing different values for a and b in (2.1.16) we reach different string theories. The bosonic case corresponds to $a = b = 1$, for heterotic string we have $a = 1, b = 0$ and for type II they both vanish: $a = b = 0$. Comparing the terms in bracket of (2.1.16) with (2.1.15) we can read the coefficients a_i as fixed by string theory computations, where a_i with $i > 4$ are zero. The last term in (2.1.16) is only present in heterotic string theory and it originates from a Green-Schwarz mechanism that modifies the B-field Lorentz transformation [51].

As can be seen from (2.1.16), the higher we go in α' the more complex the corresponding higher-derivative Lagrangian gets. This is a direct consequence of having more ways of combining the fundamental building blocks like $\mathcal{R}_{\mu\nu\rho}$, $H_{\mu\nu\rho}$ and $\nabla_{\mu}\phi$. For instance, at order α'^2 a generic effective Lagrangian $\mathcal{L}^{(2)}$ restricted to the dilaton-gravity subsector (i.e. $B_{\mu\nu} = 0$), contains 44 independent couplings (up to integration by parts and Bianchi identities). Among these 44 arbitrary coefficients, 42 are ambiguous, 2 are unambiguous and 5 are essential [52]. Therefore, field redefinitions allow us to set 37 out of 44 coefficients to zero while the remaining 7 coefficients parameterize the dilaton-gravity theory at order α'^2, which must be fixed by string computations. These numbers keep growing for higher orders in α' until a point where, not only the coefficients coming from string calculations are unknown, but the very classification of a minimal basis becomes intractable! This is a consequence of the huge number of integration by parts identities, Bianchi identities, and field redefinitions one can use to connect physically-equivalent theories. What we lack here is a systematic implementation of these identities. We will see later in Chapter 3 how this problem was solved by Hohm and Zwiebach for cosmological backgrounds, for which they provided a classification to all orders in α'.

2.1.3 Dilaton-gravity theories at order α'^3

Even though a systematic procedure is absent for generic D-dimensional backgrounds, the current state of the art for string low energy effective actions gets up to order α'^3, where (2.1.7) takes the form

$$\mathcal{I} = \mathcal{I}^{(0)} + \alpha'\mathcal{I}^{(1)} + \alpha'^2\mathcal{I}^{(2)} + \alpha'^3\mathcal{I}^{(3)} + \mathcal{O}(\alpha'^4)\,, \tag{2.1.17}$$

and each action has its associated Lagrangian via (2.1.7).

Instead of considering the most general case in (2.1.17) we restrict to the dilaton-gravity subsector, truncating the B-field to zero. There are two reasons for this: first, we do not really have access to all B-field contributions up to order α'^3, since they are only partially known. Secondly, in this thesis we are ultimately interested in α' corrections in one dimensional effective backgrounds, and so we will show in Section 3.2 that the dilaton-gravity subsector is sufficient to reconstruct the full theory for cosmological backgrounds via duality invariance. On top of this truncation, we also omit dilaton contributions in $\mathcal{L}^{(3)}$ since they are currently unknown. As we will explain in Section 3.2, this will not be a problem when going to cosmological backgrounds because duality invariance allows us to still reconstruct the full theory up to order α'^3. Finally, since we are neglecting $\mathcal{O}(\alpha'^4)$ effects, we can forget about Ricci contributions in $\mathcal{L}^{(3)}$ because we can always redefine them away by appropriate field redefinitions. Apart from higher-order effects, these redefinitions produce dilaton couplings at order α'^3, which we are already ignoring.

All in all, we restrict to dilaton-gravity models of the form (2.1.17) with

$$\mathcal{L}^{(3)} = \text{Riemann terms} + \dots\,, \tag{2.1.18}$$

where $\dots$ account for terms containing dilaton couplings, Ricci tensors or Ricci scalars. Moreover, the zeroth and first order Lagrangians, (2.1.1) and (2.1.16), reduce to

$$\mathcal{L}^{(0)} = \mathcal{R} + 4\nabla_\mu\phi\nabla^\mu\phi\,, \tag{2.1.19a}$$

$$\mathcal{L}^{(1)} = \frac{\gamma}{4}\mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma}\,, \quad \text{with} \quad \gamma = \begin{cases} 1 & \text{Bosonic} \\ \frac{1}{2} & \text{Heterotic} \\ 0 & \text{Type II} \end{cases} \tag{2.1.19b}$$

For higher orders there is no simple unified formulation, so we present the theories for each string separately.

The effective action for Type II strings contain no order α' nor α'^2. The corrections to order α'^3 were computed from four-point scattering amplitudes in [31], and later from the sigma-model beta function in [29, 33, 34]. They take the compact form

$$J(c) \equiv t_8 t_8 \mathcal{R}^4 + \frac{c}{8} \epsilon_{10}\epsilon_{10}\mathcal{R}^4 \ , \tag{2.1.20}$$

where $c = 1$ has been determined in the literature, but here we keep it more general because we will see later in Section 3.2.2 that it can be fixed by duality arguments. The first term in (2.1.20) is

$$\begin{aligned} t_8 t_8 \mathcal{R}^4 &= t^{\mu_1\dots\mu_8} t_{\nu_1\dots\nu_8} \mathcal{R}_{\mu_1\mu_2}{}^{\nu_1\nu_2} \mathcal{R}_{\mu_3\mu_4}{}^{\nu_3\nu_4} \mathcal{R}_{\mu_5\mu_6}{}^{\nu_5\nu_6} \mathcal{R}_{\mu_7\mu_8}{}^{\nu_7\nu_8} \\ &= 3\cdot 2^7 \Big[\mathcal{R}_{\alpha\beta\mu\nu}\mathcal{R}^{\beta\gamma\nu\rho}\mathcal{R}^{\sigma\mu}{}_{\gamma\delta}\mathcal{R}^{\delta\alpha}{}_{\rho\sigma} + \frac{1}{2}\mathcal{R}_{\alpha\beta\mu\nu}\mathcal{R}^{\beta\gamma\nu\rho}\mathcal{R}_{\gamma\delta\rho\sigma}\mathcal{R}^{\delta\alpha\sigma\mu} \\ &\quad - \frac{1}{2}\mathcal{R}_{\alpha\beta\mu\nu}\mathcal{R}^{\beta\gamma\mu\nu}\mathcal{R}_{\gamma\delta\rho\sigma}\mathcal{R}^{\delta\alpha\rho\sigma} - \frac{1}{4}\mathcal{R}_{\alpha\beta\mu\nu}\mathcal{R}^{\beta\gamma\rho\sigma}\mathcal{R}^{\mu\nu}{}_{\gamma\delta}\mathcal{R}^{\delta\alpha}{}_{\rho\sigma} \\ &\quad + \frac{1}{16}\mathcal{R}_{\alpha\beta\mu\nu}\mathcal{R}^{\beta\alpha\rho\sigma}\mathcal{R}^{\gamma\delta\mu\nu}\mathcal{R}_{\delta\gamma\rho\sigma} + \frac{1}{32}\mathcal{R}_{\alpha\beta\mu\nu}\mathcal{R}^{\alpha\beta\mu\nu}\mathcal{R}_{\gamma\delta\rho\sigma}\mathcal{R}^{\gamma\delta\rho\sigma} \Big] \ , \end{aligned} \tag{2.1.21}$$

where the t_8 tensor can be defined by its action over generic matrices [53, 54]

$$\begin{aligned} t^{\alpha\beta\gamma\delta\mu\nu\rho\sigma} M^1_{\alpha\beta} M^2_{\gamma\delta} M^3_{\mu\nu} M^4_{\rho\sigma} = &\ 8\,\mathrm{Tr}\left(M^1M^2M^3M^4 + M^1M^3M^2M^4 + M^1M^3M^4M^2\right) \\ &-2\,\mathrm{Tr}\left(M^1M^2\right)\mathrm{Tr}\left(M^3M^4\right) - 2\,\mathrm{Tr}\left(M^1M^3\right)\mathrm{Tr}\left(M^2M^4\right) \\ &-2\,\mathrm{Tr}\left(M^1M^4\right)\mathrm{Tr}\left(M^2M^3\right) \ . \end{aligned} \tag{2.1.22}$$

For the second term in (2.1.20) we have the following convention for the Levi-Civita tensor

$$\begin{aligned} \epsilon_{10}\epsilon_{10}\mathcal{R}^4 &= \epsilon^{\alpha\beta\mu_1\dots\mu_8}\epsilon_{\alpha\beta\nu_1\dots\nu_8} \mathcal{R}_{\mu_1\mu_2}{}^{\nu_1\nu_2} \mathcal{R}_{\mu_3\mu_4}{}^{\nu_3\nu_4} \mathcal{R}_{\mu_5\mu_6}{}^{\nu_5\nu_6} \mathcal{R}_{\mu_7\mu_8}{}^{\nu_7\nu_8} \\ &= -2\cdot 8!\,\mathcal{R}_{[\alpha\beta}{}^{\alpha\beta}\mathcal{R}_{\gamma\delta}{}^{\gamma\delta}\mathcal{R}_{\mu\nu}{}^{\mu\nu}\mathcal{R}_{\rho\sigma]}{}^{\rho\sigma} \\ &= 3.2^{10}\left[\mathcal{R}^{\alpha\beta}{}_{\gamma\delta}\mathcal{R}^{\gamma\nu}{}_{\mu\beta}\mathcal{R}^{\sigma\mu}{}_{\alpha\rho}\mathcal{R}^{\delta\rho}{}_{\sigma\nu} + \mathcal{R}^{\gamma\delta}{}_{\alpha\beta}\mathcal{R}^{\mu\nu}{}_{\gamma\delta}\mathcal{R}^{\alpha\rho}{}_{\sigma\mu}\mathcal{R}^{\sigma\beta}{}_{\nu\rho}\right. \\ &\quad - \frac{1}{2}\mathcal{R}_{\alpha\beta\mu\nu}\mathcal{R}^{\beta\gamma\nu\rho}\mathcal{R}_{\gamma\delta\rho\sigma}\mathcal{R}^{\delta\alpha\sigma\mu} + \frac{1}{2}\mathcal{R}_{\alpha\beta\mu\nu}\mathcal{R}^{\beta\gamma\mu\nu}\mathcal{R}_{\gamma\delta\rho\sigma}\mathcal{R}^{\delta\alpha\rho\sigma} \\ &\quad \left. - \frac{1}{16}\mathcal{R}_{\alpha\beta\mu\nu}\mathcal{R}^{\beta\alpha\rho\sigma}\mathcal{R}^{\gamma\delta\mu\nu}\mathcal{R}_{\delta\gamma\rho\sigma} - \frac{1}{32}\mathcal{R}_{\alpha\beta\mu\nu}\mathcal{R}^{\alpha\beta\mu\nu}\mathcal{R}_{\gamma\delta\rho\sigma}\mathcal{R}^{\gamma\delta\rho\sigma} + \dots \right] \ , \end{aligned} \tag{2.1.23}$$

where the dots stand for terms containing Ricci tensors and Ricci scalars, which can be eliminated by using field redefinitions at the expense of introducing dilaton couplings, that we are ignoring.

The couplings given by $t_8 t_8$ have nonzero contribution at four-graviton level [31], while the $\epsilon_{10}\epsilon_{10}$ interactions have nonzero contributions starting only at five-graviton level [55]. The presence of this term in the tree-level effective action was inferred by the beta-function approach in [29, 33, 34], predicting $c = 1$. This prediction was confirmed in [56] through sphere-level scattering amplitudes of five gravitons. The literature also suggests that this value for c is required by supersymmetry [57, 58] and the emergence of T-duality symmetry in a circle compactification [59, 60]. Specifically for $c = 1$, it can be shown using Bianchi identities that the corrections are given by only two terms [34]

$$J(1) = -3\cdot 2^6\left[\mathcal{R}^{\alpha\beta\mu\nu}\mathcal{R}_{\mu\nu}{}^{\gamma\delta}\mathcal{R}_{\alpha\gamma}{}^{\rho\sigma}\mathcal{R}_{\rho\sigma\beta\delta} - 4\mathcal{R}_{\alpha\beta}{}^{\gamma\delta}\mathcal{R}_{\delta\mu}{}^{\alpha\nu}\mathcal{R}_{\nu\rho}{}^{\beta\sigma}\mathcal{R}_{\sigma\gamma}{}^{\mu\rho}\right] + \dots \tag{2.1.24}$$

The consensus is that these are the unique purely gravitational terms appearing in the leading α' corrections in Type II string theory. Therefore, up to this order, the theory is defined by the following Lagrangians

$$\begin{aligned}
\mathcal{L}_{\rm II}^{(1)} &= \mathcal{L}_{\rm II}^{(2)} = 0\ , \\
\mathcal{L}_{\rm II}^{(3)} &= \frac{\zeta(3)}{3\cdot 2^{11}} J(1) + \ldots \\
&= -\frac{\zeta(3)}{32}\left[\mathcal{R}^{\alpha\beta\mu\nu}\mathcal{R}_{\mu\nu}{}^{\gamma\delta}\mathcal{R}_{\alpha\gamma}{}^{\rho\sigma}\mathcal{R}_{\rho\sigma\beta\delta} - 4\mathcal{R}_{\alpha\beta}{}^{\gamma\delta}\mathcal{R}_{\delta\mu}{}^{\alpha\nu}\mathcal{R}_{\nu\rho}{}^{\beta\sigma}\mathcal{R}_{\sigma\gamma}{}^{\mu\rho}\right] + \ldots
\end{aligned} \tag{2.1.25}$$

where, as stated before, we are omitting Ricci and dilaton terms encoded in ...

For the bosonic string, the 26-dimensional action for the purely metric sector up to and including order α'^2 was obtained in [35], based on the string 3- and 4-point amplitude calculations. It was later extended to include the dilaton contribution in [37] from the 3-loop metric beta function and a consistency condition proposed in [18, 19]. Finally, the α'^3 action for the purely metric sector was determined in [38] from the 4-loop beta function. In [38] two different schemes were used. Even though the order α'^3 action was obtained only for the metric sector, both schemes contain terms involving the dilaton, Ricci tensors and Ricci scalars at intermediate orders. Therefore, we found it useful to present the result in an alternative scheme in which those intermediate dilaton and Ricci contributions are redefined away at the expense of changing the α'^3 couplings. We refer the reader to [3] for a detailed derivation of the connection between our new scheme and the one in [38]. The resulting action in the new base is given by [3]

$$\mathcal{L}_B^{(1)} = \frac{1}{4}\mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma}\ , \tag{2.1.26a}$$

$$\mathcal{L}_B^{(2)} = \frac{1}{16}\mathcal{R}_{\mu\nu}{}^{\alpha\beta}\mathcal{R}_{\alpha\beta}{}^{\rho\sigma}\mathcal{R}_{\rho\sigma}{}^{\mu\nu} - \frac{1}{12}\mathcal{R}_{\mu\nu}{}^{\alpha\beta}\mathcal{R}_{\alpha\rho}{}^{\mu\sigma}\mathcal{R}_{\beta}{}^{\rho\nu}{}_{\sigma}\ , \tag{2.1.26b}$$

$$\mathcal{L}_B^{(3)} = \frac{1}{32}\mathcal{R}^{\alpha\beta\mu\nu}\mathcal{R}_{\mu\nu}{}^{\gamma\delta}\mathcal{R}_{\alpha\gamma}{}^{\rho\sigma}\mathcal{R}_{\rho\sigma\beta\delta} - \frac{1}{16}\mathcal{R}_{\mu\nu\alpha\beta}\mathcal{R}^{\mu\nu\alpha\lambda}\mathcal{R}_{\lambda\delta\rho\sigma}\mathcal{R}^{\beta\delta\rho\sigma} + \mathcal{L}_{\rm II}^{(3)} + \ldots\ . \tag{2.1.26c}$$

where the terms $\mathcal{L}_{\rm II}^{(3)}$ are exactly those of type II string theory, with a coefficient proportional to the transcendental $\zeta(3)$ that is the same for all string theories.

Finally, the 10-dimensional low energy effective action for the Heterotic string up to and including order α'^3 is given by

$$\mathcal{L}_H^{(1)} = \frac{1}{8}\mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma}\ , \tag{2.1.27a}$$

$$\mathcal{L}_H^{(2)} = -\frac{3}{16}\Omega_{\mu\nu\rho}\Omega^{\mu\nu\rho}\ , \tag{2.1.27b}$$

$$\begin{aligned}
\mathcal{L}_H^{(3)} = \frac{1}{2^6}\Big[&18\Omega^{\mu\nu\rho}\,\mathrm{Tr}\left(\omega_\mu\partial_\nu\Omega_\rho + \Omega_\mu\partial_\nu\omega_\rho + 2\Omega_\mu\omega_\nu\omega_\rho\right) \\
&+ 18\mathcal{R}^{\mu\nu\rho\sigma}\Omega_{\rho\mu}{}^{\lambda}\Omega_{\nu\sigma\lambda} + 18\nabla_{[\mu}\Omega_{\nu]\rho\sigma}\nabla^{\mu}\Omega^{\nu\rho\sigma} \\
&- \mathcal{R}_{\mu\alpha\beta\gamma}\mathcal{R}^{\nu\alpha\beta\gamma}\mathcal{R}^{\mu\rho\sigma\lambda}\mathcal{R}_{\nu\rho\sigma\lambda} - \mathcal{R}_{\mu\nu}{}^{\rho\sigma}\mathcal{R}_{\rho\sigma}{}^{\alpha\beta}\mathcal{R}_{\alpha\beta}{}^{\gamma\delta}\mathcal{R}_{\gamma\delta}{}^{\mu\nu} \\
&-2\mathcal{R}_{\mu\sigma}{}^{\alpha\beta}\mathcal{R}_{\nu\rho\alpha\beta}\mathcal{R}^{\mu\nu\gamma\delta}\mathcal{R}_{\gamma\delta}{}^{\rho\sigma}\Big] + \mathcal{L}_{\rm II}^{(3)}\ ,
\end{aligned} \tag{2.1.27c}$$

where $\mathcal{L}_{\rm II}^{(3)}$ is defined in (2.1.25) and $\Omega_{\mu\nu\rho}$ is the Lorentz Chern-Simons three-form. Up to order α'^2, (2.1.27a) and (2.1.27b) coincide with the action calculated in [35] by 3 and 4-point amplitude methods. Excluding $\mathcal{L}_{\rm II}^{(3)}$, we are using the cubic order $\mathcal{L}_H^{(3)}$ obtained in [49] by supersymmetry. However, the α'^3 action, including $\mathcal{L}_{\rm II}^{(3)}$, was first found by 4-point scattering amplitude methods in [61] and [32].

From the point of view of this section, the above α'^3 actions can be taken as mere examples of higher-derivative effective theories. Within the thesis bigger picture, they are the starting point for a dimensional reduction that will be performed in Section 3.2 to arrive at the cosmological classification and read the defining coefficients for each string theory.

2.1.4 Non-critical dimensions

So far we restricted the analysis to strings propagating in D-dimensional backgrounds where D is the string critical dimension, namely $D = 10$ or $D = 26$. Things change considerably if one moves to non-critical dimensions. In particular, as we will see in Chapter 5, a beta-function calculation reveals that the theory incorporates an extra zero-derivative term in the action, proportional to the failure of D being critical, and inversely proportional to α'. The latter comes from dimensional analysis. In a theory of gravity, such a term is analogous to a cosmological constant. For bosonic strings, it is given by

$$Q^2 \equiv -\frac{2(D-26)}{3\alpha'}\,, \tag{2.1.28}$$

which is positive for $D < 26$.

This seemingly inoffensive term raises some surprising problems with the interpretation of higher-derivative terms as perturbative corrections to the leading-order action [6]. To explain this, let us add the cosmological term (2.1.28) to the two-derivative theory (2.1.1), together with a tower of generic α' corrections

$$\mathcal{I} = \int d^D x\sqrt{-G}e^{-2\phi}\left(Q^2 + \mathcal{R} + 4\partial_\mu\phi\partial^\mu\phi + \sum_{n=1}^{\infty}(\alpha')^n F^{(n)}[G,\phi]\right)\,, \tag{2.1.29}$$

where we are setting the B-field to zero for simplicity and considering $D < 26$ so $Q^2 > 0$. The $F^{(n)}$ denote possible terms of order $2n+2$ in derivatives.

As we saw in previous sections, field redefinitions can be used to find a scheme in which the action gets simpler. When $D = 26$ we can use these field redefinitions order-by-order in α' to bring the theory to a minimal basis parameterized by unambiguous plus essential coefficients. When $D \neq 26$, however, any term in the action other than the cosmological term can be traded, by means of field redefinitions, for a term with an arbitrary number of derivatives! To see this, let us recall that while informally one refers to the higher-derivative corrections of string theory as "α' corrections", α' itself, being dimensionful, is not a small expansion parameter. In fact, α' can be eliminated from the action (2.1.29) by rewriting it in terms of dimensionless coordinates

$$\bar{x}^\mu \equiv Q x^\mu \,, \quad \bar{\partial}_\mu \equiv \frac{1}{Q}\partial_\mu \,, \tag{2.1.30}$$

and rescaling the action by Q^{D-2}:

$$\bar{\mathcal{I}} \equiv Q^{D-2}\mathcal{I} = \int d^D\bar{x}\,\sqrt{-G}\,e^{-2\phi}\left(1+\bar{\mathcal{R}}+4\bar{\partial}_\mu\phi\bar{\partial}^\mu\phi+\sum_{n=1}^{\infty}(\alpha' Q^2)^n \bar{F}^{(n)}[G,\phi]\right)\,, \tag{2.1.31}$$

where the bar over $\mathcal{R}$ or F indicates that all derivatives ∂_μ have been replaced by $\bar{\partial}_\mu$. In this formulation there is no α' left (notice from (2.1.28) that $\alpha' Q^2$ does not depend on α'), and there is no expansion in α'. Rather, one should think of the higher-derivative corrections as an expansion in terms of small derivatives of the fields. While this can make sense in critical-dimension string theory, in non-critical dimension string theory generic solutions feature fields whose dimensionless derivatives are of order $\bar{\partial} \sim \mathcal{O}(1)$, so that all higher-derivative terms can have significant effects.

Let's see how we can trade any term in (2.1.31) for one with more derivatives by using field redefinitions. We begin by considering the exact field redefinition of the metric

$$G_{\mu\nu} \to G_{\mu\nu} + \Delta G_{\mu\nu}\;, \tag{2.1.32}$$

where we will take $\Delta G_{\mu\nu}$ to be a local function given by a derivative expansion. This implies

$$\begin{aligned}
G^{\mu\nu} &\to\; G^{\mu\nu} - \Delta G^{\mu\nu} + \mathcal{O}((\Delta G)^2)\,,\\
\sqrt{-G} &\to\; \sqrt{-G}\Big(1+\tfrac{1}{2}G^{\mu\nu}\Delta G_{\mu\nu} + \mathcal{O}((\Delta G)^2)\Big)\,,\\
\bar{\mathcal{R}} &\to\; \bar{\mathcal{R}} + \Delta G^{\mu\nu}\Big(\bar{\mathcal{R}}_{\mu\nu} - \tfrac{1}{2}G_{\mu\nu}\bar{\mathcal{R}}\Big) + G^{\mu\nu}(\bar{\nabla}_\rho \Delta\bar{\Gamma}^\rho_{\mu\nu} - \bar{\nabla}_\mu\Delta\bar{\Gamma}^\rho_{\rho\nu}) + \;\mathcal{O}((\Delta G)^2)\,.
\end{aligned} \tag{2.1.33}$$

Indices on $\Delta G_{\mu\nu}$ are raised with the unperturbed $G^{\mu\nu}$. We take $\Delta G_{\mu\nu}$ to be given by a derivative expansion:

$$\Delta G_{\mu\nu} = \Delta^{(1)}G_{\mu\nu} + \Delta^{(2)}G_{\mu\nu} + \cdots\;, \tag{2.1.34}$$

where $\Delta^{(n)}G_{\mu\nu}$ is of order $2n+2$ in derivatives.

The redefined action $\bar{\mathcal{I}}'$ is given by (2.1.31) with G replaced by $G+\Delta G$:

$$\begin{aligned}
\bar{\mathcal{I}}' \equiv \bar{\mathcal{I}}[G+\Delta G,\phi] = &\int d^D\bar{x}\sqrt{-G}e^{-2\phi}\Big(1+\bar{\mathcal{R}}+4\bar{\partial}_\mu\phi\bar{\partial}^\mu\phi\Big)\\
&+\int d^D\bar{x}\sqrt{-G}e^{-2\phi}\left[\tfrac{1}{2}G^{\mu\nu}\Delta^{(1)}G_{\mu\nu} + (\alpha' Q^2)\bar{F}^{(1)}[G,\phi]\right]\\
&+\cdots\;,
\end{aligned} \tag{2.1.35}$$

where we used (2.1.33) and where the ellipsis denote terms with *more than four derivatives*. This follows from $\Delta^{(1)}G_{\mu\nu}$ being already of fourth order in derivatives, so that in particular new terms induced from the two-derivative action are already of order six in derivatives. We can cancel the four-derivative term $\bar{F}^{(1)}$ by choosing

$$\Delta^{(1)}G_{\mu\nu} = -\frac{2\alpha' Q^2}{D} G_{\mu\nu}\bar{F}^{(1)}[G,\phi]\,. \tag{2.1.36}$$

The six-derivative terms encoded in $\bar{F}^{(2)}$ receive further contributions from the field redefinition, and we denote the totality of all such terms by $\bar{F}'^{(2)}$. The redefined action then reads

$$\mathcal{I}' = \int d^D\bar{x}\sqrt{-G}e^{-2\phi}\left(1+\bar{\mathcal{R}}+4\bar{\partial}_\mu\phi\bar{\partial}^\mu\phi+(\alpha' Q^2)^2\bar{F}'^{(2)}[G,\phi]+\cdots\right)\,, \tag{2.1.37}$$

where the ellipsis denotes all terms with more than six derivatives. Of course, we could have instead cancelled the Einstein-Hilbert term $\bar{\mathcal{R}}$ or even the whole two-derivative Lagrangian by including a two-derivative term $\Delta^{(0)}G$ in the ΔG expansion (2.1.34), and setting $\Delta^{(0)}G_{\mu\nu} = -\frac{2}{D}G_{\mu\nu}\left(\bar{\mathcal{R}}+4\bar{\partial}_\mu\phi\bar{\partial}^\mu\phi\right)$. In that case the action would have the cosmological term followed by terms with four derivatives.

The procedure above can be iterated. Looking at the action (2.1.37) we can just repeat the procedure by setting

$$\Delta^{(2)}G_{\mu\nu} = -\frac{2(\alpha' Q^2)^2}{D} G_{\mu\nu}\bar{F}'^{(2)}[G,\phi]\,, \tag{2.1.38}$$

so as to cancel the terms with six derivatives. Thus, all terms in the action can be moved to arbitrary high order in derivatives!

This result is puzzling, because adopting the usual perturbative mindset one would view terms with large numbers of derivatives as sub-leading compared to a term with two derivatives, and hence one would feel free to drop them. This is indeed the standard procedure of bringing higher-derivative terms to a minimal form, but using this procedure literally in (2.1.31) one would conclude that only the cosmological term is non-trivial. What is really happening is that in these noncritical string backgrounds generic higher-derivative terms are not actually sub-leading relative to terms with lesser number of derivatives. Thus, while field redefinitions like (2.1.36) or (2.1.38) are perfectly legal, it is the *second step* of dropping induced terms with more derivatives that is generally illegal. This sheds doubt on attempts to get string corrected solutions in a consistent way, which is indeed problematic considering that black hole backgrounds are of this kind, and it is precisely in these scenarios where string corrections are expected to be crucial. We will see in Section 4.1 how we circumvent this obstacle and arrive at a classification of the most general higher-derivative interactions for two-dimensional black hole backgrounds.

2.2 Dimensional Reduction and $O(d, d)$

So far we have described the bosonic sector of supegravities in generic D-dimensional backgrounds. Apart from very special non-critical cases, it is not possible in general to make contact with our lower-dimensional observable universe right-away. Instead, one needs to implement a mechanism called compactification or dimensional reduction that relates theories in different dimensions. In this section we introduce this method, restricting the analysis to the two-derivative theory presented in (2.1.1) in backgrounds with $d < D$ abelian isometries, leading to a global $O(d, d|\mathbb{R})$ symmetry [2] [20–24]. The latter is directly connected with the stringy origin of these field theories [25]. We also present two particular extreme cases where the compactified theory is effectively one-dimensional: cosmological backgrounds [20–22] and black holes [44–46].

While here we only treat the two-derivative case, the concepts introduced in this section are essential for the development of the main results of this work in the context of α' corrections. More precisely, relying on the fact that the duality group is preserved to all orders in α' [23], in the following chapters we will see how to use $O(d, d)$ to arrive at a classification of string low energy effective actions in cosmological and black hole backgrounds to all orders in α'.

2.2.1 The Maharana-Schwarz action

We will refer to the original D-dimensional theory as the parent theory and the resulting n-dimensional spacetime, with $n < D$, as the effective or external theory. The remaining $d = D - n$ "extra" dimensions form the so-called internal space. For now, apart from the condition $D = n + d$, we will keep d and n generic. There are numerous ways to perform a compactification depending on the manifold we choose as our internal space. The way these extra internal dimensions are curled-up heavily modifies the physics we perceive in the effective n-dimensional theory. In this work we consider one of the simplest scenarios where the parent theory posses d abelian isometries, meaning that effective fields do not depend on the internal coordinates. In what follows, we introduce this dimensional reduction procedure through a series of steps, taking (2.1.1) as our starting point. However, all steps can be equally applied to more general parent theories, and this is exactly what we will do in Section 3.2.

1. **Split coordinates:** The coordinates of the parent space X^μ with $\mu = 0, \dots, D-1$ split into

$$X^\mu = \left(x^i, y^m\right) \,, \tag{2.2.1}$$

[2]Unless specified otherwise, we will always work with real-valued elements, so we drop the $\mathbb{R}$ label from now on. We will reinstate it only when discussing the discrete version $O(d, d|\mathbb{Z})$ later on.

where $x^i, i = 0, \ldots, n-1$ and $y^m, m = 1, \ldots, d$ parameterize the external and internal space respectively. In our particular case, nothing will depend on y^m, being the isometric directions, regardless they being compact or not.

2. **Break parent symmetry group:**

The parent theory (2.1.1) enjoys the symmetries (2.1.5) and $G_{\mu\nu}, B_{\mu\nu}$ and Φ belong to representations of the corresponding symmetry group. Upon compactification, this symmetry group is broken into the ones for the effective theory. Therefore, fields must be decomposed into representations of these lower-dimensional symmetry groups.

$$G_{\mu\nu}(X) = \begin{pmatrix} g_{ij} + A_i{}^p g_{pq} A_j{}^q & A_i{}^p g_{pn} \\ g_{mp} A_j{}^p & g_{mn} \end{pmatrix}, \tag{2.2.2a}$$

$$B_{\mu\nu}(X) = \begin{pmatrix} b_{ij} - A_{[i}{}^p V_{j]p} + A_i{}^p b_{pq} A_j{}^q & V_{in} + A_i{}^p b_{pn} \\ -V_{jm} + b_{mp} A_j{}^p & b_{mn} \end{pmatrix}, \tag{2.2.2b}$$

$$\phi(X) = \Phi + \frac{1}{4} \log g\,, \tag{2.2.2c}$$

where in the dilaton's decomposition g states for the determinant of the internal components of the metric g_{mn}. The gauge parameters of the parent theory need to be split as well

$$\xi^\mu(X) = \left(\xi^i, \Lambda^m\right)\,, \quad \lambda_\mu(X) = \left(\lambda_i, \Lambda_m\right)\,, \tag{2.2.3}$$

This decomposition allows us to identify the field content of the effective theory. Interestingly, they can be presented in a compact way by rewriting the internal directions in terms of double indices $M = 1, \ldots, 2d$

$$g_{ij}(X)\,, \quad b_{ij}(X), \quad \Phi(X)\,, \tag{2.2.4a}$$

$$A_i^M(X) \equiv \left(A_i{}^m, V_{im}\right)\,, \tag{2.2.4b}$$

$$\mathcal{H}_{MN}(X) \equiv \begin{pmatrix} g_{mn} - b_{mp} g^{pq} b_{qn} & b_{mp} g^{pn} \\ -g^{mp} b_{pn} & g^{mn} \end{pmatrix}. \tag{2.2.4c}$$

The same can be done for gauge parameters

$$\xi^i(X)\,, \quad \lambda_i(X)\,, \quad \Lambda^M(X) \equiv \left(\lambda^m, \lambda_m\right)\,. \tag{2.2.5}$$

While at this point this can be seen just as a curious fact, soon we will see that, packed in this form, fields and parameters can be identified as representations of an $O(d, d)$ global duality group.

3. **Reduction ansatz:**

The previous decomposition has nothing to do with the coordinate dependence of the effective fields. At this stage, they still depend on x^i and y^m arbitrarily. By proposing a reduction ansatz we specify the dependence of the external fields on the internal manifold. In our case, denoting collectively $\Psi \in \{g_{ij}, b_{ij}, \Phi, A_i{}^M, \mathcal{H}_{MN}, \xi^i, \lambda_i, \Lambda^M\}$, we pick the simplest ansatz where nothing depends on the internal space, namely

$$\Psi(X) = \Psi(x, y) = \Psi(x) \quad \Rightarrow \quad \frac{\partial \Psi}{\partial y^m} = 0\,. \tag{2.2.6}$$

In this case, the theory possesses d abelian isometries.

4. **Gauge transformations of the effective theory:**

   We already saw in (2.2.2) how fields in the parent theory decompose into representations of the symmetry group of the effective theory. To see how these symmetries act on the effective fields we need to decompose (2.1.5) by using (2.2.2) and (2.2.3) as well as the reduction ansatz (2.2.6). In terms of the duality covariant objects (2.2.4) and (2.2.5), the infinitesimal transformations read

$$\begin{aligned} \delta g_{ij} &= L_\xi g_{ij}\,, \quad \delta b_{ij} = L_\xi b_{ij} + 2\partial_{[i}\lambda_{j]} + A_{[i}{}^P \partial_{j]}\Lambda_P\,, \quad \delta\Phi = L_\xi \Phi\,, \\ \delta A_i{}^M &= L_\xi A_i{}^M + \partial_i \Lambda^M\,, \quad \delta\mathcal{H}_{MN} = L_\xi \mathcal{H}_{MN}\,. \end{aligned} \tag{2.2.7}$$

   ξ^i generates infinitesimal n-dimensional diffeomorphisms through the effective Lie derivative L_ξ. Under such transformation g_{ij} and b_{ij} are $\binom{0}{2}$-tensors, $A_i{}^M$ is a 1-form and Φ and $\mathcal{H}_{MN}$ are scalars. λ_i generates b-field gauge transformations and Λ^M the $U(1)^{2d}$ gauge transformations, which requires an additional transformation for the b-field.

5. **Effective action:**

   The n-dimensional effective action is obtained by plugging the ansatz (2.2.2) into the parent theories' action and killing all internal dependency. In the case of the two-derivative sector of $\mathcal{N} = 0$ supergravity, we have (2.1.1), and the compactified action is given by the Maharana-Schwarz action [24]

$$\begin{aligned} I^{(0)} &= \int d^n x \sqrt{-g} e^{-2\Phi} L^{(0)}\,, \\ L^{(0)} &\equiv R + 4\partial_i\Phi\partial^i\Phi - \frac{1}{12} H_{ijk}H^{ijk} - \frac{1}{4}\mathcal{H}_{MN}F^{ijM}F_{ij}{}^N + \frac{1}{8}\partial_i\mathcal{H}_{MN}\partial^i\mathcal{H}^{MN}\,, \end{aligned} \tag{2.2.8}$$

   where R is the n-dimensional Ricci scalar built from g_{ij} and the field strengths for the vectors and b-field are defined as

$$\begin{aligned} F_{ij}{}^M &= 2\partial_{[i}A_{j]}{}^M\,, \\ H_{ijk} &= 3\partial_{[i}b_{jk]} - 3A_{[i}{}^P\partial_j A_{k]P}\,. \end{aligned} \tag{2.2.9}$$

   Both tensors are invariant under the gauge transformations generated by λ_i and Λ^M in (2.2.7). The latter requires the Chern-Simons modification in H.

   In the effective theory, external indices are raised with g^{ij} while the double indices are contracted with

$$\eta_{MN} = \begin{pmatrix} 0 & \delta_m{}^n \\ \delta^m{}_n & 0 \end{pmatrix}\,. \tag{2.2.10}$$

As expected, the action (2.2.8) is invariant under the residual gauge transformations (2.2.7).

The $O(d,d)$ group

On top of the local symmetries (2.2.7), a novel $O(d,d)$ global symmetry emerged upon compactification that has no counterpart in the parent theory (see [25] for a review on the $O(d,d)$ group and T-duality). Elements of this duality invariant group can be represented as $2d \times 2d$ matrices that preserve the duality invariant metric (2.2.10). In matrix notation they read

$$h_{\bullet}{}^{\bullet} = \begin{pmatrix} a_{\cdot}{}^{\cdot} & b_{\cdot\cdot} \\ c^{\cdot\cdot} & d^{\cdot}{}_{\cdot} \end{pmatrix} \in O(d,d) \quad \Rightarrow \quad h\eta h^t = \eta \,, \tag{2.2.11}$$

where the bullets represent the index structure and the matrices $a, b, c, d \in \mathbb{R}^{d\times d}$ must satisfy

$$a^t c + c^t a = b^t d + d^t b = 0 \,, \quad a^t d + c^t b = 1 \,. \tag{2.2.12}$$

Here and it what follows we make an abuse of notation by using 1 as the matrix notation for $\delta_m{}^n, \delta^m{}_n, \delta_{mn}$ and δ^{mn} altogether. The should be no ambiguities by doing this since the index structure should be clear from context.

Any element of the duality group can be decomposed as successive products of the following transformations:

- **Change of basis** $A \in GL(d, \mathbb{R})$

$$h_{GL} = \begin{pmatrix} A & 0 \\ 0 & (A^{-1})^t \end{pmatrix} \,. \tag{2.2.13}$$

- **b-shifts**

$$h_b = \begin{pmatrix} 1 & \Xi \\ 0 & 1 \end{pmatrix} \,, \tag{2.2.14}$$

  where $\Xi_{mn} = -\Xi_{nm}$.

- **Factorized dualities**

$$h_{t_m} = \begin{pmatrix} 1 - t_m & t_m \\ t_m & 1 - t_m \end{pmatrix} \,, \quad (t_m)_{np} \equiv \delta_{mn}\delta_{mp} \,. \tag{2.2.15}$$

From successive products of these basic transformations one can build any $h \in O(d,d)$. One induced transformation is particularly interesting in the context of duality transformations:

⊙ **Full factorized duality:** This transformation is obtained by applying factorized dualities over all directions

$$h_f = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} . \tag{2.2.16}$$

Under this duality group g_{ij}, b_{ij} and Φ are scalars while $A_i{}^M$ is an $O(d,d)$ vector and $\mathcal{H}_{MN}$ a symmetric $\binom{0}{2}$ -tensor. The action (2.2.8) is invariant under such duality transformation. $\mathcal{H}_{MN}$ is called the generalized metric and it plays a crucial role in duality-invariant theories. The scalar Φ is also an important quantity and we will often call it the duality-invariant dilaton, to distinguish it from the scalar dilaton ϕ, which does transform under $O(d,d)$. Indeed, this can be seen from their relation via (2.2.2c), which involves the internal metric g_{mn}.

Properties of the generalized metric

Interestingly, the generalized metric $\mathcal{H}_{MN}$ leads to an $O(d,d)$ element itself upon raising one index. Indeed, defining

$$\mathcal{S}_M{}^N \equiv \mathcal{H}_{MP}\eta^{PN} , \tag{2.2.17}$$

it satisfies

$$\mathcal{S}_M{}^P \eta_{PQ} \mathcal{S}_N{}^Q = \eta_{MN} , \tag{2.2.18}$$

which implies that $\mathcal{S}_M{}^N \in O(d,d)$. Using (2.2.4c), (2.2.17) can be given in matrix notation as

$$\mathcal{S} = \mathcal{H}\eta^{-1} = \begin{pmatrix} bg^{-1} & g - bg^{-1}b \\ g^{-1} & -g^{-1}b \end{pmatrix} , \tag{2.2.19}$$

which can be shown to be traceless

$$\mathrm{Tr}\,(\mathcal{S}) = 0 , \tag{2.2.20}$$

upon introducing the notation $\mathrm{Tr}\,(\cdot)$ for the trace of $2d \times 2d$ matrices. The condition (2.2.18) makes the generalized metric $\mathcal{S}$ a constrained object. This constraint can be written in a very succinct fashion as

$$\mathcal{S}^2 = 1 , \tag{2.2.21}$$

where $\mathcal{S}^2$ stands for $\mathcal{S}_M{}^P \mathcal{S}_P{}^N$ and 1 for $\delta_M{}^N$.

We now list a series of identities that can be deduced from (2.2.21), as they will become very useful in later applications. To this end, we introduce the projectors

$$P \equiv \frac{1}{2}(1 - \mathcal{S}) , \quad \bar{P} \equiv \frac{1}{2}(1 + \mathcal{S}) , \tag{2.2.22}$$

satisfying

$$P^2 = P , \quad \bar{P}^2 = \bar{P} , \quad P\bar{P} = \bar{P}P = 0 . \tag{2.2.23}$$

Furthermore, we have the following useful identities

$$P\mathcal{S} = \mathcal{S}P = -P\,, \quad \bar{P}\mathcal{S} = \mathcal{S}\bar{P} = \bar{P}\,. \tag{2.2.24}$$

With these projectors we can now split any $O(d,d)$ matrix $A = A_\bullet{}^\bullet$ into $\pm$ components, defined as

$$A_+ \equiv PAP + \bar{P}A\bar{P}\,, \quad A_- \equiv PA\bar{P} + \bar{P}AP\,, \quad A = A_+ + A_-\,. \tag{2.2.25}$$

These $\pm$ spaces are nothing but the spaces of $O(d,d)$ tensors that commute and anti-commute with $\mathcal{S}$, as it can be seen from the relations

$$A_\pm = \frac{1}{2}\left(A \pm \mathcal{S}A\mathcal{S}\right)\,, \quad A_\pm\mathcal{S} = \pm\mathcal{S}A_\pm\,. \tag{2.2.26}$$

It is also easy to show that products of projected fields have definite projection following the rules of sign multiplication, namely A_+B_- is a minus-projected tensor, while A_-B_- is plus-projected.

With these relations we can prove that traces of minus-projected tensors vanish:

$$(A_-)_M{}^M \equiv \text{Tr}\,(A_-) = \text{Tr}\left(A_-\mathcal{S}^2\right) = -\text{Tr}\,(\mathcal{S}A_-\mathcal{S}) = -\text{Tr}\left(A_-\mathcal{S}^2\right) = -\text{Tr}\,(A_-) = 0\,, \tag{2.2.27}$$

where we used $\mathcal{S}^2 = 1$, (2.2.26) and the cyclicity of the trace. Another consequence of the generalized metric being a constrained object is that a small variation of it is also constrained. To see this, we take a variation of (2.2.21) to get

$$\delta\mathcal{S}\mathcal{S} + \mathcal{S}\delta\mathcal{S} = 0 \quad \Rightarrow \quad \delta\mathcal{S} = -\mathcal{S}\delta\mathcal{S}\mathcal{S} \quad \Rightarrow \delta\mathcal{S} = [\delta\mathcal{S}]_-\,. \tag{2.2.28}$$

where in the second equality we used $\mathcal{S}^2 = 1$ and in the third (2.2.26). This identity will become useful when computing equations of motion.

As a final remark, it is worth noticing that all properties and identities deduced for the generalized metric (2.2.19) are valid for generic d internal dimensions. They will become useful later in the context of cosmological backgrounds where $d = D - 1$.

2.2.2 Two-derivative cosmological backgrounds

Cosmological reductions stand for a particular case of dimensional reduction where all spatial directions are compactified and the effective space depends on a single time coordinate [20–22]. Using the notation of (2.2.1), $d = D - 1$, $n = 1$ and the only external direction is parameterized by $X_0 = t$. In this "extreme" case, the ansatz (2.2.2) reduces to its simplest version

$$G_{\mu\nu} = \begin{pmatrix} -n^2 & 0 \\ 0 & g_{mn} \end{pmatrix}\,, \quad B_{\mu\nu} = \begin{pmatrix} 0 & 0 \\ 0 & b_{mn} \end{pmatrix}\,, \quad \phi = \frac{1}{2}\Phi + \frac{1}{4}\log\det g\,, \tag{2.2.29}$$

where we changed conventions for the effective dilaton adding an extra $\frac{1}{2}$ factor and wrote the time-time component of the metric G_{00} in terms of the so-called lapse function n.

While, from (2.2.2) is clear that we should also have vector components A^M, we set them to zero because in cosmological backgrounds they carry no degrees of freedom. More precisely, we can always pick a gauge for Λ^M to set A^M to zero. While doing this gauge fixing procedure, one should be careful to do it in a consistent way, meaning that by setting $A^M = 0$, the equations of motion of these components should also vanish identically. Put differently, the information contained in the equations of motion of all remaining fields after setting $A^M = 0$, is the same as the one obtained by setting these components to zero in the action and then computing the equations of motion of the fields we kept. If this is not the case, the gauge-fixing is not consistent and by setting $A^M = 0$ we are missing extra constraints for the remaining effective fields.

In principle, this could happen if we had a linear term on the vectors in the Lagrangian of the form $A^M F_M(n, g, b, \Phi)$ where F_M is a generic function of the other effective fields. In this hypothetical case, by setting A^M to zero we would be missing the extra constraint $F_M(n, g, b, \Phi) = 0$ for the other fields, which comes from the vector's equation of motion. Luckily, such linear term simply does not exist in the one-dimensional theory because there are no $O(d, d)$ vectors other than A^M that could form the function F_M. Moreover, one should also check that the gauge transformations acting on these components vanish when the components themselves are set to zero. From (2.2.7) one can check that this indeed the case if we also eliminate the gauge parameters $\Lambda^M = 0$. All this tells us that, indeed, setting $A^M = 0$ is consistent.

As for general backgrounds (2.2.4), the field content can be packed into representations of the global duality group $O(d, d)$ where $n(t)$ and $\Phi(t)$ are scalars and $\mathcal{H}_{MN}(t)$ the same symmetric tensor as in (2.2.4c). From now on, however, we will use the $\mathcal{S}(t)$ rewriting of the generalized metric, as given in (2.2.19). Due to the reduction ansatz (2.2.6), all fields and parameters depend on the only external coordinate t:

$$\partial_m \Psi = 0 \,, \quad \partial_0 \Psi = \partial_t \Psi \equiv \dot{\Psi} \,. \tag{2.2.30}$$

The effective theory now has a much smaller gauge symmetry than the one for general backgrounds (2.2.7). In this case Λ^M and λ_0 just drop out from the theory. The former due to the gauge fixing of the vector fields and the second one because there is no external b-field in one dimension. The only remnant gauge symmetry is time-reparameterization invariance generated by $\xi(t) \equiv \xi^0(t)$ via $t \to t - \xi(t)$, under which the fields transform infinitesimally as follows

$$\delta n = \partial_t(\xi n) \,, \quad \delta\Phi = \xi\dot{\Phi} \,, \quad \delta g_{mn} = \xi \dot{g}_{mn} \,, \quad \delta b_{mn} = \xi \dot{b}_{mn} \quad \Rightarrow \quad \delta\mathcal{S} = \xi\dot{\mathcal{S}} \,. \tag{2.2.31}$$

This tells us that n is a scalar density while all the other fields are scalars. In particular, the transformations for the internal metric and b-field imply that the generalized metric is also a scalar. The presence of a density allows us to introduce a covariant derivative

$$\mathcal{D} \equiv \frac{1}{n}\partial_t \,, \tag{2.2.32}$$

which implies that if Ψ is a scalar, then $\mathcal{D}\Psi$ is also an scalar. In principle we could use this residual one-dimensional symmetry to remove the only degree of freedom carried by the lapse function. More precisely, we could use $\xi(t)$ to gauge fix, let's say, $n(t) = 1$, in the same way we used Λ^M to set the vector components of (2.2.29) to zero. However, as opposed to the vector case, we will see that this gauge-fixing is not consistent at the level of the action, so we better keep $n(t)$ for now.

The analogous to the Maharana-Schwarz action (2.2.8) in cosmological backgrounds is obtained by plugging (2.2.29) into (2.1.1) [20–22]. To this end we introduce the following $d \times d$ matrices

$$L \equiv \mathcal{D}g g^{-1} \,, \quad M \equiv \mathcal{D}b g^{-1} \,, \tag{2.2.33}$$

which are scalars under diffeomorphisms. Plugging (2.2.29) into the expressions for the Christoffel symbols, Riemann tensor and $H_{\mu\nu\rho}$

$$\begin{aligned} \Gamma_{\mu\nu}{}^{\rho} &= \frac{1}{2}G^{\rho\sigma}\left(\partial_\mu G_{\nu\sigma} + \partial_\nu G_{\mu\sigma} - \partial_\sigma G_{\mu\nu}\right) \,, \\ \mathcal{R}^{\rho}{}_{\sigma\mu\nu} &= \partial_\mu \Gamma_{\nu\sigma}{}^{\rho} - \partial_\nu \Gamma_{\mu\sigma}{}^{\rho} + \Gamma_{\mu\lambda}{}^{\rho}\Gamma_{\nu\sigma}{}^{\lambda} - \Gamma_{\nu\lambda}{}^{\rho}\Gamma_{\mu\sigma}{}^{\lambda} \,, \\ H_{\mu\nu\rho} &= 3\partial_{[\mu}B_{\nu\rho]} \,, \end{aligned} \tag{2.2.34}$$

and using (2.2.30) we get

$$\begin{aligned} \Gamma_{0m}{}^{n} &= \frac{n}{2}L_m{}^n \,, \quad \Gamma_{mn}{}^{0} = \frac{1}{2n}L_{mn} \,, \quad \Gamma_{00}{}^{0} = \mathcal{D}n \,, \\ \mathcal{R}_{mnpq} &= 2L_{p[m}L_{n]q} \,, \quad \mathcal{R}_{0m0n} = -\frac{n^2}{4}\left(L^2_{mn} + 2\mathcal{D}L_{mn}\right) \,, \\ \nabla_0\phi &= \frac{n}{2}\mathcal{D}\Phi + \frac{n}{4}\text{tr}\,(L) \,, \quad H_{0mn} = nM_{mn} \,, \end{aligned} \tag{2.2.35}$$

and all the other components are zero. On top of (2.2.33), we introduced further notation: internal indices are raised with g^{-1} and lower with g, namely $L_{mn} = L_m{}^p g_{pn}$, $\mathcal{D}L_{mn} = \mathcal{D}L_m{}^p g_{pn}$, etc. We also denote products of identical matrices as powers, namely $L^2_{mn} = L_m{}^p L_p{}^q g_{qn}$. Finally, $\text{tr}\,(\cdot)$ stands for the trace of $d \times d$ matrices.

From the second line of (2.2.35) we can build the scalar curvature and from the last line the kinetic terms for the dilaton and b-field

$$\begin{aligned} \mathcal{R} &= \text{tr}\,(\mathcal{D}L) + \frac{1}{4}\text{tr}\left(L^2\right) + \frac{1}{4}\text{tr}\,(L)^2 \,, \\ \nabla_\mu\phi\nabla^\mu\phi &= -\frac{1}{4}(\mathcal{D}\Phi)^2 - \frac{1}{4}\mathcal{D}\Phi\text{tr}\,(L) - \frac{1}{16}\text{tr}\,(L)^2 \,, \\ H_{\mu\nu\rho}H^{\mu\nu\rho} &= 3\text{tr}\left(M^2\right) \,. \end{aligned} \tag{2.2.36}$$

Combining these three couplings as given in (2.1.1) we get the effective action in cosmological backgrounds:

$$
\begin{aligned}
I^{(0)} &= \int dt\, n\, e^{-\Phi} \left[\mathrm{tr}\left(\mathcal{D}L\right) - \mathcal{D}\Phi \mathrm{tr}\left(L\right) - (\mathcal{D}\Phi)^2 + \frac{1}{4}\mathrm{tr}\left(L^2 - M^2\right) \right] \\
&= \int dt\, n\, e^{-\Phi} \left[-(\mathcal{D}\Phi)^2 + \frac{1}{4}\mathrm{tr}\left(L^2 - M^2\right) \right] ,
\end{aligned}
\tag{2.2.37}
$$

where to arrive to the second line we integrated by parts the covariant derivative in the first term. This action can be given in an even more succinct form by remembering that the internal fields must combined into the generalized metric. Indeed, using the definitions for $\mathcal{S}$, (2.2.19) and L and M, (2.2.33), one can show that

$$
\mathcal{D}\mathcal{S}_M{}^P \mathcal{D}\mathcal{S}_P^N = \mathrm{Tr}\left((\mathcal{D}\mathcal{S})^2\right) = -2\mathrm{tr}\left(L^2 - M^2\right) . \tag{2.2.38}
$$

Therefore, the cosmological action reaches its final form [21, 22]

$$
I^{(0)} = \int dt\, n\, e^{-\Phi} \left[-(\mathcal{D}\Phi)^2 - \frac{1}{8}\mathrm{Tr}\left((\mathcal{D}\mathcal{S})^2\right) \right] , \tag{2.2.39}
$$

which is manifestly invariant under one-dimensional diffeomorphisms (2.2.31) and global $O(d,d)$. It is a particular one-dimensional case of (2.2.8) where $x^i \to t$, the fields of the external sector, R and H_{ijk}, vanish, and vectors decouple.

Taking the variaton of (2.2.39) with respect to the fields, we get

$$
\delta I^{(0)} = \int dt\, n\, e^{-\Phi} \left[\mathrm{Tr}\left(F_{\mathcal{S}} \delta\mathcal{S}\right) + E_\Phi \delta\Phi + E_n \frac{\delta n}{n} \right] , \tag{2.2.40}
$$

While $E_\Phi = 0$ and $E_n = 0$ correspond to the equations of motion of the dilaton and lapse, respectively, $F_{\mathcal{S}} = 0$ is not the equation of motion for $\mathcal{S}$. The reason is that, since $\mathcal{S}$ is a constrained object satisfying $\mathcal{S}^2 = 1$, its variation is also a constrained object satisfying $\delta\mathcal{S} = [\delta\mathcal{S}]_-$ (see (2.2.28)). Therefore, we get

$$
\mathrm{Tr}\left(F_{\mathcal{S}}\delta\mathcal{S}\right) = \mathrm{Tr}\left(([F_{\mathcal{S}}]_+ + [F_{\mathcal{S}}]_-)[\delta\mathcal{S}]_-\right) = \mathrm{Tr}\left([F_{\mathcal{S}}]_-[\delta\mathcal{S}]_-\right) = \mathrm{Tr}\left([F_{\mathcal{S}}]_-\delta\mathcal{S}\right) , \tag{2.2.41}
$$

where in the first equality we decomposed $F_{\mathcal{S}}$ into $\pm$ projections and used (2.2.28). In the second equality we used that $[F_{\mathcal{S}}]_+[\delta\mathcal{S}]_-$ is a minus-projected tensor so from (2.2.27) its trace vanishes. In the last equality we used again (2.2.28). Plugging (2.2.41) into (2.2.40) we get

$$
\delta I^{(0)} = \int dt\, n\, e^{-\Phi} \left[\mathrm{Tr}\left([F_{\mathcal{S}}]_-\delta\mathcal{S}\right) + E_\Phi \delta\Phi + E_n \frac{\delta n}{n} \right] . \tag{2.2.42}
$$

Since now the constrained nature of $\mathcal{S}$ (and $\delta\mathcal{S}$) was already taken into account, $E_{\mathcal{S}} \equiv [F_{\mathcal{S}}]_- = 0$ do correspond to the genuine equation for the generalized metric. Following this procedure, the three equations of motion are given by

$$
E_{\mathcal{S}} = \frac{1}{4}\left[\Box_\Phi \mathcal{S}\right]_- = 0 , \tag{2.2.43a}
$$

$$
E_\Phi = 2\mathcal{D}^2\Phi - (\mathcal{D}\Phi)^2 + \frac{1}{8}\mathrm{Tr}\left((\mathcal{D}\mathcal{S})^2\right) = 0 , \tag{2.2.43b}
$$

$$
E_n = (\mathcal{D}\Phi)^2 + \frac{1}{8}\mathrm{Tr}\left((\mathcal{D}\mathcal{S})^2\right) = 0 , \tag{2.2.43c}
$$

where we introduced the linear operator

$$\Box_\Phi \equiv \mathcal{D}^2 - \mathcal{D}\Phi\mathcal{D}\,. \tag{2.2.44}$$

Using the definition of projected-objects (2.2.26) one can check

$$\begin{aligned} [\mathcal{D}\mathcal{S}]_- &= \mathcal{D}\mathcal{S}\,, \\ [\mathcal{D}^2\mathcal{S}]_- &= \mathcal{D}^2\mathcal{S} + \mathcal{S}(\mathcal{D}\mathcal{S})^2\,, \end{aligned} \tag{2.2.45}$$

and so

$$[\Box_\Phi \mathcal{S}]_- = \mathcal{D}^2\mathcal{S} + \mathcal{S}(\mathcal{D}\mathcal{S})^2 - \mathcal{D}\Phi\mathcal{D}\mathcal{S}\,. \tag{2.2.46}$$

By using (2.2.46), combining E_Φ and E_n, and omitting multiplicative factors, the cosmological equations can be written in an equivalent simpler form:

$$\mathcal{D}^2\mathcal{S} = \mathcal{D}\Phi\mathcal{D}\mathcal{S} - \mathcal{S}(\mathcal{D}\mathcal{S})^2\,, \tag{2.2.47a}$$

$$\mathcal{D}^2\Phi = -\frac{1}{8}\mathrm{Tr}\left((\mathcal{D}\mathcal{S})^2\right)\,, \tag{2.2.47b}$$

$$(\mathcal{D}\Phi)^2 = -\frac{1}{8}\mathrm{Tr}\left((\mathcal{D}\mathcal{S})^2\right)\,. \tag{2.2.47c}$$

At this point we can see why fixing $n(t) = 1$ at the level of the action is inconsistent. This is clear from setting $n = 1$ in the lapse equation (2.2.47c):

$$\dot{\Phi}^2 = -\frac{1}{8}\mathrm{Tr}\left(\dot{\mathcal{S}}^2\right)\,, \tag{2.2.48}$$

where we used $\mathcal{D} \to \partial_t$ and the dot-notation. The lapse equation now leads to an extra constraint on Φ and $\mathcal{S}$, that we would have never seen if we would have fixed $n = 1$ in the action. Once at the level of equations we can set $n = 1$ without problems. The resulting equations are obtained by replacing $\mathcal{D} \to \partial_t$ in (2.2.47).

Bianchi type I universes

Bianchi type I (BI) cosmologies are a particular case of cosmological backgrounds (2.2.29) where the metric is homogeneous but generically anisotropic, and the b-field vanishes:

$$g_{mn}(t) = a_m(t)^2\delta_{mn}\,, \quad b_{mn}(t) = 0\,. \tag{2.2.49}$$

Here the internal indices are not summed over. In general, the a_m are d independent scale factors, but we will consider the case where there are only $q \leq d$ different scale factors. We then have groups of N_i scale factors a_i with $i = 1, \ldots, q$ such that $\sum_{i=1}^q N_i = d$. By definition all N_i are non-zero positive integers. The case where all scale factors are different is included for $q = d$ and $N_i = 1$ for all i, while the fully isotropic case is included for $q = 1$ and $N_1 = d$ and it corresponds to a Friedmann-Robertson-Walker (FRW) universe. For each of these q scale factors a_i we define the corresponding Hubble parameter H_i as follows:

$$H_i \equiv \frac{\mathcal{D}a_i}{a_i}\,, \quad i = 1, \ldots, q\,, \tag{2.2.50}$$

where $\mathcal{D}$ is the covariant derivative introduced in (2.2.32).

For this specific ansatz, the generalized metric, as defined in (2.2.19), and its derivative take the simpler form

$$\mathcal{S}_M{}^N = \begin{pmatrix} 0 & a_m^2\delta_{mn} \\ a_m^{-2}\delta^{mn} & 0 \end{pmatrix}, \quad (\mathcal{D}\mathcal{S})_M{}^N = 2\begin{pmatrix} 0 & H_m a_m^2\delta_{mn} \\ -H_m a_m^{-2}\delta^{mn} & 0 \end{pmatrix}, \tag{2.2.51}$$

where there is no sum of repeated indices. With (2.2.51), the trace appearing on the action (2.2.39) is given by

$$\mathrm{Tr}\left((\mathcal{D}\mathcal{S})^2\right) = -8\mathrm{tr}\left(H_m^2\delta_m{}^n\right) = -8\sum_{i=1}^q N_i H_i^2\,, \tag{2.2.52}$$

where we noted that there are only q different directions and each of them is repeated N_i times. Thus, plugging (2.2.52) into (2.2.39), we obtain the two-derivative action for BI backgrounds

$$I_{BI}^{(0)} = \int dt\, n\, e^{-\Phi}\left[-(\mathcal{D}\Phi)^2 + \sum_{i=1}^q N_i H_i^2\right]. \tag{2.2.53}$$

The equations of motion can be obtained by varying (2.2.53) with respect to a_i, Φ and n or by specifying (2.2.47) to BI backgrounds. In both cases we get

$$\mathcal{D}H_i = \mathcal{D}\Phi H_i\,, \quad i = 1, \ldots, q\,, \tag{2.2.54a}$$

$$\mathcal{D}^2\Phi = \sum_{i=1}^q N_i H_i^2\,, \tag{2.2.54b}$$

$$(\mathcal{D}\Phi)^2 = \sum_{i=1}^q N_i H_i^2\,. \tag{2.2.54c}$$

While for generic cosmological backgrounds the theory is invariant under the full duality group, in the context of BI backgrounds, $O(d,d)$ is broken to a $(\mathbb{Z}_2)^q$ invariance under which the fields transform as follow[3]

$$\Phi \to \Phi\,, \quad n \to n\,, \quad a_i \to a_i^{-1} \quad \Rightarrow \quad H_i \to -H_i\,, \quad i = 1, \ldots, q\,, \tag{2.2.55}$$

where the last transformation follows from the ones for a_i and n and from the definition (2.2.50). It is easy to check that (2.2.53) and (2.2.54) are duality invariant. These residual symmetry corresponds to the one generated only by factorized T-duality transformations (2.2.15). Note that these duality transformations act on each different scale factor individually. Because of this, while H_1^2 or H_2^2 are duality invariant, terms like $H_1 H_2$ are not. The latter, however, is invariant under a subset of duality transformations, full-factorized T-dualities (2.2.16), which transform all scale factors simultaneously.

As already mentioned, FRW backgrounds are a particular case of BI universes. Nevertheless, they are interesting enough to study them separately. In this

[3]There is also a $GL(d)$ sector that we omit because it only rescales the a_i.

degenerate case, $q = 1, N_1 = d$, and the ansatz for metric, b-field and dilaton is given by

$$g_{mn}(t) = a(t)^2\delta_{mn}\,, \quad b_{mn}(t) = 0\,, \quad \phi(t) = \frac{1}{2}\Phi(t) + \frac{d}{2}\log a(t)\,, \tag{2.2.56}$$

in terms of a single scale factor, which leads to a single Hubble parameter

$$H(t) \equiv \frac{\mathcal{D}a}{a}\,. \tag{2.2.57}$$

In this case (2.2.51) and (2.2.52) reduce to

$$\mathcal{S} = \begin{pmatrix} 0 & a^2 \\ a^{-2} & 0 \end{pmatrix}, \quad \mathcal{D}\mathcal{S} = 2H\begin{pmatrix} 0 & a^2 \\ -a^{-2} & 0 \end{pmatrix}, \quad \text{Tr}\left((\mathcal{D}\mathcal{S})^2\right) = -8dH^2\,, \tag{2.2.58}$$

where we used a tensor notation for simplicity. The action (2.2.53) reduces to

$$I^{(0)}_{FRW} = \int dt\, n\, e^{-\Phi}\left[-(\mathcal{D}\Phi)^2 + d\,H^2\right]\,. \tag{2.2.59}$$

and its equations of motion are given as a particular case of (2.2.54)

$$\mathcal{D}H = \mathcal{D}\Phi H\,, \tag{2.2.60a}$$
$$\mathcal{D}^2\Phi = dH^2\,, \tag{2.2.60b}$$
$$(\mathcal{D}\Phi)^2 = dH^2\,. \tag{2.2.60c}$$

In this fully-degenerate case, duality transformations reduce to a single $\mathbb{Z}_2$ transformation which acts on fields as

$$\Phi \to \Phi\,, \quad n \to n\,, \quad a \to a^{-1} \quad \Rightarrow \quad H \to -H\,. \tag{2.2.61}$$

2.2.3 Two-derivative $D = 2$ black holes

In Section 2.1.4 we introduced the low energy effective actions for bosonic strings propagating in dilaton-gravity backgrounds in non-critical dimensions. When restricting to $D = 2$ and assuming fields do not depend on one of the coordinates, the two-derivative theory admits a black hole solution [44–46]. In this section, we revisit this black hole background as coming from the dimensional reduction of the two-derivative two-dimensional theory. This review will serve as a basis for an all-order extension in Section 4.1.

We begin from the two-derivative part of (2.1.29), which we rewrite here for convenience, specifying to $D = 2$ dimensions

$$\mathcal{I}^{(0)} = \int d^2x\sqrt{-G}e^{-2\phi}\left(Q^2 + \mathcal{R} + 4\partial_\mu\phi\partial^\mu\phi\right)\,, \tag{2.2.62}$$

with $Q^2 \equiv \frac{16}{\alpha'}$. From the point of view of dimensional reduction, this $D = 2$ space corresponds to the parent theory, analogous to the dilaton-gravity sector of (2.1.1) for the critical case. The target space is parameterized by $x^\mu = (x^0, x^1) =$

(t, x) and from now on we assume fields do not depend on time t. We thus make the ansatz

$$G_{\mu\nu}(x) = \begin{pmatrix} -m^2(x) & 0 \\ 0 & n^2(x) \end{pmatrix} , \qquad \phi(x) = \frac{1}{2}\Phi(x) + \frac{1}{2}\log m(x) \,. \tag{2.2.63}$$

It is worthwhile to compare with the cosmological case studied in the previous section, where all fields depend on time and are independent of d internal spatial coordinates. In that case one has a global $O(d, d)$ duality symmetry. Here, spacetime is two dimensional, and the fields do not depend on time. Time is then the one "internal" coordinate and the duality group is just $O(1, 1)$. In the cosmological setting, the component of the metric in the time-time direction is the lapse function (see (2.2.29)). Here, the component $n(x)$ of the metric in the space-space direction is the analog of the cosmological lapse function. Moreover, (2.2.63) is related to FRW backgrounds (2.2.56) with $d = 1$, after a change of signature. In such a case, $m(x)$ plays the role of "scale-factor".

Analogous to the time-translation invariance of the cosmological backgrounds (2.2.31), in the non-critical case the effective theory enjoys x-reparameterization invariance $x \to x - \xi(x)$ which acts on the fields as follows:

$$\delta n = \partial_x(\xi n) \,, \quad \delta\Phi = \xi\partial_x\Phi \,, \quad \delta m = \xi\partial_x m \,, \tag{2.2.64}$$

As in the cosmological case, the dilaton and internal metric are scalars while $n(x)$ is a density. Here we can also define a covariant derivative

$$\mathcal{D} \equiv \frac{1}{n}\frac{d}{dx} \,, \tag{2.2.65}$$

where we use the same notation as in the cosmological case, since both operators will never appear together in the same equations. As we will see, the metric degrees of freedom enters the effective action trough a Hubble-like parameter

$$M(x) \equiv \frac{\mathcal{D}m(x)}{m(x)} \,. \tag{2.2.66}$$

The group of dualities here is $O(1, 1)$ which contains, apart from $GL(1)$, a $\mathbb{Z}_2$ transformation analogous to the one in FRW backgrounds (2.2.61), which acts on the fields as

$$\begin{aligned} &\Phi(x) \to \hat{\Phi}(x) = \Phi(x) \,, \quad n(x) \to \hat{n}(x) = n(x) \,, \quad m(x) \to \hat{m}(x) = \frac{1}{m(x)} \,, \\ &M(x) \to \hat{M}(x) = -M(x) \,. \end{aligned} \tag{2.2.67}$$

where the last line follows from the transformation for m and n.

