CONSTRUCTIONS OF LIE (SUPER)ALGEBRAS FROM TRIPLE SYSTEMS*

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ABSTRACT

The construction of Lie algebras and Lie superalgebras from Freudenthal-Kantor (super)pairs by using derivations and pairs of homomorphisms satisfying so called the condition (K) is given. The construction of Lie (super)algebras from the commutative associative triple systems and the Freudenthal-Kantor (super)triple systems is also given.

This is a continuation of the previous talking at the XIVth International Colloquium on Group Theoretical Methods in Physics, Seoul, 1985[14].

1. GENERALIZED LIE (SUPER)TRIPLE SYSTEMS

A Lie algebra is the tangent algebra of Lie group at the identity element e. A Lie triple system is the tangent algebra of symmetric space G/H at eH. A general Lie triple system is the tangent algebra at eH of reductive homogeneous space G/H due to K. Nomizu[11],[cf.12]. Given a (anti-) Lie triple system T, let $\mathcal{D}(T)$ be the derivation Lie algebra of T, then $\mathcal{L}=V_0+V_1$, $V_0=\mathcal{D}(T)$, $V_1=T$, becomes a Lie algebra

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and Lie superalgebra according to V_1 is a Lie triple system and anti-Lie triple system respectively[8,3]. This result can be generalized to a general Lie (super)triple system [14].

The examples of general Lie triple sytems are given in [14], some of others are given here.

(1) The Virasoro algebra $\mathcal{L} = \sum_{n\in\mathbf{Z}} \mathbf{C}e_n \oplus \mathbf{C}e_0'$ is a Lie algebra defined by $[e_m,e_n]=(m-n)e_{m+n}+(1/12)(m^3-m)\delta_{m+n,0}e_0'$, $[e_0',\mathcal{L}]=\{0\}$. Put $\mathcal{H}=\mathbf{C}e_0\oplus\mathbf{C}e_0'$, $\mathcal{N}=\sum_{n\in\mathbf{Z}\setminus\{0\}} \mathbf{C}e_n$, then $\mathcal{N}$ becomes a general Lie triple system. Conversely, the algebraic system $\mathcal{N}=\sum_{n\in\mathbf{Z}\setminus\{0\}} \mathbf{C}e_n$ with products $e_me_n=(m-n)(1-\delta_{m+n,0})e_{m+n}$ and $[e_\ell\ e_m\ e_n]= -2\ell n\delta_{\ell+m,0}e_n$ is a general Lie triple system and its standard imbedding Lie algebra is isomorphic with $\sum_{n\in\mathbf{Z}} \mathbf{C}e_n$ such that $[e_m,e_n]=(m-n)e_{m+n}$.

(2) Let $(J,U(\varepsilon),\sigma,f)$ be a JT-ternary algebra[15], which is a generalization of J-ternary algebra due to B.N. Allison [1] and W. Hein[5]. Put $T=J\oplus\overline{J}\oplus U(\varepsilon)\oplus\overline{U(\varepsilon)}$, where $\overline{J}$, $\overline{U(\varepsilon)}$ being the copies of J, $U(\varepsilon)$ respectively. Define a bilinear product uv and a trilinear product $[uvw]$ in T by

$$u_1u_2={}^t(-\varepsilon f(b_1,b_2),f(a_1,a_2),\sigma(y_1)b_2-\sigma(y_2)b_1,\sigma(x_1)a_2-\sigma(x_2)a_1),$$

$$[u_1\ u_2\ u_3] = \begin{pmatrix} L(x_1,y_2)x_3-L(x_2,y_1)x_3+\varepsilon f(\sigma(x_3)a_1,b_2)-\varepsilon f(\sigma(x_3)a_2,b_1) \\ -L(y_2,x_1)y_3+L(y_1,x_2)y_3+f(\sigma(y_3)b_2,a_1)-f(\sigma(y_3)b_1,a_2) \\ -\sigma(y_2)\sigma(x_1)a_3+\sigma(y_1)\sigma(x_2)a_3+L(a_1,b_2)a_3-L(a_2,b_1)a_3 \\ \sigma(x_1)\sigma(y_2)b_3-\sigma(x_2)\sigma(y_1)b_3+\varepsilon L(b_2,a_1)b_3-\varepsilon L(b_1,a_2)b_3 \end{pmatrix},$$

where u_i denotes ${}^t(x_i,y_i,a_i,b_i)$, $i=1,2,3$. Then, T becomes a general Lie triple system with respect to the product defined above[15].

