Particle Production at Space-time Horizons

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'To infinity and beyond!'
– Buzz Lightyear in '*Toy Story*'

Abstract

The classical derivation of Hawking radiation produces a completely thermal spectrum. This violates energy conservation. In this paper, two alternative methods that overcome this limitation are reviewed. One is a quantum field WKB technique developed by Per Kraus and Frank Wilczek. [2] The other is a technique that treats particle production as though it were a quantum tunnelling phenomenon using WKB techniques to determine the particle flux. This method was developed by Maulik Parikh and Frank Wilczek. [3]

Both methods produce a different non-thermal correction term. We review the criticisms on the tunnelling approach by Borun Chowdhury [15] and additional contributions proposed by Akhmedov et al. [17].

We also apply the tunnelling approach to De Sitter radiation in $3 + 1$ dimensions, producing the familiar result with a non-thermal first-order correction term that restores conservation of mass-energy. Finally, we use the tunnelling method to derive Unruh radiation measured by an accelerated observer in flat Minkowski space-time.

Contents

1 Introduction

Isaac Newton said that 'Absolute, true and mathematical time, of itself, and from its own nature, flows equably, without relation to anything external.'. The work of Hendrik Lorentz, Hermann Minkowski and, most importantly, Albert Einstein provided physics with a new insight: space and time form a continuum and we cannot view one as though it were completely separate from the other. A space-time is a collective entity that lives in a manifold and any path within the space-time must obey its metric equation. Observed time intervals, lengths and velocities are dependent on the space-time coordinates of the observer as well as the observed.

In classical quantum mechanics, time is in fact completely separate from space. To apply quantum mechanics within a non-Euclidean space-time, we need quantum field theory and thereby relinquish the concept of a point particle altogether. From a field theory perspective, particles are just excitations of a field and the observed particle numbers need not be the same for all observers. A vacuum is only properly defined in the context of an inertial system.

This disparity with human intuition leads to some counter-intuitive phenomena. A relatively well-known example is the Unruh effect: one observer moves with a constant acceleration as compared to the other. Both will observe a thermal particle flux coming from the other, with a temperature that depends linearly on the magnitude of acceleration.

The most famous result, however, is the Bekenstein-Hawking radiation originating from the horizon of a black hole. A black hole is a rather peculiar piece of space-time with a matter density so large that its gravity irrevocably confines any particle that comes within a certain radius - there is no physically possible path starting within that radius that leads to any point outside of it. At this radius lies the event horizon. Even though no matter or radiation can escape the black hole, an observer at a distance will still measure particles that appear to be coming from inside. This is due to the different vacua and thereby, different observed particle numbers at both sides of the horizon. From the perspective of an outside observer, particles are produced precisely at the horizon.

As with the Unruh effect, the flux can be described by a thermal distribution. In fact, one of the ways to derive classical Bekenstein-Hawking radiation is remarkably similar to the Unruh calculation. Stephen Hawking derived [1] that a black hole of mass M described externally by the Schwarzchild metric produces radiation with a temperature

$$T_{\mathrm{H}} = \frac{\hbar c^3}{8\pi k G M} \tag{1.1}$$

where π is Archimedes' constant, c is the velocity of light in a vacuum, $\hbar$ is the reduced Planck constant and G is the gravitational constant. The presence of all these constants immediately reveals that both general relativity and quantum mechanics are involved. This means that interesting physics is happening, but also that great care must be taken.

By the first law of thermodynamics, the entropy of a black hole must then equal

$$S_{\rm BH} = \frac{4\pi k G M^2}{\hbar c} = \frac{c^3 k A}{4\hbar G} \tag{1.2}$$

where A is the surface of the event horizon. This is the Bekenstein-Hawking entropy [22].

Moreover, production of radiation implies that the black hole loses energy in some way. One might say that it radiates away its mass. As time progresses, more energy is radiated away and the black hole becomes less massive and smaller in radius. Since the temperature depends inversely on mass, the rate of emission increases with time. As a result, the heat capacity, defined as the rate of change of the energy in the system with respect to change in temperature, is negative.

Classical methods effectively assume the black hole itself is static, in the sense that the radiation does not alter it in any way. Mass-energy conservation, however, requires that any particle production at the horizon be compensated for in some way within the horizon. This in conflict with the very physical foundations any derivation of Bekenstein-Hawking radiation builds upon. Moreover, the spectrum only depends on the initial mass, that is, the mass of the black hole before emission. This may not be unphysical, but it does seem unlikely intuitively.

There is another issue with the Hawking calculation. Since the result consists of a thermal spectrum independent of anything within the horizon that may contain information, the distribution is completely random. No amount of knowledge of previously radiated quanta can be used to predict the next quantum or more specific distribution of probabilities. From an information point of view, it is pure noise. However, the particles that entered the black hole up to that point did in general carry information. Quantum mechanics requires that any physical transition from one state to another be reversible. Yet the information to reconstruct the original state appears to have been lost or in some way captivated irrecoverably behind the Schwarzschild horizon of the black hole.

This problem is referred to as the black hole information loss paradox. Different solutions have been proposed, but they usually lead to other physical problems or contradict well-established theories. Modification of the Bekenstein-Hawking radiation spectrum by accounting for effects due to back-reaction might pose a step towards solution, for such a spectrum might in fact contain information.

To overcome the limitations of the classical Hawking derivation, one needs to allow for back-reaction in the black hole due to the particle production at its horizon. This can be done by considering a shell that is emitted and taking both the influence on the black hole and its self-gravitation into account. This is a rather involved calculation and to actually reach a result, one needs to do some assumptions and approximations.

A different method has been introduced that reproduces Hawking's result with a correction term. In this model, a shell of energy is formed along with a shell of negative energy that is equal in magnitude precisely at the horizon. While the anti-particle reduces the total amount of mass-energy enclosed in the black hole, the event horizon shrinks and the positive energy shell tunnels through the shrinking horizon. In this paper we will pursue both methods and comment on their relation. We will also review some of the criticisms of the tunnelling method and the implications for the results.

Although the presence of astrophysical black holes as such has been observed frequently by astronomers, the Bekenstein-Hawking radiation of such an object is negligible in intensity compared to other astrophysical radiation sources and it remains unconfirmed by experimental evidence at the time of this writing.

We will start by exploring the geometry of spherically symmetric curved space-times and, more specifically, the Schwarzschild metric and the Schwarzschild black hole in section 2. Then, in section 3, we determine the action for a black hole incorporating back-reaction, we derive a method to quantise an action expressed in terms of canonical momenta and, finally, we derive the radiation flux observed from a Schwarzschild Black hole by an observer at a distance. This method was developed Per Kraus and Frank Wilczek [2].

Maulik Parikh and Frank Wilczek introduced another method involving semi-classical tunnelling [3], which is derived and discussed in section 4. We review some of the criticisms it has drawn, we apply the tunnelling approach to De Sitter space-time in $3+1$ dimensions and we derive the Unruh effect observed by an accelerated observer in Minkowski space-time.

Some of the more laborious calculations and derivations are performed in appendices, for the sake of readability. These are grouped in a section containing calculations concerning space-time (appendix A) and a section concerning the particle flux calculation (appendix B).

1.1 Notation

We will in general use Planck units, where Einstein's constant c, the reduced Planck constant $\hbar$ and the gravitational constant G equal 1. This greatly reduces clutter in equations and it helps focus on the fundamental mathematical relations between physical concepts and entities.

All metric tensors will have Riemannian signature $(-,+,\ldots,+)$. 4-vectors will have Greek indices and 3-vectors will have Latin indices. The

Einstein summation convention

$$A_\mu B^\mu \equiv \sum_{\mu=0}^{3} A_\mu B^\mu, \quad A_j B^j \equiv \sum_{j=1}^{3} A_j B^j \tag{1.3}$$

is implied when a symbol is used as a subscript index as well as a superscript index on the same side of an equation.

We will often use a shorthand notation for partial derivatives ('comma derivatives') of a tensor's elements

$$T^{\nu_1\ldots\nu_n,\nu_k}_{\mu_1\ldots\mu_m,\mu_j} \equiv \partial_{\mu_j}\partial^{\nu_k} T^{\nu_1\ldots\nu_n}_{\mu_1\ldots\mu_m} \tag{1.4}$$

and for covariant derivatives ('semicolon derivatives')

$$T^{\nu_1\ldots\nu_n;\nu_k}_{\mu_1\ldots\mu_m;\mu_j} \equiv \nabla_{\mu_j}\nabla^{\nu_k} T^{\nu_1\ldots\nu_n}_{\mu_1\ldots\mu_m} \tag{1.5}$$

We will also frequently use the Newtonian notation for derivatives of scalar functions with respect to the time coordinate denoted as t and the radial coordinate denoted as r.

$$\dot{A} \equiv \partial_t A, \quad A' \equiv \partial_r A \tag{1.6}$$

Although it is common to denote the Riemann tensor, the Ricci tensor and the Ricci scalar by the symbol R, this paper uses the somewhat non-standard symbol $\mathcal{R}$ to avoid confusion with the function $R(t, r)$ in the components of a space-time metric that we will use frequently.

2 ADM space-time

In the world as we view it around us, time appears to have a character separate from that of spatial dimensions. We often view different spatial configurations as manifestations of a some function of time. When we say something has changed, we invariably mean it has changed as compared to the situation at some earlier time. We pick the time coordinate as our axis of reference in space-time.

The General Theory of Relativity does not single out any coordinate. One should, at least in principle, be able to describe any relativistically correct physical analysis in a covariant manner, that is, we should be able to pick any set of coordinates, irrespective of which coordinate we like to call 'time'.

However, in many cases we can still choose to describe the metric in which our physical system lives as an infinite set of spatial metrics that change as time proceeds. Such a description is closer to human intuition, but more importantly, it allows for a well-defined Hamiltonian formalism, which requires separate notions of space and time.

It can also be significantly easier to determine curvature properties of the metric in this manner, provided there is a proper mathematical framework. Firstly, one needs a well-defined description for both a constant-time slice of space and, secondly, a prescription to go from one instant in time to another.

