

A Model of Macroscopic Quantum Geometry

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A model quantum system is proposed to describe states of flat space on large scales, excluding all standard quantum and gravitational degrees of freedom. It is based on standard quantum spin commutators, with operators interpreted as positions instead of spin, and a Planck-scale length ℓ_P in place of Planck's constant $\hbar$. The algebra is used to derive a new quantum geometrical uncertainty in direction, with variance given by $\langle \Delta\theta^2 \rangle = \ell_P/L$ at separation L , that dominates over standard quantum position uncertainty for bodies greater than the Planck mass. The system is discrete and holographic, and agrees with gravitational entropy if $\ell_P = l_P/\sqrt{4\pi}$, where $l_P \equiv \sqrt{\hbar G/c^3}$ denotes the standard Planck length. A physical interpretation is proposed that accounts for properties of classical space-time in the macroscopic limit; to approximate locality and causality in macroscopic systems, position states of multiple bodies must be entangled by proximity. The model predicts directional fluctuations on timescale $\tau \approx L/c$ that lead to correlated signals between adjacent interferometers.

INTRODUCTION

In general relativity, space-time is a dynamical classical system that exchanges energy and information with classical matter. Of course, experiments prove[1, 2] that matter is really a quantum system, and as such, does not respect foundational assumptions of relativity, such as locality[3–6]. One resolution of this long-standing tension could be that space-time geometry is also a quantum system, and that classical features, such as locality, only emerge as approximate macroscopic behaviors[7]. This idea appears to be consistent with the observed classical dynamical behavior of bodies, and indeed it has been shown that classical motion, Newtonian gravity, and general relativity can be derived from a statistical theory[8, 9]. On the other hand the detailed character of the states of quantum geometry, and the fidelity of the classical approximation to this statistical behavior, are not known.

This paper presents a macroscopic quantum model of geometrical states— an effective description of new geometrical degrees of freedom of position. It starts with an approximation, based on a classical massive body embedded in a quantum geometry, complementary to quantum field theory, which is based on quantized matter fields embedded in a classical background geometry. The model here cannot describe elementary particles— indeed, it does not even have any dynamics— but offers a candidate quantum description for macroscopic geometrical states, and the departures they generate from classical behavior.

The model here offers a new description of a very simple system: the position of a distant, massive body at rest in flat space-time. In standard physics this system is

completely trivial— quantum effects are negligible, and nothing at all happens classically. However, in the framework proposed here, where macroscopic directional information is limited, the new geometrical degrees of freedom display a new form of observable, coherent uncertainty and fluctuation in direction, whose properties can be related to classical gravity, and possibly even be observed.

Motivation and Context

The model described here is not a fundamental theory, but an effective or emergent description that applies only on large scales. We conjecture that the residual Planckian quantum effects of geometry on large scales can be described by certain directional degrees of freedom with specific properties. In this spirit, it is similar to quantum descriptions of collective excitations in condensed matter systems (e.g., ref. [10, 11]).

Quantum Character of Dynamical Geometry

Since dynamical geometry exchanges energy and information with quantum matter, it must itself also be a quantum system. Several hints from theory suggest that that we should consider the possibility of new, small but non negligible of quantum behaviors of geometry, even on large scales.

It is well known that the general-relativistic paradigm of dynamical geometry responding to matter actually becomes inconsistent on scales smaller than the Planck length, $l_P \equiv \sqrt{\hbar G/c^3}$, where c denotes the speed of light, $\hbar$ denotes Planck's constant, and G denotes Newton's

gravitational constant (see Figure 1). Beyond that scale, classical geometry is not a useful approximation[7].

Many theories address possible new forms of physics at the Planck scale[12]. Quantum field theory predicts that Planck scale effects are highly suppressed at the much longer length scales accessible with direct experiments, but on the other hand, it *assumes* a classical geometry and classical spatial structure for its quantized modes, and this assumption may be incorrect, without conflicting with current experiments[13, 14]. General states and formal observables in field theory are nonlocal[15], so in principle, new forms of entanglement with quantum geometry are possible even in large systems.

To show that quantum response of geometry to matter is not confined to small scales only, it is useful to consider the wave function of a quantum particle state that extends over a macroscopic region, for example, the wave function of a single photon that arrives from a distant star, and is captured by a telescope before it is focused to create a sharp image. Suppose a photon is radiated from an atom in a typical state, with an indeterminate direction. In general, the photon state extends many light-years without having a determinate direction. Now suppose that a measurement, say an absorption by an atom in a detector, “collapses” the location of the photon to a definite, localized position state. The spatial state changes acausally and non locally from a scale of light years to the scale of an absorbing atom. This occurrence is not only instantaneously acausal everywhere at the moment of detection, it is actually *global and retroactive*: the entire historical trajectory previous to the “collapse event” is suddenly much more localized than it was before the measurement. In the case of a photon landing on a detector, the entire previous path is localized to the size of the telescopic aperture, as opposed to a region extending many light years.

Now consider the geometry that accompanies this photon. General relativity couples to classical matter, so there is no standard theory for geometry in this situation. For the geometry to match the gravity of the photon, the geometry needs to match the state where the mass is. Instantaneous, retroactive rearrangement of the geometrical quantum state throughout a macroscopic 4-volume of space and time needs to follow processes that shape the matter wave function. Quantum geometry must display the same “spooky” nonlocal quantum entanglement as that of the matter that couples to it.

For practical purposes such nonclassical features of geometry can usually be disregarded because the indeterminate gravity of quantum particles is small. But it is clear that the states of an evolving quantum geometry are non-local and macroscopically indeterminate in both space and time, even though any new quantum-geometrical physics may originate at the Planck scale. Apparently, the separation of scales in field theory does not apply to geometry, particularly when quantum measurement and

collapse are considered. This example shows that geometry is a quantum system that depends on measurement and observations at all scales.