By using a representation of Lie algebra, a notion of general Lie triple system can be more generalized. A quadruple $(A,B(\delta),\sigma,f)$, $\delta=\pm1$, is called a general Lie (super) triple system if

A is a Lie algebra with product $[,]$,

$B(\delta)$ is an algebraic system with a bilinear product $ab=L(a)b$ and a trilinear product $[abc]=L(a,b)c$ such that

$$ab=-\delta ba,\quad L(a,b)=-\delta L(b,a),$$
$$[abc]+[bca]+[cab]+(ab)c+(bc)a+(ca)b=0,$$

σ is a special representation of Lie algebra A into a vector space $B(\delta)$, that is, $\sigma([x,y])=\sigma(x)\sigma(y)-\sigma(y)\sigma(x)$,

f is a bilinear mapping of $B(\delta)$ into A such that $f(a,b) = -\delta f(b,a)$,

$$[x,f(a,b)]=f(\sigma(x)a,b)+f(a,\sigma(x)b),$$
$$[\sigma(x),L(a)]=L(\sigma(x)a),$$
$$\sigma f(a,b)=L(a,b),$$
$$f(ab,c)+f(bc,a)+f(ca,b)=0,$$

$x,y,z\in A$, $a,b,c\in B(\delta)$.

PROPOSITION 1. Let $(A,B(\delta),\sigma,f)$ be a general Lie (super)triple system. Put $\mathcal{L}=A\oplus B(\delta)$. Define a product in $\mathcal{L}$ by

(*) $\quad [x+a,y+b]=[x,y]+f(a,b)+\sigma(x)b-\sigma(y)a+ab,$

$x,y\in A$, $a,b\in B(\delta)$, then $\mathcal{L}=V_0\oplus V_1$, $V_0=A$, $V_1=B(\delta)$, is a Lie algebra and Lie quasisuperalgebra with respect to the product (*) according to $\delta=1$ and -1 respectively, that is,

$$[V_0,V_0]\subset V_0,\quad [V_0,V_1]\subset V_1,\quad [u_i,v_j]=-(-1)^{ij}[v_j,u_i],$$
$$(-1)^{ik}[[u_i,v_j]w_k]+(-1)^{ji}[[v_j,w_k]u_i]+(-1)^{kj}[[w_k,u_i]v_j]=0,$$

$u_i\in V_i$, $v_j\in V_j$, $w_k\in V_k$.

As a corollary of the Proposition, it follows the known result: if $ab=0$ for all $a,b\in V_1$, then $\mathcal{L}$ becomes the Lie algebra and Lie superalgebra according to $\delta=1$ and -1 respectively.

2. CONSTRUCTION OF LIE (SUPER)ALGEBRAS FROM FREUDENTHAL-KANTOR (SUPER)PAIRS

Let $\varepsilon,\delta=1$ or -1. A pair $U(\varepsilon,\delta)=(U(\varepsilon,\delta)^+,U(\varepsilon,\delta)^-)$ of vector spaces $U(\varepsilon,\delta)^\sigma$, $\sigma=\pm$, with product $\langle a_\sigma\ b_{-\sigma}\ c_\sigma\rangle:=$

$L(a_\sigma, b_{-\sigma})c_\sigma$ is called a Freudenthal-Kantor pair or Freudenthal-Kantor superpair according to $\delta=1$ or -1, if

$$\text{(U1)}\quad \begin{aligned}&[L(a_\sigma,b_{-\sigma}),L(c_\sigma,d_{-\sigma})]\\ &\quad = L(L(a_\sigma,b_{-\sigma})c_\sigma,d_{-\sigma})+\varepsilon L(c_\sigma,L(b_{-\sigma},a_\sigma)d_{-\sigma}),\end{aligned}$$

$$\text{(U2)}\quad \begin{aligned}&K(K(a_\sigma,b_\sigma)c_{-\sigma},d_\sigma)\\ &\quad = L(d_\sigma,c_{-\sigma})K(a_\sigma,b_\sigma)-\varepsilon K(a_\sigma,b_\sigma)L(c_{-\sigma},d_\sigma),\end{aligned}$$