In this section, we derive such a framework for a general metric with spherical symmetry. This is written in ADM (Arnowitt, Deser and Misner) form and we derive the metric properties required for a Hamiltonian description. We also introduce new coordinates for the Schwarzschild metric that describes a non-rotating, charge-free black hole that are used extensively in chapters 3 and 4.

2.1 The ADM form metric

Any space-time metric is defined by the line element, which determines the infinitesimal difference in path length for an infinitesimal displacement dx^μ. If a metric is spherically symmetric in the sense that it is invariant under rotations, it can be written in ADM form, short for Arnowitt, Deser and Misner. In coordinates (t, r, ϕ, θ), the quadratic line element equation is written as

$$ds^2 = -N^t(t,r)^2 dt^2 + L(t,r)^2 [dr + N^r(t,r)dt]^2 + R(t,r)^2 d\Omega_2^2 \qquad (2.1.1)$$

[6, 9] where $d\Omega_2^2$ is the quadratic line element of a 2-sphere

$$d\Omega_2^2 := d\theta^2 + \sin^2\theta \, d\phi^2 \qquad (2.1.2)$$

$N^t(t,r)$ is called the lapse function and $N^r(t,r)$ the shift function [6, 7]. They are independent functions of the time coordinate t and the radial coordinate r., as are L and R.

$$L(t,r) = \frac{ds}{dr} \tag{2.1.3}$$

$$R(t,r) = \frac{ds}{d\Omega_2} \tag{2.1.4}$$

These four functions can be used to express any metric that observes spherical symmetry in the sense that all angular dependence is in the form of a scaled 2-sphere contribution. As we will see in section 3.4, this allows for a well-defined Hamiltonian principle.

The metric tensor $g_{\mu\nu}$ can be read from (2.1.1). For brevity, we will omit the arguments of the four functions.

$$\begin{aligned} &g_{tt} = -N^{t2} + L^2 N^{r2} \qquad &&g_{rr} = L^2 \qquad &&g_{tr} = g_{rt} = L^2 N^r \\ &g_{\theta\theta} = R^2 &&g_{\phi\phi} = R^2 \sin^2\theta \end{aligned} \tag{2.1.5}$$

with all other components equalling 0. The inverse metric $g^{\mu\nu}$ is obtained by inverting the matrix formed by the components of $g_{\mu\nu}$. Its components are

$$\begin{aligned} &g^{tt} = -N^{t-2} \qquad &&g^{rr} = L^{-2} - N^{r2} N^{t-2} \qquad &&g^{tr} = g^{rt} = N^r N^{t-2} \\ &g^{\theta\theta} = R^{-2} &&g^{\phi\phi} = R^{-2} \sin^{-2}\theta \end{aligned} \tag{2.1.6}$$

To compute integrals involving ADM space-time, one needs the integration weight $\sqrt{-g}$, where g denotes the determinant of the metric when expressed as a 4×4 matrix.

$$\begin{aligned} \det g &= \begin{vmatrix} L^2 N^{r2} - N^{t2} & L^2 N^r & 0 & 0 \\ L^2 N^r & L^2 & 0 & 0 \\ 0 & 0 & R^2 & 0 \\ 0 & 0 & 0 & R^2 \sin^2\theta \end{vmatrix} \\ &= \left[L^4 N^{r2} - L^2 N^{t2} - L^4 N^{r2} \right] R^4 \sin^2\theta \\ &= -N^{t2} L^2 R^4 \sin^2\theta \end{aligned} \tag{2.1.7}$$

Hence

$$\sqrt{-g} = N^t L R^2 \sin\theta \tag{2.1.8}$$

These are the basic properties of a general ADM form metric. They are valid for any choice of the four defining functions that appear in equation (2.1.1).

2.2 Curvature

Most choices of the four ADM metric functions describe a space-time that is subject to curvature. Those that describe the Schwarzschild black hole are no exception. It is a massive object that severely curves space-time in its vicinity. To describe it, we first need to remind ourselves of the mathematical description of curvature.

The intrinsic curvature of a space-time is determined by the connection coefficients of the metric. In General Relativity, the connection is given by the Christoffel symbols $\Gamma^{\rho}_{\mu\nu}$

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\lambda} \left[g_{\mu\lambda,\nu} + g_{\nu\lambda,\mu} - g_{\mu\nu,\lambda} \right] \tag{2.2.1}$$

It is in a sense the deviation from a flat space-time: it contains the difference between the conformal derivative and the partial derivative of a tensor. Note that we need not worry about index placement for the connection symbol, as it is not a proper tensor; it is merely one of the non-tensorial terms that form a new tensor together. The covariant derivative of a tensor $T^{\mu_1 \ldots \mu_m}{}_{\nu_1 \ldots \nu_n}$ with respect to ρ is given by

$$\begin{aligned} T^{\mu_1 \ldots \mu_m}{}_{\nu_1 \ldots \nu_n ; \rho} = T^{\mu_1 \ldots \mu_m}{}_{\nu_1 \ldots \nu_n , \rho} + \sum_{j=1}^{m} \Gamma^{\mu_j}_{\rho\lambda} T^{\mu_1 \ldots \mu_{j-1} \lambda \mu_{j+1} \ldots \mu_m}{}_{\nu_1 \ldots \nu_n} \\ - \sum_{j=1}^{n} \Gamma^{\lambda}_{\rho\nu_j} T^{\mu_1 \ldots \mu_m}{}_{\nu_1 \ldots \nu_{j-1} \lambda \nu_{j+1} \ldots \nu_n} \end{aligned} \tag{2.2.2}$$

Armed with the Christoffel connection, we can define the Riemann curvature tensor, which determines the change in a vector due to curvature when parallel transported around a loop through space-time.

$$\mathcal{R}^{\rho}{}_{\sigma\mu\nu} := \Gamma^{\rho}_{\nu\sigma,\mu} - \Gamma^{\rho}_{\mu\sigma,\nu} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma} \tag{2.2.3}$$

The Ricci curvature scalar is a measure for the of curvature of a space-time. It describes how much the volume contained within a geodesic sphere deviates from Euclidian space-time. It is created by the contraction of the Ricci tensor, which in turn is a contraction of the Riemann tensor.

$$\begin{aligned} \mathcal{R} := g^{\mu\nu}\mathcal{R}_{\mu\nu} &= g^{\mu\nu}\mathcal{R}^{\lambda}{}_{\mu\lambda\nu} \\ &= g^{\mu\nu}\left[\Gamma^{\lambda}_{\nu\mu,\lambda} - \Gamma^{\lambda}_{\lambda\mu,\nu} + \Gamma^{\lambda}_{\lambda\sigma}\Gamma^{\sigma}_{\nu\mu} - \Gamma^{\lambda}_{\nu\sigma}\Gamma^{\sigma}_{\lambda\mu}\right] \end{aligned} \tag{2.2.4}$$

The scalar curvature tensor plays an important role in the action formalism in curved space-time. In section 2.5, we will determine the value for ADM space-time and in section 3.3 we will use this result to write down the action of a black hole emitting shells of energy.

2.3 A constant-time hypersurface

ADM space-time can be viewed as an infinite set of constant-time spaces paired with a recipe to go from one instant in time to another. While a two-dimensional slice of a three-dimensional object is called a surface, an $(n-1)$-dimensional submanifold of an n-dimensional manifold is called a hypersurface. If we call the manifold of the four-dimensional space-time M, a slice of it at a time t lives on the hypersurface Σ_t.

One can define a vector perpendicular to the hypersurface that points towards the future, i.e., to the 'next' hypersurface Σ_{t+dt}.

$$\xi^\mu := g^{\mu\nu} t_{;\nu} = g^{\mu\nu} t_{,\nu} = g^{\mu\nu} \delta^t_\nu \tag{2.3.1}$$

With the inverse metric components from (2.1.6), the vector's components are

$$\xi^t = g^{tt} = -N^{t\,-2}, \quad \xi^r = g^{tr} = N^r N^{t\,-2}, \quad \xi^\theta = \xi^\phi = 0 \tag{2.3.2}$$

The magnitude of ξ^μ, however, is not relevant. It is the orientation we need. Let us define a new vector n^μ by normalising the perpendicular time vector

$$\begin{aligned} n^\mu :&= \frac{\xi^\mu}{\sqrt{|\xi_\nu \xi^\nu|}} = \frac{\xi^\mu}{\sqrt{|g_{\nu\rho}\xi^\rho \xi^\nu|}} \\ &= \frac{\xi^\mu}{\sqrt{\left|\left(-N^{t2} + L^2 N^{r2}\right) N^{t-4} + L^2 N^{r2} N^r N^{t-4} - 2L^2 N^{r2} N^{t-4}\right|}} \\ &= N^t \xi^\mu \end{aligned} \tag{2.3.3}$$

The components of the normal time vector are then

$$n^t = -\frac{1}{N^t}, \quad n^r = \frac{N^r}{N^t}, \quad n^\theta = n^\phi = 0 \tag{2.3.4}$$

Since n^μ is normalised to 1, the inner product $n_\mu n^\mu$ can be either $+1$, representing a spacelike vector, or -1, representing a timelike vector. Let us verify that the vector pointing into the future really is timelike

$$\sigma := n_\mu n^\mu = -1 + L^2 \frac{N^{r2}}{N^{t2}} + L^2 \frac{N^{r2}}{N^{t2}} - 2L^2 \frac{N^{r2}}{N^{t2}} = -1 \tag{2.3.5}$$

On the hypersurface, we can define an induced metric h_{ij}, following the common convention of using Latin indices for three-dimensional tensors. If we use coordinates $\{x^\mu\}$ on M and $\{y^i\}$ on Σ_t, the elements of the induced metric are given by the coordinate transformation

$$h_{ij} := \frac{\partial x^\mu}{\partial y^i} \frac{\partial x^\nu}{\partial y^j} g_{\mu\nu} \tag{2.3.6}$$

The coordinate t is by definition constant on the hypersurface, but the spatial coordinates r, θ and ϕ are still very well suited for their jobs, so the most convenient choice for y^i is in fact x^i. The induced metric h_{ij} of an ADM timeslice is then

$$h_{rr} = L^2, \quad h_{\theta\theta} = R^2, \quad h_{\phi\phi} = R^2 \sin^2\theta \tag{2.3.7}$$

with all other components equalling zero. Since this metric is diagonal, the corresponding inverse metric h^{ij} consists of the inverse of the elements of h_{ij}.