To get the effective one-dimensional action in terms of m and Φ, we need to work out the Ricci scalar for the above metric ansatz. The non-vanishing Christoffel symbols and Riemann tensor components this time are given by

$$\begin{aligned} &\Gamma^0_{\ 10} = nM \,, \quad \Gamma^1_{\ 00} = \frac{1}{n}m^2 M \,, \quad \Gamma^1_{\ 11} = \mathcal{D}n \,, \\ &\mathcal{R}_{1010} = n^2 m^2 \left(M^2 + \mathcal{D}M\right) \,. \end{aligned} \tag{2.2.68}$$

The Ricci scalar is therefore

$$\mathcal{R} = -2\left(M^2 + \mathcal{D}M\right) = -2\frac{\mathcal{D}^2 m}{m}\,. \tag{2.2.69}$$

Inserting this into (2.2.62), together with the ansatz for the dilaton in (2.2.63), and integrating by parts in order to have only first-order derivatives we obtain

$$I^{(0)} = \int dx\, n\, e^{-\Phi}\left[Q^2 + (\mathcal{D}\Phi)^2 - M^2\right]\,, \tag{2.2.70}$$

which is manifestly diffeomorphism and duality invariant. The latter is a consequence of (2.2.70) containing only even powers of M.

The equations of motion follow from the general variation

$$\delta I^{(0)} = \int dx\, n\, e^{-\Phi}\left[E_m \frac{\delta m}{m} + E_\Phi \delta\Phi + E_n \frac{\delta n}{n}\right]\,. \tag{2.2.71}$$

with

$$E_m = 2\left(\mathcal{D}M - \mathcal{D}\Phi M\right) = 0\,, \tag{2.2.72a}$$

$$E_\Phi = -2\mathcal{D}^2\Phi + (\mathcal{D}\Phi)^2 + M^2 - Q^2 = 0\,, \tag{2.2.72b}$$

$$E_n = -(\mathcal{D}\Phi)^2 + M^2 + Q^2 = 0\,. \tag{2.2.72c}$$

Combining the dilaton and lapse equations and dropping overall numerical factors we get the equivalent system

$$\mathcal{D}M = \mathcal{D}\Phi M\,, \tag{2.2.73a}$$

$$\mathcal{D}^2\Phi = M^2\,, \tag{2.2.73b}$$

$$(\mathcal{D}\Phi)^2 = M^2 + Q^2\,. \tag{2.2.73c}$$

It is interesting to notice the similarities between the two-dimensional black hole background and FRW. In particular, the difference between (2.2.73) and (2.2.60) is just the additional Q^2 factor in the lapse equation. While seemingly innocent, the cosmological term is the only reason why (2.2.60) lead to cosmology while (2.2.73) admits a black hole solution. The latter takes the form [44–46]:

$$\begin{aligned} ds^2 &= -m^2(x)dt^2 + \frac{1}{m^2(x)}dx^2\,, \quad m^2(x) = 1 - a\, e^{Qx}\,, \\ \Phi(x) &= Qx - \frac{1}{2}\log\left|1 - a\, e^{Qx}\right| \;+\; \hat{\Phi}_0\,, \end{aligned} \tag{2.2.74}$$

where $a > 0$ and $\hat{\Phi}_0$ are integration constants. This solution describes the exterior region of a black hole for $x < x_H$ and the interior region for $x > x_H$, with $x_H \equiv -\frac{\log a}{Q}$. The latter corresponds to the position of the black hole horizon, which is a coordinate singularity. For the exterior, $x = -\infty$ corresponds to the asymptotically flat region, while for the interior, $x = \infty$ leads to the black hole singularity. As opposed to the position of the horizon, this singular point cannot be removed by a change of coordinates. It anticipates that the two-derivative black hole solution coming from string theory cannot be the end of the story for a complete quantum theory of gravity, and so new ingredients should come into play. We will see later in Section 4.3.2 how higher-derivative corrections to (2.2.70), already present in classical string theory, become essential in the resolution of the black hole singularity.

2.3 Double Field Theories

From previous sections we convinced ourselves of the ubiquitous presence of a global duality group $O(d,d|\mathbb{R})$ in string low energy effective theories. This duality group is a residual effect of the stringy origin from which these field theories come. More precisely, its presence is rooted in the one-dimensional nature of strings, a crucial difference to the standard point-particle approach of field theories. When strings propagate on d-dimensional toroidal backgrounds, this novel feature allows strings not only to move along the internal space with momentum p, like particles, but also to wrap around the compact toroidal directions. The latter introduces the so-called winding number w, which, as opposed to the continuum momentum p, is quantized. When the string is quantized, the internal momentum p becomes discrete and the full string theory enjoys a novel duality (T-duality) generated by the discrete group $O(d,d|\mathbb{Z})$ [46], which maps winding modes in a compact space to momentum, or Kaluza-Klein, modes in another (dual) compact space. We talk about a duality rather than a proper symmetry because $O(d,d|\mathbb{Z})$ changes backgrounds, but both of them, the original and the dual one, lead to physically equivalent theories in which momentum and winding are exchanged. This $O(d,d|\mathbb{Z})$ duality of the full quantum worldsheet theory in d-dimensional toroidal backgrounds is enhanced to its continuous version $O(d,d|\mathbb{R})$ when the theory is truncated to be independent of the d-dimensional internal space. In this extreme scenario, all memory of Kaluza-Klein or winding modes is lost.

At the level of the target space, the enhanced duality group $O(d,d|\mathbb{R})$ behaves as a genuine global symmetry for string low energy effective theories in arbitrary backgrounds with d abelian isometries. This is exactly the duality group $O(d,d)$ with which we have been dealing so far. We recall from Section 2.2.1 that this symmetry emerged only upon dimensional reduction and assuming independence on the internal space. This symmetry is manifest for the effective theory as it can be seen from (2.2.8) but is highly non obvious from the parent theory (2.1.1)! This is why sometimes we refer to T-duality as a hidden symmetry of the D-dimensional parent theory. Moreover, while the compactified theory can encode information about Kaluza-Klein modes upon relaxing the internal independence condition (2.2.6), there is no way to say anything about winding modes since already the parent theory (2.1.1) lacks that information.

The hidden nature of T-duality and the absence of winding modes, makes the D-dimensional theory (2.1.1) not the best formulation to deal with stringy effects. It is this very need that triggered the formulation of Double Field Theory (DFT) [26–28] (see [62] and [63] for reviews), a proposal to incorporate T-duality as a manifest symmetry of a field theory. In what follows we give a brief introduction to the background independent formulation of the theory restricted to the universal massless sector [28].

2.3.1 A brief introduction to DFT

DFT lives in a $2D$-dimensional double target space parameterized by coordinates

$$\hat{X}^{\hat{M}} = (X^\mu, \tilde{X}_\mu)\,, \tag{2.3.1}$$

with $\hat{M} = 1, \ldots, 2D$. The presence of dual coordinates is reminiscent of the string winding modes. To see why this is the case, we can reorder the coordinates in a way analogous to dimensional reduction (2.2.1)

$$\hat{X}^{\hat{M}} = (x^i, \tilde{x}_i, y^m, \tilde{y}_m)\,, \tag{2.3.2}$$

where $i = 0, \ldots, n-1$ and $m = 1, \ldots, d$ such that $d + n = D$. Here, $\tilde{y}_m$ are interpreted as dual conjugate to the winding modes, in the same way that y^m are the dual conjugate to momenta. It is in this sense that DFT is a "doubled" theory: it doubles the coordinates of the compact space

$$Y^M \equiv (y^m, \tilde{y}_m)\,, \tag{2.3.3}$$

with $M = 1, \ldots, 2d$. The doubling in the external direction $\tilde{x}_i$ has no real physical interpretation but it enters the theory just for purely aesthetic reasons. If wanted, we could just fix $\tilde{x}_i = 0$ and the theory would have an internal continuous $O(d, d|\mathbb{R})$ symmetry. This leads to the split-formulation of DFT [64]. However, to present the theory in here we find useful to keep the dual external coordinates, in which case the theory has a bigger fictitious $O(D, D)$ symmetry.

When it comes to the field content, there are different formulations of DFT depending on which string and sector of its spectrum we are interested in. Here we are concerned with the one capturing the universal massless sector studied in Section 2.1.1, which historically, is also the first background independent formulation of the theory. DFT introduces the degrees of freedom for the metric $G_{\mu\nu}$, B-field $B_{\mu\nu}$ and dilaton ϕ in an $O(D, D)$ covariant fashion through the fields

$$\hat{\mathcal{H}}_{\hat{M}\hat{N}}(\hat{X}) \equiv \begin{pmatrix} G_{\mu\nu} - B_{\mu\rho}G^{\rho\sigma}B_{\sigma n} & B_{\mu\rho}G^{\rho\nu} \\ -G^{\mu\rho}B_{\rho\nu} & G^{\mu\nu} \end{pmatrix}, \tag{2.3.4a}$$

$$d(\hat{X}) \equiv \phi - \frac{1}{2}\log\sqrt{-G} \quad \Rightarrow \quad \sqrt{-G}e^{-2\phi} = e^{-2d}\,, \tag{2.3.4b}$$

where the first one is nothing but a higher-dimensional version of the internal generalized metric of (2.2.4c) and $d(\hat{X})$ is called the generalized dilaton. Analogous to their $2d$-dimensional counterpart, d is a scalar under $O(D, D)$ while $\hat{\mathcal{H}}$ is a symmetric 2-tensor satisfying $\hat{\mathcal{H}}_{\hat{M}\hat{P}}\hat{\eta}^{\hat{P}\hat{Q}}\hat{\mathcal{H}}_{\hat{Q}\hat{N}} = \hat{\eta}_{\hat{M}\hat{N}}$ with the $O(D, D)$ invariant metric

$$\hat{\eta}_{\hat{M}\hat{N}} = \begin{pmatrix} 0 & \delta_\mu{}^\nu \\ \delta^{\mu\nu} & 0 \end{pmatrix}, \tag{2.3.5}$$

which is the D-dimensional version of (2.2.10).

While formally all fields and gauge parameters depend on the doubled coordinates $\hat{X}$, consistency of the theory demands to impose an additional constraint named strong constraint or section condition, given by

$$\hat{\eta}^{\hat{M}\hat{N}}\partial_{\hat{M}}\partial_{\hat{N}}(\cdots) = 0\,, \tag{2.3.6}$$

where $\cdots$ stands for any field or product of fields. The fact that (2.3.6) holds also over products of fields gives the constraint the strong character and is the responsible for making DFT a mere duality-invariant reformulation of the bosonic massless sector of supergravity. More precisely, while in principle the field content (2.3.4) depends on doubled coordinates, (2.3.6) half the degrees of freedom as it can be seen by writing it in components

$$\tilde{\partial}^{\mu}\partial_{\mu}(\cdots) = 0\,, \tag{2.3.7}$$

with $\tilde{\partial}^{\mu} \equiv \frac{\partial}{\partial \tilde{X}_{\mu}}$. Obviously,

$$\tilde{\partial}^{\mu}(\cdots) = 0 \tag{2.3.8}$$

is a solution to the constraint, where fields do not depend on dual coordinates, and it can be proven that any other solution is physically equivalent to it.

In (2.1.5) we introduced how diffeomorphisms and gauge transformation for the B-field acts on the massless fields. Now G and B are unified in the generalized metric (2.3.4a) and so the symmetries are also unified into generalized diffeomorphisms. Infinitesimally, these novel symmetries are generated by double parameters

$$\hat{\xi}^{\hat{M}} \equiv (\xi^{\mu}, \tilde{\xi}_{\mu})\,, \tag{2.3.9}$$

and they act on the fields (2.3.4) as follows

$$\delta\hat{\mathcal{H}}_{\hat{M}\hat{N}} = \hat{\mathcal{L}}_{\hat{\xi}}\hat{\mathcal{H}}_{\hat{M}\hat{N}} \equiv \mathcal{L}_{\hat{\xi}}\hat{\mathcal{H}}_{\hat{M}\hat{N}} + \mathcal{K}_{\hat{M}}{}^{\hat{P}}\hat{\mathcal{H}}_{\hat{P}\hat{N}} + \mathcal{K}_{\hat{N}}{}^{\hat{P}}\hat{\mathcal{H}}_{\hat{M}\hat{P}}\,, \qquad \mathcal{K}_{\hat{M}\hat{N}} \equiv 2\partial_{[\hat{M}}\hat{\xi}_{\hat{N}]}\,, \tag{2.3.10a}$$

$$\delta\left(e^{-2d}\right) = \partial_{\hat{M}}\left(\hat{\xi}^{\hat{M}}e^{-2d}\right)\,. \tag{2.3.10b}$$

Here, $\mathcal{L}_{\hat{\xi}}$ is the standard Lie-derivative in $2D$ dimensions and $\mathcal{K}_{\hat{M}}{}^{\hat{N}}$ measures the departure from conventional Riemannian geometry. Together, they form the generalized Lie derivative $\hat{\mathcal{L}}_{\hat{\xi}}$ [65]. In the context of generalized geometry [66, 67], $\hat{\mathcal{L}}_{\hat{\xi}}$ corresponds to an $O(D,D)$ extension of the Dorfman bracket, and so sometimes is also called D-bracket. From the perspective of generalized geometry, $\hat{\mathcal{H}}_{\hat{M}\hat{N}}$ is a symmetric $\binom{0}{2}$ tensor, while e^{-2d} transforms as a generalized density and so it corresponds to the integration measure in the DFT action. Note, however, that the generalized dilaton itself transforms as

$$\delta d = \hat{\xi}^{\hat{P}}\partial_{\hat{P}}d - \frac{1}{2}\partial_{\hat{P}}\hat{\xi}^{\hat{P}}\,, \tag{2.3.11}$$

and so it does not belong to any $O(D,D)$ representation. It can be shown that upon solving the strong constraint via (2.3.8), and rewriting the $O(D,D)$ objects in terms of supergravity fields using (2.3.4), (2.3.10) reduce to the standard transformations for the massless sector (2.1.5)!

Consistency of the theory requires generalized diffeomorphisms to form an algebra [65] and so they must satisfy a closure condition: the commutator of two successive transformations gives again a transformation in terms of a new parameter determined by the bracket of the algebra. In the case of generalized Lie derivatives we obtain

$$\left[\hat{\mathcal{L}}_{\hat{\xi}_1}, \hat{\mathcal{L}}_{\hat{\xi}_2}\right] = \hat{\mathcal{L}}_{[\hat{\xi}_1,\hat{\xi}_2]_{(C)}}\,, \tag{2.3.12}$$

with the C-bracket defined as

$$[\hat{\xi}_1, \hat{\xi}_2]^{\hat{M}}_{(C)} \equiv 2\hat{\xi}^{\hat{P}}_{[1}\partial_{\hat{P}}\hat{\xi}^{\hat{M}}_{2]} - \hat{\xi}^{P}_{[1}\partial^{\hat{M}}\hat{\xi}_{2]\hat{P}}\,. \tag{2.3.13}$$

The first term corresponds to the bracket of standard diffeomorphisms in $2D$ dimensions. The closure of the algebra (2.3.12) requires the strong constraint (2.3.6). Upon solving the strong constrain with $\tilde{\partial}^{\mu} = 0$, (2.3.13) reduces to the bracket for not only diffeomorphisms but also B-field gauge transformations. In generalized geometry this is the Courant bracket and (2.3.13) is its duality-covariant extension.

The action of DFT can be written in terms of the generalized Ricci scalar $\hat{\mathcal{R}}$

$$\begin{aligned}
\mathcal{I}_{\rm DFT} \equiv & \int d^{2D}\hat{X}e^{-2d}\hat{\mathcal{R}}\,,\\
\hat{\mathcal{R}} \equiv & \,4\hat{\mathcal{H}}^{\hat{M}\hat{N}}\partial_{\hat{M}}\partial_{\hat{N}}d - \partial_{\hat{M}}\partial_{\hat{N}}\hat{\mathcal{H}}^{\hat{M}\hat{N}} - 4\hat{\mathcal{H}}^{\hat{M}\hat{N}}\partial_{\hat{M}}d\partial_{\hat{N}}d + 4\partial_{\hat{M}}\hat{\mathcal{H}}^{\hat{M}\hat{N}}\partial_{\hat{N}}d\\
&+\frac{1}{8}\hat{\mathcal{H}}^{\hat{M}\hat{N}}\partial_{\hat{M}}\hat{\mathcal{H}}^{\hat{P}\hat{Q}}\partial_{\hat{N}}\hat{\mathcal{H}}_{\hat{P}\hat{Q}} - \frac{1}{2}\hat{\mathcal{H}}^{\hat{M}\hat{N}}\partial_{\hat{M}}\hat{\mathcal{H}}^{\hat{P}\hat{Q}}\partial_{\hat{P}}\hat{\mathcal{H}}_{\hat{N}\hat{Q}}\,,
\end{aligned} \tag{2.3.14}$$

which is a generalized scalar, namely $\delta\hat{\mathcal{R}} = \hat{\mathcal{L}}_{\hat{\xi}}\hat{\mathcal{R}} = \hat{\xi}^{\hat{P}}\partial_{\hat{P}}\hat{\mathcal{R}}$. While this is a well-defined geometric quantity in the context of generalized geometry, there is no well-defined notion of Riemann tensor [68]. After some tedious algebra, one can check that the action is invariant under generalized diffeomorphisms (2.3.10). This is only true, however, once (2.3.6) is imposed. By picking the supergravity solution (2.3.8) and using the ansatz (2.3.4), (2.3.14) reduces to (2.1.1), the action of $\mathcal{N} = 0$ supergravity. We will not do it here, but in principle one can vary (2.3.14) with respect to $\hat{\mathcal{H}}$ and d to get the DFT equations of motion. As expected, upon the section condition, these reduce to a system equivalent to (2.1.3).

2.3.2 HSZ theory and its cosmological reduction

While DFT allows us to describe the massless sector of supegravity in a unified geometric picture, when it comes to α' corrections the double theory does not represent a major simplification of the problem[4]. However, in the quest of an α'-complete spacetime theory, Hohm, Siegel and Zwiebach took inspiration from DFT to develop a double α'-geometry, also referred as HSZ theory [72]. Its

[4] Although see [50] and [69] for a construction of DFT at order α', and [70, 71] for a higher-order proposal of heterotic DFT.

construction was based on a non-standard chiral CFT and is thus not a conventional string theory [5]. Nevertheless, HSZ theory shares crucial features of any string theory, such as duality invariance under $O(d,d|\mathbb{R})$ for backgrounds with d abelian isometries, and the presence of infinitely many higher-derivative corrections for the massless fields. While in its original formulation HSZ theory carries only up to six derivatives, it also features extra massive fields, in addition to the universal massless sector, and integrating out these extra fields induces an infinite tower of α' corrections for the massless fields that are kept. These higher-derivative corrections include a Green-Schwarz-type deformation at order α' and a Riemann-cube invariant at order α'^2 [69, 76, 77], but beyond that only very little is known. Truncating to the zeroth order in α', the theory reduces to DFT or, upon solving the strong constraint, to supergravity (2.1.1).

As DFT, HSZ also lives in a $2D$-dimensional space with coordinates $\hat{X}^{\hat{M}}$ and double indices are also contracted with the $O(D,D)$ invariant metric (2.3.5). The field content includes the "double metric" $\hat{\mathcal{M}}_{\hat{M}\hat{N}}$ and a generalized dilaton field d. This theory is also subjected to the strong constraint (2.3.6). A crucial difference with DFT, is that the double metric is symmetric but otherwise unconstrained, namely $\hat{\mathcal{M}}$ is not an element of $O(D,D)$ in general. This implies that $\hat{\mathcal{M}}$ is not of the form (2.3.4a), and so it contains more degrees of freedom corresponding to some novel massive states.

Under infinitesimal generalized diffeomorphisms with parameter $\hat{\xi}^{\hat{M}}$, the generalized dilaton transforms as in conventional DFT (2.3.11) while the double metric receives linear and quadratic corrections in α'

$$\delta\hat{\mathcal{M}}_{\hat{M}\hat{N}} = \hat{\mathcal{L}}_\xi \hat{\mathcal{M}}_{\hat{M}\hat{N}} + \alpha' \mathcal{J}^{(1)}_{\hat{M}\hat{N}} + \alpha'^2 \mathcal{J}^{(2)}_{\hat{M}\hat{N}} \,. \tag{2.3.15}$$

Here $\hat{\mathcal{L}}_\xi$ is the generalized Lie derivative as given in (2.3.10a) and the higher-derivative contributions are given by

$$\begin{aligned} \mathcal{J}^{(1)}_{\hat{M}\hat{N}} &\equiv -\frac{1}{2}\partial_{\hat{M}}\hat{\mathcal{M}}^{\hat{P}\hat{Q}}\partial_{\hat{P}}\mathcal{K}_{\hat{Q}\hat{N}} - \partial_{\hat{P}}\hat{\mathcal{M}}_{\hat{Q}\hat{M}}\partial_{\hat{N}}\mathcal{K}^{\hat{Q}\hat{P}} + (\hat{M} \rightleftarrows \hat{N}) \,, \\ \mathcal{J}^{(2)}_{\hat{M}\hat{N}} &\equiv -\frac{1}{4}\partial_{\hat{M}\hat{K}}\hat{\mathcal{M}}^{\hat{P}\hat{Q}}\partial_{\hat{N}\hat{P}}\mathcal{K}_{\hat{Q}}{}^{\hat{K}} + (\hat{M} \rightleftarrows \hat{N}) \,. \end{aligned} \tag{2.3.16}$$

These transformations close under a deformation of the C-bracket (2.3.13) which we call the C'-bracket:

$$\left[\hat{\xi}_1, \hat{\xi}_2\right]^{\hat{M}}_{(C')} \equiv \left[\hat{\xi}_1, \hat{\xi}_2\right]^{\hat{M}}_{(C)} + \alpha' \partial_{\hat{P}}\hat{\xi}^{\hat{Q}}_{[1}\partial^{\hat{M}}\partial_{\hat{Q}}\hat{\xi}^{\hat{P}}_{2]} \,. \tag{2.3.17}$$

The dynamics of the theory is encoded in an action, that can be written compactly as

$$\mathcal{I}_{\text{HSZ}} = \int d^{2D}\hat{X}\, e^{-2d} \left\langle \hat{\mathcal{M}} \,\middle|\, \hat{\eta} - \frac{1}{6}\hat{\mathcal{M}} \star \hat{\mathcal{M}} \right\rangle \,. \tag{2.3.18}$$

The definitions for the inner product $\langle \cdot | \cdot \rangle$ and star-product $\star$ involve long expressions in terms of $\hat{\mathcal{M}}$, d and combinations of them up to and including

[5] It also appears to be closely related to the "chiral string theory" of [73] and/or to the ambitwistor string [74, 75], but to our knowledge the precise connections have not been established.

six derivatives. Since these explicit expressions are not very illuminating, we refer the reader to equations (2.11), (2.12) and (2.14) of [78] where the definitions are given in detail, or to appendix D therein where the complete action is exposed. The exact gauge symmetry under (2.3.11) and (2.3.15) can be checked once the definitions of the products are used, together with the strong constraint (2.3.6).

Cosmological reduction

We will now truncate the above theory to a cosmological ansatz in which fields depend only on time, and we will prove that the truncation is consistent [5]. The steps to follow are very similar to the ones for the cosmological reduction of Section 2.2.2 but this time the parent theory lives in a doubled space so some small modifications are required. To this end we assume a split of the coordinates and indices as follows

$$\hat{X}^{\hat{M}} = \left(t\,,\ \tilde{t}\,,\ Y^M\right)\,, \qquad {}_{\hat{M}} = \left({}_0\,,\ {}^0\,,\ {}_M\right)\,, \qquad M = 1,\ldots,2d\,, \tag{2.3.19}$$

with $d = D-1$. This ansatz breaks the manifest $O(D,D)$ invariance to $O(1,1)\times O(d,d)$. Furthermore, we will solve the strong constraint by selecting a frame in which the fields do not depend on $\tilde{t}$ nor Y^M, which breaks the $O(1,1)$ factor and importantly *preserves* the internal $O(d,d)$. We will thus set

$$\partial^0 = \partial_M = 0 \tag{2.3.20}$$

everywhere in the field equations and gauge transformations.

Let us now turn to the decompositions of the fields and the $O(D,D)$ metric, which are given by

$$\hat{\eta}_{\hat{M}\hat{N}} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \eta_{MN} \end{pmatrix}\,, \qquad \hat{\mathcal{M}}_{\hat{M}\hat{N}} = \begin{pmatrix} -n^2 B & A & 0 \\ A & -\frac{1}{n^2} & 0 \\ 0 & 0 & \mathcal{M}_{MN} \end{pmatrix}\,, \qquad d = \frac{1}{2}\Phi - \frac{1}{2}\ln n\,, \tag{2.3.21}$$

where all fields depend only on time t. Here we recognize the $O(d,d)$ invariant metric (2.2.10), the lapse function of cosmological backgrounds $n(t)$ and the duality-invariant dilaton $\Phi(t)$. Additionally, we have two extra scalar fields $A(t)$ and $B(t)$ and the internal sector of the double metric is now parameterized by $\mathcal{M}_{MN}(t)$. The latter is the analogous to the internal generalized metric $\mathcal{H}_{MN}$ in (2.2.4c), but this time $\mathcal{M}_{MN}$ is not an element of $O(d,d)$, so it contains more degrees of freedom.

Apart from the exclusive time dependence, the only truncation applied in (2.3.21) is given by the vanishing of the vector components of the double metric. This is a consistent choice and the proof is analogous to the one given for cosmological backgrounds below (2.2.29). In short, fixing the off-diagonal components $\hat{\mathcal{M}}_{0M} = \hat{\mathcal{M}}^0{}_M = 0$ is consistent because there are no $O(d,d)$ vectors apart from these components themselves. Additionally, one would be tempted to further reduce the external 2×2 block in the double metric to be diagonal and $O(1,1)$ valued by setting $A = 0$, $B = 1$, but this turns out to be inconsistent.

By inspecting the action of generalized diffeomorphisms (2.3.15) for this ansatz, with the gauge parameter decomposed as

$$\hat{\xi}^{\hat{M}} = \left(\xi^0, \xi_0, \Lambda^M\right) \equiv \left(\xi, 0, 0\right) , \tag{2.3.22}$$

we find the following gauge transformations for the component fields

$$\begin{aligned}
\delta n &= \xi \dot{n} + n\dot{\xi} \, , \\
\delta A &= \xi \dot{A} - 3\alpha' \ddot{\xi} \frac{\dot{n}}{n^3} \, , \\
\delta B &= \xi \dot{B} + \alpha' \frac{1}{n^2} \ddot{\xi} \dot{A} + \alpha'^2 \frac{1}{n^5} \dddot{\xi} \left(\ddot{n} - 3\frac{\dot{n}^2}{n}\right) , \\
\delta \mathcal{Z}_M{}^N &= \xi \dot{\mathcal{Z}}_M{}^N \, , \\
\delta \Phi &= \xi \dot{\Phi} \, ,
\end{aligned} \tag{2.3.23}$$

where we introduced a bottom-up index notation for the double metric

$$\mathcal{Z}_M{}^N \equiv \mathcal{M}_{MP} \eta^{PN} \, , \tag{2.3.24}$$

and used the dot notation for time derivatives $\partial_t \Psi = \partial_0 \Psi \equiv \dot{\Psi}$. To zeroth order in α' we recognize the familiar transformations under time reparametrizations $t \to t - \xi(t)$, but these transformations receive α'-corrections. The fact that the corrections to the transformations of A and B contain corrections not depending on A and B themselves, prevents us from setting them to a constant, so both A and B *must* be kept for consistency of the truncation. Instead, as already stated, setting to zero the vectorial components of the double metric is perfectly consistent. In fact, the transformations (2.3.22) acting on these components vanish when the components themselves are set to zero, contrary to what happens with A and B. As a consequence of this truncation, the full original gauge symmetry of HSZ (2.3.15) is now broken to time reparametrizations (2.3.22).

It is instructive to inspect also the gauge algebra under this cosmological reduction. One obtains with (2.3.17)

$$\left[\hat{\xi}_1, \hat{\xi}_2\right]_{0\,(C')} = 2\xi_{[1}\dot{\xi}_{2]} \, , \tag{2.3.25}$$

$$\left[\hat{\xi}_1, \hat{\xi}_2\right]^0_{(C')} = \alpha' \dot{\xi}_{[1}\ddot{\xi}_{2]} = \left[\sqrt{\tfrac{\alpha'}{2}}\dot{\xi}_1, \sqrt{\tfrac{\alpha'}{2}}\dot{\xi}_2\right]_{0\,(C')} \, , \tag{2.3.26}$$

$$\left[\hat{\xi}_1, \hat{\xi}_2\right]^M_{(C')} = 0 \, . \tag{2.3.27}$$

Given the relationship in the second line, one may suspect that the only surviving algebra is that of standard one-dimensional diffeomorphisms, suggesting that there should be a field basis in which this symmetry is realized in the standard way. Indeed, we can find an explicit field redefinition that removes the higher-derivative terms in δA and δB. To this end, it is convenient to remember the covariant derivative $\mathcal{D} \equiv \frac{1}{n}\partial_t$. Specifically, writing the original fields in terms of new primed fields as

$$\begin{aligned} A &= A' - \frac{3}{2}\alpha'(\mathcal{D}\ln n)^2 \, , \\ B &= B' + \alpha'(\mathcal{D}\ln n)\mathcal{D}A' - \alpha'^2 \left[\frac{1}{4}(\mathcal{D}\ln n)^4 + (\mathcal{D}^2\ln n)(\mathcal{D}\ln n)^2 - \frac{1}{2}(\mathcal{D}^2\ln n)^2\right] \, , \end{aligned} \tag{2.3.28}$$

it is straightforward to verify that for A' and B' being reparametrization scalars, the higher-derivative terms induce precisely the higher-order corrections in (2.3.23). Furthermore, it is immediate that the above relations can be inverted hence proving that this is a legal field redefinition. All in all, we can express the theory in terms of fields given by the lapse function n and a number of reparametrization scalars, with transformation rules

$$\delta n = \xi\dot{n} + n\dot{\xi}\, , \quad \delta A = \xi\dot{A}\, , \quad \delta B = \xi\dot{B}\, , \quad \delta \mathcal{Z}_M{}^N = \xi\dot{\mathcal{Z}}_M{}^N\, , \quad \delta\Phi = \xi\dot{\Phi}\, , \tag{2.3.29}$$

where we removed the primes from A' and B'.

We now give the HSZ action in this cosmological reduction, which is obtained by plugging the ansatz (2.2.2) together with the field redefinitions (2.3.28) into (2.3.18). More precisely, we performed the reduction at the level of the inner and star products using their explicit definitions as given in [78] and then combined these results back into the form of the action (2.3.18). As a consistency check we can use that in this field basis the diffeomorphisms act in the usual way, which implies that the derivatives of the lapse function should combine to form covariant derivatives $\mathcal{D} = \frac{1}{n}\partial_t$ of the scalar fields.[6] Upon integration by parts, we find that the final manifestly gauge-invariant action is given by

$$I = \int dt\, n\, e^{-\Phi} \left[\frac{1}{\alpha'}L^{(-1)} + L^{(0)} + \alpha' L^{(1)} + \alpha'^2 L^{(2)}\right] \, , \tag{2.3.30}$$

where

$$\begin{aligned} L^{(-1)} &= \frac{1}{2}\mathrm{Tr}\,(\mathcal{Z}) - \frac{1}{6}\mathrm{Tr}\left(\mathcal{Z}^3\right) + A - \frac{1}{3}A^3 - AB\, , \\ L^{(0)} &= -\frac{1}{8}\mathrm{Tr}\left((\mathcal{D}\mathcal{Z})^2\right) - \frac{3}{2}(\mathcal{D}\Phi)^2 + \frac{1}{4}(\mathcal{D}A)^2 + \frac{3}{2}A^2\mathcal{D}^2\Phi + \frac{1}{2}B\mathcal{D}^2\Phi\, , \\ L^{(1)} &= \frac{1}{2}A\left[\mathcal{D}^4\Phi - \mathcal{D}\Phi\mathcal{D}^3\Phi - 3(\mathcal{D}^2\Phi)^2\right] \, , \\ L^{(2)} &= \frac{1}{4}(\mathcal{D}^3\Phi)^2 + \frac{1}{2}(\mathcal{D}^2\Phi)^3\, . \end{aligned} \tag{2.3.31}$$

Finally, we can bring the full action to its simplest form by performing the

[6] As a second consistency check, we also performed the reduction directly at the level of the α'-expanded action as given in Appendix D of [78]. We used Cadabra2 [79] for the reductions and consistency checks.

following field redefinition

$$\begin{aligned}
B' =& B - 1 + \frac{1}{3}A^2 - \alpha' \left[\frac{4}{3}A\mathcal{D}^2\Phi + \frac{1}{4}\mathcal{D}\Phi\mathcal{D}A - \frac{1}{4}\mathcal{D}^2 A\right] \\
&\qquad\qquad - \frac{\alpha'^2}{2}\left[\frac{3}{4}\mathcal{D}^4\Phi - \frac{3}{4}\mathcal{D}\Phi\mathcal{D}^3\Phi - \frac{5}{3}(\mathcal{D}^2\Phi)^2\right] , \\
A' =& - A + \frac{\alpha'}{2}\mathcal{D}^2\Phi ,
\end{aligned} \tag{2.3.32}$$

to get (omitting the primes for A' and B') [5]

$$\begin{aligned}
I = \int dt\, n\, e^{-\Phi} \Big\{ \frac{1}{\alpha'}\Big[AB + \frac{1}{2}\mathrm{Tr}\,(\mathcal{Z}) - \frac{1}{6}\mathrm{Tr}\,(\mathcal{Z}^3)\Big] - \frac{1}{8}\mathrm{Tr}\left((\mathcal{D}\mathcal{Z})^2\right) - (\mathcal{D}\Phi)^2 \\
+ \frac{\alpha'^2}{4}\Big[\frac{1}{4}(\mathcal{D}^3\Phi)^2 + \frac{1}{3}(\mathcal{D}^2\Phi)^3\Big]\Big\} .
\end{aligned} \tag{2.3.33}$$

We observe that after the above series of field redefinitions A and B completely trivialize in the sense that their equations of motion simply set them to zero. Therefore, the original theory given by (2.3.31) is equivalent to an effective theory for $\mathcal{Z}, n$ and Φ only, whose action is given by (2.3.33) after setting $A = B = 0$.

At the beginning of this section we introduced HSZ theory as an α'-complete extension of DFT. In particular, we mentioned that expanding the HSZ action (2.3.18) in powers of α', the zeroth order corresponds to the action for DFT (2.3.14). Indeed, one can see how this works for cosmological backgrounds by taking (2.3.33) and assuming that, to leading order, the double metric and generalized metric (2.2.19) coincide, namely $\mathcal{Z} = \mathcal{S} + \mathcal{O}(\alpha')$. Neglecting higher orders, the resulting action is exactly (2.2.39)! This computation, together with a thorough analysis of HSZ in cosmological backgrounds, will be elaborated in Section 3.3. In there, we will take (2.3.33) as starting point, not only to make contact with supergravity and its α' corrections, but also to analyze non-perturbative features of HSZ.

Chapter 3

Cosmological Backgrounds and α'

The previous chapter sets the stage for the development of the main results of this work. There we introduced the low energy limit of string theories for the massless sector, the concept of α' corrections, dimensional reduction and the duality group $O(d,d)$ that emerges in toroidal compactifications. All these concepts reappear in this chapter in a cooperative way to analyze α'-corrections in cosmological backgrounds. To this end, we begin in Section 3.1 reviewing the systematic approach developed in [39, 40] to arrive at the cosmological classification, and then revisit the algorithm for BI universes. In Section 3.2 we study dilaton-gravity theories in cosmological backgrounds up to and including order α'^3. Finally, Section 3.3 is devoted to analyze the α' structure of HSZ in cosmological backgrounds.

Section 3.1.3 contains results of [6], Section 3.2 is largely based on [3] and Section 3.3 follows closely [5].

3.1 The Cosmological Classification

As shown in Section 2.2.1, upon toroidal compactifications the two-derivative $\mathcal{N}=0$ supergravity action (2.1.1) reduces to (2.2.8), which is manifestly $O(d,d)$ invariant. This feature is not unique to the two-derivative action, but to the full string low energy effective theory. This was proven by Sen, who showed that classical (tree-level) string theory truncated to states of zero momentum along d directions admits an $O(d,d|\mathbb{R})$ invariance to all orders in α' [23]! A subtlety in here is that one should allow for the $O(d,d)$ transformations themselves to receive α' corrections [80]. This was indeed found in Meissner's seminal work on the cosmological reduction to first order in α' when the theory was written in terms of standard supegravity fields [42]. However, in [39, 40] it was proven that in one dimension these corrections can be removed by changing to another field basis[1]. In those works, Hohm and Zwiebach developed a general framework that systematically uses field redefinitions to bring cosmological actions to a form that involves only first-order derivatives. The claim is that this procedure eliminates all ambiguities resulting from the freedom to perform integrations by part and to use lower-order equations of motion to modify

[1]For dimensional reduction to a generic number of dimensions, however, the $O(d,d)$ transformations receive non-trivial Green-Schwarz-type α'-deformations [81, 82], which has a precursor in double field theory [50, 69, 72, 83], see also [78, 84–89].

higher-derivative terms. Upon passing to this canonical field basis the $O(d,d)$ invariance, if present, should take the standard form.

This systematic procedure introduces a classification for the most general duality-invariant theory to all orders containing metric, b-field and dilaton in cosmological backgrounds:

$$\begin{aligned} I = \int dt\, n\, e^{-\Phi} \Big[& -(\mathcal{D}\Phi)^2 - \frac{1}{8}\text{Tr}\left((\mathcal{D}\mathcal{S})^2\right) \\ & + \alpha' c_{1,0}\text{Tr}\left((\mathcal{D}\mathcal{S})^4\right) + \alpha'^2 c_{2,0}\text{Tr}\left((\mathcal{D}\mathcal{S})^6\right) \\ & + \alpha'^3 \left(c_{3,0}\text{Tr}\left((\mathcal{D}\mathcal{S})^8\right) + c_{3,1}\left[\text{Tr}\left((\mathcal{D}\mathcal{S})^4\right)\right]^2 \right) \\ & + \alpha'^4 \left(c_{4,0}\,\text{Tr}\left((\mathcal{D}\mathcal{S})^{10}\right) + c_{4,1}\,\text{Tr}\left((\mathcal{D}\mathcal{S})^6\right)\,\text{Tr}\left((\mathcal{D}\mathcal{S})^4\right)\right) + \ldots \Big] \,. \end{aligned} \tag{3.1.1}$$

In here, due to $O(d,d)$, fields enter the classification through $\mathcal{S}, \Phi$ and their covariant derivatives $\mathcal{D}$, which contain the lapse function n. The first line in (3.1.1) corresponds to the lowest order action $I^{(0)}$ in (2.2.39). All ambiguities were removed in (3.1.1) and in this minimal basis all higher-deritivative corrections contain only first derivatives of $\mathcal{S}$, no traces $\text{Tr}\left((\mathcal{D}\mathcal{S})^2\right)$, and the only dilaton contribution comes from the term in the measure $e^{-\Phi}$. The $\ldots$ stand for higher α' orders which follow the same pattern: terms at order α'^p contain traces with $2(p+1)$ factors of $\mathcal{D}\mathcal{S}$, where each trace involves an even number of them except for $(\mathcal{D}\mathcal{S})^2$. The tower of higher-derivative corrections is now tamed and fully determined up to coefficients $c_{i,j}$, which cannot be determined by duality principles and must be computed from a beta-function computation or string scattering amplitudes. Their values are only partially known [1, 3, 39, 42, 90, 91].

In what follows we begin by introducing the systematic of field redefinitions in an abstract fashion. We then specialize it to the string universal massless sector in cosmological backgrounds and summarize the algorithm developed in [40] to arrive at (3.1.1). Restricting the space of backgrounds even further, in Section 3.1.3 we revisit the classification for Bianchi type I universes with q different scale factors [6].

3.1.1 Systematic of field redefinitions

We consider a generic action $I[\Psi]$ depending on fields that we collectively denote as Ψ, and admitting the following infinite expansion in α'

$$I[\Psi] = \sum_{p=0}^{\infty} \alpha'^p I^{(p)}[\Psi] \,, \tag{3.1.2}$$

where the term $I^{(p)}$ contains $2(p+1)$ derivatives of Ψ. From our earlier discussion in Section 2.1.2 we learned that all higher terms $I^{(p)}$ with $p \geq 1$ can be modified via field redefinitions. In particular, there is a canonical or minimal basis where the action takes its simpler form. We now introduce a systematic method to get this canonical form by removing ambiguities with field redefinitions order-by-order in α'.

We begin by performing a generic field redefinition

$$\Psi \to \Psi' = \Psi + \delta\Psi \,, \tag{3.1.3}$$

under which a generic action $I[\Psi]$ transforms as follows:

$$I'[\Psi'] \equiv I[\Psi + \delta\Psi] = I[\Psi] + \sum_{n=1}^{\infty} \frac{1}{n!} \Delta_n I \cdot (\delta\Psi)^n \,. \tag{3.1.4}$$

Here we use a symbolic notation in which the integral is not displayed explicitly. This equation defines implicitly the n-th variational derivatives $\Delta_n I$, the first of which is nothing but the equations of motion

$$\Delta_n I \equiv \frac{\delta^n I}{\delta\Psi^n} \,, \quad \Delta_1 I = \frac{\delta I}{\delta\Psi} \equiv E_\Psi \,. \tag{3.1.5}$$

If we now consider a perturbative action of the form (3.1.2), the variational derivatives admit the following expansion

$$\Delta_n I = \sum_{p=0}^{\infty} \alpha'^p \Delta_n I^{(p)} \,, \quad E_\Psi = \sum_{p=0}^{\infty} \alpha'^p E_\Psi^{(p)} \,. \tag{3.1.6}$$

We also assume an expansion for the field redefinitions beginning at order α'

$$\delta\Psi = \sum_{p=1}^{\infty} \alpha'^p \delta\Psi^{(p)} \,. \tag{3.1.7}$$

Inserting (3.1.6) and (3.1.7) into (3.1.4), we end up with an infinite expansion in α' for the redefined action I', which to lowest orders reads

$$\begin{aligned}
I' = {} & I^{(0)} + \alpha' \Big(I^{(1)} + E_\Psi^{(0)} \cdot \delta\Psi^{(1)} \Big) \\
& + \alpha'^2 \Big(I^{(2)} + E_\Psi^{(1)} \cdot \delta\Psi^{(1)} + E_\Psi^{(0)} \cdot \delta\Psi^{(2)} + \frac{1}{2} \Delta_2 I^{(0)} \cdot \big(\delta\Psi^{(1)}\big)^2 \Big) \\
& + \alpha'^3 \Big(I^{(3)} + E_\Psi^{(2)} \cdot \delta\Psi^{(1)} + E_\Psi^{(1)} \cdot \delta\Psi^{(2)} + E_\Psi^{(0)} \cdot \delta\Psi^{(3)} \\
& + \frac{1}{2} \Delta_2 I^{(1)} \cdot \big(\delta\Psi^{(1)}\big)^2 + \Delta_2 I^{(0)} \cdot \delta\Psi^{(1)} \cdot \delta\Psi^{(2)} + \frac{1}{3!} \Delta_3 I^{(0)} \cdot \big(\delta\Psi^{(1)}\big)^3 \Big) + \mathcal{O}(\alpha'^4) \,.
\end{aligned} \tag{3.1.8}$$

The natural method of bringing the action into a canonical form then proceeds order-by-order in α' in an algorithmic way: one first picks a $\delta\Psi^{(1)}$ to remove all ambiguities from the four-derivative action $I^{(1)}$. This in turn induces new terms proportional to $E_\Psi^{(1)}$ and $\Delta_2 I^{(0)}$ into the action of second order in α'. These terms together with the original $I^{(2)}$ can then be brought to a canonical form by picking a suitable $\delta\Psi^{(2)}$. Both $\delta\Psi^{(1)}$ and $\delta\Psi^{(2)}$ then induce new terms into the action of order α'^3, which finally can be brought to a canonical form by picking a suitable $\delta\Psi^{(3)}$. This process keeps going analogously to all orders in α'. In practice, however, one implements the algorithm to a certain finite order in α', beyond which induced effects are neglected. This relies on the perturbative nature of the problem, where each order in α' is a small correction to the previous one.

Let us work out a simple example at order α'^2 to see how this algorithm is implemented in practice: we begin at order α', and we assume $I^{(1)}$ contains ambiguous terms that can be removed by field redefinitions. Without loss of generality, we can write this ambiguous contribution as a term multiplying the lowest order equations of motion, i.e.,

$$I = I^{(0)} + \alpha' \left[E_\Psi^{(0)} \cdot X(\Psi) + \cdots \right] + \alpha'^2 I^{(2)} + \mathcal{O}(\alpha'^3)\,. \tag{3.1.9}$$

Here $X(\Psi)$ is an arbitrary function of the fields Ψ with two derivatives, the ellipsis denote the remaining terms in $I^{(1)}$, all containing four derivatives, and $\mathcal{O}(\alpha'^3)$ represents higher-order effects that we are neglecting. By performing a field redefinition of the form

$$\delta\Psi^{(1)} = -X(\Psi)\,, \tag{3.1.10}$$

from (3.1.8) we then infer that, in the redefined action I', (3.1.9) is replaced by

$$I' = I^{(0)} + \alpha' [\cdots] + \alpha'^2 \left[I^{(2)} + E_\Psi^{(1)} \cdot (-X(\Psi)) + \frac{1}{2}\Delta_2 I^{(0)} \cdot \big(- X(\Psi)\big)^2 \right] + \mathcal{O}(\alpha'^3)\,, \tag{3.1.11}$$

where the ellipsis denote the same four-derivative terms as in (3.1.9), which are unaffected by the redefinition. By comparing (3.1.9) and (3.1.11) we see that we managed to remove a term at order α' in the original action, at expenses of inducing higher-order contributions.

Identifying the exact $\delta\Psi^{(1)}$ that does the job can become very tedious in practice. In view of this, we now introduce a simple trick that allows us to implement the same transformation in a more pragmatic way: we first notice that, if we forget about induced terms, the net effect of applying (3.1.10) on (3.1.9) is equivalent to have used the lowest-order equations of motion as a replacement rule at level of the action

$$E_\Psi^{(0)} = 0\,. \tag{3.1.12}$$

Since we do care about induced terms, however, we cannot simply use (3.1.12). Instead, we artificially extend (3.1.12) to the rule

$$E_\Psi^{(0)} = -\alpha' \Delta_\Psi\,, \tag{3.1.13}$$

where Δ_Ψ is just an auxiliary time-dependent object that later will be set to zero. Both substitutions coincide at leading order, but now from (3.1.13) we can read $\delta\Psi^{(1)}$ as the term that multiplies Δ_Ψ.[2]

Using the rule (3.1.13) repeatedly to all other ambiguous terms in $I^{(1)}$ we end up with a transformed action $I'^{(1)}$ in the canonical basis. For each use of (3.1.13) we read the corresponding $\delta\Psi^{(1)}$. All these $\delta\Psi^{(1)}$ are then added together and, combining them with $E_\Psi^{(1)}$ and $\Delta_2 I^{(0)}$ as dictated by the general

[2]For instance, using (3.1.13) in (3.1.9), it is easy to see that the linear term in Δ_Ψ is exactly $-X = \delta\Psi^{(1)}$.

structure in (3.1.8), we get the corresponding induced effects. After this step, we do not need the auxiliary object Δ_Ψ anymore so we set it to zero.

We now move to the next order in α'. We repeat the algorithm but this time the starting point contains $I'^{(1)}$ already in the canonical form, and at order α'^2 we have $I^{(2)} + E_\Psi^{(1)} \cdot \delta\Psi^{(1)} + \frac{1}{2}\Delta_2 I^{(0)} \cdot \left(\delta\Psi^{(1)}\right)^2$. A redundant term at order α'^2 now takes the form

$$I = I^{(0)} + \alpha' I'^{(1)} + \alpha'^2 \left[E_\Psi^{(0)} \cdot Y(\Psi) + \cdots\right] + \mathcal{O}(\alpha'^3)\,, \tag{3.1.14}$$

where $Y(\Psi)$ is a four-derivative term and $\cdots$ stand for the rest of the six-derivative couplings that complete the second-order action. In order to remove the ambiguity, one option is to perform the redefinition explicitly using

$$\delta\Psi^{(2)} = -Y(\Psi)\,. \tag{3.1.15}$$

From (3.1.8) we see that, since (3.1.15) starts at order α'^2, it preserves the already canonical $I'^{(1)}$, it removes the ambiguous term in (3.1.14), and induce α'^3 effects. Another option, is to use the trick presented before: apply (3.1.13) to remove the ambiguity, read $\delta\Psi^{(2)}$ from the linear term in Δ_Ψ, add the corresponding induced terms to the next order, and then set $\Delta_\Psi = 0$. However, since this example truncates at order α'^2, we do not need to keep track of induced terms. In particular, instead of (3.1.13), we can simply use $E_\Psi^{(0)} = 0$ ((3.1.12)) because we do not care about the exact form of $\delta\Psi^{(2)}$! Repeating this last step with all other ambiguous terms at order α'^2 we end up with a transformed $I'^{(2)}$ in canonical form. Since we are not interested in $\mathcal{O}(\alpha'^3)$, the cleaning procedure is finished.

Extrapolating the lessons learned in this simple α'^2 example, we can describe a general algorithm to implement field redefinitions in practice up to a generic α'^p order: we begin at order α' with $I^{(1)}$ and remove ambiguous terms using the rule

$$E_\Psi^{(0)} = -\alpha'\Delta_\Psi\,, \tag{3.1.16}$$

as many times as needed. After applying this rule, we will generate a polynomial in $\alpha'\Delta_\Psi$, from which we can read $\delta\Psi^{(1)}$ as the coefficient of the linear term. We save this value for later, since we need it to build the induced terms up to order α'^p (see (3.1.8)). We now set Δ_Ψ to zero since it is not needed anymore. At this step, the transformed $I'^{(1)}$ is in canonical form.

We move up to order α'^2, where the new second-order action is composed of the original $I^{(2)}$ together with the induced terms proportional to $\delta\Psi^{(1)}$. This combination can be brought to a canonical form via (3.1.16). The variation $\delta\Psi^{(2)}$ can be read from whatever ends up next to Δ_Ψ. We save also this value for later and set $\Delta_\Psi = 0$.

We keep repeating these steps moving up in α' until we reach order α'^p, where the transformed action at that order depends on the original $I^{(p)}$ together with all $\delta\Psi^{(n)}$ with $n = 1, \ldots, p-1$, which we have been saving along the way. At this last stage, we can remove ambiguities by just using the rule $E_\Psi^{(0)} = 0$ ((3.1.12)), since there is no need to keep track of $\delta\Psi^{(p)}$.

This systematic approach of implementing field redefinitions is by no means restricted to cosmological backgrounds. Notice that we did not make any assumption on the field content nor spacetime dimensions. However, when it comes to practical purposes, applying this algorithm on generic backgrounds becomes very tedious and it is not always clear that the resulting theory does not hide more redundancies that were overlooked. The challenge in this method is how to implement the substitution rules (3.1.16) in a "smart" or "unidirectional" way, where at the end of the procedure a canonical form is reached. For the one-dimensional case, Hohm and Zwiebach cracked this problem in [39, 40] where the equations of motion are used in a very organized and systematic way. In the next subsection we introduce this method.

3.1.2 Classification of $O(d,d)$ cosmologies

The all-α'-order extension of (2.2.39) takes the generic form

$$I = \sum_{p=0}^{\infty} \alpha'^p I^{(p)}\,, \tag{3.1.17}$$

where each order $I^{(p)}$ can contain a sum of different terms

$$I^{(p)} = \sum_k c_{p,k}\, I_k^{(p)}\,. \tag{3.1.18}$$

The coefficients $c_{p,k}$ are just constant real numbers and each $I_k^{(p)}$ contains $2(p+1)$ derivatives. For instance, from (2.2.39) we see that $I^{(0)}$ is composed of just two two-derivative terms, proportionals to $(\mathcal{D}\Phi)^2$ and $\mathrm{Tr}\left((\mathcal{D}\mathcal{S})^2\right)$ with coefficients -1 and $-\frac{1}{8}$, respectively. By demanding diffeomorphism and duality invariance there is a finite number of couplings we can build at each order in α' in terms of $\mathcal{S}$, Φ and their covariant derivatives $\mathcal{D}$. A generic term has the form

$$I_k^{(p)} = \int dt\, n\, e^{-\Phi} \left[\prod_i \mathrm{Tr}\left(\prod_l (\mathcal{D}^{a_i^l}\mathcal{S})^{b_i^l}\right) \prod_j (\mathcal{D}^{c_j+1}\Phi)^{d_j}\right], \tag{3.1.19}$$

where the exponents $a_i^l, b_i^l, c_j, d_j, \in \mathbb{N}_0\ \forall\, i, j, l$ need to satisfy the condition

$$\sum_i \sum_l a_i^l b_i^l + \sum_j (c_j+1) d_j = 2(p+1)\,, \tag{3.1.20}$$

because all derivatives in $I_k^{(p)}$ must add up to $2(p+1)$. There are many configurations of exponents that satisfy (3.1.20), each inequivalent configuration corresponds to a different label k. We are omitting the boundaries of the sums since they are convoluted expressions of p and the exponents themselves, in practice, however, the different configurations of (3.1.20) are easy to build.

Notice we are not considering dilaton terms without derivatives in $I_k^{(p)}$ and so the non-derivative dependence only enters through $e^{-\Phi}$ in the measure. On the other hand, we are including the possibility of having non-derivative contributions of $\mathcal{S}$, which corresponds to $a_i^l = 0$ and since $\mathcal{S}^2 = 1$, it must be accompanied by $b_i^l = 1$. Moreover, the following trace configurations vanish for all $q \in \mathbb{N}_0$

$$\begin{aligned} \mathrm{Tr}\,(\mathcal{D}^q\mathcal{S}) &= 0\,, \\ \mathrm{Tr}\left((\mathcal{D}\mathcal{S})^{2q+1}\right) &= 0\,, \\ \mathrm{Tr}\,(\mathcal{S}(\mathcal{D}\mathcal{S})^q) &= 0\,, \end{aligned} \tag{3.1.21}$$

and so any $I_k^{(p)}$ containing them also vanishes. The first identity follows from the traceless condition (2.2.20) and its derivatives. The second line is a consequence of odd powers of $\mathcal{DS}$ being minus-projected objects (see (2.2.26)) and the identity (2.2.27). The last identity in (3.1.21) is proven as follows:

$$\mathrm{Tr}\,(\mathcal{S}(\mathcal{D}\mathcal{S})^q) = \mathrm{Tr}\left(\mathcal{S}\mathcal{D}\mathcal{S}(\mathcal{D}\mathcal{S})^{q-1}\right) = -\mathrm{Tr}\left(\mathcal{D}\mathcal{S}\mathcal{S}(\mathcal{D}\mathcal{S})^{q-1}\right) = -\mathrm{Tr}\,(\mathcal{S}(\mathcal{D}\mathcal{S})^q) = 0\,, \tag{3.1.22}$$

where in the second equality we used $\mathcal{SDS} = -\mathcal{DSS}$ and in the third equality cyclicity of the trace.

Equation (3.1.19) parameterizes the most general term in the action at order α'^p, we now want to remove all ambiguities coming from field redefinitions and integration by parts in a systematic way. The claim is that at order α'^p, the action $I^{(p)}$ can be brought to a form where the most general term $I_k^{(p)}$ contains only traces of even powers of $\mathcal{DS}$, excluding $\mathrm{Tr}\,((\mathcal{DS})^2)$, namely

$$I_k^{(p)} \approx \int dt\, n\, e^{-\Phi} \left[\prod_{i=1}^{p} \left[\mathrm{Tr}\left((\mathcal{D}\mathcal{S})^{2(i+1)}\right)\right]^{k_i}\right]\,, \quad k_i = 1, 2, \ldots \quad \sum_i k_i(i+1) = p+1 \tag{3.1.23}$$

where now we are using a different convention for the exponents compared to (3.1.19), in which i labels the powers of $\mathcal{DS}$ inside the trace and k_i labels the powers of traces. In this notation, k is a single label characterizing the vector of exponents $\{k_i\}$. The first orders together with their respective coefficients can be read from (3.1.1) and the k labels are given by

$$\begin{aligned} p = 1: &\qquad (k_1 = 1) \to k = 0\,, \\ p = 2: &\qquad (k_1 = 0,\ k_2 = 1) \to k = 0\,, \\ p = 3: &\qquad (k_1 = 0,\ k_2 = 0,\ k_3 = 1) \to k = 0\,, \\ &\qquad (k_1 = 2,\ k_2 = 0,\ k_3 = 0) \to k = 1\,. \end{aligned} \tag{3.1.24}$$

In (3.1.23) we introduced the symbol $\approx$ which will be heavily used from now on. It stands for *equivalence at the level of the action, up to integration by parts, field redefinitions and higher α' effects*. As it can be seen, this notation encodes a very dense meaning, but it is quite powerful when doing computations and it makes intermediate steps clearer. Apart from practical purposes, it is also motivated by the fact that, even though the original action $I^{(p)}$ and the one after the classification, let's say $I'^{(p)}$, are not equal, they are really physically equivalent up to order α'^p! The $\approx$ symbol hides all these ambiguities or redundancies that relate both actions.

To prove the statement of the classification we proceed inductively: we first assume that up to order α'^p the action at each order takes the form (3.1.18) with all $I_k^{(1)}, I_k^{(2)}, \ldots, I_k^{(p-1)}$ already in the minimal basis (3.1.23), and then prove that this is also satisfied for $I_k^{(p)}$ up to field redefinitions.

As we saw in the previous subsection, applying a field redefinition of the form $\Psi \to \Psi' = \Psi + \delta\Psi$ with $\delta\Psi$ and $I[\Psi]$ admitting an infinite expansion in α', the transformed action takes the form (3.1.8), where $\Psi \in \{\mathcal{S}, \Phi, n\}$. If we now implement the inductive procedure mentioned above, all actions $I^{(q)}$ with $q \leq p-1$ are already in the minimal basis where $I_k^{(q)}$ take the form (3.1.23). Therefore, we need field redefinitions that do not modify these previous orders, but do modify $I^{(p)}$. To this end we use

$$\delta\Psi = \alpha'^p \delta\Psi^{(p)} , \tag{3.1.25}$$

where $\delta\Psi^{(p)}$ is built from covariant objects containing $2p$ derivatives. Under this transformation, (3.1.8) reads

$$\begin{aligned} I' &= \sum_{q=0}^{p-1} \alpha'^q I^{(q)} + \alpha'^p \Big(I^{(p)} + E_\Psi^{(0)} \cdot \delta\Psi^{(p)} \Big) + \alpha'^{p+1} \Big(I^{(p+1)} + E_\Psi^{(1)} \cdot \delta\Psi^{(p)} \Big) + \cdots \\ &= \sum_{q=0}^{p-1} \alpha'^q I^{(q)} + \alpha'^p \left\{ I^{(p)} + \int dt\, n\, e^{-\Phi} \left[\mathrm{Tr}\left(E_{\mathcal{S}}^{(0)} \delta\mathcal{S}^{(p)} \right) + E_\Phi^{(0)} \delta\Phi^{(p)} + E_n^{(0)} \frac{\delta n^{(p)}}{n} \right] \right\} \\ &\quad + \mathcal{O}(\alpha'^{p+1}) , \end{aligned} \tag{3.1.26}$$

where in the first line we kept the implicit notation of (3.1.8) and the $\cdots$ represent infinitely-many induced terms. In the second line we specified to the cosmological background and hid higher orders in $\mathcal{O}(\alpha'^{p+1})$. The zeroth-order equations of motion are the ones coming from the two-derivative theory $I^{(0)}$. They were given in (2.2.43) but we rewrite them here for convenience

$$E_{\mathcal{S}}^{(0)} = \frac{1}{4} \left[\mathcal{D}^2\mathcal{S} - \mathcal{D}\Phi\mathcal{D}\mathcal{S} + \mathcal{S}(\mathcal{D}\mathcal{S})^2 \right] = \mathcal{O}(\alpha') , \tag{3.1.27a}$$

$$E_\Phi^{(0)} = 2\mathcal{D}^2\Phi - (\mathcal{D}\Phi)^2 + \frac{1}{8}\mathrm{Tr}\left((\mathcal{D}\mathcal{S})^2 \right) = \mathcal{O}(\alpha') , \tag{3.1.27b}$$

$$E_n^{(0)} = (\mathcal{D}\Phi)^2 + \frac{1}{8}\mathrm{Tr}\left((\mathcal{D}\mathcal{S})^2 \right) = \mathcal{O}(\alpha') . \tag{3.1.27c}$$

Notice that now in the right hand side of (3.1.27) we do not have zeros due to the presence of $E_\Psi^{(1)}, E_\Psi^{(2)}, \ldots$.

So far, $I^{(p)}$ in (3.1.26) contains terms of the most general form (3.1.19). We can now bring it to the canonical form by implementing the algorithm introduced in the previous subsection. The idea of the systematic procedure was to choose specific $\delta\mathcal{S}^{(p)}, \delta\Phi^{(p)}$ and $\delta n^{(p)}$ so to remove all ambiguities of $I^{(p)}$. In practice, we saw that this could be implemented by using the equations of motion as substitution rules at the level of the action, while keeping track of induced effects proportional to $\delta\Psi^{(p)}$. In the particular case of the classification, the inductive nature of the proof allows us to implement the algorithm in its simplest form, where we can simply use the rules $E_\Psi^{(0)} \approx 0$ ((3.1.12)) and forget about $\delta\Psi^{(p)}$ or any other induced term! The reason being that higher-order effects will appear at the next step of the inductive procedure but, whatever is induced, it is already contemplated in the general form of (3.1.19).

Let's see how this substitution rule works explicitly for cosmological backgrounds treating a particular example at order α', i.e. $I = I^{(0)} + \alpha' I^{(1)} + \mathcal{O}(\alpha'^2)$.

This will be an explicit realization of the abstract computation made below equation (3.1.9). Suppose the action takes the form

$$I = I^{(0)} + \alpha' \int dt\, n\, e^{-\Phi} \left[c\,\mathrm{Tr}\left(\mathcal{D}^2\mathcal{S}(\mathcal{D}\mathcal{S})^2\right) + \cdots \right] + \mathcal{O}(\alpha'^2)\,, \tag{3.1.28}$$

where $\mathrm{Tr}\,(\mathcal{D}^2\mathcal{S}(\mathcal{D}\mathcal{S})^2)$ is the term we want to redefine away, c an arbitrary coefficient, and $\cdots$ contain all the other four-derivative terms that complete $I^{(1)}$. By considering a redefinition of the form $\mathcal{S} \to \mathcal{S}' = \mathcal{S} + \alpha'\delta\mathcal{S}^{(1)}$ we get the transformed action

$$\begin{aligned} I' = I^{(0)} + \alpha' \int dt\, n\, e^{-\Phi} \Big[& c\,\mathrm{Tr}\left(\mathcal{D}^2\mathcal{S}(\mathcal{D}\mathcal{S})^2\right) \\ & + \frac{1}{4}\mathrm{Tr}\left(\left[\mathcal{D}^2\mathcal{S} - \mathcal{D}\Phi\mathcal{D}\mathcal{S} + \mathcal{S}(\mathcal{D}\mathcal{S})^2\right]\delta\mathcal{S}^{(1)}\right) + \cdots \Big] + \mathcal{O}(\alpha'^2)\,, \end{aligned} \tag{3.1.29}$$

where we are neglecting higher orders. From (3.1.29) we see that by choosing $\delta\mathcal{S}^{(1)} = -4c(\mathcal{D}\mathcal{S})^2$ we end up with

$$I' = I^{(0)} + \alpha' \int dt\, n\, e^{-\Phi} \left[c\,\mathrm{Tr}\left(\left[\mathcal{D}\Phi\mathcal{D}\mathcal{S} - \mathcal{S}(\mathcal{D}\mathcal{S})^2\right](\mathcal{D}\mathcal{S})^2\right) + \cdots \right] + \mathcal{O}(\alpha'^2)\,, \tag{3.1.30}$$

where the $\cdots$ remains unchanged. By comparing (3.1.28) with (3.1.30) we see that the net effect of applying a field redefinition was to just use $E_{\mathcal{S}}^{(0)} = 0$ as a substitution rule, meaning

$$\mathcal{D}^2\mathcal{S} \approx \mathcal{D}\Phi\mathcal{D}\mathcal{S} - \mathcal{S}(\mathcal{D}\mathcal{S})^2\,. \tag{3.1.31}$$

The example in (3.1.28) has nothing special, the same result holds for other removable terms and for the other equations of motion in (3.1.27), which can also be used as substitution rules.

All in all, to bring $I^{(p)}$ to the minimal form (3.1.23), we just use the lowest-order equations as substitution rules, knowing that behind the scenes there is a precise change of variables that was carried on. To this end, we use:

$$\mathcal{D}^2\mathcal{S} \approx \mathcal{D}\Phi\mathcal{D}\mathcal{S} - \mathcal{S}(\mathcal{D}\mathcal{S})^2\,, \tag{3.1.32a}$$

$$\mathcal{D}^2\Phi \approx -\frac{1}{8}\mathrm{Tr}\left((\mathcal{D}\mathcal{S})^2\right)\,, \tag{3.1.32b}$$

$$(\mathcal{D}\Phi)^2 \approx -\frac{1}{8}\mathrm{Tr}\left((\mathcal{D}\mathcal{S})^2\right)\,, \tag{3.1.32c}$$

where for the second rule we just combined the dilaton's and lapse's equations.

Now we proceed to show that using these rules in a systematic way we can bring any term in $I^{(p)}$ of the form (3.1.19), to one of the form (3.1.23). The algorithm goes as follows:

1. **Eliminate $\mathcal{D}^2\mathcal{S}$:**

   This first step is conceptually simple. For each $I_k^{(p)}$ of the form (3.1.19) in $I^{(p)}$, we just use (3.1.32a) to remove $\mathcal{D}^2\mathcal{S}$ factors. If there are n of these factors, we use the rule n times. By the end of this step, an action at order $I^{(p)}$ containing the most general terms (3.1.19) is equivalent in the sense of $\approx$ (i.e. up to field redefinitions, boundary terms, and higher-order effects) to another action $I^{(p)}$ where the most general term $I_k^{(p)}$ does not contain second derivatives over $\mathcal{S}$.

2. **Eliminate $\mathcal{D}^2\Phi$:**

   This step follows almost identically to the previous one. For each of the new $I_k^{(p)}$ terms, namely (3.1.19) but without $\mathcal{D}^2\mathcal{S}$ contributions, we use (3.1.32b) repeatedly to remove all $\mathcal{D}^2\Phi$ factors. After this step, we end up with an equivalent action where $I_k^{(p)}$ have no second-order contributions.

3. **Eliminate higher derivatives of $\mathcal{S}$:**

   In this step we begin writing any higher-derivative of $\mathcal{S}$ as $\mathcal{D}^{q+2}\mathcal{S} = \mathcal{D}^q(\mathcal{D}^2\mathcal{S})$ where $q > 0$ because the case $q = 0$ was already treated in step 1. We now integrate by parts all $\mathcal{D}^q$, leaving along a $\mathcal{D}^2\mathcal{S}$ factor that we can replace for first-order contributions using (3.1.32a). After this, we integrate back the $\mathcal{D}^q$ one-by-one. At each iteration, we distribute, apply Leibniz rule and eliminate any generated second-order factor by using step 1. and 2. When the iteration is over, we managed to remove all higher derivatives of $\mathcal{S}$ at expenses of generating other terms containing first-order derivatives of $\mathcal{S}$ and generic dilaton contributions. This iterative procedure must be repeated for each term in ${I_k}^{(p)}$ containing higher derivatives of $\mathcal{S}$. By the end of this step, an action $I^{(p)}$ is equivalent to one where the most general term now takes the simpler form:

$$I_k^{(p)} \approx \int dt\, n\, e^{-\Phi} \left[\prod_{i=0}^{p} \left[\mathrm{Tr}\left((\mathcal{D}\mathcal{S})^{2(i+1)} \right) \right]^{b_i} \prod_{j=0, j\neq 1}^{2p+1} (\mathcal{D}^{j+1}\Phi)^{d_j} \right] . \tag{3.1.33}$$

   Here we are already using the conventions for the exponents of (3.1.23). The k labels the different configurations of exponents $b_i, d_j \in \mathbb{N}_0$ that must satisfy

$$\sum_{i=0}^{p} 2b_i(i+1) + \sum_{j=0, j\neq 1}^{2p+1} d_j(j+1) = 2(p+1) . \tag{3.1.34}$$

   Notice that (3.1.33) does not include traces containing $\mathcal{S}$ without derivatives, neither odd powers of $\mathcal{D}\mathcal{S}$. This is a consequence of the identities (3.1.21).