where $K(a_\sigma,b_\sigma)c_{-\sigma}:=\langle a_\sigma c_{-\sigma} b_\sigma\rangle-\delta\langle b_\sigma c_{-\sigma} a_\sigma\rangle$[cf.2,4,9,13,17]. A pair $D=(D_+,D_-)$ of endomorphisms D_σ of $U(\varepsilon,\delta)^\sigma$ is a derivation of $U(\varepsilon,\delta)$ if $[D_\sigma,L(a_\sigma,b_{-\sigma})]=L(D_\sigma a_\sigma,b_{-\sigma})+L(a_\sigma, D_{-\sigma}b_{-\sigma})$. A pair $C=(C_+,C_-)$, $C_\sigma\in\mathrm{Hom}(U(\varepsilon,\delta)^{-\sigma},U(\varepsilon,\delta)^\sigma)$ is said to satisfy the condition (K) if

$$\text{(K1)}\quad K(C_\sigma a_{-\sigma},b_\sigma)-L(b_\sigma,a_{-\sigma})C_\sigma+\varepsilon C_\sigma L(a_{-\sigma},b_\sigma)=0,$$

$$\text{(K2)}\quad \varepsilon C_{-\sigma}K(a_\sigma,b_\sigma)+L(C_{-\sigma}a_\sigma,b_\sigma)-\delta L(C_{-\sigma}b_\sigma,a_\sigma)=0,$$

$$\text{(K3)}\quad K(a_{-\sigma},b_{-\sigma})C_\sigma-L(b_{-\sigma},C_\sigma a_{-\sigma})+\delta L(a_{-\sigma},C_\sigma b_{-\sigma})=0.$$

The pair $(L(a_+,b_-),\varepsilon L(b_-,a_+))$ is a derivation of $U(\varepsilon,\delta)$ and $(\delta K(a_+,b_+),-\varepsilon K(a_-,b_-))$ satisfies the condition (K). Let $\mathcal{D}$, $\mathcal{K}$ be the vector spaces spanned by derivations and pairs of homomorphisms satisfying the condition (K). Put $\mathcal{L}=\mathcal{D}\oplus\mathcal{K}\oplus U(\varepsilon,\delta)$ and define a product in $\mathcal{L}$ by

$$\begin{aligned}
&[(D_+,D_-),(D_+',D_-')]=([D_+,D_+'],[D_-,D_-']),\\
&[(D_+,D_-),(C_+,C_-)]=(D_+C_+-C_+D_-,D_-C_--C_-D_+),\\
&[(D_+,D_-),(a_+,a_-)]=(D_+a_+,D_-a_-)= -[(a_+,a_-),(D_+,D_-)],\\
&[(C_+,C_-),(C_+',C_-')]=(C_+C_-'-C_+'C_-,-C_-'C_++C_-C_+'),\\
&[(C_+,C_-),(a_+,a_-)]=(C_+a_-,C_-a_+)= -[(a_+,a_-),(C_+,C_-)],\\
&[(a_+,0),(b_+,0)]=(\delta K(a_+,b_+),0),\\
&[(0,a_-),(0,b_-)]=(0,-\varepsilon K(a_-,b_-)),\\
&[(a_+,0),(0,b_-)]=(L(a_+,b_-),\varepsilon L(b_-,a_+))=-\delta[(0,b_-),(a_+,0)].
\end{aligned}$$

THEOREM 1. For a Freudenthal-Kantor (super)pair $U(\varepsilon,\delta)$, $\mathcal{L}=V_0\oplus V_1$, $V_0=\mathcal{D}\oplus\mathcal{K}$, $V_1=U(\varepsilon,\delta)$, becomes a Lie algebra and Lie superalgebra with respect to the above product according to $\delta=1$ and -1 respectively.

Remark. W. Hein[6] constructed a universal enveloping

Z-graded Lie algebra from a pair algebra A only satisfying the condition (U1) with $\varepsilon=-1$ and showed that the condition $S_{\sigma 3}(A)=\{0\}$ gives an identity, which is equivalent to (U2) with $\delta=1$ under (U1)[6,p.325,Example 2]. The above construction is a pair generalization of the construction in [14].

3. CONSTRUCTION OF LIE (SUPER)ALGEBRAS FROM ASSOCIATIVE TRIPLE SYSTEMS AND FREUDENTHAL-KANTOR (SUPER)TRIPLE SYSTEMS

Recall the definition of commutative associative triple systems. A triple system A with product $\langle abc\rangle=\ell(a,b)c:=m(a,c)b$ is called a commutative associative triple system if $\langle ab\langle cde\rangle\rangle=\langle a\langle dcb\rangle e\rangle=\langle\langle abc\rangle de\rangle$ and $\langle abc\rangle=\langle cba\rangle$[10]. The linear span of $m(a,b)$'s becomes a Jordan triple system.