$$h^{rr} = L^{-2}, \quad h^{\theta\theta} = R^{-2}, \quad h^{\phi\phi} = R^{-2}\sin^{-2}\theta \tag{2.3.8}$$

Using these results and equation (2.2.1), the Christoffel symbols $\widehat{\Gamma}^i_{jk}$ of the submanifold are

$$\begin{aligned}
&\widehat{\Gamma}^r_{rr} = \frac{L'}{L} && \widehat{\Gamma}^r_{\theta\theta} = -\frac{RR'}{L^2} && \widehat{\Gamma}^r_{\phi\phi} = -\frac{RR'}{L^2}\sin^2\theta \\
&\widehat{\Gamma}^\theta_{r\theta} = \widehat{\Gamma}^\theta_{\theta r} = \frac{R'}{R} && \widehat{\Gamma}^\theta_{\phi\phi} = -\sin\theta\cos\theta && \widehat{\Gamma}^\phi_{r\phi} = \widehat{\Gamma}^\phi_{\phi r} = \frac{R'}{R} \\
&\widehat{\Gamma}^\phi_{\theta\phi} = \widehat{\Gamma}^\phi_{\phi\theta} = \frac{\cos\theta}{\sin\theta}
\end{aligned} \tag{2.3.9}$$

with all other symbol elements $\widehat{\Gamma}^i_{jk} = 0$. The Ricci curvature scalar $\hat{\mathcal{R}}$ for the hypersurface is determined in appendix A.3. The result is

$$\widehat{\mathcal{R}} = -4\frac{R''}{L^2R} + 4\frac{L'R'}{L^3R} - 2\frac{R'^2}{L^2R^2} + \frac{2}{R^2} \tag{2.3.10}$$

We have now determined the scalar curvature of a constant-time hypersurface and we know how the hypersurfaces at different times relate to each other.

2.4 Embedding the hypersurface

Now that we have established the properties of a constant-time slice of space, let us examine how it fits in space-time. We will follow the analyses on hypersurfaces in Carroll [8] and Wald [7].

The first object we need is the projection tensor

$$P_{\mu\nu} := g_{\mu\nu} - \sigma n_\mu n_\nu \tag{2.4.1}$$

It projects any vector V^μ tangential to the hypersurface. Such a projection is obviously orthogonal to n^μ

$$P_{\mu\nu}V^\mu n^\nu = V^\mu n_\mu - \sigma^2 n_\mu V^\mu = 0 \tag{2.4.2}$$

Vectors that are tangential to the hypersurface are by definition orthogonal to n^μ, so when applied to such a vector, the second term equals zero. The projection tensor thus acts on these vectors in a manner identical to the metric tensor. To determine the components of $P_{\mu\nu}$ for ADM space-time, one needs the one-form corresponding to the normal vector in our metric.

Armed with the projection tensor we can determine the extrinsic curvature tensor $K_{\mu\nu}$. This object describes the curvature arising from the embedding of the hypersurface into the manifold.

$$K_{\mu\nu} := \frac{1}{2}\mathcal{L}_n P_{\mu\nu} \tag{2.4.3}$$

where $\mathcal{L}_n$ denotes the Lie derivative with respect to normal vector n^μ. The Lie derivative is in a sense the tensorial generalisation of the directional derivative $V^\mu f_{,\mu}$ of a scalar function f with respect to a vector V^μ. For a general tensor $T^{\mu_1 \dots \mu_m}{}_{\nu_1 \dots \nu_n}$, it is given by

$$\begin{aligned}\mathcal{L}_V T^{\mu_1 \dots \mu_m}{}_{\nu_1 \dots \nu_n} = V^\rho T^{\mu_1 \dots \mu_m}{}_{\nu_1 \dots \nu_n;\rho} - \sum_{j=1}^{m} V^{\mu_j}{}_{;\rho} T^{\mu_1 \dots \mu_{j-1}\rho\mu_{j+1} \dots \mu_m}{}_{\nu_1 \dots \nu_n} \\ + \sum_{j=1}^{n} V^\rho{}_{;\nu_j} T^{\mu_1 \dots \mu_m}{}_{\nu_1 \dots \nu_{j-1}\rho\nu_{j+1} \dots \nu_n}\end{aligned} \tag{2.4.4}$$

[7]. However, the connection terms introduced by the first covariant derivative are exactly cancelled by those from the other covariant derivatives. We might as well calculate the Lie derivative using regular partial derivatives.

$$\begin{aligned}\mathcal{L}_V T^{\mu_1 \dots \mu_m}{}_{\nu_1 \dots \nu_n} = V^\rho T^{\mu_1 \dots \mu_m}{}_{\nu_1 \dots \nu_n,\rho} - \sum_{j=1}^{m} V^{\mu_j}{}_{,\rho} T^{\mu_1 \dots \mu_{j-1}\rho\mu_{j+1} \dots \mu_m}{}_{\nu_1 \dots \nu_n} \\ + \sum_{j=1}^{n} V^\rho{}_{,\nu_j} T^{\mu_1 \dots \mu_m}{}_{\nu_1 \dots \nu_{j-1}\rho\nu_{j+1} \dots \nu_n}\end{aligned} \tag{2.4.5}$$

In explicit form, the extrinsic curvature tensor is then

$$K_{\mu\nu} = \frac{1}{2}\left[n^\lambda P_{\mu\nu,\lambda} + n^\lambda{}_{,\mu} P_{\lambda\nu} + n^\lambda{}_{,\nu} P_{\mu\lambda}\right] \tag{2.4.6}$$

It is clear that $K_{\mu\nu}$ is symmetric in μ, ν. Alternatively, the extrinsic curvature tensor may be expressed in terms of the time normal vector and its derivatives as

$$K_{\mu\nu} = n_{\nu;\mu} - \sigma n_\mu n^\lambda n_{\nu;\lambda} \tag{2.4.7}$$

This is derived in appendix A.1. Since this form contains explicit covariant derivatives, we will not use it to derive the

Gauss' equation relates the Riemann curvature tensor of a hypersurface to its counterpart on the space-time manifold

$$\widehat{\mathcal{R}}^\rho{}_{\sigma\mu\nu} = P^\rho{}_\alpha P^\beta{}_\sigma P^\gamma{}_\mu P^\delta{}_\nu \mathcal{R}^\alpha{}_{\beta\gamma\delta} + \sigma\left[K^\rho{}_\mu K_{\sigma\nu} - K^\rho{}_\nu K_{\sigma\mu}\right] \tag{2.4.8}$$

It is derived in A.2, as is the equation

$$\mathcal{R}_{\mu\nu}n^\mu n^\nu = K^{\mu\nu}K_{\mu\nu} - K^2 + \left(n^\rho{}_{;\rho}n^\mu - n^\mu{}_{;\nu}n^\nu\right)_{;\mu} \tag{2.4.9}$$

for the scalar product of the Ricci tensor with the product of two normal vectors. K^2 denotes the scalar quantity $K^\mu{}_\nu$. Contraction of (2.4.8) yields

$$\widehat{\mathcal{R}} = \mathcal{R} - 2\sigma\mathcal{R}_{\mu\nu}n^\mu n^\nu + \sigma\left[K^2 - K^{\mu\nu}K_{\mu\nu}\right] \tag{2.4.10}$$

The Ricci curvature scalar for a four-dimensional space-time containing three-dimensional hypersurfaces with a perpendicular normal vector n^μ is then given by

$$\mathcal{R} = \widehat{\mathcal{R}} + \sigma\left[K^{\mu\nu}K_{\mu\nu} - K^2\right] + 2\sigma\left(n^\mu{}_{;\nu}n^\nu - n^\nu{}_{;\nu}n^\mu\right)_{;\mu} \tag{2.4.11}$$

which again is derived in appendix A.2.

This result provides us with a straightforward way to determine the scalar curvature of ADM space-time without needing to calculate all 32 unique Christoffel connection components explicitly.

2.5 Curvature properties of an ADM form metric

In the preceding section, we introduced a few objects that relate hypersurfaces to their embedding manifold and we derived a method to determine the Ricci scalar curvature for such a space-time. Armed with the results of section 2.3, we can now determine those properties for a space-time with an ADM-form metric consisting of hypersurfaces of constant time.

From equations (2.3.4) and (2.1.5), we have

$$n_t = N^t - L^2\frac{N^{r2}}{N^t} + L^2\frac{N^{r2}}{N^t} = N^t \tag{2.5.1}$$

$$n_r = L^2\frac{N^{r2}}{N^t} - L^2\frac{N^{r2}}{N^t} = 0 \tag{2.5.2}$$

$$n_\theta = n_\phi = 0 \tag{2.5.3}$$

Now the projection tensor for our metric is

$$\begin{aligned} &P_{tt} = L^2N^{r2} \qquad &&P_{rr} = L^2 \qquad &&P_{tr} = P_{rt} = L^2N^r \\ &P_{\theta\theta} = R^2 &&P_{\phi\phi} = R^2\sin^2\theta && \end{aligned} \tag{2.5.4}$$

with the other components equalling 0. None of the components depend on N^t, which is reasonable, since the hypersurface does not extend in the t dimension.