4. **Eliminate higher derivatives of Φ:**

   This step is very similar to the previous one. In this case we start from $\mathcal{D}^{q+2}\Phi = \mathcal{D}^q(\mathcal{D}^2\Phi)$ where $q > 0$, integrate by parts all $\mathcal{D}^q$, and apply (3.1.32b) in the remaining $\mathcal{D}^2\Phi$ factor. Integrating back the q derivatives one-by-one, at each step we distribute, apply Leibniz and use steps 1. and 2. When the iteration is over, we are guaranteed to converge to a sum of products between traces of multiple $\mathcal{D}\mathcal{S}$ and, at most, single derivatives of the dilaton, $\mathcal{D}\Phi$. At this stage, any $I_k^{(p)}$ can be brought to the form

$$I_k^{(p)} \approx \int dt\, n\, e^{-\Phi} \left[\prod_{i=0}^{p} \left[\mathrm{Tr}\left((\mathcal{D}\mathcal{S})^{2(i+1)} \right) \right]^{b_i} (\mathcal{D}\Phi)^{2j} \right] , \tag{3.1.35}$$

where we noticed that, since the total number of derivatives is even, we need an even number of $\mathcal{D}\Phi$ factors. Moreover, in order $I_k^{(p)}$ to have $2(p+1)$ derivatives, we need $j = p + 1 - \sum_{i=0}^{p} b_i(i+1)$.

5. **Eliminate Φ:**

   At this point, apart from $e^{-\Phi}$, the only dilaton contribution comes from even powers of $\mathcal{D}\Phi$. These can be redefined away easily by using (3.1.32c) repeatedly. More precisely:

$$(\mathcal{D}\Phi)^{2j} = \left[(\mathcal{D}\Phi)^2 \right]^j \approx \left[-\frac{1}{8} \mathrm{Tr}\left((\mathcal{D}\mathcal{S})^2 \right) \right]^j . \tag{3.1.36}$$

   Having eliminated all dilaton couplings, the most general $I_k^{(p)}$ is equivalent to

$$I_k^{(p)} \approx \int dt\, n\, e^{-\Phi} \left[\prod_{i=0}^{p} \left[\mathrm{Tr}\left((\mathcal{D}\mathcal{S})^{2(i+1)} \right) \right]^{k_i} \right] , \tag{3.1.37}$$

   where k now labels the vector k_i.

6. **Eliminate Tr $((\mathcal{D}\mathcal{S})^2)$:**

   While the last step lead to a formidable simplification, there is a further highly non-trivial step that allows us to remove terms containing Tr $((\mathcal{D}\mathcal{S})^2)$. This last part of the algorithm is the longest one and involves the use of the three rules (3.1.32) together with integration by parts.

   We begin with a term of the form (3.1.37). If there is no Tr $((\mathcal{D}\mathcal{S})^2)$, there is nothing to remove, and so we move to the next $I_k^{(p)}$, labeled by another k. If there is at least one factor of Tr $((\mathcal{D}\mathcal{S})^2)$, we separate one of them from the rest of the traces and we perform a series of operations:

$$\begin{aligned} I_k^{(p)} &= \int dt\, n\, e^{-\Phi} \left[\prod_{i=0}^{p} \left[\mathrm{Tr}\left((\mathcal{D}\mathcal{S})^{2(i+1)} \right) \right]^{k_i} \right] \\ &= \int dt\, n\, e^{-\Phi} \left[\mathrm{Tr}\left((\mathcal{D}\mathcal{S})^2 \right)\, F(\mathcal{D}\mathcal{S}) \right] \\ &\approx -8 \int dt\, n\, e^{-\Phi} \left[(\mathcal{D}\Phi)^2\, F(\mathcal{D}\mathcal{S}) \right] \\ &= 8 \int dt\, n\, \mathcal{D}(e^{-\Phi}) \left[\mathcal{D}\Phi\, F(\mathcal{D}\mathcal{S}) \right] \\ &\approx -8 \int dt\, n\, e^{-\Phi} \left[\mathcal{D}^2\Phi\, F(\mathcal{D}\mathcal{S}) + \mathcal{D}\Phi\, \mathcal{D}F(\mathcal{D}\mathcal{S}) \right] , \end{aligned} \tag{3.1.38}$$

   where in the second line we collected all other traces in the function

$$F(\mathcal{D}\mathcal{S}) \equiv \prod_{i=0}^{p} \left[\mathrm{Tr}\left((\mathcal{D}\mathcal{S})^{2(i+1)} \right) \right]^{k'_i} , \quad k'_0 = k_0 - 1 , \quad k'_i = k_i \quad \forall i \geq 1 , \tag{3.1.39}$$

which contains $2p$ derivatives, and can have other $\text{Tr}\left((\mathcal{DS})^2\right)$ factors as well. We introduced the label k'_i to account for the missing $\text{Tr}\left((\mathcal{DS})^2\right)$ factor. In the third line of (3.1.38) we used (3.1.32c) but in the opposite direction. In the fourth line we rewrote a dilaton derivative as $e^{-\Phi}\mathcal{D}\Phi = -\mathcal{D}(e^{-\Phi})$, and in the last line we integrated by parts. The factor $\mathcal{D}F(\mathcal{DS})$ can be simplified by using field redefinitions:

$$\begin{aligned}
&\mathcal{D}F(\mathcal{DS}) = \mathcal{D}\left[\prod_{i=0}^{p}\left[\text{Tr}\left((\mathcal{DS})^{2(i+1)}\right)\right]^{k'_i}\right] \\
&= \sum_{j=0}^{p} 2k'_j(j+1)\left[\text{Tr}\left((\mathcal{DS})^{2(j+1)}\right)\right]^{k'_j-1}\text{Tr}\left((\mathcal{DS})^{2j+1}\mathcal{D}^2\mathcal{S}\right)\prod_{i=0,i\neq j}^{p}\left[\text{Tr}\left((\mathcal{DS})^{2(i+1)}\right)\right]^{k'_i} \\
&\approx \sum_{j=0}^{p} 2k'_j(j+1)\left[\text{Tr}\left((\mathcal{DS})^{2(j+1)}\right)\right]^{k'_j-1}\mathcal{D}\Phi\,\text{Tr}\left((\mathcal{DS})^{2(j+1)}\right)\prod_{i=0,i\neq j}^{p}\left[\text{Tr}\left((\mathcal{DS})^{2(i+1)}\right)\right]^{k'_i} \\
&= \sum_{j=0}^{p} 2k'_j(j+1)\mathcal{D}\Phi\,\left[\text{Tr}\left((\mathcal{DS})^{2(j+1)}\right)\right]^{k'_j}\prod_{i=0,i\neq j}^{p}\left[\text{Tr}\left((\mathcal{DS})^{2(i+1)}\right)\right]^{k'_i} \\
&= \sum_{j=0}^{p} 2k'_j(j+1)\mathcal{D}\Phi\,F(\mathcal{DS}) \\
&= 2p\,\mathcal{D}\Phi\,F(\mathcal{DS})\,.
\end{aligned} \tag{3.1.40}$$

In the second line we used Leibniz and in the third line the rule (3.1.32a) and drop the $-\mathcal{S}(\mathcal{DS})^2$ contribution because of (3.1.21). In the fifth line we recognized the common factor of the sum as $F(\mathcal{DS})$ and for the last step we identified the sum as the total number of derivatives of F.

Inserting this result back into (3.1.38), we keep going with the proof:

$$\begin{aligned}
I_k^{(p)} &\approx -8\int dt\,n\,e^{-\Phi}\left[\mathcal{D}^2\Phi\,F(\mathcal{DS}) + 2p\,(\mathcal{D}\Phi)^2\,F(\mathcal{DS})\right] \\
&\approx (2p+1)\int dt\,n\,e^{-\Phi}\left[\text{Tr}\left((\mathcal{DS})^2\right)\,F(\mathcal{DS})\right] \\
&= (2p+1)I_k^{(p)}\,,
\end{aligned} \tag{3.1.41}$$

where in the second line we used (3.1.32b) and (3.1.32c) and in the last equality we identified the original $I_k^{(p)}$ term (see (3.1.38)). The final outcome of this long computation is that any $I_k^{(p)}$ containing $\text{Tr}\left((\mathcal{DS})^2\right)$ is physically equivalent to $(2p+1)I_k^{(p)}$. Since $p \geq 1$, $I_k^{(p)}$ must be physically equivalent to zero!

After repeating this chain of operations to all $I_k^{(p)}$ in $I^{(p)}$ we can conclude that the order α'^p action is equivalent to one where the most general term is given by (3.1.23), which is what we wanted to prove.

3.1.3 Bianchi type I universes and FRW

Classification of higher-derivative terms depends on the specific background and field configuration. The above mentioned "cosmological" classification with fields depending only on time includes the G_{00} component of the metric, the purely spatial components of the metric and B-field, and the dilaton. By restricting this space of backgrounds further, there will generally be a more refined classification. For instance, we may assume the consistent truncation where the B-field vanishes and the spatial metric is diagonal, with d "scale factors" on the diagonal that may or may not be equal. These are the Bianchi type I (BI) universes studied in Section 2.2.2 [3]. For this smaller space of backgrounds there are fewer higher-derivative terms that one can write, but also fewer field redefinitions, so that the classification problem has to be reconsidered. Here we perform such classification and show that for one of the scale factors (that can be picked arbitrarily) all higher-derivative terms can be removed by field redefinitions. In particular, specializing further to the case that all scale factors are equal, corresponding to FRW backgrounds, it follows that *all* higher-derivative terms are removable by field redefinitions! [6]

As we saw in (2.2.53), the lowest-order action for BI backgrounds with $q \leq d$ anisotropic directions, is given by

$$I^{(0)} = \int dt\, n\, e^{-\Phi} \left[-(\mathcal{D}\Phi)^2 + \sum_{i=1}^{q} N_i H_i^2 \right] , \qquad (3.1.42)$$

with the Hubble parameters $H_i \equiv \frac{\mathcal{D}a_i}{a_i}$ built from the scale factors a_i for $i = 1, \ldots, q$. As for the cosmological classification, we now consider the infinite α' extension of (3.1.42) to take the form

$$I = \sum_{p\geq 0} \alpha'^p I^{(p)} , \quad I^{(p)} = \sum_k c_{p,k}\, I_k^{(p)} . \qquad (3.1.43)$$

Our goal now is to use field redefinitions to reduce these higher-order terms to a minimal set of couplings in the same spirit of the previous section. To this end we will use the lowest-order equations of motion (2.2.54) simply as substitution rules in the action which we rewrite here for convenience

$$\mathcal{D}H_i \approx \mathcal{D}\Phi H_i , \quad i = 1, \ldots, q , \qquad (3.1.44a)$$

$$\mathcal{D}^2\Phi \approx \sum_{i=1}^{q} N_i H_i^2 , \qquad (3.1.44b)$$

$$(\mathcal{D}\Phi)^2 \approx \sum_{i=1}^{q} N_i H_i^2 , . \qquad (3.1.44c)$$

These are the direct reduction to BI universes of the rules (3.1.32)!

For BI universes, all steps of the classification are similar to those applied in Section 3.1.2 and so we will omit intermediate steps. Specifically, the same inductive step-by-step proof with the same itemization, proceeds as follows. We assume that to any order in α' any term in the action is writable as a product of factors $\mathcal{D}^k\Phi$ and $\mathcal{D}^l H_i$ with $i = 1, \ldots, q$. We can now do field redefinitions of a_i, Φ and n that yields rules (3.1.44a), (3.1.44b) and (3.1.44c) in order to establish:

[3] Higher-derivative corrections to BI backgrounds in the context of the cosmological classification were also considered in [92], where matter fields were included.

1. A factor in an action including $\mathcal{D}H_i$ can be replaced by a factor with only first derivatives. This follows directly from the first substitution rule (3.1.44a).

2. A factor in an action including $\mathcal{D}^2\Phi$ can be replaced by a factor with only first derivatives. This follows directly from the substitution rule in (3.1.44b).

3. Any action can be reduced so that it only contains products of H_i, not their derivatives. The proof proceeds as in step 3. of the cosmological classification: We write any higher derivative as $\mathcal{D}^{p+1}H_i = \mathcal{D}^p(\mathcal{D}H_i)$, and then integrate by parts the $\mathcal{D}^p$. Then we use (3.1.44a), after which we integrate back one-by-one, eliminating any generated second derivative, using steps 1. and 2. At the end, we are left with only first-order derivatives of scale factors, encoded in H_i.

4. Any action can be reduced so that it only has first derivatives of Φ. The proof is identical to the one from the previous step.

5. Any higher-derivative term is equivalent to one without any appearance of $\mathcal{D}\Phi$. To see this we notice that up to this point, any $I_k^{(p)}$ is of the form

$$I_k^{(p)} = \int dt\, n\, e^{-\Phi}(\mathcal{D}\Phi)^{2j}\prod_{i=1}^{q} H_i^{2l_i}\,. \tag{3.1.45}$$

   Here k labels the different combinations of exponents that must satisfy $j + \sum_{i=1}^{q} l_i = p+1$. This comes from $I_k^{(p)}$ having $2(p+1)$ derivatives. The absence of odd powers in (3.1.45) is a consequence of invariance under duality transformations: $H_i \to -H_i$. All these even products of $\mathcal{D}\Phi$ can be easily removed by applying repeatedly (3.1.44c).

The above chain of arguments proved that there is a field basis in which all higher-derivative terms are of the form

$$I_k^{(p)} = \int dt\, n\, e^{-\Phi}\prod_{i=1}^{q} H_i^{2k_i}\,. \tag{3.1.46}$$

Here k labels the different vectors of exponents k_i, such that $\sum_{i=1}^{q} k_i = p+1$.

We now want to prove that one of the scale factors can be completely removed from higher-order terms, appearing only in the two-derivative theory (3.1.42). Without loss of generality, we chose the scale factor a_q. To this end, it is convenient to reorder the rule (3.1.44c) such that H_q is distinguished over the others Hubble parameters:

$$H_q^2 \approx \frac{1}{N_q}\left((\mathcal{D}\Phi)^2 - \sum_{i=1}^{q-1} N_i H_i^2\right). \tag{3.1.47}$$

The computations are similar to what we did in step 6. of the cosmological classification. We begin defining the analogous to (3.1.39)

$$F(H_i) \equiv \prod_{i=1}^{q} H_i^{2k_i'}\,, \quad k_q' = k_q - 1\,, \quad k_i' = k_i\,, \quad i \neq q\,, \quad \sum_{i=1}^{q} k_i' = p\,, \tag{3.1.48}$$

separating a H_q^2 factor in (3.1.46), and using (3.1.47) to get

$$I_k^{(p)} = \int dt\, n\, e^{-\Phi} H_q^2 F(H_i) \approx \frac{1}{N_q} \int dt\, n\, e^{-\Phi} \Big((\mathcal{D}\Phi)^2 - \sum_{j=1}^{q-1} N_j H_j^2 \Big) F(H_i)\,. \tag{3.1.49}$$

For the term containing $(\mathcal{D}\Phi)^2$ we use $e^{-\Phi}\mathcal{D}\Phi = -\mathcal{D}(e^{-\Phi})$ and integrate by parts

$$I_k^{(p)} \approx \frac{1}{N_q} \int dt\, n\, e^{-\Phi} \Big(\mathcal{D}^2\Phi F(H_i) + \mathcal{D}\Phi\mathcal{D}F(H_i) - \sum_{j=1}^{q-1} N_j H_j^2 F(H_i) \Big)\,. \tag{3.1.50}$$

One can easily show that $\mathcal{D}F(H_i) \approx 2p\mathcal{D}\Phi F(H_i)$. This follows from a much simpler version of the operations performed in (3.1.40). Using this rule in (3.1.50) we get

$$\begin{aligned} I_k^{(p)} &\approx \frac{1}{N_q} \int dt\, n\, e^{-\Phi} \Big(\mathcal{D}^2\Phi + 2p(\mathcal{D}\Phi)^2 - \sum_{j=1}^{q-1} N_j H_j^2 \Big) F(H_i) \\ &\approx \frac{1}{N_q} \int dt\, n\, e^{-\Phi} \Big((2p+1) \sum_{j=1}^{q} N_j H_j^2 - \sum_{j=1}^{q-1} N_j H_j^2 \Big) F(H_i) \\ &= \frac{1}{N_q} \int dt\, n\, e^{-\Phi} \Big((2p+1) N_q H_q^2 + 2p \sum_{j=1}^{q-1} N_j H_j^2 \Big) F(H_i) \\ &= (2p+1) I_k^{(p)} + \frac{2p}{N_q} \int dt\, n\, e^{-\Phi} \sum_{j=1}^{q-1} N_j H_j^2 F(H_i)\,, \end{aligned} \tag{3.1.51}$$

where in the second line we used (3.1.44b) and (3.1.44c). In the third line we separated the q factor of the first sum, and in the last equality we recognized the original action (see (3.1.49)). Bringing the latter to the left-hand side, we get:

$$I_k^{(p)} \approx -\frac{1}{N_q} \int dt\, n\, e^{-\Phi} \sum_{j=1}^{q-1} N_j H_j^2 F(H_i)\,. \tag{3.1.52}$$

The right-hand side of (3.1.52) contains $2(k_q - 1)$ powers of H_q (see (3.1.48)). Therefore, after this series of operations, we showed that a term like (3.1.46) containing $2k_q$ factors of H_q is equivalent to one with $2(k_q - 1)$ factors of H_q. Repeating this procedure iteratively, we can reduce by two units the power of H_q, at the expense of increasing the powers of the other Hubble parameters. At the end of the iteration, we are able to redefine away any appearance of H_q at higher orders in α', being its only appearance in the two-derivative action (3.1.42)! The most general higher-derivative term is therefore physically equivalent to

$$I_k^{(p)} \approx \int dt\, n\, e^{-\Phi} \prod_{i=1}^{q-1} H_i^{2k_i}\,. \tag{3.1.53}$$

As a corollary of this result, we get the absence of α' corrections for two particular cases of BI universes:

⊙ **FRW:** In this case we have only one independent scale factor:

$$q = 1, \quad N_1 = d, \quad a_1(t) \equiv a(t), \quad H_1(t) \equiv H(t)\,. \tag{3.1.54}$$

Since from (3.1.53) we can always remove one Hubble parameter completely, there is a scheme where there are no α' corrections in the action at all. The action to all orders in α' is just given by the two-derivative theory:

$$I_{\text{FRW}} = \int dt\, n\, e^{-\Phi}\left[-(\mathcal{D}\Phi)^2 + d\, H^2\right]\,. \tag{3.1.55}$$

⊙ **Isotropic and static directions:** A slightly more general case than FRW corresponds to the case

$$\begin{aligned} &q = 2, \quad N_1 \equiv N, \quad N_2 = d - N, \\ a_1(t) \equiv a(t), \quad &H_1(t) \equiv H(t), \quad a_2(t) = \text{const.} \quad \Rightarrow \quad H_2 = 0\,. \end{aligned} \tag{3.1.56}$$

This is one of the simplest anisotropic backgrounds where we have N isotropic directions and $d - N$ static ones. As in FRW, there is only one Hubble parameter, that we can redefine away. From (3.1.53) we see that there are no higher-order corrections at all. In this case the full action coincides with the lowest order one:

$$I_{\text{static}} = \int dt\, n\, e^{-\Phi}\left[-(\mathcal{D}\Phi)^2 + N\, H^2\right]\,. \tag{3.1.57}$$

A few comments are in order concerning the FRW case, which appears to be in conflict with the the classification of subsection (3.1.2). There we saw that for generic cosmological backgrounds there is a minimal field basis such that the action is given by (3.1.1). The coefficients $c_{2,0}$, $c_{3,0}$, etc., *cannot* be changed by field redefinitions and hence have an invariant meaning (and are certainly non-zero). Specializing (3.1.1) then to FRW backgrounds with a single scale factor $a(t)$ one obtains corrections of (3.1.55) with higher powers of H^2. As it was shown in [40, 47], depending on the coefficients the resulting theory may exhibit, for instance, non-perturbative de Sitter vacua, which are not visible in (3.1.55). So how is this result consistent with our above statement that for FRW backgrounds all higher-derivative corrections are removable by field redefinitions?

To understand the subtlety let us consider the first correction in α' and let us add a term proportional to $\text{Tr}\left((\mathcal{D}\mathcal{S})^2\right)$:

$$I^{(1)}[\mathcal{S}] = \int dt\, n\, e^{-\Phi}\left\{c_{2,0}\text{Tr}\left((\mathcal{D}\mathcal{S})^4\right) + \xi\left[\text{Tr}\left((\mathcal{D}\mathcal{S})^2\right)\right]^2\right\}\,. \tag{3.1.58}$$

As recalled above and shown in step 6. of the cosmological classification, the new term in here can be removed by field redefinitions: the coefficient ξ has no invariant meaning and we may choose $\xi = 0$, as done in (3.1.1). We can, however, also choose it to be non-zero and adjust it so that for FRW backgrounds it cancels the contribution from the single trace term. Specifically,

$$\xi \equiv -\tfrac{c_{2,0}}{2d} \qquad \Rightarrow \qquad I^{(1)}[\mathcal{S}_{\text{FRW}}] = 0\,, \tag{3.1.59}$$

where $\mathcal{S}_{\rm FRW}$ denotes the generalized metric (2.2.51) for a single scale factor. Thus, there is a field basis also for $\mathcal{S}$ so that, *when evaluated on FRW backgrounds*, the first-order α' corrections disappear. Similar remarks apply to all higher-derivative corrections.

So what, in view of the above discussion, is the fate of potential non- perturbative de Sitter vacua? It must be emphasized that the above manipulations using field redefinitions order-by-order in α' are strictly perturbative. There is no reason why a non-perturbative solution that is visible in one perturbative scheme must also be visible in another perturbative scheme. Although we are still missing a better understanding of this perturbative vs non-perturbative issue, the non-perturbative approach recently introduced in [48] represents a promising direction in these regards. Relatedly, whether a given solution physically exhibits the properties of de Sitter space depends on how one probes the spacetime with matter and how such content is coupled to the background fields (see, for instance, [92, 93] for a study of the classification in presence of matter fields).

3.2 General String Cosmologies at Order α'^3

The result of the classification of Section 3.1.2 states that any duality-invariant action in cosmological backgrounds can be brought to the minimal form (3.1.1). However, starting from a generic (not necessarily $O(d,d)$-invariant) cosmological action, the classification does not tell us how to perform such field redefinitions in practice to test for duality-invariance, and, if present, find the coefficients $c_{p,k}$ in (3.1.1). In this section we refine the original method of [40] to account for exactly these limitations, restricting to dilaton-gravity theories.

We begin in Section 3.2.1 by taking the general algorithm introduced in Section 3.1.1 and specifying it to dilaton-gravity theories in cosmological backgrounds. Here, the main difference with the systematic approach used in the classification of Section 3.1.2 is that the procedure is not inductive. Instead, one needs to keep track of higher-order effects because they can break duality invariance. If the latter is preserved after bringing the theory to canonical form, we give a prescription of how to read the coefficients of the classification in (3.1.1), coming from a background with vanishing b-field. Then, in Section 3.2.2 we perform a cosmological reduction of the D-dimensional dilaton-gravity theories presented in Section 2.1.3 and apply the aforementioned method to bring the theories to canonical form up to order α'^3. We find that these theories are compatible with duality-invariance, which allows us to read the first coefficients for bosonic, heterotic and Type II strings, collected in Table 3.1 [1, 3]

	$c_{1,0}$	$c_{2,0}$	$c_{3,0}$	$c_{3,1}$
Bosonic	$\frac{1}{2^6}$	$-\frac{1}{3.2^7}$	$\frac{1}{2^{12}}-\frac{3}{2^{12}}\zeta(3)$	$\frac{1}{2^{16}}+\frac{1}{2^{12}}\zeta(3)$
Heterotic	$\frac{1}{2^7}$	0	$-\frac{3}{2^{12}}\zeta(3)$	$-\frac{15}{2^{19}}+\frac{1}{2^{12}}\zeta(3)$
Type II	0	0	$-\frac{3}{2^{12}}\zeta(3)$	$\frac{1}{2^{12}}\zeta(3)$

Table 3.1: First coefficients of the classification for string theories.

3.2.1 Systematic of field redefinitions for dilaton-gravity theories

In Section 3.1.1 we described a general procedure to implement field redefinitions perturbatively order-by-order in α'. The algorithm is valid for any field configuration, in any background and to any order in α'. Here we specify it to dilaton-gravity theories in cosmological backgrounds up to and including order α'^3, with field content given by $\Psi(t) = \{n(t), g_{mn}(t), \Phi(t)\}$.

The cosmological ansatz for dilaton-gravity theories reads

$$e_\mu{}^\alpha = \begin{pmatrix} n & 0 \\ 0 & e_m{}^a \end{pmatrix} , \quad G_{\mu\nu} = \begin{pmatrix} -n^2 & 0 \\ 0 & g_{mn} \end{pmatrix} , \quad \phi = \frac{1}{2}\Phi + \frac{1}{4}\log\det g \,, \tag{3.2.1}$$

where we included the D-dimensional vielbein $e_\mu{}^\alpha$ and its internal counterpart $e_m{}^a$, spliting flat indices as $\alpha = (\bar{0}, a)$. Cosmological actions are obtained by plugging this ansatz in the parent action and truncating all but the time derivative.

Up to order α'^3, the dilaton-gravity D-dimensional action takes the form $\mathcal{I} = \mathcal{I}^{(0)} + \alpha'\mathcal{I}^{(1)} + \alpha'^2\mathcal{I}^{(2)} + \alpha'^3\mathcal{I}^{(3)} + \mathcal{O}(\alpha'^4)$. Upon compactification, the one-dimensional action is given by

$$I = \int dt\, n\, e^{-\Phi} \left[L^{(0)} + \alpha' L^{(1)} + \alpha'^2 L^{(2)} + \alpha'^3 L^{(3)}\right] \,, \tag{3.2.2}$$

whose leading contribution can be read from (2.2.36) by truncating M to zero

$$L^{(0)} = -(\mathcal{D}\Phi)^2 + \frac{1}{4}\text{tr}\left(L^2\right) \,, \tag{3.2.3}$$

with $L \equiv \mathcal{D}g g^{-1}$. The variation of (3.2.2) with respect to the fields gives rise to the equations of motion

$$E_\Psi = E_\Psi^{(0)} + \alpha' E_\Psi^{(1)} + \alpha'^2 E_\Psi^{(2)} + \mathcal{O}(\alpha'^3) = 0 \,, \tag{3.2.4}$$

where $\Psi = \{n, g_{mn}, \Phi\}$ and we omitted $E_\Psi^{(3)}$ because it will not be needed in our study. The zeroth-order equation is given by

$$\begin{aligned} E_g^{(0)} &= \frac{1}{2}\mathcal{D}\Phi L - \frac{1}{2}\mathcal{D}L \,, \\ E_n^{(0)} &= (\mathcal{D}\Phi)^2 - \frac{1}{4}\text{tr}\left(L^2\right) \,, \\ E_\Phi^{(0)} &= 2\mathcal{D}^2\Phi - (\mathcal{D}\Phi)^2 - \frac{1}{4}\text{tr}\left(L^2\right) \,. \end{aligned} \tag{3.2.5}$$

Following Section 3.1.1, after performing generic field redefinitions $\Psi \to \Psi' = \Psi + \delta\Psi$ with $\delta\Psi$ expanded up to order α'^3 the transformed action takes the form (3.1.8), in which we neglect $\mathcal{O}(\alpha'^4)$ effects or higher. We now want to bring the action to canonical form order-by-order in α'. As opposed to the classification in Section 3.1, here we do care about induced terms because we want the exact form of the action up to order α'^3. This includes the coefficients next to the couplings in the minimal basis. Therefore, we cannot simply use $E_\Psi^{(0)} \approx 0$ as a substitution rule (as we did in (3.1.32)), this time we need

$$E_\Psi^{(0)} \approx -\alpha' \Delta_\Psi \,. \tag{3.2.6}$$

Here we are using again the symbol $\approx$, which means equality at the level of the action up to field redefinitions, integration by part and higher-order effects. Using (3.2.5), (3.2.6) can be written in a convenient way

$$\mathcal{D}L \approx \mathcal{D}\Phi L + 2\alpha'\Delta_g \,, \tag{3.2.7a}$$

$$(\mathcal{D}\Phi)^2 \approx \frac{1}{4}\text{tr}\left(L^2\right) - \alpha'\Delta_n \,, \tag{3.2.7b}$$

$$\mathcal{D}^2\Phi \approx \frac{1}{4}\text{tr}\left(L^2\right) - \frac{1}{2}\alpha'(\Delta_n + \Delta_\Phi) \,. \tag{3.2.7c}$$

As explained in Section 3.1.1, the use of field redefinitions in the form of substitution rules allows us to remove ambiguities in a systematic way. Using (3.2.7) we now give a step-by-step procedure to remove any appearance of $\mathcal{D}L, \mathcal{D}\Phi$ and $\text{tr}\,(L^2)$ and their derivatives, at each order in α', at the expense of inducing new terms at higher orders. In the following we will refer to terms containing $\mathcal{D}L, \mathcal{D}\Phi$ and $\text{tr}\,(L^2)$ and their time derivatives as removable. At a technical level, the procedure is very similar to the one for the classification so we will not describe it thoroughly here (for more details see [3]). The only difference is the additional terms Δ_Ψ in (3.2.7) that let us keep track of $\delta\Psi$, which are then used to build the induced terms at each order. The main steps of the algorithm are summarized as follow:

1. We remove higher derivatives of L by using (3.2.7a) as many times as needed.
2. We do the same for higher derivatives of $\mathcal{D}\Phi$ using (3.2.7c). In this step, higher derivatives of L may reappear, which can be eliminated using step 1. This could produce again higher derivatives of $\mathcal{D}\Phi$ that are eliminated via (3.2.7c). This internal loop may be repeated a few times but it is guaranteed to coverge to an action with only first-order derivatives.
3. Using (3.2.7b) we remove any higher powers of $\mathcal{D}\Phi$. If higher-derivative terms are produced, we use steps 1. and 2. and iterate.
4. Reached this point, the only possible contribution from the dilaton comes from a linear term in $\mathcal{D}\Phi$. This can be eliminated by an integration by parts together with (3.2.7a).
5. Finally, in order to eliminate terms with $\text{tr}\,(L^2)$ one has to use (3.2.7b) to make a $(\mathcal{D}\Phi)^2$ factor reappear, followed by an integration by parts of

the dilaton. This will induce higher derivatives of $\mathcal{D}\Phi$ and L that can be traded for $(\mathcal{D}\Phi)^2$ and $\mathrm{tr}\,(L^2)$ by using (3.2.7a) and (3.2.7c). Finally, one uses (3.2.7b) once more to bring everything to exactly the same form as the original term that we started from, but with a different coefficient.

The idea is to apply these five steps at the level of the action order-by-order, starting at order α'. After completing this procedure to first order, the resulting $I'^{(1)}$ will be in canonical form, which in this context it means that the action contains no terms with $\mathcal{D}L, \mathcal{D}\Phi, \mathrm{tr}\,(L^2)$ and their derivatives. On top of $I'^{(1)}$ in the minimal basis, the five-step procedure generates Δ_Ψ-dependent contributions. The terms linearly coupled to Δ_Ψ correspond to $\delta g^{(1)}, \delta n^{(1)}$, and $\delta\Phi^{(1)}$, which we save to build induced terms at the next orders using the general structure (3.1.8).

We can then apply the algorithm to second order in α', taking as the starting point the dimensionally reduced action $I^{(2)}$ supplemented with the terms that were induced by the field redefinitions of the previous iteration of the algorithm. Again, the resulting action at order α'^2 will be in canonical form and from the linear terms in Δ_Ψ we can read $\delta g^{(2)}, \delta n^{(2)}$, and $\delta\Phi^{(2)}$.

Finally, we apply the algorithm to the action to third order in α', including the induced terms to this order. In this final step we do not have to keep track of $\delta\Psi^{(3)}$, as these contribute only to fourth order in α'. Therefore, we can use (3.2.7) with $\Delta_\Psi = 0$. In practice, this implies that at order α'^3 we can simply set to zero all terms containing $\mathcal{D}L, \mathcal{D}\Phi$ or $\mathrm{tr}\,(L^2)$! This motivates a new notation for functions of these removable terms that appear at order α'^3: inside a Lagrangian these will be denoted as $\mathbb{L}$, while for removable terms in the equations of motion we use $\mathbb{E}$.

At this point, the field redefinition procedure has been completed and the action up to and including order α'^3 is in the minimal basis, containing only the structures $\mathrm{tr}\,(L^p)$ with $p \neq 2$.

Connection to the cosmological classification

We can now relate the resulting dilaton-gravity theory to the cosmological classification of (3.1.1). To this end, we need to write the theory, if possible, in terms of duality-invariant objects, using the following relation between $L_m{}^n$ and the generalized metric $(\mathcal{S}_g)_M{}^N$

$$\mathcal{S}_g = \begin{pmatrix} 0 & g \\ g^{-1} & 0 \end{pmatrix} , \qquad \mathcal{D}(\mathcal{S}_g^{2k}) = \begin{pmatrix} (-1)^k L^{2k} & 0 \\ 0 & (-1)^k \left[L^{2k}\right]^t \end{pmatrix} , \tag{3.2.8}$$

which implies

$$\mathrm{Tr}\left((\mathcal{D}\mathcal{S}_g)^{2k}\right) = (-1)^k\, 2\, \mathrm{tr}\left(L^{2k}\right) . \tag{3.2.9}$$

Here we introduced a subscript in $\mathcal{S}_g$ to differentiate it from the full generalized metric (2.2.19), with non-vanishing b-field. This is a rather important distinction, since when $b = 0$ the duality group $O(d,d)$ is broken to $GL(d) \times \mathbb{Z}_2$, in which the $\mathbb{Z}_2$ sector acts on the metric via $g \to g^{-1}$ (these are the full-factorized dualities of (2.2.16)). From (3.2.9) we see that this residual duality-invariance requires that odd powers such as $\mathrm{tr}\,(L^3)^2$ and $\mathrm{tr}\,(L^3)\,\mathrm{tr}\,(L^5)$ are actually absent, which in turn poses constraints on the D-dimensional higher-derivative corrections.

Having said that, it is important to keep in mind that dilaton-gravity backgrounds allow us to test only this smaller subsector of duality invariance. The remaining information, completing the group to $O(d,d)$, comes from the b-field couplings we are ignoring. However, if we assume $O(d,d)$-invariance must hold, we can use this to bootstrap the b-field contributions in the cosmological theory simply by extending $\mathcal{S}_g \to \mathcal{S}$.

3.2.2 Cosmological reductions and canonical form

Here we apply the algorithm of the previous subsection to the cosmological reduction of the dilaton-gravity theories of Section 2.1.3, corresponding to bosonic, heterotic and Type II strings. [3]

Before doing this, however, an important comment is in order: in Section 2.1.3 we omitted dilaton contributions in the parent theory at order α'^3 and we claimed that would not prevent us from getting the full duality-invariant action in cosmological backgrounds. We can now explain why. Compactification of dilaton terms in $\mathcal{L}^{(3)}$, if present, would lead to $\mathcal{D}\Phi$ and $\mathrm{tr}\,(L)$ factors in the effective theory (see (2.2.35) or (3.2.11) below). The dilaton contributions are removable, and so they can be set to zero via field redefinitions at expense of higher-order effects. The $\mathrm{tr}\,(L)$ cannot be eliminated, but they are incompatible with $\mathbb{Z}_2 \subset O(d,d)$ as seen by (3.2.9). Assuming duality invariance, all these odd-trace terms should cancel among themselves. Therefore, as well as for the missing B-field couplings, without the parent dilatons at order α'^3 we cannot fully test $O(d,d)$-invariance. If such unknown couplings turned out to be compatible with $O(d,d)$, the cosmological actions should coincide with the ones found here [4].

The D-dimensional actions (2.1.17) are built from Riemann tensors, Chern-Simons connections and derivatives of the dilaton defined by

$$
\begin{aligned}
\Gamma_{\mu\nu}{}^{\rho} &= \frac{1}{2} G^{\rho\sigma}\left(\partial_\mu G_{\nu\sigma} + \partial_\nu G_{\mu\sigma} - \partial_\sigma G_{\mu\nu}\right) , \\
\omega_{\mu\alpha}{}^{\beta} &= e_\alpha{}^{\nu} \nabla_\mu e_\nu{}^{\beta} = e_\alpha{}^{\nu}\partial_\mu e_\nu{}^{\beta} - e_\alpha{}^{\nu}\Gamma_{\mu\nu}{}^{\rho} e_\rho{}^{\beta} , \\
\mathcal{R}^{\rho}{}_{\sigma\mu\nu} &= \partial_\mu \Gamma_{\nu\sigma}{}^{\rho} - \partial_\nu \Gamma_{\mu\sigma}{}^{\rho} + \Gamma_{\mu\lambda}{}^{\rho}\Gamma_{\nu\sigma}{}^{\lambda} - \Gamma_{\nu\lambda}{}^{\rho}\Gamma_{\mu\sigma}{}^{\lambda} , \\
\mathcal{R}_{\mu\nu\alpha}{}^{\beta}(\omega) &= \partial_\mu \omega_{\nu\alpha}{}^{\beta} - \partial_\nu \omega_{\mu\alpha}{}^{\beta} + \omega_{\mu\alpha}{}^{\gamma}\omega_{\nu\gamma}{}^{\beta} - \omega_{\nu\alpha}{}^{\gamma}\omega_{\mu\gamma}{}^{\beta} = -e_\alpha{}^{\sigma} e_\rho{}^{\beta}\mathcal{R}^{\rho}{}_{\sigma\mu\nu} , \\
\Omega_{\mu\nu\rho}(\omega) &= \mathrm{Tr}\left(\omega_{[\mu}\partial_\nu\omega_{\rho]} + \frac{2}{3}\omega_{[\mu}\omega_\nu\omega_{\rho]}\right) .
\end{aligned}
\tag{3.2.10}
$$

[4]This is indeed the case for type II string at order α'^3 and bosonic string at order α'^2, for which the parent effective actions including all massless fields were determined by demanding T-duality invariance in [60] and [94], respectively. These full theories were then reduced to cosmological backgrounds in [90] and [91], determining the coefficients $c_{3,0}$ for type II and $c_{2,0}$ for bosonic string, being in perfect agreement with the results obtained here.

Their non-vanishing components under cosmological reduction are

$$
\begin{aligned}
\Gamma_{0m}{}^{n} &= \frac{n}{2} L_m{}^n \,, & \Gamma_{mn}{}^{0} &= \frac{1}{2n} L_{mn} \,, & \Gamma_{00}{}^{0} &= \mathcal{D}n \,, \\
\omega_{ma}{}^{\bar{0}} &= -\frac{1}{2} L_{ma} \,, & \omega_{0a}{}^{b} &= n\, e_a{}^p \mathcal{D} e_p{}^b - \frac{n}{2} L_a{}^b \,, \\
\mathcal{R}_{mnpq} &= \frac{1}{2} L_{p[m} L_{n]q} \,, & \mathcal{R}_{0m0n} &= -\frac{n^2}{2} \mathcal{D} L_{mn} - \frac{n^2}{4} L^2_{mn} \,, \\
\Omega_{0m}{}^{n}(\omega) &= \frac{n}{6} \dot{L}_{[m}{}^p L_{n]p} \,, & \nabla_0 \phi &= \frac{n}{2} \mathcal{D}\Phi + \frac{n}{4} \mathrm{tr}\,(L) \,,
\end{aligned}
\tag{3.2.11}
$$

where internal indices are raised with g^{mn}, lower with g_{mn} (see below (2.2.35)), and flattened with $e_m{}^a$ and $e_a{}^m$ such that $L_{ma} = L_m{}^n g_{np} e_a{}^p$.

Upon compactification, the one-dimensional action for dilaton-gravity theories takes the generic form (3.2.2). Since the zeroth- and first-order Lagrangians of the D-dimensional theory admit the universal forms (2.1.19) for all strings, the same is true for their cosmological counterpart. The leading contribution was given in (3.2.3) while

$$
L^{(1)} = \frac{\gamma}{4} \left(\frac{1}{8} \mathrm{tr}\,(L^4) + \frac{1}{8} \mathrm{tr}\,(L^2)^2 + \mathrm{tr}\,(L^2 \mathcal{D} L) + \mathrm{tr}\,((\mathcal{D}L)^2) \right) , \tag{3.2.12}
$$

with $\gamma = 1, \frac{1}{2}, 0$ for bosonic, heterotic, and Type II strings, respectively. The $\mathcal{O}(\alpha')$ contribution to the equations of motion comes from (3.2.12) and is given by

$$
\begin{aligned}
E_g^{(1)} = \gamma \Big\{ & \frac{1}{4} \left[(\mathcal{D}\Phi)^2 - \mathcal{D}^2\Phi \right] \left[L^2 + 2\mathcal{D}L \right] \\
& + \frac{1}{8} \mathcal{D}\Phi \left[L^3 + 2L\mathcal{D}L - 8\mathcal{D}^2 L - 6(\mathcal{D}L)L + \mathrm{tr}\,(L^2)\, L \right] \\
& - \frac{1}{8} \left[L^2 \mathcal{D}L + L(\mathcal{D}L)L + (\mathcal{D}L)L^2 \right] + \frac{1}{2} \left[(\mathcal{D}^2 L)L + \mathcal{D}^3 L - L\mathcal{D}^2 L \right] \\
& - \frac{1}{4} \mathrm{tr}\,(L\mathcal{D}L)\, L - \frac{1}{8} \mathrm{tr}\,(L^2)\, \mathcal{D}L \Big\} \,,
\end{aligned}
\tag{3.2.13a}
$$

$$
\begin{aligned}
E_n^{(1)} = \gamma \Big\{ & -\frac{3}{32} \mathrm{tr}\,(L^4) - \frac{3}{32} \mathrm{tr}\,(L^2)^2 - \frac{1}{4} \mathrm{tr}\,((\mathcal{D}L)^2) + \frac{1}{2} \mathrm{tr}\,(L\mathcal{D}^2 L) \\
& - \frac{1}{4} \mathcal{D}\Phi \mathrm{tr}\,(L^3) - \frac{1}{2} \mathcal{D}\Phi \mathrm{tr}\,(L\mathcal{D}L) \Big\} \,,
\end{aligned}
\tag{3.2.13b}
$$

$$
E_\Phi^{(1)} = \gamma \left\{ -\frac{1}{32} \mathrm{tr}\,(L^4) - \frac{1}{32} \mathrm{tr}\,(L^2)^2 - \frac{1}{4} \mathrm{tr}\,((\mathcal{D}L)^2) - \frac{1}{4} \mathrm{tr}\,(L^2 \mathcal{D}L) \right\} . \tag{3.2.13c}
$$

The higher-order Lagrangians and equations of motion depend on the specific string theory under consideration.

We now begin with the algorithm to bring (3.2.2) to canonical form. For the first order in α', since $I^{(1)}$ and $E_\Psi^{(1)}$ are common to all strings, we can treat all cases simultaneously. The idea is to redefine away all removable terms from (3.2.12) (i.e. those containing $\mathcal{D}L, \mathcal{D}\Phi, \mathrm{tr}\,(L^2)$ and their derivatives). By following the five-step procedure of the previous section we find

$$
L^{(1)} \approx \frac{\gamma}{32} \mathrm{tr}\,(L^4) + \alpha' \left(\Delta_n \frac{\delta n^{(1)}}{n} + \Delta_\Phi \delta\Phi^{(1)} + \mathrm{tr}\,(\Delta_g \delta g^{(1)} g^{-1}) \right) + \mathcal{O}(\alpha'^2 \Delta_\Psi^2) \,, \tag{3.2.14}
$$

where the first term is the transformed $L'^{(1)}$ in canonical form and the rest are artificial Δ_Ψ-dependent contributions, from which we obtained

$$\frac{\delta^{(1)}n}{n} = \frac{\gamma}{32}\text{tr}\left(L^2\right)\ , \tag{3.2.15a}$$

$$\delta^{(1)}\Phi = \frac{3\gamma}{32}\text{tr}\left(L^2\right)\ , \tag{3.2.15b}$$

$$\delta^{(1)}g\,g^{-1} = \gamma\left[-\frac{1}{4}\mathcal{D}\Phi L + \frac{1}{2}\mathcal{D}L - \frac{1}{4}L^2\right]\ . \tag{3.2.15c}$$

Obtaining (3.2.15) was the only purpose of introducing Δ_Ψ, so we can now set them to zero in (3.2.14).

For the next and following orders we need to analyze each string separately.

Type II strings

The action for Type II strings contains no order α' nor α'^2 deformations. The Lagrangians of the parent theory is given by (2.1.25), rewritten here for convenience

$$\begin{aligned}
\mathcal{L}^{(1)}_{\text{II}} &= \mathcal{L}^{(2)}_{\text{II}} = 0\ ,\\
\mathcal{L}^{(3)}_{\text{II}} &= \frac{\zeta(3)}{3\cdot 2^{11}}J(1) + \dots\\
&= -\frac{\zeta(3)}{32}\left[\mathcal{R}^{\alpha\beta\mu\nu}\mathcal{R}_{\mu\nu}{}^{\gamma\delta}\mathcal{R}_{\alpha\gamma}{}^{\rho\sigma}\mathcal{R}_{\rho\sigma\beta\delta} - 4\mathcal{R}_{\alpha\beta}{}^{\gamma\delta}\mathcal{R}_{\delta\mu}{}^{\alpha\nu}\mathcal{R}_{\nu\rho}{}^{\beta\sigma}\mathcal{R}_{\sigma\gamma}{}^{\mu\rho}\right] + \dots
\end{aligned} \tag{3.2.16}$$

with $\dots$ referring to Ricci and dilaton contributions that we are ignoring for reasons explained above. The order α'^3 is written in terms of the function

$$J(c) = t_8 t_8\mathcal{R}^4 + \frac{c}{8}e_{10}e_{10}\mathcal{R}^4\,, \tag{3.2.17}$$

with t_8 and e_{10} defined in (2.1.21) and (2.1.23), respectively. The point of introducing such a function is the following: the term containing t_8 can be calculated from four-point scattering amplitudes, whereas the Gauss-Bonnet term with e_{10} starts at fifth order in a field expansion. The cosmological reduction of $J(c)$ in (3.2.17) gives

$$J(c) = \frac{1}{4}(9-45c)\text{tr}\left(L^8\right) + \frac{1}{16}(51+45c)\text{tr}\left(L^4\right)^2 - 6(1-c)\text{tr}\left(L^3\right)\text{tr}\left(L^5\right) + \mathbb{L}^{(3)}_{\text{II}}\,, \tag{3.2.18}$$

where $\mathbb{L}^{(3)}_{\text{II}}$ denotes a function of removable terms. We then see that the requirement of $O(9,9)$ symmetry fixes the coefficient c to its expected value $c=1$, as it forbids the presence of the interaction $\text{tr}\,(L^3)\,\text{tr}\,(L^5)$.

The cosmological reduction of (3.2.16) in the form (3.2.2) is then given by

$$\begin{aligned}
L^{(1)}_{\text{II}} &= L^{(2)}_{\text{II}} = 0\ ,\\
L^{(3)}_{\text{II}} &= \frac{\zeta(3)}{2^{11}}\left[-3\text{tr}\left(L^8\right) + 2\text{tr}\left(L^4\right)^2\right] + \mathbb{L}^{(3)}_{\text{II}}\ .
\end{aligned} \tag{3.2.19}$$

The procedure to bring the action to canonical form is trivial for the first two orders, so we have $\delta\Psi^{(1)} = \delta\Psi^{(2)}$. At order α'^3, there are no induced terms so we only need to remove ambiguous terms from $L^{(3)}_{\rm II}$ by using (3.2.7) with $\Delta_\Psi = 0$. As explained before, at this order this accounts for just setting to zero all removable contributions. The latter are all packed in $\mathbb{L}^{(3)}_{\rm II}$ and so the field redefinition procedure finalizes with (3.2.19) with $\mathbb{L}^{(3)}_{\rm II} = 0$.

The final action can then be written in terms of the generalized metric by using (3.2.9) together with the bootstrap step $\mathcal{S}_g \to \mathcal{S}$ to arrive at [1]

$$I'_{\rm II} = \int dt\, n\, e^{-\Phi} \left\{ -(\mathcal{D}\Phi)^2 - \frac{1}{8}{\rm Tr}\left((\mathcal{D}\mathcal{S})^2\right) + \alpha'^3 \frac{\zeta(3)}{2^{12}} \left[-3\,{\rm Tr}\left((\mathcal{D}\mathcal{S})^8\right) + {\rm Tr}\left((\mathcal{D}\mathcal{S})^4\right)^2 \right] \right\} . \tag{3.2.20}$$

Bosonic strings

For the bosonic string, the 26-dimensional dilaton-gravity Lagrangian was given in (2.1.26) in the scheme of [3]. The compactified theory reads

$$\begin{aligned} L^{(2)}_B &= \frac{1}{192}{\rm tr}\left(L^6\right) + \frac{1}{16}{\rm tr}\left((\mathcal{D}L)^3\right) - \frac{1}{768}{\rm tr}\left(L^2\right)^3 + \frac{3}{32}{\rm tr}\left(L^2(\mathcal{D}L)^2\right) \\ &\quad + \frac{1}{16}{\rm tr}\left(L^4\mathcal{D}L\right) + \frac{1}{256}{\rm tr}\left(L^2\right){\rm tr}\left(L^4\right) - \frac{1}{64}{\rm tr}\left(L^3\right){\rm tr}\left(L\mathcal{D}L\right) \\ &\quad - \frac{1}{64}{\rm tr}\left(L\mathcal{D}L\right)^2 + \frac{1}{64}{\rm tr}\left(L\mathcal{D}LL\mathcal{D}L\right) , \end{aligned} \tag{3.2.21}$$

$$L^{(3)}_B = \frac{1}{2^{11}}\left[{\rm tr}\left(L^8\right) - {\rm tr}\left(L^4\right)^2\right] - \frac{\zeta(3)}{2^{11}}\left[3{\rm tr}\left(L^8\right) - 2{\rm tr}\left(L^4\right)^2\right] + \mathbb{L}^{(3)}_B , \tag{3.2.22}$$

where $\mathbb{L}^{(3)}_B$ contains removable terms, and $L^{(1)}_B$ is given in (3.2.12) with $\gamma = 1$. The first order equations of motion were given in (3.2.13) while the second-order ones read

$$E^{(2)}_g = \mathbb{E}^{(2)}_g , \tag{3.2.23}$$

$$E^{(2)}_n = -\frac{5}{192}{\rm tr}\left(L^6\right) + \mathbb{E}^{(2)}_n , \tag{3.2.24}$$

$$E^{(2)}_\Phi = -\frac{1}{192}{\rm tr}\left(L^6\right) + \mathbb{E}^{(2)}_\Phi . \tag{3.2.25}$$

Here $\mathbb{E}^{(2)}_\Psi$ collectively denote functions that depend on the removable quantities $\mathcal{D}L, \mathcal{D}\Phi$, and ${\rm tr}\,(L^2)$. We do not write their explicit form, simply because we do not need them. As it can be seen from (3.1.8), their first contribution appear at order α'^3, and we know that at that order we can simply set removable terms to zero.

We can now bring the action to its minimal form using field redefinitions. The leading term was already considered in (3.2.14), obtaining

$$L^{(1)}_B \approx L'^{(1)}_B = \frac{1}{32}{\rm tr}\left(L^4\right) , \tag{3.2.26}$$

together with the fluctuations (3.2.15) with $\gamma = 1$. Moving to the second order, we need to remove ambiguities from the combination $I_B^{(2)} + E_\Psi^{(1)} \cdot \delta\Psi^{(1)} + \frac{1}{2}\Delta_2 I_B^{(0)} \cdot (\delta\Psi^{(1)})^2$, which consists of all known quantities: $I_B^{(2)}$ is built from (3.2.21), $E_\Psi^{(1)}$ is given in (3.2.13), $\delta\Psi^{(1)}$ was computed in (3.2.15), and $\Delta_2 I^{(0)}$ can be easily computed from a second variation of (3.2.3). Using (3.2.7) we get

$$
\begin{aligned}
&I_B^{(2)} + E_\Psi^{(1)} \cdot \delta\Psi^{(1)} + \frac{1}{2}\Delta_2 I_B^{(0)} \cdot (\delta\Psi^{(1)})^2 \\
&\approx \int dt\, n\, e^{-\Phi} \left[\frac{1}{192} \text{tr}\left(L^6\right) + \alpha' \left(\Delta_n \frac{\delta n^{(2)}}{n} + \Delta_\Phi \delta\Phi^{(2)} + \text{tr}\left(\Delta_g \delta g^{(2)} g^{-1}\right) \right) + \mathcal{O}(\alpha'^2 \Delta_\Psi^2) \right] ,
\end{aligned} \tag{3.2.27}
$$

from which we read the second-order Lagrangian in canonical form from the first term

$$
L_B'^{(2)} = \frac{1}{192} \text{tr}\left(L^6\right) , \tag{3.2.28}
$$

and $\delta\Psi^{(2)}$ from the terms in parenthesis. These second-order fluctuations are required for the induced terms at order α'^3, yet we do not display them here since their exact expressions are long, and not particularly illuminating.

Finally, following (3.1.8), at order α'^3 we need to consider the induced terms that we have been collecting so far

$$
\begin{aligned}
&E_\Psi^{(2)} \cdot \delta\Psi^{(1)} + E_\Psi^{(1)} \cdot \delta\Psi^{(2)} + \frac{1}{2}\Delta_2 I_B^{(1)} \cdot \left(\delta\Psi^{(1)}\right)^2 + \Delta_2 I_B^{(0)} \cdot \delta\Psi^{(1)} \cdot \delta\Psi^{(2)} \\
&\qquad + \frac{1}{3!}\Delta_3 I_B^{(0)} \cdot \left(\delta\Psi^{(1)}\right)^3 = \int dt\, n\, e^{-\Phi} \left[\frac{9}{2^{14}} \text{tr}\left(L^4\right)^2 + \tilde{\mathbb{L}}_B^{(3)} \right] .
\end{aligned} \tag{3.2.29}
$$

Here $\tilde{\mathbb{L}}_B^{(3)}$ contains only removable terms, that can be trivially removed by field redefinitions at this order. The first term of the right-hand side, however, corresponds to a non-trivial contribution at order α'^3. It shifts the coefficient next to $\text{tr}\left(L^4\right)^2$ in the original $L_B^{(3)}$ in (3.2.22). Therefore, setting removable terms to zero we end up with

$$
L_B^{(3)} \approx L_B'^{(3)} = \frac{1 - 3\zeta(3)}{2^{11}} \text{tr}\left(L^8\right) + \frac{1 + 16\zeta(3)}{2^{14}} \text{tr}\left(L^4\right)^2 . \tag{3.2.30}
$$

The resulting action is written in terms of traces of even powers of L, so it can be cast in terms of the generalized metric using (3.2.9). The manifestly $O(25, 25)$ invariant expression is given by

$$
\begin{aligned}
I_B' = \int dt\, n\, e^{-\Phi} \Big\{ &-(\mathcal{D}\Phi)^2 - \frac{1}{8}\text{Tr}\left((\mathcal{D}\mathcal{S})^2\right) + \alpha' \frac{1}{2^6}\text{Tr}\left((\mathcal{D}\mathcal{S})^4\right) - \alpha'^2 \frac{1}{3 \cdot 2^7}\text{Tr}\left((\mathcal{D}\mathcal{S})^6\right) \\
&+ \alpha'^3 \left[\frac{1 - 3\zeta(3)}{2^{12}} \text{Tr}\left((\mathcal{D}\mathcal{S})^8\right) + \frac{1 + 16\zeta(3)}{2^{16}} \text{Tr}\left((\mathcal{D}\mathcal{S})^4\right)^2 \right] \Big\} .
\end{aligned} \tag{3.2.31}
$$

Heterotic strings

The 10-dimensional low energy effective action for the Heterotic string up to and including order α'^3 was given in (2.1.27). In a cosmological background, the theory reduces to

$$L_H^{(2)} = \frac{3}{128}\text{tr}\left(L^2(\mathcal{D}L)^2\right) - \frac{3}{128}\text{tr}\left(L\mathcal{D}LL\mathcal{D}L\right) \ , \tag{3.2.32}$$

$$L_H^{(3)} = -\frac{1}{2^{12}}\text{tr}\left(L^4\right)^2 + \frac{\zeta(3)}{2^{11}}\left[-3\text{tr}\left(L^8\right) + 2\text{tr}\left(L^4\right)^2\right] + \mathbb{L}_H^{(3)} \, , \tag{3.2.33}$$

with $L_H^{(1)}$ given in (3.2.12) with $\gamma = \frac{1}{2}$. To order α' the deformations to the equations of motion are given in (3.2.13), while at second-order we have

$$E_g^{(2)} = \mathbb{E}_g^{(2)} \ , \tag{3.2.34}$$

$$E_n^{(2)} = \mathbb{E}_n^{(2)} \ , \tag{3.2.35}$$

$$E_\Phi^{(2)} = \mathbb{E}_\Phi^{(2)} \, , \tag{3.2.36}$$

which shows that, under field redefinitions, they contribute to the transformed action at order α'^3 only with removable terms.

Bringing the action to a canonical form follows the same steps we performed several times already, so we do not repeat them here. At the end of the field redefinition procedure, the transformed action in canonical form has the following Lagrangians

$$\begin{aligned} L'^{(1)}_H &= \frac{1}{64}\text{tr}\left(L^4\right) \\ L'^{(2)}_H &= 0 \\ L'^{(3)}_H &= -\frac{3\zeta(3)}{2^{11}}\,\text{tr}\left(L^8\right) - \frac{15 - 2^7\zeta(3)}{2^{17}}\text{tr}\left(L^4\right)^2 \, . \end{aligned} \tag{3.2.37}$$

Since they all contain traces of only even powers of L, the action can be written in a manifestly $O(9,9)$ form using (3.2.9). The final result is

$$\begin{aligned} I_H = \int dt\, n\, e^{-\Phi} \Big\{ & -(\mathcal{D}\Phi)^2 - \frac{1}{8}\text{Tr}\left((\mathcal{D}\mathcal{S})^2\right) + \alpha' \frac{1}{2^7}\text{Tr}\left((\mathcal{D}\mathcal{S})^4\right) \\ & + \alpha'^3 \left[-\frac{3\zeta(3)}{2^{12}}\,\text{Tr}\left((\mathcal{D}\mathcal{S})^8\right) - \frac{15 - 2^7\zeta(3)}{2^{19}}\text{Tr}\left((\mathcal{D}\mathcal{S})^4\right)^2 \right] \Big\} \, . \end{aligned} \tag{3.2.38}$$

3.3 HSZ and α'

In Section 2.3.2 we introduced HSZ theory as an α'-complete extension of Double Field Theory. Its spectrum contains, as a subsector, the fields of the D-dimensional universal massless sector of string theory: the metric, B-field and dilaton. The field content is completed by some novel massive modes. In its original formulation, the action of HSZ has only a finite number of derivatives, which is a direct consequence of the presence of the massive modes. These extra fields can be integrated out, leaving an effective theory for the massless sector with an infinite tower of α' corrections. From here one can relate HSZ to conventional supergravity. Another option is to keep the massive fields and analyze the non-perturbative α'-exact theory directly.

In this section, we carry out both procedures for cosmological backgrounds [5]. We begin integrating out the massive fields perturbatively in α', obtaining the effective action up to order α'^4. We then bring it to canonical form by using the method introduced in Section 3.1.1. In particular, this allows us to read the coefficients of the classification (3.1.1) for HSZ. We then come back to the original formulation where the massive fields are kept. We find a two-derivative reformulation of the theory, take the tensionless limit $\alpha' \to \infty$, and find solutions in FRW-like backgrounds upon perturbation theory in $\frac{1}{\alpha'}$.

3.3.1 Supergravity limit up to order α'^4

We start from the action in the form (2.3.33) setting $A = B = 0$, as implied by their own field equations. This leaves us with the following action for $\mathcal{Z}, n$ and Φ:

$$I = \int dt\, n\, e^{-\Phi} \left\{ \frac{1}{2\alpha'} \left[\mathrm{Tr}\left(\mathcal{Z}\right) - \frac{1}{3}\mathrm{Tr}\left(\mathcal{Z}^3\right) \right] - \frac{1}{8}\mathrm{Tr}\left((\mathcal{D}\mathcal{Z})^2\right) - (\mathcal{D}\Phi)^2 \right.$$
$$\left. + \frac{\alpha'^2}{4} \left[\frac{1}{4}(\mathcal{D}^3\Phi)^2 + \frac{1}{3}(\mathcal{D}^2\Phi)^3 \right] \right\} . \tag{3.3.1}$$

The equations of motion can be obtained from the general variation of this action

$$\delta I = \int dt\, n\, e^{-\Phi} \left[\mathrm{Tr}\left(\delta\mathcal{Z} E_{\mathcal{Z}}\right) + \delta\Phi E_\Phi + \frac{\delta n}{n} E_n \right] , \tag{3.3.2}$$

with

$$0 = E_{\mathcal{Z}} \equiv \frac{1}{2\alpha'}\left(1 - \mathcal{Z}^2\right) - \frac{1}{4}\mathcal{D}\Phi\mathcal{D}\mathcal{Z} + \frac{1}{4}\mathcal{D}^2\mathcal{Z} , \tag{3.3.3a}$$

$$0 = E_\Phi \equiv -\frac{1}{2\alpha'} \left[\mathrm{Tr}\left(\mathcal{Z}\right) - \frac{1}{3}\mathrm{Tr}\left(\mathcal{Z}^3\right) \right] + \frac{1}{8}\mathrm{Tr}\left((\mathcal{D}\mathcal{Z})^2\right) - (\mathcal{D}\Phi)^2 + 2\mathcal{D}^2\Phi$$
$$- \frac{\alpha'^2}{8} \left[\mathcal{D}^6\Phi - 3\mathcal{D}\Phi\mathcal{D}^5\Phi - 7\mathcal{D}^2\Phi\mathcal{D}^4\Phi + 3(\mathcal{D}\Phi)^2\mathcal{D}^4\Phi - \frac{9}{2}(\mathcal{D}^3\Phi)^2 \right.$$
$$\left. + 11\mathcal{D}\Phi\mathcal{D}^2\Phi\mathcal{D}^3\Phi - (D\Phi)^3\mathcal{D}^3\Phi \quad + \frac{8}{3}(\mathcal{D}^2\Phi)^3 - 2(\mathcal{D}\Phi)^2(\mathcal{D}^2\Phi)^2 \right] , \tag{3.3.3b}$$

$$0 = E_n \equiv \frac{1}{2\alpha'} \left[\mathrm{Tr}\left(\mathcal{Z}\right) - \frac{1}{3}\mathrm{Tr}\left(\mathcal{Z}^3\right) \right] + \frac{1}{8}\mathrm{Tr}\left((\mathcal{D}\mathcal{Z})^2\right) + (\mathcal{D}\Phi)^2$$
$$- \frac{\alpha'^2}{8} \left[\mathcal{D}\Phi\mathcal{D}^5\Phi - \mathcal{D}^2\Phi\mathcal{D}^4\Phi - 2(\mathcal{D}\Phi)^2\mathcal{D}^4\Phi + \frac{1}{2}(\mathcal{D}^3\Phi)^2 - 4\mathcal{D}\Phi\mathcal{D}^2\Phi\mathcal{D}^3\Phi \right.$$
$$\left. + (D\Phi)^3\mathcal{D}^3\Phi + \frac{4}{3}(\mathcal{D}^2\Phi)^3 + 2(\mathcal{D}\Phi)^2(\mathcal{D}^2\Phi)^2 \right] . \tag{3.3.3c}$$

We now decompose the double metric into the generalized metric plus extra fields [95]

$$\mathcal{Z} = \mathcal{S} + \mathcal{F} . \tag{3.3.4}$$

Here $\mathcal{S}$ encodes the internal metric and b-field (2.2.19), and $\mathcal{F}$ contains massive fields that will be integrated out perturbatively. While the decomposition (3.3.4) is totally generic, in perturbation theory we think of $\mathcal{F}$ as being of one order in α' higher than $\mathcal{S}$. This is motivated by the fact that, in this case, the leading order contribution to (3.3.1) does not depend on $\mathcal{F}$ and it corresponds to standard supergravity in cosmological backgrounds (2.2.39).

As mentioned several times before, the generalized metric is a constrained object satisfying $\mathcal{S}^2 = 1$. This allows us to define $\pm$-projected spaces via (2.2.26). We can then assume $\mathcal{F}$ to be a constrained field belonging to the $+$ subspace, i.e. $\mathcal{F} = \mathcal{F}_+$, because any part in $\mathcal{F}$ belonging to the $-$ subspace can be removed by a field redefinition $\mathcal{S} \to \mathcal{S} + \delta\mathcal{S}$ since $\delta\mathcal{S} = [\delta\mathcal{S}]_-$ ((2.2.28)). Thus, without loss of generality, in perturbation theory we can write

$$\mathcal{Z} = \mathcal{S} + \mathcal{F}_+ \,. \tag{3.3.5}$$

Let us now inspect the equation of motion (3.3.3a) after using this decomposition of $\mathcal{Z}$. Inserting (3.3.5) into (3.3.3a) one obtains

$$\mathcal{S}\mathcal{F}_+ = \frac{\alpha'}{4}\Box_\Phi(\mathcal{S} + \mathcal{F}_+) - \frac{1}{2}\mathcal{F}_+^2 \,, \tag{3.3.6}$$

where we used the definition $\Box_\Phi \equiv \mathcal{D}^2 - \mathcal{D}\Phi\mathcal{D}$ we introduced in (2.2.44). From this we can obtain the equations for $\mathcal{F}_+$ or $\mathcal{S}$ by projecting into $\pm$ subspaces. For the generalized metric we project to the $-$ subspace, and use $[\mathcal{F}_+]_- = [\mathcal{F}_+^2]_- = 0$. This yields

$$[\Box_\Phi(\mathcal{S} + \mathcal{F}_+)]_- = 0 \,. \tag{3.3.7}$$

For the extra fields we take the $+$ projection:

$$\mathcal{F}_+ = \frac{\alpha'}{4}[\Box_\Phi\mathcal{S}]_+\mathcal{S} + \frac{\alpha'}{4}[\Box_\Phi\mathcal{F}_+]_+\mathcal{S} - \frac{1}{2}\mathcal{F}_+^2\mathcal{S} \,. \tag{3.3.8}$$

Since this is the equation of motion for $\mathcal{F}_+$, we can solve for it perturbatively in α' following the iterative procedure we now describe.

Equation (3.3.8) is the starting point for integrating out the extra fields F_+. We assume a perturbative expansion in α', namely

$$\mathcal{F}_+ = \sum_{i\geq 1} \alpha'^i \mathcal{F}_+^{(i)} \,. \tag{3.3.9}$$

Inserting this expansion into (3.3.8) we obtain the recursive relations

$$\begin{aligned}
\mathcal{F}_+^{(1)} &= \frac{1}{4}[\Box_\Phi\mathcal{S}]_+\mathcal{S} = -\frac{1}{4}(\mathcal{D}\mathcal{S})^2 \,, \\
\mathcal{F}_+^{(i)} &= \frac{1}{4}[\Box_\Phi\mathcal{F}_+^{(i-1)}]_+\mathcal{S} - \frac{1}{2}\sum_{j=1}^{i-1}\mathcal{F}_+^{(j)}\mathcal{F}_+^{(i-j)}\mathcal{S} \,, \quad i \geq 2 \,.
\end{aligned} \tag{3.3.10}$$

By solving these equations recursively, we can express all $\mathcal{F}_+^{(i)}$ in terms of $\mathcal{S}$, Φ and n. Plugging these expressions back into the action =(3.3.1) (after splitting

$\mathcal{Z} + \mathcal{S} + \mathcal{F}$) yields the effective action for the conventional fields. We performed this computation explicitly up to order α'^4 obtaining [5]

$$I = I^{(0)} + \alpha' I^{(1)} + \alpha'^2 I^{(2)} + \alpha'^3 I^{(3)} + \alpha'^4 I^{(4)} + \mathcal{O}(\alpha'^5)\,, \quad I^{(p)} = \int dt\, n\, e^{-\Phi} L^{(p)}\,, \tag{3.3.11}$$

with the Lagrangians

$$\begin{aligned} L^{(0)} &= -\frac{1}{8}\mathrm{Tr}\left((\mathcal{D}\mathcal{S})^2\right) - (\mathcal{D}\Phi)^2\,,\\ L^{(1)} &= \;\; 0\,,\\ L^{(2)} &= \;\; \frac{1}{3\cdot 2^7}\mathrm{Tr}\left((\mathcal{D}\mathcal{S})^6\right) - \frac{1}{2^6}\mathrm{Tr}\left(\mathcal{D}^2\mathcal{S}\mathcal{D}\mathcal{S}\mathcal{D}^2\mathcal{S}\mathcal{D}\mathcal{S}\right) - \frac{1}{2^6}\mathrm{Tr}\left((\mathcal{D}^2\mathcal{S})^2(\mathcal{D}\mathcal{S})^2\right)\\ &\quad + \frac{1}{16}(\mathcal{D}^3\Phi)^2 + \frac{1}{12}(\mathcal{D}^2\Phi)^3\,,\\ L^{(3)} &= \;\; \mathbb{L}^{(3)}\,,\\ L^{(4)} &= \;\; \mathbb{L}^{(4)}\,. \end{aligned} \tag{3.3.12}$$

The notation $\mathbb{L}^{(3)}$ and $\mathbb{L}^{(4)}$ is to indicate that these terms are functions of $\mathcal{D}^2\mathcal{S}, \mathcal{D}\Phi$, and $\mathrm{Tr}\left((\mathcal{D}\mathcal{S})^2\right)$, whose specific form is irrelevant for our purposes. These are removable terms, the analogous to $\mathcal{D}L, \mathcal{D}\Phi$, and $\mathrm{tr}\left(L^2\right)$ in the dilaton-gravity theories of Section 3.2. As we learned in that section, removable terms in the Lagrangian can be redefined away at the expense of introducing higher order terms.