Macroscopic Quantum Geometry

Suppose then that classical spatial relationships, including notions of spatial dimension, position, direction, and locality, emerge as average approximate behaviors in the macroscopic limit of a quantum system. The states of this system can include new quantum-geometrical degrees of freedom and forms of quantum position uncertainty and entanglement that may produce new physical effects on macroscopic scales, that are not included in standard theory.

A simple model of a macroscopic quantum geometry with these properties is proposed here. As a simple approximation, we dispense with both the conventional gravity and quantum properties of matter fields altogether, and use an approximation that describes only the information in the quantum geometry of flat space. The model is not a fundamental theory, indeed *it ignores all standard quantum degrees of freedom*, and cannot describe elementary particles and fields. Instead, *it is a model quantum system that includes only new quantum geometrical degrees of freedom*. Position is represented not as classical coordinates, but as a set of operators with a noncommutative algebra. Basic quantum mechanical principles then lead to an model system that predicts new physical behavior: a qualitatively new form of coherent quantum indeterminacy of angular position of massive bodies on macroscopic scales.

This hybrid of quantum physics and geometry applies in a regime complementary to that of field theory in a nearly-flat space-time. The mass must be larger than the Planck mass so that standard quantum uncertainty is subdominant, while the separation is large enough that space-time curvature effects are negligible (see Fig. 1).

Relation to Noncommutative Geometry

The new geometrical degrees of freedom are introduced here using a noncommutative quantum algebra of position operators. Noncommutative geometries have been previously applied to fields and other systems, by deforming the classical background geometry to a quantum one[16, 17]. The application here differs from those approaches.

First, to preserve covariance in the classical limit, as explained below, we use position and 4-velocity operators to contract the 4D antisymmetric tensor to the two indices required for a commutator coefficient. Usually, this contraction is accomplished with an auxiliary field that is not a part of the dynamical quantum system, and

which provides additional macroscopic coherence. Here, we posit a different, self-contained algebra of dynamical entities, which reduces the number of degrees of freedom and increases the geometrical uncertainty.

Second, no attempt is made to blend the geometry with fields or other continuous functions on a deformed space: the operators here simply refer to positions of classical massive bodies. Thus, the physical application of the theory is different.

These differences lead to significantly different physical effects. Quantum positions in this framework do not respect locality or separation of scales. They approximate classical positions in the macroscopic limit, and approximate locality emerges in a consistent way from the quantum system. However, quantum departures from classical directionality on macroscopic scales, albeit very small, are much larger in this system than in previous approaches. Because classical geometrical coherence is not enforced by assumption, the macroscopic quantum departures from classical geometry are also more macroscopically coherent.

Relation to Thermodynamic Gravity

Internal evidence from gravitational theory has accumulated that elements of geometry, such as manifolds, events, paths and metrics, may not be fundamental physical entities, and that classical properties of motion—including inertia, the principle of equivalence, and gravity—may be emergent statistical behaviors.

Early suspicion of a statistical or thermodynamic origin of geometry came from the laws of black hole thermodynamics, derived from the behavior of classical black hole solutions[18, 19]; for example, the entropy of a black hole was identified with one quarter of the area of its event horizon in Planck units. It became more concrete with the theoretical discovery of Hawking quantum radiation from black holes.[20]

Jacobson[8] explained these results and generalized them by showing that the equations of general relativity themselves can be derived entirely from thermodynamic arguments. He argued that the gravitational lensing caused by matter must distort the causal structure of space-time in a particular way—that is, such that the Einstein equations hold—in order that any accelerated observer see the correct relation $\delta Q = TdS$ between energy flux Q , Unruh temperature T , and entropy S just inside any Rindler causal horizon.

Thus, general relativity emerges statistically from states of some holographic quantum system. The fundamental geometrical objects in Jacobson’s reasoning are not events, but 2D null surfaces. The key physical effect of curvature is gravitational lensing, which can also be interpreted (via Fermat’s theorem) as a distortion of null wavefronts.

More recently, Verlinde[9] derived gravity from a statistical theory, but in a different way. Entropy is again holographic, and information about position is encoded on two dimensional spacelike surfaces. Emergent behavior of bodies in space, including position, inertia, and Newtonian gravity, is derived from positional degrees of freedom. The positional relationship between bodies is equivalent to coarse-grained information on surfaces between them.

In both cases, the derivations connect one classical theory with another: thermodynamics with gravity. They show that geometry derives from statistical behavior of some underlying quantum system. The fundamental degrees of freedom of the geometrical states are not characterized in detail, but the Jacobson and Verlinde treatments highlight some important properties of the quantum system. The states appear holographic in the emergent system, and fundamentally respect causal structure. Somehow, the encoded geometrical information respects a nearly-classical set of null relationships, while limiting the amount of information overall.

The states of the quantum geometrical system described below are also nonlocal and holographic, and could serve as a model basis of such a statistical or emergent theory of emergent gravity. If the macroscopic states of the underlying quantum system resemble those proposed here, these classical properties of the system are accompanied by specific new quantum effects. They create a new quantum-geometrical ambiguity in directional position on 2D spacelike surfaces, while preserving radial separation and causal structure. Thermodynamics provide an emergent theory for the motion of mean position; the quantum geometry here provides a quantum model of a new variance from that classical behavior.

The model here does not extend to curved geometry; in thermodynamic language, it treats only the zero temperature case. Although the geometry is a quantum system, there is no “quantum gravity”, since curvature degrees of freedom are not quantized. However, if we normalize the model to agree with the Planckian information content suggested by low-energy gravity, the zero-temperature quantum fluctuations can then be predicted with no free parameters.