In appendix A.4, we use equation (2.4.6) to show that the components of the extrinsic curvature tensor for an ADM form metric are

$$K_{tt} = \frac{N^{r2}}{N^t} N^{r\prime} L^2 + \frac{N^{r3}}{N^t} L L' - \frac{N^{r2}}{N^t} L \dot{L} \tag{2.5.5}$$

$$K_{rr} = \frac{N^r}{N^t} L L' - \frac{1}{N^t} L \dot{L} + \frac{N^{r\prime}}{N^t} L^2 \tag{2.5.6}$$

$$K_{rt} = K_{tr} = \frac{N^r N^{r\prime}}{N^t} L^2 - \frac{N^r}{N^t} L \dot{L} + \frac{N^{r2}}{N^t} L L' \tag{2.5.7}$$

$$K_{\theta\theta} = -\frac{R\dot{R}}{N^t} + \frac{N^r}{N^t} R R' \tag{2.5.8}$$

$$K_{\phi\phi} = -\frac{R\dot{R}}{N^t} \sin^2\theta + \frac{N^r}{N^t} R R' \sin^2\theta \tag{2.5.9}$$

while all other components equal zero. With these results, the Ricci scalar for a general spherically symmetric metric in ADM form can be determined using equation (2.4.11). This calculation is carried out in appendix A.5

$$\begin{aligned}
\mathcal{R} = & -4\frac{R''}{L^2 R} + 4\frac{L' R'}{L^3 R} - 2\frac{R'^2}{L^2 R^2} + \frac{2}{R^2} - \frac{2}{N^{t2}} \frac{\dot{R}^2}{R^2} - 2\frac{N^{r2}}{N^{t2}} \frac{R'^2}{R^2} \\
& + 4\frac{N^r}{N^{t2}} \frac{\dot{R} R'}{R^2} + 4\frac{N^r}{N^{t2}} \frac{L'}{L} \frac{\dot{R}}{R} - \frac{4}{N^{t2}} \frac{\dot{L}}{L} \frac{\dot{R}}{R} + 4\frac{N^{r\prime}}{N^{t2}} \frac{\dot{R}}{R} - 4\frac{N^{r2}}{N^{t2}} \frac{L'}{L} \frac{R'}{R} \\
& + 4\frac{N^r}{N^{t2}} \frac{\dot{L}}{L} \frac{R'}{R} - 4\frac{N^r N^{r\prime}}{N^{t2}} \frac{R'}{R} + 2\left(n^\nu{}_{;\nu} n^\mu - n^\mu{}_{;\nu} n^\nu\right)_{;\mu}
\end{aligned} \tag{2.5.10}$$

All terms are in explicit form for a general ADM metric, except for the covariant derivative at the end. Although it would be possible in principle to determine this contribution, this would be rather laborious and, moreover, it would be pointless, for we will only use this result in a form where it makes no physical contribution.

2.6 The Schwarzschild metric

The Schwarzschild metric obeyed by a non-rotating, charge-free black hole of mass M is

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt_S^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega_2{}^2 \tag{2.6.1}$$

where t_S is the Schwarzschild time, denoted as such to prevent confusion with a new time coordinate we are about to introduce. It is an ADM spacetime with the metric functions

$$L(t,r) = \frac{1}{\sqrt{1 - \frac{2M}{r}}}, \qquad R(t,r) = r \tag{2.6.2}$$

$$N^t(t,r) = \sqrt{1 - \frac{2M}{r}}, \qquad N^r(t,r) = 0 \tag{2.6.3}$$

Inspection of the line element equation reveals that there is a coordinate horizon at the Schwarzschild radius r_S

$$r_S := 2M \tag{2.6.4}$$

This is a well-known and important aspect of the Schwarzschild black hole. Within this horizon, the space-time curvature due to the enclosed mass is so strong that no particle can escape within a finite amount of time. Put more explicitly, no paths exist that start with a radial component smaller than r_S and end with a radial coordinate larger than r_S within a finite, increasing time interval.

Even though particles cannot escape the black hole horizon, there is nothing that prevents particle paths from originating precisely *at* the horizon. To describe the paths of such particles, we need an equivalent formulation with coordinates that are still valid at the coordinate singularity. With the substitution

$$dt := dt_S + \sqrt{\frac{2M}{r}}\left(1 - \frac{2M}{r}\right)^{-1} dr \tag{2.6.5}$$

we have

$$\begin{aligned} dt_S^2 &= \left(dt - \sqrt{\frac{2M}{r}}\left(1 - \frac{2M}{r}\right)^{-1}\right) dr^2 \\ &= dt^2 - 2\sqrt{\frac{2M}{r}}\left(1 - \frac{2M}{r}\right)^{-1} dt\, dr + \frac{2M}{r}\left(1 - \frac{2M}{r}\right)^{-2} dr^2 \end{aligned} \tag{2.6.6}$$

Plugging this into (2.6.1) yields the Gullstrand-Painlevé metric

$$\begin{aligned} ds^2 &= -\left(1 - \frac{2M}{r}\right) dt^2 + 2\sqrt{\frac{2M}{r}} dt\, dr - \frac{2M}{r}\left(1 - \frac{2M}{r}\right)^{-1} dr^2 \\ &\quad + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d{\Omega_2}^2 \\ &= -\left(1 - \frac{2M}{r}\right) dt^2 + 2\sqrt{\frac{2M}{r}} dt\, dr + dr^2 + r^2 d{\Omega_2}^2 \end{aligned} \tag{2.6.7}$$

The corresponding ADM metric functions are

$$L_{\mathrm{GP}}(t, r) = 1, \qquad R_{\mathrm{GP}}(t, r) = r \tag{2.6.8}$$

$$N^t_{\mathrm{GP}}(t, r) = 1, \qquad N^r_{\mathrm{GP}}(t, r) = \sqrt{\frac{2M}{r}} \tag{2.6.9}$$

The Gullstrand-Painlevé metric is still stationary, like the Schwarzschild metric, in the sense that there exists a timelike Killing vector

$$t^\mu = \delta^\mu_t, \tag{2.6.10}$$

but, unlike the Schwarzschild metric, it is not static. Time reversal changes the sign of the second term in the line element equation.

By the substitution in equation (2.6.5), we have implicitly defined a new time coordinate t in terms of t_S and r. Explicitly,

$$\begin{aligned} t &= \int dt_S + \int dr \sqrt{\frac{2M}{r}} \left(1 - \frac{2M}{r}\right)^{-1} \\ &= t_S + \sqrt{2M} \int du \frac{2u}{u(1-2M/u^2)} = t_S + 2\sqrt{2M} \int du \frac{u^2}{u^2 - 2M} \\ &= t_S - 4M \tanh^{-1}\left(\frac{u}{\sqrt{2M}}\right) = t_S - 4M \tanh^{-1}\left(\sqrt{\frac{r}{2M}}\right) \\ &= t_S - 2M \ln\left(\frac{1+\sqrt{\frac{r}{2M}}}{1-\sqrt{\frac{r}{2M}}}\right) = t_S + 2M \ln\left(\frac{\sqrt{2M}-\sqrt{r}}{\sqrt{2M}+\sqrt{r}}\right) \end{aligned} \tag{2.6.11}$$

were we have chosen the integration constants to equal zero and used the substitution $u^2 := r$.

We assume that any radiation originating at the horizon has a radial path, i.e. $d{\Omega_2}^2 = 0$. Radial null geodesics in the Gullstrand-Painlevé metric obey the equation

$$\begin{aligned} 0 &= -\left(1 - \frac{2M}{r}\right) dt^2 + 2\sqrt{\frac{2M}{r}} dt dr + dr^2 \\ &= \left(dr - \left(1 - \sqrt{\frac{2M}{r}}\right) dt\right) \left(dr + \left(1 + \sqrt{\frac{2M}{r}}\right) dt\right) \end{aligned} \tag{2.6.12}$$

This equation is solved by

$$dr = \left(1 - \sqrt{\frac{2M}{r}}\right) dt \vee dr = -\left(1 + \sqrt{\frac{2M}{r}}\right) dt \tag{2.6.13}$$

whence we obtain that a null geodesic follows a path described by

$$\frac{dr}{dt} = \pm 1 - \sqrt{\frac{2M}{r}} \tag{2.6.14}$$

Within the black hole, i.e., for radii less than the Schwarzschild radius (2.6.4), a particle always moves inwards as the time t increases. However, massless particles outside the black hole, always move towards the horizon if they follow the negative-sign path, whereas they always move away from the horizon if they follow the positive-sign path. Introducing

$$\eta = \begin{cases} -1 & \text{Inward trajectories} \\ +1 & \text{Outward trajectories} \end{cases} \tag{2.6.15}$$

we may write that

$$\frac{dr}{dt} = \eta - \sqrt{\frac{2M}{r}} \tag{2.6.16}$$

Since we have an equation for $\frac{dr}{dt}$ in terms of r, the null geodesics are most easily parametrised by integrating the inverse of $\frac{dr}{dt}$ over r

$$\begin{aligned} t(r) &= \int_0^r d\bar{r}\, \frac{dt}{dr}\bigg|_{r=\bar{r}} = \int_0^r \frac{d\bar{r}}{\eta - \sqrt{\frac{2M}{\bar{r}}}} \qquad (2.6.17) \\ &= \eta r + 2\sqrt{2Mr} - 2M \log\left(\frac{\sqrt{2M} + \sqrt{r}}{\sqrt{2M} - \sqrt{r}}\right) + 2\eta M \log(r - 2M) \end{aligned}$$

With a few lines of algebra (performed in appendix A.6),

$$t_\eta(r) = \eta r + 2\sqrt{2Mr} + 4\eta M \log\left(\sqrt{2M} - \eta\sqrt{r}\right) \tag{2.6.18}$$

We now have a mathematical description the path of particles emitted radially by a Schwarzschild black hole as well as particles travelling inward from the event horizon.

3 Hawking radiation with back-reaction

In the original derivation of radiation emitted by black hole horizons by Stephen Hawking, the geometry is considered completely static in the sense that emitting a particle does not change the geometry and it does not influence the spectrum of the next particle. The resulting spectrum is completely thermal. This allows for the emission of particles that contain more energy than the black hole contains. Moreover, energy conservation is clearly violated: particle production at the horizon does not result in reduction of the energy contents of the black hole.

To address these problems, we need to consider back-reaction, the contribution of the shell to the gravity of the black hole. Per Kraus and Frank Wilczek have introduced a method that reproduces the Hawking result with a correction term [2]. In this section, the calculation in that paper is reproduced.

3.1 The particle flux from a Schwarzschild horizon

To derive Hawking radiation, the entity we need to determine is the particle flux an observer at a distance would measure if he were to look at the horizon of a Schwarzschild black hole with mass M. We will need to consider two kinds of observers: those at infinity and those freely falling through the horizon of the black hole.