Classification

The action containing the Lagrangians (3.3.12) can be brought to a canonical form, the form of the classification (3.1.1). At that point, we can read the specific coefficients that define unambiguously the massless sector of HSZ in cosmological backgrounds. The process of implementing field redefinitions in a systematic way was introduced in Section 3.1.1 and applied to dilaton-gravity theories in Section 3.2. We will not repeat the algorithm here since it is very similar to the one implemented there. More precisely, the five-step procedure below equation (3.2.7) follows almost identically, except that this time we want to remove $\mathcal{D}^2\mathcal{S}, \mathcal{D}\Phi$, and $\mathrm{Tr}\left((\mathcal{D}\mathcal{S})^2\right)$ contributions. For the implementation of that procedure to HSZ in cosmological backgrounds we refer the reader to the appendix of [5]. Here, instead, we summarize some intermediate steps and give the main result.

The fact that the first order in (3.3.12) is zero simplifies considerably the process. The general structure in (3.1.8) up to order α'^4 for this particular case

[5]See [5] for a detailed derivation.

reads

$$\begin{aligned} I' = I^{(0)} &+ \alpha'^2\Big(I^{(2)} + E_\Psi^{(0)}\cdot\delta\Psi^{(2)}\Big) \\ &+ \alpha'^3\Big(I^{(3)} + E_\Psi^{(0)}\cdot\delta\Psi^{(3)}\Big) \\ &+ \alpha'^4\Big(I^{(4)} + E_\Psi^{(2)}\cdot\delta\Psi^{(2)} + \frac{1}{2}\Delta_2 I^{(0)}\cdot\big(\delta\Psi^{(2)}\big)^2 + E_\Psi^{(0)}\cdot\delta\Psi^{(4)}\Big) + \mathcal{O}(\alpha'^5)\,, \end{aligned} \tag{3.3.13}$$

where the order α'^3 does not receive any induced terms from lower orders as a consequence of the original action having no $I^{(1)}$. Moreover, $\delta\Psi^{(3)}$ does not propagate to the next order, which makes the process even simpler. As explained in Section 3.1.1, field redefinitions are implemented order-by-order via the substitution rule $E_\Psi^{(0)} \approx -\alpha'\Delta_\Psi$ with Δ_Ψ an artificial time-dependent factor that allows us to keep track of induced terms. In this case $E_\Psi^{(0)}$ is given in (3.1.27) and so the rules read

$$\mathcal{D}^2\mathcal{S} \approx \mathcal{D}\Phi\mathcal{D}\mathcal{S} - (\mathcal{D}\mathcal{S})^2\mathcal{S} - 4\alpha'\Delta_\mathcal{S}\,, \tag{3.3.14a}$$

$$(\mathcal{D}\Phi)^2 \approx -\frac{1}{8}\mathrm{Tr}\left((\mathcal{D}\mathcal{S})^2\right) - \alpha'\Delta_n\,, \tag{3.3.14b}$$

$$\mathcal{D}^2\Phi \approx -\frac{1}{8}\mathrm{Tr}\left((\mathcal{D}\mathcal{S})^2\right) - \frac{\alpha'}{2}\left(\Delta_n + \Delta_\Phi\right)\,. \tag{3.3.14c}$$

The field redefinition procedure begins at order α', where the action is already in canonical form, in particular it is zero. Therefore, there are no ambiguous terms to remove and so $\delta\Psi^{(1)} = 0$. We then move up to order α'^2, where we have no induced terms and so we only have to remove ambiguities from the original $L^{(2)}$ in (3.3.12). This is achieved by the use of (3.3.14), mimicking the five-step procedure applied for dilaton-gravity theories. At the end of this process, the resulting $I'^{(2)}$ is in canonical form, it has no $\mathcal{D}^2\mathcal{S}, \mathcal{D}\Phi$, or $\mathrm{Tr}\left((\mathcal{D}\mathcal{S})^2\right)$. We find a non-trivial $\delta\Psi^{(2)}$ that can be read from the linear term in Δ_Ψ. At order α'^3 there are no induced terms (see (3.3.13)), the only contribution is the original $L^{(3)}$ in (3.3.12), which contains only removable terms. These can be freely set to zero since we do not have to keep track of $\delta\Psi^{(3)}$. Finally, at order α'^4 we have the last line of (3.3.12) with $L^{(4)} = \mathbb{L}^{(4)}$ and non-trivial induced terms coming from $\delta\Psi^{(2)}$. While the former can be simply set to zero, the induced terms generate a non-trivial contribution proportional to $\mathrm{Tr}\left((\mathcal{D}\mathcal{S})^4\right)\mathrm{Tr}\left((\mathcal{D}\mathcal{S})^6\right)$.

The final result of this systematic procedure is the HSZ action in the cosmological classification (3.1.1) with coefficients [3, 5]

$c_{1,0}$	$c_{2,0}$	$c_{3,0}$	$c_{3,1}$	$c_{4,0}$	$c_{4,1}$
0	$\frac{1}{3\cdot 2^7}$	0	0	0	$-\frac{1}{3\cdot 2^{13}}$

3.3.2 Two-derivative reformulation and α'-exact Friedmann equations

As we just saw, (2.3.33) corresponds to an effective theory for $\mathcal{Z}, n$ and Φ where A and B can be set to zero for free. However, even though they encode no extra information for the physical fields, they can still be used to bring the theory to a formulation without positive powers of α', i.e., without more than two derivatives. To this end, we perform the following field redefinition

$$\begin{aligned} B &= B' - \frac{2}{3}A'^2 + \alpha'\left[\frac{1}{4}\mathcal{D}A'\mathcal{D}\Phi - \frac{1}{4}\mathcal{D}^2A' + \frac{1}{3}A'\mathcal{D}^2\Phi\right] \\ &\qquad - \frac{\alpha'^2}{2}\left[-\frac{1}{4}\mathcal{D}^4\Phi + \frac{1}{4}\mathcal{D}\Phi\mathcal{D}^3\Phi + \frac{1}{3}(\mathcal{D}^2\Phi)^2\right], \\ A &= A' + \frac{\alpha'}{2}\mathcal{D}^2\Phi\,, \end{aligned} \tag{3.3.15}$$

under which the action (2.3.33) transforms to (omitting the primes)

$$\begin{aligned} I = \int dt\, n\, e^{-\Phi}\Big\{&\frac{1}{\alpha'}\Big[AB - \frac{2}{3}A^3 + \frac{1}{2}\mathrm{Tr}\,(\mathcal{Z}) - \frac{1}{6}\mathrm{Tr}\left(\mathcal{Z}^3\right)\Big] \\ &- \frac{1}{8}\mathrm{Tr}\left((\mathcal{D}\mathcal{Z})^2\right) - (\mathcal{D}\Phi)^2 + \frac{1}{4}(\mathcal{D}A)^2 + \frac{1}{2}B\mathcal{D}^2\Phi\Big\}\,. \end{aligned} \tag{3.3.16}$$

In this formulation, all higher orders in α' are "hidden" in the on-shell value of A and B. Indeed, we note that B enters in the action just linearly, so it plays the role of a Lagrange multiplier, which imposes a condition on the scalar field A:

$$\frac{\delta I}{\delta B} = 0 \quad \Rightarrow \quad A = -\frac{\alpha'}{2}\mathcal{D}^2\Phi\,. \tag{3.3.17}$$

Reinserting this value of A into (3.3.16), the A^3 and $(\mathcal{D}A)^2$ couplings are replaced by the α'^2 terms $(\mathcal{D}^2\Phi)^3$ and $(\mathcal{D}^3\Phi)^2$, respectively. The resulting action is exactly the effective action for $\mathcal{Z}, n$ and Φ obtained from (2.3.33) after fixing $A = B = 0$.

The fact that HSZ admits a formulation like (3.3.16) is an important and rare feature of the theory. This allows us to take the so-called tensionless limit $\alpha' \to \infty$. Nor the two-derivative reformulation nor the tensionless limit have counterpart in any other target-space string effective theory! Interestingly, this is not the first time these characteristics are studied in HSZ. In [96], the full HSZ theory (2.3.18) was considered up to quadratic order in field perturbations about flat space and it was shown that the higher-derivative terms can be removed by introducing certain auxiliary fields. One can then take the tensionless limit $\alpha' \to \infty$ smoothly, for which one finds an enhanced gauge symmetry whose corresponding gauge fields are the auxiliary fields. It is worth enumerating the differences between our results and the ones in [96]:

- ⊙ In the cosmological setting, to arrive at the two-derivative reformulation (3.3.16) we did not need auxiliary fields as in [96]. The already present A and B fields were sufficient to reabsorb higher-derivative terms.
- ⊙ While the two-derivative reformulation of [96] was found up to quadratic order in field perturbation, the action (3.3.16) is valid to all orders in fields.
- ⊙ Upon taking the tensionless limit, the enhanced gauge symmetry found in [96] trivializes for cosmological backgrounds, so we cannot see it in our analysis.

One can check that all our results are consistent with the ones in [96]. The precise details for this connection can be found in [5].

α'-exact Friedmann equations

We now analyze the two-derivative equations for FRW backgrounds, bringing them into the form of string cosmology with "matter fields", which here are the extra fields A, B and $\mathcal{F}$. We begin by considering the two-derivative form of the action (3.3.16) and expanding $\mathcal{Z} = \mathcal{S} + \mathcal{F}$, which yields

$$I = I^{(0)} + I_m\,, \tag{3.3.18}$$

with

$$\begin{aligned} I^{(0)} &\equiv \int dt\, n\, e^{-\Phi} \left\{ -\frac{1}{8}\text{Tr}\left((\mathcal{D}\mathcal{S})^2\right) - (\mathcal{D}\Phi)^2 \right\}, \\ I_m &\equiv \int dt\, n\, e^{-\Phi} \left\{ \frac{1}{\alpha'}\left[AB - \frac{2}{3}A^3 - \frac{1}{2}\text{Tr}\left(\mathcal{S}\mathcal{F}^2\right) - \frac{1}{6}\text{Tr}\left(\mathcal{F}^3\right)\right] \right. \\ &\qquad \left. + \frac{1}{4}(\mathcal{D}A)^2 + \frac{1}{2}B\mathcal{D}^2\Phi - \frac{1}{4}\text{Tr}\left(\mathcal{D}\mathcal{S}\mathcal{D}\mathcal{F}\right) - \frac{1}{8}\text{Tr}\left((\mathcal{D}\mathcal{F})^2\right) \right\}. \end{aligned} \tag{3.3.19}$$

In this split $I^{(0)}$ is the standard lowest order, two-derivative gravity action of (2.2.39), and I_m encodes what we will call "the matter content" parameterized by A, B and $\mathcal{F}$. It is important to point out that here we make no a priori assumptions on the α'-dependence of $\mathcal{F}$. From $I^{(0)}$ one obtains the equations of motion for the massless fields, as computed in (2.2.43). For the matter action, we encode the variations in terms of an $O(d,d)$-covariant energy momentum tensor $\mathcal{T}_M{}^N$, an energy density ρ, and a dilatonic charge σ [93, 97], defined as:[6]

$$\mathcal{T}_M{}^N \equiv -2\frac{e^{\Phi}}{n}\mathcal{S}_M{}^P\frac{\delta I_m}{\delta \mathcal{S}^P{}_N}\,, \tag{3.3.20a}$$

$$\rho \equiv -e^{\Phi}\frac{\delta I_m}{\delta n}\,, \tag{3.3.20b}$$

$$\sigma \equiv -2\frac{e^{\Phi}}{n}\frac{\delta I_m}{\delta \Phi}\,. \tag{3.3.20c}$$

The equations following from (3.3.18) for $\mathcal{S}, \Phi$ and n are then given by

$$[\Box_\Phi \mathcal{S}]_-\,\mathcal{S} = -2[\mathcal{T}]_-\,, \tag{3.3.21a}$$

$$2\mathcal{D}^2\Phi - (\mathcal{D}\Phi)^2 + \frac{1}{8}\text{Tr}\left((\mathcal{D}\mathcal{S})^2\right) = \frac{1}{2}\sigma\,, \tag{3.3.21b}$$

$$(\mathcal{D}\Phi)^2 + \frac{1}{8}\text{Tr}\left((\mathcal{D}\mathcal{S})^2\right) = \rho\,, \tag{3.3.21c}$$

[6] All quantities are defined with a $\sqrt{|g|}e^{\Phi}$ re-scaling compared to standard definitions.

with

$$[\mathcal{T}]_- = \mathcal{S}\left[-\frac{1}{2}\square_\Phi \mathcal{F} + \frac{1}{\alpha'}\mathcal{F}^2\right]_- , \tag{3.3.22a}$$

$$\begin{aligned}\rho &= \frac{1}{2\alpha'}\text{Tr}\left(\mathcal{S}\mathcal{F}^2\right) + \frac{1}{6\alpha'}\text{Tr}\left(\mathcal{F}^3\right) - \frac{1}{4}\text{Tr}\left(\mathcal{D}\mathcal{S}\mathcal{D}\mathcal{F}\right) - \frac{1}{8}\text{Tr}\left((\mathcal{D}\mathcal{F})^2\right)\\ &\quad - \frac{1}{\alpha'}\left(AB - \frac{2}{3}A^3\right) + \frac{1}{4}(\mathcal{D}A)^2 + \frac{1}{2}B(\mathcal{D}\Phi)^2 - \frac{1}{2}\mathcal{D}B\mathcal{D}\Phi\,,\end{aligned} \tag{3.3.22b}$$

$$\begin{aligned}\sigma &= -\frac{1}{\alpha'}\text{Tr}\left(\mathcal{S}\mathcal{F}^2\right) - \frac{1}{3\alpha'}\text{Tr}\left(\mathcal{F}^3\right) - \frac{1}{2}\text{Tr}\left(\mathcal{D}\mathcal{S}\mathcal{D}\mathcal{F}\right) - \frac{1}{4}\text{Tr}\left((\mathcal{D}\mathcal{F})^2\right)\\ &\quad + \frac{2}{\alpha'}\left(AB - \frac{2}{3}A^3\right) + \frac{1}{2}(\mathcal{D}A)^2 + 2B\mathcal{D}^2\Phi - B(\mathcal{D}\Phi)^2 + 2\mathcal{D}B\mathcal{D}\Phi - \mathcal{D}^2B\,.\end{aligned} \tag{3.3.22c}$$

Where we wrote the result in terms of the $\square_\Phi$ operator of (2.2.44) and projected-objects (2.2.26). It is worth remembering that the minus-projection in (3.3.21a) comes from $\mathcal{S}$ being a constrained object (see (2.2.28)). In addition, the EOM for the matter fields are given by

$$0 = \frac{1}{4}\square_\Phi(\mathcal{S}+\mathcal{F}) - \frac{1}{2\alpha'}\left(\mathcal{S}\mathcal{F}+\mathcal{F}\mathcal{S}\right) - \frac{1}{2\alpha'}\mathcal{F}^2\,, \tag{3.3.23a}$$

$$A = -\frac{\alpha'}{2}\mathcal{D}^2\Phi\,, \tag{3.3.23b}$$

$$B = 2A^2 + \frac{\alpha'}{2}\square_\Phi A = \alpha'^2\left[-\frac{1}{4}\mathcal{D}^4\Phi + \frac{1}{2}(\mathcal{D}^2\Phi)^2 + \frac{1}{4}\mathcal{D}\Phi\mathcal{D}^3\Phi\right] , \tag{3.3.23c}$$

where in the last equality of (3.3.23c) we used the on-shell value of A. Indeed, we see here that A and B can be eliminated completely, but we find it convenient to keep them in order to be able to work with second-order equations. It can be shown that reparametrization invariance implies the following continuity equation:

$$\mathcal{D}\rho + \frac{1}{2}\text{Tr}\left(\mathcal{S}\mathcal{D}\mathcal{S}\mathcal{T}\right) - \mathcal{D}\Phi(\rho + \frac{1}{2}\sigma) = 0\,. \tag{3.3.24}$$

Let us now specify to FRW backgrounds, characterized by the ansatz (2.2.56), containing a single scale factor $a(t)$ and with it a single Hubble parameter $H(t) = \frac{\mathcal{D}a}{a}$. Using (2.2.58), we can write out the l.h.s of (3.3.21) for FRW backgrounds

$$\begin{aligned}[\square_\Phi\mathcal{S}]_-\,\mathcal{S} \;&\rightarrow\; 2(\mathcal{D}H - \mathcal{D}\Phi H)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} ,\\ 2\mathcal{D}^2\Phi - (\mathcal{D}\Phi)^2 + \frac{1}{8}\text{Tr}\left((\mathcal{D}\mathcal{S})^2\right) \;&\rightarrow 2\,\mathcal{D}^2\Phi - (\mathcal{D}\Phi)^2 - dH^2\,,\\ (\mathcal{D}\Phi)^2 + \frac{1}{8}\text{Tr}\left((\mathcal{D}\mathcal{S})^2\right) \;&\rightarrow\; (\mathcal{D}\Phi)^2 - dH^2\,.\end{aligned} \tag{3.3.25}$$

For the r.h.s of (3.3.21) we need to choose a parameterization for $\mathcal{F}$. Instead of considering the extra fields in full generality, however, we will truncate the theory to the subsector $\mathcal{F}_- = 0$. While we find that in general this is not a consistent truncation of the full theory (3.3.19), it can be proven (see [5]) that it is a consistent truncation for FRW backgrounds. Thus, from now on we consider $\mathcal{F}_- = 0$ and we proceed to parameterize $\mathcal{F}_+$. The most general ansatz for a +-projected $O(d,d,\mathbb{R})$ tensor consistent with the FRW background (2.2.58) is given in terms of two symmetric matrices $f_{1m}{}^n(t)$ and $f_{2m}{}^n(t)$. However, demanding homogeneity and isotropy, as for standard cosmology, we set $f_{1mn} = f_1\delta_{mn}\,, \quad f_{2mn} = f_2\delta_{mn}$. This is also a consistent truncation. In this simplified scheme, $\mathcal{F}_+$ takes the simple form

$$\mathcal{F} = \mathcal{F}_+ = \begin{pmatrix} 0 & f_1 a^2 \\ a^{-2} f_1 & 0 \end{pmatrix} + \begin{pmatrix} f_2 & 0 \\ 0 & f_2 \end{pmatrix} = f_1 \mathcal{S} + f_2 \mathbf{1}\,, \tag{3.3.26}$$

where $\mathbf{1}$ denotes the unit matrix with components $\delta_M{}^N$. With this truncation, the matter content is described by just four scalar fields, A, B, f_1, and f_2.

By taking derivatives of (3.3.26) and using the definition for the generalized energy momentum tensor (3.3.22a) with $\mathcal{F} = \mathcal{F}_+$ it follows

$$[\mathcal{T}]_- = -\frac{1}{2}\mathcal{S}\,[\Box_\Phi \mathcal{F}_+]_- = [2H\mathcal{D}f_1 + (\mathcal{D}H - \mathcal{D}\Phi H)\, f_1] \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,. \tag{3.3.27}$$

In order to give a physical interpretation to (3.3.27) notice that for FRW backgrounds (3.3.20a) gives

$$\mathcal{T}_M{}^N = -2\frac{e^\Phi}{n}\mathcal{S}_M{}^P \frac{\delta I_m}{\delta \mathcal{S}^P{}_N} = \begin{pmatrix} -T_m{}^n & 0 \\ 0 & T^m{}_n \end{pmatrix}\,, \quad \text{with} \quad T_m{}^n = p\,\delta_m{}^n\,, \tag{3.3.28}$$

which describes a perfect fluid with pressure p. Comparison with (3.3.27) then motivates us to view the extra fields of HSZ as describing an effective perfect fluid with an effective pressure determined by f_1,

$$p = -2H\mathcal{D}f_1 - (\mathcal{D}H - \mathcal{D}\Phi H)\, f_1\,. \tag{3.3.29}$$

For the energy density and the dilatonic charge we just need to insert our expressions for $\mathcal{S}$ and $\mathcal{F}$ and their derivatives, (2.2.58) and (3.3.26), into the definition of ρ and σ in (3.3.22b) and (3.3.22c). After some straightforward yet tedious algebra, one obtains

$$\begin{aligned} \rho = {} & \frac{d}{\alpha'}\left(2f_2 f_1 + \frac{1}{3}f_2{}^3 + f_2 f_1{}^2\right) + dH^2 f_1(f_1+2) - \frac{d}{4}\left((\mathcal{D}f_1)^2 + (\mathcal{D}f_2)^2\right) \\ & - \frac{1}{\alpha'}\left(AB - \frac{2}{3}A^3\right) + \frac{1}{4}(\mathcal{D}A)^2 + \frac{1}{2}B(\mathcal{D}\Phi)^2 - \frac{1}{2}\mathcal{D}B\mathcal{D}\Phi\,, \\ \sigma = {} & -\frac{2d}{\alpha'}\left(2f_2 f_1 + \frac{1}{3}f_2{}^3 + f_2 f_1{}^2\right) + 2dH^2 f_1(f_1+2) - \frac{d}{2}\left((\mathcal{D}f_1)^2 + (\mathcal{D}f_2)^2\right) \\ & + \frac{2}{\alpha'}\left(AB - \frac{2}{3}A^3\right) + \frac{1}{2}(\mathcal{D}A)^2 + 2B\mathcal{D}^2\Phi - B(\mathcal{D}\Phi)^2 + 2\mathcal{D}B\mathcal{D}\Phi - \mathcal{D}^2 B\,. \end{aligned} \tag{3.3.30}$$

With (3.3.29) and (3.3.30) the right-hand sides of the equations of motion (3.3.21) are completely determined.

Next, for the equations of motion of the extra fields (3.3.23) we can consider just the $+$ projection of (3.3.23a), because the minus projection is exactly the equation for $\mathcal{S}$ in (3.3.21a) and so it vanishes on-shell. By taking $\mathcal{F}_- = 0$, the $+$ projection of (3.3.23a) reduces to

$$\begin{aligned} 0 &= -\frac{\alpha'}{4}\left[\Box_\Phi(\mathcal{S}+\mathcal{F}_+)\right]_+ + \mathcal{S}\mathcal{F}_+ + \frac{1}{2}\mathcal{F}_+^2 \\ &= \left[\frac{1}{2}{f_1}^2 + \frac{1}{2}{f_2}^2 + f_1 - \frac{\alpha'}{4}\Box_\Phi f_2\right]\mathbf{1} + \left[(f_2 - \alpha' H^2)(f_1+1) - \frac{\alpha'}{4}\Box_\Phi f_1\right]\mathcal{S}\,, \end{aligned} \tag{3.3.31}$$

which imply two inequivalent equations (one for each scalar field)

$$\begin{aligned} \frac{1}{2}{f_1}^2 + \frac{1}{2}{f_2}^2 + f_1 - \frac{\alpha'}{4}\Box_\Phi f_2 = 0\,, \\ (f_2 - \alpha' H^2)(f_1+1) - \frac{\alpha'}{4}\Box_\Phi f_1 = 0\,. \end{aligned} \tag{3.3.32}$$

When combining (3.3.25), (3.3.29), (3.3.30) and (3.3.32), we end up with a non-linear system of coupled second-order differential equations for $a, \Phi, n, A, B, f_1, f_2$:

$$p = \mathcal{D}H - \mathcal{D}\Phi H\,, \tag{3.3.33a}$$

$$\frac{1}{2}\left(\frac{1}{2}\sigma + \rho\right) = \mathcal{D}^2\Phi - dH^2\,, \tag{3.3.33b}$$

$$\rho = (\mathcal{D}\Phi)^2 - dH^2\,, \tag{3.3.33c}$$

$$0 = (f_2 - \alpha' H^2)(f_1+1) - \frac{\alpha'}{4}\Box_\Phi f_1\,, \tag{3.3.33d}$$

$$0 = \frac{1}{2}{f_1}^2 + \frac{1}{2}{f_2}^2 + f_1 - \frac{\alpha'}{4}\Box_\Phi f_2\,, \tag{3.3.33e}$$

$$A = -\frac{\alpha'}{2}\mathcal{D}^2\Phi\,, \tag{3.3.33f}$$

$$B = 2A^2 + \frac{\alpha'}{2}\Box_\Phi A\,, \tag{3.3.33g}$$

with the "effective matter sources" given in (3.3.29) and (3.3.30).

It is instructive to check that these equations are invariant under duality transformations, which in the case of FRW backgrounds reduce to a simple $\mathbb{Z}_2$ transformation $a \to a^{-1}$, while the rest of the fields, including f_1 and f_2, behave as scalars. On top of this consistency check, one can also see that, if $f_1 = \mathcal{O}(\alpha')$ and $f_2 = \mathcal{O}(\alpha')$, the whole system (3.3.33) reduces to the standard Friedmann equations in vacuum (2.2.60) upon neglecting higher orders in α'.

Remarkably, the above system represents a non-perturbative and α'-complete set of equations for a consistent truncation of a theory sharing many features of genuine string theory. Unfortunately, without making any further assumptions on the field content, finding analytic solutions of (3.3.33) in general seems to be difficult. An exception is a rather degenerate solution given by

$$f_1(t) = f_2(t) = -1\,, \quad \Phi(t) = \sqrt{\frac{2d}{3\alpha'}}\,t + \Phi_0\,, \quad A = B = 0\,, \tag{3.3.34}$$

with Φ_0 constant. Here, using (3.3.26), $f_1 = f_2 = -1$ implies that $\mathcal{F} = -\mathcal{S} - \mathbf{1}$ and so $\mathcal{Z} = \mathcal{S} + \mathcal{F} = -\mathbf{1}$, which makes $\mathcal{S}$ disappears completely. In other words, there is no gravity contribution to the solution, a fact that can be corroborated by using $f_1 = f_2 = -1$ in (3.3.33) to see that all $H(t)$ contributions just cancel out.

Apart from this simple case, looking for exact and more complex solutions of the system (3.3.33) is a complicated task and one should look for simplifications. One option would be to use the split into massless sector and matter content more seriously and not as mere notation. While the massive fields f_1, f_2, A and B from HSZ theory are rather abstract objects, the notion of pressure, energy density and dilatonic charge are not. Therefore, we could take inspiration from known examples or cosmological scenarios already studied in the literature in other contexts, not directly related to HSZ, to restrict the space of solutions. For instance, one could be interested in backgrounds whose matter content describes a barotropic fluid $p = w\rho$ and/or no dilatonic charge, $\sigma = 0$. These cosmology-driven conditions would impose constraints on the extra fields that could help in finding solutions. A second option is to study particular configurations for the dilaton and Hubble parameter and ask if there exist any configuration of the extra fields $A(t), B(t), f_1(t)$ and $f_2(t)$ such that the equations are satisfied. Finally, a third option is to look for solutions perturbatively. While the first cosmology-inspired approach seems promising, we will not follow that approach here. Instead, we study briefly the second option but then move right away to the perturbative approach. Since for none of these methods we need the matter-content interpretation, from now on we treat p, ρ and σ in (3.3.29) and (3.3.30) as pure notation to encode what is in the l.h.s of (3.3.33).

We begin by ruling out the somewhat degenerate branch of solutions (3.3.34) by demanding $f_1 \neq -1$. On top of that we also exclude the flat Minkowski background by demanding $H \neq 0$. From now on we will gauge fix the lapse to $n(t) = 1$ and adopt the following notation for the extra fields

$$x \equiv 1 + f_1 \neq 0\,, \quad y \equiv f_2\,. \tag{3.3.35}$$

We now observe that under these assumptions equation (3.3.33a) can be solved exactly: inserting the definition for p in the l.h.s of (3.3.33a) and using (3.3.35) we can see that (3.3.33a) is equivalent to

$$-2H\partial_t \ln x = \dot{H} - \dot{\Phi}H \quad \Rightarrow \quad \partial_t \ln(x^{-2}) = \partial_t(\ln H - \Phi)\,, \tag{3.3.36}$$

where we inverted H to arrive at the second expression. This equation can be integrated exactly to arrive at

$$H(t) = Qe^{\Phi(t)}x(t)^{-2}\,, \qquad Q = \text{constant} \neq 0\,. \tag{3.3.37}$$

This relation tells us that the Hubble parameter is completely determined from the dilaton and one of the extra fields. For the rest of the system we cannot do much without considering particular truncations or certain limits of the theory and so here we just rewrite them in terms of x and y:

$$H = Qe^{\Phi}x^{-2}\,, \tag{3.3.38a}$$

$$\ddot{\Phi} = dH^2x^2 - \frac{d}{4}\left(\dot{x}^2+\dot{y}^2\right) + \frac{1}{4}(\dot{A})^2 + \frac{1}{2}B\ddot{\Phi} + \frac{1}{4}\dot{B}\dot{\Phi} - \frac{1}{4}\ddot{B}\,, \tag{3.3.38b}$$

$$\begin{aligned}\dot{\Phi}^2 = &\frac{d}{\alpha'}\left(x^2 - 1 + \frac{1}{3}y^2\right)y + dH^2x^2 - \frac{d}{4}\left(\dot{x}^2+\dot{y}^2\right)\\ &- \frac{1}{\alpha'}\left(AB - \frac{2}{3}A^3\right) + \frac{1}{4}(\dot{A})^2 + \frac{1}{2}B(\dot{\Phi})^2 - \frac{1}{2}\dot{B}\dot{\Phi}\,,\end{aligned} \tag{3.3.38c}$$

$$0 = -\frac{\alpha'}{4}\ddot{x} + \frac{\alpha'}{4}\dot{\Phi}\dot{x} + xy - \alpha' H^2 x\,, \tag{3.3.38d}$$

$$0 = -\frac{\alpha'}{2}\ddot{y} + \frac{\alpha'}{2}\dot{\Phi}\dot{y} + x^2 + y^2 - 1\,, \tag{3.3.38e}$$

$$A = -\frac{\alpha'}{2}\ddot{\Phi}\,, \tag{3.3.38f}$$

$$B = 2A^2 + \frac{\alpha'}{2}\ddot{A} - \frac{\alpha'}{2}\dot{\Phi}\dot{A}\,. \tag{3.3.38g}$$

Most of the complexity of the system comes from the terms involving A and B, since they are the only ones implicitly encoding up to order six in derivatives.

Once the matter-content interpretation is abandoned, (3.3.38) is just system of coupled second-order ordinary differential equations. We did not succeed in finding solutions of the full theory for specific ansätze of the dilaton and Hubble parameter. In the process we ruled out some possible configurations. Among these no-go results, probably the most interesting ones are the absence of solutions with constant duality-invariant dilaton $\Phi(t) = \Phi_0$, and the absence of de Sitter background in Einstein frame with constant scalar dilaton $\phi(t) = \phi_0$, which is related to Φ via (2.2.56). Other explored ansätze that failed to solve (3.3.33) can be consulted in [5]. This analysis showed that some of the simplest backgrounds one can propose for the standard fields H and Φ are not solutions of the HSZ equations for any configuration of the extra fields A, B, f_1 and f_2. It is worth mentioning, however, that this study just scratched the surface of the whole landscape of possible backgrounds one could propose, and we expect that upon a more exhaustive analyses exact (analytic or numerical) solutions could be found.

3.3.3 Tensionless limit, de Sitter solution and $\frac{1}{\alpha'}$ expansion

Due to the complexity of (3.3.33), finding exact analytic solutions is not an easy task. However, we can exploit the two-derivative nature of the system (3.3.33), a feature that has no analog in string low energy effective theories, where equations of motion are only known perturbatively in α', and each order increase the number of derivatives by two. In particular, we can take the tensionless limit $\alpha' \to \infty$ and look for solutions of the resulting simpler system. In this limit case, the equations are simple enough to obtain the general exact solutions! By sending α' to infinity, all non-derivative contributions disappear, and (3.3.38) reduces to

$$H = Qe^{\Phi}x^{-2}\,,\,, \tag{3.3.39a}$$

$$\ddot{\Phi} = dH^2x^2 - \frac{d}{4}\left(\dot{x}^2+\dot{y}^2\right) + \frac{1}{4}(\dot{A})^2 + \frac{1}{2}B\ddot{\Phi} + \frac{1}{4}\dot{B}\dot{\Phi} - \frac{1}{4}\ddot{B}\,, \tag{3.3.39b}$$

$$\dot{\Phi}^2 = dH^2x^2 - \frac{d}{4}\left(\dot{x}^2+\dot{y}^2\right) + \frac{1}{4}(\dot{A})^2 + \frac{1}{2}B(\dot{\Phi})^2 - \frac{1}{2}\dot{B}\dot{\Phi}\,, \tag{3.3.39c}$$

$$0 = \ddot{x} - \dot{\Phi}\dot{x} + 4H^2x\,, \tag{3.3.39d}$$

$$0 = \ddot{y} - \dot{\Phi}\dot{y}\,, \tag{3.3.39e}$$

$$0 = \ddot{\Phi}\,, \tag{3.3.39f}$$

$$0 = \ddot{A} - \dot{\Phi}\dot{A}\,. \tag{3.3.39g}$$

We now turn to (3.3.39f), which implies a linear dilaton profile:

$$\Phi(t) = -\omega(t-t_0)\,, \tag{3.3.40}$$

where ω is an integration constant. From now on the solutions are different depending whether ω vanishes or not. Since the procedure to get both family of solutions is almost identical, we describe in detail only the $\omega \neq 0$ case and just give the final result for vanishing ω.

Equations (3.3.39g) and (3.3.39e) take exactly the same form and, upon using (3.3.40), they can be solved exactly

$$A(t) = A_0 + A_1 e^{-\omega(t-t_0)}\,, \quad y(t) = y_0 + y_1 e^{-\omega(t-t_0)}\,. \tag{3.3.41}$$

Then, by subtracting (3.3.39c) from (3.3.39b), using (3.3.40) and reordering terms we get a second order differential equation for B,

$$\ddot{B} + 3\omega\dot{B} + 2\omega^2 B - 4\omega^2 = 0\,, \tag{3.3.42}$$

which is exactly solved by

$$B(t) = 2 + B_1 e^{-\omega(t-t_0)} + B_2 e^{-2\omega(t-t_0)}\,. \tag{3.3.43}$$

At this point we have two remaining equations for $x(t)$, namely (3.3.39d) and (3.3.39b) (or equivalent (3.3.39c)). We found it easier to solve (3.3.39b) because it is a first order differential equation, and then check (3.3.39d). Inserting (3.3.39a), (3.3.40), (3.3.41) and (3.3.43) into (3.3.39b) we arrive at the first order equation

$$\dot{x}^2 - \left(4Q^2x^{-2} + C_1\right)e^{-2\omega(t-t_0)} = 0\,, \qquad C_1 \equiv \frac{\omega^2}{d}\left(A_1^2 - dy_1^2 - 2B_2\right)\,, \tag{3.3.44}$$

where we defined the constant C_1 to simplify the notation. By multiplying both sides with x^2 and changing variables to $z(t) \equiv x(t)^2$ we arrive at the equation

$$\dot{z}^2 - \left(16Q^2 + 4C_1 z\right)e^{-2\omega(t-t_0)} = 0\,, \tag{3.3.45}$$

which has different solutions depending whether C_1 vanishes or not,

$$z(t) = \pm\frac{4Q}{\omega}e^{-\omega(t-t_0)} + x_0 \quad \text{if} \quad C_1 = 0\,, \tag{3.3.46}$$

$$z(t) = \frac{C_1}{\omega^2}\left(e^{-\omega(t-t_0)} + x_0\right)^2 - \frac{4Q^2}{C_1} \quad \text{if} \quad C_1 \neq 0\,. \tag{3.3.47}$$

Returning to the original variable $x(t) = \pm\sqrt{z(t)}$ and plugging the result together with (3.3.39a) and (3.3.40) into (3.3.39d) one can verify that the last equation of the system is also satisfied.

All in all, combining the above results we conclude that, for $\omega \neq 0$, the most general solution to the system (3.3.39) is given by:

$$\Phi(t) = -\,\omega(t-t_0)\,, \quad \omega \neq 0\,, \tag{3.3.48a}$$

$$H(t) = Qe^{-\omega(t-t_0)}x(t)^{-2}\,, \quad Q \neq 0\,, \tag{3.3.48b}$$

$$A(t) = A_0 + A_1 e^{-\omega(t-t_0)}\,, \tag{3.3.48c}$$

$$B(t) = 2 + B_1 e^{-\omega(t-t_0)} + B_2 e^{-2\omega(t-t_0)}\,, \tag{3.3.48d}$$

$$y(t) = y_0 + y_1 e^{-\omega(t-t_0)}\,, \tag{3.3.48e}$$

$$x(t) = \begin{cases} \pm 2\sqrt{\pm\frac{Q}{\omega}e^{-\omega(t-t_0)} + x_0} & \text{if} \quad C_1 = 0\,, \\ \pm\sqrt{\frac{C_1}{\omega^2}\left(e^{-\omega(t-t_0)} + x_0\right)^2 - \frac{4Q^2}{C_1}} & \text{if} \quad C_1 \neq 0\,, \end{cases} \quad C_1 \equiv \frac{\omega^2}{d}\left(A_1^2 - dy_1^2 - 2B_2\right)\,. \tag{3.3.48f}$$

Repeating identical steps for the $\omega = 0$ case, we get a second set of solutions:

$$\Phi(t) = 0\,, \tag{3.3.49a}$$

$$H(t) = Qx(t)^{-2}\,, \quad Q \neq 0\,, \tag{3.3.49b}$$

$$A(t) = A_0 + A_1(t-t_0)\,, \tag{3.3.49c}$$

$$B(t) = B_0 + B_1(t-t_0)\,, \tag{3.3.49d}$$

$$y(t) = y_0 + y_1(t-t_0)\,, \tag{3.3.49e}$$

$$x(t) = \begin{cases} \pm 2\sqrt{\pm Q(t-t_0) + x_0} & \text{if} \quad C_2 = 0\,, \\ \pm\sqrt{C_2[(t-t_0)+x_0]^2 - \frac{4Q^2}{C_2}} & \text{if} \quad C_2 \neq 0\,, \end{cases} \quad C_2 \equiv \frac{1}{d}\left(A_1^2 - dy_1^2\right)\,. \tag{3.3.49f}$$

Each of these families is parameterized by several independent free parameters. In particular, for the $\omega \neq 0$ case one can analyze the simplest solution of this family obtained by taking $A_0 = A_1 = B_1 = B_2 = y_0 = y_1 = x_0 = 0$ and so arriving at

$$\Phi(t) = -\omega(t-t_0)\,, \quad H(t) = \text{sign}(Q)\frac{|\omega|}{4}\,, \quad x(t) = \pm 2\sqrt{\left|\frac{Q}{\omega}\right|}e^{-\frac{1}{2}\omega(t-t_0)}\,, \tag{3.3.50a}$$

$$A(t) = y(t) = 0\,, \quad B(t) = 2\,, \tag{3.3.50b}$$

where we kept only the real $x(t)$ branch. Remarkably, the Hubble parameter is constant, and hence this solution corresponds to a de Sitter background in string frame. Note that the de Sitter scale here is simply an integration constant and not determined by a bare parameter in the action, which means that H is fixed by the initial conditions. Furthermore, it can be shown that (3.3.50) also admits a de Sitter solution in Einstein frame with constant dilaton for the particular case of $d = 4$, corresponding to five spacetime dimensions.

The tensionless limit can be interpreted as the zeroth order of a perturbative expansion in small $\frac{1}{\alpha'}$. Therefore, in the remainder of this section we explore the first order correction in $\frac{1}{\alpha'}$ to the system (3.3.39). More precisely, we return to the full equations (3.3.38), write for all fields

$$\Psi(t) = \Psi^{(0)}(t) + \frac{1}{\alpha'}\Psi^{(1)}(t) + \mathcal{O}\left(\frac{1}{\alpha'^2}\right) , \tag{3.3.51}$$

and expand all equations up to first order in $\frac{1}{\alpha'}$. By doing so each equation will split into two, one for each order, the leading one corresponding to the tensionless limit studied in (3.3.39). We will not consider corrections to the most general zeroth-order solutions found in (3.3.48) and (3.3.49) but we will restrict to the particular case of (3.3.50) with $Q > 0$ and $\omega > 0$ for simplicity. However, the following steps should be equally applicable to the general solutions.

Rather than going into each detail, here we show some examples of the procedure described above for the simplest equations. Taking (3.3.38f), for instance, we expand and keep only up to first order in the string's tension,

$$\begin{aligned} 0 &= \ddot{\Phi} + \frac{2}{\alpha'}A , \\ 0 &= \ddot{\Phi}^{(0)} + \frac{1}{\alpha'}\left(\ddot{\Phi}^{(1)} + 2A^{(0)}\right) + \mathcal{O}\left(\frac{1}{\alpha'^2}\right) . \end{aligned} \tag{3.3.52}$$

This splits into two equations, one for the tensionless limit and a new first order equation that determines $\Phi^{(1)}$ in terms of $A^{(0)}$. Inserting the solution (3.3.50) we see that (3.3.52) is solved by

$$\Phi^{(1)}(t) = -\omega_1(t - t_0) , \tag{3.3.53}$$

where ω_1 is a new integration constant, and we omitted a possible constant shift for simplicity. For the second and last explicit calculation, we consider (3.3.38g)

$$\begin{aligned} 0 &= \ddot{A} - \dot{\Phi}\dot{A} + \frac{1}{\alpha'}\left(4A^2 - 2B\right) , \\ 0 &= \ddot{A}^{(0)} - \dot{\Phi}^{(0)}\dot{A}^{(0)} + \frac{1}{\alpha'}\left(\ddot{A}^{(1)} - \dot{\Phi}^{(0)}\dot{A}^{(1)} - \dot{\Phi}^{(1)}\dot{A}^{(0)} + 4(A^{(0)})^2 - 2B^{(0)}\right) + \mathcal{O}\left(\frac{1}{\alpha'^2}\right) . \end{aligned} \tag{3.3.54}$$

By inserting (3.3.50) the leading order is automatically solved while the $\frac{1}{\alpha'}$ contribution determines $A^{(1)}$ in terms of the zeroth-order solutions to be

$$A^{(1)}(t) = A_0 + A_1 e^{-\omega(t-t_0)} + \frac{4}{\omega}(t - t_0) . \tag{3.3.55}$$

Following the same procedure, $y^{(1)}(t)$ can be determined by expanding equation (3.3.38e) and $B^{(1)}(t)$ by expanding the combination of (3.3.38b) and

(3.3.38c). Inserting these and all previous results into the expansion of (3.3.38b) we get a first order differential equation for $x^{(1)}(t)$ which can be solved exactly. Finally, at this point all first order corrections were determined, yet we still have to check that the expansion of (3.3.38d) holds up to first order in the string's tension. We performed all these steps and found the most general extension to the solution (3.3.50) (with positive Q and ω) up to and including first order in $\frac{1}{\alpha'}$. This family of corrections is parameterized in terms of integration constants and, for generic values of them, the solutions are not de Sitter vacua. However, there are particular integration constants for which the zeroth order de Sitter solution (3.3.50) is indeed preserved:

$$\Phi(t) = -\omega(t-t_0)\,, \quad H(t) = \frac{\omega}{4}\,, \quad x(t) = \pm 2\sqrt{\frac{Q}{\omega}}e^{-\frac{1}{2}\omega(t-t_0)}\,, \tag{3.3.56a}$$

$$A(t) = \frac{4}{\alpha'\omega}(t-t_0)\,, \quad B(t) = 2\,, \tag{3.3.56b}$$

$$y(t) = -\frac{8Q}{\alpha'\omega^2}e^{-\omega(t-t_0)}(t-t_0) - \frac{2}{\alpha'\omega}(t-t_0)\,, \quad \omega > 0\,,\ Q > 0\,. \tag{3.3.56c}$$

Here we absorbed ω_1 into a new $\frac{1}{\alpha'}$-corrected constant ω. This is the general behavior of integration constants in perturbation theory, at each order the new constants should renormalize previous ones. For $\Phi(t), H(t), x(t)$ and $B(t)$ the solutions take the same structural form as in the tensionless limit (3.3.50), except that the integration constants are $\frac{1}{\alpha'}$-corrected. It remains as an important open question whether the de Sitter vacua are preserved perturbatively at higher orders in $\frac{1}{\alpha'}$.

Chapter 4

Two-Dimensional Black Holes and α'

Black holes are perhaps the most mysterious objects predicted by general relativity. They are characterized by an event horizon, dividing spacetime into two causally disconnected regions, and a singularity in the interior. We expect general relativity to break down close to the singularity, but to be still applicable at the event horizon, since the curvature there might be relatively small. This then leads to the problem of resolving the black hole singularity: finding a new theory that replaces general relativity in the appropriate regime so that there are solutions with an event horizon but no spacetime singularity in the interior [98–102].

In view of our journey so far, string theory seems like a good candidate for such a task, and so we can use it to ask the concrete question: Can the black hole singularity be resolved in string theory by means of higher-derivative α' corrections? Providing an answer to this exciting question requires us to break the problem into smaller pieces. The first obstacle would be to identify black hole backgrounds inside the vast string landscape. Once such a candidate is identified, we face the problem of incorporating the infinite tower of α' corrections in a controllable manner. Suppose that such all-order formulation exists, then it comes the challenge of solving the corresponding α'-complete equations of motion. At this point, we should be able to confirm whether a regular black hole is part of this space of solutions.

This chapter is devoted to give an answer to the previous question, breaking the problem into the steps mentioned above. We will consider string theory in $D = 2$ with one isometric direction. Apart from its simplicity, this case is particularly promising in that there is an exact worldsheet CFT whose target space interpretation is that of a two-dimensional (2D) black hole (BH) [44–46]. Thus there should be an exact BH solution to all orders in α'. Since this corresponds to a background in non-critical dimensions, we will need to sort out the problem anticipated in Section 2.1.4 regarding the peturbative nature of higher-derivative corrections in the presence of a cosmological term. To this end, we will assume that higher-derivative terms in the action come with numerical coefficients that fall off in such a way that terms with more derivatives are sub-leading compared to terms with less derivatives. Under this assumption, we classify all possible higher-derivative corrections that are compatible with T-duality, in the same spirit of the cosmological classification [40, 47]. This provides us with a full space of duality-invariant α'-complete theories, a space that contains a point representing string theory.

We are now in possession of an α'-complete set of equations that we can attempt to solve. To this end, we first introduce a systematic method to solve the equations perturbatively in a derivative expansion, and we give an all-order proposal for the corrections to the two-derivative BH. We will not explore the physics behind these perturbative corrections. Instead, we will move right away to the more interesting scenario of non-perturbative solutions. We will use a powerful representation recently introduced by Gasperini and Veneziano [48], which will allow us to obtain a large class of solutions. We show that not all these solutions admit a black hole interpretation, yet there is a big family corresponding to α'-deformed black holes. Some members of this family include, just as the two-derivative solution, a horizon and a singularity. Furthermore, we identify another region in this solution space in which the black hole geometry exhibits a horizon but no singularity. This is a regular black hole whose interior is a regular cosmology.

This chapter is largely based on [6] and [7], and certain computations and figures are taken from these references.

4.1 Duality-Invariant Theories in $D = 2$

The two-derivative 2D BH studied in Section 2.2.3 is an example of a background in non critical dimensions and it corresponds to the leading term in an infinite α'-expansion. As illustrated in Section 2.1.4, these higher-derivative corrections behave very differently for theories in non-critical dimensions. In general, higher-derivative corrections are not small compared to lower-derivative terms, and so a perturbative description is misleading. This sheds doubt on attempts to find a more accurate black hole solution by means of higher-derivative corrections. In particular, looking for a classification for the most general α'-corrected 2D backgrounds, as we did for cosmological case in critical string theory, sounds like an ill-posed problem. This is due to the fact that in these non-critical scenarios field redefinitions can be used to trade any term in the action to one with an arbitrarily high number of derivatives.

In this section we circumvent this obstacle by showing that if the higher-derivative terms are suppressed in a particular way, some simplifications of the effective action are valid [6]. To see this, suppose an oracle gives us an action of the form (2.1.29) with infinitely many higher-derivative terms. A priori, general higher-derivative terms are all of the same order. Let us suppose, however, that the higher-derivative terms come with numerical coefficients that fall off in such a way that terms with four or more derivatives are subleading compared to terms with less derivatives. Since two-derivative terms come with order one coefficients, this could happen if the terms of order $(\alpha')^n$, with $n \geq 1$, come with coefficients of order ϵ^n with $\epsilon < 1$. In this situation we can ask and answer the following question: What are the most general field redefinitions that preserve this pattern, and what are the most general higher-derivative corrections modulo these restricted field redefinitions? We will show that these additional requirements eliminate those field redefinitions that allow one to remove arbitrary terms, and we will arrive at a minimal non-trivial set of higher-derivative terms that resembles the cosmological classification for critical string theory. Given our current knowledge on α' corrections, we cannot know if such classification applies to the 2D BH coming from string theory.

In Section 4.1.1 we recap the two-derivative black hole. In Section 4.1.2 we study field redefinitions in a 2D dilaton-lapse toy model with time independence. This allows us to identify "allowed" redefinitions, such that the fall-off structure of the multiplicative coefficients is preserved. These consist of generic redefinitions of the metric component $m(x)$, and a specific linear combination of the lapse function $n(x)$ and duality-invariant dilaton $\Phi(x)$. Generic redefinitions of n or Φ are not allowed for the classification, those are the ones that can remove any higher-derivative term. The general action in this setup, where fields are time independent, is obtained in Section 4.1.3 and takes the form [6]

$$I = \int dx\, n\, e^{-\Phi} \left[Q^2 + (\mathcal{D}\Phi)^2 - M^2 + \sum_{i\geq 1} \frac{\epsilon_i}{Q^{2i}} M^{2i+2} \right] . \tag{4.1.1}$$

Here the metric is $ds^2 = -m^2(x)dt^2 + n^2(x)dx^2$, and we defined $M \equiv \mathcal{D}\ln m, \mathcal{D} \equiv \frac{1}{n}\partial_x$ and $Q^2 = 16/\alpha'$. This is the most general higher-derivative extension to (2.2.70), up to field redefinitions that preserve the assumed fall-off structure of the coefficients $\epsilon_i \sim \epsilon^i$ with $\epsilon \ll 1$. We finish this section by giving the corresponding α'-complete set of equations for interior and exterior regions.

4.1.1 Two-derivative black hole solution

In Section 2.1.4 we considered the target-space theory of bosonic strings in 2D for the dilaton-gravity sector. Later in Section 2.2.3 we analyzed the two-derivative truncation in time-independent backgrounds, where the metric is given by

$$ds^2 = -m^2(x)dt^2 + n^2(x)dx^2 , \tag{4.1.2}$$

and the duality-invariant dilaton $\Phi(x)$ is related to the scalar dilaton $\phi(x)$ via

$$\phi(x) = \frac{1}{2}\Phi(x) + \frac{1}{2}\log|m(x)| . \tag{4.1.3}$$

The latter is the one coming from the string worldsheet action, and so it determines the string coupling as $g_s = e^{\phi}$. The 2D curvature for a metric of the form (4.1.2) is given by

$$\mathcal{R} = -2(M^2 + \mathcal{D}M) = -2\,\frac{\mathcal{D}^2 m}{m} . \tag{4.1.4}$$

and so the effective action takes the form

$$I^{(0)} = \int dx\, n\, e^{-\Phi} \left[Q^2 + (\mathcal{D}\Phi)^2 - M^2 \right] . \tag{4.1.5}$$

The covariant derivative $\mathcal{D}$ guarantees the invariance of (4.1.5) under x- reparameterization. The action is also manifestly duality invariant, since it only contains even powers of M, and T-duality sends $M \to \hat{M} = -M$ while leaving Φ and n unchanged (see (2.2.67)).

The equations of motion come from a general variation of (4.1.5), which can be combined in a convenient way to arrive at the following equivalent system

$$\mathcal{D}M = \mathcal{D}\Phi M \,, \tag{4.1.6a}$$

$$\mathcal{D}^2\Phi = M^2 \,, \tag{4.1.6b}$$

$$(\mathcal{D}\Phi)^2 = M^2 + Q^2 \,. \tag{4.1.6c}$$

The solution to (4.1.6) is unique up to coordinate reparameterization and it admits a BH interpretation [44–46]. Different coordinate patches cover different regions of such black hole. For instance, the field configuration given in (2.2.74) was obtained by picking the gauge $n(x) = \frac{1}{m(x)}$, and it covered the exterior and interior region of the BH, separated by a coordinate singularity. In this section, we study the two-derivative solution in more detail, exploiting the gauge freedom to get different regions of the black hole. We first pick a gauge to get the exterior region, then we change coordinates to get the interior region, and finally we recover (2.2.74) from yet another coordinate transformation. We finish by studying how these solutions are modified under duality transformations.

Exterior and Interior solutions

If we pick the gauge

$$n(x) = 1 \,, \tag{4.1.7}$$

the solution to (4.1.6) for metric and dilaton is given by [44]

$$ds^2 = -m^2(x)dt^2 + dx^2 \,, \quad m(x) = -\tanh\tfrac{Qx}{2} \,, \quad \Phi(x) = -\log|\sinh Qx| + \Phi_0 \,, \tag{4.1.8}$$

where Φ_0 is an integration constant. The scalar curvature (4.1.4) takes the form

$$\mathcal{R} = \frac{Q^2}{\cosh^2\frac{Qx}{2}} \,, \tag{4.1.9}$$

while the scalar dilaton $\phi(x)$ in (4.1.3) reads

$$\phi = -\frac{1}{2}\log\cosh^2\tfrac{Qx}{2} + \phi_0 \,, \quad \phi_0 = \phi(0) = \frac{1}{2}\left(\Phi_0 - \log 2\right) . \tag{4.1.10}$$

The solution is valid for $x \in \mathbb{R}$. However, the metric becomes singular at $x = 0$ since $m(0) = 0$ and so $\det G = 0$. Therefore, the solution splits into $x < 0$ and $x > 0$ regions. The former describes the outside region of a two-dimensional black hole [44–46] while the latter is a totally disconnected universe which forms part of the maximally extended BH solution. For the exterior region, $x = 0$ corresponds to the position of the black hole horizon and $x = -\infty$ to the asymptotically flat region. Indeed, at $x = 0$ the metric vanishes and the curvature is finite

$$m(0) = 0\,, \qquad \mathcal{R}(0) = Q^2\,, \tag{4.1.11}$$

which proves that $x = 0$ is just a coordinate singularity. Asymptotically we have

$$\lim_{x\to-\infty} m(x) = 1\,, \qquad \lim_{x\to-\infty} \mathcal{R}(x) = 0\,. \tag{4.1.12}$$

On the other hand, the duality-invariant dilaton $\Phi(x)$ becomes infinite at both extremes

$$\lim_{x\to 0^-} \Phi(x) = \infty\,, \qquad \lim_{x\to-\infty} \Phi(x) = -\infty\,. \tag{4.1.13}$$

This does not necessarily indicate a pathology since Φ is not the scalar dilaton ϕ that determines the string coupling as $g_s = e^{\phi}$. From (4.1.10) we can see that the scalar dilaton and hence the string coupling are finite at the horizon $\phi(0) = \phi_0$. We also infer $\lim_{x\to-\infty}\phi(x) = -\infty$, so the string coupling g_s goes to zero in the asymptotically flat region, consistent with a weak coupling regime.

The solution (4.1.8) describes only the exterior region of the black hole. Therefore, it has no information on the BH singularity, which lies in the interior. The latter can be obtained from the following change of coordinates

$$x \to \tilde{x} = ix\,, \tag{4.1.14}$$

which acts in the metric and dilaton as follows

$$\begin{aligned} m^2(x) &= \tanh^2 \tfrac{Qx}{2} = -\tan^2 \tfrac{Q\tilde{x}}{2}\,, \\ \Phi(x) &= -\log\left(-\sinh Qx\right) + \Phi_0 = -\log\sin Q\tilde{x} + \Phi_0 - \log i\,. \end{aligned} \tag{4.1.15}$$

The interior solution then takes the form[1]

$$ds^2 = \tilde{m}^2(\tilde{x})dt^2 - d\tilde{x}^2\,, \quad \tilde{m}(\tilde{x}) = \tan\tfrac{Q\tilde{x}}{2}\,, \quad \tilde{\Phi}(\tilde{x}) = -\log\sin Q\tilde{x} + \Phi_0\,. \tag{4.1.16}$$

The line element of (4.1.16) describes a cosmological backgrounds where $\tilde{x}$ plays the role of "time" and $\tilde{m}(\tilde{x})$ corresponds to the scale factor [43]. The curvature is obtained by applying (4.1.14) to (4.1.9)

$$\tilde{\mathcal{R}}(\tilde{x}) = \frac{Q^2}{\cos^2\frac{Q\tilde{x}}{2}}\,, \tag{4.1.17}$$

while for the scalar dilaton we get

$$\tilde{\phi}(\tilde{x}) = -\tfrac{1}{2}\log\cos^2\tfrac{Q\tilde{x}}{2} + \phi_0\,. \tag{4.1.18}$$

This solution is now valid for the finite range $\tilde{x} \in (0,\pi)$. For $\tilde{x} = 0$, the metric, scalar dilaton and curvature take the values

$$\tilde{m}(0) = 0\,, \qquad \tilde{\phi}(0) = \phi_0\,, \qquad \tilde{\mathcal{R}}(0) = Q^2\,, \tag{4.1.19}$$

which is consistent with a horizon interpretation. More importantly, the value for the curvature at the horizon coincides with the one obtained from the

[1] The complex shift in the definition of $\tilde{\Phi}(\tilde{x})$ guarantees the measure in the action stays real under (4.1.14).

exterior, (4.1.11). This must happen since both solutions are related by a coordinate transformation and $\mathcal{R}$ is a scalar quantity. The same holds for the scalar dilaton, since exterior and interior solutions attain the same value at the horizon, $\tilde{\phi}(0) = \phi(0) = \phi_0$.

At the other end $\tilde{x} = \pi$, all fields and curvature diverge

$$\tilde{m}(\pi) = \infty\,, \qquad \tilde{\phi}(\pi) = \infty\,, \qquad \tilde{\mathcal{R}}(\pi) = \infty\,, \tag{4.1.20}$$

signaling that this is the position of the BH singularity.

These exterior and interior black hole solutions can be seen as two regions of a single coordinate system. To this end, we can start from the exterior solution for the metric (4.1.8) and introduce a new coordinate $x'(x)$ via

$$dx'^2 = m^2(x)dx^2\,, \tag{4.1.21}$$

so that the metric takes the form

$$ds^2 = -m^2(x)dt^2 + \frac{dx'^2}{m^2(x)}\,. \tag{4.1.22}$$

Here, $m^2(x)$ is still the one given in (4.1.8) but it must be written in terms of x'. Equation (4.1.21) can be easily integrated to get

$$x \to x' = -\tfrac{1}{Q}\ln\left(a\cosh^2\tfrac{Qx}{2}\right) \quad \Rightarrow \quad \cosh^2\tfrac{Qx}{2} = \tfrac{1}{a}e^{-Qx'}\,, \quad a > 0\,, \tag{4.1.23}$$

where a is an integration constant and we choose the minus sign of the square root in (4.1.21) so x and x' are correlated. As a consequence, with $a, Q > 0$, the exterior region $x \in (-\infty, 0)$ is mapped to $x' \in (-\infty, -\frac{\log a}{Q})$. As a result,

$$m(x) = -\tanh\tfrac{Qx}{2} = \sqrt{1 - \cosh^{-2}\tfrac{Qx}{2}} = \sqrt{1 - ae^{Qx'}}\,, \tag{4.1.24}$$

which is always real in the justified region for x'. For the dilaton, one gets

$$\Phi(x) = -\log\left(-\sinh Qx\right) + \Phi_0 = Qx' - \tfrac{1}{2}\log\left(1 - ae^{Qx'}\right) + \hat{\Phi}_0\,, \tag{4.1.25}$$

with $\hat{\Phi}_0 = \Phi_0 + \log\frac{a}{2}$. All in all, the transformed solution can be written as [45]

$$\begin{aligned} &ds^2 = -m^2(x')dt^2 + \frac{1}{m^2(x')}dx'^2\,, \quad m^2(x') = 1 - a\,e^{Qx'}\,, \\ &\Phi(x') = Qx' - \frac{1}{2}\log|1 - a\,e^{Qx'}| \;+\; \hat{\Phi}_0\,. \end{aligned} \tag{4.1.26}$$

The curvature and scalar dilaton this time read

$$\mathcal{R}(x') = aQ^2e^{Qx'}\,, \quad \phi(x') = \frac{1}{2}\left(\hat{\Phi}_0 + Qx'\right)\,. \tag{4.1.27}$$

This solution is exactly the one presented in (2.2.74), which can be obtained by solving the equations of motion (4.1.6) in the gauge $n(x) = \frac{1}{m(x)}$. A step-by-step derivation of this solution can be found in [6].

In this solution, the position of the horizon $x = 0$ was mapped to $x' = x_H \equiv -\frac{\log a}{Q}$ while the asymptotically flat region was preserved at $x = x' = -\infty$. This covers the entire range in which the coordinate transformation (4.1.23) was valid. Therefore, it seems that from this change of coordinates we gained nothing, but it is just the same exterior solution in a different system. While this is formally correct, the advantage of having the new expression (4.1.26) is that now we can perform an analytic extension of the solution. This means that, while originally (4.1.26) was valid only for $x' < -\frac{\log a}{Q}$, we notice there is no real obstacle in extending it to also cover $x' > -\frac{\log a}{Q}$ and so we do it. In this analytically extended solution, $x' > -\frac{\log a}{Q}$ corresponds to the interior region found in (4.1.16). Indeed, for $x' > x_H$, the time-time component of the metric becomes negative, and at the horizon we have

$$m^2(x_H) = 0\,, \quad \mathcal{R}(x_H) = Q^2\,, \quad \phi(x_H) = \frac{1}{2}\left(\hat{\Phi}_0 - \log a\right) = \phi_0\,, \tag{4.1.28}$$

which coincide with the values found from both, exterior and interior regions. This region of the spacetime ends at $x' = \infty$ where metric, curvature and scalar dilaton diverges, consistent with the BH singularity found in (4.1.20).

Dual solutions

We finalize this section by studying how the solutions we found change under T-duality. More precisely, we will revisit briefly the phenomena found in [46], where the exterior region of the 2D BH is mapped to a region beyond the singularity, while the interior region is mapped into itself.

Starting from the exterior solution (4.1.8) and performing the duality transformation (2.2.67), one obtains the new solution, also valid for $x \in (-\infty, 0)$:

$$\hat{m}(x) = -\coth\tfrac{Qx}{2}\,, \quad \hat{\Phi}(x) = -\log|\sinh Qx| + \Phi_0\,, \quad \hat{\phi}(x) = -\tfrac{1}{2}\log\sinh^2\tfrac{Qx}{2} + \phi_0\,, \tag{4.1.29}$$

where the dual metric is one over the original one and the duality-invariant dilaton remains the same. Inserting $\hat{m}(x)$ in (4.1.4) we get the dual curvature

$$\hat{\mathcal{R}}(x) = -\frac{Q^2}{\sinh^2\frac{Qx}{2}}\,. \tag{4.1.30}$$

In this dual solution, the former horizon $x = 0^-$ is mapped to a curvature singularity

$$\lim_{x\to 0^-}\hat{m}(x) = \infty\,, \quad \lim_{x\to 0^-}\hat{\phi}(x) = \infty\,, \quad \lim_{x\to 0^-}\hat{\mathcal{R}}(x) = -\infty\,, \tag{4.1.31}$$

while the asymptotic region is again a flat spacetime which lies beyond the BH singularity.

The dual solution to the interior solution (4.1.16) is given by

$$\hat{\tilde{m}}(\tilde{x}) = \cot\tfrac{Q\tilde{x}}{2}\,, \quad \hat{\tilde{\Phi}}(\tilde{x}) = -\log\sin Q\tilde{x} + \Phi_0\,, \quad \hat{\tilde{\phi}}(\tilde{x}) = -\tfrac{1}{2}\log\sin^2\tfrac{Q\tilde{x}}{2} + \phi_0\,, \tag{4.1.32}$$

together with the curvature

$$\hat{\tilde{\mathcal{R}}}(x) = \frac{Q^2}{\sin^2 \frac{Q\tilde{x}}{2}} \,. \tag{4.1.33}$$

This geometry still corresponds to the interior region, with the horizon and the singularity exchanged. This can be made clear by noting that duality plus the "time reversal" transformation $\tilde{x} \to \pi - \tilde{x}$ leaves the interior solution invariant: $\hat{\tilde{m}}(\pi - \tilde{x}) = \tilde{m}(\tilde{x})$.

4.1.2 Two classes of field redefinitions

We are ultimately interested in classifying all higher-derivative corrections to the two-derivative black hole background (4.1.5). However, since we are dealing with non-critical backgrounds, as anticipated in Section 2.1.4 we need to be cautious when considering perturbative α'-corrections. In view fo this, in this section we analyze a simpler dilaton-lapse model with a cosmological term. This toy model, whose two-derivative action is obtained by setting $M = 0$ in (4.1.5), will allow us to understand better the role of field redefinitions in non-critical theories. More precisely, we will see clearly that a conventional derivative expansion is problematic. All terms seem equally important and field redefinitions used to classify interactions do not operate as usual.

In order to avoid these obstacles, we will assume theories in which there is an in-built suppression of the higher-derivative terms, a suppression due to coefficients ϵ_i that multiply the interactions and become smaller as the number of derivatives increase. Even under these assumptions complications arise due to the cosmological term. We will argue that there are two classes of field redefinitions:

1. Separate field redefinitions of the lapse or dilaton function, which generate new interactions via the variation of the cosmological term as well as the variation of other terms.
2. Simultaneous field redefinitions of the lapse and dilaton for which no terms arise from the variation of the cosmological term.

We will show that redefinitions of type 1 do not preserve the structure of in-built suppression: higher derivative terms induced by the redefinitions are not suppressed appropriately and thus cannot be neglected. Redefinitions of type 2, however, respect the structure of in-built suppression and thus can be used in the conventional setting of effective field theory to classify interactions.

While this toy model is considerably simpler than the 2D black hole, the results obtained here will apply almost without change to the case where $M \neq 0$.

Two-derivative dilaton-lapse model

The two-derivative sector of the dilaton-lapse model is obtained by setting $M = 0$ in the action (4.1.5)

$$I^{(0)} = \int dx\, n\, e^{-\Phi} \left[Q^2 + (\mathcal{D}\Phi)^2\right] . \tag{4.1.34}$$

Its equations of motion are given by

$$\begin{aligned} E_n^{(0)} &\equiv Q^2 - (\mathcal{D}\Phi)^2 = 0\,, \\ E_\Phi^{(0)} &\equiv -Q^2 + (\mathcal{D}\Phi)^2 - 2\mathcal{D}^2\Phi = 0\,. \end{aligned} \tag{4.1.35}$$

These are solved by the linear dilaton background

$$n^{(0)} = 1, \quad \Phi^{(0)} = Qx\,. \tag{4.1.36}$$

The simplest higher-derivative extension to (4.1.34) is given by

$$I = \int dx\, n\, e^{-\Phi} \left[Q^2 + (\mathcal{D}\Phi)^2 + c\alpha'(\mathcal{D}\Phi)^4\right] . \tag{4.1.37}$$

Note that for the solution $\Phi^{(0)} = Qx$, all three terms in the action are of the same order Q^2 (since $\alpha' \sim 1/Q^2$). The lapse equation following from the above action is given by

$$E_n \equiv Q^2 - (\mathcal{D}\Phi)^2 - 3c\alpha'(\mathcal{D}\Phi)^4 = 0\,, \tag{4.1.38}$$

and the dilaton equation is automatically satisfied when the lapse equation holds due to a Bianchi identity. The above equation admits a unique real solution of the form

$$n = 1, \quad \Phi = \omega x, \quad \omega^2 \equiv \frac{\sqrt{1 + 12c\alpha' Q^2} - 1}{6c\alpha'}\,. \tag{4.1.39}$$

This solution can be considered a small correction to (4.1.36) when ω^2 has a convergent perturbative expansion in powers of $c\alpha' Q^2$. A sufficient condition for such scenario is given by

$$\epsilon \equiv c\alpha' Q^2 \ll 1\,. \tag{4.1.40}$$

If satisfied, ω^2 can be expanded in powers of ϵ and so

$$\Phi = \omega x = \left(1 - \tfrac{3}{2}\epsilon + \mathcal{O}(\epsilon^2)\right) Qx = \Phi^{(0)} - \tfrac{3}{2}\epsilon Q x + \mathcal{O}(\epsilon^2)\,, \tag{4.1.41}$$

where the leading term is the two-derivative solution (4.1.36) and the rest are truly small corrections to it. If (4.1.40) does not hold, the four-derivative term in the action contributes with terms comparable to the two-derivative solution, making a derivative expansion meaningless. We assume that (4.1.40) holds.