Planck Bandwidth Broadcast

Previous effective theories with similar macroscopic behavior have been based on the idea that information about relative position of bodies propagates as a broadcast wave function with a finite Planck bandwidth[21–23]. The quantum uncertainty of direction is almost the same as the present model, because in both cases it originates from the limited information in the system. The uncertainty corresponds to the non-uniqueness of classical geodesics that describe a system of bandwidth-limited

waves.

The model proposed here has the advantage that it is fixed just by basic principles of covariance and quantum mechanics in the macroscopic limit, and is not derived from any particular microscopic theory. Fewer assumptions are needed to connect the directional uncertainty with the number of degrees of freedom, and thence to thermodynamic gravity. Although the model is connected to new Planckian physics, it need not be assumed to apply in this form at the Planck scale; rather, it is a hypothesis, in the form of a quantum mechanical system, about how attributes of position might depart slightly from classical behavior in the macroscopic limit.

Classical Massive Bodies

The physical quantity identified with the position operator here is the mean position of an assembly of matter, which in keeping with classical use is called a “body”. This concept does not have a clear microscopic meaning, but has a clear macroscopic physical interpretation according to the usual meaning of position in the classical limit. The model is based on this emergent behavior, rather than a bottoms-up derivation from a fundamental theory. No attempt is made here to relate quantum geometry to a more fundamental theory, such as string theory, apart from the normalization based on a holographic origin of gravity. As shown below, the approach is self-consistent because the spatial locality used to define the assembly of particles emerges with increasing precision on macroscopic scales; indeed, it can be used as a framework to describe how approximate spatial locality emerges as a property of states of massive systems.

Unlike familiar quantum mechanical systems, the commutator considered here does not depend on the mass of the body, only on its position. As shown below, the new geometrical uncertainty dominates over standard quantum-mechanical uncertainty for assemblies of matter above a few Planck masses. The model here thus serves as a self-consistent approximation for a quantum theory of position in systems that are both much larger than a Planck length in size, and much larger than a Planck mass in total energy. It cannot serve as a theory of elementary particles, but may be a better approximation than quantum field theory for directional degrees of freedom in large systems (see Figure 1).

MODEL

Covariant Commutation Relations

Suppose that the position of a body in each direction μ is a quantum observable, represented by a self-adjoint

operator $\hat{x}_\mu$. The commutators of these operators represent the quantum deviations of a body from a classical trajectory. The body is assumed to be massive enough that we ignore the conventional position commutators—the usual quantum effects associated with its motion. To describe the quantum geometry, we posit the following commutators relating positions in different directions:

$$[\hat{x}_\mu, \hat{x}_\nu] = \hat{x}^\kappa \hat{U}^\lambda \epsilon_{\mu\nu\kappa\lambda} i\ell_P, \quad (1)$$

where indices $\mu, \nu, \kappa, \lambda$ run from 0 to 3 with the usual summation convention, $\hat{U}^\lambda \equiv \dot{\hat{x}}^\lambda (\dot{\hat{x}}_\alpha \dot{\hat{x}}^\alpha)^{-1/2}$ represents an operator with the same form as the dimensionless 4-velocity of the body (normalized to preserve reparametrization invariance, where $\dot{x} \equiv \partial x / \partial \tau$ and τ denotes proper time), and $\epsilon_{\mu\nu\kappa\lambda}$ is the antisymmetric 4-tensor.

The departure of positions from classical behavior—the commutator—depends on position and 4-velocity in a specific and unusual way. The explicit form of dependence adopted in Eq.(1) on position and velocity is driven by the requirement of covariance. The quantum commutator of two vectors requires two antisymmetric indices that must be matched by indices on the right side. Thus we require a nonvanishing antisymmetric tensor, which in four dimensions has four indices, $\epsilon_{\mu\nu\kappa\lambda}$. Two of its antisymmetric indices match those of the noncommuting positions. The other two must contract with two different vectors to avoid vanishing. The unique geometrically defined options are the 4-velocity and position of the body being measured.

In the limit $\ell_P \rightarrow 0$, the commutator vanishes, and positions in different spatial directions behave independently and classically. It is interesting to ask what happens if the scale ℓ_P is not zero, in particular if it is of order the Planck length, $\ell_P \approx \ell_P$. The information density of geometrical states then agrees with holographic entropy bounds for gravitating systems, as explained below.

Equation (1) is manifestly covariant: the two sides transform in the same way under the homogeneous Lorentz group, as a direct product of vectors. The algebra of the quantum position operators respects the transformation properties of corresponding coordinates in an emergent classical Minkowski space-time, in a limit where the operators are interpreted as the usual space-time coordinates. The theory itself thus defines no preferred direction in space.

On the other hand, Eq. (1) is not translation invariant. The commutator does depend on the position and 4-velocity of the body being measured, or equivalently, on the origin and rest frame of the coordinate system. We interpret this to mean that *the commutator describes a quantum relationship between world lines that depends on their relative positions and velocities, but not on any other properties of the bodies being compared.* The Hilbert space of the model quantum geometry de-

depends on a causal structure defined by a choice of time-like trajectory[7], and the quantum-geometrical position state is defined in relation to a particular world line, the origin of the coordinates. Unlike a classical geometry, which is defined independently of any observer, the state of a quantum geometry and indeed the definition of position are shaped by this choice, so they cannot obey translation invariance. As discussed below, if ℓ_P is of the order of the Planck length, this subtle new quantum effect would have escaped detection.