For any Freudenthal-Kantor (super)triple system(F-K-(S)TS) $U(\varepsilon,\delta)$ with product $L(x,y)z$, $A\otimes U(\varepsilon,\delta)$ becomes F-K-(S)TS with respect to the trilinear product $\langle a\otimes x\ b\otimes y\ c\otimes z\rangle:=\ell(a,b)c\otimes L(x,y)z$, $a,b,c\in A$, $x,y,z\in U(\varepsilon,\delta)$, in which we have $K(a\otimes x,b\otimes y)=m(a,b)\otimes K(x,y)$.

Put $s(a,b)=\frac{1}{2}(\ell(a,b)+\ell(b,a))$, $t(a,b)=\frac{1}{2}(\ell(a,b)-\ell(b,a))$. Assume the endomorphisms s^*,t^* of A satisfy the conditions $s^*\ell(a,b)=\ell(s^*a,b)=\ell(a,s^*b)=\ell(a,b)s^*$, $t^*\ell(a,b)=\ell(t^*a,b)=-\ell(a,t^*b)=\ell(a,b)t^*$. If D,B are the derivation and anti-derivation of $U(\varepsilon,\delta)$ respectively, then $s^*\otimes D$ and $t^*\otimes B$ are derivations of F-K(S)TS $A\otimes U(\varepsilon,\delta)$. Especially, $s(a,b)\otimes D$ and $t(a,b)\otimes B$ are derivations of $A\otimes U(\varepsilon,\delta)$. The endomorphism m^* of A is said to satisfy the condition (K) if $m^*\ell(a,b)=\ell(b,a)m^*=m(m^*a,b)$, $m^*m(a,b)=\ell(m^*a,b)$, and $m(a,b)m^*=\ell(a,m^*b)$ [17]. It is shown that $m(a,b)\in \mathrm{End}(A)$ satisfies the condition (K). If m^*(resp.C) satisfies the condition (K) in A (resp.$U(\varepsilon,\delta)$), then $m^*\otimes C$ satisfies the condition (K) in F-K(S)TS $A\otimes U(\varepsilon,\delta)$.

In F-K(S)TS $U(\varepsilon,\delta)$, define a trilinear product [,,] by $[xyz]:=S(x,y)z+\delta K(x,y)z$, where $S(x,y):=L(x,y)+\varepsilon L(y,x)$. Then, if $\varepsilon\delta=-1$, this product defines a Lie triple product or anti-Lie triple product on $U(\varepsilon,\delta)$ according to $\delta=1$ or -1, that is, $[xyz]= -\delta[yxz]$, $[xyz]+[yzx]+[zxy]=0$, $[L(x,y), L(u,v)]=L(L(x,y)u,v)-\delta L(u,L(y,x)v)$. The last relation follows from that if D is a derivation of $U(\varepsilon,\delta)$, then

$$[D,S(x,y)+\delta K(x,y)]=S(Dx,y)+S(x,Dy)+\delta K(Dx,y)+\delta K(x,Dy),$$

and if C satisfies the condition (K), then

$$[C,S(x,y)+\delta K(x,y)]=S(Cx,y)+S(x,Cy)+\delta K(Cx,y)+\delta K(x,Cy).$$

Therefore, a F-K(S)TS $A\otimes U(-\delta,\delta)$ becomes (anti-)Lie triple system with respect to the product:

$$[a\otimes x\ b\otimes y\ c\otimes z]:=(L(a\otimes x,b\otimes y)-\delta L(b\otimes y,a\otimes x)+\delta K(a\otimes x,b\otimes y))(c\otimes z)$$
$$=(s(a,b)\otimes S(x,y)+t(a,b)\otimes T(x,y)+\delta m(a,b)\otimes K(x,y))(c\otimes z),$$

where $S(x,y)=L(x,y)-\delta L(y,x)$, $T(x,y)=L(x,y)+\delta L(y,x)$.

If $D,B\in \mathrm{End}(U(-\delta,\delta))$ are derivation, anti-derivation respectively and C satisfies the condition (K), then $s^*\otimes D$, $t^*\otimes B$, $m^*\otimes C$ are the derivations of (anti-)Lie triple system $A\otimes U(-\delta,\delta)$, especially, so are $s(a,b)\otimes D, t(a,b)\otimes B, m(a,b)\otimes C$.