At this point it is convenient to introduce the unit-free notation used in (2.1.30) where we define

$$\bar{x} \equiv Qx\,, \quad \bar{\mathcal{D}} \equiv \frac{1}{n}\frac{d}{d\bar{x}} = \frac{1}{Q}\mathcal{D}\,. \tag{4.1.42}$$

In terms of these derivatives, the action (4.1.37) becomes

$$\bar{I} \equiv \frac{1}{Q} I = \int d\bar{x}\, n\, e^{-\Phi} \left[1 + (\bar{\mathcal{D}}\Phi)^2 + \epsilon(\bar{\mathcal{D}}\Phi)^4\right] . \tag{4.1.43}$$

In this notation a variation of the lapse $n \to n + \delta n$ gives to leading order

$$\bar{I} \to \bar{I} + \int d\bar{x}\, n\, e^{-\Phi} \frac{\delta n}{n} \bar{E}_n + \mathcal{O}((\delta n)^2), \quad \text{with} \quad \bar{E}_n = 1 - (\bar{\mathcal{D}}\Phi)^2 - 3\epsilon(\bar{\mathcal{D}}\Phi)^4 . \tag{4.1.44}$$

We now do a redefinition to remove the four derivative term. We take

$$\frac{\delta n}{n} = -\epsilon(\bar{\mathcal{D}}\Phi)^4 . \tag{4.1.45}$$

The associated field redefinition is $n = n' + \delta n(n')$ and $\Phi = \Phi'$. The redefined action, called $\bar{I}'$, and written in terms of the new (primed) fields is given by

$$\bar{I}' = \int d\bar{x}\, n'\, e^{-\Phi'} \left[1 + (\bar{\mathcal{D}}\Phi')^2 + \epsilon(\bar{\mathcal{D}}\Phi')^6 + \mathcal{O}(\epsilon^2) + \mathcal{O}((\delta n)^2)\right] . \tag{4.1.46}$$

The field redefinition eliminated the four-derivative term at the cost of introducing a six-derivative term, *also* at order ϵ. Since derivatives $\bar{\mathcal{D}}$ are of order one, this new term is not parametrically smaller than the original four-derivative term. This shows that pure lapse transformations do not allow us to classify interactions in the sense of effective field theory. They are redefinitions of type 1, and as claimed do not respect the structure of in-built suppression. We would have wanted the new six-derivative interaction to appear at a higher order in ϵ.

We now discuss type 2 transformations; those that preserve the structure of in-built suppression and thus can be used to remove higher-derivative terms without inducing same-order effects. In this case, it will allow us to eliminate the four-derivative term consistently. The redefinition is of the form

$$\Phi = \Phi' + \delta\Phi(n', \Phi') , \quad n = n' + \delta n(n', \Phi') , \quad \text{with} \;\; \delta\Phi = \frac{\delta n}{n} . \tag{4.1.47}$$

For such a correlated redefinition of the dilaton and the lapse, the variation of the action to linearized order is, from (2.2.71),

$$\delta I = \int d\bar{x}\, n\, e^{-\Phi} \left[\, (E_n + E_\Phi)\delta\Phi \,\right] , \quad E_n + E_\Phi = -2\bar{D}^2\Phi + \mathcal{O}(\epsilon) . \tag{4.1.48}$$

An important fact is that this linear combination of equations of motion has no constant term. It is an straightforward computation to show that, to order ϵ, picking

$$\frac{\delta n}{n} = \delta\Phi = \tfrac{3}{2}\epsilon(\bar{\mathcal{D}}\Phi')^2 , \tag{4.1.49}$$

the transformed action takes the form

$$\bar{I}' = \int d\bar{x}\, n'\, e^{-\Phi'} \left[1 + (\bar{\mathcal{D}}\Phi')^2 + \mathcal{O}(\epsilon^2)\right] , \tag{4.1.50}$$

which has no four-derivative terms and induced effects are of order ϵ^2!

The general dilaton-lapse model

The lessons of the above discussion can be refined by considering the general version of the dilaton model, which in dimensionless units read

$$\bar{I} = \int d\bar{x}\, n\, e^{-\Phi}\Big[1 + (\bar{\mathcal{D}}\Phi)^2 + \sum_{i\geq 1}\epsilon_i \bar{L}^{(2i+2))}(\bar{\mathcal{D}};\Phi)\ \Big]\,,\quad \epsilon_i \equiv c_i(\alpha' Q^2)^i\,, \tag{4.1.51}$$

which includes infinitely many α' corrections depending only on covariant derivatives of Φ, with $L^{(2i+2)}(\bar{\mathcal{D}};\Phi)$ containing $2i+2$ derivatives.

Again, this action has no meaningful derivative expansion unless the coefficients ϵ_i decay fast enough. This can be formalized by extending the condition (4.1.40) to:

$$\epsilon \equiv \epsilon_1 \ll 1\,,\quad \epsilon_i \sim (\epsilon)^i\,,\ i\geq 1\,, \tag{4.1.52}$$

where the symbol $\sim$ denotes proportionality up to factors of order one. The above condition guarantees that each term in the derivative expansion is parametrically smaller than the previous one. In order to explore the effect of perturbative field redefinitions, we will assume that this condition is satisfied. Note that the condition above also implies that

$$\epsilon_p\epsilon_k \sim \epsilon_{p+k}\,. \tag{4.1.53}$$

A type 1 transformation will break condition (4.1.52), the correlation between the number of derivatives and the power of ϵ. This happens, because such redefinitions generate variations from the zero-derivative term in the action and the two-derivative term in the action, and the powers of ϵ are no longer correlated with the number of derivatives.

In order to preserve the correlation between derivatives and powers of ϵ we must do a redefinition of both the lapse and the dilaton

$$\Phi = \Phi' + \delta\Phi(\Phi',n')\,,\quad n = n' + \delta n(\Phi',n')\,, \tag{4.1.54}$$

with the relation

$$\frac{\delta n}{n'} = e^{\delta\Phi} - 1\,, \tag{4.1.55}$$

which at leading order coincides with (4.1.47). These redefinitions are constructed such that the measure is kept invariant, namely $ne^{-\Phi} = n'e^{-\Phi'}$. As a result, the cosmological term does not generate variations and so condition (4.1.52) is preserved. These are the type 2 transformations.

To see how these redefinitions are consistent with the fall-off condition, we consider (4.1.55) with

$$\delta\Phi = \sum_{i\geq 1}\epsilon_i F_\Phi^{(2i)}\,,\quad \frac{\delta n}{n} = \sum_{i\geq 1}\epsilon_i F_n^{(2i)}\,, \tag{4.1.56}$$

with $F_\Phi^{(2i)}$ and $F_n^{(2i)}$ generic gauge- and duality-invariant terms depending on $\bar{\mathcal{D}}\Phi$, containing $2i$ derivatives, and satisfying (4.1.55).

Applying the variation to the action (4.1.51), we note that we must only vary the terms inside the brackets. We concentrate on the effect of $\delta\Phi$ for now. Beginning with the two-derivative Lagrangian, each term in the variation will have an ϵ_i accompanied with $2i+2$ derivatives, $2i$ of them from $F_\Phi^{(2i)}$, and the other two from the two derivative term being varied. This is indeed consistent with the structure of the suppression. Continuing with the higher-derivative terms, varying $\bar{L}^{(2j+2)}$ in the action with the $F_\Phi^{(2k)}$ term of the dilaton variation, one gets the product $\epsilon_j\epsilon_k \sim \epsilon_{j+k}$ multiplying terms with $2(j+k)+2$ derivatives, which is also consistent with the constraint. Identical remarks hold for the variation of n in terms of $F_n^{(2i)}$.

Therefore, we can use (4.1.55) order-by-order in ϵ so to remove terms consistently. This is the procedure developed in [40] and revisited in Section 3.1 for critical strings, with the role of α' played here by the ϵ_i's satisfying (4.1.52). Using that logic, we can implement field redefinitions as simple substitution rules in the action. In particular, the rule corresponding to (4.1.55) is given by

$$\bar{E}_n + \bar{E}_\Phi = -2\bar{\mathcal{D}}^2\Phi \quad \Rightarrow \quad \bar{\mathcal{D}}^2\Phi \approx 0 + \mathcal{O}(\epsilon)\,, \tag{4.1.57}$$

and so we can recursively eliminate any term containing higher derivatives of the dilaton. It can be easily shown that, upon integration by parts, all higher-derivative terms in the dilaton-lapse model are of this form. Therefore, we can conclude that the all-order theory (4.1.51) is totally equivalent to the lowest order one (4.1.34)!

4.1.3 Duality-invariant theories to all orders

The final conclusion of the previous subsection can be easily extended to the case when m propagates. To see this, we consider the analogue to (4.1.51) in the presence of M

$$I = \int dx\, n\, e^{-\Phi}\Big[1 + (\mathcal{D}\Phi)^2 - M^2 + \sum_{i\geq 1}\epsilon_i L^{(2i+2)}(n, \mathcal{D}\Phi, M)\Big]\,, \tag{4.1.58}$$

where $L^{(2i+2)}$ contains $2i+2$ derivatives, and the in-built suppression condition (4.1.52) holds.

We are still working with dimensionless coordinates but we abandoned the bar notation so to keep the following computations cleaner. From now on, unless specifically stated otherwise, we will always work in dimension-free units.

We notice that by extending (4.1.55) to

$$\begin{gathered}\Phi = \Phi' + \delta\Phi(n',\Phi',m')\,, \qquad n = n' + \delta n(n',\Phi',m')\,, \qquad m = m' + \delta m(n',\Phi',m')\,,\\ \frac{\delta n}{n} = e^{\delta\Phi} - 1\,, \quad \delta\Phi = \sum_{i\geq 1}\epsilon_i F_\Phi^{(2i)}\,, \qquad \frac{\delta m}{m} = \sum_{i\geq 1}\epsilon_i F_m^{(2i)}\,,\end{gathered} \tag{4.1.59}$$

the in-built suppression feature is satisfied for the induced terms since transformations of m do not affect the measure $ne^{-\Phi}$ and so it remains invariant under the redefinitions (4.1.59). Finally, in the same way we could use (4.1.57) to perform a classification for the dilaton model (4.1.51), here we can apply the rules

$$
\begin{aligned}
E_m = 0 \quad &\Rightarrow \quad \mathcal{D}M \approx \mathcal{D}\Phi M + \mathcal{O}(\epsilon)\,, &\text{(4.1.60a)}\\
E_n + E_\Phi = 0 \quad &\Rightarrow \quad \mathcal{D}^2\Phi \approx M^2 + \mathcal{O}(\epsilon)\,, &\text{(4.1.60b)}
\end{aligned}
$$

where we used the lowest order equations of motion given in (4.1.6).

The classification here is very similar to the one implemented in Section 3.1.3 for Bianchi type I universes with $H_i(t) \to M(x)$. The algorithm therein goes through up to the point where redefinitions of the lapse function are needed, which in this case are not allowed because of the extra condition $n'e^{-\Phi'} = ne^{-\Phi}$. Specifically, the same itemized step-by-step proof of Section 3.1.3 holds up to and including step 4. More precisely, we assume that to any order in ϵ any term in the action is writable as a product of factors $\mathcal{D}^k\Phi$ and $\mathcal{D}^l M$. We can now do field redefinitions of the form (4.1.59), which in practice consist of applying the rules (4.1.60) in the action, in order to establish that higher derivatives of M and higher derivatives of $\mathcal{D}\Phi$ can be removed at each order.

At this point, a generic higher-derivative term at order ϵ^p, which we call $I^{(p)}$, is given by

$$
I^{(p)} = \int dx\, n\, e^{-\Phi}\, (\mathcal{D}\Phi)^{2j} M^{2l}\,, \tag{4.1.61}
$$

where even powers are a consequence of duality invariance together with the fact that the total number of derivatives must be even. Moreover, in order to preserve the fall-off condition (4.1.52), the total number of derivatives must add up to $2p+2$, which means $j + l = p + 1$.

In the case of cosmological backgrounds we would remove all $(\mathcal{D}\Phi)^{2j}$ very easily, simply by using the rule $E_n^{(0)} \approx 0$ as in (3.1.44c). For the two-dimensional case, however, this is not allowed since it would violate the condition $n'e^{-\Phi'} = ne^{-\Phi}$ and, as a consequence, would break (4.1.52). Luckily, there is another way of removing these terms which goes as follow: using $e^{-\Phi}\mathcal{D}\Phi = -\mathcal{D}e^{-\Phi}$ for one of the $\mathcal{D}\Phi$ factors in (4.1.61), and then integrating by parts we find

$$
\begin{aligned}
I^{(p)} &= -\int dx\, n\, \mathcal{D}(e^{-\Phi})\, (\mathcal{D}\Phi)^{2j-1} M^{2l}\,,\\
&= \int dx\, n\, e^{-\Phi} \Big((2j-1)(\mathcal{D}\Phi)^{2j-2}\mathcal{D}^2\Phi M^{2l} + (\mathcal{D}\Phi)^{2j-1} 2l M^{2l-1}\mathcal{D}M \Big)\,,\\
&\approx \int dx\, n\, e^{-\Phi} \Big((2j-1)(\mathcal{D}\Phi)^{2j-2} M^{2l+2} + 2l(\mathcal{D}\Phi)^{2j} M^{2l} \Big)\,,
\end{aligned} \tag{4.1.62}
$$

where we used (4.1.60) in the last line. The second term in the last line is a multiple of the original term. Thus, bringing it to the left-hand side and using $l \neq \frac{1}{2}$, we get

$$
I^{(p)} \approx \frac{2j-1}{1-2l} \int dx\, n\, e^{-\Phi}\, (\mathcal{D}\Phi)^{2j-2} M^{2l+2}\,. \tag{4.1.63}
$$

Thus, we can systematically reduce the powers of $\mathcal{D}\Phi$ in steps of two until removing all $\mathcal{D}\Phi$ factors!

The above chain of arguments proved that there is a field basis in which all higher-derivative terms involve only powers of M^2 and so the most general action is given by [6]

$$I = \int dx\, n\, e^{-\Phi} \left[1 + (\mathcal{D}\Phi)^2 - M^2 + \sum_{i\geq 1} \epsilon_i M^{2(i+1)} \right] . \tag{4.1.64}$$

If (4.1.64) is truncated at order ϵ^{N-1} we know from (4.1.52) that the remaining terms are of order $\mathcal{O}(\epsilon^N \mathcal{D}^{2N+2})$ and therefore contribute with small corrections to the solutions of the truncated theory.

As for the cosmological classification (3.1.1), we managed to parameterize our ignorance of the higher-derivative corrections in terms of a countably infinite set of coefficients ϵ_i satisfying (4.1.52). It is worth emphasizing that (4.1.64) encodes a whole family of theories and string theory would represent just a single point in this theory space. Moreover, with our current knowledge, we cannot even know if string theory belongs to this space, since all theories in (4.1.64) assume the fall-off condition (4.1.52). This could be not the case for the α' corrections coming from string theory.

Equations of motion for exterior and interior regions

The all-order action (4.1.64) can be written as

$$I = \int dx\, n\, e^{-\Phi} \Big[1 + (\mathcal{D}\Phi)^2 + F(M) \Big] , \tag{4.1.65}$$

where

$$F(M) \equiv \sum_{i=0}^{\infty} \epsilon_i M^{2i+2} = -M^2 + \epsilon M^4 + \mathcal{O}(\epsilon^2) \,, \quad \epsilon_0 \equiv -1 \,. \tag{4.1.66}$$

Taking the variation of (4.1.65) with respect to m, Φ and n, gives the following set of equations

$$\begin{aligned} \mathcal{D}\big(e^{-\Phi} f(M)\big) &= 0 \,, \\ (\mathcal{D}\Phi)^2 + \breve{g}(M) - 1 &= 0 \,, \\ -2\mathcal{D}^2\Phi + (\mathcal{D}\Phi)^2 - 1 - F(M) &= 0 \,. \end{aligned} \tag{4.1.67}$$

Here we introduced,

$$\begin{aligned} f(M) &\equiv \ F'(M) = 2\sum_{i=0}^{\infty} (i+1)\epsilon_i M^{2i+1} = -2M + 4\epsilon M^3 + \mathcal{O}(\epsilon^2) \,, \\ \breve{g}(M) &\equiv \sum_{i=0}^{\infty} (2i+1)\epsilon_i M^{2(i+1)} = -M^2 + 3\epsilon M^4 + \mathcal{O}(\epsilon^2) \,, \end{aligned} \tag{4.1.68}$$

which satisfy the following relation:

$$\breve{g}'(M) = M f'(M) \,, \tag{4.1.69}$$

with primes denoting derivative with respect to the arguments.

Combining the second and third equation in (4.1.67) and using the identity

$$F(M) = Mf(M) - \breve{g}(M)\,, \tag{4.1.70}$$

we can replace the third equation for a simpler one and get the equivalent system:

$$\begin{aligned} \mathcal{D}\big(e^{-\Phi}f(M)\big) &= 0\,,\\ (\mathcal{D}\Phi)^2 + \breve{g}(M) - 1 &= 0\,,\\ \mathcal{D}^2\Phi + \frac{1}{2}Mf(M) &= 0\,. \end{aligned} \tag{4.1.71}$$

Using (4.1.68), it is easy to see that for $\epsilon_i = 0\ \forall\ i$, (4.1.71) reduce to the two-derivative equations of motion (4.1.6) with $Q \to 1$.

Once at the level of the equations of motion, we can safely pick a particular gauge for the lapse function without missing extra constraints on the remaining fields. In our quest to solve (4.1.71), it will become very useful to work in the $n = 1$ gauge, in which $\mathcal{D} = \frac{1}{n}\frac{d}{dx} \to \frac{d}{dx}$ and so (4.1.71) reads

$$\frac{d}{dx}\left(e^{-\Phi}f(M)\right) = 0\,, \tag{4.1.72a}$$

$$\left(\frac{d\Phi}{dx}\right)^2 + \breve{g}(M) - 1 = 0\,, \tag{4.1.72b}$$

$$\frac{d^2\Phi}{dx^2} + \tfrac{1}{2}Mf(M) = 0\,, \tag{4.1.72c}$$

with

$$M(x) = \frac{d\log m}{dx} \quad\Rightarrow\quad \frac{m(x_2)}{m(x_1)} = \exp\Big(\int_{x_1}^{x_2} M(x')dx'\Big)\,. \tag{4.1.73}$$

For the two-derivative theory it was straightforward to solve (4.1.6). Finding solutions to (4.1.72) for generic $f(M)$ and $\breve{g}(M)$ is definitely much harder. Such is the case that we will devote all Section 4.2 and Section 4.3.2 just to solve that system of equations.

Regardless the specific form of the solution to (4.1.72), we will assume it admits an interpretation as the "exterior region" of some maximally extended solution. Clearly this is motivated from the two-derivative case (4.1.6), whose solution (4.1.8) corresponds to the exterior region of a black hole. Under this assumption, it is natural to assume there is also an "interior region" (analogous to (4.1.16) for the two-derivative BH). Since we do not have access to the exterior solution, we do not know if (4.1.14) is valid for mapping regions. Instead, we implement a "signature change" trick. This consists of looking for a transformation at the level of the action that preserves the measure and maps a metric with time-like isometry to a metric with space-like isometry as follows:

$$ne^{-\Phi} \;\to\; \tilde{n}e^{-\tilde{\Phi}}\,, \tag{4.1.74}$$

$$ds^2 = -m^2(x)dt^2 + n^2(x)dx^2 \quad\to\quad ds^2 = \tilde{m}^2(x)dt^2 - \tilde{n}^2(x)dx^2\,. \tag{4.1.75}$$

This can be achieved by setting

$$m = i\tilde{m}\,, \quad n = -i\tilde{n}\,, \quad \Phi = \tilde{\Phi} - \log i\,, \tag{4.1.76}$$

which preserves the measure and the product mn. From the definition of $\mathcal{D}$ and M, we have

$$\mathcal{D} = i\tilde{\mathcal{D}}\,, \quad M = i\tilde{M}\,, \tag{4.1.77}$$

with $\tilde{\mathcal{D}} = \frac{1}{\tilde{n}}\frac{d}{dx}$ and $\tilde{M} = \frac{1}{\tilde{m}\tilde{n}}\partial_x\tilde{m}$.

Applying (4.1.76) to (4.1.65) we get

$$\tilde{I} = \int dx\, \tilde{n} e^{-\tilde{\Phi}} \left[1 - (\tilde{\mathcal{D}}\tilde{\Phi})^2 + \tilde{F}(\tilde{M})\right]\,, \tag{4.1.78}$$

where we have defined

$$\tilde{F}(\tilde{M}) \equiv F(i\tilde{M}) = \sum_{i=0}^{\infty}(-1)^{i+1}\epsilon_i \tilde{M}^{2i+2} = \tilde{M}^2 + \epsilon\tilde{M}^4 + \mathcal{O}(\epsilon^2) \tag{4.1.79}$$

The equations of motion in the $\tilde{n} = 1$ gauge are now given by

$$\frac{d}{dx}\left(e^{-\tilde{\Phi}}\tilde{f}(\tilde{M})\right) = 0\,, \tag{4.1.80a}$$

$$\left(\frac{d\tilde{\Phi}}{dx}\right)^2 - \tilde{\breve{g}}(\tilde{M}) + 1 = 0\,, \tag{4.1.80b}$$

$$\frac{d^2\tilde{\Phi}}{dx^2} + \tfrac{1}{2}\tilde{M}\tilde{f}(\tilde{M}) = 0\,. \tag{4.1.80c}$$

The new function $\tilde{f}(\tilde{M})$ is defined analogously to (4.1.68):

$$\tilde{f}(\tilde{M}) \equiv \tilde{F}'(\tilde{M})\,, \tag{4.1.81}$$

while $\tilde{\breve{g}}(\tilde{M})$ can be defined via the identity

$$\tilde{\breve{g}}'(\tilde{M}) = \tilde{M}\tilde{f}'(\tilde{M})\,. \tag{4.1.82}$$

One can check that these functions are related to the ones in (4.1.68) via

$$\tilde{f}(\tilde{M}) = i f(i\tilde{M})\,, \quad \tilde{\breve{g}}(\tilde{M}) = \breve{g}(i\tilde{M})\,. \tag{4.1.83}$$

Equations (4.1.72) and (4.1.80) encode higher-derivative solutions for 2D backgrounds with time-like and space-like isometries, respectively. We interpret these solutions as the exterior and interior regions of a maximally extended spacetime. It is worth emphasizing that while in both cases we use the label x to denote the coordinate, these two solutions do not form together a single solution over the full real line. The two x's are really different. As a consistency check, one can corroborate that the two-derivative truncation of (4.1.72) and (4.1.80) have (4.1.8) and (4.1.16) as unique solutions, respectively.

4.2 α'-Corrected Black Holes

In the previous section we found a canonical formulation of the action, following our classification of possible duality-invariant terms up to field redefinitions. This lead to the action (4.1.65), whose equations of motion in $n = 1$ gauge are given by

$$\frac{d}{dx}\left(e^{-\Phi} f(M)\right) = 0\,, \tag{4.2.1a}$$

$$\left(\frac{d\Phi}{dx}\right)^2 + \breve{g}(M) - 1 = 0\,, \tag{4.2.1b}$$

$$\frac{d^2\Phi}{dx^2} + \tfrac{1}{2} M f(M) = 0\,, \tag{4.2.1c}$$

with $f(M)$ and $\breve{g}(M)$ given in (4.1.68) and $m(x)$ obtained from $M(x)$ via

$$\frac{m(x_2)}{m(x_1)} = \exp\left(\int_{x_1}^{x_2} M(x')dx'\right). \tag{4.2.2}$$

The corresponding curvature and scalar dilaton are obtained from the same formulas we used for the two-derivative case, since they are independent of the specific action or equations of motion. They are given by

$$\mathcal{R} = -2(M^2 + \partial_x M) = -2\,\frac{\partial_x^2 m}{m}\,, \tag{4.2.3}$$

and

$$\phi(x) = \frac{1}{2}\left(\Phi(x) + \log|m(x)|\right)\,. \tag{4.2.4}$$

We interpreted solutions coming from (4.2.1) as describing the "exterior region" of a maximally extended solution. In view of this, we introduced the "interior" version of the equations of motion given by

$$\frac{d}{dx}\left(e^{-\tilde{\Phi}} \tilde{f}(\tilde{M})\right) = 0\,, \tag{4.2.5a}$$

$$\left(\frac{d\tilde{\Phi}}{dx}\right)^2 - \tilde{\breve{g}}(\tilde{M}) + 1 = 0\,, \tag{4.2.5b}$$

$$\frac{d^2\tilde{\Phi}}{dx^2} + \tfrac{1}{2} \tilde{M} \tilde{f}(\tilde{M}) = 0\,. \tag{4.2.5c}$$

Here $\tilde{f}$ and $\tilde{\breve{g}}$ are related to the original functions via (4.1.83) and $\tilde{m}(x)$ is given by

$$\frac{\tilde{m}(x_2)}{\tilde{m}(x_1)} = \exp\left(\int_{x_1}^{x_2} \tilde{M}(x')dx'\right). \tag{4.2.6}$$

The interior curvature and scalar dilaton are obtained by applying (4.1.76) to (4.1.4) and (4.1.3) respectively, and then setting $\tilde{n}(x) = 1$. We get

$$\tilde{\mathcal{R}} = 2(\tilde{M}^2 + \partial_x \tilde{M}) = 2\,\frac{\partial_x^2 \tilde{m}}{\tilde{m}}\,, \tag{4.2.7}$$

and

$$\tilde{\phi}(x) = \frac{1}{2}\left(\tilde{\Phi}(x) + \log|\tilde{m}(x)|\right) . \tag{4.2.8}$$

In this section we solve these systems from two very different approaches. In Section 4.2.1 we introduce a systematic method to get perturbative solutions in a derivative (or ϵ) expansion and give the general structure of those solutions to all orders [6]. We then move into non-perturbative regimes in Section 4.2.2 [7] by considering a novel parameterization of the problem recently introduced by Gasperini and Veneziano in [48]. In Section 4.2.4 we use the parameterization to get a family of α'-corrected black hole solutions in which the singularity is still present [7].

4.2.1 Perturbative solutions

We now present a systematic method to solve (4.2.1) order-by-order in ϵ^2 [6]. To this end we focus on the first two equations, solve for $\frac{d\Phi}{dx}$ in both and equate the result. We get

$$\frac{d\Phi}{dx} = \frac{f'(M)}{f(M)}\frac{dM}{dx} = \pm\sqrt{1-\breve{g}(M)} , \tag{4.2.9}$$

or, equivalently,

$$\frac{f'(M)\, dM}{f(M)\sqrt{1-\breve{g}(M)}} = \pm dx . \tag{4.2.10}$$

By integration we have

$$\int^{M} \frac{f'(M')dM'}{f(M')\sqrt{1-\breve{g}(M')}} = \pm(x - x_0) , \tag{4.2.11}$$

where x_0 is an integration constant. This equation fixes a relation between a function of M (the left-hand side) and a function of x (the right-hand side).

If we have a function $W(M)$ such that

$$dW \equiv \frac{f'(M)}{f(M)\sqrt{1-\breve{g}(M)}} dM , \tag{4.2.12}$$

the general solution to (4.2.11) is given by

$$W(M) = \pm(x - x_0) , \tag{4.2.13}$$

a relation that can be inverted to determine $M(x)$, and with it $m(x)$. The dilaton is then found from (4.2.1a),

$$e^{-\Phi} f(M) = q , \tag{4.2.14}$$

with q some constant. By the Bianchi identity, the last equation in (4.2.1) holds when the first two hold.

[2]This method is inspired in the one used in [40] for FRW backgrounds.

Recovering the two-derivative black hole solution

Let us now re-derive the lowest-order BH solution (4.1.8) from the general formula (4.2.11). For the two-derivative action we have (see (4.1.68))

$$f(M) = -2M\,, \quad f'(M) = -2\,, \quad \breve{g}(M) = -M^2\,, \tag{4.2.15}$$

and so (4.2.12) takes the form

$$dW = \frac{dM}{M\sqrt{1+M^2}} \quad \Rightarrow \quad W(M) = -\mathrm{arcsch}\, M\,. \tag{4.2.16}$$

By inserting this result into (4.2.13) and inverting $W(M)$ we end up with

$$M = -\mathrm{csch}\, x = \partial_x \ln m\,. \tag{4.2.17}$$

where we chose the plus sign and fixed $x_0 = 0$, without loss of generality. This is easily integrated and we obtain

$$m(x) = -\tanh \tfrac{x}{2}\,. \tag{4.2.18}$$

The dilaton solution is obtained from (4.2.14). Using $f(M) = -2M$ and (4.2.17) we get

$$e^{\Phi(x)} = \tfrac{2}{q}\,\mathrm{csch}\, x \quad \Rightarrow \quad \Phi(x) = -\log \tfrac{q}{2} \sinh x\,, \tag{4.2.19}$$

which requires q to have the same sign as x. As expected, equations (4.2.18) and (4.2.19) coincide with the black hole solution presented in (4.1.8), upon changing to dimensionfull coordinates $x \to Qx$ and identifying $\Phi_0 = \log \frac{2}{|q|}$. As before, $x < 0$ corresponds to the exterior region, while $x > 0$ is a causally disconnected universe.

Systematics of perturbative solutions

An analytic expression for the non-perturbative $W(M)$ may not exist in general. However, when considering perturbative solutions in ϵ, equation (4.2.12) becomes a power series in ϵ, where each term is easier to integrate than the non-perturbative dW[3]. In this perturbative regime, a systematic approach exists such that solutions to any order in ϵ can be obtained from lowest-order ones. This algorithm takes (4.2.12) as the starting point and expand it around small ϵ to get

$$dW = \frac{f'(M)dM}{f(M)\sqrt{1-\breve{g}(M)}} = dW^{(0)} + \epsilon dW^{(1)} + \epsilon^2 dW^{(2)} + \mathcal{O}(\epsilon^3)\,. \tag{4.2.20}$$

Integrating each of these terms we arrive at the perturbative version of (4.2.13),

$$W(M) = W^{(0)}(M) + \epsilon W^{(1)}(M) + \epsilon^2 W^{(2)}(M) + \mathcal{O}(\epsilon^3) = -x\,, \tag{4.2.21}$$

[3]We will see later in Section 4.2.2 how to deal with these non-perturbative cases!

where this time we pick the minus sign just for simplicity (we can always change to the plus convention by sending $x \to -x$ at the end of the computation). From (4.2.16) we already know that $W^{(0)}(M) = -\text{arcsch}\, M$, which clearly can be inverted. By doing so, (4.2.21) becomes

$$\begin{aligned} M &= \text{csch}\left(x + \epsilon W^{(1)}(M) + \epsilon^2 W^{(2)}(M)\right) + \mathcal{O}(\epsilon^3) \\ &= \text{csch}\, x + \epsilon\, \text{csch}' x\ W^{(1)}(M) \\ &\qquad + \epsilon^2 \left[\tfrac{1}{2}\text{csch}'' x\ (W^{(1)}(M))^2 + \text{csch}' x\ W^{(2)}(M)\right] + \mathcal{O}(\epsilon^3)\,, \end{aligned} \tag{4.2.22}$$

where $'$ means derivative with respect to the argument. Then, we expand

$$M = M^{(0)} + \epsilon M^{(1)} + \epsilon^2 M^{(2)} + \mathcal{O}(\epsilon^3)\,, \tag{4.2.23}$$

on both sides of the last equality to read the solution order by order in ϵ

$$M^{(0)}(x) = \text{csch}\, x\,, \tag{4.2.24a}$$

$$M^{(1)}(x) = \text{csch}' x\ \ W^{(1)}(M^{(0)}(x))\,, \tag{4.2.24b}$$

$$\begin{aligned} M^{(2)}(x) &= \tfrac{1}{2}\text{csch}'' x\ \ (W^{(1)}(M^{(0)}(x)))^2 + \text{csch}' x\ \ W^{(2)}(M^{(0)}(x)) \\ &\qquad + \text{csch}' x\ \ W^{(1)\prime}(M^{(0)}(x)) M^{(1)}(x)\,. \end{aligned} \tag{4.2.24c}$$

We can see that each order $M^{(i)}(x)$ is determined from the lowest-order ones. From $M = \partial_x \ln m$ and (4.2.14) we can get the perturbative solutions for $m(x)$ and $\Phi(x)$. The resulting solution will be in the $n = 1$ gauge. If needed, it can be mapped to the $mn = 1$ gauge with the use of (4.1.21) as we did for the two-derivative case. This time, however, the $m(x)$ in (4.1.21) is the corrected solution.

Just as a demonstration of how the above algorithm works in practice, we work out the first ϵ-order explicitly. Up to first order we have (see (4.1.68))

$$f(M) = -2M + 4\epsilon M^3\,, \quad f'(M) = -2 + 12\epsilon M^2\,, \quad \check{g}(M) = -M^2 + 3\epsilon M^4\,. \tag{4.2.25}$$

Inserting these quantities into (4.2.20), and expanding up to first order in ϵ we can read

$$\begin{aligned} dW^{(0)} &= \frac{1}{M\sqrt{1+M^2}} dM\,, \\ dW^{(1)} &= -\frac{M\,(8+5M^2)}{2\,(1+M^2)^{\frac{3}{2}}} dM\,. \end{aligned} \tag{4.2.26}$$

Each order can be integrated independently to obtain

$$\begin{aligned} W^{(0)}(M) &= -\text{arcsch} M\,, \\ W^{(1)}(M) &= -\frac{1+\frac{5}{2}M^2}{\sqrt{1+M^2}}\,. \end{aligned} \tag{4.2.27}$$

Finally, by using (4.2.24a) and (4.2.24b) we can read $M^{(1)}(x)$:

$$M^{(1)}(x) = \text{csch}' x\ W^{(1)}(M^{(0)}(x)) = \text{csch}\, x + \tfrac{5}{2}\,\text{csch}^3 x\,. \tag{4.2.28}$$

All in all, up to order ϵ, $M(x)$ is given by

$$M(x) = \text{csch}\, x + \epsilon\, \left(\text{csch}\, x + \tfrac{5}{2}\,\text{csch}^3 x\right) + \mathcal{O}(\epsilon^2)\,. \tag{4.2.29}$$

We easily find $m(x)$ from

$$\begin{aligned} m(x) =&\ e^{\int^x M(x')dx'} = e^{\int^x M_0(x')dx'}\left(1+\epsilon\int^x M_1(x')dx'\right)+\mathcal{O}(\epsilon^2) \\ =&\ \tanh\tfrac{x}{2}\left(1+\epsilon\int^x\left(\operatorname{csch}x'+\tfrac{5}{2}\operatorname{csch}^3x'\right)dx'\right)+\mathcal{O}(\epsilon^2)\,, \end{aligned} \tag{4.2.30}$$

and doing the integral,

$$m(x) = \tanh\tfrac{x}{2}\left(1-\tfrac{1}{4}\epsilon\left[\ln\tanh\tfrac{x}{2}+5\coth x\,\operatorname{csch}x\right]\right)+\mathcal{O}(\epsilon^2)\,. \tag{4.2.31}$$

Finally, the dilaton profile comes from combining (4.2.14), (4.2.25) and (4.2.29) and is given by

$$e^{\Phi(x)} = \tfrac{1}{q}f(M) = -\tfrac{2}{q}\left[\operatorname{csch}x+\epsilon\left(\operatorname{csch}x+\tfrac{1}{2}\operatorname{csch}^3x\right)\right]+\mathcal{O}(\epsilon^2)\,. \tag{4.2.32}$$

In view of the pattern emerging at order ϵ in (4.2.29) and (4.2.32), it feels natural to ask whether such structure persists perturbatively to all orders. Indeed, by following an inductive procedure we confirmed that the following ansatz can be used to solve (4.2.1) to all orders in ϵ[4]:

$$M = \sum_{p\geq 0} M^{(p)}\epsilon^p\,,\qquad M^{(p)} = \sum_{k=0}^{p} a_k^{(p)}\operatorname{csch}^{2k+1}x\,, \tag{4.2.33a}$$

$$e^{\Phi} = \sum_{p\geq 0}\left[e^{\Phi}\right]^{(p)}\epsilon^p\,,\qquad \left[e^{\Phi}\right]^{(p)} = \sum_{k=0}^{p} b_k^{(p)}\operatorname{csch}^{2k+1}x\,. \tag{4.2.33b}$$

Here $a_k^{(p)}$ and $b_k^{(p)}$ are some order-one coefficients determined completely from the ϵ_i coefficients in the action. For instance, from (4.2.29) we can read $a_0^{(0)} = 1$, $a_0^{(1)} = 1$ and $a_1^{(1)} = \frac{5}{2}$.

The solution (4.2.33) corresponds to an all-order perturbative correction to the *exterior region* of the two-derivative black hole (4.1.8). The equivalent corrected solution for the interior region should come from an identical systematic procedure, but taking (4.2.5) as starting point instead of (4.2.1). In the interior region, however, we expect quantum-gravity effects to be rather strong, since we know that general relativity breaks close to the singularity. Therefore, a perturbative analysis sounds insufficient in these regimes. We need a non-perturbative approach to solve (4.2.5). This is exactly the topic of all remaining sections in this chapter.

4.2.2 The Gasperini-Veneziano parameterization

In order to study solutions of (4.2.1) and (4.2.5) we will use a useful parameterization inspired by a recent work by Gasperini and Veneziano in the context of pre-big bang string cosmology [48]. Instead of describing the fields

[4] A brief description of the inductive procedure can be found in Appendix A of [6].

as functions of the coordinate x, they are described as functions of an arbitrary parameter. This is conveniently taken to be f, from (4.1.68), by inverting $f(M)$ in (4.1.68) to get $M(f)$. We will see how the new parameterization for M induces a new parameterization for all other quantities, including the dilaton $\Phi(f)$ and the original coordinate $x(f)$. The latter serves to connect the f parameterization with the standard spacetime dependency.

From this new perspective, the solution space is described by the set of functions $M(f)$ instead of $f(M)$. This leads to some invertibility issues that we will explain in detail at the end of this subsection. For now, however, we concentrate on solving the equations of motion in terms of the f parameterization. We will distinguish the parameter for each region, using f to describe the exterior and $\tilde{f}$ for the interior.

Exterior solution

We begin from the second equation (4.2.1b). In terms of f, it gives

$$\left(\frac{d\Phi}{dx}\right)^2 = 1 - \int_0^M \breve{g}'(\tilde{M})d\tilde{M} = 1 - \int_0^M \tilde{M} f'(\tilde{M})d\tilde{M} = 1 - \int_0^f M(\tilde{f})d\tilde{f}\,, \tag{4.2.34}$$

where in the first equality we used that $\breve{g}(M=0)=0$ (see (4.1.68)) and in the second equality we used (4.1.69). This yields a parameterization of the dilaton derivative in terms of f:

$$\frac{d\Phi}{dx}(f) = \pm\sqrt{P(f)}\,, \quad \text{with} \quad P(f) \equiv 1 - \int_0^f M(f')df'\,, \tag{4.2.35}$$

where the plus/minus signs correspond to using different branches of the square root. The allowed range of f is determined by the condition $P(f) \geq 0$. We see that $P(f=0) = 1 > 0$, so by continuity there is at least an interval around $f = 0$ where the solution exists. Moreover, if the integral appearing above is negative for all values of f (as will be the case for the standard BH), then the solution is valid for the whole real line $f \in \mathbb{R}$.

Luckily, we do not need to solve (4.2.35) to get $\Phi(f)$. The latter can be easily obtained from integrating exactly (4.2.1a)

$$\Phi(f) = \log|f| + \Phi_1\,, \tag{4.2.36}$$

with $\Phi_1 \equiv \Phi(f=1)$ an integration constant that absorbs the sign of f and encodes the mass of the black hole. If instead of integrating (4.2.1a) we distribute the x derivative we get the relation between f and x:

$$dx = \pm\frac{df}{f\sqrt{P(f)}}\,, \tag{4.2.37}$$

where we used (4.2.35). Using (4.2.37) in (4.2.2) we obtain an expression for the metric components

$$\frac{m(x_2)}{m(x_1)} = \frac{m(f_2)}{m(f_1)} = \exp\left(\pm\int_{f_1}^{f_2}\frac{df}{f}\frac{M(f)}{\sqrt{P(f)}}\right). \tag{4.2.38}$$

With $m(f)$, we can relate $\Phi(f)$ to the scalar dilaton $\phi(f)$ via (4.2.4), where now all quantities are parameterized by f. For the curvature, we have from (4.2.3),

$$\mathcal{R}(f) = -2\Big(M^2 + \frac{dM}{df}\frac{df}{dx}\Big) = -2\Big(M^2 \pm f\,\frac{dM}{df}\sqrt{P(f)}\;\Big)\,. \tag{4.2.39}$$

Collecting the results, for ease of reference, we have [7]

$$\frac{d\Phi}{dx}(f) = \;\pm\sqrt{P(f)}\,, \tag{4.2.40a}$$

$$\Phi(f) = \log|f| + \Phi_1\,, \tag{4.2.40b}$$

$$dx = \;\pm\frac{df}{f\sqrt{P(f)}}\,, \tag{4.2.40c}$$

$$\frac{m_\pm(f_2)}{m_\pm(f_1)} = \exp\Big(\pm\int_{f_1}^{f_2}\frac{df}{f}\frac{M(f)}{\sqrt{P(f)}}\;\Big)\,, \tag{4.2.40d}$$

$$\mathcal{R}_\pm(f) = -2\Big(M^2 + \frac{dM}{df}\frac{df}{dx}\Big) = -2\Big(M^2 \pm f\,\frac{dM}{df}\sqrt{P(f)}\;\Big)\,, \tag{4.2.40e}$$

$$\phi_\pm(f) = \frac{1}{2}\left(\Phi(f) + \log|m_\pm(f)|\right)\,. \tag{4.2.40f}$$

Here the argument $P(f)$ of the square roots is

$$P(f) \equiv 1 - \int_0^f M(f')df'\,, \tag{4.2.41}$$

and we used the plus/minus subscript to differentiate branches.

Equations (4.2.40) encode the general solution to (4.2.1) in terms of f as a parameter! The solution is completely determined once an ansatz for $M(f)$ is given, and the physics depends on such choice. Without $M(f)$ we cannot say much about the specifics of each solution, but we can still analyze some global aspects of them: the first thing to notice is that, from our starting point in (4.1.68), since $f(M)$ has a power series expansion in M, its inverse $M(f)$, at least perturbatively, is expected to have the following expansion

$$M(f) = -\tfrac{1}{2}f\left[1 + \frac{\epsilon}{2}f^2 + \mathcal{O}(\epsilon^2 f^4)\right] = -\tfrac{1}{2}\,f\,\left[1 + h(f^2)\right]\,, \tag{4.2.42}$$

where we introduced the function h as follows

$$h(\xi) \equiv \frac{\epsilon}{2}\,\xi + \mathcal{O}(\epsilon^2\xi^2)\,. \tag{4.2.43}$$

Here $M(f) = -\frac{f}{2}$, obtained for $h(\xi) = 0$, corresponds to the standard two-derivative case, for which we recover the exterior BH solution of (4.1.8), as we will show in Section 4.2.3. The expansion (4.2.42) implies the following two properties for generic $M(f)$

$$M(f=0) = 0\,, \quad M(-f) = -M(f)\,, \tag{4.2.44}$$

and

$$h(\xi=0) = 0\,, \quad h'(\xi=0) = \text{finite}\,. \tag{4.2.45}$$

These observations reveals that $f = 0$ corresponds to the asymptotically flat region in all solutions.

To see this, we study the behavior of (4.2.40) and (4.2.41) near $f = 0$ for generic $M(f)$. Using (4.2.42), (4.2.44) and $P(f = 0) = 1$, we get

$$\Phi(f) \simeq \log|f|\,, \quad dx \simeq \pm\frac{df}{f}\,, \quad m_\pm(f) \simeq m_\pm(0)e^{\mp\frac{1}{2}f}\,, \quad \mathcal{R}_\pm(0) = 0\,. \tag{4.2.46}$$

From the second relation we infer $x \simeq \pm\ln f + \text{const}$ and hence that $f = 0$ corresponds to an asymptotic region with infinite x. From the third relation we see that we can choose the integration constant such that $m_\pm(0) = 1$. Using this together with $\Phi(0) = -\infty$ in (4.2.40f) we obtain $\phi_\pm(0) = -\infty$. All these results are compatible with the interpretation as a faraway region.

This makes $f = 0$ the end of the spacetime, which implies that solutions with $f < 0$ and with $f > 0$ should be treated separately, not as two regions of the same exterior solution. On top of this distinction, we have the two branches of the square root of (4.2.40a) corresponding to the $\pm$ choices. While this seems to suggest we have four different solutions, all of them should be physically equivalent in string theory. More precisely, the negative and positive regions of f are related by T-duality since $M \to -M$ and (4.2.42) imply

$$f \to \hat{f} = -f\,. \tag{4.2.47}$$

Moreover, solutions in the different plus/minus branches are connected via a trivial sign flip of x. More precisely, the solution with minus sign and $f < 0$ is identical to the solution with plus sign and $f > 0$, upon changing $x \to -x$. Their T-dual solutions can be obtained by (4.2.47).

Interior solution

In order to get the interior solutions in the Gasperini-Veneziano parameterization, we repeat the procedure we just performed for the exterior case, but this time using the equations (4.2.5). Without going into details, the solutions read [7]

$$\frac{d\tilde{\Phi}}{dx}(\tilde{f}) = \pm\sqrt{\tilde{P}(\tilde{f})}\,, \tag{4.2.48a}$$

$$\Phi(\tilde{f}) = \log|\tilde{f}| + \tilde{\Phi}_1\,, \tag{4.2.48b}$$

$$dx = \pm\frac{d\tilde{f}}{\tilde{f}\sqrt{\tilde{P}(\tilde{f})}}\,, \tag{4.2.48c}$$

$$\frac{\tilde{m}_\pm(\tilde{f}_2)}{\tilde{m}_\pm(\tilde{f}_1)} = \exp\left(\pm\int_{\tilde{f}_1}^{\tilde{f}_2}\frac{df}{f}\frac{\tilde{M}(f)}{\sqrt{\tilde{P}(f)}}\right), \tag{4.2.48d}$$

$$\tilde{\mathcal{R}}_\pm(\tilde{f}) = 2\left(\tilde{M}^2 + \frac{d\tilde{M}}{d\tilde{f}}\frac{d\tilde{f}}{dx}\right) = 2\left(\tilde{M}^2 \pm \tilde{f}\,\frac{d\tilde{M}}{d\tilde{f}}\sqrt{\tilde{P}(\tilde{f})}\;\right), \tag{4.2.48e}$$

$$\tilde{\phi}_\pm(\tilde{f}) = \frac{1}{2}\left(\tilde{\Phi}(\tilde{f}) + \log|\tilde{m}_\pm(\tilde{f})|\right)\,. \tag{4.2.48f}$$

Here the argument of the square roots, called $\tilde{P}(\tilde{f})$, is defined as follows

$$\tilde{P}(\tilde{f}) \equiv -1 + \int_0^{\tilde{f}} \tilde{M}(\tilde{f}')d\tilde{f}' . \tag{4.2.49}$$

Since $\tilde{P}(\tilde{f} = 0) = -1 < 0$, this time the solution excludes the point $\tilde{f} = 0$. Moreover, in general it is not even guaranteed that the interior solution exist at all. There would be no interior solution if $\tilde{P}(\tilde{f})$ is negative for all $\tilde{f}$.

We can determine the function $\tilde{M}(\tilde{f})$ for the interior region from the function $M(f)$ for the exterior region by inverting the relation $\tilde{f}(\tilde{M}) = if(i\tilde{M})$ we found in (4.1.83). This is given by

$$\tilde{M}(\tilde{f}) = -iM(-i\tilde{f}) . \tag{4.2.50}$$

In particular, the perturbative expansion of (4.2.42) is mapped to the following perturbative expansion for $\tilde{M}(\tilde{f})$

$$\tilde{M}(\tilde{f}) = \tfrac{1}{2}\tilde{f}\left[1 - \tfrac{\epsilon}{2}\tilde{f}^2 + \mathcal{O}(\epsilon^2\tilde{f}^4)\right] = \tfrac{1}{2}\tilde{f}\left[1 + h(-\tilde{f}^2)\right] , \tag{4.2.51}$$

with the same $h(\xi)$ defined in (4.2.43).

The range of validity of the interior solution is very different from the one of the exterior solution. From now on we assume there is always at least one positive branch point $f_0 > 0$ such that $\tilde{P}(f_0) = 0$ and $\tilde{P}(\tilde{f}) > 0$ for some interval $\tilde{f} \in (f_0, f_1)$, where $f_1 > f_0$ can be another branch point or the point at infinity. If this is the case, due to the even parity of $\tilde{P}(\tilde{f})$, the interior solution exists at least for an interval $\tilde{f} \in (-f_1, -f_0) \cup (f_0, f_1)$. However, since for the interior solution T-duality also acts as

$$\tilde{f} \to \hat{\tilde{f}} = -\tilde{f} , \tag{4.2.52}$$

both negative and positive regions are related via a duality transformation. We are imposing to have at least one branch point for positive f. Multiple branch points are also possible in principle. They would lead to multiple separated domains within $\tilde{f} > 0$ such that $\tilde{P}(\tilde{f})$ is positive in each of those disconnected regions.

In the next sections we will use the Gasperini-Veneziano parameterization to get several black hole solutions with different properties. Before doing that, however, we dedicate a few paragraphs to clarify the invertibility issues arising when going from $f(M)$ to the $M(f)$ picture.

On the non-invertibility of $M(f)$

As found in Section 4.1.3, the most general higher-derivative corrections can be encoded in a function $F(M)$ given as an infinite power series expansion in (even) powers of M ((4.1.66)). Each successive term contains an additional power of ϵ or, in dimensionful units, α'. It is quite possible, perhaps even likely, that this series only has a finite radius of convergence, making the definition of the action incomplete. From $F(M)$ one defines $f(M) \equiv F'(M)$ and in the standard approach one usually works with $f(M(x))$. Here, we change the perspective and describe α' corrections via $M(f)$, which we assume to be a well defined (single valued) function, and $f(M)$ is just its inverse. From this approach, it follows that the inverse $f(M)$ may not exists or, in order words, may be multivalued.

An example of this situation is given by assuming a function $M(f)$ that grows monotonically from $M(f=0)=0$ to some maximum $M(f=f_*)=M_*$ with $f_*>0$, and then falls down. Then the inverse $f(M)$ is not single valued. In fact, if one expands $M(f)$ as a series in f and perturbatively inverts it, $f(M)$ will only converge up to $M=M_*$, where it has a branch point. Departing from this specific example, in general let M_* denote the maximum value of M for which the series $f(M)$ converges. In these scenarios, the multivalued nature of $f(M)$ propagates to all other quantities in the equations of motions.

One of these quantities is $\breve{g}(M)$, defined by the relations (4.1.68) and (4.1.69) which are translated to

$$\frac{d\breve{g}}{dM}=M\frac{df}{dM} \quad \text{and} \quad \breve{g}(0)=0\,. \tag{4.2.53}$$

Both of these are implemented by the integral definition

$$\breve{g}(M)\equiv\int_{M'=0}^{M'=M}M'\frac{df(M')}{dM'}dM'\,,\quad M\in[0,M_*]\,. \tag{4.2.54}$$

The condition on the range of validity of the definition is needed because $\breve{g}$ is a multivalued function, a property it inherits from $f(M)$. But now, in this range we can change the variable of integration from M' to f' to get

$$\breve{g}(M)\equiv\int_0^{f(M)}M(f')df'\,,\quad M\in[0,M_*]\,, \tag{4.2.55}$$

where we used $f(M=0)=0$. The above definition suggests the following construction of a *new function* $g(f)$ well defined for *all* f:

$$g(f)\equiv\int_0^f M(f')df'\,. \tag{4.2.56}$$

It follows immediately from the last two equations that

$$\breve{g}(M)=g(f(M))\,,\quad M\in[0,M_*]\,, \tag{4.2.57}$$

which states that $\breve{g}(M)$ and $g(f(M))$ coincide in the domain of definition of $\breve{g}(M)$.

We can now see how this change in perspective from $f(M),g(M)$ to $M(f),g(f)$ affects the equations of motion (4.2.1). For instance, taking the lapse equation (4.2.1b) and using (4.2.57) we get

$$\left(\frac{d\Phi}{dx}\right)^2=1-\breve{g}(M)=1-g(f(M))\,,\quad M\in[0,M_*]\,. \tag{4.2.58}$$

Here the dilaton derivative is effectively set equal to a function of $f(M)$. But then we may as well forget M, declaring that this equation sets the dilaton derivative equal to a *function of* f:

$$\left(\frac{d\Phi}{dx}\right)^2(f) = 1 - g(f) = 1 - \int_0^f M(f')df' = P(f)\,, \qquad \forall\, f\,, \tag{4.2.59}$$

which coincides with (4.2.58) in the range $M \in [0, M_*]$, but it is also valid for *all* f.

What started as a subtle observation regarding the non-invertibility of $M(f)$, ended up revealing the true implications of the new parameterization: the original set of equations (4.2.1) and (4.2.5) are not exactly the same as (4.2.40) and (4.2.48). Both descriptions coincide in the range of validity of the perturbative series coming from $F(M)$ or $f(M)$. However, now the f parameterization represents an extension of the original equations, going beyond the limitations of the power series definition of the α' corrections! We will assume these provide the non-perturbative definition of the theory with α' corrections. Therefore, solutions of (4.2.40) or (4.2.48) can be non-perturbative, having no analog to solutions of (4.2.1) and (4.2.5). This will be the case for the regular black hole treated in Section 4.3.2.

4.2.3 The two-derivative black hole in the f parameterization

Here we work out the standard two-derivative BH solution of Section 4.1.1 in the Gasperini-Veneziano parametrization. A visual representation of the black hole regions in the f and x parameterization can be found in Fig. (4.1). Each patch will be explained in detail in this section. The experience gained here will prove essential for the later generalizations.

Exterior solution

The $M(f)$ ansatz for the exterior region of the two-derivative theory corresponds to the limit $\epsilon \to 0$ in (4.2.42):

$$M(f) = -\tfrac{1}{2}f\,. \tag{4.2.60}$$

Associated to M we have $P(f)$ given in (4.2.41):

$$P(f) = 1 + \tfrac{1}{4}f^2 \;>\; 0 \qquad \forall f \in \mathbb{R}\,, \tag{4.2.61}$$

which implies that all f are in principle allowed. The duality-invariant dilaton does not depend on the specific ansatz for $M(f)$, so it is still given by (4.2.40b). Using (4.2.61) in (4.2.40a), its x-derivative is given by

$$\frac{d\Phi}{dx} = \sqrt{1 + \tfrac{1}{4}f^2}\,, \tag{4.2.62}$$

where we choose the plus sign to fit the conventions of (4.1.8), where $x = 0^-$ is the horizon and the exterior region is $x < 0$, with the asymptotically flat region at $x \to -\infty$. To stick to this convention we also need to pick $f \geq 0$.

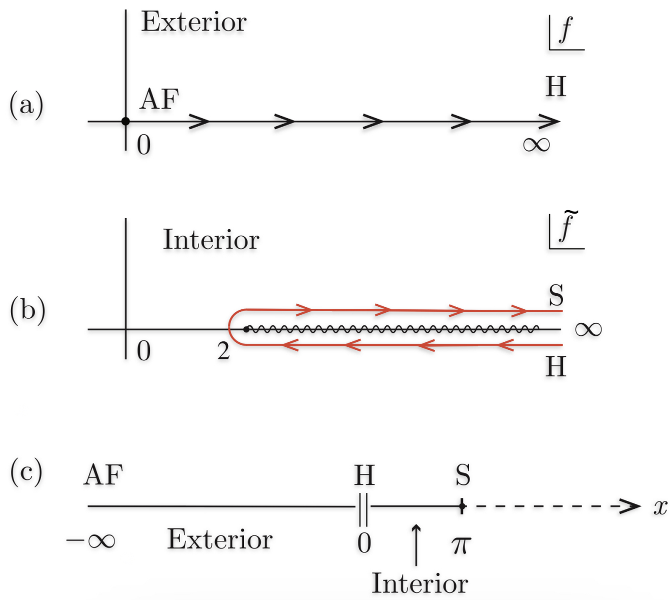


Figure 4.1: The two-derivative black hole. (a) The exterior is produced by a contour $f \in (0, \infty)$ joining the asymtotically flat (AF) region to the horizon (H). (b) The interior requires a contour going under and then over the cut, with branch point $\tilde{f} = 2$ and the horizon and singularity at $\tilde{f} = \infty$, under and over the cut, respectively. (c) The x-space representation of the black hole. The exterior is in $(-\infty, 0)$ and the interior in $(0, \pi)$.

The relation between x and f can be obtained from (4.2.40c), which for this ansatz takes the form

$$dx = \frac{df}{f} \frac{1}{\sqrt{1 + \frac{1}{4} f^2}} = -d \,\text{arcsinh}\, \tfrac{2}{f} \,. \tag{4.2.63}$$

Integrating and choosing the integration constant to be zero we get

$$x(f) = -\text{arcsinh} \tfrac{2}{f} \quad \Rightarrow \quad \tfrac{f}{2} = -\tfrac{1}{\sinh x} \,, \tag{4.2.64}$$

From here we see that x and f are positively correlated. Moreover, $x = -\infty$ is mapped to $f = 0^+$ and it corresponds to the faraway region, and $x = 0^-$ is mapped to $f = \infty$, the position of the horizon (see Fig. (4.1) (a) and (c)).

In order to determine the metric we take the plus sign in (4.2.40d), use (4.2.60) and (4.2.61), and choose the integration limits to be $f_2 = f$ and $f_1 = 0$:

$$\frac{m(f)}{m(f=0)} = \exp\left(-\frac{1}{2} \int_0^f \frac{df'}{\sqrt{1 + \frac{f'^2}{4}}} \right) = \exp\left(-\text{arcsinh} \tfrac{f}{2} \right) = -\tfrac{f}{2} + \sqrt{1 + \tfrac{f^2}{4}} \,. \tag{4.2.65}$$

Here we dropped the plus subscript so not to clutter the notation. From the last equality we see that we can simply take $m(f) = -\frac{f}{2} + \sqrt{1 + \frac{f^2}{4}}$ with $m(f = 0) = 1$ consistent with the asymptotically flat region interpretation. For large f, $m(f) \simeq 1/f$, so that $m(f = \infty) = 0$, consistent with $f = \infty$ being the position of the horizon.

We now compute the curvature from (4.2.40e), using the top sign and finding

$$\mathcal{R}(f) = f\left(-\frac{f}{2} + \sqrt{1+\frac{f^2}{4}}\right) = f m(f)\,. \tag{4.2.66}$$

Note that $\mathcal{R}(0) = 0$, as befits the asymptotically flat region, and $\mathcal{R}(f) \simeq 1$ for $f \to \infty$, which is the value of the curvature at the horizon as obtained in (4.1.11) for $Q \to 1$.

At this point we have $M(f), \Phi(f), m(f)$ and $\mathcal{R}(f)$, and so the full f-parameterized solution is determined. As a consistency check one may use the relation between f and x to recover the BH solution in the familiar form as given in (4.1.8). For the metric, for example, we have, using (4.2.64), and recalling that x is negative,

$$m(x) = \tfrac{1}{\sinh x} + \sqrt{1+\tfrac{1}{\sinh^2 x}} = \tfrac{1}{\sinh x}\left(1-\sqrt{1+\sinh^2 x}\right) = -\tanh\tfrac{x}{2}\,. \tag{4.2.67}$$

For the dilaton we have, since $f > 0$,

$$\Phi(x) = \log f(x) + \Phi_1 = \log\left(\tfrac{-2}{\sinh x}\right) + \Phi_1 = -\log|\sinh x| + \Phi_0\,. \tag{4.2.68}$$

The exterior solution can be visualized in a M-vs-$\frac{d\Phi}{dx}$ plot, both as a function of f, as shown in Fig. (4.2).

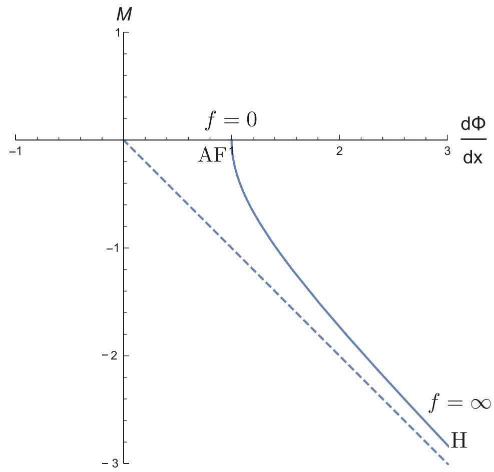


Figure 4.2: Two-derivative black hole exterior. The solid line is the plot of the dilaton derivative $d\Phi/dx$ and M as a function of $f \in [0,\infty)$. The point on the horizontal axis is $f = 0$ and represents the asymptotically flat region. As $f \to \infty$ we reach the horizon and the curve asymptotes to the dashed line $|M| = |\frac{d\Phi}{dx}|$.

Interior solution

The BH interior solution of (4.1.16) can also be recovered in the $\tilde{f}$ parameterization, using (4.2.48). The ansatz for $\tilde{M}(\tilde{f})$ is obtained by using the formula (4.2.50) on the ansatz for the exterior (4.2.60):

$$\tilde{M}(\tilde{f}) = \tfrac{1}{2}\tilde{f}\,. \tag{4.2.69}$$

The argument of the square root this time takes the form

$$\tilde{P}(\tilde{f}) = -1 + \tfrac{1}{4}\tilde{f}^2 \,, \tag{4.2.70}$$

and so the square root has branch cuts on the real line with $|\tilde{f}| > 2$, with branch points at $\tilde{f} = \pm 2$. Therefore, the interior solution is valid for the interval $\tilde{f} \in (-\infty, -2) \cup (2, \infty)$, where negative and positive regions are related via T-duality (4.2.52). From now on we choose $\tilde{f} \geq 2$.

In order to describe all of the internal region in the $\tilde{f}$-parameterization we need to cover the range $\tilde{f} \geq 2$ twice. We begin from $\tilde{f} = \infty$ to $\tilde{f} = 2$ traveling under the branch cut, and then we return from $\tilde{f} = 2$ to $\tilde{f} = \infty$ over the cut. The difference between paths is given by the choice of $\pm$ sign in front of the square root in the solution (4.2.48). The square root is assumed to take negative values below the cut ($-$ sign) and positive values above the cut ($+$ sign).

When going under the cut, the dilaton derivative is given by

$$\frac{d\tilde{\Phi}}{dx}(\tilde{f}) = -\sqrt{-1 + \tfrac{1}{4}\tilde{f}^2} \,. \tag{4.2.71}$$

The relation between x and $\tilde{f}$ in (4.2.48c) takes the form

$$dx = -\frac{d\tilde{f}}{\tilde{f}\sqrt{-1 + \frac{1}{4}\tilde{f}^2}} = -\frac{d\tilde{f}}{|\tilde{f}|\sqrt{-1 + \frac{1}{4}\tilde{f}^2}} = d\,\mathrm{arccsc}\,\tfrac{\tilde{f}}{2} \,, \quad \tilde{f} \geq 2 \,. \tag{4.2.72}$$

Integrating this equation and setting the integration constant to zero we get

$$x(\tilde{f}) = \mathrm{arccsc}\,\tfrac{\tilde{f}}{2} \quad \Rightarrow \quad \tfrac{\tilde{f}(x)}{2} = \frac{1}{\sin x} \,. \tag{4.2.73}$$

In this parameterization as $\tilde{f}$ decreases x increases. Indeed, we have:

$$\tilde{f} \in [\infty, 2]_- \;\to\; x \in [0, \tfrac{\pi}{2}] \,, \tag{4.2.74}$$

with the minus subscript indicating that we are traveling under the cut. As we will see, $\tilde{f} = \infty$ ($x = 0$) corresponds to the position of the horizon, while $\tilde{f} = 2$ ($x = \frac{\pi}{2}$) is just an intermediate point in the interior solution.

For the metric we need (4.2.48d) with the minus sign. Choosing the lower boundary to be $\tilde{f}_1 = 2$ and leaving the upper boundary arbitrary we get

$$\frac{\tilde{m}_-(\tilde{f})}{\tilde{m}_-(\tilde{f}=2)} = \exp\left(-\tfrac{1}{2}\int_2^{\tilde{f}} \frac{df}{\sqrt{-1 + \frac{1}{4}f^2}}\right) = \exp\left(-\mathrm{arccosh}\tfrac{\tilde{f}}{2}\right) = \tfrac{\tilde{f}}{2} - \sqrt{-1 + \tfrac{\tilde{f}^2}{4}} \,, \tag{4.2.75}$$

For the curvature one can show with (4.2.48e) that

$$\tilde{\mathcal{R}}_-(\tilde{f}) = \tilde{f}\left(\tfrac{\tilde{f}}{2} - \sqrt{-1 + \tfrac{\tilde{f}^2}{4}}\right) = \tilde{f}\tilde{m}_-(\tilde{f}) \,. \tag{4.2.76}$$

From (4.2.75) we see that $\tilde{f} = 2$ is just an intermediate point in the interior solution. In particular we can pick the values

$$\tilde{m}_-(2) = 1 \,, \quad \tilde{\mathcal{R}}_-(2) = 2 \,. \tag{4.2.77}$$

On the other hand, as anticipated, $\tilde{m}_-(\tilde{f})$ and $\tilde{\mathcal{R}}_-(\tilde{f})$ are consistent with $\tilde{f} = \infty$ being the location of the horizon since $\tilde{m}_-(\tilde{f}) \simeq 1/\tilde{f}$ for large $\tilde{f}$ and so

$$\lim_{\tilde{f}\to\infty} \tilde{m}_-(\tilde{f}) = 0\,, \qquad \lim_{\tilde{f}\to\infty} \tilde{\mathcal{R}}_-(\tilde{f}) = 1\,. \tag{4.2.78}$$

The other half of the solution is recovered by going over the cut. In this situation, the dilaton derivative (4.2.48b) and the relation between x and $\tilde{f}$ from (4.2.48c) have the opposite sign as in (4.2.71) and (4.2.72). For the latter we have

$$dx = -d\,\text{arccsc}\,\tfrac{\tilde{f}}{2} = d\,\text{arccos}\,\tfrac{2}{\tilde{f}} \quad \Rightarrow \quad x(\tilde{f}) = \text{arccos}\,\tfrac{2}{\tilde{f}} + c_0\,, \tag{4.2.79}$$

where after integration we picked a non-trivial integration constant c_0. This is necessary since at $\tilde{f} = 2$ we already had $x = \frac{\pi}{2}$ from under the cut, so by continuity we must have

$$\tfrac{\pi}{2} = \text{arccos}\,1 + c_0 = c_0 \quad \Rightarrow \quad c_0 = \tfrac{\pi}{2}\,. \tag{4.2.80}$$

Therefore,

$$x(\tilde{f}) - \tfrac{\pi}{2} = \text{arccos}\,\tfrac{2}{\tilde{f}} \quad \Rightarrow \quad \tfrac{2}{\tilde{f}(x)} = \cos(x - \tfrac{\pi}{2}) = \sin x\,. \tag{4.2.81}$$

Indeed we have $x(\tilde{f} = 2) = \frac{\pi}{2}$ and $x(\tilde{f} = \infty) = \pi$, since x grows with $\tilde{f}$. So we have

$$\tilde{f} \in [2, \infty]_+ \;\to\; x \in [\tfrac{\pi}{2}, \pi]\,, \tag{4.2.82}$$

where the subscript $+$ indicates going on top of the cut. Interestingly, comparing (4.2.73) and (4.2.81) we can see that the relation $\tilde{f}(x)$ is the same for under and over the cut. The former covers $x \in (0, \frac{\pi}{2})$ and the latter the remaining half of the interior solution $x \in (\frac{\pi}{2}, \pi)$.