States of a Distant Massive Body at Rest

The physical interpretation of Eq.(1) is most straightforward when converted into a quantum geometry of spatial position in the classically defined rest frame of a body— a non-relativistic limit. In that frame, the classical 4-velocity is $U^\lambda = (1, 0, 0, 0)$ so the non-vanishing terms of Eq. (1) are those multiplied by $\epsilon_{\mu\nu\kappa\lambda}$ with $\lambda = 0$. The remaining terms describe a noncommutative geometry in three dimensions:

$$[\hat{x}_i, \hat{x}_j] = \hat{x}_k \epsilon_{ijk} i \ell_P, \quad (2)$$

where indices i, j, k now run from 1 to 3, and the operators $\hat{x}_i$ correspond to positions at a single time, in the rest frame of the body. Eq. (2) describes a quantum-geometrical relationship between positions of two trajectories (or massive bodies) that have proper 3-separation $\hat{x}_k$, and whose world lines have the same 4-velocity.

The algebra in Eq. (2) defines a consistent quantum theory; indeed, it is the same as the standard quantum description of angular momentum, albeit in an unconventional physical application. Here, position in units of a fundamental length ℓ_P takes the place usually assigned to angular momentum in units of $\hbar$. The algebra itself is of course well known[24], and leads to precise results on the properties of the quantum-geometrical system.

For example, the operators obey the Jacobi identities

$$[\hat{x}_i, [\hat{x}_j, \hat{x}_k]] + [\hat{x}_k, [\hat{x}_i, \hat{x}_j]] + [\hat{x}_j, [\hat{x}_k, \hat{x}_i]] = 0, \quad (3)$$

which shows that in the non-relativistic limit (Eq. 2), the effective quantum theory is self-consistent. Note that this conclusion does not require positing new quantum-geometrical commutators for the operators $\hat{x}_\mu$, or knowing the Hamiltonian for the system. Usually the quantum conditions (Eq. 2) are derived from a correspondence principle with classical Poisson brackets; here, they are derived from general considerations of covariance, since there is no classical system corresponding to the quantum geometrical degrees of freedom.

As estimated below, in laboratory applications on a scale of $\approx 10^{36} \ell_P$, the quantum geometrical transverse positions fluctuate extremely slowly, equivalent to about $10^{-18} c$, so the nonrelativistic approximation is excellent.

More generally, for a body at rest the terms in Eq. (1) multiplied by 1, 2, 3 components of $\hat{U}^\lambda$ are smaller in magnitude than the 0 component by a factor of order $|\hat{x}|^{-1/2}$, which is tiny for a macroscopic system, so they can be consistently neglected. The discussion below is confined to this limit, and thereby avoids confronting the correspondence of $\hat{U}^\lambda$ with its classical counterpart, or the subtle relationship of the time operator, $\hat{x}_0$, with the classical time coordinate.

Radial Separation Eigenstates

Consider an operator

$$|\hat{x}|^2 \equiv \hat{x}_i \hat{x}_i, \quad (4)$$

corresponding to the squared modulus of separation, analogous to the square of total angular momentum. In the same way that total angular momentum commutes with all of its components, the radial separation between trajectories behaves classically:

$$[|\hat{x}|^2, \hat{x}_i] = 0, \quad (5)$$

so that emergent causal structure, with light cones defined by $|\hat{x}|^2$, has no quantum uncertainty. For any single direction— for example, a direction of a plane wave mode in field theory— the system behaves classically. It is only in comparison of positions in transverse directions that new quantum-geometric effects appear.

We adapt conventional notation used for angular momentum. Let l denote positive integers corresponding to the quantum numbers of radial separation, analogous to total angular momentum. The separation operator takes discrete eigenvalues:

$$|\hat{x}|^2 |l\rangle = l(l+1) \ell_P^2 |l\rangle. \quad (6)$$

We denote the discrete eigenvalues corresponding to classical separation by

$$L \equiv \sqrt{l(l+1)} \ell_P. \quad (7)$$

Directional Eigenstates and Uncertainty

A body can be in a state of definite radial separation, and also definite position in any single direction. However, a body cannot be in a definite position state in more than one direction, and this leads to a new kind of uncertainty that can also be understood from the angular momentum algebra.

If a system is in a state of definite angular momentum in one direction, the components of angular momentum in the transverse directions are indeterminate. Similarly, if a body is in a state of definite separation in one direction, the position components in transverse directions are

indeterminate— in the macroscopic limit, very slightly so.

Now consider projections of the operator $\hat{x}_i$. Let l_i denote the eigenvalues of position in direction i . In a state $|l\rangle$ of separation number l , the position operator $\hat{x}_i$ can have eigenvalues in units of ℓ_P ,

$$l_i = l, l-1, \dots, -l, \quad (8)$$

giving $2l+1$ possible values. In an eigenstate with a definite value of position in direction i ,

$$\hat{x}_i |l, l_i\rangle = l_i \ell_P |l, l_i\rangle. \quad (9)$$

We now calculate the variance for the position wave function in transverse directions. Define raising and lowering operators for each direction:

$$\hat{x}_{3\pm} \equiv \hat{x}_1 \pm i\hat{x}_2, \quad (10)$$

with equivalent expressions for cyclic permutations of the indices. These can be used to show[24] that for any i ,

$$|\hat{x}|^2 = \hat{x}_{i+}\hat{x}_{i-} + \hat{x}_i^2 + \hat{x}_i = \hat{x}_{i-}\hat{x}_{i+} - \hat{x}_i^2 + \hat{x}_i. \quad (11)$$

Direct calculation (e.g., ref.[24]) then leads to the following product of amplitudes for measurements of either of the transverse components $\hat{x}_j$, with $j \neq i$:

$$\langle l_i | \hat{x}_j | l_i - 1 \rangle \langle l_i - 1 | \hat{x}_j | l_i \rangle = (l + l_i)(l - l_i + 1) \ell_P^2 / 2, \quad (12)$$

again for any i .