For observers of the first type, we will expand the field in terms of modes $u_k(r)e^{-i\omega_k t}$ and $u_k^*(r)e^{i\omega_k t}$ with operators $\hat{a}_k$ and $\hat{a}_k^\dagger$

$$\hat{\phi}(t) = \int dk \left[\hat{a}_k u_k(r)e^{-i\omega_k t} + \hat{a}_k^\dagger u_k^*(r)e^{+i\omega_k t}\right] \tag{3.1.1}$$

whereas for the horizon perspective we will use modes v_k and v_k^* with operators $\hat{b}_k$ and $\hat{b}_k^\dagger$, respectively

$$\hat{\phi}(t) = \int dk \left[\hat{b}_k v_k(t,r) + \hat{b}_k^\dagger v_k^*(t,r)\right] \tag{3.1.2}$$

$\hat{b}_k$ and $\hat{b}_k^\dagger$ are related to $\hat{a}_k$ and $\hat{a}_k^\dagger$ by means of a Bogoliubov transformation

$$\hat{a}_k = \int dk' \left[\alpha_{kk'}\hat{b}_{k'} + \beta_{kk'}\hat{b}_{k'}^\dagger\right] \tag{3.1.3}$$

$$\hat{b}_k = \int dk' \left[\alpha_{k'k}^*\hat{a}_{k'} - \beta_{k'k}\hat{a}_{k'}^\dagger\right] \tag{3.1.4}$$

where the Bogoliubov coefficients $\alpha_{kk'}$ and $\beta_{kk'}$ are determined by

$$\alpha_{kk'} = \frac{1}{2\pi}\frac{1}{u_k(t,r)}\int_{-\infty}^{\infty} dt\, e^{i\omega_k t} v_{k'}(t,r) \tag{3.1.5}$$

$$\beta_{kk'} = \frac{1}{2\pi}\frac{1}{u_k(t,r)}\int_{-\infty}^{\infty} dt\, e^{i\omega_k t} v_{k'}^*(t,r) \tag{3.1.6}$$

We are interested in the particle number N_k of the mode k in the vacuum $|0_S\rangle$ of an observer travelling through the horizon

$$\begin{aligned} N_k &= \langle 0_S | \hat{a}_k^\dagger \hat{a}_k | 0_S \rangle \\ &= \langle 0_S | \int dk_1' dk_2' \left[\alpha_{kk_1'}^* \hat{b}_{k_1'}^\dagger + \beta_{kk_1'}^* \hat{b}_{k_1'} \right] \left[\alpha_{kk_2'} \hat{b}_{k_2'} + \beta_{kk_2'} \hat{b}_{k_2'}^\dagger \right] | 0_D \rangle \\ &= \langle 0_S | \int dk_1' dk_2' \, \beta_{kk_1'}^* \beta_{kk_2'} \hat{b}_{k_2'}^\dagger \delta\left(k_2' - k_1'\right) | 0_S \rangle = \int dk' |\beta_{kk'}|^2 \end{aligned} \quad (3.1.7)$$

With the completeness relation of the Bogoliubov coefficients,

$$\int dk' \left[|\alpha_{kk'}|^2 - |\beta_{kk'}|^2 \right] = 1, \quad (3.1.8)$$

we can rewrite the particle number into a form where the integrand does not depend on k' if $\left|\frac{\alpha_{kk'}}{\beta_{kk'}}\right|^2$ does not depend on k'

$$N_k = \int dk' \frac{|\beta_{kk'}|^2}{|\alpha_{kk'}|^2 - |\beta_{kk'}|^2} = \int dk' \frac{1}{\left|\frac{\alpha_{kk'}}{\beta_{kk'}}\right|^2 - 1}$$

The particle flux over the frequency interval $[\omega_k, \omega_k + d\omega_k]$ is

$$\Phi(\omega_k) = \frac{d\omega_k}{2\pi} \frac{1}{\left|\frac{\alpha_{kk'}}{\beta_{kk'}}\right|^2 - 1} \quad (3.1.9)$$

Some fraction of all radiated particles will follow paths that end back into the black hole due to curvature. The remaining fraction $\kappa(\omega)$ will continue to propagate to infinite r. In general, this fraction will be a function of the frequency ω.

$$\Phi_\infty(\omega_k) = \frac{d\omega_k}{2\pi} \frac{\kappa(\omega_k)}{\left|\frac{\alpha_{kk'}}{\beta_{kk'}}\right|^2 - 1} \quad (3.1.10)$$

To determine the flux as a function of wavelength, we will need to derive the ratio of Bogoliubov coefficients. We will perform the calculation within the WKB (Wentzel-Kremers-Brillouin) or semi-classical approximation. That is, we will assume that the classical terms are dominant. Put more explicitly, we assume that we can write the field as

$$\phi(t, r) \simeq e^{iS(t,r)} \quad (3.1.11)$$

where $S(t, r)$ is the action as an explicit function of time and the radial coordinate. In the following sections, we will derive this action.

3.2 Quantisation in the WKB regime

In general relativity, the action S of a field ϕ on a manifold M is defined as the integral of a Lagrangian density $\mathcal{L}[\phi]$, that is in general a function of ψ and a finite number of its covariant derivatives [7]

$$S[\phi] = \int_M \mathcal{L}[\phi] \tag{3.2.1}$$

While classical and quantum mechanics have a time coordinate that is considered separately from spatial coordinates, there is no such thing in a covariant action formulation. Unlike nonrelativistic mechanics, general relativity does not in general have a unique global time coordinate. Fortunately, in ADM metrics there is a straightforward global time coordinate: the one we called t in equation (2.1.1). The metric is stationary with respect to t and within certain limits the general metric reduces to a Minkowski metric with t as the zeroth 'timelike' coordinate.

In section 2 we used foliation of the space-time manifold as a nifty trick to determine curvature properties. In this section, it is fundamental to the method of quantisation and to our notion of energy.

If by evaluation of ϕ on each constant-time slice Σ_t the Lagrangian density can be expressed in terms of a set of canonical coordinates q^j, their first derivatives with respect to time t and perhaps directly on t, the action can be written as

$$S\left[\{q^j\};\{\dot{q}^j\};t\right] = \int dt \int_{\Sigma_t} \mathcal{L}\left[\{q^j\};\{\dot{q}^j\};t\right] \tag{3.2.2}$$

where the integrand defines the Lagrangian L

$$L\left[\{q^j\};\{\dot{q}^j\};t\right] := \int_{\Sigma_t} \mathcal{L}\left[\{q^j\};\{\dot{q}^j\};t\right] \tag{3.2.3}$$

Note that we can only do this because we have well-defined notions of time and space. Moreover, doing this explicitly breaks covariance.

To go from Lagrangian mechanics to Hamiltonian mechanics, we define the Hamiltonian as the Legendre transform of L with respect to $\{\dot{q}^j\}$

$$H\left[\{q^j\};\{\dot{q}^j\};t\right] := \sum_j \frac{\partial L}{\partial \dot{q}^j}\dot{q}^j - L\left[\{q^j\};\{\dot{q}^j\};t\right] \tag{3.2.4}$$

Defining the canonical momenta p_j as

$$p_j := \frac{\partial L}{\partial \dot{q}^j} \tag{3.2.5}$$

the Lagrangian and Hamiltonian can be rewritten to obtain

$$H\left[\{q^j\};\{p_j\};t\right] = p_j\dot{q}^j - L\left[\{q^j\};\{p_j\};t\right] \tag{3.2.6}$$

The total differential operator of the Hamiltonian is

$$\begin{aligned} dH &= p_j d\dot{q}^j + \dot{q}^j dp_j - \frac{dL}{dq^j} dq^j - \frac{dL}{d\dot{q}^j} d\dot{q}^j - \frac{dL}{dt} dt \\ &= \dot{q}^j dp_j - \dot{p}_j dq^j - \frac{dL}{dt} dt \end{aligned} \tag{3.2.7}$$

leading to the Hamilton equations

$$\frac{\partial H}{\partial p_j} = \dot{q}^j, \quad \frac{\partial H}{\partial q^j} = -\dot{p}_j \tag{3.2.8}$$

Let us now consider the action as an explicit function of time t with coordinates $\{q^j(t)\}$ that are expressed as functions of time. This is often called Hamilton's principal function. Suppose we have two 'neighbouring' paths, both starting out at $\{q^j(t_1)\}$, t_1 but ending at the different endpoints $\{q_a^j(t_1)\}$ and $\{q_b^j(t_1)\}$. The difference in the action function between those two paths is

$$\delta S = \sum_j \frac{\partial L}{\partial \dot{q}^j} \delta q^j \Big|_{t_1}^{t_2} + \sum_j \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q^j} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^j} \right] \delta q^j \tag{3.2.9}$$

For a physical path, the Euler-Lagrange equation in the integrand should equal zero. Furthermore, the derivative in the boundary term is just the canonical momentum defined in equation (3.2.5). Evidently,

$$\frac{\partial S}{\partial q^j} = p_j \tag{3.2.10}$$

By definition, the total derivative of Hamilton's principal function with respect to time is the Lagrangian. Let us determine it explicitly

$$L = \frac{dS}{dt} = \frac{\partial S}{\partial t} + \sum_j \frac{\partial S}{\partial q^j} \dot{q}^j = \frac{\partial S}{\partial t} + p_j \dot{q}^j \tag{3.2.11}$$

Subtracting $L(t)$ from both sides yields the Hamilton-Jacobi equation

$$\frac{\partial S}{\partial t} + H = 0 \tag{3.2.12}$$

As it happens, equations (3.2.12) and (3.2.10) can also be obtained from the Schrödinger equation for a wave function that only depends on the classical action (i.e., first order WKB regime),

$$H \Psi_0 e^{iS} = i \partial_t \left(\Psi_0 e^{iS} \right) \tag{3.2.13}$$

Carrying out the time derivative reproduces the Hamilton-Jacobi equation multiplied by the wave function

$$H \Psi_0 e^{iS} = -\frac{\partial S}{\partial t} \Psi_0 e^{iS} \tag{3.2.14}$$

The momentum equations (3.2.10) are obtained in a similar manner. Applying the momentum operator $\hat{p}_j$ on such a wave function yields

$$\hat{p}_j \Psi = -i\partial_j \left(\Psi_0 e^{iS}\right) = \Psi_0(\partial_j S)e^{iS} = (\partial_j S)\Psi \tag{3.2.15}$$

Since the latter result does not rely on the Schrödinger equation, it can also be applied to classical fields obeying e.g. the Klein-Gordon equation. For a field of the form e^{iS}, as suggested in equation (3.1.11), application of the momentum operator results in

$$\hat{p}_j e^{iS} = -i\partial_j \left(e^{iS}\right) = (\partial_j S)e^{iS} \tag{3.2.16}$$

Hence quantisation within the WKB regime is obtained by the substitution

$$p_j \to \frac{\partial S}{\partial q^j} \tag{3.2.17}$$

We have now derived an effective method to quantise a known action function on a stationary ADM form metric within the WKB regime. This will prove very useful.