This time, taking the plus sign choice in (4.2.48d) and (4.2.48e), we obtain the expressions for the metric and curvature over the cut:

$$\tilde{m}_+(\tilde{f}) = \tfrac{\tilde{f}}{2} + \sqrt{-1 + \tfrac{\tilde{f}^2}{4}}\,, \quad \tilde{\mathcal{R}}_+(\tilde{f}) = \tilde{f}\tilde{m}_+(\tilde{f})\,. \tag{4.2.83}$$

The intermediate point $\tilde{f} = 2$ ($x = \frac{\pi}{2}$) gives again (4.2.77), but this time the boundary $\tilde{f} = \infty$ ($x = \pi$) corresponds to the singularity, where

$$\lim_{\tilde{f}\to\infty} \tilde{m}_+(\tilde{f}) = \infty\,, \qquad \lim_{\tilde{f}\to\infty} \tilde{\mathcal{R}}_+(\tilde{f}) = \infty\,. \tag{4.2.84}$$

As a consistency check one may again verify that the solution in the form (4.1.16) follows from the relation between $\tilde{f}$ and x.

The $\tilde{f}$ contours and their relation with the different parts of the interior region can be found in Fig. (4.1) (b). A plot of the dilaton derivative $d\tilde{\Phi}/dx$ and the metric derivative $\tilde{M}$, both as a function of $\tilde{f}$ helps to visualize the solution (Fig. 4.3).

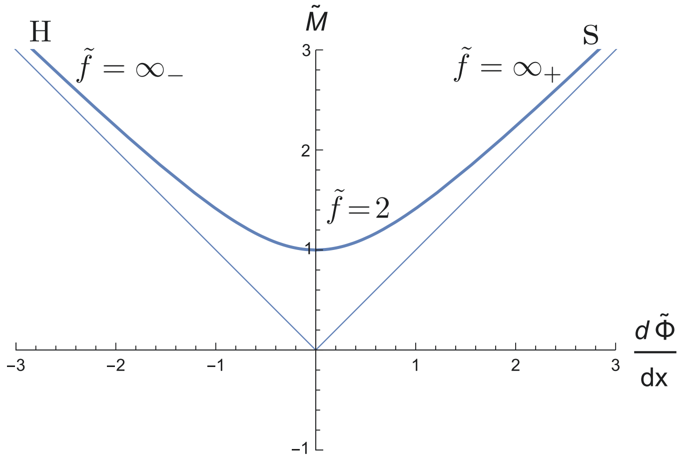


Figure 4.3: Two-derivative black hole interior. The solid line plots the relation between the dilaton spatial derivative and $\tilde{M}$. The horizon H is on the upper left, the singularity S on the upper right (both at $\tilde{f} = \infty$, below and above the cut). The branch point at $\tilde{f} = 2$ corresponds to the point with minimal $\tilde{M} = 1$. The $\tilde{f}$ contour is that shown in Fig.4.1 (b).

4.2.4 A family of singular black hole solutions

We now present a class of functions $M(f)$ that lead to BH solutions in the conventional sense [7]. They have an exterior and an interior region. The exterior includes the asymptotically flat region at $f = 0$ and the horizon sits at $f = \infty$. For the interior region (Fig. (4.4) (a)) we have, as for the standard BH, a branch point in $\tilde{f}$ space and a cut running from this point to infinity. We mimic the parameterization used for the two-derivative case in which the contour begins under the cut at infinity, goes down to the branch point, and then returns to infinity at the top of the cut. The horizon and the singularity, that is not removed in this class of solutions, are at $\tilde{f} = \infty$, below and above the cut, respectively. Both regions, we believe, form a single solution because the curvature is continuous across the horizon.

This class of $M(f)$'s serve as a proof of concept that α'-corrected BH solutions, in the conventional sense, can be obtained in this new parameterization. However, we do not claim to have found the most general solutions of this kind.

This family of solutions is parameterized by a function $h(\xi)$ via the relation

$$M(f) = -\tfrac{f}{2}\left[1 + h(f^2)\right] , \tag{4.2.85}$$

where we take

$$h(0) = 0 , \quad h'(0) < \infty , \tag{4.2.86}$$

conditions that allow making $f = 0$ the far away region of the black hole. We also require the correction to $M = -f/2$ implied by h to be "small". For this we impose[5]

$$|h(\xi)| < 1 , \qquad \forall \xi \in \mathbb{R} . \tag{4.2.87}$$

[5] A relaxation of this condition where $h(\xi)$ reaches 1 at some finite number of points is also possible. We do not explore that case here. We refer the interested reader to [7].

This implies, in particular, that $M(f) < 0$ for all $f > 0$. To get a horizon we need conditions on the behavior of h for large argument:

$$\lim_{\xi\to\infty} h(\xi) = 0\,, \qquad \int_0^\infty h(\xi)d\xi = \alpha < \infty\,, \qquad \lim_{\xi\to\infty} \xi h'(\xi) = 0\,. \tag{4.2.88}$$

The first condition implies that $M(f) \simeq -f/2$ for very large f, the second condition imposes the integral over positive arguments to be finite, and the third condition follows from the first for a regular function at infinity.

For the interior of the black hole, we need conditions on $h(\xi)$ for large negative ξ. At the cost of some generality, we simply demand $h(\xi)$ to be an odd function:

$$h(-\xi) = -h(\xi)\,. \tag{4.2.89}$$

Exterior solution

For the exterior solution, we have equations (4.2.40). Picking the plus sign and inserting the function $M(f)$ as given in (4.2.85), the equations for the metric and curvature reduce to

$$\frac{m(f_2)}{m(f_1)} = \exp\left(\int_{f_1}^{f_2} df\, I_e(f)\right), \qquad I_e(f) \equiv -\frac{1+h(f^2)}{2\sqrt{P(f)}}\,, \tag{4.2.90a}$$

$$\mathcal{R}(f) = -\frac{f}{2}\left[f(1+h(f^2))^2 - 2\left(1+h(f^2)+2f^2\frac{dh(f^2)}{df^2}\right)\sqrt{P(f)}\right], \tag{4.2.90b}$$

where we dropped the $+$ subscript. The location of branches is now determined by:

$$P(f) = 1 + \tfrac{1}{4}f^2 + \tfrac{1}{4}\int_0^{f^2} d\xi\, h(\xi)\,. \tag{4.2.91}$$

From the bound (4.2.87) on $h(\xi)$, we infer that

$$-f^2 < \int_0^{f^2} d\xi\, h(\xi) < f^2 \quad \Rightarrow \quad P(f) \geq 1 \quad \forall\, f \in \mathbb{R}\,, \tag{4.2.92}$$

showing that there are no branch cuts in the f plane and we can work with

$$f \in (0,\infty)\,. \tag{4.2.93}$$

We now consider the behavior of the integrand I_e. Using the vanishing of $h(\xi = 0)$ and the first two conditions in (4.2.88), we conclude

$$\lim_{f\to 0} I_e(f) = -\tfrac{1}{2}\,, \quad \lim_{f\to\infty} I_e(f) \simeq -\tfrac{1}{f} \quad \Rightarrow \quad \int_0^f df' I_e(f') \simeq -\log f\,, \text{ as } f \to \infty\,. \tag{4.2.94}$$

For the ratio of metric values, these limits imply

$$\frac{m(\infty)}{m(0)} = \exp\left(\int_0^\infty df\, I_e(f)\right) = \exp(-\infty) = 0\,, \tag{4.2.95}$$

which allows us to pick the integration constant such that

$$m(0) = 1\,, \quad m(\infty) = 0\,, \tag{4.2.96}$$

the former consistent with $f = 0$ being the faraway region and the latter consistent with $f = \infty$ being the horizon.

Additionally, as required, the curvature goes to zero for $f \to 0$ and is finite for $f \to \infty$. While the former is a general feature of solutions coming from the classification, the behavior at the horizon is a consequence of (4.2.88), which imply

$$\lim_{f\to\infty} \mathcal{R}(f) \simeq -\frac{f^2}{2}\left[1-\sqrt{1+\frac{4+\alpha}{f^2}}\right] \simeq 1+\frac{\alpha}{4}\,. \tag{4.2.97}$$

The scalar dilaton behaves as in the standard two-derivative black hole. For the far away region, we have $m(0) = 1$ and so from (4.2.40b) and (4.2.40f) we get $\phi(0) = -\infty$, indicating weak coupling. On the other hand, for large f we have the behaviors

$$\Phi(f) = \log f + \Phi_1\,, \quad m(f) \simeq -\log f + c'\,, \tag{4.2.98}$$

where the former is just the definition of the duality-invariant dilaton (4.2.40b) and the latter comes from the last relation in (4.2.94). Here c' is just a constant. Therefore, at the horizon $f = \infty$, from (4.2.40f) we have

$$\phi(\infty) \simeq \tfrac{1}{2}(\log f + \Phi_1 - \log f + c') \simeq c = \text{const.}\,, \tag{4.2.99}$$

a constant as in the standard black hole.

Interior solution

For the interior solution, we use (4.2.50) to find that the ansatz (4.2.85) gives

$$\tilde{M}(\tilde{f}) = -iM(-i\tilde{f}) = \tfrac{\tilde{f}}{2}\left(1+h(-\tilde{f}^2)\right) = \tfrac{\tilde{f}}{2}\left(1-h(\tilde{f}^2)\right)\,, \tag{4.2.100}$$

where the last equality follows because h is odd by assumption. The expressions for the metric ratios in (4.2.48d) and the curvature in (4.2.48e), become

$$\frac{\tilde{m}_\pm(\tilde{f}_2)}{\tilde{m}_\pm(\tilde{f}_1)} = \exp\left(\pm\int_{\tilde{f}_1}^{\tilde{f}_2} df\, I_i(\tilde{f})\right), \quad I_i(\tilde{f}) \equiv \frac{1-h(\tilde{f}^2)}{2\sqrt{\tilde{P}(\tilde{f})}}\,, \tag{4.2.101a}$$

$$\tilde{\mathcal{R}}_\pm(\tilde{f}) = \frac{\tilde{f}}{2}\left[\tilde{f}(1-h(\tilde{f}^2))^2 \pm 2\left(1-h(\tilde{f}^2)-2\tilde{f}^2\frac{dh(\tilde{f}^2)}{d\tilde{f}^2}\right)\sqrt{\tilde{P}(\tilde{f})}\,\right]\,, \tag{4.2.101b}$$

Regarding branch cuts, this time the range of $\tilde{f}$ depends on the roots of the function

$$\tilde{P}(\tilde{f}) = -1 + \tfrac{1}{4}\tilde{f}^2 - \tfrac{1}{4}\int_0^{\tilde{f}^2} d\xi\, h(\xi)\,. \tag{4.2.102}$$

This function picks the following values at the boundaries

$$\tilde{P}(0) = -1 < 0\,, \qquad \lim_{\tilde{f}\to\pm\infty} \tilde{P}(\tilde{f}) = \lim_{\tilde{f}\to\pm\infty} \left(-1 - \tfrac{1}{4}\alpha + \tfrac{1}{4}\tilde{f}^2\right) = \infty\,, \tag{4.2.103}$$

where we used the second equation of (4.2.88). This shows that $\tilde{P}(\tilde{f})$ must change sign as $\tilde{f}$ grows from zero. Moreover, $\tilde{P}(\tilde{f})$ grows monotonically since

$$\frac{d\tilde{P}}{d\tilde{f}} = \frac{\tilde{f}}{2}\left(1 - h(\tilde{f}^2)\right) > 0 \quad \forall\, \tilde{f} \in \mathbb{R}\,, \tag{4.2.104}$$

as a consequence of the "small-correction" assumption $|h| < 1$. Therefore, the range of validity of the solution is taken to be

$$\tilde{f} \in (f_0, \infty)\,, \tag{4.2.105}$$

where $f_0 > 0$ is the unique point at which $\tilde{P}(\tilde{f})$ changes signs, namely $\tilde{P}(f_0) = 0$, a branch point. As for the standard BH, the interior is parameterized by the range:

$$\tilde{f} \in [f_0, \infty]_- \,\cup\, [f_0, \infty]_+\,. \tag{4.2.106}$$

Here the minus subscript indicates going under the cut, and corresponds to the minus sign choice in (4.2.101). The plus subscript corresponds to traveling over the cut and corresponds to the plus sign in (4.2.101).

For the metric, from (4.2.101a) we have

$$\frac{\tilde{m}_\pm(\infty)}{\tilde{m}_\pm(f_0)} = \exp\left(\pm \int_{f_0}^{\infty} d\tilde{f}\, I_i(\tilde{f})\right). \tag{4.2.107}$$

The first half of the interior region is covered by traveling under the cut. This contains the horizon at $\tilde{f} = \infty$, and so we need the right-hand side of (4.2.107) to be zero for the minus sign in order to obtain $\tilde{m}_-(\infty) = 0$ and $\tilde{m}_-(f_0)$ finite. To show that this is indeed the case, let us consider the integral of I_i. At the branch point f_0 there is an integrable square-root singularity: $\tilde{P}(f_0) = 0$ and the derivative $\tilde{P}'(f_0)$ is nonzero, see (4.2.104). We thus need only focus on the behavior of I_i for large $\tilde{f}$. For the $\tilde{f} \to \infty$ limit, using (4.2.88) we find

$$I_i(\tilde{f}) \simeq \frac{1}{\tilde{f}} \quad \Rightarrow \quad \int_{f_0}^{\tilde{f}} d\tilde{f}'\, I_i(\tilde{f}') \simeq \ln \tilde{f}\,. \tag{4.2.108}$$

Finally, inserting this into (4.2.107) we get $\frac{\tilde{m}_\pm(\infty)}{\tilde{m}_\pm(f_0)} = e^{\pm\infty}$, from where we can choose a finite value at the branch point $\tilde{m}_+(f_0) = \tilde{m}_-(f_0) \equiv \tilde{m}_0$, so that we have

$$\tilde{m}_-(\infty) = 0\,, \quad \text{and} \quad \tilde{m}_+(\infty) = \infty\,. \tag{4.2.109}$$

The former corresponds to the desired behavior at the horizon. In the process, we got a singularity at $\tilde{f} = \infty$ over the cut, which is a unavoidable in this class of solutions. We will understand better this point later in Section 4.3.1.

The curvature at the branch point f_0 is easily evaluated since $\tilde{P}(f_0)$ vanishes:

$$\tilde{\mathcal{R}}_-(f_0) = \tilde{\mathcal{R}}_+(f_0) = 2\tilde{M}^2(f_0) = \tfrac{1}{2} f_0^2 \left(1 - h(f_0^2)\right)^2\,. \tag{4.2.110}$$

For the infinite limit we have, from (4.2.101b)

$$\tilde{\mathcal{R}}_{\pm}(\tilde{f}) \simeq \tfrac{1}{2}\tilde{f}^2\left[1 \pm \sqrt{1-\tfrac{4+\alpha}{\tilde{f}^2}}\;\right], \quad \tilde{f}\to\infty\,. \tag{4.2.111}$$

As a result we find

$$\lim_{\tilde{f}\to\infty}\tilde{\mathcal{R}}_{-}(\tilde{f}) = 1+\tfrac{\alpha}{4}\,, \quad \lim_{\tilde{f}\to\infty}\tilde{\mathcal{R}}_{+}(\tilde{f}) = \infty\,. \tag{4.2.112}$$

The first is the curvature at the horizon as computed from the interior. As required, it coincides with the curvature at the horizon computed from the exterior in (4.2.97). The second result is the infinite curvature at the singularity.

For the duality-invariant dilaton in the interior region we have $\tilde{\Phi} = \log\tilde{f} + \tilde{\Phi}_1$, which is well defined for $\tilde{f}\in(f_0,\infty)$. $\tilde{\Phi}$ does not differentiate between lower or upper branches. It diverges at $\tilde{f}\to\infty$ but it is finite for $\tilde{f}=f_0$. For the scalar dilaton we use

$$\tilde{\phi}_{\pm}(\tilde{f}) = \tfrac{1}{2}\left(\tilde{\Phi}(\tilde{f}) + \log\tilde{m}_{\pm}(\tilde{f})\right)\,, \tag{4.2.113}$$

which does distinguish branches through $\tilde{m}_{\pm}$. While it is clear that $\tilde{\phi}_{+}(f_0) = \tilde{\phi}_{-}(f_0) =$ finite, for $\tilde{f}\to\infty$ under and over the cut we need $\log\tilde{m}_{\pm}$. To this end, we combine (4.2.101a) with the value of the integral from (4.2.108) to get

$$\log\tilde{m}_{\pm}(\tilde{f}) \simeq \pm\log\tilde{f} + c_{\pm}\,, \quad \tilde{f}\to\infty\,, \tag{4.2.114}$$

where $c_{\pm}$ are constants.Plugging this expansion back into (4.2.113), together with the definition for $\tilde{\Phi}(\tilde{f})$ we find that for $\tilde{f}\to\infty$

$$\tilde{\phi}_{-}(\tilde{f}) \simeq \tilde{c}\,, \quad \tilde{\phi}_{+}(\tilde{f}) \simeq \log\tilde{f}\,. \tag{4.2.115}$$

The first one is consistent with the horizon interpretation, where we need $\tilde{c}=c$ with c in (4.2.99), in order to match exterior and interior regions. The behavior of $\tilde{\phi}_{+}$ close to infinity signals, once more, the presence of a singularity there.

To sum up, the general ansatz (4.2.85) parameterized with a function $h(\xi)$ satisfying (4.2.87), (4.2.88) and (4.2.89) lead to BH solutions. There are infinitely many functions satisfying these conditions. For example

$$h(\xi) = \xi\exp(-\xi^2) \tag{4.2.116}$$

is one of them, satisfying $\int_0^\infty h(\xi)d\xi = \frac{1}{2}$ and $|h(\xi)| \le \frac{1}{\sqrt{2e}} < 1$.

The way the more general solutions are parameterized in terms of f mimics the standard BH. The exterior corresponds to $f\in(0,\infty)$ with $f=0$ the faraway region and $f=\infty$ the horizon. The interior region is covered by two patches: $\tilde{f}\in(f_0,\infty)_{-}$ and $\tilde{f}\in(f_0,\infty)_{+}$, where the first goes under the branch cut (picking the minus sign in (4.2.101)) and the second over it (with the plus sign in (4.2.101)). We begin under the branch at $\tilde{f}=\infty$ (the horizon) until $\tilde{f}=f_0$, the branch point, where all quantities are finite. From there, we return to $\tilde{f}=\infty$ over the branch cut. This time, the metric and curvature diverge at this point, which is identified as the singularity.

4.3 On Black Hole Singularity Resolution

In the previous sections we combined the higher-derivative classification and the Gasperini-Veneziano parameterization to arrive at a non-perturbative all-order description of duality-invariant 2D backgrounds. This model is encoded in equations (4.2.40) and (4.2.48). From them, we recovered the standard 2D BH and found a family of α'-corrected extensions of such background. These are black holes in the conventional sense: they contain a horizon but also a singularity. Considering how broad the theory space contained in (4.2.48) is, at first sight it feels almost obvious that regular black holes should be part of such a landscape. We will demonstrate in this section that regular black holes are indeed possible, but they are not that easy to build as one could have thought [7].

We begin this section by showing that for dilaton-gravity theories in 2D time-independent backgrounds, T-duality implies that the dual of a solution with a regular horizon must have a curvature singularity. This result is a generalization of the early observation of Giveon [46] noting this fact for the two-derivative 2D BH. This result implies that there are two kind of BH solutions:

- ⊙ If the maximally extended BH is invariant under T-duality (self-dual), it must have a curvature singularity somewhere if it has a horizon.
- ⊙ If the maximally extended BH has no singularity, then it cannot be self-dual.

As we already analyzed, the two-derivative black hole solution is of the first kind: it is self-dual, and T-duality maps the horizon to the singularity. We will see that the family of α'-corrected black holes are also of the first kind, as a consequence of the branch structure used for the interior solution.

The solutions of the second kind would be "regular" but they would still be physically equivalent (in the sense of string theory) to its T-dual, which necessarily would exhibit a curvature singularity. From the point of view of general relativity, however, the regular solution would be disconnected of the singular region and so a point-particle in the regular region could never reach the singularity. In Section 4.3.2 we build, what we believe it is, a solution of the second kind, a regular black hole, obtained by engineering an $M(f)$ that makes dual regions disconnected. The representation of such regular solutions in $\tilde{f}$-and x-space can be visualized in Fig (4.4). We will explain these contours in detail in Section 4.3.2.

4.3.1 A horizon implies a singularity

We consider dilaton-gravity 2D backgrounds with a generic real metric of the form:

$$ds^2 = -m^2(x)dt^2 + n^2(x)dx^2 \,. \tag{4.3.1}$$

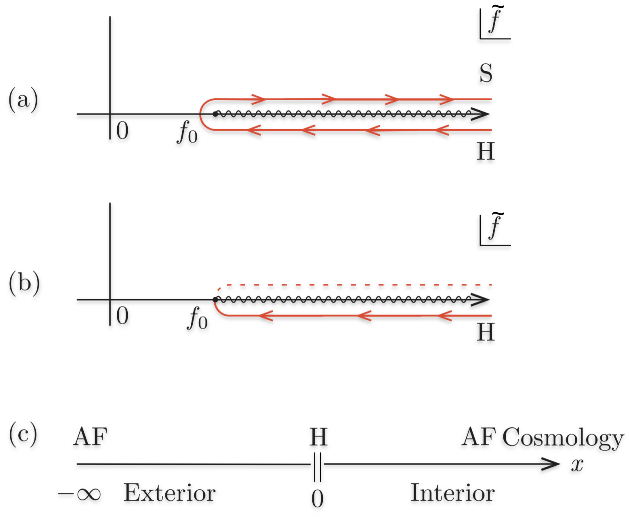


Figure 4.4: (a) The interior of a deformed black hole with a full contour containing a horizon H and a singularity S. (b) The interior of the regularized black hole, without a singularity. The bottom contour containing the horizon ends at f_0 and the top contour (dotted) is not needed for completeness. (c) The x space representation of the regularized black hole.

We assume the action and equations of motions describing these backgrounds are duality-invariant, but otherwise completely generic. We assume T-duality acts in the standard form, sending $m \to 1/m$ and leaving the lapse and dilaton invariant.

To prove that a horizon implies a singularity in the dual geometry we consider a configuration where: 1) the horizon is the point x_h at which $m(x)$ vanishes and 2) regularity means the scalar curvature is finite everywhere. Without loss of generality, we take $x_h = 0$. Moreover, since the conclusions should be gauge independent, we pick $n = 1$ for simplicity. In this gauge, the curvature and its dual are given by

$$\mathcal{R}(x) = -2\frac{m''(x)}{m(x)}\,, \quad \hat{\mathcal{R}}(x) = -\mathcal{R}(x) - 4\left(\frac{m'(x)}{m(x)}\right)^2\,, \tag{4.3.2}$$

where prime denotes derivative with respect to the argument. The second identity arises from computing the dual curvature via $\hat{\mathcal{R}}(m) \equiv \mathcal{R}(\hat{m}) = \mathcal{R}(1/m)$, and comparing it with $\mathcal{R}(m)$.

We define the neighborhoods $N_\epsilon = (0, \epsilon)$, and $\bar{N}_\epsilon = [0, \epsilon)$ where we include the point 0. We assume that there is an ϵ sufficiently small such that the following hold:

1. The function $m(x)$ is continuous in $\bar{N}_\epsilon$ and vanishing at $x = 0$. This is an isolated zero: the neighborhood $\bar{N}_\epsilon$ does not contain another zero of m.

2. The curvature $\mathcal{R}(x)$ in $\bar{N}_\epsilon$ is finite.

These are the natural conditions satisfied by a conventional horizon.

Claim: *The dual curvature $\hat{\mathcal{R}}$ cannot be finite in $\bar{N}_\epsilon$.*

Proof: Since $\mathcal{R}$ and $\hat{\mathcal{R}}$ are related via (4.3.2) and $\mathcal{R}$ is assumed finite in $\bar{N}_\epsilon$ this means that the finiteness condition

$$\frac{m'(x)}{m(x)} < \infty\,, \qquad \forall x \in \bar{N}_\epsilon\,, \tag{4.3.3}$$

cannot hold. Let us assume it holds and derive a contradiction.

Since the sign of $m(x)$ is merely conventional, we can consider $m(x) > 0$ for $x \in N_\epsilon$. This follows from the continuity of m and the isolated zero. Given the positivity of $m(x)$ in $\bar{N}_\epsilon$ we can write $m(x)$ in terms of its logarithm $L(x)$:

$$m(x) = e^{L(x)}\,, \quad \text{with} \;\; m(0) = 0 \quad \Rightarrow \quad L(0) = -\infty\,. \tag{4.3.4}$$

Since $m(x) \neq 0$ in N_ϵ, then $L(x)$ is finite in N_ϵ. Taking a derivative of m we have

$$\frac{m'(x)}{m(x)} = L'(x)\,. \tag{4.3.5}$$

The hypothesis (4.3.3) implies $L'(x) < \infty$ in $\bar{N}_\epsilon$. Now consider the relation:

$$L(0) = L(x) - \int_0^x L'(u)du\,, \quad \forall\; x \in N_\epsilon\,, \tag{4.3.6}$$

and examine the right-hand side above. For $x \in N_\epsilon$, the function $L(x)$ is finite. Moreover, since $L'(x)$ is finite in $\bar{N}_\epsilon$ and x is finite, the integral is also finite. But with both terms on the right-hand side finite, we cannot have $L(0) = -\infty$ as required. This contradiction means that (4.3.3) does not hold, $L'(x)$ needs to diverge so the integral above gives an infinite $L(0)$. This is what we wanted to show. □

The result above is reasonable: In one dimension, T-duality maps $m(x) \to 1/m(x)$, which makes a vanishing metric unavoidably dual to a divergent one. All we had to prove was that such coordinate singularity was in fact a curvature singularity.

Corollary

This result actually gives a simple proof that the interior solutions coming from (4.2.48), based on the contour extending below and above the branch cut in $\tilde{f}$ space, must lead to a singularity. The main observation here is simple. Recall that $\tilde{\mathcal{R}}$ is given in (4.2.7) and its dual $\hat{\tilde{\mathcal{R}}}$ is obtained by letting $\tilde{M} \to -\tilde{M}$:

$$\tilde{\mathcal{R}} \;=\; 2\tilde{M}^2 + 2\partial_x \tilde{M}\,, \quad \hat{\tilde{\mathcal{R}}} \;=\; 2\tilde{M}^2 - 2\partial_x \tilde{M}\,. \tag{4.3.7}$$

Compare with equation (4.2.48e) for the curvatures above (+) and below (−) the cut:

$$\tilde{\mathcal{R}}_\pm(\tilde{f}) = 2\tilde{M}^2 \pm 2\tilde{f}\,\frac{d\tilde{M}}{d\tilde{f}}\sqrt{\tilde{P}(\tilde{f})}\;\;. \tag{4.3.8}$$

Since $\tilde{M}(\tilde{f})$ and $\tilde{P}(\tilde{f})$ are functions without branches, each term in the above curvature formula takes the same value above or below the cut; the only difference being that they enter with different sign combinations for the curvature and for its dual. It thus follows that we can identify $\tilde{\mathcal{R}} = \tilde{\mathcal{R}}_+$ and $\hat{\tilde{\mathcal{R}}} = \tilde{\mathcal{R}}_-$, and so the curvatures above and below the cut are related by T-duality! This means that interior solutions where the $\tilde{f}$ contour defining the x domain extends both above and below the cut are actually self-dual. From our recent proof, having a horizon below the cut (as we conventionally set it up) will imply a curvature singularity above the cut.

From this simple corollary, it is clear now why the standard BH and the family of α'-corrected BH solutions found in Section 4.2.4 were singular. In the following section we will circumvent this seemingly no-go result for duality-invariant regular black holes, by building solutions whose interior regions are not self-dual. To this end, we will cover the interior with just one $\tilde{f}$ contour in a consistent way.

4.3.2 Regular black hole solutions

We begin from the same ansatz for $M(f)$ as in Section 4.2.4

$$M(f) = -\tfrac{1}{2} f \left[1 + h(f^2)\right] , \tag{4.3.9}$$

with $h(\xi)$ still satisfying (4.2.86), (4.2.88), and the odd property (4.2.89). In this case, the interior solution is parameterized by

$$\tilde{M}(\tilde{f}) = \tfrac{1}{2} \tilde{f} \left[1 - h(\tilde{f}^2)\right] . \tag{4.3.10}$$

The difference with the singular BH's studied so far comes from modifying condition (4.2.87). This time, we will allow for $h(\xi)$ to reach one at a single point $\xi_0 = f_0^2$, the same point at which the function $\tilde{P}(f)$ first goes from negative to positive:

$$|h(\xi)| < 1 , \quad \forall \, \xi \in \mathbb{R} - \{\xi_0\} , \quad h(\xi_0) = 1 . \tag{4.3.11}$$

On top of this, we demand that the extra conditions

$$h(\xi_0) = 1 , \quad h'(\xi_0) = 0 , \quad h''(\xi_0) < 0 , \quad \text{and} \quad \int_0^{\xi_0} d\xi \, h(\xi) = \xi_0 - 4 . \tag{4.3.12}$$

We will confirm that with these conditions $\tilde{P}(f_0) = 0$. This fact and the motivation for the above set of conditions will be explained below.

We will now see how the equations (4.2.40) and (4.2.48) are indeed solved by an $M(f)$ satisfying the above conditions, and that solution is a regular black hole. We will show that the interior region is a cosmology that at late times is asymptotic to Minkowski space with a constant dilaton. This behavior suggests strongly that this is not a solution in string theory.

Exterior solution

The additional conditions (4.3.12) preserve all good features of the exterior BH solution. In particular, for the exterior region f_0 has no special meaning, it is just a finite intermediate point in the solution's domain. The object that determines the latter is the argument of the square root:

$$P(f) = 1 + \tfrac{1}{4}f^2 + \tfrac{1}{4}\int_0^{f^2} d\xi\, h(\xi)\,, \tag{4.3.13}$$

and the domain $f \in (0, \infty)$ we used for the singular black holes remains unchanged since $P(f)$ is still positive for all reals, including f_0. The analysis of the exterior solution in Section (4.2.4) remains unchanged, and the results for the metric, curvature, and dilaton derived there hold here as well.

Interior solution

For the interior solution, the role of f_0 is crucial to get a regular solution. The idea is that the conditions (4.3.12) render the position $x(f_0)$ infinitely faraway. Therefore, after going from the horizon ($\tilde{f} = \infty$) to $\tilde{f} = f_0$ under the cut, we do not need to go back to $\tilde{f} = \infty$ on the top of the cut, where the singularity always was. Now, we can just stop at f_0 because in terms of x this is the end of spacetime (see Fig (4.4) (b)).

We begin with the function $\tilde{P}(\tilde{f})$ which takes the form

$$\tilde{P}(\tilde{f}) = -1 + \int_0^{\tilde{f}} \tilde{M}(\tilde{f}')d\tilde{f}' = -1 + \tfrac{1}{4}\tilde{f}^2 - \tfrac{1}{4}\int_0^{\tilde{f}^2} d\xi\, h(\xi)\,, \tag{4.3.14}$$

and attains the following values

$$\tilde{P}(0) = -1\,, \quad \tilde{P}(f_0) = 0\,, \quad \tilde{P}(\tilde{f}) \simeq \tfrac{1}{4}\tilde{f}^2 \;\text{ for }\; \tilde{f} \to \infty\,. \tag{4.3.15}$$

The first equality is manifest, the second equality follows from the last condition in (4.3.12), and the last one from the convergence of $\int_0^\infty h(\xi)d\xi$. To understand the nature of the interior solution we must consider derivatives of $\tilde{P}(\tilde{f})$. The first derivative coincides with $\tilde{M}(\tilde{f})$:

$$\tilde{P}'(\tilde{f}) = \tilde{M}(\tilde{f}) = \tfrac{\tilde{f}}{2}\left[1 - h(\tilde{f}^2)\right] \geq 0 \quad \forall\, \tilde{f} \geq 0\,, \tag{4.3.16}$$

where the inequality follows from $|h| \leq 1$. We then have the following values

$$\tilde{P}'(0) = \tilde{M}(0) = 0\,, \quad \tilde{P}'(f_0) = \tilde{M}(f_0) = 0\,, \quad \tilde{P}'(\tilde{f}) = \tilde{M}(\tilde{f}) \simeq \tfrac{1}{2}\tilde{f} \;\text{ for }\; \tilde{f} \to \infty\,. \tag{4.3.17}$$

The first equality is manifest, the second follows from $h(\xi_0) = 1$, the third from the vanishing of h for large argument. The second derivative is given by

$$\tilde{P}''(\tilde{f}) = \tilde{M}'(\tilde{f}) = \tfrac{1}{2}\big(1 - h(\xi)\big) - \xi\, h'(\xi)\,, \tag{4.3.18}$$

using $\xi = \tilde{f}^2$. We obtain

$$\tilde{P}''(\tilde{f}=0) = \tilde{M}'(\tilde{f}=0) = \tfrac{1}{2}\,, \quad \tilde{P}''(f_0) = \tilde{M}'(f_0) = 0\,. \tag{4.3.19}$$

The second equality holds because $h(\xi_0) = 1$ and $h'(\xi_0) = 0$. For the third derivative we have

$$\tilde{P}'''(\tilde{f}) = \tilde{M}''(\tilde{f}) = -3\tilde{f}h'(\xi) - 2\tilde{f}^3 h''(\xi)\,. \tag{4.3.20}$$

The special values here are

$$\tilde{P}'''(\tilde{f}=0) = \tilde{M}''(\tilde{f}=0) = 0\,, \quad \tilde{P}'''(f_0) = \tilde{M}''(f_0) = -2f_0^3\, h''(\xi_0) > 0\,, \tag{4.3.21}$$

the second relation following from the assumption $h''(\xi_0) < 0$.

All these results can be represented collectively in the two plots of Fig.4.5, where we show $\tilde{M}(\tilde{f})$ and $\tilde{P}(\tilde{f})$.

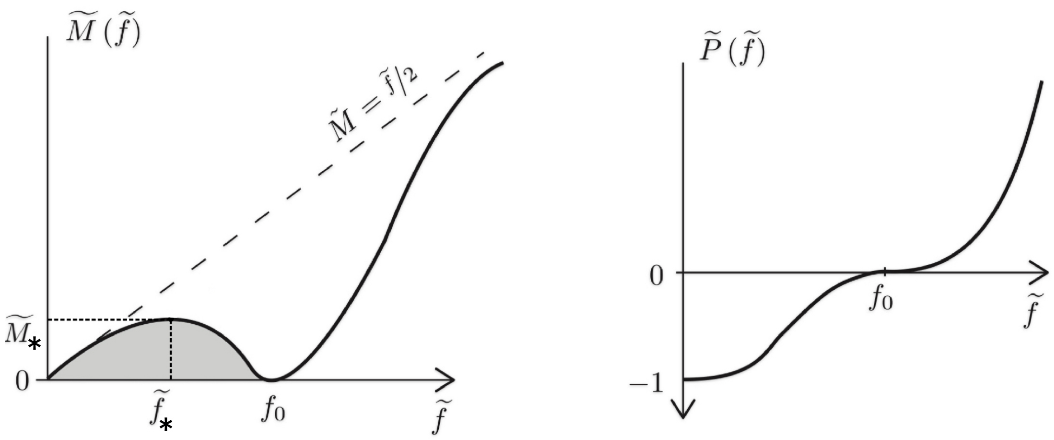


Figure 4.5: Left: Sketch of $\tilde{M}(\tilde{f})$. The shaded region must have unit area, the point $\tilde{f}_*$ indicates the position of the first local maximum $\tilde{M}(\tilde{f}_*) = \tilde{M}_*$, the point f_0 is a minimum, and for large $\tilde{f}$ the curve approaches the line $\tilde{f}/2$. Right: Sketch of $\tilde{P}(\tilde{f})$. This function changes sign at f_0, which is an inflection point.

The plot of $\tilde{M}$ (left) shows a function that begins at zero with positive slope and remains strictly positive until f_0 where $\tilde{M}(f_0) = \tilde{M}'(f_0) = 0$ and $M''(f_0) > 0$, which makes f_0 a minimum. The area under the curve in the interval $(0, f_0)$ is exactly one because $\tilde{P}(f_0) = -1 + \int_0^{f_0} d\tilde{f}\tilde{M}(\tilde{f}) = 0$. For $\tilde{f} > f_0$ the function remains always positive (see (4.3.16)) and approaches infinity exactly as the standard BH.

Let us now turn to the graph of $\tilde{P}(\tilde{f})$. This function equals -1 for $\tilde{f} = 0$, the minimum of $\tilde{P}(\tilde{f})$. The function then increases with $\tilde{f}$ until f_0, where it vanishes together with its first and second derivative. Since the third derivative is nonzero, in fact, $\tilde{P}'''(f_0) > 0$, we see that f_0 is an inflection point. Afterwards, $\tilde{P}(\tilde{f})$ keeps increasing monotonically ((4.3.16)), and at infinity it behaves as $\tilde{f}^2/4$.

Note now that with $h(\xi)$ as discussed, the domain for the interior solution is $\tilde{f} \in (f_0, \infty)$. Since $\tilde{P}(f_0) = \tilde{P}'(f_0) = \tilde{P}''(f_0) = 0$, the function $\tilde{P}(\tilde{f})$ near f_0 is given by

$$\tilde{P}(\tilde{f}) \simeq \tfrac{1}{6}\tilde{P}'''(f_0)(\tilde{f}-f_0)^3 \;=\; -\tfrac{1}{3}f_0^3 h''(f_0^2)(\tilde{f}-f_0)^3 > 0 \qquad \text{for} \quad \tilde{f} \sim f_0\,. \tag{4.3.22}$$

This behavior will allow us to regularize the BH's interior.

We first confirm the presence of the horizon in this interior solution. To this end, consider equation (4.2.48d), pick the minus sign and integrate to get

$$\frac{\tilde{m}(\infty)}{\tilde{m}(f_0)} = \exp\left(-\int_{f_0}^{\infty} d\tilde{f}\,\frac{1-h(\tilde{f}^2)}{2\sqrt{\tilde{P}(\tilde{f})}}\right), \tag{4.3.23}$$

where we dropped the minus subscript since this interior solution is covered by a single branch. The integral is convergent at the lower limit since the integrand there goes as $\sqrt{\tilde{f}-f_0}$. At the upper limit we get a logarithmic divergence. Thus, the integral is infinite and the right-hand side of (4.3.23) vanishes. Since $\tilde{m}(f_0)$ is finite we can choose

$$\tilde{m}(f_0) = 1\,, \quad \tilde{m}(\infty) = 0\,, \tag{4.3.24}$$

which is the desired behavior for a horizon at $\tilde{f} = \infty$.

For the curvature, we can use (4.3.8) with the lower sign, as we are working below the cut,

$$\tilde{\mathcal{R}}_-(\tilde{f}) = 2\Big[(\tilde{P}'(\tilde{f}))^2 - \tilde{f}\tilde{P}''(\tilde{f})\sqrt{\tilde{P}(\tilde{f})}\,\Big]\,, \tag{4.3.25}$$

where we rewrote the dependence on $\tilde{M}$ in terms of derivatives of $\tilde{P}$. The curvature vanishes at $\tilde{f} = f_0$, because $\tilde{P}, \tilde{P}'$, and $\tilde{P}''$ all vanish at f_0. For $\tilde{f} \to \infty$ the behavior of $\tilde{P}$ is the same as we had for the singular solutions and therefore we get again the first expression in (4.2.112). In summary, we have

$$\tilde{\mathcal{R}}_-(f_0) = 0\,, \quad \tilde{\mathcal{R}}_-(\infty) = 1 + \tfrac{\alpha}{4}\,. \tag{4.3.26}$$

For the dilaton we have, as usual, $\tilde{\Phi}(\tilde{f}) = \log \tilde{f} + \tilde{\Phi}_1$, which is finite for f_0 and diverges for $\tilde{f} \to \infty$. The scalar dilaton is given by $\tilde{\phi} = \frac{1}{2}(\tilde{\Phi} + \log \tilde{m})$. As opposed to $\tilde{\Phi}$, it has no divergence as $\tilde{f} \to \infty$. The argument follows identically as the one used in Section 4.2.4 to arrive at (4.2.115). For $\tilde{f} = f_0$, $\tilde{m}(f_0) = 1$ and so the scalar dilaton is finite

$$\tilde{\phi}(f_0) = \tfrac{1}{2}\tilde{\Phi}(f_0)\,. \tag{4.3.27}$$

Finally, the relation between x and $\tilde{f}$ is very different from that in the family of singular black holes: the point half-way from the horizon to the singularity, f_0, is moved to infinite distance. Using the behavior (4.3.22) of $\tilde{P}(\tilde{f})$ close to f_0, the differential equation (4.2.40c) relating x and $\tilde{f}$ takes the form

$$dx = -d\tilde{f}\frac{1}{\tilde{f}\sqrt{\tilde{P}(\tilde{f})}} \simeq -d\tilde{f}\sqrt{\frac{3}{f_0^5|h''(\xi_0)|}}\,(\tilde{f}-f_0)^{-\frac{3}{2}} \;\;\text{for}\;\; \tilde{f} \sim f_0\,. \tag{4.3.28}$$

This integral diverges as $\tilde{f} \to f_0$. On the other hand, for $\tilde{f} \to \infty$ we get a finite integral. This allows us to define $x(\tilde{f})$ with the following values at the boundaries

$$x(f_0) = \infty\,, \quad x(\infty) = 0\,. \tag{4.3.29}$$

These results make the interior region to be a regular cosmology, identifying x with a time coordinate τ and t with a spatial coordinate w, as required by the signature of the metric

$$ds^2 = \tilde{m}^2(x)dt^2 - dx^2 = -d\tau^2 + \tilde{m}^2(\tau)dw^2\,. \tag{4.3.30}$$

This cosmology begins at time $\tau = 0$ with finite curvature $\tilde{\mathcal{R}}(\tau = 0) = 1 + \frac{\alpha}{4}$ and evolves as $\tau \to \infty$ to a geometry with zero curvature and hence to flat space. In this cosmology there is no big-bang, and time begins at the position of the horizon.

In this cosmology, the asymptotically flat space has a *constant* dilaton (4.3.27). This may seem surprising, given the well-known fact that the "cosmological" term in the 2D action requires a linear dilaton when the spacetime becomes flat. Indeed, this is the case for the exterior region of all BH solutions we explored so far, including the regular black hole. Let's now see how the non-perturbative nature of the f parameterization, anticipated at the end of Section 4.2.2, is the main responsible of this exotic behavior at infinity.

We begin considering the relations at the exterior (4.2.59) and (4.2.57)

$$\left(\frac{d\Phi}{dx}\right)^2 = 1 - g(f) \ = P(f)\,, \tag{4.3.31a}$$

$$\breve{g}(M) \ = \ g(f(M))\,, \qquad M \in [0, M_*]\,, \tag{4.3.31b}$$

and their interior counterpart:

$$\left(\frac{d\tilde{\Phi}}{dx}\right)^2 = -1 + \tilde{g}(\tilde{f}) = \tilde{P}(\tilde{f})\,. \tag{4.3.32a}$$

$$\tilde{\breve{g}}(\tilde{M}) \ = \ \tilde{g}(\tilde{f}(\tilde{M}))\,, \qquad \tilde{M} \in [0, \tilde{M}_*]\,, \tag{4.3.32b}$$

Here $\tilde{M}_*$ is the value beyond which $\tilde{f}(\tilde{M})$ is not expected to have a convergent series. Looking at Fig. 4.5, $\tilde{M}_*$ is the maximum of $\tilde{M}(\tilde{f})$ attained at $\tilde{f}_* \in (0, f_0)$.

Consider first the exterior, whose asymptotic region lies at $f \sim 0$ and $M \sim 0$. Here $P(0) = 1$ (see (4.3.13)), and therefore $g(0) = 0$ ((4.3.31a)). This is consistent with (4.3.31b): perturbatively $\breve{g}(M) = -M^2 + \cdots$ which gives (left-hand side) $\breve{g}(0) = 0$ consistent with (right-hand side) $g(f(0)) = g(0) = 0$. Indeed, this requires a rolling dilaton ((4.3.31a)).

Consider now the interior, and its asymptotically flat region at $\tilde{f} \sim f_0$ and $\tilde{M} \sim 0$. Here $\tilde{P}(f_0) = 0$ ((4.3.15)), therefore $\tilde{g}(f_0) = 1$ ((4.3.32a)), and there is no need for a rolling dilaton. The value $\tilde{g}(f_0) = 1$ may sound surprising given that $\tilde{M}(f_0) = 0$ and $\tilde{\breve{g}}(\tilde{M} = 0) = 0$. There is no contradiction, however, with (4.3.32b): $\tilde{\breve{g}}(\tilde{M}) = \tilde{M}^2 + \cdots$ gives (left-hand side) $\tilde{\breve{g}}(\tilde{M} = 0) = 0$ consistent with (right-hand side) $\tilde{g}(\tilde{f}(0)) = \tilde{g}(0) = 0$. Since both $\tilde{M}(0)$ and $\tilde{M}(f_0)$ are zero, the inverse function $\tilde{f}(\tilde{M})$ is necessarily multivalued. In the domain of definition of the perturbative expansion $\tilde{f}(0) = 0$, so this expansion cannot see what is happening for $\tilde{f} \sim f_0$.

The above discussion shows how this regular BH convincingly emerges from a suitable choice of $M(f)$. Still, the interior solution approaches a flat background with fixed dilaton, which is associated with a $c = 2$ CFT. Since classical backgrounds of (bosonic) string field theory are $c = 26$ CFT's, one does not expect the regular BH to be a string theory solution. More likely, the choice of $M(f)$ leading to it defines higher-derivative corrections that do not occur in string theory.

A particular example

As a proof of the existence of such a regular solution, we built a concrete example:

$$h(\xi) = \frac{16\,\xi_0^3\,\xi}{(\xi^2 + 3\xi_0^2)^2}\,. \tag{4.3.33}$$

Here ξ_0 is a parameter that will be adjusted to make h satisfies all required conditions. The derivatives of h are

$$h'(\xi) = \frac{48\,\xi_0^3\,(\xi_0^2 - \xi^2)}{(\xi^2 + 3\xi_0^2)^2}\,, \quad h''(\xi) = \frac{192\,\xi_0^3\,\xi(\xi^2 - 3\xi_0^2)}{(\xi^2 + 3\xi_0^2)^4}\,. \tag{4.3.34}$$

The following conditions are straightforward to check: $h(\xi)$ is odd, $h(0) = 0$, $h'(0)$ is finite and $h(\xi) < 1$ for all ξ, except $|h(\pm\xi_0)| = 1$. The integral of h is easily calculated

$$\int_0^{\xi} d\xi'\,h(\xi') = \frac{8\xi_0}{3}\frac{\xi^2}{\xi^2 + 3\xi_0^2} \quad \Rightarrow \quad \int_0^{\infty} d\xi'\,h(\xi') = \frac{8\xi_0}{3} = \alpha < \infty\,, \tag{4.3.35}$$

and with it the second condition in (4.2.88) is obeyed. The first and third conditions (vanishing of h and $\xi h'$ as $\xi \to \infty$) are also satisfied.

We now need to check the extra conditions (4.3.12) concerning the point ξ_0, where $h(\xi_0) = 1$. The first three are easily corroborated, in particular $h''(\xi_0) = -\frac{3}{2}\frac{1}{\xi_0^2} < 0$, as required. The integral constraint in (4.3.12) is not automatically satisfied, but it can be fulfilled by fixing the value of ξ_0, which also fixes α

$$\int_0^{\xi_0} d\xi'\,h(\xi') = \tfrac{2\xi_0}{3} = \xi_0 - 4 \quad \Rightarrow \quad \xi_0 = 12 \quad \Rightarrow \alpha = 32\,. \tag{4.3.36}$$

Fig. 4.6 shows the plot of $d\tilde{\Phi}/dx$ and $\tilde{M}$ as a function of $\tilde{f}$ for this regular black hole (continuous line). Superposed in the figure we see the analogous plot (Fig. 4.3) for the two-derivative black hole (dashed lines).

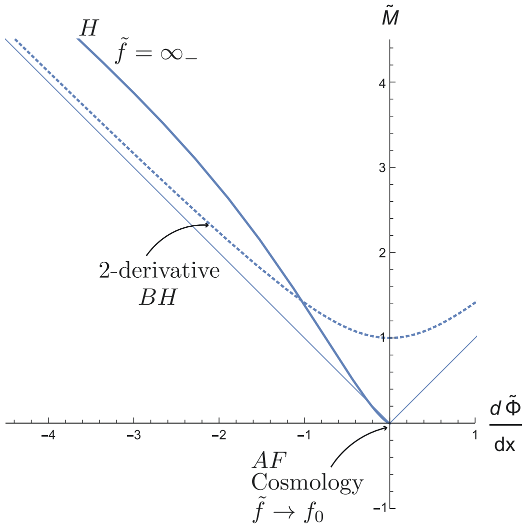


Figure 4.6: Parametric plot for $\frac{d\tilde{\Phi}}{dx}$ and $\tilde{M}$ both as functions of $\tilde{f} \in (12, \infty)$ for the interior of the black hole with resolved singularity (continuous line). Shown dashed is the analogous plot for the two-derivative black hole. The faint lines are asymptotes at $45°$ and $135°$. The horizon is far on the $135°$ asymptote, and the late-time asymptotically Minkowski part of the cosmology is around the origin.

Chapter 5

String Worldsheet and β Functions

So far we have been dealing with string low energy effective actions. These target-space theories can be derived from the string worldsheet via string scattering amplitudes or beta function computations. In this chapter we study the bosonic string worldsheet action and develop the beta function approach in detail. We begin in Section 5.1 introducing a general procedure to read the Weyl anomaly coefficients of a renormalizable two-dimensional sigma model, whose vanishing determines the target-space equations for the background fields. These coefficients depend on the beta functions of the sigma model, which can be computed using the background field method. After introducing such procedure, we finish the section by computing the one-loop beta function for dilaton-gravity worldsheet and obtaining the corresponding two-derivative target-space action. In Section 5.2 we perform a dimensional reduction of Polyakov action to cosmological backgrounds, present the general form of the Weyl anomaly coefficients and restrict the background field method to the cosmological case. We compute the one-loop beta functions and with them the target-space equations, which coincide with the ones coming from the two-derivative cosmological action presented in Section 2.2.2. In Section 5.3 we present the main result of this chapter: the two-loop beta function of cosmological Polyakov. With it, we derive the target-space cosmological action (3.1.1) for bosonic string up to order α' [4]. This provides an independent check of the $\mathcal{O}(\alpha')$ coefficient in the cosmological classification computed in [3, 39, 42] (and revisited in Section 3.2.2), where this coefficient was computed by direct dimensional reduction.

Some parts of Section 5.1 were already published in [2], while Section 5.2 and Section 5.3 contain results from [4].

5.1 Weyl Anomaly and β Functions

String theory is described by a two-dimensional worldsheet action encoding the dynamics of one-dimensional objects moving in a D-dimensional target space. For bosonic strings propagating on a flat background, we have the Polyakov action:

$$I \equiv \frac{1}{2\lambda} \int d^2\sigma \sqrt{h} \left[h^{\mathbf{ab}} \partial_{\mathbf{a}} X^\mu \partial_{\mathbf{b}} X^\nu \eta_{\mu\nu}\right] , \tag{5.1.1}$$

where σ^{a} with $\mathrm{a}, \mathrm{b} = 1, 2$ parameterize the two-dimensional worldsheet, h_{ab} is a Euclidean metric, h its determinant, and $\lambda \equiv 2\pi\alpha'$. The coordinates $X^\mu(\sigma)$ define a flat D-dimensional target space with Minkowski metric $\eta_{\mu\nu}$, where $\mu, \nu = 0, \ldots, D-1$. The Polyakov action is invariant under global Poincaré, two-dimensional reparameterization of σ^{a} and Weyl transformations, which rescale the worldsheet metric with a conformal factor $h_{\mathrm{ab}}(\sigma) \to \Omega^2(\sigma) h_{\mathrm{ab}}(\sigma)$. From the point of view of the worldsheet, (5.1.1) represents a theory of D scalar fields X^μ coupled to two-dimensional gravity. Using the gauge symmetries of the theory we can fix $h_{\mathrm{ab}}(\sigma) = \delta_{\mathrm{ab}}$, the flat Euclidean metric, and so (5.1.1) reduces to a free theory from which we can read the string spectrum. It contains infinitely many string excitations, among which we find the already-familiar universal massless sector: $G_{\mu\nu}(X), B_{\mu\nu}(X)$ and $\phi(X)$.

We can use these same fields as sources of curvature for the target-space in which the string propagates, extending the Polyakov action (5.1.1) to:

$$I = \frac{1}{2\lambda} \int d^2\sigma \left[\sqrt{h} h^{\mathrm{ab}} \partial_{\mathrm{a}} X^\mu \partial_{\mathrm{b}} X^\nu G_{\mu\nu}(X) + i\epsilon^{\mathrm{ab}} \partial_{\mathrm{a}} X^\mu \partial_{\mathrm{b}} X^\nu B_{\mu\nu}(X) + \alpha' \sqrt{h}\, R^{(2)} \phi(X) \right] , \tag{5.1.2}$$

giving rise to a non-linear sigma model where the background fields play the role of infinitely many coupling constants. The B-field enters the action through the antisymmetric density ϵ^{ab}, normalized such that $\epsilon^{12} = 1$[1]. The dilaton couples via $R^{(2)}$, the Ricci scalar for the two-dimensional worldsheet metric h_{ab}, and by dimensional counting it requires an extra factor of α'[2]. This theory retains worldsheet reparameterization invariance and Poincaré symmetry is promoted to invariance under target-space diffeomorphisms. On top of that, the B-field introduces an abelian gauge symmetry parameterized by a one-form: $B \to B + d\lambda$. While the graviton and B-field preserve Weyl invariance, the dilaton term in (5.1.2) breaks it. However, this happens at a higher order in α' which, as we will see, plays the role of loop-counting parameter in the non-linear sigma model. This means that the violation of Weyl invariance due to the dilaton at a classical level can be cured by a one-loop effect coming from the metric and B-field. We can then safely state that (5.1.2) is conformally invariant at the classical level, up to leading order in α'.

This symmetry, however, does not survive quantum-mechanically unless we impose the vanishing of the so-called Weyl anomaly $T_{\mathrm{a}}{}^{\mathrm{a}} = 0$, with T_{ab} the sigma-model energy-momentum tensor. This imposes certain conditions on the the background fields which take the form:

$$\bar{\beta}^I \equiv \beta^I + \delta_\xi \Psi^I = 0 \,, \tag{5.1.3}$$

where I is an index labeling the massless fields $\Psi^I \in \{G_{\mu\nu}, B_{\mu\nu}, \phi\}$, making (5.1.3) really three equations in one. In here, β^I are the renormalization group beta functions coming from (5.1.2) and $\delta_\xi \Psi^I$ encode gauge transformations of

[1] The i factor is because we are in Euclidean signature and $\epsilon^{\mathrm{ab}} \partial_{\mathrm{a}} X^\mu \partial_{\mathrm{b}} X^\nu B_{\mu\nu}$ has only one σ^1 ("time") derivative.

[2] In natural units, the action has length-dimension zero $[I] = 0$. Since $[\lambda] = [\alpha'] = 2$ and $[\sigma^{\mathrm{a}}] = 1$, this implies that the terms in brackets in (5.1.2) have dimension zero. For the metric coupling this is achieved by taking $[X^\mu] = 1$ and $[G_{\mu\nu}(X)] = 0$. For the dilaton coupling we have $[\phi] = 0$ and $[R^{(2)}] = -2$, which forces the presence of α'.

the corresponding field via a *specific* field-dependent parameter ξ. From the point of view of the target space, (5.1.3) are the equations of motion for the background fields, giving rise to the low-energy effective actions with which we have being working so far!

With the main goal of making the worldsheet discussion self-contained, we begin in Section 5.1.1 giving a brief overview of how imposing the vanishing of the Weyl anomaly gives rise to (5.1.3) for generic renormalizable two-dimensional sigma models. For a more detailed explanation we refer the reader to one of the foundational works [18] or to a recent review in [2]. We then introduce the background-field method in Section 5.1.2, a useful tool to compute beta functions, and in Section 5.1.3 we apply it to the dilaton-gravity worldsheet at one loop and read the corresponding two-derivative target-space action. All concepts introduced in this "review" section will prove essential for the one- and two-loop computations in cosmological backgrounds, carried out in the next sections.

5.1.1 Weyl anomaly coefficients

Renormalizable two-dimensional sigma models like (5.1.2) can be written in a generic form as

$$I = \int d^2\sigma A_I(h, X) \cdot \Psi^I(X)\,. \tag{5.1.4}$$

Here Ψ^I are the background fields, playing the role of coupling constants in the sigma model, and A_I are composite operators. We assume an implicit sum over repeated indices I, which run over the different couplings. $X(\sigma)$ are the coordinates of the target-space and $h^{\mathbf{ab}}(\sigma)$ the metric of the two-dimensional manifold. For later use, we introduced the dot product

$$f \cdot g = \int d^D x\, f(x)\, g(x)\,. \tag{5.1.5}$$

In this notation, the Polyakov action with just a metric field, for instance, is written as

$$\Psi^I = G_{\mu\nu}\,, \quad A_I = A_G^{\mu\nu} = \frac{1}{2\lambda}\sqrt{h}h^{\mathbf{ab}}\partial_{\mathbf{a}}X^\mu\partial_{\mathbf{b}}X^\nu\,\delta^D(x - X(\sigma))\,. \tag{5.1.6}$$

Equation (5.1.4) corresponds to the classical action, which is assumed to be Weyl invariant. As a consequence, the trace of the energy-momentum tensor must vanish. The latter can be computed from the standard formula:

$$T_{\mathbf{a}}{}^{\mathbf{a}} = \frac{2}{\sqrt{h}} h^{\mathbf{ab}} \frac{\delta I}{h^{\mathbf{ab}}} = 0\,. \tag{5.1.7}$$

Upon quantization, this identity does not hold in general but it requires the vanishing of the Weyl anomaly coefficients (5.1.3). To see this, we need to renormalize the theory, which requires us to distinguish between bare (divergent) couplings and renormalized (finite) ones. The bare action takes the form

$$I_0 = \int d^n\sigma\, A_{I0} \cdot \Psi_0^I\,, \tag{5.1.8}$$

where Ψ_0^I and A_{I0} are the bare couplings and composite operators, respectively. In here we already implemented dimensional regularization by taking $n = 2 + \epsilon$, where ϵ is the regularization parameter. The bare couplings Ψ_0^I are taken to have mass dimension $\epsilon = n - 2$, implying that the bare operators A_{I0} have dimension 2. The bare action (5.1.8) can be written as a sum of the renormalized action and counterterms,

$$I_0 = I_{\text{ren}} + I_{\text{c.t.}} \,, \quad I_{\text{ren}} = \int d^n\sigma\, \mu^\epsilon A_{I0} \cdot \Psi^I \,, \qquad I_{\text{c.t.}} = \int d^n\sigma\, \mu^\epsilon A_{I0} \cdot T^I(\Psi) \,, \tag{5.1.9}$$

in terms of the renormalized couplings denoted by Ψ^I. Here μ is the renormalization scale that is introduced in order to make the renormalized couplings dimensionless. The action (5.1.9) is used to derive the Feynman rules, and the counterterms T^I are obtained by demanding they cancel the divergent contributions to the quantum effective action. The partition function of the theory is given by the path integral

$$Z(\Psi, \mu) = e^{-W(\Psi,\mu)} = \int \mathcal{D}X\, e^{-I_0} \,, \tag{5.1.10}$$

and quantum expectation values are denoted by $\langle \cdots \rangle = \frac{1}{Z}\int \mathcal{D}X \cdots e^{-I_0}$. Given that the theory is renormalizable, the bare action (5.1.9) contains all the required counterterms to render the above path integral finite. Counterterms can also incorporate finite contributions, whose specific form defines different renormalization schemes. In here, we perform renormalization via minimal subtraction (MS), in which the counterterms are purely divergent. This, combined with dimensional regularization, allow us to express the counterterms as a Laurent series in ϵ, $T^I(\Psi) = \sum_{n=1}^{\infty} \frac{1}{\epsilon^n} T_n^I(\Psi)$.

Comparison of (5.1.9) with the canonical form (5.1.8) of the bare action gives the bare couplings in terms of the renormalized ones:

$$\Psi_0^I = \mu^\epsilon \left[\Psi^I + \sum_{n=1}^{\infty} \frac{1}{\epsilon^n} T_n^I(\Psi) \right] . \tag{5.1.11}$$

With this relation we may compute the beta functions associated to the couplings Ψ^I in $n = 2 + \epsilon$ dimensions, which are defined by

$$\beta^I \equiv \mu \frac{d\Psi^I}{d\mu} + \epsilon\, \Psi^I \,. \tag{5.1.12}$$

The derivatives of Ψ^I are obtained by requiring that the bare couplings Ψ_0^I do not depend on the renormalization scale, $\frac{d\Psi_0^I}{d\mu} = 0$. Indeed, differentiating (5.1.11), using (5.1.12), and matching $\mathcal{O}(\epsilon^0)$ and $\mathcal{O}(\epsilon)$ one obtains

$$\beta^I = -T_1^I + \Psi^J \cdot \frac{\partial}{\partial \Psi^J} T_1^I \,, \tag{5.1.13}$$

which is an exact expression in terms of the counterterm T_1^I and the usual perturbative evaluation arises from the loop expansion of the latter. The higher-order terms in ϵ^{-n} provide the so-called pole relations between higher-order counterterms [103, 104]. The above operator $\Psi^J \cdot \frac{\partial}{\partial \Psi^J}$ has to be understood in terms of the functional derivative as

$$f \cdot \frac{\partial F}{\partial \Psi} = \int d^D x \, f(x) \, \frac{\delta F}{\delta \Psi(x)} = F[\Psi + f]|_{\text{linear part in } f} \,, \tag{5.1.14}$$

where the last identity provides us with a practical way of computing such functional derivatives.

The next ingredient we need is a prescription to define renormalized composite operators A_I for the bare couplings in (5.1.8). Recalling (5.1.8), the integral of the bare operator A_{I0} is given by

$$\int d^n \sigma \, A_{I0} = \frac{\partial I_0}{\partial \Psi_0^I} \,. \tag{5.1.15}$$

Accordingly, we now define the renormalized composite operators A_I by demanding

$$\int d^n \sigma A_I = \frac{\partial I_0}{\partial \Psi^I} \,. \tag{5.1.16}$$

The quantum expectation value is then given by

$$\left\langle \int d^n \sigma A_I \right\rangle = \frac{\partial W}{\partial \Psi^I} \,, \tag{5.1.17}$$

that is guaranteed to be finite, because it is the derivative of a finite quantity by a finite parameter. Given (5.1.16) and (5.1.15) the relation between bare and renormalized operators is known up to possible total derivative terms:

$$A_I = A_{J0} \cdot \frac{\partial \Psi_0^J}{\partial \Psi^I} + \partial_{\mathbf{a}} \Omega_I^{\mathbf{a}} \,. \tag{5.1.18}$$

Assuming that the set $\{A_{I0}\}$ is a complete basis of dimension-two operators (modulo the bare equations of motions $\frac{\delta I_0}{\delta X^\mu}$ that we shall always discard, having zero expectation value), the total derivative part can also be expanded in terms of A_{I0}, namely

$$\partial_{\mathbf{a}} \Omega_I^{\mathbf{a}} = A_{J0} \cdot \Lambda_I^J \,, \quad \Lambda_I^J = \mu^\epsilon \sum_{n=1}^{\infty} \frac{1}{\epsilon^n} \, Q_{nI}^J(\Psi) \,. \tag{5.1.19}$$

This allows us to define the renormalization matrix as

$$A_I = A_{J0} \cdot Z_I^J \,, \quad Z_I^J = \frac{\partial \Psi_0^J}{\partial \Psi^I} + \Lambda_I^J \,. \tag{5.1.20}$$

Given the relation (5.1.11) between bare and renormalized couplings, we infer

$$Z_J^I = \mu^\epsilon \left[\delta_J^I + \sum_{n=1}^{\infty} \frac{1}{\epsilon^n} \, X_{nJ}^I(\Psi) \right] \,, \quad X_{nJ}^I = \frac{\partial T_n^I}{\partial \Psi^J} + Q_{nJ}^I \,. \tag{5.1.21}$$

It should be emphasized that while for ordinary field theories the Z_J^I are finite-dimensional matrices, for the string sigma model they are actually differential operators.

Now that we have related bare quantities with renormalized ones, we can get an operator expression for the trace of the energy-momentum tensor. Mimicking the classical definition (5.1.7), at the quantum level one can choose

$$T_{\mathbf{a}}{}^{\mathbf{a}} = \frac{2}{\sqrt{h}} h^{\mathbf{ab}} \frac{\delta I_0}{\delta h^{\mathbf{ab}}} \,, \tag{5.1.22}$$

which is not zero in general, and its expectation value

$$\langle T_{\mathbf{a}}{}^{\mathbf{a}} \rangle = \frac{2}{\sqrt{h}} h^{\mathbf{ab}} \frac{\delta W}{\delta h^{\mathbf{ab}}} \tag{5.1.23}$$

is finite. Using the decomposition (5.1.8), the Weyl variation $\frac{\delta}{\delta h^{\mathbf{ab}}}$ only sees the composite operators and acts as follows [18]:

$$2\, h^{\mathbf{ab}} \frac{\partial}{\partial h^{\mathbf{ab}}} A_{I0} = -\epsilon\, A_{I0} + \partial_{\mathbf{a}} \omega_I^{\mathbf{a}} \,, \tag{5.1.24}$$

where the second term accounts for possible total derivative terms. This can be verified by a direct computation in the case of (5.1.2), where A_G and A_B generate no total derivative terms, but A_ϕ does. It should be noted that since the bare operators A_{I0} enter (5.1.24) this total derivative is given directly by the Weyl variation of the classical Lagrangian, implying that it requires no quantum computation. We can, as before, use the completeness of the basis $\{A_{I0}\}$ modulo equations of motion to expand

$$\partial_{\mathbf{a}} \omega_I^{\mathbf{a}} = A_{J0} \cdot \lambda_I^J(\Psi_0) \,, \quad \lambda_I^J(\Psi_0) = \lambda_I^J(\Psi) + \mathcal{O}(\epsilon^{-1}) \,. \tag{5.1.25}$$

The trace operator is then computed as

$$\begin{aligned}
\sqrt{h}\, T_{\mathbf{a}}{}^{\mathbf{a}} &= 2\, h^{\mathbf{ab}} \frac{\delta}{\delta h^{\mathbf{ab}}} I_0 \\
&= 2\, h^{\mathbf{ab}} \frac{\delta}{\delta h^{\mathbf{ab}}} \int d^n\sigma\, A_{I0} \cdot \Psi_0^I \\
&= (-\epsilon\, A_{I0} + \partial_{\mathbf{a}} \omega_I^{\mathbf{a}}) \cdot \Psi_0^I \\
&= A_{I0} \cdot \left(-\epsilon\, \Psi_0^I + \lambda_J^I(\Psi_0) \cdot \Psi_0^J\right) \\
&= A_{I0} \mu^\epsilon \Big[-\epsilon\, \Psi^I - T_1^I(\Psi) + \lambda_J^I(\Psi) \cdot \Psi^J + \mathcal{O}(\epsilon^{-1}) \Big] \,,
\end{aligned} \tag{5.1.26}$$

where we used (5.1.11) and (5.1.25). The last step consists in rewriting the bare operators in terms of the renormalized ones. Using (5.1.21) we have:

$$A_{I0} = A_J \cdot (Z^{-1})_I^J \,, \quad (Z^{-1})_I^J = \mu^{-\epsilon} \left[\delta_I^J - \frac{1}{\epsilon} X_{1I}^J + \mathcal{O}(\epsilon^{-2}) \right] . \tag{5.1.27}$$

Combining this expression with the last line of (5.1.26) all the divergent terms must cancel out, since by definition both $T_{\mathbf{a}}{}^{\mathbf{a}}$ and A_I are finite. Upon letting $\epsilon = 0$ the Weyl anomaly operator can thus be written as

$$\sqrt{h}\, T_{\mathbf{a}}{}^{\mathbf{a}} = A_I \cdot \bar{\beta}^I \,, \tag{5.1.28}$$

giving rise to the Weyl anomaly coefficients defined by

$$\bar{\beta}^I \equiv \beta^I + \left(\lambda_J^I + Q_{1J}^I\right) \cdot \Psi^J \,. \tag{5.1.29}$$

From (5.1.28) we can see how the vanishing of the Weyl anomaly is equivalent to the vanishing of each $\bar{\beta}^I$ independently.