The left side of equation (12) can be interpreted as the expected value for the operator

$$\hat{x}_j |l_i - 1\rangle \langle l_i - 1| \hat{x}_j \quad (13)$$

for components with $j \neq i$, in a state $|l_i\rangle$ of definite $\hat{x}_i$. For $l \gg 1$ and $l_i \approx l$, this corresponds to the expected variance in components of position transverse to separation:

$$\langle \hat{x}_j | l_i - 1 \rangle \langle l_i - 1 | \hat{x}_j \rangle \rightarrow \langle \hat{x}_j^2 \rangle. \quad (14)$$

We conclude that position in any direction transverse to separation is indeterminate, with a variance given by the right hand side of Eq. (12) in the limit of $l \gg 1$:

$$\langle \hat{x}_\perp^2 \rangle = \langle \hat{x}_j^2 \rangle = L \ell_P. \quad (15)$$

Transverse positions show the same kind of quantum indeterminacy as transverse components of angular momentum in the nearly-classical, large-angular-momentum limit. The physical analog is measurement of a component transverse to the angular momentum vector; such a measurement reveals a superposition of transverse direction states, with a range of eigenvalues. In our application, position can similarly have a definite value in the radial direction— corresponding to its classical value—

but is indeterminate in the transverse directions. The new quantum-geometrical uncertainty increases with separation, and at macroscopic separations, it is much larger than ℓ_P .

As usual in quantum mechanics, the state can be represented by a wave function, and the commutator leads to an uncertainty relation between the variances of the wave functions of the conjugate variables. It can be derived from Eq.(2) in the usual way, although the conjugate variables are now positions in different directions, instead of angular momentum components in different directions. In the rest frame, the uncertainty relations for a body at positions $\hat{x}_k$ are

$$\Delta x_i \Delta x_j \geq |\bar{x}_k \epsilon_{ijk}| \ell_P / 2, \quad (16)$$

where $\Delta x_i = \langle |x_i - \bar{x}_i|^2 \rangle^{1/2}$ represents the spread of the wave function in each direction about the expected (classical) position $\bar{x}_i$. Note that this result is consistent with, but less constraining than Eq. (15), which does not allow uncertainty to be squeezed out of one transverse direction and into the other.

Total number of eigenstates

The geometrical position states have a discrete spectrum. The number of position eigenstates within a 3-sphere of radius R can be counted in the same way as discrete angular momentum eigenstates, as follows.

Recall that radial position has discrete quantum numbers l . Using Eq. (7) with $R = L$, the number l_R of radial position eigenstates for a radius R is given by setting $l_R(l_R + 1) = (R/\ell_P)^2$, that is, they are discrete with approximately Planckian separation. For each of these, from Eq. (24), there are $2l + 1$ eigenstates of direction. The total number of quantum position eigenstates in a 3-sphere is then

$$\mathcal{N}_{Q3S}(R) = \sum_{l=1}^{l_R} (2l + 1) = l_R(l_R + 2) = (R/\ell_P)^2, \quad (17)$$

where the last equality applies in the large l limit. Thus, the number of quantum-geometrical position eigenstates in a volume scales holographically, as the surface area in Planck units.

Normalization from Gravity

Although gravity does not enter explicitly into any of the arguments here, it is natural to fix the physical value of ℓ_P by equating the number of geometrical eigenstates with the number of states deduced from thermodynamic arguments applied to gravitational systems[8, 9]. An exact calibration is possible, because the number of states

in a 3-sphere can be counted exactly for both geometry and gravity.

The number of position states for a massive body enclosed in a 3-sphere of radius R that statistically reproduces nonrelativistic Newtonian gravity (cf. [9], Eq. 3.10) is given by:

$$\mathcal{N}_{G3S}(R) = 4\pi(R/ct_P)^2, \quad (18)$$

with the usual definition of Planck time, $ct_P \equiv \sqrt{\hbar G/c^3} = 1.616 \times 10^{-35} \text{m}$. Note that this is four times larger than the entropy of a black hole event horizon of the same radius. The quantum-geometrical and gravitational estimates agree, $\mathcal{N}_{Q3S} = \mathcal{N}_{G3S}$, if the numerical value of the effective commutator coefficient ℓ_P is

$$\ell_P = ct_P/\sqrt{4\pi}. \quad (19)$$

If Eq. (19) holds, bodies in the space-time that emerges from the quantum geometry move as described by Newtonian gravity.

As discussed below, an interesting and attainable experimental goal is to detect or rule out a commutator with this value. From Eq. (15), the theory predicts quantum-geometrical uncertainty in physical units,

$$\langle \hat{x}_\perp^2 \rangle = Lct_P/\sqrt{4\pi} = (2.135 \times 10^{-18} \text{m})^2 (L/1\text{m}), \quad (20)$$

with no free parameters. Experimental consequences are discussed below.

The numerical value of the coefficient in Eq. (20) is slightly lower (by a factor of $\sqrt{\pi}/4$) than the value previously estimated by a different method, using a wave theory normalized to black hole entropy (Eq. 29 of ref. [23].) The value computed here is better controlled, because the quantum and gravitational numbers both refer to the same physical system, a spherical 3-volume at rest with no curvature.

EMERGENCE OF LOCALITY

The effect of the quantum geometry can be characterized as an uncertainty in angular position or direction. Consider the quantum-geometrical variance in angular position of a body at separation L , from Eq. (15):

$$\langle \Delta\theta^2 \rangle \equiv \langle \hat{x}_\perp^2 \rangle / L^2 = \ell_P / L. \quad (21)$$

Unlike the position uncertainty, the directional uncertainty decreases with separation. Classical geometry and locality thus emerge as excellent approximations to describe relative positions of trajectories with separations much larger than the Planck length.