3.3 The action for a black hole

Now we have determined how to quantise the action within the relevant regime, let us proceed to deriving it. Even though most of the analysis only pertains to Schwarzschild black holes, we will formulate the action for a general ADM form metric in order to have a well-defined variation principle.

For a Schwarzschild black hole releasing a shell of energy, the action consists of three parts. The first is a contribution arising from the bare space-time. The appropriate Lagrangian density for a vacuum in General Relativity is [7]

$$\mathcal{L}_G := \frac{1}{16\pi}\sqrt{-g}\mathcal{R} \tag{3.3.1}$$

where $\mathcal{R}$ is the Ricci curvature scalar and g denotes the determinant of a matrix with the components of the metric tensor $g_{\mu\nu}$ as its elements. By integration over the space-time, we obtain the Einstein-Hilbert action

$$S_G = \frac{1}{16\pi}\int d^4x\sqrt{-g}\mathcal{R} \tag{3.3.2}$$

The choice of this Lagrangian is motivated by the analysis in appendix B.1.

To obtain a covariant theory with a well-defined variational principle in the Hamiltonian formulation of ADM space-time one also needs to include a contribution due to the ADM mass M_+ [2, 13]. This is all the mass-energy in the system as measured by an observer at infinity.

$$S_M = -\int dt\, M_+ \tag{3.3.3}$$

The third term is the energy shell that is emitted through the horizon

$$S_s = -m \int \sqrt{-g_{\mu\nu}(\hat{t},\hat{r}) d\hat{x}^\mu d\hat{x}^\nu} \tag{3.3.4}$$

where $\hat{x}^\mu$ are the coordinates of the shell and m is the rest mass of the shell. The total action for our system is then

$$\begin{aligned} S_T &= S_G + S_M + S_s \\ &= \frac{1}{16\pi} \int d^4x \sqrt{-g}\mathcal{R} + \int dt\, M_+ - m \int \sqrt{-g_{\mu\nu}(\hat{t},\hat{r}) d\hat{x}^\mu d\hat{x}^\nu} \end{aligned} \tag{3.3.5}$$

A convenient choice for parametrisation in the shell contribution is the time coordinate t. For a metric of the form (2.1.1) with spherical symmetry, this becomes

$$S_s = -m \int dt \sqrt{N^t(t,\hat{r})^2 - L(t,\hat{r})^2 \left(\frac{d\hat{r}}{dt} + N^r(t,\hat{r}) \right)^2} \tag{3.3.6}$$

Using the results in equations (2.1.8) and (2.5.10), the Einstein-Hilbert action for the ADM metric is

$$\begin{aligned} S_G = \frac{1}{16\pi} \int d^4x\, N^t L R^2 \sin\theta \Bigg[&-4\frac{R''}{L^2 R} + 4\frac{L'R'}{L^3 R} - 2\frac{R'^2}{L^2R^2} + \frac{2}{R^2} \\ &- \frac{2}{N^{t2}}\frac{\dot{R}^2}{R^2} - 2\frac{N^{r2}}{N^{t2}}\frac{R'^2}{R^2} + 4\frac{N^r}{N^{t2}}\frac{\dot{R}R'}{R^2} + 4\frac{N^r}{N^{t2}}\frac{L'}{L}\frac{\dot{R}}{R} - \frac{4}{N^{t2}}\frac{\dot{L}}{L}\frac{\dot{R}}{R} \\ &+ 4\frac{N^{r\prime}}{N^{t2}}\frac{\dot{R}}{R} - 4\frac{N^{r2}}{N^{t2}}\frac{L'}{L}\frac{R'}{R} + 4\frac{N^r}{N^{t2}}\frac{\dot{L}}{L}\frac{R'}{R} - 4\frac{N^r N^{r\prime}}{N^{t2}}\frac{R'}{R} \\ &+ 2\left(n^\nu{}_{;\nu} n^\mu - n^\mu{}_{;\nu} n^\nu\right)_{;\mu} \Bigg] \\ = \int dt\, dr\, N^t \Bigg[&-\frac{RR''}{L} + \frac{L'RR'}{L^2} - \frac{R'^2}{2L} + \frac{L}{2} - \frac{1}{2N^t} L\dot{R}^2 \\ &- \frac{N^{r2}}{2N^t} R'^2 + \frac{N^r}{N^t} L\dot{R}R' + \frac{N^r}{N^t} L'\dot{R}R - \frac{1}{N^t}\dot{L}R\dot{R} + \frac{N^{r\prime}}{N^t} LR\dot{R} \\ &- \frac{N^{r2}}{N^t} L'RR' + \frac{N^r}{N^t}\dot{L}RR' - \frac{N^r N^{r\prime}}{N^t} LRR' \Bigg] \end{aligned} \tag{3.3.7}$$

where we have used that

$$\int d\theta\, d\phi\, \sin\theta f(t,r) = 4\pi f(t,r) \tag{3.3.8}$$

for any function $f(t,r)$ that does not depend on θ and ϕ and we have neglected the covariant derivative that results in a boundary integral. This is the explicit form of the gravitational contribution. It only depends on ADM metric functions.

We now know which parts constitute the action in explicit form in terms of the shell's position and momentum, the ADM mass, the ADM metric functions and their derivatives.

3.4 The canonical action

Since we need to quantise our system using the Hamilton-Jacobi formalism developed in section 3.2, we need to write the action in a canonical form

$$S = \int d^4x \left[\pi_j \dot{q}^j - \mathcal{H}\left(\{q^j\};\{\pi_j\};t\right)\right] \tag{3.4.1}$$

The form of the metric equation (2.1.1) suggests that a good choice of canonical coordinates for the geometric terms is (L, R), while for the shell term the radial shell coordinate $\hat{r}$ is the most appealing candidate. Then the total action takes the form

$$S_T = \int dt\, p\dot{\hat{r}} + \int dt\, dr \left[\pi_L \dot{L} + \pi_R \dot{R} - \mathcal{H}\right] - \int dt\, M_+ \tag{3.4.2}$$

The canonical momenta in the L and R directions are obtained from the derivatives of the Lagrangian density in equation (3.3.7) with respect to $\dot{L}$ and $\dot{R}$, respectively.

$$\pi_L = \frac{\partial \mathcal{L}}{\partial \dot{L}} = -\frac{1}{N^t} R\dot{R} + \frac{N^r}{N^t} RR' \tag{3.4.3}$$

$$\pi_R = \frac{\partial \mathcal{L}}{\partial \dot{R}} = \frac{1}{N^t}\left[-L\dot{R} - \dot{L}R + N^{r\prime} LR\right] + \frac{N^r}{N^t}\left[LR' + L'R\right] \tag{3.4.4}$$

In appendix B.2 we show that by inversion, the derivatives of L and R can be expressed in terms of the canonical momenta as

$$\dot{L} = -\frac{N^t}{R}\pi_R + N^t \frac{L}{R^2}\pi_L + N^{r\prime} L + N^r L' \tag{3.4.5}$$

$$\dot{R} = -\frac{N^t}{R}\pi_L + N^r R' \tag{3.4.6}$$

whence we obtain the action expressed in terms of (3.4.3) and (3.4.4)

$$S_G = \int dt\, dr\, N^t \left[-\frac{RR''}{L} + \frac{L'RR'}{L^2} - \frac{R'^2}{2L} + \frac{L}{2} - \frac{\pi_L \pi_R}{R} - \frac{L{\pi_L}^2}{2R^2}\right] \tag{3.4.7}$$

All necessary ingredients to determine the gravitational part of the Hamiltonian density $\mathcal{H}$ as defined in equation (3.2.4) are now present.

$$\begin{aligned}
\mathcal{H}^G &= \pi_L \dot{L} + \pi_R \dot{R} - \mathcal{L}_G \\
&= -\frac{N^t}{R}\pi_L\pi_R + N^t\frac{L}{R^2}{\pi_L}^2 + N^{r\prime} L\pi_L + N^r L'\pi_L - \frac{N^t}{R}\pi_L\pi_R \\
&\quad + N^r R'\pi_R - N^t\left[-\frac{RR''}{L} + \frac{L'RR'}{L^2} - \frac{R'^2}{2L} + \frac{L}{2} - \frac{\pi_L\pi_R}{R} - \frac{L{\pi_L}^2}{2R^2}\right] \\
&= N^t\left[\left(\frac{RR'}{L}\right)' - \frac{R'^2}{2L} - \frac{L}{2} - \frac{\pi_L\pi_R}{R} + \frac{L{\pi_L}^2}{2R^2}\right] \\
&\quad + N^{r\prime} L\pi_L + N^r L'\pi_L + N^r R'\pi_R
\end{aligned} \tag{3.4.8}$$

In the last equality, we have used that

$$\left(\frac{RR'}{L}\right)' = \frac{RR''}{L} + \frac{R'^2}{L} - \frac{L'RR'}{L^2} \tag{3.4.9}$$

In the action functional, the Hamiltonian is to be integrated over the radial coordinate r. Integration by parts yields

$$\int dr\, N^{r\prime} L\pi_L = -\int dr\, N^r \left[L'\pi_L + L\pi_L{}'\right] \tag{3.4.10}$$

where we have neglected the boundary terms. We might as well replace $N^{r\prime}L\pi_L$ with $-N^r\left[L'\pi_L + L\pi_L{}'\right]$ and write

$$\mathcal{H}^G = N^t\left[\left(\frac{RR'}{L}\right)' - \frac{R'^2}{2L} - \frac{L}{2} - \frac{\pi_L\pi_R}{R} + \frac{L\pi_L{}^2}{2R^2}\right] + N^r\left[R'\pi_R - L\pi_L'\right] \tag{3.4.11}$$