It is worth noticing that $\bar{\beta}^I$ differ from the β functions (5.1.12) by two total derivative terms: the λ-contribution can be found simply by varying the classical action, while the Q-contribution, which is much harder to compute, can be found by direct renormalization of the dimension-two operators A_I. It can be shown [18] that both total derivative terms take the form of gauge transformations of the corresponding field Ψ^I. Because of this, in practice one computes the λ^I_J operators once and for all from the classical theory, while Q^I_{1J} are not computed explicitly but just parameterized in terms of generic field-dependent gauge parameters and fixed later by other means, such as imposing the symmetries of the theory.

For the Polyakov action (5.1.2), (5.1.29) take the form [18]:

$$\begin{aligned} \bar{\beta}^G_{\mu\nu} &= \beta^G_{\mu\nu} + 2\alpha' \nabla_\mu \nabla_\nu \phi + \nabla_{(\mu} W_{\nu)} \,, \\ \bar{\beta}^B_{\mu\nu} &= \beta^B_{\mu\nu} + \alpha' H^\lambda{}_{\mu\nu} \nabla_\lambda \phi + \tfrac{1}{2} H^\lambda{}_{\mu\nu} W_\lambda + \partial_{[\mu} L_{\nu]} \,, \\ \bar{\beta}^\phi &= \beta^\phi + \alpha' (\nabla \phi)^2 + \tfrac{1}{2} \nabla^\mu \phi \, W_\mu \,. \end{aligned} \tag{5.1.30}$$

where the $\mathcal{O}(\alpha')$ terms correspond to the λ-contributions and the terms involving the gauge-parameters W and L are the Q-contributions. The former are exact in α' while the Q-terms receive corrections to all orders. This way of decomposing the Weyl anomaly coefficients into beta-function and total-derivative terms becomes very convenient. Indeed, $W_\mu(G,B)$ and $L_\mu(G,B)$ do not depend on ϕ and so the dilaton dependence displayed in (5.1.30) coming from the total-derivative contributions is exact to all orders in α' [18]! Moreover $\beta^G_{\mu\nu}$ and $\beta^B_{\mu\nu}$ do not depend on the dilaton neither, and β^ϕ itself is only linear in ϕ, i.e. $\beta^\phi = \Delta(G,B)\,\phi + \omega(G,B)$. Finally, the Q-terms can be fixed by covariance and dimensional counting (see *e.g.* [36]): by expanding in powers of α', the most general form of W is $W_\mu(G,B) = \sum_{L=1}^\infty \alpha'^L W^{(L)}_\mu(\mathcal{R},\nabla,H)$, where $W^{(L)}_\mu$ contains $2L-1$ derivatives of the metric and B-field. This already implies that at one-loop $W^{(1)}_\mu = 0$ and

$$W_\mu(\mathcal{R},\nabla,H) = \alpha'^2 \Big(a_1 \nabla_\mu \mathcal{R} + a_2 \nabla_\mu (H_{\nu\rho\sigma} H^{\nu\rho\sigma}) + a_3 H_{\mu\nu\lambda} \nabla_\rho H^{\rho\nu\lambda} \Big) + \mathcal{O}(\alpha'^3) \,, \tag{5.1.31}$$

and similarly for L_μ, respectively.

5.1.2 Background-field method and quantum effective action

Once the general structure of the Weyl anomaly coefficients (5.1.29) is specified to a particular background and field configuration, the only remaining peaces are the beta functions. These are obtained from the divergencies of the quantum effective action. We now revisit the standard QFT techniques to get these divergencies and then introduce the background-field method.

We consider a generic field theory with Euclidean action $I[\Psi]$ defined on an n-dimensional manifold parameterized by σ. The generating functional of all disconnected correlators is given by

$$Z[J] = \int \mathcal{D}\Psi\, e^{-I[\Psi]+J\cdot\Psi} \quad \Rightarrow \quad \langle \Psi_1 ... \Psi_n \rangle = \frac{1}{Z}\frac{\delta}{\delta J_1}\cdots\frac{\delta}{\delta J_n} Z[J]\bigg|_{J=0}, \tag{5.1.32}$$

where we introduced the notation $\Psi_i \equiv \Psi(\sigma_i)$, $J_i \equiv J(\sigma_i)$, and

$$J\cdot\Psi \equiv \int d^n\sigma\, J(\sigma)\,\Psi(\sigma)\,. \tag{5.1.33}$$

We also used the definition for the expectation value

$$\langle \cdots \rangle = \frac{1}{Z}\int \mathcal{D}\Psi \cdots e^{-I[\Psi]}, \tag{5.1.34}$$

with the factor $Z \equiv Z[0]$ ensuring that the correlators are normalized, $\langle 1 \rangle = 1$.

The generating functional of the connected correlators, $W[J]$, is defined by

$$W[J] \equiv \log \frac{Z[J]}{Z}, \tag{5.1.35}$$

which ensures that $W[0] = 0$, meaning that all the vacuum bubbles are subtracted and so we have

$$\langle \Psi_1 ... \Psi_n \rangle_{\text{connected}} = \frac{\delta}{\delta J_1}\cdots\frac{\delta}{\delta J_n} W[J]\bigg|_{J=0}. \tag{5.1.36}$$

The generating functional of one-particle-irreducible (1PI) diagrams, namely those that cannot be split into two disjoint pieces by cutting a single internal line, is given by the quantum effective action

$$\Gamma[\varphi] = J\cdot\varphi - W[J]\,, \tag{5.1.37}$$

where

$$\varphi(J) \equiv \frac{\delta W}{\delta J} = \langle \Psi \rangle_J \tag{5.1.38}$$

is nothing but the quantum expectation value in the presence of J. Assuming tadpole cancellation one has $\varphi(0) = \langle \Psi \rangle_{\text{connected}} = 0$. Here, $\Gamma[\varphi]$ is the Legendre transformation of $W[J]$ so it does not depend on J but only on φ. Formally, one needs to invert the relationship by expressing $J[\varphi]$, which can be obtained from

$$\frac{\delta \Gamma}{\delta \varphi} = J\,. \tag{5.1.39}$$

which is the quantum-mechanical field equation for φ. The 1PI diagrams are obtained by taking functional derivatives of $\Gamma[\varphi]$. Using (5.1.32) and the definitions above one can express the quantum effective action in terms of the path integral:

$$e^{-\Gamma[\varphi]} = \frac{1}{Z}\int \mathcal{D}\Psi\, e^{-I[\Psi]+J\cdot(\Psi-\varphi)}\,. \tag{5.1.40}$$

So far all these were conventional QFT techniques for calculating correlators, we now introduce the background-field method, which is particularly useful in the context of sigma models (see [105] and references therein for a nice presentation of the method). To this end, we use (5.1.39) in (5.1.40), together with a shift in the integration variable by $\Psi = \varphi + \pi$ to obtain:

$$e^{-\frac{1}{\hbar}\Gamma[\varphi]} = \frac{1}{Z}\int \mathcal{D}\pi\, e^{-\frac{1}{\hbar}\left(I[\varphi+\pi]-\frac{\delta\Gamma}{\delta\varphi}\cdot\pi\right)}\,, \tag{5.1.41}$$

where we have reinstated Planck's constant $\hbar$ as a loop counting parameter. Here φ is viewed as the classical background and π as the quantum fluctuation. From a diagramatic perspective, contribution from φs appear as external legs while πs represent internal ones. Note that there is an all-order perturbative subtraction in (5.1.41) given implicitly by the term $-\frac{\delta\Gamma}{\delta\varphi}\cdot\pi$. Luckily we do not have to compute this contribution at each order since its only role is to remove non-1PI contributions from Γ. Knowing this, we can work with the formula

$$e^{-\frac{1}{\hbar}\Gamma[\varphi]} = \frac{1}{Z}\int \mathcal{D}\pi\, e^{-\frac{1}{\hbar}I[\varphi+\pi]}\Big|_{\text{1PI}}\,, \tag{5.1.42}$$

and compute just the 1PI vacuum diagrams of π omitting all the rest.

As usual, we compute $\Gamma[\varphi]$ perturbatively order-by-order in loops. To this end, one starts by writing Γ as a power series in $\hbar$:

$$\Gamma[\varphi] = I[\varphi] + \Delta\Gamma\,, \quad \Delta\Gamma[\varphi] \equiv \hbar\,\Gamma_{1l}[\varphi] + \hbar^2\,\Gamma_{2l}[\varphi] + \mathcal{O}(\hbar^3)\,, \tag{5.1.43}$$

where we separated the classical contribution to the effective action ($I[\varphi]$) from the quantum corrections ($\Delta\Gamma$), in which $\Gamma_{il}[\varphi]$ corresponds to the i-loop effective action. Equation (5.1.43) takes care of the left-hand side of (5.1.42). For the right-hand side, one needs to expand the action in powers of π

$$I[\varphi+\pi] = I_{0\pi} + I_{1\pi} + I_{2\pi} + \mathcal{O}(\pi^3)\,. \tag{5.1.44}$$

The zeroth order just gives the original classical action in terms of the background field $I_{0\pi} = I[\varphi]$. The linear term in π always contribute to non-1PI diagrams, so we can ignore it. This is because something of the form $\pi\varphi^n$ will give a diagram with n external lines and one internal line, that, attached to any other piece of the diagram, can be simply cut. An alternative way of seeing this is from (5.1.41), where $-\frac{\delta\Gamma}{\delta\varphi}\cdot\pi$ clearly cancels linear contributions to lowest order. The next term in (5.1.44), $I_{2\pi}$, can be further split into

$$I_{2\pi} = I_{\text{kin}} + V_2\,, \tag{5.1.45}$$

in which I_{kin} is the kinetic part of the original action but in terms of π rather than φ, giving rise to the propagator. The remaining quadratic peace V_2 is part of the interaction terms, which can be grouped with all other higher-power contributions coming from (5.1.47) into

$$I_{\text{int}} = V_2 + V_3 + \mathcal{O}(\pi^4)\,. \tag{5.1.46}$$

In here, $V_i = I_{i\pi}$ with $i = 3, 4, \ldots$ corresponds to the ith-order contribution to the expansion in (5.1.44). Using the definition for kinetic and interaction terms we can rewrite the expansion (5.1.44) in an alternative form

$$I[\varphi+\pi] = I[\varphi] + I_{1\pi} + I_{\text{kin}} + I_{\text{int}}\,. \tag{5.1.47}$$

Inserting (5.1.43) and (5.1.47) into (5.1.42) gives us

$$-\tfrac{1}{\hbar}\Delta\Gamma[\varphi] = \langle e^{-\frac{1}{\hbar}I_{\text{int}}}\rangle_{1\text{PI}}\,, \tag{5.1.48}$$

where the expectation values in this case are computed from I_{kin} and we picked a normalized such that $\langle 1\rangle = 1$ with respect to this new partition function:

$$\langle\cdots\rangle = \frac{1}{Z_0}\int\mathcal{D}\pi\,\cdots e^{-\frac{1}{\hbar}I_{\text{kin}}[\pi]}\,,\quad Z_0 \equiv \int\mathcal{D}\pi\, e^{-\frac{1}{\hbar}I_{\text{kin}}[\pi]}\,. \tag{5.1.49}$$

Equation (5.1.48) is the starting point for any loop expansion, in which, as usual, we compute Feynman diagrams using Wick's theorem. The internal lines and propagators are given by πs while φs represent external legs. Because of this, we end up computing exactly the same 1PI diagrams that we would have computed by conventional methods. However, the background-field method has the advantage of preserving explicit gauge invariance, as we now see with a sigma-model example.

5.1.3 An example: dilaton-gravity at one loop

In this last part of the review section, we revisit the general techniques introduced above for a particular example, the dilaton-gravity sigma model:

$$I = \frac{1}{2\lambda}\int d^2\sigma\sqrt{h}\left[h^{\mathbf{ab}}\partial_{\mathbf{a}}X^\mu\partial_{\mathbf{b}}X^\nu G_{\mu\nu}(X) + \alpha' R^{(2)}\phi(X)\right]\,, \tag{5.1.50}$$

obtained from (5.1.2) by setting to zero the B-field. Our goal here is to obtain the $\mathcal{O}(\alpha')$ target-space equations of motion for $G_{\mu\nu}$ and ϕ from the vanishing of the Weyl anomaly coefficients $\bar\beta^G_{\mu\nu} = 0$ and $\bar\beta^\phi = 0$ at one-loop, respectively. The general structure of the latter was given already in (5.1.30) which for $B = 0$ reads:

$$\begin{aligned}\bar\beta^G_{\mu\nu} &= \beta^G_{\mu\nu} + 2\alpha'\,\nabla_\mu\nabla_\nu\phi + \mathcal{O}(\alpha'^2)\,,\\ \bar\beta^\phi &= \beta^\phi + \alpha'\,(\nabla\phi)^2 + \mathcal{O}(\alpha'^2)\,.\end{aligned} \tag{5.1.51}$$

In here, we took into account that at one loop W_μ vanishes (see (5.1.31)) and so the only missing peaces are the β functions. These are related to the counterterms T_1^G and T_1^ϕ via (5.1.13), obtained upon renormalization of the couplings from the single pole divergent part of the quantum effective action. We now compute $\Gamma[\varphi]$ at one loop using the background-field method and deduce the corresponding β functions. We first do it in detail for the metric coupling and then sketch the general idea for the dilaton contribution. We finish showing how the latter can be obtained more easily from β^G by imposing consistency of (5.1.51).

Metric sector

Let's consider

$$I = \int d^2\sigma\,\mathcal{L}[X(\sigma)]\,,\quad \mathcal{L}[X(\sigma)] \equiv \frac{1}{2\lambda}G_{\mu\nu}(X)\partial_{\mathbf{a}}X^\mu\partial^{\mathbf{a}}X^\nu\,, \tag{5.1.52}$$

where we used the gauge freedom to fix h_{ab} to be the flat worldsheet metric. In principle, it is possible to apply the background-field method in the form just described in Section 5.1.2. However, splitting $X^\mu = \varphi^\mu + \pi^\mu$ leads to a perturbative expansion lacking manifest target space covariance, since the fluctuation field π^μ is a coordinate difference and thus has no geometric meaning. To remedy this we employ a field redefinition of π^μ as follows [104, 106]: We consider a geodesic $X^\mu(\tau)$, where τ is the affine parameter (and we suppress the dependence on the worldsheet coordinates σ^{a} for now), such that

$$X^\mu(\tau = 0) = \varphi^\mu\,, \quad \text{and} \quad X^\mu(\tau = 1) = \varphi^\mu + \pi^\mu\,. \tag{5.1.53}$$

The derivative along the geodesic defines a tangent vector $\xi^\mu(\tau)$

$$\frac{d}{d\tau} = \xi^\mu(\tau)\partial_\mu\,, \quad \xi^\mu(\tau) \equiv \frac{dX^\mu}{d\tau}\,, \tag{5.1.54}$$

which satisfies the geodesic equation

$$\frac{D}{D\tau}\xi^\mu(\tau) = 0\,, \quad \frac{D}{D\tau} \equiv \xi^\mu(\tau)\nabla_\mu\,, \tag{5.1.55}$$

written in terms of the covariant derivative along the geodesic $\frac{D}{D\tau}$. We can then use the tangent vector at $\tau = 0$

$$\xi^\mu \equiv \xi^\mu(0) \tag{5.1.56}$$

as the quantum field for the background-field expansion. Since it is a genuine vector, this ensures manifest target-space covariance.

It is possible to derive the exact nonlinear relation $\pi^\mu(\xi) = \xi^\mu - \frac{1}{2}\Gamma^\mu_{\nu\lambda}\xi^\nu\xi^\lambda + \mathcal{O}(\xi^3)$ implementing the field redefinition and use it to write down the covariant expansion. This procedure, however, becomes very cumbersome after a few orders. A considerable simplification was found in [106] by noting that one usually needs to expand only the Lagrangian, which is just a scalar. We thus consider a scalar field evaluated along the geodesic: $\Phi(\tau) \equiv \Phi(X(\tau))$. We can expand it around $\tau = 0$, yielding

$$\Phi(\tau) = \sum_{n=0}^{\infty} \frac{\tau^n}{n!} \left.\frac{d^n\Phi}{d\tau^n}\right|_{\tau=0}\,. \tag{5.1.57}$$

Since Φ is a scalar field, the derivative along the geodesic is already covariant:

$$\frac{D\Phi}{D\tau} = \xi^\mu(\tau)\nabla_\mu\Phi = \xi^\mu(\tau)\partial_\mu\Phi = \frac{d\Phi}{d\tau}\,, \tag{5.1.58}$$

where we used (5.1.54) and (5.1.55). Given that any application of $\frac{d}{d\tau}$ maps the scalar into a scalar, it immediately follows by induction that

$$\frac{d^n\Phi}{d\tau^n} = \frac{d^{n-1}}{d\tau^{n-1}}\left(\xi(\tau)\cdot\nabla\Phi\right) = \left(\xi(\tau)\cdot\nabla\right)^n\Phi = \frac{D^n\Phi}{D\tau^n}\,. \tag{5.1.59}$$

This observation turns (5.1.57) into a manifestly-covariant expansion

$$\Phi(\tau) = \sum_{n=0}^{\infty} \frac{\tau^n}{n!} \left.\frac{D^n\Phi}{D\tau^n}\right|_{\tau=0}\,, \tag{5.1.60}$$

which takes a particularly simple form when evaluated at $\tau = 1$:

$$\Phi(1) = e^{\frac{D}{D\tau}}\Phi(0)\,. \tag{5.1.61}$$

Using (5.1.61) for the Lagrangian $\Phi(\tau) \to \mathcal{L}[X(\tau)]$, and recalling (5.1.53), one obtains [107]

$$\mathcal{L}[\varphi + \pi(\xi)] = e^{\frac{D}{D\tau}}\mathcal{L}[\varphi]\,. \tag{5.1.62}$$

In order to apply this to the sigma model (5.1.52) we need to consider the pullback to the worldsheet. To this end we use the worldsheet-dependent geodesic $X^\mu(\sigma;\tau)$ and $\xi^\mu(\sigma;\tau)$ to compute

$$\frac{D}{D\tau}\partial_{\mathbf{a}}X^\mu = \frac{d}{d\tau}\partial_{\mathbf{a}}X^\mu + \xi^\nu\Gamma^\mu_{\nu\lambda}\partial_{\mathbf{a}}X^\lambda = \partial_{\mathbf{a}}\xi^\mu + \partial_{\mathbf{a}}X^\nu\Gamma^\mu_{\nu\lambda}\xi^\lambda \equiv D_{\mathbf{a}}\xi^\mu\,, \tag{5.1.63}$$

where we have defined the covariant derivative on the worldsheet

$$D_{\mathbf{a}} \equiv \partial_{\mathbf{a}}X^\mu\nabla_\mu\,, \tag{5.1.64}$$

which acts as on the pullback of target-space tensors. To act further with $\frac{D}{D\tau}$ we need the commutator

$$\begin{aligned}\left[\frac{D}{D\tau}, D_{\mathbf{a}}\right] &= \left[\xi^\mu\nabla_\mu, \partial_{\mathbf{a}}X^\nu\nabla_\nu\right] \\ &= \xi^\mu\partial_{\mathbf{a}}X^\nu\left[\nabla_\mu,\nabla_\nu\right] + \left(\frac{D}{D\tau}\partial_{\mathbf{a}}X^\nu\right)\nabla_\nu - (D_{\mathbf{a}}\xi^\mu)\nabla_\mu \\ &= \xi^\mu\partial_{\mathbf{a}}X^\nu\,\mathcal{R}^\#_{\mu\nu}\,,\end{aligned} \tag{5.1.65}$$

where $\mathcal{R}^\#_{\mu\nu}$ is the Riemann tensor acting as an operator, *e.g.* $\mathcal{R}^\#_{\mu\nu}V^\rho = \mathcal{R}_{\mu\nu}{}^\rho{}_\lambda V^\lambda$. One can thus determine, for instance,

$$\frac{D^2}{D\tau^2}\partial_{\mathbf{a}}X^\mu = \frac{D}{D\tau}D_{\mathbf{a}}\xi^\mu = \left[\frac{D}{D\tau}, D_{\mathbf{a}}\right]\xi^\mu = \xi^\nu\partial_{\mathbf{a}}X^\lambda\,\mathcal{R}_{\nu\lambda}{}^\mu{}_\rho\,\xi^\rho\,. \tag{5.1.66}$$

These tools allow one to systematically expand the sigma model action in a simple and recursive manner. The expansion is carried in powers of ξ, namely:

$$I[\varphi + \pi(\xi)] = I_{0\xi} + I_{1\xi} + I_{2\xi} + \mathcal{O}(\xi^3) = I[\varphi] + I_{1\xi} + I_{\text{kin}} + I_{\text{int}}\,, \tag{5.1.67}$$

with

$$I_{2\xi} = I_{\text{kin}} + V_2\,, \quad I_{\text{int}} = V_2 + V_3 + \cdots\,, \quad V_i = I_{i\xi}\,, \quad i = 3, 4, \ldots \tag{5.1.68}$$

Using (5.1.62) and the compatibility condition for the metric $\frac{D}{D\tau}G_{\mu\nu} = \xi^\rho\nabla_\rho G_{\mu\nu} = 0$ the first few orders of the expansion are easily obtained as

$$\begin{aligned} I_{0\xi} = I[\varphi] &= \frac{1}{2\lambda}\int d^2\sigma\left[G_{\mu\nu}(\varphi)\partial_{\mathbf{a}}\varphi^\mu\partial^{\mathbf{a}}\varphi^\nu\right]\,, \\ I_{1\xi} &= \frac{1}{2\lambda}\int d^2\sigma\,\frac{D}{D\tau}\left[G_{\mu\nu}(X)\,\partial_{\mathbf{a}}X^\mu\partial^{\mathbf{a}}X^\nu\right]\Big|_{\tau=0} = \frac{1}{\lambda}\int d^2\sigma\left[G_{\mu\nu}(\varphi)\,D_{\mathbf{a}}\xi^\mu\partial^{\mathbf{a}}\varphi^\nu\right]\,, \\ I_{2\xi} &= \frac{1}{\lambda}\int d^2\sigma\,\frac{1}{2}\frac{D}{D\tau}\left[G_{\mu\nu}(X)\,D_{\mathbf{a}}\xi^\mu\partial^{\mathbf{a}}X^\nu\right]\Big|_{\tau=0} \\ &= \frac{1}{2\lambda}\int d^2\sigma\left[G_{\mu\nu}(\varphi)\,D^{\mathbf{a}}\xi^\mu D_{\mathbf{a}}\xi^\nu + \mathcal{R}_{\mu\nu\lambda\rho}(\varphi)\,\partial^{\mathbf{a}}\varphi^\mu\partial_{\mathbf{a}}\varphi^\rho\,\xi^\nu\xi^\lambda\right]\,, \end{aligned} \tag{5.1.69}$$

where we recall that, after taking the derivatives, evaluating at $\tau = 0$ amounts to replacing X^μ by φ^μ. This already exhausts the terms needed at one loop.

From the quadratic term we can read the kinetic and interacting part by expanding $D_{\mathbf{a}}\xi^\mu = \partial_{\mathbf{a}}\xi^\mu + \partial_{\mathbf{a}}\varphi^\nu \Gamma^\mu_{\nu\lambda}(\varphi)\xi^\lambda$. However, the kinetic term in this case, $G_{\mu\nu}(\varphi)\partial^{\mathbf{a}}\xi^\mu\partial_{\mathbf{a}}\xi^\nu$, has a non-standard form, since $G_{\mu\nu}(\varphi)$ is not constant on the worldsheet. To overcome this difficulty it is customary to introduce vielbeins $e_\mu{}^\alpha(\varphi)$ such that $G_{\mu\nu} = e_\mu{}^\alpha \eta_{\alpha\beta} e_\nu{}^\beta$ and flatten the fluctuation by introducing $\xi^\alpha = e_\mu{}^\alpha\, \xi^\mu$. We then have the covariant derivative acting on the flatted vectors

$$e_\mu{}^\alpha D_{\mathbf{a}}\xi^\mu = D_{\mathbf{a}}\xi^\alpha = \partial_{\mathbf{a}}\xi^\alpha + \partial_{\mathbf{a}}\varphi^\mu\, \omega_\mu{}^\alpha{}_\beta(\varphi)\, \xi^\beta\,, \tag{5.1.70}$$

in terms of the spin connection $\omega_{\mu\alpha}{}^\beta = e_\alpha{}^\nu \nabla_\mu e_\nu{}^\beta$. The quadratic term $I_{2\xi}$ then takes the form:

$$I_{2\xi} = \frac{1}{2\lambda}\int d^2\sigma \left[D^{\mathbf{a}}\xi^\alpha D_{\mathbf{a}}\xi_\alpha + \mathcal{R}_{\mu\alpha\beta\nu}\, \partial^{\mathbf{a}}\varphi^\mu \partial_{\mathbf{a}}\varphi^\nu\, \xi^\alpha\xi^\beta\right] = I_{\text{kin}} + V_2\,, \tag{5.1.71}$$

which has a standard kinetic term

$$I_{\text{kin}} = \frac{1}{2\lambda}\int d^2\sigma\, \partial^{\mathbf{a}}\xi^\alpha \partial_{\mathbf{a}}\xi_\alpha\,. \tag{5.1.72}$$

The quadratic interacting terms are then given by

$$\begin{aligned} V_2 = \frac{1}{2\lambda}\int d^2\sigma \Big[2\, \partial^{\mathbf{a}}\varphi^\mu\, \omega_{\mu\,\alpha\beta}\, \xi^\beta \partial_{\mathbf{a}}\xi^\alpha + \partial^{\mathbf{a}}\varphi^\mu \partial_{\mathbf{a}}\varphi^\nu\, \omega_{\mu\,\gamma\alpha}\, \omega_\nu{}^\gamma{}_\beta\, \xi^\alpha\xi^\beta + \mathcal{R}_{\mu\alpha\beta\nu}\, \partial^{\mathbf{a}}\varphi^\mu \partial_{\mathbf{a}}\varphi^\nu\, \xi^\alpha\xi^\beta\Big] \\ \equiv V_\omega + V_{\omega\omega} + V_{\mathcal{R}}\,, \end{aligned} \tag{5.1.73}$$

where we distinguished different tensor structures. The propagator can be derived from (5.1.72) and it takes the typical form for massless scalar fields

$$\langle \xi^\alpha(\sigma_1)\, \xi^\beta(\sigma_2)\rangle = \lambda \delta^{\alpha\beta} G(\sigma_1 - \sigma_2)\,, \quad G(\sigma) = \int \frac{d^2p}{(2\pi)^2}\frac{e^{ip\cdot\sigma}}{p^2}\,. \tag{5.1.74}$$

We have now all the ingredients to compute the one-loop quantum effective action. To do this, we note that for the worldsheet sigma model, $\lambda = 2\pi\alpha'$ is playing the role of loop-counting parameter and so (5.1.48) reads

$$-\tfrac{1}{\lambda}\Delta\Gamma[\varphi] = \langle e^{-I_{\text{int}}}\rangle_{1\text{PI}}\,, \tag{5.1.75}$$

where the effective action at L loops is of order $(\alpha')^{L-1}$ [3] Let us note that the general form of the effective action is

$$\Delta\Gamma[\varphi] = \frac{1}{2\lambda}\int d^2\sigma \left[\mathcal{M}_{\mu\nu}(G)\, \partial^{\mathbf{a}}\varphi^\mu \partial_{\mathbf{a}}\varphi^\nu + \cdots\right]\,, \tag{5.1.76}$$

where $\mathcal{M}_{\mu\nu}$ is some target space tensor, which prior to renormalization contains divergent coefficients and the ellipsis denote terms with more than two derivatives of φ. The divergent part of $\mathcal{M}_{\mu\nu}$ corresponds to minus the counterterms $T^G_{\mu\nu}(G)$, whose simple pole in ϵ, $T^G_{\mu\nu,1}$ determines the renormalization of

[3] Notice that here we are using a slightly different convention compared to Section 5.1.2 by absorbing a $\frac{1}{\lambda}$ factor inside I_{kin} and I_{int}. See, for instance, (5.1.72) and (5.1.73).

the metric and hence determines the β function. For this reason, it is sufficient for our purposes to consider contributions to Γ with only two factors of $\partial_{\mathbf{a}}\varphi^\mu$.

Since every propagator rises the power of λ by one, it is easy to see from (5.1.75) that in order to compute the L-loop quantum effective action one needs to expand the shifted action up to order $2L$ in fluctuations. Therefore, the one-loop effective action comes only from quadratic interactions

$$\Gamma_{1l} = -\langle e^{-V_2}\rangle_{\text{1PI}} \,. \tag{5.1.77}$$

Moreover, the only terms contributing to the renormalization of the metric are

$$\Gamma_{1l} = \langle V_R\rangle_{\text{1PI}} + \langle V_{\omega\omega}\rangle_{\text{1PI}} - \tfrac{1}{2}\,\langle V_\omega^2\rangle_{\text{1PI}} + \cdots\ , \tag{5.1.78}$$

where dots stand for terms with more than two factors of $\partial_{\mathbf{a}}\varphi^\mu$. On dimensional grounds, and using gauge invariance, it follows that the terms V_ω and $V_{\omega\omega}$ involving the spin connection cannot contribute to ultraviolet (UV) divergences (see [2] for an explicit computation). We are then left with the single divergent contribution

$$\begin{aligned}\Gamma_{1l}^{\text{div}} = \langle I_R\rangle_{\text{1PI}} &= -\frac{1}{2\lambda}\int d^2\sigma\,\mathcal{R}_{\mu\alpha\beta\nu}\,\partial^{\mathbf{a}}\varphi^\mu\partial_{\mathbf{a}}\varphi^\nu\langle\xi^\alpha(\sigma)\xi^\beta(\sigma)\rangle\\ &= -\frac{G(0)}{2}\int d^2\sigma\,\mathcal{R}_{\mu\nu}\,\partial^{\mathbf{a}}\varphi^\mu\partial_{\mathbf{a}}\varphi^\nu\ .\end{aligned} \tag{5.1.79}$$

In order to regularize the propagator at coinciding points $G(0) = \int \frac{d^2p}{(2\pi)^2}\frac{1}{p^2}$ we continue to $n = 2+\epsilon$ dimensions and introduce an infrared (IR) mass regulator m^2 by changing the propagator[4] as $\frac{1}{p^2} \to \frac{1}{p^2+m^2}$. This yields:

$$G(0)_{\text{reg}} = \tilde{\mu}^{2-n}\int\frac{d^np}{(2\pi)^n}\,\frac{1}{p^2+m^2} = \frac{1}{4\pi}\left(\frac{m^2}{4\pi\tilde{\mu}^2}\right)^{\frac{\epsilon}{2}}\Gamma(-\epsilon/2)\ , \tag{5.1.80}$$

where we introduced the arbitrary mass parameter $\tilde{\mu}$ to keep $G(0)_{\text{reg}}$ dimensionless and solve the one-loop integral. In order to extract the pole, we expand the gamma function $\Gamma(x) = \frac{1}{x} - \gamma + \mathcal{O}(\sigma)$, where γ is the Euler-Mascheroni constant, and we obtain

$$\Gamma_{1l}^{\text{div}} = \frac{1}{4\pi}\left(\frac{1}{\epsilon} + \log\frac{m}{\mu}\right)\int d^2\sigma\,\mathcal{R}_{\mu\nu}\,\partial^{\mathbf{a}}\varphi^\mu\partial_{\mathbf{a}}\varphi^\nu\ . \tag{5.1.81}$$

Here we redefined the renormalization scale as $\mu^2 \equiv 4\pi e^{-\gamma}\tilde{\mu}^2$, as it is customary in the minimal-subtraction scheme. At this point we can fix the one-loop counterterm by demanding that it cancels the divergence:

$$I_{\text{c.t.}} = -\frac{1}{4\pi\epsilon}\int d^2\sigma\,\mathcal{R}_{\mu\nu}\,\partial^{\mathbf{a}}\varphi^\mu\partial_{\mathbf{a}}\varphi^\nu\,, \tag{5.1.82}$$

[4] This amounts to adding the mass term $\frac{m^2}{2\lambda}\int d^2\sigma\,\xi^\alpha\xi_\alpha$ to the action, that suffices to regularize IR divergences at one-loop.

Following the method outlined in Section 5.1.1, we can use equation (5.1.9) to write the bare action in terms of renormalized quantities and counterterms as

$$I_0 = I_{\text{ren}} + I_{\text{c.t.}} = \frac{1}{2\lambda}\int d^n\sigma\, G^0_{\mu\nu}\,\partial^{\mathbf{a}}\varphi^\mu\partial_{\mathbf{a}}\varphi^\nu = \frac{1}{2\lambda}\int d^n\sigma\,\mu^\epsilon\left(G_{\mu\nu} + T^G_{\mu\nu}(G)\right)\partial^{\mathbf{a}}\varphi^\mu\partial_{\mathbf{a}}\varphi^\nu\;, \tag{5.1.83}$$

where, using (5.1.82), we read

$$T^G_{1,\mu\nu} \equiv -\frac{\lambda}{2\pi}\mathcal{R}_{\mu\nu} = -\alpha'\mathcal{R}_{\mu\nu}\,. \tag{5.1.84}$$

The β function is then completely determined from (5.1.84) via (5.1.13):

$$\beta^G_{\mu\nu} = \alpha'\left(1 - G\cdot\frac{\partial}{\partial G}\right)\mathcal{R}_{\mu\nu}\;, \tag{5.1.85}$$

where we recall that the operator $G\cdot\frac{\partial}{\partial G}$ should be regarded as the integrated functional derivative as in (5.1.14). In this case, however, the operator simply counts the number of $G_{\mu\nu}$ minus the number of $G^{\mu\nu}$, giving $G\cdot\frac{\partial}{\partial G}\mathcal{R}_{\mu\nu} = 0$. Taking this into account, we can finally read off the one-loop beta function [12]

$$\beta^G_{\mu\nu} = \alpha'\,\mathcal{R}_{\mu\nu}\,. \tag{5.1.86}$$

For this particular purely-metric sigma model, $\beta^G_{\mu\nu}$ coincides exactly with the Weyl anomaly coefficient $\bar{\beta}^G_{\mu\nu}$ as it can be seen from (5.1.51). The target-space equations of motion at leading order are then nothing but Einstein's equations in vacuum.

Including the dilaton

We now include the dilaton coupling in the metric sigma model

$$I = \frac{1}{2\lambda}\int d^2\sigma\sqrt{h}\left[h^{\mathbf{ab}}\partial_{\mathbf{a}}X^\mu\partial_{\mathbf{b}}X^\nu G_{\mu\nu}(X) + \alpha' R^{(2)}\phi(X)\right]\;, \tag{5.1.87}$$

and we are interested in the one-loop β function for G and ϕ. Treating the dilaton term is technically more involved, since it requires computations on a curved worldsheet. On top of this, one can see that the full $\mathcal{O}(\alpha'^L)$ dilaton β function requires an $(L+1)$-loop computation. This is because the dilaton coupling in the sigma model (5.1.87) appears with one extra order of α' as compared to the other couplings. For these two reasons, it is often preferable to fix the dilaton equation from consistency [18, 19, 108–110]. We will now sketch the idea behind the computation from a curved worldsheet, but then we move right away to the consistency method.

The background-field expansion of (5.1.87) can be implemented identically as we did in the flat case (5.1.52) since the expansion $X^\mu = \varphi^\mu + \pi^\mu(\xi)$ does not see whether the metric is curved or flat. This leads immediately to the quadratic action

$$I_{2\xi} = \frac{1}{2\lambda}\int d^2\sigma\,\sqrt{h}h^{\mathbf{ab}}\Big[D_{\mathbf{a}}\xi^\alpha D_{\mathbf{b}}\xi_\alpha + \mathcal{R}_{\mu\alpha\beta\nu}\,\partial_{\mathbf{a}}\varphi^\mu\partial_{\mathbf{b}}\varphi^\nu\,\xi^\alpha\xi^\beta\Big] + \frac{1}{8\pi}\int d^2\sigma\sqrt{h}R^{(2)}\,\xi^\alpha\xi^\beta\nabla_\alpha\nabla_\beta\phi\;, \tag{5.1.88}$$

where we denoted $\nabla_\alpha\nabla_\beta\phi \equiv e_\alpha{}^\mu e_\beta{}^\nu\nabla_\mu\nabla_\nu\phi$. Following conventional perturbation theory, we shall expand $h_{\mathbf{ab}}$ around flat space: $h_{\mathbf{ab}} = \delta_{\mathbf{ab}} + \gamma_{\mathbf{ab}}$, and consider the one-loop effective action perturbatively in powers of $\gamma_{\mathbf{ab}}$. In particular, the propagators are still extracted from the flat-space free theory I_{kin} in (5.1.72), and terms with any powers of $\gamma_{\mathbf{ab}}$ are treated as interactions.

Before starting any computation, there is one immediate consequence that can be derived from the structure of the action: thanks to the coupling with the scalar curvature $R^{(2)}$, every term involving the dilaton appears with at least one factor of $\gamma_{\mathbf{ab}}$. Since the lowest order coupling of the metric $G_{\mu\nu}$ (and B-field if present) only involves the flat background $\delta_{\mathbf{ab}}$ one can immediately show that the dilaton *cannot* renormalize the metric nor the B-field at any order in perturbation theory! This implies that the β functions $\beta^G_{\mu\nu}$ and $\beta^B_{\mu\nu}$ do not depend on the dilaton at any order in α', as anticipated below (5.1.30).

Having shown that β^G is not affected by the dilaton, we are left to determine the β function of the dilaton itself. To this end, we need to extract from $V_2 = I_{2\xi} - I_{\text{kin}}$ the terms that can renormalize the coupling $\int d^2\sigma\sqrt{h}R^{(2)}\phi$ in $\Gamma_{1l} = -\left\langle e^{-V_2}\right\rangle_{1\text{PI}}$, which in particular do not contain $\partial_{\mathbf{a}}\varphi^\mu$ factors. Without going into details (see e.g. [2]), it can be shown that the only contributions are given by

$$\Gamma^\phi_{1l} = \langle I_\phi\rangle - \tfrac{1}{2}\,\langle I^2_{\gamma\partial\xi\partial\xi}\rangle_{1\text{PI}} + \mathcal{O}(\gamma^3)\;, \qquad I_{\gamma\partial\xi\partial\xi} = -\frac{1}{2\lambda}\int d^2\sigma\,\bar{\gamma}^{\mathbf{ab}}\partial_{\mathbf{a}}\xi^\alpha\partial_{\mathbf{b}}\xi_\alpha\;, \quad I_\phi = \frac{1}{8\pi}\int d^2\sigma\sqrt{h}R^{(2)}\,\xi^\alpha\xi^\beta\nabla_\alpha\nabla_\beta\phi\;, \tag{5.1.89}$$

where the superscript ϕ means possible contributions proportional to $\int d^2\sigma\sqrt{h}R^{(2)}$ and we defined $\bar{\gamma}_{\mathbf{ab}} = \gamma_{\mathbf{ab}} - \frac{1}{2}\,\delta_{\mathbf{ab}}\gamma_{\mathbf{c}}{}^{\mathbf{c}}$. The divergent part coming from $\langle I_\phi\rangle$ is straightforward to evaluate since it involves the same one-loop integral performed in (5.1.80)

$$\langle I_\phi\rangle = \frac{1}{8\pi}\int d^2\sigma\sqrt{h}R^{(2)}\,\langle\xi^\alpha\xi^\beta\rangle\nabla_\alpha\nabla_\beta\phi = \frac{\lambda}{8\pi}\,G(0)\int d^2\sigma\sqrt{h}R^{(2)}\,\nabla^2\phi = -\frac{\lambda}{16\pi^2}\frac{1}{\epsilon}\int d^2\sigma\sqrt{h}R^{(2)}\,\nabla^2\phi + \mathcal{O}(\epsilon^0)\;. \tag{5.1.90}$$

The double contraction term, on the other hand, it is a bit more involved. Not only because of the more complicated loop integral, but also because at the end of the computation one needs to recognize $\sqrt{h}R^{(2)}$ from its quadratic expansion in $\gamma_{\mathbf{ab}}$. After this tedious procedure one obtains

$$-\tfrac{1}{2}\,\langle I^2_{\gamma\partial\xi\partial\xi}\rangle_{1\text{PI}} = \frac{D}{24\pi}\,\frac{1}{\epsilon}\int d^2\sigma\sqrt{h}R^{(2)} + \mathcal{O}(\gamma^3) + \dots\;, \tag{5.1.91}$$

where the spacetime dimension D comes from a trace of the target-space flat metric $\eta_{\alpha\beta}$, the ellipses stand for UV-finite contributions, and $\mathcal{O}(\gamma^3)$ encodes the higher-order terms required to complete $\sqrt{h}R^{(2)}$.

In order to cancel the divergences one needs the counterterm

$$I_{\text{c.t.}} = \frac{1}{\epsilon}\int d^2\sigma\sqrt{h}R^{(2)}\left[-\frac{D}{24\pi}+\frac{\lambda}{16\pi^2}\nabla^2\phi\right] \quad\Rightarrow\quad T_1^\phi = -\frac{D}{6}+\frac{\alpha'}{2}\nabla^2\phi\,, \tag{5.1.92}$$

where we used (5.1.9) and $\lambda = 2\pi\alpha'$. Using the relation between β and T_1 (5.1.13) we can read the one-loop beta function for the dilaton

$$\beta^\phi = \frac{D}{6}-\frac{\alpha'}{2}\nabla^2\phi\,. \tag{5.1.93}$$

This was the last missing peace of (5.1.51) at one loop. Before presenting its final form, however, one needs to recall that β^g and β^ϕ receive an extra contribution coming from the reparametrization ghost system corresponding to the worldsheet metric being a dynamical field that needs to be integrated over in the path integral. As is well-known [111], for bosonic string this only produces a shift in the constant term $D\to D-26$. All in all, (5.1.51) now takes the form

$$\bar\beta^G_{\mu\nu} = \alpha'\left(\mathcal{R}_{\mu\nu}+2\,\nabla_\mu\nabla_\nu\phi\right)\,,\qquad \bar\beta^\phi = \frac{D-26}{6}-\frac{\alpha'}{2}\left(\nabla^2\phi-2\,\nabla^\mu\phi\nabla_\mu\phi\right)\,. \tag{5.1.94}$$

Setting these functions to zero provides the correct field equations associated to the target-space effective action

$$\mathcal{I} = \int d^DX\sqrt{-G}\,e^{-2\phi}\left[-\frac{2(D-26)}{3\,\alpha'}+\mathcal{R}+4\,\nabla^\mu\phi\nabla_\mu\phi\right]\,. \tag{5.1.95}$$

This is exactly the two-derivative low energy effective action used in (2.1.29) in the context of non-critical backgrounds. For critical dimensions, i.e. $D=26$, (5.1.95) reduces to the $B=0$ truncation of (2.1.1).

Dilaton beta function from consistency

As we pointed out at the beginning, and should have became clear form the sketch above, computing the dilaton beta function is more involved than other fields. This is due to requirement of a curved worldsheet and the fact that getting β^ϕ at L loops requires an $(L+1)$-loop computation.[5] For this reason, in practice one fixes β^ϕ by consistency [18, 19, 108–110]. Rather than discussing the general procedure, the idea is easily understood by giving the details in the simplest case at hand. Let us suppose that we do not know the dilaton β function. The metric β function, together with the general relation (5.1.51) fixes the metric field equation to be

$$\mathcal{R}_{\mu\nu}+2\,\nabla_\mu\nabla_\nu\phi = 0 \quad\Rightarrow\quad \mathcal{R}+2\,\nabla^2\phi = 0\,. \tag{5.1.96}$$

[5] The one-loop β^ϕ for the purely-metric sector (5.1.93), however, is not modified at two-loops. This higher-loop effect would be visible upon including the B-field, which would result in a further contribution $-\frac{\alpha'}{24}H^2$ arising at two loops.

Taking the divergence of the first equation one obtains

$$\begin{aligned}\nabla^\nu \mathcal{R}_{\mu\nu} &= -2\,\nabla^2\nabla_\mu\phi = -2\left(\nabla_\mu\nabla^2\phi + \mathcal{R}_{\mu\nu}\,\nabla^\nu\phi\right) \\ &= -2\,\nabla_\mu\nabla^2\phi + 4\left(\nabla_\mu\nabla_\nu\phi\right)\nabla^\nu\phi = -2\,\nabla_\mu\left(\nabla^2\phi - (\nabla\phi)^2\right) .\end{aligned} \tag{5.1.97}$$

Consistency with the Bianchi identity $\nabla^\nu \mathcal{R}_{\mu\nu} = \frac{1}{2}\,\nabla_\mu \mathcal{R}$ gives the constraint

$$0 = \nabla_\mu\left(\nabla^2\phi - (\nabla\phi)^2 + \tfrac{1}{4}\,\mathcal{R}\right) = \tfrac{1}{2}\,\nabla_\mu\left(\nabla^2\phi - 2\,(\nabla\phi)^2\right) , \tag{5.1.98}$$

that can be integrated to get

$$\nabla^2\phi - 2\,(\nabla\phi)^2 = C\,, \tag{5.1.99}$$

for an undetermined constant C. This shows that the dilaton equation in (5.1.94) is correctly reproduced by (5.1.99), apart from the constant C that can be easily fixed by matching with the one-loop result (5.1.94) to be $C = \frac{D-26}{3\alpha'}$. Here we use that, on dimensional grounds, the constant term in (5.1.94) cannot receive α' corrections.

5.2 Polyakov Action in Cosmological Backgrounds

Traditionally, in order to get the target-space action in an effective $(D-d)$-dimensional background one would first determine the higher-derivative corrections for the full target-space string theory by requiring vanishing of the Weyl anomaly and, second, dimensionally reduce along d directions. This is indeed the approach we followed in Section 3.2 where we took the D-dimensional effective actions for various strings up to order α'^3 (already present in the literature) and then reduced them to cosmological backgrounds. Now we aim to circumvent the need for a two-step procedure by computing the beta function and Weyl anomaly coefficients of a worldsheet theory already in cosmological backgrounds [4].

To this end, in this section we begin by presenting the purely metric sector of the Polyakov action in cosmological backgrounds and revisiting the general structure of the Weyl anomaly coefficients and the background field expansion. We then compute the beta functions at one loop and obtain the corresponding two-derivative target-space equations. While the worldsheet action is not $O(d,d)$ invariant, we show how the duality-covariant beta functions and equations of motion can be derived rather directly from the purely metric calculation. We thus finish recovering the two-derivative (duality-invariant) cosmological action (2.2.39).

5.2.1 Weyl anomaly coefficients and background-field method

Our starting point is the worldsheet action for bosonic string in curved D-dimensional background

$$I = \frac{1}{2\lambda}\int d^2\sigma\, G_{\mu\nu}(X)\partial_{\mathrm{a}}X^\mu\partial^{\mathrm{a}}X^\nu\,, \tag{5.2.1}$$

but this time we choose a target space with Euclidean signature. We now perform a dimensional reduction to purely time-dependent backgrounds following Section 2.2.2: we split coordinates as $X^\mu = (t, y^m)$ and make the ansatz that target-space fields only depend on (Euclidean) time t:

$$e_\mu{}^\alpha = \begin{pmatrix} n & 0 \\ 0 & e_m{}^a \end{pmatrix}, \quad G_{\mu\nu} = \begin{pmatrix} n^2 & 0 \\ 0 & g_{mn} \end{pmatrix}, \quad \phi = \frac{1}{2}\Phi + \frac{1}{4}\log\det g\,, \tag{5.2.2}$$

Having set the b-field to zero, the duality group $O(d,d)$ is reduced to $GL(d)\times\mathbb{Z}_2$ but we will assume that the full $O(d,d)$ group would be restored by switching on $b_{mn}(t)$. (Notice that we followed this same procedure in Section 3.2.) Using this ansatz into (5.2.1) gives the worldsheet action in cosmological background

$$I = \int d^2\sigma\,\mathcal{L}[t,y]\,, \quad \mathcal{L}[t,y] \equiv \frac{1}{2\lambda}\left[n^2(t)\,\partial^{\mathsf{a}} t\,\partial_{\mathsf{a}} t + g_{mn}(t)\,\partial^{\mathsf{a}} y^m \partial_{\mathsf{a}} y^n\right], \tag{5.2.3}$$

Since the dilaton target-space equation will be determined by consistency we can safely work on a flat worldsheet, thereby discarding the dilaton coupling. At this point, one may think that it would be more convenient to work directly with $G_{\mu\nu}(X)$ and use the ansatz (5.2.2) at the end of the computation. It turns out, instead, that considering $n^2(t)$ as a one-dimensional metric and $g_{mn}(t)$ as a $GL(d)$ multiplet of scalars from the beginning is crucial in order to produce manifest duality-invariant field equations.

The target-space equations are given by the vanishing of the anomaly coefficients. The generic form of these quantities can be derived by following the steps presented in Section 5.1.1 for generic two-dimensional sigma models or, more easily, they can be obtained from a dimensional reduction of $\bar\beta^G_{\mu\nu}$ in (5.1.30) in which case we obtain

$$\begin{aligned} \bar\beta &= \beta + \alpha' n^2\,\mathcal{D}^2\Phi + n^2\,\mathcal{D}\left(\mathcal{W} + \tfrac{1}{2}\mathrm{tr}\,(L)\right), \\ \bar\beta_{mn} &= \beta_{mn} + \alpha'\,\frac{1}{2}\mathcal{D}\Phi\mathcal{D}g_{mn} + \tfrac{1}{2}\left(\mathcal{W} + \tfrac{1}{2}\mathrm{tr}\,(L)\right)\mathcal{D}g_{mn}\,. \end{aligned} \tag{5.2.4}$$

In here we defined $\bar\beta = \bar\beta^G_{00}$ and $\bar\beta = \bar\beta^G_{mn}$, and analogously for the beta function, and $\mathcal{W} \equiv \frac{1}{n}W_0$. We also used the expression for the covariant derivative $\mathcal{D} = \frac{1}{n}\partial_0$ as well as the definition for $L_m{}^n = \mathcal{D}g_{mp}g^{pn}$ giving

$$\mathcal{D}\big(\log\det g\big) = \mathrm{tr}\left(\mathcal{D}g g^{-1}\right) = \mathrm{tr}\,(L)\,. \tag{5.2.5}$$

The one-dimensional vector $\mathcal{W}$ has an expansion in α' of the form $\mathcal{W} = \sum_{n=1}^{\infty}\alpha'^{\,n}\,\mathcal{W}^{(n)}$, where $\mathcal{W}^{(n)}$ contains $2n-1$ derivatives $\mathcal{D}$. These $\mathcal{W}^{(n)}$ can be obtained, in principle, by renormalization of the operators $\partial^{\mathsf{a}} t\partial_{\mathsf{a}} t$ and $\partial^{\mathsf{a}} y^m\partial_{\mathsf{a}} y^n$, but it is more convenient to determine them by other means. We will see later how this is done for the one- and two-loop case.

Apart from this one-dimensional vector, the only missing peaces in (5.2.4) are the beta functions. These are obtained from the divergent contributions of the quantum effective action, which can be computed using the background-field method. In what follows we revisit such method for the sigma model (5.2.3).

We begin by shifting the fields as $t \to t + \pi$ and $y^m \to y^m + \pi^m$, where now t and y^m are viewed as classical backgrounds, and integrating over the quantum fluctuations π and π^m. Since we want to preserve manifest one-dimensional diffeomorphism covariance, we shall redefine $\pi = \pi(\xi)$ in terms of the covariant fluctuation ξ, which is a genuine one-dimensional vector. We do not need to redefine the internal fluctuations π^m since they are already scalars under time reparameterization. We now follow similar steps as the ones carried for the D-dimensional case in Section 5.1.3. In cosmological backgrounds, however, such procedure is much simpler. We define the tangent vector

$$\xi \equiv n \frac{dt}{d\tau} \,, \tag{5.2.6}$$

in which the extra n factor plays the role of an einbein, making ξ already flat (the analogous to $\xi^\alpha = e_\mu{}^\alpha \xi^\mu$ in D-dimensions). The derivative along the geodesics is already manifestly covariant

$$\frac{d}{d\tau} = \xi \mathcal{D} \,, \tag{5.2.7}$$

with $\mathcal{D} = \frac{1}{n}\partial_t$ the covariant derivative under one-dimensional diffeomorphisms. The expansion of the shifted Lagrangian then takes the form (see (5.1.62))

$$\mathcal{L}[t + \pi(\xi), y^m + \pi^m] = e^{\xi \mathcal{D}} \mathcal{L}[t, y^m + \pi^m] \,. \tag{5.2.8}$$

It is a simple exercise to check that the following results hold

$$\mathcal{D}\partial_{\mathbf{a}} t = \frac{1}{n}\partial_{\mathbf{a}}\xi \,, \quad [\xi\mathcal{D}, \partial_{\mathbf{a}}] = 0 \,, \tag{5.2.9}$$

as a consequence of the absence of a spin connection and a Riemann tensor in one dimension. Using these identities together with $\mathcal{D}\partial_{\mathbf{a}} y^m = 0$ and the geodesic equation $\xi\mathcal{D}\xi = 0$ we can read the expanded action to all orders

$$\begin{aligned} I[t + \pi(\xi), y^m + \pi^m] = I[t, y^m] + I_1 \\ + \frac{1}{2\lambda}\int d^2\sigma \Big[\partial^{\mathbf{a}}\xi\partial_{\mathbf{a}}\xi + \sum_{p=2}^{\infty} \frac{1}{p!}\xi^p \mathcal{D}^p g_{mn} \partial^{\mathbf{a}} y^m \partial_{\mathbf{a}} y^n \\ + 2\sum_{p=1}^{\infty} \frac{1}{p!}\xi^p \mathcal{D}^p g_{mn}\partial^{\mathbf{a}} y^m \partial_{\mathbf{a}}\pi^n + \sum_{p=0}^{\infty}\frac{1}{p!}\xi^p\mathcal{D}^p g_{mn}\partial^{\mathbf{a}}\pi^m\partial_{\mathbf{a}}\pi^n\Big] \,. \end{aligned} \tag{5.2.10}$$

In here, $I[t, y^m]$ is the classical action, and I_1 contains linear terms in ξ or π^m that do not contribute to 1PI diagrams. The remaining terms encode kinetic as well as interaction terms and all target-space fields are evaluated at t. While the kinetic term for ξ already takes the canonical form, the one for π^m involves $g_{mn}(t)$, which depends on $\sigma^{\mathbf{a}}$. We solve this by flattening the internal fluctuation as $\pi^m = e^m{}_a \pi^a$. This introduces a local $SO(d)$ symmetry realizing the coset $GL(d)/SO(d)$, where the symmetric tensor $g_{mn} = e_m{}^a e_n{}^b \delta_{ab}$ is viewed as a standard representative. The worldsheet derivatives indeed covariantize as

$$\partial_{\mathbf{a}}\pi^m = e_a{}^m D_{\mathbf{a}}\pi^a \,, \quad D_{\mathbf{a}}\pi^a = \partial_{\mathbf{a}}\pi^a + n\,\partial_{\mathbf{a}} t\, W^a{}_b \pi^b \,, \quad W^{ab} = e_m{}^a \mathcal{D} e^{bm} \,. \tag{5.2.11}$$

The action (5.2.10) then splits into a kinetic term plus interaction vertices $I = I_{\text{kin}} + I_{\text{int}}$, with

$$I_{\text{kin}} = \frac{1}{2\lambda}\int d^2\sigma \left[\partial^{\mathbf{a}}\xi\partial_{\mathbf{a}}\xi + \partial^{\mathbf{a}}\pi^a\partial_{\mathbf{a}}\pi_a\right] , \tag{5.2.12}$$

from which we read the two-point functions

$$\langle\xi(\sigma_1)\xi(\sigma_2)\rangle = \lambda\int \frac{d^2p}{(2\pi)^2}\frac{e^{ip\cdot(\sigma_1-\sigma_2)}}{p^2} , \quad \langle\pi^a(\sigma_1)\pi^b(\sigma_2)\rangle = \lambda\,\delta^{ab}\int \frac{d^2p}{(2\pi)^2}\frac{e^{ip\cdot(\sigma_1-\sigma_2)}}{p^2} . \tag{5.2.13}$$

The interaction terms can be grouped, as usual, according to the power of ξ plus π^a fluctuations: $I_{\text{int}} = V_2 + V_3 + V_4 + \cdots$. These can be easily read by covariantizing the worldsheet derivatives on π^a in (5.2.10).

The quantum effective action is then computed from the normalized expectation value

$$-\tfrac{1}{\lambda}\Delta\Gamma[\varphi] = \langle e^{-I_{\text{int}}}\rangle_{1\text{PI}} . \tag{5.2.14}$$

Renormalizability of the sigma model (5.2.3) ensures that the only divergent parts of $\Delta\Gamma$ are local and proportional to either $\partial^{\mathbf{a}}t\partial_{\mathbf{a}}t$ or $\partial^{\mathbf{a}}y^m\partial_{\mathbf{a}}y^n$. Therefore, (5.2.14) takes the generic form

$$\Delta\Gamma[t,y] = \frac{1}{2\lambda}\int d^2\sigma \left[\mathcal{M}\,\partial^{\mathbf{a}}t\partial_{\mathbf{a}}t + \mathcal{M}_{mn}\,\partial^{\mathbf{a}}y^m\partial_{\mathbf{a}}y^n\right] + \cdots , \tag{5.2.15}$$

where the ellipses denote terms with other external-leg structure, which are UV-finite. The divergences of $\mathcal{M}$ and $\mathcal{M}_{mn}$ are minus the counterterms T and T_{mn} respectively, and their $1/\epsilon$ parts T_1 and $T_{1,mn}$ determine respectively the beta functions β and β_{mn} via equation (5.1.13).

5.2.2 Cosmological Polyakov at One loop

The full one-loop effective action is given by the normalized expectation value

$$\Gamma_{1l} = -\big\langle e^{-V_2}\big\rangle_{1\text{PI}} , \tag{5.2.16}$$

where only quadratic vertices are required, given by

$$\begin{aligned} V_2 &= \tilde{V}_2 + V_{\partial t\partial y} \\ \tilde{V}_2 &= \frac{1}{2\lambda}\int d^2\sigma \Big[2n\,\partial^{\mathbf{a}}t\,W_{ab}\,\partial_{\mathbf{a}}\pi^a\pi^b + n^2\partial^{\mathbf{a}}t\partial_{\mathbf{a}}t\,W^c{}_a\,W_{cb}\,\pi^a\pi^b \\ &\qquad + 2\,\xi\,\mathcal{D}g_{mn}e_a{}^j\partial^{\mathbf{a}}y^m\partial_{\mathbf{a}}\pi^a + \tfrac{1}{2}\,\xi^2\,\mathcal{D}^2g_{mn}\,\partial^{\mathbf{a}}y^m\partial_{\mathbf{a}}y^n\Big] . \end{aligned} \tag{5.2.17}$$

The extra term $V_{\partial t\partial y}$, which depends on the connection W^{ab}, is proportional to $\partial_{\mathbf{a}}t\partial^{\mathbf{a}}y^m$ and hence it cannot contribute to UV divergences. In order to renormalize n^2 and g_{mn}, at one-loop order we only need to compute

$$\Gamma^{\text{div}}_{1l} = \big\langle\tilde{V}_2\big\rangle - \tfrac{1}{2}\big\langle\tilde{V}_2^2\big\rangle_{1\text{PI}} . \tag{5.2.18}$$

The resulting Feynman integrals contain both UV and IR divergences. Using dimensional regularization, we substitute every integral by

$$\int \frac{d^2k}{(2\pi)^2} \longrightarrow \tilde{\mu}^{2-n} \int \frac{d^nk}{(2\pi)^n} , \quad n = 2 + \epsilon . \tag{5.2.19}$$

IR divergences can be regularized in various ways. One option is to substitute every massless propagator with $\frac{1}{p^2+m^2}$. This, however, requires to add a mass term to the Lagrangian that has to be renormalized and used in the background-field expansion, which makes higher-loop computations harder. We instead choose to regularize Feynman integrals by putting masses only in those propagators which actually cause infrared divergences at zero external momenta. The basic one-loop integral requiring both UV and IR regularization is the tadpole[6] $\int \frac{d^2k}{(2\pi)^2} \frac{1}{k^2}$, which is regularized as

$$\begin{aligned} G_{\text{tad}} &\equiv \tilde{\mu}^{2-n} \int \frac{d^nk}{(2\pi)^n} \frac{1}{k^2+m^2} = \frac{1}{4\pi} \left(\frac{m^2}{4\pi\tilde{\mu}^2} \right)^{\epsilon/2} \Gamma(-\epsilon/2) \\ &= -\frac{1}{2\pi} \left(\frac{1}{\epsilon} + \frac{1}{2} \log \frac{m^2}{\mu^2} \right) + \mathcal{O}(\epsilon) , \end{aligned} \tag{5.2.20}$$

where $\mu^2 = 4\pi\, e^{-\gamma} \tilde{\mu}^2$. We will also make frequent use of the identity

$$\int \frac{d^nk}{(2\pi)^n} 1 = 0 , \tag{5.2.21}$$

which can be derived by computing

$$\tilde{\mu}^{2-n} \int \frac{d^nk}{(2\pi)^n} \frac{k^2}{k^2+m^2} = -m^2 G_{\text{tad}} . \tag{5.2.22}$$

With these rules in place one can easily compute the tadpole diagrams in $\langle \tilde{V}_2 \rangle$, yielding

$$\langle \tilde{V}_2 \rangle = \frac{G_{\text{tad}}}{4} \int d^2\sigma\, \mathcal{D}^2 g_{mn}\, \partial^{\mathbf{a}} y^m \partial_{\mathbf{a}} y^n + \frac{G_{\text{tad}}}{2} \int d^2\sigma\, n^2\, W^{ab} W_{ab}\, \partial^{\mathbf{a}} t \partial_{\mathbf{a}} t . \tag{5.2.23}$$

The bubble diagrams contained in $\langle \tilde{V}_2^2 \rangle$ are generally non-local, being of the schematic form

$$\int \frac{d^2p}{(2\pi)^2} A(p)\, \Pi(p)\, B(-p) , \tag{5.2.24}$$

where A and B represent the Fourier transform of products of fields, *e.g.* $\partial_{\mathbf{a}} t\, n(t)\, W^{ab}(t)$, while $\Pi(p)$ is the one-loop bubble integral. The only UV divergent bubbles at one loop are given by

$$\Pi_{\mathbf{ab}}(p) = \int \frac{d^2k}{(2\pi)^2} \frac{k_{\mathbf{a}} k_{\mathbf{b}}}{k^2(p-k)^2} , \quad \Pi'_{\mathbf{ab}}(p) = \int \frac{d^2k}{(2\pi)^2} \frac{k_{(\mathbf{a}}(p-k)_{\mathbf{b})}}{k^2(p-k)^2} , \tag{5.2.25}$$

[6]In higher dimensions massless tadpoles are zero in dimensional regularization, but not in two dimensions.

and one can see by power counting that their UV divergence only comes from the zero-momentum contribution $\Pi(0) = \Pi'(0)$, which gives the local expression

$$\Pi(0) \int d^2\sigma \, A(x) \, B(x) \, . \tag{5.2.26}$$

The divergent part $\Pi(0)$ is regularized as

$$\begin{aligned} \Pi_{\mathbf{ab}}(0) &= \tilde{\mu}^{2-n} \int \frac{d^n k}{(2\pi)^n} \frac{k_{\mathbf{a}} k_{\mathbf{b}}}{k^2(k^2+m^2)} = \frac{1}{n} \, \delta_{\mathbf{ab}} \, \tilde{\mu}^{2-n} \int \frac{d^n k}{(2\pi)^n} \frac{k^2}{k^2(k^2+m^2)} \\ &= \frac{1}{n} \, \delta_{\mathbf{ab}} \, G_{\text{tad}} = -\frac{1}{4\pi\epsilon} \, \delta_{\mathbf{ab}} + \mathcal{O}(\epsilon^0) \, . \end{aligned} \tag{5.2.27}$$

With these techniques we can compute the divergent part of $-\frac{1}{2} \langle \tilde{V}_2^2 \rangle_{\text{1PI}}$, which reads

$$\begin{aligned} -\tfrac{1}{2} \langle \tilde{V}_2^2 \rangle_{\text{1PI}} = &- \frac{G_{\text{tad}}}{4} \int d^2\sigma \, \left(\mathcal{D}g g^{-1} \mathcal{D}g\right)_{mn} \partial^{\mathbf{a}} y^m \partial_{\mathbf{a}} y^n \\ &- \frac{G_{\text{tad}}}{4} \int d^2\sigma \, n^2 \, W^{ab} \big(W_{ab} - W_{ba}\big) \partial^{\mathbf{a}} t \partial_{\mathbf{a}} t \, . \end{aligned} \tag{5.2.28}$$

Summing the contributions of (5.2.23) and (5.2.28) one obtains the full one-loop divergences. These have to be canceled, in the MS scheme, by purely divergent counterterms. In the y-sector we obtain

$$I_{y,\text{c.t.}} = \frac{1}{8\pi\epsilon} \int d^2\sigma \, \left(\mathcal{D}^2 g - \mathcal{D}g g^{-1} \mathcal{D}g\right)_{mn} \partial^{\mathbf{a}} y^m \partial_{\mathbf{a}} y^n \, , \tag{5.2.29}$$

while the t-sector gives

$$\begin{aligned} I_{t,\text{c.t.}} &= \frac{1}{8\pi\epsilon} \int d^2\sigma \, n^2 \, W^{ab} \big(W_{ab} + W_{ba}\big) \, \partial^{\mathbf{a}} t \partial_{\mathbf{a}} t \\ &= -\frac{1}{16\pi\epsilon} \int d^2\sigma \, n^2 \, \text{tr} \left(\mathcal{D}g \mathcal{D}g^{-1}\right) \, \partial^{\mathbf{a}} t \partial_{\mathbf{a}} t \, . \end{aligned} \tag{5.2.30}$$

From here we read the simple poles

$$T_1 = -\frac{\lambda}{8\pi} n^2 \text{tr} \left(\mathcal{D}g \mathcal{D}g^{-1}\right) \, , \quad T_{1,mn} = \frac{\lambda}{4\pi} \big(\mathcal{D}^2 g - \mathcal{D}g g^{-1} \mathcal{D}g\big)_{mn} \, , \tag{5.2.31}$$

which in turn fixes the one-loop beta functions to be (see (5.1.13)):

$$\beta = \frac{\alpha'}{4} \, n^2 \, \text{tr} \left(\mathcal{D}g \mathcal{D}g^{-1}\right) \, , \quad \beta_{mn} = -\frac{\alpha'}{2} \left(\mathcal{D}^2 g - \mathcal{D}g g^{-1} \mathcal{D}g\right)_{mn} \, . \tag{5.2.32}$$

Duality invariance

Let us discuss the duality covariance of these beta functions. In the general case where the internal b-field $b_{mn}(t)$ is non vanishing, g_{mn} and b_{mn} can be combined into the manifestly $O(d,d)$ covariant generalized metric ${\mathcal{S}_M}^N$ (2.2.19). When b_{mn} is zero, the generalized metric simplifies to

$${\mathcal{S}_M}^N = \begin{pmatrix} 0 & g_{mn} \\ g^{mn} & 0 \end{pmatrix} \, , \tag{5.2.33}$$

which is covariant only under $GL(d) \times \mathbb{Z}_2$. It is now quite simple to see that strings of $GL(d)$ matrix products of the form

$$\left(\mathcal{D}^{p_1} g g^{-1} \mathcal{D}^{p_2} g g^{-1} \cdots g^{-1} \mathcal{D}^{p_n} g\right)_{mn} \tag{5.2.34}$$

coincide with the $_{mn}$ component of the duality covariant tensor

$$\left(\mathcal{D}^{p_1} \mathcal{S}\, \mathcal{S} \mathcal{D}^{p_2} \mathcal{S}\, \mathcal{S} \cdots \mathcal{S} \mathcal{D}^{p_n} \mathcal{S}\right)_M{}^N , \tag{5.2.35}$$

in terms of the simplified $\mathcal{S}$ in (5.2.33). Traces require more care: in general, $GL(d)$ traces will not combine into $O(d,d)$ ones prior to computing the Feynman integrals. This can be seen by remembering that $GL(d)$ traces of an even power of $L = \mathcal{D} g g^{-1}$ are duality invariant (see (3.2.9))

$$\text{Tr}\left((\mathcal{D}\mathcal{S})^{2k}\right)\big|_{b=0} = (-1)^k\, 2\, \text{tr}\left(L^{2k}\right) , \tag{5.2.36}$$

while odd powers cannot be written in terms of $O(d,d)$ traces. At this level, the trace in β (5.2.32) is indeed duality invariant since

$$\text{tr}\left(\mathcal{D} g \mathcal{D} g^{-1}\right) = \tfrac{1}{2}\, \text{Tr}\left(\left(\mathcal{D}\mathcal{S}\right)^2\right)\big|_{b=0} . \tag{5.2.37}$$

We have thus shown that the one-loop beta functions can be written in the form

$$\beta = \frac{\alpha'}{8}\, n^2\, \text{Tr}\left(\left(\mathcal{D}\mathcal{S}\right)^2\right) , \quad \beta_{mn} = -\frac{\alpha'}{2}\left(\mathcal{D}^2\mathcal{S} - \mathcal{D}\mathcal{S}\mathcal{S}\mathcal{D}\mathcal{S}\right)_{mn} . \tag{5.2.38}$$

This is not enough to prove covariance of β_{mn} under the $\mathbb{Z}_2$ T-duality: when the b-field is zero, a generic covariant tensor $\mathcal{A}_M{}^N$ has only components A_{mn} and $\widetilde{A}^{mn}$

$$\mathcal{A}_M{}^N = \begin{pmatrix} 0 & A_{mn} \\ \widetilde{A}^{mn} & 0 \end{pmatrix} , \tag{5.2.39}$$

and the $\mathbb{Z}_2$ duality acts by swapping $A_{mn} \leftrightarrow \widetilde{A}^{mn}$, which is induced by the $\mathbb{Z}_2$ operation $g \leftrightarrow g^{-1}$. Given only A_{mn}, as it is the case for the beta function β_{mn}, we want to construct the dual $\widetilde{A}^{mn}$ with $GL(d)$ operations. In order to do so, it is useful to remember that the property $\mathcal{S}^2 = 1$ allows to decompose $O(d,d)$ tensors into $\pm$ spaces with definite parity under conjugation by $\mathcal{S}$

$$\mathcal{A} = \mathcal{A}_+ + \mathcal{A}_- , \quad \mathcal{A}_\pm = \frac{1}{2}\left(\mathcal{A} \pm \mathcal{S}\mathcal{A}\mathcal{S}\right) , \quad \mathcal{A}_\pm \mathcal{S} = \pm \mathcal{S}\mathcal{A}_\pm . \tag{5.2.40}$$

From the decomposition (5.2.39), one can see that the projected components are simply related by conjugation by g, namely

$$A_{\pm\, mn} = \pm g_{mp}\, \widetilde{A}^{pq}_\pm\, g_{qn} . \tag{5.2.41}$$

While $\mathcal{D}\mathcal{S} = [\mathcal{D}\mathcal{S}]_-$, higher derivatives of $\mathcal{S}$ do not have definite parity under conjugation. For instance

$$\mathcal{K} \equiv [\mathcal{D}^2\mathcal{S}]_- = \mathcal{D}^2\mathcal{S} + \mathcal{S}\left(\mathcal{D}\mathcal{S}\right)^2 , \quad \mathcal{K} = \begin{pmatrix} 0 & K_{mn} \\ \widetilde{K}^{mn} & 0 \end{pmatrix} . \tag{5.2.42}$$

From (5.2.32) we see that $\beta_{mn} = -\frac{\alpha'}{2} K_{mn}$. Since the tensor $\mathcal{K}$ has odd parity, we see that the component β_{mn} is sufficient in order to define $\boldsymbol{\beta}_M{}^N$ by

$$\boldsymbol{\beta}_M{}^N = \begin{pmatrix} 0 & \beta_{mn} \\ \widetilde{\beta}^{mn} & 0 \end{pmatrix} , \quad \widetilde{\beta}^{mn} = -g^{mp}\beta_{pq}g^{qn} . \tag{5.2.43}$$

Target-space equations

Having computed the beta functions, the Weyl anomaly coefficients (5.2.4) are completely determined up to $\mathcal{W}^{(1)}$, which in principle could be obtained by renormalization of the dimension-two operators. A simpler way to do this, however, is to impose the symmetries of the theory. One-dimensional covariance and rigid $GL(d)$ invariance fix $\mathcal{W}^{(1)}$ up to a constant:

$$\mathcal{W}^{(1)} = k_1 \,\mathbf{tr}\left(\mathcal{D}gg^{-1}\right) = k_1 \mathbf{tr}\left(L\right) . \tag{5.2.44}$$

In order to fix the constant k_1, we *demand* duality covariance of the target-space equations, which requires $k_1 = -\frac{1}{2}$. This is a consequence of β and β_{mn} being already duality-covariant! The target-space field equations are then given by $\bar{\beta} = \bar{\beta}_{mn} = 0$

$$\begin{aligned} \frac{1}{8}\, n^2\, \mathrm{Tr}\left(\left(\mathcal{DS}\right)^2\right) + n^2\, \mathcal{D}^2\Phi = 0 \\ E_{mn} = \left(\mathcal{K} - \mathcal{D}\Phi\mathcal{DS}\right)_{mn} = 0\, . \end{aligned} \tag{5.2.45}$$

The scalar equation is manifestly $\mathbb{Z}_2$ invariant. The $GL(d)$ tensor equation $E_{mn} = 0$ instead requires the dual equation $\widetilde{E}^{mn} = 0$ to be satisfied. Since $\mathcal{K}$ and $\mathcal{DS}$ are both parity odd, this is indeed the case, given that $\widetilde{E}^{mn} = -g^{mp}E_{pq}g^{qn}$. We have thus shown that the equations (5.2.45) are duality invariant and we shall write them in covariant matrix form:

$$\mathrm{Tr}\left(\left(\mathcal{DS}\right)^2\right) + 8\,\mathcal{D}^2\Phi = 0\, , \quad \mathcal{D}^2\mathcal{S} + \mathcal{S}\left(\mathcal{DS}\right)^2 - \mathcal{D}\Phi\mathcal{DS} = 0\, . \tag{5.2.46}$$

As a further consistency check, one should notice that a field equation for $\mathcal{S}$ must be parity odd in order to descend from an action principle. This is due to $\delta\mathcal{S}$ being odd ($\delta\mathcal{S} = [\delta\mathcal{S}]_-$), and it can easily seen to be the case for (5.2.46). The target-space equations must be $O(d,d)$ invariant once b_{mn} is turned on. Given that the equations (5.2.46) are manifestly covariant, we invoke $O(d,d)$ symmetry to extend (5.2.46) to the case with non-vanishing b_{mn}, where now $\mathcal{S}$ is the full generalized metric (2.2.19). The field equations (5.2.46) coincide with the ones of the cosmological two-derivative low-energy effective action (2.2.39):

$$I^{(0)} = \int dt\, n\, e^{-\Phi}\left[-(\mathcal{D}\Phi)^2 - \frac{1}{8}\mathrm{Tr}\left((\mathcal{DS})^2\right)\right] , \tag{5.2.47}$$

Let us notice that the cosmological action used all over Chapter 3 is written for a Lorentzian target-space metric. The mapping to this Euclidean metric is given, at the worldsheet level, by $n^2 \to -n^2$. Given the same two-derivative action (5.2.47), the coefficients of the α'^n corrections will have a relative $(-1)^n$ sign compared to the notation used in Chapter 3.