As a remarkable Lie (super)algebra between the standard imbedding Lie (super)algebra and the Lie (super)algebra $\mathcal{D}\oplus A\otimes U(-\delta,\delta)$, where $\mathcal{D}$ denotes the derivation Lie algebra of (anti-)Lie triple system $A\otimes U(-\delta,\delta)$, we obtain a Lie (super) algebra $\mathcal{G}$ in the following theorem, which is a generalization of the result due to U. Hirzebruch[7].

THEOREM 2. Let A be a commutative associative triple system and $U(\varepsilon,\delta)$ be a Freudenthal-Kantor (super)triple system with $\varepsilon\delta= -1$. Let $\mathcal{L},\mathcal{M},\mathcal{N}$ be the vector spaces spanned by $s(a,b)\otimes D$, $t(a,b)\otimes B$, $m(a,b)\otimes C$ respectively, where D(resp. B) is the derivation(resp. anti-derivation) of $U(\varepsilon,\delta)$ and C satisfies the condition (K) in $U(\varepsilon,\delta)$. Then, the vector space $\mathcal{G}=V_0\oplus V_1$, $V_0=\mathcal{L}+\mathcal{M}+\mathcal{N}$, $V_1=A\otimes U(-\delta,\delta)$, becomes a Lie algebra and Lie superalgebra according to $\delta=1$ and -1 respectively, which contains the standard imbedding Lie (super) algebra as a sub(super)algebra. The commutations in $\mathcal{L},\mathcal{M},\mathcal{N}$ satisfy the relations:

$[\mathcal{L},\mathcal{L}]\subset\mathcal{L},\ [\mathcal{L},\mathcal{M}]\subset\mathcal{M},\ [\mathcal{L},\mathcal{N}]\subset\mathcal{N},\ [\mathcal{M},\mathcal{M}]\subset\mathcal{L},\ [\mathcal{M},\mathcal{N}]\subset\mathcal{N},\ [\mathcal{N},\mathcal{N}]\subset\mathcal{L}+\mathcal{M}.$

The vector space direct sum $T(\varepsilon,\delta)=U(\varepsilon,\delta)\oplus U(\varepsilon,\delta)$ becomes the (anti-)Lie triple system by defining a triple product as

$$\left[\begin{pmatrix}a\\b\end{pmatrix}\begin{pmatrix}c\\d\end{pmatrix}\begin{pmatrix}e\\f\end{pmatrix}\right]=\begin{pmatrix}L(a,d)-\delta L(c,b) & \delta K(a,c)\\ -\varepsilon K(b,d) & \varepsilon(L(d,a)-\delta L(b,c))\end{pmatrix}\begin{pmatrix}e\\f\end{pmatrix}\ [9,16].$$

Put $J=\begin{pmatrix}0 & \delta\\ -\varepsilon & 0\end{pmatrix}$, then J is an automorphism of (anti-)Lie triple system $T(\varepsilon,\delta)$ such that $J^2=-\varepsilon\delta Id$, especially, if $\varepsilon\delta=-1$, then J is an involutive automorphism. For any $X,Y\in T(\varepsilon,\delta)$, define a Nijenhuis operator in $T(\varepsilon,\delta)$ by

$$N(X,Y):=[JX,JY]-J[JX,Y]-J[X,JY]+J^2[X,Y].$$

If $N(X,Y)=0$ for all X,Y in (anti-)Lie triple system $A\otimes U(\varepsilon,\delta)\oplus A\otimes U(\varepsilon,\delta)$ with $\varepsilon\delta=-1$, then $L(a\otimes x,b\otimes y)-\delta L(b\otimes y,a\otimes x)+\varepsilon K(a\otimes x,b\otimes y)=0$, that is, $\ell(a,b)\otimes L(x,y)-\delta\ell(b,a)\otimes L(y,x)+\varepsilon m(a,b)\otimes K(x,y)=0$. Then, we have

PROPOSITION 2. When $\varepsilon\delta=-1$, the Nijenhuis operator vanishes identically in (anti-)Lie triple system $A\otimes U(\varepsilon,\delta)\oplus A\otimes U(\varepsilon,\delta)$ if and only if $[a\otimes x\ b\otimes y\ c\otimes z]=2\delta m(a,b)c\otimes K(x,y)z$ in (anti-)Lie triple system $A\otimes U(\varepsilon,\delta)$.

As a corollary we see that let $U(-1,1)$ be a Jordan triple system, then $N(X,Y)=0$ for all X,Y in the Lie triple system $A\otimes U(-1,1)\oplus A\otimes U(-1,1)$ if and only if the induced Lie triple system $A\otimes U(-1,1)$ is abelian.

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