Evidently, the gravitational part of the Hamiltonian density can be written in terms of a generator $\mathcal{H}_t^G$ for the lapse function N^t and a generator $\mathcal{H}_r^G$ for the shift function N^r. More generally, we wish to write that

$$\mathcal{H} = N^t\mathcal{H}_t + N^r\mathcal{H}_r \tag{3.4.12}$$

where

$$\mathcal{H}_t = \mathcal{H}_t^G + \mathcal{H}_t^s \tag{3.4.13}$$

$$\mathcal{H}_r = \mathcal{H}_r^G + \mathcal{H}_r^s \tag{3.4.14}$$

The superscripts g and s indicate the gravitational part due to the Einstein-Hilbert action term and for the part due to the presence of the shell, respectively. The gravitational generators are easily obtained from equation (3.4.11)

$$\mathcal{H}_t^G = \frac{L\pi_L^2}{2R^2} - \frac{\pi_L\pi_R}{R} + \left(\frac{RR'}{L}\right)' - \frac{R'^2}{2L} - \frac{L}{2} \tag{3.4.15}$$

$$\mathcal{H}_r^G = R'\pi_R - L\pi_L' \tag{3.4.16}$$

It remains to determine the shell contributions to the Hamiltonian density. In equation (3.3.6) we determined the action due to the shell in generic form. The resulting Lagrangian density is

$$\mathcal{L}_s = -m\sqrt{\hat{N^t}^2 - \hat{L}^2\left(\dot{\hat{r}} + \hat{N^r}\right)^2} \tag{3.4.17}$$

Note that this is a density in time, whereas the 5Lagrangian density for the gravitational terms was a density in the radial space direction as well as time. The canonical momentum in the direction of the radial coordinate of the shell is

$$p = \frac{\partial \mathcal{L}_s}{\partial \dot{\hat{r}}} = \frac{m\hat{L}^2\left(\dot{\hat{r}} + \hat{N}^r\right)}{\sqrt{\hat{N^t}^2 - \hat{L}^2\left(\dot{\hat{r}} + \hat{N}^r\right)^2}} \tag{3.4.18}$$

Hence the Lagrangian density also equals

$$\mathcal{L}_s = -\frac{m^2\hat{L}^2}{p}\left(\dot{\hat{r}} + \hat{N}^r\right) \tag{3.4.19}$$

If we observe that

$$p^2 = \frac{m^2\hat{L}^4\left(\dot{\hat{r}} + \hat{N}^r\right)^2}{\hat{N^t}^2 - \hat{L}^2\left(\dot{\hat{r}} + \hat{N}^r\right)^2} \tag{3.4.20}$$

then

$$\hat{L}^2\left(\dot{\hat{r}} + \hat{N}^r\right)^2 = \frac{\hat{N^t}^2 p^2}{m^2\hat{L}^2 + p^2} \tag{3.4.21}$$

and from there we can determine $\dot{\hat{r}}$ in terms of p

$$\dot{\hat{r}} = \sqrt{\frac{\hat{N^t}^2 p^2}{m^2\hat{L}^4 + \hat{L}^2 p^2}} - \hat{N}^r = \frac{\hat{N}^t p}{\hat{L}\sqrt{m^2\hat{L}^2 + p^2}} - \hat{N}^r \tag{3.4.22}$$

The Hamiltonian density is determined from the generic recipe outlined in equation (3.2.4).

$$\begin{aligned}
\hat{\mathcal{H}}^s &= p\dot{\hat{r}} - \mathcal{L}_s \\
&= \frac{\hat{N}^t p^2}{\hat{L}\sqrt{m^2\hat{L}^2 + p^2}} - \hat{N}^r p + \frac{m^2\hat{L}^2}{p}\left(\frac{\hat{N}^t p}{\hat{L}\sqrt{m^2\hat{L}^2 + p^2}} - \hat{N}^r + \hat{N}^r\right) \\
&= \left(p^2 + m^2\hat{L}^2\right)\frac{\hat{N}^t}{\hat{L}\sqrt{m^2\hat{L}^2 + p^2}} - \hat{N}^r p = \hat{N}^t\sqrt{m^2 + \frac{p^2}{\hat{L}^2}} - \hat{N}^r p
\end{aligned} \tag{3.4.23}$$

Again, this is a density in time. In fact, the contribution only exists at precisely $r = \hat{r}$, since the lapse and shift functions are taken at the shell coordinate. In order to include it as part of the integral over t and r, we need to weigh it with the Dirac delta distribution $\delta(r - \hat{r})$. This leaves us with the following contributions

$$\mathcal{H}_t^s = \sqrt{\left(\frac{p}{L}\right)^2 + m^2}\,\delta(r - \hat{r}) \qquad \mathcal{H}_r^s = -p\,\delta(r - \hat{r}) \tag{3.4.24}$$

Adding these to the results for the gravitational terms expressed in equation (3.4.15) and (3.4.16) finally yields

$$\mathcal{H}_t = \frac{L\pi_L^2}{2R^2} - \frac{\pi_L \pi_R}{R} + \left(\frac{RR'}{L}\right)' - \frac{R'^2}{2L} - \frac{L}{2}$$
$$+ \sqrt{\left(\frac{p}{L}\right)^2 + m^2}\, \delta(r - \hat{r}) \tag{3.4.25}$$
$$\mathcal{H}_r = R'\pi_R - L\pi_L' - p\,\delta(r - \hat{r}) \tag{3.4.26}$$

We have now determined the canonical momenta and the Hamiltonian for the Einstein-Hilbert action with an added shell contribution for a general ADM-form metric.

3.5 Determining the constraints

The only explicit dependence on N^r and N^t in the action is in the Hamiltonian density contains and it does not depend on derivatives of these functions, so if we vary by δN^r, we obtain

$$\delta S_T = \int dt\, dr\, \delta N^t \mathcal{H}_t, \quad \delta S_T = 0 \Rightarrow \mathcal{H}_t = 0 \tag{3.5.1}$$

Similarly for $N^r \to N^r + \delta N^r$,

$$\delta S_T = \int dt\, dr\, \delta N^r \mathcal{H}_r, \quad \delta S_T = 0 \Rightarrow \mathcal{H}_r = 0 \tag{3.5.2}$$

If we consider an infinitesimal shell of space-time that is static, i.e., with $\pi_L = \pi_R = 0$, we require the contribution to the action (3.4.2) to equal zero. Hence

$$-N^t \mathcal{H}_t^G - N^r \mathcal{H}_r^G - \mathcal{M}' = 0 \tag{3.5.3}$$

where $\mathcal{M}(r)$ is the enclosed mass at radius r. At $r \neq \hat{r}$, $\mathcal{H}_t^G = \mathcal{H}_t = 0$ and $\mathcal{H}_r^G = \mathcal{H}_r = 0$, so

$$\mathcal{M}'(r \neq \hat{r}) = 0 \tag{3.5.4}$$

or, in other words, $\mathcal{M}$ is constant except precisely at the shell, where a discontinuity exists. Within the shell, the static mass $\mathcal{M}$ is just the mass contained within the horizon (the black hole mass). Outside of the shell, the shell contributes to the mass and the static mass equals the total ADM mass M_+ of the system

$$\mathcal{M} = \begin{cases} M & (r < \hat{r}) \\ M_+ & (r > \hat{r}) \end{cases} \tag{3.5.5}$$

In the Schwarzschild and Schwarzschild-Painlevé metrics, N^t can be written in terms of L, R and their derivatives as

$$N^t = 1 = \frac{R'}{L} \tag{3.5.6}$$

With equations (3.4.3) and (3.4.4),

$$N^r = \frac{N^t}{RR'}\left(\pi_L + R\dot{R}\right) = \frac{\pi_L}{LR} + \frac{\dot{R}}{L} \tag{3.5.7}$$

Since $\dot{R} = 0$ for Schwarzschild and Schwarzschild-Painlevé metrics, the second term can be dropped to obtain

$$0 = -\frac{R'}{L}\mathcal{H}_t^G - \frac{\pi_L}{LR}\mathcal{H}_r^G - \mathcal{M}' \tag{3.5.8}$$

Integration yields (Appendix B.3)

$$\mathcal{M}(r) = \frac{\pi_L^2}{2R} - \frac{RR'^2}{2L^2} + \frac{R}{2} \tag{3.5.9}$$

where we have used that $R(t,0)$ equals zero for any suitable coordinate set. This expression can be inverted to express π_L in terms of $\mathcal{M}$ where it is properly defined, i.e. at $r \neq \hat{r}$

$$\pi_L = \sqrt{2R\left(\mathcal{M} - \frac{R}{2} + \frac{R(R')^2}{2L^2}\right)} = R\sqrt{\left(\frac{R'}{L}\right)^2 + \frac{2\mathcal{M}}{R} - 1} \tag{3.5.10}$$

With the help of equation (3.4.16), π_R can be expressed in terms of the derivative of π_L with respect to the radial coordinate

$$\pi_R = \frac{L}{R'}\pi_L' \tag{3.5.11}$$

Although we will not need the explicit form in the following, for the sake of completeness

$$\pi_R = L\sqrt{\left(\frac{R'}{L}\right)^2 + \frac{2\mathcal{M}}{R} - 1} + \frac{RR'' - \frac{L'}{L^2} + \frac{L}{R}\mathcal{M}}{\sqrt{\left(\frac{R'}{L}\right)^2 + \frac{2\mathcal{M}}{R} - 1}} \tag{3.5.12}$$

Integration of $\mathcal{H}_t$ around the shell leads to an additional constraint

$$\begin{aligned}
0 &= \int_{\hat{r}-\epsilon}^{\hat{r}+\epsilon} dr\, \mathcal{H}_t = \int_{\hat{r}-\epsilon}^{\hat{r}+\epsilon} dr\, \mathcal{H}_t^G - p \\
&= \int_{\hat{r}-\epsilon}^{\hat{r}+\epsilon} dr\, \left[R'\pi_R - L\pi_L'\right] - p \\
&= -\, L\pi_L\big|_{r=\hat{r}-\epsilon}^{\hat{r}+\epsilon} + \int_{\hat{r}-\epsilon}^{\hat{r}+\epsilon} dr\, \left[R'\pi_R + L'\pi_L\right] - p
\end{aligned} \tag{3.5.13}$$