For any experimentally accessible scale, the deviation from classicality is fractionally negligible. However, as separations approach the Planck scale, directions become

indeterminate. The classical approximation to geometry then breaks down, consistent with the idea of a space-time emerging from a Planckian quantum system. That system does not resemble a traditional quantum-gravitational “space-time foam” with quantum equivalent of strong Planck scale gravity and virtual black holes, inspired by extrapolation of quantum field theory. Instead, positions, curvature and gravity only emerge as statistical behaviors on large scales.

Angular Fluctuations and Coherence

The uncertainty describes properties of a wave function in a stationary state that respects the symmetries of a body at rest. As usual, the wave function itself is not observed in any physical measurement, since it describes a superposition of histories and position states. Instead, the spread of the wave function manifests as random results, or noise, in a time series of nonlocal position measurements in more than one direction. The effect of the uncertainty on measurements of position is constrained by requiring the system to approach standard physics in the macroscopic limit.

The state described by the geometrical wave function cannot independently describe the trajectories of individual particles, since that would result in unacceptable, locally measurable fluctuations dependent on an arbitrary choice of origin for the coordinate system. However, the uncertainty is consistent with emergence of locality on large scales, if the quantum-geometrical deviations from a classical trajectory are entangled coherently and non-locally, so that all matter in a small region (where “small” means much smaller than L) shares approximately the same deviation $x_i - \bar{x}_i$ in an actual position measurement. The scaling of geometrical uncertainty with L then allows a hierarchical recursive emergence of locality on large scales, as illustrated in Figure (2).

Colloquially, we can say that matter which is close together, moves together. In a Copenhagen interpretation of quantum mechanics, one would say: *The geometry “collapses” into the same state for events on the same null surfaces defined by the causal diamond of the observer.* Then, in spite of the observer-dependent branching of the wave function, classical causal structure emerges as a symmetry of the macroscopic system.

This model is consistent with the classical approximations used to interpret Eqs. (1): the quantum operators refer to a collective position and 4-velocity of an arbitrary assembly of mass-energy in a compact region of space, and average over time to the mean classical space-time position, as long as the region is much smaller than L .

In this model, the positional quantum states of bodies in emergent space-time possess a new kind of nonlocal coherence not describable by states of standard quantum

theory in classical space-time. The coherent entanglement of geometrical position states creates a new correlation in the mean positions of otherwise separate bodies—including an in-common, coherent quantum-geometrical deviation from their classical trajectories. There is no such thing as a massive body exactly at rest, but a world line (and rest frame) becomes directionally better defined on large scales. The decoherence of position states in space and time—the deviation from noiseless classical behavior—is given by the uncertainty amplitude in Eq. (15), where L corresponds to the separation of the elements and L/c the timescale of the variation. Put another way, the direction to a distant body, together with its neighbors, fluctuates slowly about its classical value with variance given in Eq. (21).

This interpretation of the uncertainty opens up a way to build an experiment that probes Planck scale physics. Positions fluctuate, with a power spectrum of angular variations given approximately by the Planck time—that is, in an average over duration τ , the mean square variation is

$$\langle \Delta\theta^2 \rangle_\tau = \ell_P / c\tau. \quad (22)$$

In an experiment of size L , the variations accumulate up to durations $\tau = L/c$, ultimately leading to a total variance in position given by the overall uncertainty, Eq. (20). This prediction can be tested by making very sensitive measurements of transverse positions of massive bodies.

To detect the effect, an apparatus must compare positions of compact bodies distributed over an extended region of space in more than one direction, over a duration comparable to its size. Over longer durations, the fluctuations are smaller, as positions in a compact region average to their classical values.

Fluctuations should appear in a time series of such comparative measurements. In the time domain, the covariant formulation (Eq. 1) contains an operator $\hat{x}_0$, that approximately corresponds to standard coordinate time. However, this correspondence is not exact. In the model of emergent locality presented here, a local clock is associated with a particular direction in space. It can be synchronized perfectly in the standard way[5, 6] with other clocks in that direction at large distances, using light to compare clock ticks. However, it displays Planckian noise when compared with clocks at macroscopic distances in other directions. As discussed below, the magnitude of the effect is currently unobservable in tests on a cosmic or microscopic scale, but might be detected as a new source of noise in interferometer signals.

Quantum States and Entanglement

Nonlocal entanglement of position states is perhaps the strangest feature of this quantum geometry, but appears

to be required in order that the number of position states not exceed the holographic gravitational bound. Because of the coherence, neighboring elements of an apparatus fluctuate together, even if there is no physical connection between them apart from relative proximity. It is worth sketching how the entanglement works in a somewhat more formal way.

The full quantum state includes all possible combinations of positions in what eventually appears, to an emergent observer, as a macroscopic spatial volume of size L . It includes a superposition of all the combinations of discrete choices of position. The basis states are formed around a particular world line, the origin of classical coordinates. A special role is assigned to the radial quantum number l , which is associated with the emergent causal structure that defines that particular world line. The angular quantum number can be represented as a projection associated with any direction in the emergent space.

Based on the algebra above, let us write down a position state of a single massive body A at rest in on the z axis. Again, this description will ignore the standard quantum physics attached to A , and treat only the quantum-geometrical part. The natural basis to choose is the radial direction to the body from the origin. Any state within separation L can be written as a superposition,

$$|A\rangle = \sum_{l=1}^{l(L)} \sum_{l_z=-l}^{l_z=l} \alpha_{l,l_z}(A) |l, l_z\rangle \quad (23)$$

where $\alpha(A)$ is the wave function, the complex amplitude for a system—a massive body in a quantum spacetime—to be in each eigenstate. Each component is labeled by discrete quantum numbers l, l_z represents a corresponding eigenstate. Separation L comes in discrete eigenvalues (from Eq. 7) $L/\ell_P = \sqrt{l(l+1)}$. The eigenvalues that correspond to direction are described by projection onto any arbitrary direction; in this case we have chosen the z axis. In a state $|l\rangle$ of separation number l , the operator for position $\hat{z}$ can have eigenvalues in units of ℓ_P ,

$$l_z = l, l-1, \dots, -l, \quad (24)$$

giving $2l+1$ possible values.