In order to determine the dilaton equation, one demands mutual consistency of the other field equations (see equation (5.1.96) and below). In this case, one applies a covariant derivative $\mathcal{D}$ to the first equation in (5.2.46). Upon substituting the on-shell value of $\mathcal{D}^2\mathcal{S}$ and $\text{Tr}\left(\left(\mathcal{D}\mathcal{S}\right)^2\right)$, as well as using the identity

$$\text{Tr}\left(\mathcal{S}\left(\mathcal{D}\mathcal{S}\right)^n\right) = 0\,, \quad n \geq 0\,, \tag{5.2.48}$$

one finds $\mathcal{D}(\mathcal{D}^2\Phi - \mathcal{D}\Phi\mathcal{D}\Phi) = 0$. The three derivative equation can be integrated, with the integration constant being proportional to $\frac{d-25}{\alpha'}$ [18] (see (5.1.94)). The constant term vanishes in the critical dimension $d+1=26$, yielding

$$\mathcal{D}^2\Phi - \left(\mathcal{D}\Phi\right)^2 = 0\,, \tag{5.2.49}$$

which is equivalent to the dilaton equation derived from (5.2.47).

5.3 Cosmological Polyakov at Two Loops

In this section we turn to the duality-invariant two-loop beta function and the determination of the order-α' correction to the cosmological target-space equations [4]. We begin in the first subsection with a discussion of the ambiguities of beta functions, which reflect the ambiguities in the target-space theory due to field redefinitions. In the second subsection we compute the two-loop beta function, which will then be used in the third subsection to determine the target-space theory. We close, in the final subsection, with a discussion of possible simplifications for higher-loop computations.

5.3.1 Beta function ambiguities

Before starting the computation of the two-loop beta functions, we shall discuss a strategy to simplify the problem. The low-energy spacetime effective action has an expansion in powers of α' of the form $I = \sum_{n=0}^{\infty} \alpha'^n I^{(n)}$, where the two-derivative action $I^{(0)}$ is given by (5.2.47). The classification of [40] implies that there is a field basis in which all higher-derivative corrections take the form of the classification (3.1.1). In particular, the first-order correction $I^{(1)}$ is determined up to a single parameter c_1 as

$$I^{(1)} = c_1 \int dt\, n\, e^{-\Phi}\, \text{Tr}\left(\left(\mathcal{D}\mathcal{S}\right)^4\right)\,. \tag{5.3.1}$$

From the sigma model perspective, this implies that it is sufficient to determine the field equation of $\mathcal{S}$ in order to fix c_1. We will thus compute the two-loop beta function only for the g_{mn} coupling.

Let us suppose to have computed the beta function β_{mn} at two loops in the MS scheme. On general grounds, one expects it to be $GL(d)$ covariant. Furthermore, as we mentioned in Section 5.2.2, our perturbative expansion ensures that every term without traces can be written in terms of the duality-covariant matrix $\mathcal{S}$. This is not the case for terms containing traces, where $O(d,d)$ covariance can in general be established only after performing the Feynman integrals and possibly $GL(d)$-covariant field redefinitions. It turns out that at two loops the only $O(d,d)$-breaking trace term is $\text{tr}\,(g^{-1}\mathcal{D}^2 g)$ but, as we will show in the following, its coefficient is zero. This establishes that β_{mn} can be written in terms of $\mathcal{S}$ at two loops, without the need for any field redefinitions.

Coming now to possible ambiguities arising from field redefinitions, we shall assume that $\beta_{mn}(g,\mathcal{D}) = \beta_{mn}(\mathcal{S},\mathcal{D})$, for $b=0$. Expanding in α' one has

$$\begin{aligned} \beta_{mn} &= \alpha' \beta^{(1)}_{mn} + \alpha'^2 \beta^{(2)}_{mn} + \mathcal{O}\left(\alpha'^3\right) , \\ \beta &= \alpha' \beta^{(1)} + \mathcal{O}\left(\alpha'^2\right) , \end{aligned} \tag{5.3.2}$$

where $\alpha'\beta^{(1)}_{mn}$ and $\alpha'\beta^{(1)}$ are given by (5.2.38). The beta functions are computed in a given renormalization scheme, which we choose to be minimal subtraction. A change in the renormalization scheme is equivalent to a redefinition of the couplings n^2 and g_{mn} [36]. While this does not affect the one-loop beta functions, it introduces an ambiguity starting at two loops. Given a set of couplings $\Psi^I = (n^2, g_{mn})$, they can be viewed as a set of coordinates in the (infinite dimensional) coupling space. Since the beta functions are given by $\beta^I = \mu \frac{d\Psi^I}{d\mu}$, they are tangent vectors along renormalization group trajectories. A field redefinition of n^2 and g_{mn} can be viewed as a change of coordinates in coupling space: $\Psi^I \to \Psi^I + \delta\Psi^I$, under which β^I transforms as a vector, *i.e.* $\delta\beta^I = \mathcal{L}_{\delta\Psi}\beta^I$, where $\mathcal{L}$ denotes the Lie derivative. For the case at hand this results in

$$\delta\beta_{mn} = \delta g_{pq} \cdot \frac{\partial \beta_{mn}}{\partial g_{pq}} + \delta n^2 \cdot \frac{\partial \beta_{mn}}{\partial n^2} - \beta_{pq} \cdot \frac{\partial(\delta g_{mn})}{\partial g_{pq}} - \beta \cdot \frac{\partial(\delta g_{mn})}{\partial n^2} , \tag{5.3.3}$$

where we recall that derivatives are functional derivatives acting as

$$\begin{aligned} f_{mn} \cdot \frac{\partial A[g]}{\partial g_{mn}} &= \int dt\, f_{mn}(t)\, \frac{\delta}{\delta g_{mn}(t)}\, A[g] = A[g+f]|_{\text{linear part in } f} , \\ h \cdot \frac{\partial A[n^2]}{\partial n^2} &= \int dt\, h(t)\, \frac{\delta}{\delta n^2(t)}\, A[n^2] = A[n^2+h]|_{\text{linear part in } h} . \end{aligned} \tag{5.3.4}$$

Since the spacetime action (5.3.1) is given in a fixed field basis, it is necessary to account for the ambiguity (5.3.3) to be able to compare equations of motion. Given that $\beta^{(2)}_{mn}$ can already be written in terms of $\mathcal{S}$, we only look at field redefinitions for which $\delta\beta_{mn}$ can also be written in terms of $\mathcal{S}$. The most general redefinition with this property is given, at order α', by

$$\begin{aligned} \delta n^2 &= a_1\, \alpha' n^2\, \text{Tr}\left((\mathcal{D}\mathcal{S})^2 \right) , \\ \delta g_{mn} &= \alpha' \left[b_1\, \mathcal{K} + b_2\, \mathcal{S}(\mathcal{D}\mathcal{S})^2 \right]_{mn} . \end{aligned} \tag{5.3.5}$$

Using (5.3.5) in (5.3.3) one obtains (apart from $\delta\beta^{(1)}_{mn} = 0$)

$$\delta\beta^{(2)}_{mn} = p\left(\mathcal{K}\,\mathrm{Tr}\left((\mathcal{DS})^2\right) + \mathcal{DS}\,\mathrm{Tr}\left(\mathcal{KDS}\right)\right)_{mn} + q\left(\tfrac{1}{8}\,\mathcal{S}(\mathcal{DS})^2\mathrm{Tr}\left((\mathcal{DS})^2\right) - \mathcal{SK}^2\right)_{mn}\,, \tag{5.3.6}$$

where $p = \frac{1}{2}a_1 + \frac{1}{8}b_1$ and $q = b_2$. The remaining ingredient to write down the field equations $\bar\beta_{mn} = 0$ at two loops is the $\mathcal{W}$ vector appearing in (5.2.4). At order α'^2, $\mathcal{W}^{(2)}$ is a $GL(d)$ scalar with three derivatives $\mathcal{D}$ acting on g_{mn} and its inverse. There are several $GL(d)$-invariant possibilities, but requiring $O(d,d)$-invariance fixes $\mathcal{W}^{(2)}$ to be of the form

$$\mathcal{W}^{(2)} = k_2\,\mathrm{Tr}\left(\mathcal{KDS}\right)\ . \tag{5.3.7}$$

Having collected these ingredients, the target-space equations can be written as

$$\bar\beta_{mn} = -\tfrac{\alpha'}{2}\left(\mathcal{K} - \mathcal{D}\Phi\mathcal{DS}\right)_{mn} + \alpha'^2\left(\beta^{(2)}_{mn} + \delta\beta^{(2)}_{mn} + \tfrac{1}{2}\,\mathcal{W}^{(2)}\mathcal{D}g_{mn}\right) = 0\,, \tag{5.3.8}$$

which should be compared with the equations obtained from the action $I^{(0)} + \alpha' I^{(1)}$. As we have discussed in the previous section, any variational equation for $\mathcal{S}$ obtained from an action I must be of definite odd parity. This will provide a useful check of our computation, since any parity even term arising from $\beta^{(2)}_{mn}$ should be removable by a suitable choice of parameters in $\delta\beta^{(2)}_{mn}$. Having discussed how to relate the two-loop beta function with field equations, in the next section we will compute $\beta^{(2)}_{mn}$ by determining the divergent part of the two-loop effective action.

5.3.2 Two-loop beta function

The full effective action Γ has a meaningful expansion in powers of $\partial_{\mathrm{a}}y^m$ as

$$\Gamma = \Gamma_{0\partial y} + \Gamma_{1\partial y} + \Gamma_{2\partial y} + \cdots\ . \tag{5.3.9}$$

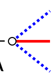
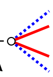
UV-divergent terms are present only in $\Gamma_{0\partial y}$, which determines the beta function for n^2, and in $\Gamma_{2\partial y}$, determining the beta function for g_{mn}. We then restrict to $\Gamma_{2\partial y}$ and compute its UV divergences. Renormalizability implies that the divergent part of $\Gamma_{2\partial y}$ contains no factors of $\partial_{\mathrm{a}}t$. We shall thus focus on the smaller subsector $\Gamma_{2\partial y, 0\partial t}$, which we name Γ_g. The relevant vertices at two loops can be obtained by applying the background-field expansion on the y-sector of (5.2.3), namely

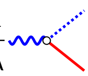
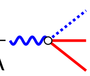
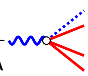
$$I_y = \frac{1}{2\lambda}\int d^2\sigma\, g_{mn}(t)\,\partial^{\mathrm{a}}y^m\partial_{\mathrm{a}}y^n\,, \tag{5.3.10}$$

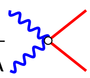
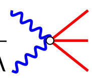
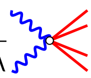
up to fourth order in fluctuations. The resulting interaction part, $I_{y\,\mathrm{int}}$, can be decomposed in terms of the number of external legs $\partial_{\mathrm{a}}y^m$ as follows:

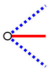
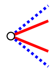
(5.3.11)

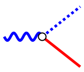
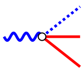
$I_{y\,\mathrm{int}} = V_{0y} + V_{1y} + V_{2y}\,,$

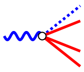
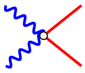
$V_{0y} = \frac{1}{2\lambda}$  $+ \frac{1}{4\lambda}$ 

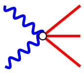
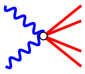
$V_{1y} = \frac{1}{\lambda}$  $+ \frac{1}{2\lambda}$  $+ \frac{1}{6\lambda}$  ,

$V_{2y} = \frac{1}{4\lambda}$  $+ \frac{1}{12\lambda}$  $+ \frac{1}{48\lambda}$  , where we choose the following representation for the vertices

 $= \int d^2\sigma\, \xi\, \mathcal{D} g_{mn} e_a{}^m e_b{}^n \partial^{\mathbf{a}}\pi^a \partial_{\mathbf{a}}\pi^b \,,$  $= \int d^2\sigma\, \xi^2\, \mathcal{D}^2 g_{mn} e_a{}^m e_b{}^n \partial^{\mathbf{a}}\pi^a \partial_{\mathbf{a}}\pi^b \,,$

 $= \int d^2\sigma\, \xi\, \mathcal{D} g_{mn} e_a{}^m \partial^{\mathbf{a}}\pi^a \partial_{\mathbf{a}} y^n \,,$  $= \int d^2\sigma\, \xi^2\, \mathcal{D}^2 g_{mn} e_a{}^m \partial^{\mathbf{a}}\pi^a \partial_{\mathbf{a}} y^n \,,$

 $= \int d^2\sigma\, \xi^3\, \mathcal{D}^3 g_{mn} e_a{}^m \partial^{\mathbf{a}}\pi^a \partial_{\mathbf{a}} y^n \,,$  $= \int d^2\sigma\, \xi^2\, \mathcal{D}^2 g_{mn} \partial^{\mathbf{a}} y^m \partial_{\mathbf{a}} y^n \,,$

 $= \int d^2\sigma\, \xi^3\, \mathcal{D}^3 g_{mn} \partial^{\mathbf{a}} y^m \partial_{\mathbf{a}} y^n \,,$  $= \int d^2\sigma\, \xi^4\, \mathcal{D}^4 g_{mn} \partial^{\mathbf{a}} y^m \partial_{\mathbf{a}} y^n \,.$

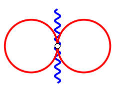
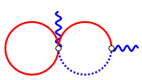
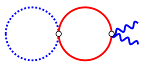
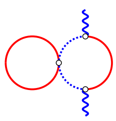
(5.3.12)

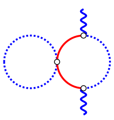
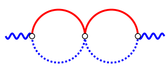
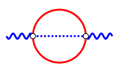
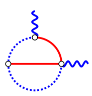
In this diagrammatic representation, straight red lines and dashed blue lines correspond to ξ and $\partial_{\mathbf{a}}\pi^a$ fluctuations, respectively, while wavy blue lines represent external legs $\partial_{\mathbf{a}} y^m$. The tensor structure of each vertex can be read from the diagram, since each red line corresponds to one derivative of g_{mn} and each dashed blue line represents an internal vielbein $e_a{}^m$. For instance, a vertex with p internal red lines and one internal blue line encodes the structure $\mathcal{D}^p g_{mn} e_a{}^m$.

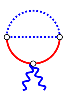
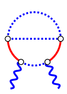
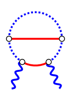
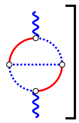
With these vertices, the relevant two-loop effective action is given by

$$\Gamma_{g,2l} = \left\langle e^{-V_{0y}} \left(V_{2y} - \tfrac{1}{2} V_{1y}^2 \right) \right\rangle_{\lambda,\text{1PI}} - \text{ subtractions} \,, \tag{5.3.13}$$

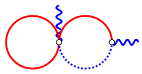
where the subscript means to keep only contributions of order λ, which correspond to two-loop diagrams. The role of subtraction terms is to remove one-loop subdivergences of two-loop diagrams, ensuring that all non-local divergences cancel. At two loops, the subtractions can be obtained by expanding the one-loop counterterms up to second order in fluctuations and using the new vertices to insert counterterms in one-loop diagrams. This procedure, however, does not seem valid at higher-loop order [107]. In view of possible future applications, we employ a different method, which consists in subtracting subdivergences diagram-by-diagram [38, 112, 113]. This also allows to show that entire classes of diagrams can be ignored when computing the beta function. We denote the subtraction procedure by an operator $\mathcal{R}$ acting on a given two-loop diagram. We also assume that all finite contributions are discarded at the end. The divergent part of $\Gamma_{g,2l}$ is then given by the sum of twelve diagrams as follows:

$\Gamma^{\text{div}}_{g,2l} = \lambda \mathcal{R} \Big[\frac{1}{16}$  $- \frac{1}{2}$  $- \frac{1}{8}$  $+ \frac{1}{4}$ 

$+ \frac{1}{4}$  $+ \frac{1}{2}$  $- \frac{1}{4}$  $+$  (5.3.14)

$+ \frac{1}{8}$  $- \frac{1}{4}$  $- \frac{1}{2}$  $- \frac{1}{2}$  $\Big] \,.$

A greatly simplifying feature of Γ_g is that its divergent part, at any loop order, can be computed from diagrams with zero external momenta. In particular, this means that the diagrams in (5.3.14) are in fact vacuum diagrams, with the external lines and white circles only denoting vertices. For the sake of compactness, the Feynman diagrams in (5.3.14) represent both the Feynman integral and the worldsheet structure, *e.g.*



$$
\begin{aligned}
&= J_{\mathbf{ab}} \int d^2\sigma \left(\mathcal{D}^3 \mathcal{S}\mathcal{S}\mathcal{D}\mathcal{S}\right)_{mn} \partial^{\mathbf{a}} y^m \partial^{\mathbf{b}} y^n \ , \\
J_{\mathbf{ab}} &= \int dk dl \, \frac{k_{\mathbf{a}} k_{\mathbf{b}}}{k^4 l^2} \ ,
\end{aligned}
\tag{5.3.15}
$$

where we introduced a shorthand notation for the dimensionally extended measure:

$$
\int dk \equiv \tilde{\mu}^{2-n} \int \frac{d^n k}{(2\pi)^n} \ , \qquad \mu^2 = 4\pi e^{-\gamma} \tilde{\mu}^2 \ . \tag{5.3.16}
$$

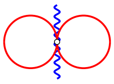
Before discussing the method, let us mention that all diagrams in (5.3.14) fall into two topological classes. The first six diagrams belong to the "chain" topology, which consists of two one-loop diagrams joined at a vertex. These are the simplest to compute since the two one-loop factors do not have common momenta. The remaining six diagrams belong instead to the so-called "sunset" topology, where the two individual loops share momentum along a common line.

We will now present the strategy to compute the two-loop diagrams. In order to better illustrate the procedure, we will include some detailed examples before giving the final result. As we have previously mentioned, the Feynman integrals corresponding to (5.3.14) are plagued by infrared divergences. In computing (the UV divergent part of) two-loop integrals, we will proceed as follows:

1. Write down each Feynman diagram with purely massless propagators.
2. Using algebraic and integration by parts identities, manipulate numerators to rewrite every *integrand* in terms of a basis of master integrals.
3. In this latter basis, put mass regulators only on propagators responsible for IR divergences.
4. Compute only the master integrals.

It should be stressed that this four-step procedure is valid at any loop order and at higher loops simplifies the computation enormously [33, 38, 112, 114]. From now on, the Feynman diagrams will represent only the integrals so that, for instance, the diagram (5.3.15) only stands for $J_{\mathbf{ab}}$. At two loops, it turns out that all twelve diagrams can be reduced to linear combinations of just two master diagrams, one for each topology.

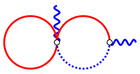
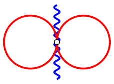
Let us start discussing the first three steps of the above list. The first diagram in (5.3.14) cannot be simplified, meaning that it is the master integral for the chain topology:

 $$= \left(\int dk \, \frac{1}{k^2} \right)^2 \xrightarrow{\text{IR reg.}} \left(\int dk \, \frac{1}{k^2 + m^2} \right)^2 . \tag{5.3.17}$$

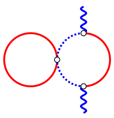
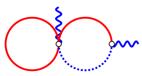
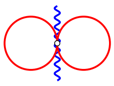
In the second step we have introduced masses as IR regulators where necessary. The second diagram in (5.3.14) can be reduced to (5.3.17) by using integration by parts (IBP) identities in momentum space. In particular, for the 2-loop case we only need

$$\int dk \, \frac{\partial}{\partial k_{\mathbf{a}}} \left(\frac{k_{\mathbf{b}}}{k^2} \right) = 0 \, , \tag{5.3.18}$$

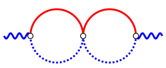
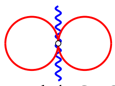
yet analogous identities exist for higher number of propagators and external momenta contributions. Using (5.3.18) we get

 $$\begin{aligned} &= \int dk dl \, \frac{k_{\mathbf{a}} k_{\mathbf{b}}}{k^4 l^2} = \left(\int dl \, \frac{1}{l^2} \right) \left[-\frac{1}{2} \int dk \, \frac{\partial}{\partial k^{\mathbf{a}}} \left(\frac{1}{k^2} \right) k_{\mathbf{b}} \right] \\ &= \left(\int dl \, \frac{1}{l^2} \right) \left(\frac{1}{2} \delta_{\mathbf{ab}} \int dk \, \frac{1}{k^2} \right) \\ &= \frac{1}{2} \, \delta_{\mathbf{ab}} \end{aligned} \tag{5.3.19}$$  .

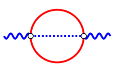
The third diagram in (5.3.14) is zero due to the identity $\int dk \, 1 = 0$, which arises in the diagram from the blue tadpole. The next diagram can be also reduced to (5.3.17) by an intermediate (almost trivial) algebraic step

 $= \int dk dl \, \frac{k_{\mathbf{a}} k_{\mathbf{b}} k^2}{k^6 l^2} = \int dk dl \, \frac{k_{\mathbf{a}} k_{\mathbf{b}}}{k^4 l^2} =$  $= \frac{1}{2} \, \delta_{\mathbf{ab}}$  , (5.3.20)

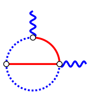
where we used (5.3.19) in the last equality. One can appreciate that starting with massless propagators is crucial for the reduction. Apart from another diagram vanishing due to a blue tadpole, the only remaining chain diagram can be reduced to (5.3.17) by using (5.3.18) twice:

 $= \int dk dl \, \frac{k_{\mathbf{a}} k^{\mathbf{c}}}{k^4} \frac{l_{\mathbf{c}} l_{\mathbf{b}}}{l^4} = \left(\frac{1}{2} \delta_{\mathbf{a}}{}^{\mathbf{c}} \int dk \, \frac{1}{k^2} \right) \left(\frac{1}{2} \delta_{\mathbf{cb}} \int dl \, \frac{1}{l^2} \right) = \frac{1}{4} \, \delta_{\mathbf{ab}}$  . (5.3.21)

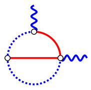
Starting with the sunset topology, the first diagram of this type is the only sunset master integral and is given by

 $= \int dk dl \, \frac{l_{\mathbf{a}} l_{\mathbf{b}}}{k^2 l^2 (k-l)^2} \xrightarrow{\text{IR reg.}} \int dk dl \, \frac{l_{\mathbf{a}} l_{\mathbf{b}}}{l^2 (k^2 + m^2) \big((k-l)^2 + m^2 \big)} \, .$ (5.3.22)

It is easy to see that the massive propagators correspond to the red lines in the diagram. Let us now give an explicit example of a reduction involving a nontrivial algebraic manipulation. We consider the next sunset diagram in (5.3.14), which is given by

 $= \int dk dl \, \frac{k_{(\mathbf{a}} l_{\mathbf{b})} k \cdot l}{k^4 l^2 (k-l)^2} \, .$ (5.3.23)

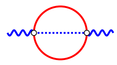
In order to proceed, we use the identity $k \cdot l = \frac{1}{2}\left(k^2 + l^2 - (k-l)^2\right)$ to rewrite (5.3.23) as



$$= \frac{1}{2}\int dkdl \left[\frac{k_{(\mathbf{a}}l_{\mathbf{b})}}{k^2l^2(k-l)^2} + \frac{k_{(\mathbf{a}}l_{\mathbf{b})}}{k^4(k-l)^2} - \frac{k_{(\mathbf{a}}l_{\mathbf{b})}}{k^4l^2}\right]$$
$$= \frac{1}{2}\int dkdl \left[\frac{k_{(\mathbf{a}}l_{\mathbf{b})}}{k^2l^2(k-l)^2} + \frac{k_{\mathbf{a}}k_{\mathbf{b}}}{k^4l^2} - 2\,\frac{k_{(\mathbf{a}}l_{\mathbf{b})}}{k^4l^2}\right]\;, \tag{5.3.24}$$

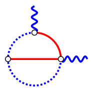
where, in going from the first to the second line, we wrote $l_{\mathbf{b}} = k_{\mathbf{b}} - (k-l)_{\mathbf{b}}$ and then renamed $k - l \to l$. This rewriting plus renaming trick corresponds to integrating by parts in configuration space (which has nothing to do with the IBP identity in momentum space (5.3.18)). By counting propagators one can see that the first term is of the form of (5.3.22), albeit with a different positioning of derivatives. The second term coincides with (5.3.19), while the last term vanishes by $SO(n)$ symmetry. At this point, we integrate by parts (in configuration space) the first term, yielding

$$\frac{1}{2}\int dkdl\,\frac{k_{(\mathbf{a}}l_{\mathbf{b})}}{k^2l^2(k-l)^2} = \frac{1}{2}\int dkdl\,\frac{k_{\mathbf{a}}k_{\mathbf{b}} - k_{(\mathbf{a}}(k-l)_{\mathbf{b})}}{k^2l^2(k-l)^2}$$
$$= \frac{1}{4}\int dkdl\,\frac{k_{\mathbf{a}}k_{\mathbf{b}}}{k^2l^2(k-l)^2} = \frac{1}{4}$$

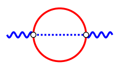


$$, \tag{5.3.25}$$

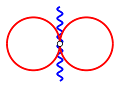
where we recognized the left-hand side in the second term of the first line. Putting the two terms together and using (5.3.19), we finally obtain the diagrammatic reduction



$$= \frac{1}{4}$$

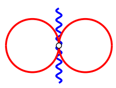


$$+ \frac{1}{4}\,\delta_{\mathbf{ab}}$$



$$. \tag{5.3.26}$$

For the remaining diagrams, the reduction procedure is completely analogous. One iteratively removes scalar products of momenta by using $k \cdot l = \frac{1}{2}\left(k^2 + l^2 - (k-l)^2\right)$ and cancels propagators when possible, integrates by parts when necessary, and further uses $\int dk\, 1 = 0$ as well as $SO(n)$ symmetry, which sets to zero all parity odd integrals. Applying this procedure to all diagrams in (5.3.14), one is left with the two master integrals



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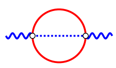
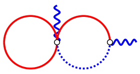
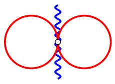
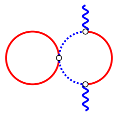
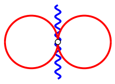
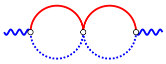
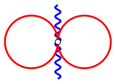


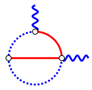
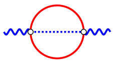
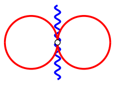
, (5.3.27)

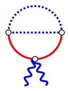
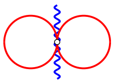
while the remaining diagrams can be reduced as follows:

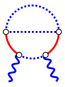
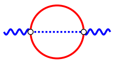
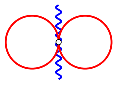
 $= \frac{1}{2}\,\delta_{\mathbf{ab}}$ 

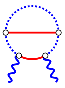
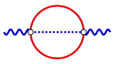
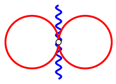
 $= \frac{1}{2}\,\delta_{\mathbf{ab}}$  ,

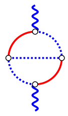
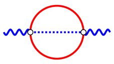
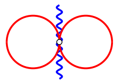
 $= \frac{1}{4}\,\delta_{\mathbf{ab}}$  ,

 $= \frac{1}{4}$  $+ \frac{1}{4}\,\delta_{\mathbf{ab}}$  , (5.3.28)

 $= -\frac{1}{4}$  ,

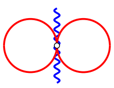
 $= \frac{1}{4}$  $- \frac{1}{4}\,\delta_{\mathbf{ab}}$  ,

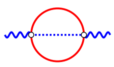
 $= \frac{1}{4}$  $+ \frac{1}{4}\,\delta_{\mathbf{ab}}$  ,

 $= -\frac{1}{4}$  $+ \frac{3}{8}\,\delta_{\mathbf{ab}}$  ,

where we have discarded the two diagrams with a blue tadpole, since they vanish in dimensional regularization. Interestingly, these are the only diagrams whose worldsheet structure is not $O(d,d)$ covariant, because of the $GL(d)$ trace $\mathrm{tr}\,(g^{-1}\mathcal{D}^2 g)$!

Having found the reduction, it is now time to compute the master integrals and subtract their subdivergences. The evaluation of the two master integrals trivially reduces to products of the basic tadpole G_{tad}, which is given in (5.2.20). While for the chain topology this can be seen already from (5.3.17), for the sunset we need to use $SO(n)$ symmetry to reduce the tensor integral $J_{\mathbf{ab}}$ to a scalar integral: $J_{\mathbf{ab}} = \frac{1}{n}\,\delta_{\mathbf{ab}}\,J^{\mathbf{c}}{}_{\mathbf{c}}$, yielding[7]

 $= (G_{\mathrm{tad}})^2 = \frac{1}{4\pi^2}\left(\frac{1}{\epsilon^2} + \frac{1}{\epsilon}\log\frac{m^2}{\mu^2}\right)$,

 $= \frac{1}{n}\,\delta_{\mathbf{ab}}\,(G_{\mathrm{tad}})^2 = \frac{\delta_{\mathbf{ab}}}{8\pi^2}\left(\frac{1}{\epsilon^2} + \frac{1}{\epsilon}\log\frac{m^2}{\mu^2} - \frac{1}{2\epsilon}\right)$. (5.3.29)

Let us now come to the $\mathcal{R}$ operation to remove subdivergences. This operation is recursive, allowing to subtract subdivergences at any loop order [113]. In the two-loop case, we shall proceed by following these steps:

1. Given a two-loop master integral, consider all possible one-loop subdiagrams obtained by cutting lines open.

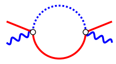
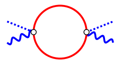
2. For each one-loop subdiagram, extract the divergent part (an operation that we denote by $\mathcal{P}$). This shrinks the original subdiagram to a vertex,

[7] Recall that $\mu^2 = 4\pi e^{-\gamma}\tilde{\mu}^2$ and that we are always discarding any finite part.

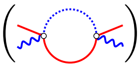
which we denote by a cross ⊗.

3. Substitute the one-loop subdiagram in the original two-loop diagram with the cross vertex found in the previous step. The substitution and insertion operation are denoted by $\star$.

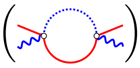
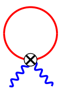
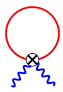
We will examine in detail the subtraction procedure for the sunset master integral, since the case of the chain master integral is simpler. First of all, the diagram (5.3.22) has two independent subdiagrams, the first of which appearing twice:

$2\times$  and  . (5.3.30)

The second subdiagram is UV finite and thus does not enter the subtraction. It should be mentioned that in (5.3.30) the wavy blue lines still carry zero momentum, but the other external lines have to be taken with arbitrary momenta, since they belong to the two-loop vacuum diagram. In this case (and in the other master integral as well), the divergence arises only at zero external momentum and is easily computed as

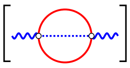
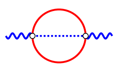
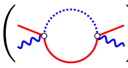
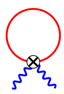
$\mathcal{P}\Big($  $\Big) = \mathcal{P}\int dk\, \frac{k_{\mathbf{a}}k_{\mathbf{b}}}{k^2(k^2+m^2)} = \delta_{\mathbf{ab}}\,\mathcal{P}\left(\frac{1}{n}\,I\right) = -\frac{1}{4\pi\epsilon}\,\delta_{\mathbf{ab}}\,,$ (5.3.31)

where we notice that the masses remain in the same propagators as in the original diagram. The insertion operator $\star$ reduces in this case to multiplication by (5.3.31). In more complicated cases, where the divergent part of a subdiagram has momentum dependence, this has to be inserted at the position of the cross vertex. In the case at hand this simply gives

$2\,\mathcal{P}\Big($  $\Big)\star$  $= -\frac{1}{2\pi\epsilon}\,\delta_{\mathbf{ab}}$  $= -\frac{1}{2\pi\epsilon}\,\delta_{\mathbf{ab}}\,G_{\mathrm{tad}}$

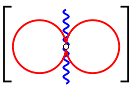
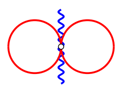
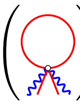
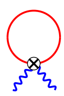
$$= \frac{1}{8\pi^2}\,\delta_{\mathbf{ab}}\left(\frac{2}{\epsilon^2}+\frac{1}{\epsilon}\log\frac{m^2}{\mu^2}\right)\,. \tag{5.3.32}$$

Subtracting (5.3.32) from the value of the sunset in (5.3.29) finally gives the subtracted result

$\mathcal{R}\Big[$  $\Big] =$  $-\,2\,\mathcal{P}\Big($  $\Big)\star$ 

$$= -\frac{\delta_{\mathbf{ab}}}{8\pi^2}\left(\frac{1}{\epsilon^2}+\frac{1}{2\epsilon}\right)\,. \tag{5.3.33}$$

Applying the same procedure to the chain master integral gives

$\mathcal{R}\Big[$  $\Big] =$  $-\,2\,\mathcal{K}\Big($  $\Big)\star$ 

$$= -\frac{1}{4\pi^2\epsilon^2}\,. \tag{5.3.34}$$

Let us pause to discuss this result. First of all, a good sanity check of the computation is that the divergent terms should be local and independent of both the mass regulator m and the renormalization scale μ. One can see from (5.3.33) and (5.3.34) that this is indeed the case. More importantly, the subtracted result of the chain diagram has no simple pole $\frac{1}{\epsilon}$. Since the beta function is computed, at all loops, from the simple pole part of the counterterms, this implies that the chain diagrams do not contribute to the beta function. This is an example of a more general result: by using this direct subtraction method it can be proven [113] that no diagram with the factorized chain topology can exhibit a $\frac{1}{\epsilon}$ pole. Besides implying that the first six diagrams in (5.3.14) can be discarded when computing the beta function, this also entails that any chain appearing in the reduction (5.3.28) can be ignored as well. We will compute the contribution of the chains nonetheless, in order to display the full two-loop divergences of the effective action.

Using the subtracted values (5.3.33) and (5.3.34) in the decomposition (5.3.28), one finds the values of all the integrals appearing in (5.3.14). Pairing them with the corresponding worldsheet structures, the two-loop divergences of the effective action can be written in the form

$$\Gamma^{\text{div}}_{g,2l} = -\frac{1}{2\lambda}\int d^2\sigma \left(\frac{1}{\epsilon^2}\,T_{2,mn} + \frac{1}{\epsilon}\,T_{1,mn}\right)\partial^{\mathsf{a}} y^m \partial_{\mathsf{a}} y^n \,, \tag{5.3.35}$$

The $GL(d)$ tensors $T_{1,mn}$ and $T_{2,mn}$ are the $_{mn}$ components of $O(d,d)$ matrices $\mathcal{T}_{1,M}{}^{N}$ and $\mathcal{T}_{2,M}{}^{N}$ constructed from $\mathcal{S}$ and $\mathcal{D}$. The direct reading of the tensor structures from the diagrams in (5.3.14) is in terms of $\mathcal{D}^n\mathcal{S}$, with n up to four. This is not a good basis, since $\mathcal{D}^n\mathcal{S}$ has no definite parity, except for $n=0,1$. We thus introduce a basis of independent odd structures, which we choose to be $\mathcal{DS}$, $\mathcal{K}$, $[\mathcal{DK}]_-$ and $[\mathcal{D}^2\mathcal{K}]_-$. All four-derivative structures, both even and odd, can be written in terms of this odd basis and $\mathcal{S}$. The manifest parity decomposition of $\mathcal{D}^n\mathcal{S}$ in this basis is given, up to fourth order, by

$$\begin{aligned}
\mathcal{DS} &= [\mathcal{DS}]_- \,,\\
\mathcal{D}^2\mathcal{S} &= [\mathcal{D}^2\mathcal{S}]_- - \mathcal{S}\big(\mathcal{DS}\big)^2 \,, \quad [\mathcal{D}^2\mathcal{S}]_- \equiv \mathcal{K}\,,\\
\mathcal{D}^3\mathcal{S} &= [\mathcal{DK}]_- - \big(\mathcal{DS}\big)^3 - \tfrac{3}{2}\,\mathcal{S}\left(\mathcal{DSK}+\mathcal{KDS}\right),\\
\mathcal{D}^4\mathcal{S} &= [\mathcal{D}^2\mathcal{K}]_- - 2\,\mathcal{S}\Big(\mathcal{DS}[\mathcal{DK}]_- + [\mathcal{DK}]_-\mathcal{DS}\Big) - 3\,\mathcal{SK}^2\\
&\quad + \mathcal{S}(\mathcal{DS})^4 - 2\,\mathcal{DSKDS} - \tfrac{3}{2}\Big(\mathcal{K}(\mathcal{DS})^2 + \mathcal{K}(\mathcal{DS})^2\Big)\,.
\end{aligned} \tag{5.3.36}$$

Using the decomposition (5.3.36) and recalling that $\lambda = 2\pi\alpha'$, the matrices $\mathcal{T}_{1,2}$ can be finally written as

$$\begin{aligned}
\mathcal{T}_2 &= \frac{\alpha'^2}{8}\left[[\mathcal{D}^2\mathcal{K}]_- - \mathcal{SK}^2 + \frac{1}{2}(\mathcal{DS})^2\mathcal{K} + \frac{1}{2}\mathcal{K}(\mathcal{DS})^2 + \frac{1}{4}\mathcal{K}\,\mathrm{Tr}\left((\mathcal{DS})^2\right)\right],\\
\mathcal{T}_1 &= \frac{\alpha'^2}{8}\left[\frac{1}{2}(\mathcal{DS})^2\mathcal{K} + \frac{1}{2}\mathcal{K}(\mathcal{DS})^2 + \mathcal{SK}^2 - \frac{1}{8}\mathcal{S}(\mathcal{DS})^2\,\mathrm{Tr}\left((\mathcal{DS})^2\right)\right].
\end{aligned} \tag{5.3.37}$$

The higher-pole term $\mathcal{T}_2$ obey the so-called pole equations [103, 104], which are typically used for consistency checks. We do not carry such computations here. The simple pole part $\mathcal{T}_1$ determines the (duality-covariant) beta function, whose explicit form at two loops is given by

$$\beta^{(2)} = -\frac{1}{4}\left[\frac{1}{2}(\mathcal{D}\mathcal{S})^2\mathcal{K} + \frac{1}{2}\mathcal{K}(\mathcal{D}\mathcal{S})^2 + \mathcal{S}\mathcal{K}^2 - \frac{1}{8}\mathcal{S}(\mathcal{D}\mathcal{S})^2\,\mathrm{Tr}\left((\mathcal{D}\mathcal{S})^2\right)\right] . \tag{5.3.38}$$

5.3.3 Target-space theory at order α'

Having computed the two-loop beta function, the field equations (5.3.8) are given by

$$\begin{aligned}\bar{\beta}_{p,q} = -\frac{\alpha'}{2}\left(\mathcal{K} - \mathcal{D}\Phi\mathcal{D}\mathcal{S}\right) - \frac{\alpha'^2}{4}\Bigg[&\frac{1}{2}(\mathcal{D}\mathcal{S})^2\mathcal{K} + \frac{1}{2}\mathcal{K}(\mathcal{D}\mathcal{S})^2 \\ &+ (1+4\,q)\left(\mathcal{S}\mathcal{K}^2 - \frac{1}{8}\mathcal{S}(\mathcal{D}\mathcal{S})^2\,\mathrm{Tr}\left((\mathcal{D}\mathcal{S})^2\right)\right) \\ &- 4p\,\mathcal{K}\,\mathrm{Tr}\left((\mathcal{D}\mathcal{S})^2\right) - 4\left(p + \tfrac{1}{2}\,k_2\right)\mathcal{D}\mathcal{S}\,\mathrm{Tr}\left(\mathcal{K}\mathcal{D}\mathcal{S}\right)\Bigg] = 0\,,\end{aligned} \tag{5.3.39}$$

where we included the ambiguity (5.3.6) and the unknown $\mathcal{W}$ vector (5.3.7). Before comparing the equation (5.3.39) with the one obtained from the target-space action, let us analyze the structure of the α' correction. We recall that the two parameters p and q are completely arbitrary, reflecting the ambiguity in the renormalization scheme. The parameter k_2, on the other hand, ought to be determined once p and q are chosen. Coming to the $O(d,d)$ tensor structures appearing in (5.3.39), the first line consists of parity odd terms with no traces. These terms are not affected by the ambiguity. The second line contains parity even terms, in a combination which is fully ambiguous, while the last line displays parity odd terms with a trace.

As we have discussed in the previous sections, any field equation for $\mathcal{S}$ must have definite odd parity in order to be duality-invariant, meaning that it should obey

$$\mathcal{S}\bar{\beta}_{p,q}\mathcal{S} + \bar{\beta}_{p,q} = 0\,. \tag{5.3.40}$$

Since the sigma model (5.2.3) only exhibits manifest $GL(d)$ symmetry, one does not expect (5.3.40) to hold for arbitrary renormalization schemes (p,q). One can see, however, that all schemes with $q = -\frac{1}{4}$ do obey (5.3.40) and are thus duality invariant. Let us stress that there are three independent parity even structures with four derivatives[8]: $\mathcal{S}\mathcal{K}^2$, $\mathcal{S}(\mathcal{D}\mathcal{S})^2\mathrm{Tr}\left((\mathcal{D}\mathcal{S})^2\right)$ and $\mathcal{S}(\mathcal{D}\mathcal{S})^4$, but only one linear combination is ambiguous, namely $\mathcal{S}\mathcal{K}^2 - \frac{1}{8}\mathcal{S}(\mathcal{D}\mathcal{S})^2\mathrm{Tr}\left((\mathcal{D}\mathcal{S})^2\right)$. It is thus a highly nontrivial check of our computation that the only parity even terms in (5.3.38) appear in the ambiguous combination! We shall thus choose

[8]From the diagrammatic expansion of the effective action one can see that only odd powers of $\mathcal{S}$ can appear.

the duality invariant scheme $q=-\frac{1}{4}$ and rewrite the field equation as

$$\bar{\beta}_{p,-\frac{1}{4}} = -\frac{\alpha'}{2}\left(\mathcal{K}-\mathcal{D}\Phi\mathcal{D}\mathcal{S}\right) - \alpha'^2\left[\frac{1}{8}(\mathcal{D}\mathcal{S})^2\mathcal{K}+\frac{1}{8}\mathcal{K}(\mathcal{D}\mathcal{S})^2 \right.$$
$$\left. - p\,\mathcal{K}\,\mathrm{Tr}\left((\mathcal{D}\mathcal{S})^2\right) - \left(p+\tfrac{1}{2}k_2\right)\mathcal{D}\mathcal{S}\,\mathrm{Tr}\left(\mathcal{K}\mathcal{D}\mathcal{S}\right)\right] = 0\,. \tag{5.3.41}$$

The equivalence of (5.3.41) with the field equation of the target-space theory is perturbative in α'. In particular, this means that one can rewrite terms by using the lower order equations, *i.e.*

$$\begin{aligned} \mathcal{K} &= \mathcal{D}\Phi\mathcal{D}\mathcal{S} + \mathcal{O}(\alpha')\,, \\ \mathcal{D}^2\Phi &= \left(\mathcal{D}\Phi\right)^2 + \mathcal{O}(\alpha')\,, \\ \mathrm{Tr}\left((\mathcal{D}\mathcal{S})^2\right) &= -8\left(\mathcal{D}\Phi\right)^2 + \mathcal{O}(\alpha')\,, \end{aligned} \tag{5.3.42}$$

only committing errors of order α'^2. In particular, upon using (5.3.42) one can see that all four-derivative odd tensors reduce to two different structures:

$$\begin{aligned} \mathcal{K}(\mathcal{D}\mathcal{S})^2 = (\mathcal{D}\mathcal{S})^2\mathcal{K} = \mathcal{D}\mathcal{S}\mathcal{K}\mathcal{D}\mathcal{S} &= \mathcal{D}\Phi(\mathcal{D}\mathcal{S})^3 + \mathcal{O}(\alpha')\,, \\ \mathcal{K}\,\mathrm{Tr}\left((\mathcal{D}\mathcal{S})^2\right) = \mathcal{D}\mathcal{S}\,\mathrm{Tr}\left(\mathcal{K}\mathcal{D}\mathcal{S}\right) &= -8\left(\mathcal{D}\Phi\right)^3\mathcal{D}\mathcal{S} + \mathcal{O}(\alpha')\,. \end{aligned} \tag{5.3.43}$$

Using the on-shell values (5.3.43), the field equation (5.3.41) is perturbatively equivalent to the simpler form

$$\bar{\beta}'_p = -\frac{\alpha'}{2}\left(\mathcal{K}-\mathcal{D}\Phi\mathcal{D}\mathcal{S}\right) - \alpha'^2\left[\frac{1}{4}\mathcal{D}\Phi(\mathcal{D}\mathcal{S})^3 + 4\left(4p+k_2\right)\left(\mathcal{D}\Phi\right)^3\mathcal{D}\mathcal{S}\right] = 0\,. \tag{5.3.44}$$

We are now ready to compare (5.3.44) with the field equation obtained from the target-space action

$$I^{(0)} + \alpha' I^{(1)} = \int dt\, n\, e^{-\Phi}\left[-\left(\mathcal{D}\Phi\right)^2 + \mathrm{Tr}\left(-\tfrac{1}{8}\left(\mathcal{D}\mathcal{S}\right)^2 + \alpha' c_1\left(\mathcal{D}\mathcal{S}\right)^4\right)\right]. \tag{5.3.45}$$

Given that the action (5.3.45) does not contain double traces, no single trace can appear in the field equations. Using (5.3.43) implies that the order α' correction does not contain terms $\left(\mathcal{D}\Phi\right)^3\mathcal{D}\mathcal{S}$. Demanding this to be consistent with (5.3.44), determines $k_2=-4p$ for a given choice of scheme p. Varying (5.3.45) with respect to $\mathcal{S}$ yields

$$\begin{aligned} \mathcal{E} &= -\frac{1}{2}\left(\mathcal{K}-\mathcal{D}\Phi\mathcal{D}\mathcal{S}\right) + 8\,\alpha' c_1\left[\mathcal{K}(\mathcal{D}\mathcal{S})^2 + \mathcal{D}\mathcal{S}\mathcal{K}\mathcal{D}\mathcal{S} + (\mathcal{D}\mathcal{S})^2\mathcal{K} - \mathcal{D}\Phi(\mathcal{D}\mathcal{S})^3\right] \\ &= -\frac{1}{2}\left(\mathcal{K}-\mathcal{D}\Phi\mathcal{D}\mathcal{S}\right) + 16\,\alpha' c_1\,\mathcal{D}\Phi(\mathcal{D}\mathcal{S})^3 + \mathcal{O}(\alpha'^2) = 0\,. \end{aligned} \tag{5.3.46}$$

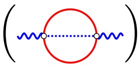
Comparing the above equation with (5.3.44) determines the c_1 coefficient to be

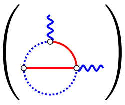
$$c_1 = -\frac{1}{64}\,, \tag{5.3.47}$$

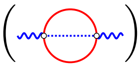
which, including the sign difference due to Euclidean signature, coincides with the known result for the bosonic string [3,39,42] which we also rederived in Section 3.2.2 (see Table 3.1).

We just computed the full two-loop beta function (5.3.38), as well as the higher pole term $\mathcal{T}_2$, including ambiguities in the renormalization scheme. We now finish this section by discussing a strategy to maximally simplify the computation in the case where one's only goal is to fix c_1. We believe this approach could be useful for future higher-loop calculations in order to fix the coefficients of the classification (3.1.1).

First of all, if one is only interested in computing the beta function, all diagrams with a product topology (which at two loops consist of the chain topology) can be ignored. In our case this already reduces the number of diagrams from twelve to six. Moreover, duality invariance of the target-space equations implies that there should be a renormalization scheme (which corresponds to $q = -\frac{1}{4}$ in our case) in which the beta function has definite odd parity. Assuming this to be the case allows to ignore the last two Feynman diagrams in (5.3.14), since they have the purely even structure $\mathcal{S}(\mathcal{DS})^4$. Finally, from the classification (3.1.1) one knows that trace terms should be removable by field redefinitions[9]. Assuming that traces do not contribute in determining c_1 allows to ignore two more diagrams with a closed blue loop in (5.3.14). At the end, c_1 is determined by two Feynman diagrams, which in turn depend on a single master integral, as follows:

$$\Gamma_{\text{relevant}} = -\frac{\lambda}{4}\mathcal{R}_{\frac{1}{\epsilon}}\Big($$  $$\Big)_{\mathbf{ab}} \int d^2\sigma \left[\mathcal{D}^2\mathcal{S}\mathcal{S}\mathcal{D}^2\mathcal{S}\right]_{-,mn} \partial^{\mathbf{a}} y^m \partial^{\mathbf{b}} y^n$$

$$- \lambda\,\mathcal{R}_{\frac{1}{\epsilon}}\Big($$  $$\Big)_{\mathbf{ab}} \int d^2\sigma \left[(\mathcal{DS})^2\mathcal{D}^2\mathcal{S}\right]_{-,mn} \partial^{\mathbf{a}} y^m \partial^{\mathbf{b}} y^n$$

$$= -\frac{\lambda}{4}\mathcal{R}_{\frac{1}{\epsilon}}\Big($$  $$\Big)_{\mathbf{ab}} \int d^2\sigma \left[\mathcal{D}^2\mathcal{S}\mathcal{S}\mathcal{D}^2\mathcal{S} + (\mathcal{DS})^2\mathcal{D}^2\mathcal{S}\right]_{-,mn} \partial^{\mathbf{a}} y^m \partial^{\mathbf{b}} y^n$$

$$= \frac{\lambda}{64\pi^2\epsilon} \int d^2\sigma \left[\mathcal{D}^2\mathcal{S}\mathcal{S}\mathcal{D}^2\mathcal{S} + (\mathcal{DS})^2\mathcal{D}^2\mathcal{S}\right]_{-,mn} \partial^{\mathbf{a}} y^m \partial_{\mathbf{a}} y^n \,, \tag{5.3.48}$$

where by $[\quad]_-$ we denoted the odd projection and by $\mathcal{R}_{\frac{1}{\epsilon}}$ we meant to discard higher poles. We have also used the reduction (5.3.28) in terms of master integrals, and the subtracted value (5.3.33). The odd projection of the above $O(d,d)$ tensor is given by

$$\left[\mathcal{D}^2\mathcal{S}\mathcal{S}\mathcal{D}^2\mathcal{S} + \tfrac{1}{2}\mathcal{D}^2\mathcal{S}(\mathcal{DS})^2 + \tfrac{1}{2}(\mathcal{DS})^2\mathcal{D}^2\mathcal{S}\right]_- = -\tfrac{1}{2}\left(\mathcal{K}(\mathcal{DS})^2 + (\mathcal{DS})^2\mathcal{K}\right) , \tag{5.3.49}$$

Finally, (5.3.48) yields the beta function

$$\beta^{(2)} = -\frac{1}{8}\left(\mathcal{K}(\mathcal{DS})^2 + (\mathcal{DS})^2\mathcal{K}\right) = -\frac{1}{4}\mathcal{D}\Phi(\mathcal{DS})^3 + \mathcal{O}(\alpha'^3) \,, \tag{5.3.50}$$

which is indeed the relevant part of (5.3.38)!

[9] This is true at two and three-loop level. For higher loops, traces can be ignored nonetheless if one is only interested in determining the coefficients of the single trace terms of the target-space action.

Chapter 6

Conclusions and Outlook

In this thesis we studied α' corrections to D-dimensional string low energy effective actions with $D-1$ isometries. To this end, we made heavy use of T-duality and the ideas behind the cosmological classification of [40]. Such construction characterizes duality-invariant cosmologies to all orders in α', but since at each order in α' there are a finite number of parameters that are not determined by $O(d,d)$, it remains an open question to which points in this theory space actual string theories belong.

We made progress in this direction by determining all free parameters up to and including α'^3 for bosonic, heterotic and type II string theory. To this end we took the dilaton-gravity actions that can be found in the literature, performed a cosmological reduction, and brought the reduced theories to a canonical field basis. We showed that they are compatible with duality-invariance, which can be promoted to the full $O(d,d)$-symmetry to predict the missing B-field and eight-derivative-dilaton couplings. The complete couplings for the massless fields in generic backgrounds at order α'^3 remain unknown, except for a few cases: in [94] the bosonic string effective action including all massless fields up to order α'^2 was determined by demanding T-duality invariance. The same method was used in [60] to obtain the complete Type II at order α'^3. These full theories were then reduced to cosmological backgrounds in [91] and [90], being in perfect agreement with the results obtained here.

The previous target-space approach relies on knowing the parent theories prior to compactification, which usually involves very challenging S-matrix or beta-function computations. We proposed a method to alleviate such complications by introducing a beta-function approach based on a worldsheet theory already adapted to cosmological backgrounds. Despite not being $O(d,d)$ invariant, we found an efficient procedure to determine the duality-invariant beta functions and tested it by computing the first two non-trivial physical coefficients in the cosmological classification for bosonic strings. This yields an independent first-principle derivation of the $\mathcal{O}(\alpha')$ corrections and set the stage for higher-loop extensions.

As a toy model of how to deal with α'-complete string theory in presence of massive modes we have explored the Hohm-Siegel-Zwiebach (HSZ) theory for purely time-dependent cosmological backgrounds. While HSZ theory, being based on a non-standard chiral CFT, is not a conventional string theory, it shares many features of string theory such as the presence of a fundamental parameter α' governing higher-derivative corrections and exact duality invariance under $O(d,d)$. We were able to provide a two-derivative yet α'-exact reformulation in which the tensionless limit $\alpha' \to \infty$ can be taken smoothly, and we set up perturbation theory with $\frac{1}{\alpha'}$ as the small expansion parameter. We found a string frame de Sitter solution at zeroth order in such expansion, and showed that it survives the first $\frac{1}{\alpha'}$ correction.

With the purpose of extending the classification to other backgrounds, we revisited the systematic method of [40] for Bianchi Type I (BI) cosmologies governed by q scale factors, and for non-critical two-dimensional (2D) backgrounds with time-like isometry. For the former, we have shown that only $q-1$ scale factors receive non-trivial higher-derivative corrections, implying that in flat Friedmann-Robertson-Walker (FRW) backgrounds *all* α' corrections are on-shell trivial. For the 2D backgrounds we first emphasized that for non-critical dimensions there are *no* invariant terms other than the cosmological term: any contribution with two or more derivatives can be traded for terms with an arbitrary number of derivatives, hence invalidating the familiar setting of perturbative α' corrections. We then circumvented this obstacle by assuming that the numerical coefficients governing the α' expansion in the action fall off in such a manner that terms with more derivatives are sub-leading relative to those with less derivatives. We gave a meaningful classification of higher-derivative corrections and applied it to the black hole (BH) solution of 2D string theory.

Using this larger space of duality-invariant theories, we explored non- perturbative solutions. We started from the perturbatively defined equations of motion, in which the infinitely many α' corrections are encoded in a function $f(M)$, defined as a series containing only even powers of the Hubble-like parameter M. We changed gears to describe the theory space in terms of the inverse function $M(f)$. Using f as a parameter, and with square roots of non-trivial functions appearing in the equations, the f space naturally becomes a space with branch points and branch cuts, which determine the underlying structure of the solution. We have argued that the f parameterization provides a non-perturbative extension of the original theory that applies when the series defining the perturbative α'-corrected action does not converge.

We saw that the BH interior, parameterized by $\tilde{f}$, leads to a branch point at some real, positive f_0 with a cut going all the way to $\tilde{f} = \infty$. For the black hole interior with a singularity (see Fig. 4.4 (a)), the x coordinate reaches a finite value at the branch point f_0, a point that can be reached in finite proper time. This means that the space cannot end there, and one must indeed return to $\tilde{f} = \infty$ over the cut. To avoid the singularity, while preserving a horizon, we altered the nature of the branch point f_0 (Fig. 4.4 (b)). Now x reaches an infinite value as we approach f_0, and it takes infinite proper time to get there. Thus we get a complete space without having to go over the branch. This yields a black hole with a horizon but a regular interior, a regular black hole (Fig. 4.4 (c)). The main caveat is that such solution does not appear to be a string theory one.

We have also discussed duality in the black hole solutions. The maximally extended geometry of the black hole of the two-derivative theory describes a self-dual solution where, as noted by Giveon [46], duality maps the horizon to the curvature singularity. We showed this is actually a general result. In particular, complete solutions that have horizons but no singularities cannot be self-dual.

With these results we contributed to a better understanding of higher- derivative corrections to the low energy limits of string theory. We did so by working with a broader class of theories constrained by T-duality, which provides us with a rich theoretical laboratory to explore extensions of classical gravity. More concretely, we made progress in three different yet related directions: restrict the possible subspaces of this duality-invariant theory space in which the known string theories must live, find solutions of α'-complete equations and analyze the physical implications of including string-like effects, and extend the classification to other backgrounds by following the same duality-invariant principles. Each of these directions branch into a huge landscape of possibilities and thus it is fair to say that our results here only scratch the surface. In what follows we present (what we think are) some interesting directions to explore in the future.

Future directions

Regarding string theories inside the cosmological classification, an important first future direction would be to cross-check our results for the coefficients by different methods. For instance, one might compute these coefficients from string scattering amplitudes already adapted to cosmological backgrounds, which must be $O(d,d)$ invariant (although the procedure would be somewhat indirect as there is no scattering in one dimension).

Alternatively, one could obtain these coefficients from the vanishing of the higher-loop beta functions of the corresponding string worldsheet theories. We already started this program in this thesis by using a worldsheet directly adapted to dimensional reduction and performing a two-loop computation. By a suitable automatization of the techniques elaborated here, one could envision that higher-loop computations are indeed possible, eventually going beyond the state of the art.

It is worth mentioning that another duality-invariant beta-function approach was studied in [115, 116] and later extended in [2, 4] by using a genuine $O(d,d)$-invariant worldsheet action [117, 118]. Unfortunately, the presence of chiral bosons and the corresponding lack of manifest Lorentz invariance of the action makes the minimal subtraction renormalization scheme not applicable [4], complicating computations beyond one-loop order.

Since the development of the cosmological classification in [40], numerous works have already implemented the general setup in different contexts such as: backgrounds incorporating (duality-invariant) matter content [92, 93], de Sitter as well as general non-singular solutions [48, 119–122], and stability analysis of certain solutions [123–126]. Determining the coefficients to higher orders could be useful in the more realistic scenarios explored in these works, for instance, by supporting or ruling out the hypothesis assumed therein. In particular, it would be interesting to see whether the first coefficients determined here can already be of any use in these regards.

While knowing these coefficients to higher orders is an interesting direction, we expect the perturbative approach to be insufficient to describe certain solutions at extreme regimes. We already observed some hints of this behavior from the regular pre-big bang scenarios proposed in [48], the regular black hole solution found here [7], and the de Sitter vacua from cosmological HSZ in the tensionless limit [5]. Those three solutions only emerge once an infinite number of higher-derivative corrections are taken into account, and so they would probably be invisible from a perturbative computation, no matter how high we go in determining these coefficients. Such non-perturbative description of the string effective theories presumably requires, on top of the massless fields, the incorporation of string massive modes. This construction is out of reach at the moment, partially because we do not have a clear understanding of how to couple massive modes consistently to the worldsheet.

It view of these observations, determining the exact location of string theories in the space of duality-invariant theories is definitely not an easy task and its resolutions may not be possible in the near future unless novel ideas come into play. In the meantime, however, we can consider an equally interesting line of research, which consists of studying directly the broader landscape of duality-invariant theories.

We already made some progress in this direction in the context of 2D backgrounds, using the Gasperini-Veneziano parameterization, where we found sufficient conditions for having singular and regular black holes. It would be important to determine the maximal extensions of these (regular) black holes and to display their Penrose diagrams. We expect such maximally extended spacetimes to be more easily constructed using the $mn = 1$ gauge, instead of the $n = 1$ gauge used here that results in disconnected f contours for the exterior and interior solutions.

On top of a further study of the solutions found in this work, it would be interesting to do a more exhaustive analysis of the solution space itself, and find a weaker set of conditions leading to a bigger class of BH backgrounds. For instance, while the regular solution found here identifies the interior region with a regular cosmology, there could be other geometries that also avoid the singularity. A promising starting point in this direction could be the study of the regular black hole ansatz proposed in [127] by Dijkgraaf, Verlinde and Verlinde (DVV), whose causal structure was studied in [128]. Such study was already started in [6], where it was shown that the DVV ansatz, in its given form, does not fit the framework of the duality-invariant classification. However, as implied by [129] and [130], there should be a different scheme in which DVV is a solution of the classification.

Other interesting directions inside the analysis of duality-invariant theories are related to one of its special points: HSZ theory. While one can integrate out the massive modes of HSZ to relate it to the classification, it is exactly the opposite direction which makes the theory so special: it is the only known point in the classification whose infinite higher-derivative corrections come from a theory with a finite number of derivatives and a finite number of extra (massive) fields. A further study of HSZ as well as an exploration of other points sharing the same special behavior are interesting future directions.

Regarding HSZ, it is important to find non-perturbative solutions for the α'-complete Friedmann equations obtained here, as well as to keep exploring the perturbative de Sitter solutions. It would be also interesting, if possible, to extend the two-derivative reformulation of HSZ to other backgrounds. In particular, to time-independent 2D backgrounds, and see whether the α'-complete equations admit black hole solutions. If this is the case, it would be very intriguing to understand the connection between these solutions and the black hole solutions found in this work in the context of the classification [6, 7].

Regardless of how much we can extend the analysis of HSZ itself, the very existence of such a special point inside the duality-invariant space of theories motivates us to look for others with similar characteristics. Starting from the classification, one could look for conditions on the higher-derivative corrections, if any, such that they arise from integrating out extra fields. Coming from the finite-derivative formulation, on the other hand, a modest first attempt could be to extend HSZ in a minimal way such that the $\alpha' \to 0$ limit (standard supergravity) is preserved. For instance, one could extend the HSZ action by promoting the fixed coefficients next to the couplings of the massive fields to arbitrary parameters, giving rise to a family of α'-complete theories.

While it is unlikely that any of these naive modifications is related to a genuine string theory, one might hope that the lessons learned here could help in the construction of a target-space theory including genuine massive string modes. Even going beyond string theory, the special points found through the methods mentioned above could be interesting theories on their own, which could be uplifted to more general backgrounds, giving rise to simpler UV completions of gravity.

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