The separation operator commutes with any of the component projections—separation acts classically—but positions in different directions do not commute with each other. The state in Eq. (23) is written in terms of radial separation and z component, and has eigenfunctions with definite values of z . Such a state can also be represented by a superposition of states in other directions, and has an indeterminacy in transverse position, given by Eq. (20).

Emergence of Locality via Hierarchical Recursive Entanglement

The quantum states of multiple bodies are not independent, but are entangled with the same quantum-geometrical system. A composite system of two bodies at rest can be written as a tensor product of subsystems,

$$|AB\rangle = |A\rangle \otimes |B\rangle \quad (25)$$

The states of two bodies close to each other are entangled, so that projections onto transverse components are coherent. When positions of A and B are both measured in the z direction, they share the same natural decomposition, and they can both be in eigenstates of z . When they are measured in a transverse (x or y) direction in this state, the transverse position measurement yields a definite value, with probability determined from the squared modulus of their transverse wave functions. They are placed in a new state projected onto the transverse direction. If world lines A and B are close together compared with L , measurements of x_A and x_B (or y_A and y_B) are highly correlated. Because they share almost the same causal structure, they share the same projected states (see Figure 2).

Geometrical and Standard Quantum Uncertainty

The standard quantum limit [25, 26] for the variance in a body's position difference measured at two times separated by a duration τ is:

$$\Delta x_{SQL}^2 \equiv \langle (x(t) - x(t + \tau))^2 \rangle \geq 2\hbar\tau/m. \quad (26)$$

This can be derived either from non relativistic quantum mechanics, where it includes the reaction force on the mass due a measurement, or from quantum field theory in the cavity modes of an interferometer with a beamsplitter of mass m , where it includes the quantum correlation of photon occupation numbers and shot-noise pressure fluctuations in the two arms.

For any L , the standard quantum uncertainty (Eq. 26) is greater than the quantum-geometrical position variance (Eq. 20) at separation $L = c\tau$ if the mass m of a body satisfies $m < 4\sqrt{\pi}m_P$, where $m_P \equiv \hbar/c^2 t_P = 2.176 \times 10^{-5}\text{g}$ denotes the Planck mass. That is, for masses below the Planck mass, the new quantum geometrical uncertainty is negligible compared to standard quantum uncertainty, for systems of any scale or configuration. This crossover is shown in figure (1).

Scope and Limitations

It is useful to summarize what this model does and does not do.

Essentially, it is a model for quantum geometrical degrees of freedom, that only applies to the position of a massive body at rest, at macroscopic separation. Although the physics of this system are completely trivial in standard physics, the model shows possible new quantum effects from the geometry, that are not included in standard approximations. The domain where the new effects dominate standard effects is shown in figure (1). A more detailed model of the relationship with quantum field theory[14] describes how field states could be directionally entangled with geometrical states on large scales.

The model provides a concrete hypothesis about how locality might emerge on macroscopic scales from a quantum theory. It also provides a concrete hypothesis about the character of the position states in an emergent space-time.

The main new useful result is to provide concrete and definite predictions about the character of the new quantum-geometrical effects. The model provides a new, exact connection between the number of discrete geometrical degrees of freedom, and the width of a transverse wave function of position. These two quantities are observable in different ways: the first from gravity, the second from measurements of fluctuations in the transverse positions of massive bodies.

At the same time, the model here is extremely limited in scope and applicability. There is no Hamiltonian, and no equations of motion. Classical motion and inertia only arise in the emergent system. At a macroscopic level this is certainly possible, as shown in the entropic view of gravity, which offers a possible route to extract a limit that looks like classical dynamical physics in classical space-time.

It is not clear how to connect with dynamical quantum theories, such as string theory or fields in noncommutative geometries (e.g., [17]). The commutator used here, based on the 3D antisymmetric tensor ϵ_{ijk} and the position of a body, differs from the 2D θ_{ij} generally applied in those studies.

These limitations reflect the fact that the model lacks dynamics, and describes only a time-invariant state. It is equivalent to a theory of angular momentum with no Hamiltonian, and no theory of transitions between states. Nevertheless, it is natural to interpret the width of the wave function as a source of noise in a time series of directional measurements.

EXPERIMENTAL TESTS

Particle Experiments

The purely transverse character implies zero quantum-geometrical uncertainty— equivalent to a classical geometry— for measurements of position that involve

interactions with particles propagating in only a single direction. This behavior is different from fluctuations or dispersion caused by gravitational waves or quantized-metric fluctuations. It applies even over cosmic distances, so the theory is unconstrained by studies of light from distant sources that constrain some forms of Planck-scale Lorentz invariance violation.[27–29]

The predicted fluctuations[23], while much larger than the Planck length, are also too small to detect with laboratory position measurements involving modest numbers of atoms or molecules. Indeed, standard quantum uncertainty (Eq. 26) dominates geometrical uncertainty even up to bodies of order the Planck mass. Quantum-geometrical uncertainty is negligible on the mass scale of Standard Model particles ($< 10^{-16}m_P$), which accounts for why classical space-time is such a good approximation for systems involving small numbers of particles, and why standard theory agrees so well with precision tests in microscopic experiments[14]. The space-time even in a single $\approx \text{TeV}^4$ collision volume encompasses $\approx 10^{32}$ degrees of freedom, and detectors do not have sufficient precision to detect the predicted tiny deviations from the phase space of quantized field modes in a classical geometry. Predicted effects in particle experiments were estimated in ref.[13] for similarly constrained modes, and found to be unobservable.

Interferometry

Detection of the quantum-geometrical uncertainty requires a nonlocal, bidirectional experiment, that measures relative positions of bodies on at least three widely separated world lines, with time sampling comparable to their separation. It can best be distinguished from other noise sources by measuring mean positions of large assemblies of particles, that is, massive bodies such as mirrors.

A Michelson interferometer creates a coherent state of many photons spatially extended along two macroscopic arms[25]. The measured signal from the interfered light depends on positions of three mirrors, a beamsplitter and two end mirrors, at three different times separated by the arm length L_a . The world lines of the mirrors are widely separated in two directions, so quantum-geometrical position uncertainty of the beamsplitter relative to the end mirrors leads to noise, or fluctuations in the signal. Quantum geometrical position fluctuations with variance given in Eq. (20), and timescale $\tau \approx L_a/c$, create noise with calculable statistical properties.

The statistical properties of the noise are estimated in ref.[23]. In a standard Michelson interferometer, the power spectral density of position noise in conventional units (variance of fractional arm length difference fluctuation per frequency interval), at frequencies low compared

with c/L_a , is given approximately by the Planck time:

$$h^2 \approx \frac{\langle \hat{x}_\perp^2 \rangle}{cL_a} = t_P/\sqrt{4\pi} = (1.23 \times 10^{-22} \text{Hz}^{-1/2})^2. \quad (27)$$

Care must be taken in using quoted limits, since the quantum-geometry effect on the signal depends on the interferometer layout in ways that significantly differ from gravitational wave response. However, it is clear that existing interferometers[30] already rule out such fluctuations for commutators significantly larger than the Planck scale.

If two separate interferometers have mirrors in nearly the same positions, separated by much less than L_a , their geometrical position states are entangled, so fluctuations in their signals are correlated in a way that depends on the experimental configuration. The effect of entangled quantum-geometrical position fluctuations can be separated in this way from other sources of environmental or quantum noise.

An experiment based on this specific signature, currently under construction at Fermilab[31], is designed to achieve the required Planckian sensitivity to detect the fluctuations, or to convincingly rule them out. The theory here suggests that the outcome of this experiment, either positive or negative, could shed light on the emergent origin of classical locality and gravity from a Planck scale theory.

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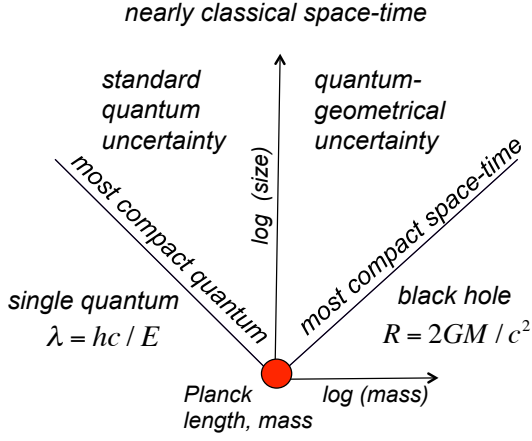


FIG. 1: Domains of applicability of approximations, in terms of size and energy of a physical system. The photoelectric relation at left relates the wavelength and energy of a single quantum. More compact systems do not exist, since quanta do not come in smaller packets. The Schwarzschild formula on the right relates the radius and mass of a black hole. Again, more compact systems do not exist, since a black hole is the most compact configuration of space-time for a given mass. Below these two lines, in particular in a system smaller than the Planck length where the two lines meet, no system based on classical dynamical geometry responding to quantum matter can exist. Above these lines, nearly-classical geometry is a useful approximation, so systems can be approximated in the usual way as quantum fields and particles on a classical background. The model here predicts new residual effects of quantum geometry that persist on macroscopic length scales; they may dominate standard quantum uncertainty for transverse positions of massive bodies above the Planck mass.

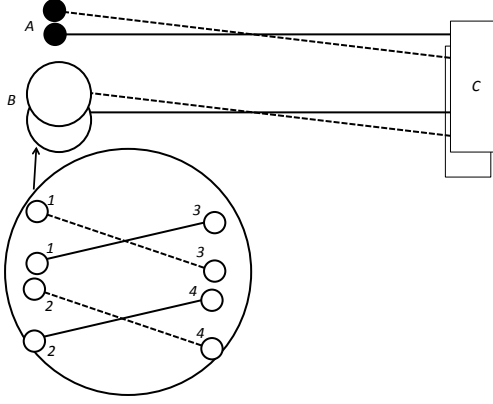


FIG. 2: Illustration of how locality can emerge via recursive hierarchical entanglement. Causally connected events share the same geometrical state: collapse of a transverse position state wavefunction spreads on a light cone from each event where a null surface that defines a measurement in a particular direction intersects a surface of constant time, in the rest frame of the body used to measure the system. This idea can be used to recursively extrapolate the behavior of a single-body state to a system of multiple bodies. A massive body is defined as a collection of smaller ones in a localized region, and its position is defined relative to distant regions. Consider a pair of nearby bodies, A and B , and their direction relative to a distant body C . The transverse separation is small, $AB \ll AC \approx BC$. The AC direction and BC direction share the same geometrical state and fluctuate coherently on timescale $\approx AC \approx BC$. Two possible orientations are shown as solid and dashed lines. Similarly, consider components of B , labeled 1, 2, 3, and 4. Their positions fluctuate coherently relative to C on the timescale BC . They also fluctuate relative to each other on the shorter timescale $13 \approx 24 \ll BC$. Again, two possible orientations are shown. The directional fluctuation is larger for the subsystem but the transverse motion is smaller. A quantum system with these properties gradually approximates classical locality on large scales. However, its spatial information content grows only holographically with scale.