If L and R are smooth at and around the shell, the integral in the second term does not contribute a finite value in the limit $\epsilon \to 0$. So

$$\pi_L(\hat{r}+\epsilon) - \pi_L(\hat{r}-\epsilon) = -\frac{p}{\hat{L}} \tag{3.5.14}$$

Likewise, integration of $\mathcal{H}_r$ reveals that

$$\begin{aligned}
0 &= \int_{\hat{r}-\epsilon}^{\hat{r}+\epsilon} dr\, \mathcal{H}_r \\
&= \int_{\hat{r}-\epsilon}^{\hat{r}+\epsilon} dr \left[\frac{L\pi_L^2}{2R^2} - \frac{\pi_L \pi_R}{R} + \left(\frac{RR'}{L}\right)' - \frac{R'^2}{2L} - \frac{L}{2} \right] + \sqrt{\left(\frac{p}{\hat{L}}\right)^2 + m^2} \\
&= \int_{\hat{r}-\epsilon}^{\hat{r}+\epsilon} dr \frac{RR''}{L} + \sqrt{\left(\frac{p}{\hat{L}}\right)^2 + m^2} \\
&= \left.\frac{RR'}{L}\right|_{\hat{r}-\epsilon}^{\hat{r}+\epsilon} + \int_{\hat{r}-\epsilon}^{\hat{r}+\epsilon} dr \left[\frac{RR'}{L} - \frac{L'R}{L^2} \right] + \sqrt{\left(\frac{p}{\hat{L}}\right)^2 + m^2} \\
&= \left.\frac{RR'}{L}\right|_{\hat{r}-\epsilon}^{\hat{r}+\epsilon} + \sqrt{\left(\frac{p}{\hat{L}}\right)^2 + m^2}
\end{aligned} \tag{3.5.15}$$

We have kept only the term in the integrand that contains a second derivative with respect to r and we have performed integration by parts on the penultimate line. In the last, line the integral over first-derivative terms is dropped again. When multiplied by $\frac{L}{R}$, this also results in a constraint for R'

$$R'(\hat{r}+\epsilon) - R'(\hat{r}-\epsilon) = -\frac{1}{\hat{R}} \sqrt{p^2 + m^2 \hat{L}^2} \tag{3.5.16}$$

For a massless particle, this is simplified to

$$R'(\hat{r}+\epsilon) - R'(\hat{r}-\epsilon) = -\eta \frac{p}{\hat{R}} \tag{3.5.17}$$

These are the constraints for the action around the shell. They will be put to good use when we vary the action.

3.6 The explicit action

In the preceding sections, we derived the canonical form action and the constraints due to the discontinuity at the radius where the shell resides and we showed that the Hamiltonian density equals zero. We now wish to express the action as an explicit function in terms of the Hamilton-Jacobi canonical momenta in order to quantise the system.

The derivative of the action with respect to the time whilst keeping π_L and π_R fixed is

$$\left.\frac{\partial S}{\partial t}\right|_{\pi_L, \pi_R} = p\dot{\hat{r}} + \int dr \left[\pi_L \dot{L} + \pi_R \dot{R} \right] - M_+ \tag{3.6.1}$$

If we first integrate the contribution due to the canonical momentum in the L direction, we can later add or subtract terms to make sure the above equation is met. Since π_L and $\mathcal{M}$ are discontinuous at $\hat{r}$ and there is a singularity at $r = 0$, we integrate over r from an arbitrary point r_0 to $r - \epsilon$ and from $r + \epsilon$ to infinity

$$\begin{aligned} S_L = \int dt \int_{r_0}^{\infty} dr \frac{dL}{dt} \pi_L &= \int_{r_0}^{\infty} dr \int dL\, \pi_L \\ &= \int_{r_0}^{\hat{r}-\epsilon} dr \int dL\, R \sqrt{\left(\frac{R'}{L}\right)^2 + \frac{2M}{R} - 1} \\ &\quad + \int_{\hat{r}+\epsilon}^{\infty} dr \int dL\, R \sqrt{\left(\frac{R'}{L}\right)^2 + \frac{2M_+}{R} - 1} \end{aligned} \tag{3.6.2}$$

where we have filled in M and M_+ for $\mathcal{M}$ at $r < \hat{r}$ and $r > \hat{r}$, respectively. In appendix B.4, we show that the result is

$$\begin{aligned} S_L = &\int_{r_0}^{\hat{r}-\epsilon} dr R \left[R' \log \left| \frac{\frac{R'}{L} - \sqrt{\left(\frac{R'}{L}\right)^2 + \frac{2M}{R} - 1}}{\sqrt{\frac{2M}{R} - 1}} \right| + L \sqrt{\left(\frac{R'}{L}\right)^2 + \frac{2M}{R} - 1} \right] \\ &+ \int_{\hat{r}+\epsilon}^{\infty} dr R \left[R' \log \left| \frac{\frac{R'}{L} - \sqrt{\left(\frac{R'}{L}\right)^2 + \frac{2M_+}{R} - 1}}{\sqrt{\frac{2M_+}{R} - 1}} \right| + L \sqrt{\left(\frac{R'}{L}\right)^2 + \frac{2M_+}{R} - 1} \right] \end{aligned} \tag{3.6.3}$$

Apparently, the terms obtained by variation of L depend on R' as well as R. Let us consider variations in R,

$$\delta_R S_L = \int_{r_0}^{\infty} dr \left[\frac{\partial S_L'}{\partial R} \delta R - \frac{d}{dr} \left(\frac{\partial S_L'}{\partial R'} \right) \delta R' \right] \tag{3.6.4}$$

The first term is just the contribution due to π_R, but the second term needs to be subtracted from the action to fulfil equation (3.6.1). We expect $\frac{\delta S_L'}{\delta R'}$ to approach zero at $r = r_0$ and $r \to \infty$. However, R' is in general discontinuous around the shell, so we need to integrate 'around' the shell again. Equation (B.4.4) in the appendix shows that

$$\begin{aligned} \delta_R S_L = \int_{r_0}^{\infty} dr \int dR\, \pi_R - \hat{R} \log &\left| \frac{\frac{R'(r+\epsilon)}{L} - \sqrt{\left(\frac{R'(r+\epsilon)}{L}\right)^2 + \frac{2M_+}{\hat{R}} - 1}}{\sqrt{\frac{2M_+}{\hat{R}} - 1}} \right| \delta \hat{R} \\ + \hat{R} \log &\left| \frac{\frac{R'(r-\epsilon)}{L} - \sqrt{\left(\frac{R'(r-\epsilon)}{L}\right)^2 + \frac{2M}{\hat{R}} - 1}}{\sqrt{\frac{2M}{\hat{R}} - 1}} \right| \delta \hat{R} \end{aligned} \tag{3.6.5}$$

The final parameter equation (3.6.1) requires us to consider variation of is M_+. In equation (3.6.3), only the second integral depends on M_+.

Variation of M_+ by δM_+ then leads to a an action modified by $\delta_{M_+} S$, as derived in appendix B.4,

$$\delta_{M_+} S_L = \int_{\hat{r}+\epsilon}^{\infty} L \frac{\sqrt{\left(\frac{R'}{L}\right)^2 + \frac{2M_+}{R} - 1}}{\frac{2M_+}{R} - 1} \delta M_+ \tag{3.6.6}$$

Finally, variation of t results in an extra term $-M_+$. $p\dot{\hat{r}}$ does not contribute, since p is already constrained explicitly.

Subtracting the superfluous terms obtained due to variation in R and M_+ from S_L results in the action

$$\begin{aligned} S = &\int_{r_0}^{\hat{r}-\epsilon} dr R \left[R' \log \left| \frac{\frac{R'}{L} - \sqrt{\left(\frac{R'}{L}\right)^2 + \frac{2M}{\hat{R}} - 1}}{\sqrt{\frac{2M}{R} - 1}} \right| + L \sqrt{\left(\frac{R'}{L}\right)^2 + \frac{2M}{\hat{R}} - 1} \right] \\ &+ \int_{\hat{r}+\epsilon}^{\infty} dr R \left[R' \log \left| \frac{\frac{R'}{L} - \sqrt{\left(\frac{R'}{L}\right)^2 + \frac{2M_+}{R} - 1}}{\sqrt{\frac{2M_+}{R} - 1}} \right| + L \sqrt{\left(\frac{R'}{L}\right)^2 + \frac{2M_+}{R} - 1} \right] \\ &+ \int dt\, \hat{R} \log \left| \frac{\frac{R'(r+\epsilon)}{L} - \sqrt{\left(\frac{R'(r+\epsilon)}{L}\right)^2 + \frac{2M_+}{\hat{R}} - 1}}{\sqrt{\frac{2M_+}{\hat{R}} - 1}} \right| \frac{d\hat{R}}{dt} \\ &- \int dt\, \hat{R} \log \left| \frac{\frac{R'(r-\epsilon)}{L} - \sqrt{\left(\frac{R'(r-\epsilon)}{L}\right)^2 + \frac{2M}{\hat{R}} - 1}}{\sqrt{\frac{2M}{\hat{R}} - 1}} \right| \frac{d\hat{R}}{dt} \\ &+ \int dt \int_{\hat{r}+\epsilon}^{\infty} dr\, L \frac{\sqrt{\left(\frac{R'}{L}\right)^2 + \frac{2M_+}{R} - 1}}{1 - \frac{2M_+}{R}} \frac{dM_+}{dt} - \int dt\, M_+ \end{aligned} \tag{3.6.7}$$

Lagrangians are typically expressed as an integral over time. Such an expression is simplified if we use the gauge freedom to fix the R coordinate for all $r > \hat{r}$. Then $R'(\hat{r} + \epsilon)$ is also fixed by that choice and with the constraints (3.5.14) and (3.5.16), $R'(\hat{r} - \epsilon)$ is then fixed too. Defining

$$R'_> := R'(r > \hat{r}) = R'(\hat{r} + \epsilon) \tag{3.6.8}$$

and

$$R'_< := R'(r < \hat{r}) \tag{3.6.9}$$

The total action as an integral over time is then (appendix B.5)

$$
\begin{aligned}
S = & \int dt \left[\int_{r_0}^{\hat{r}-\epsilon} dr \left[\pi_L \dot{L} + \pi_R \dot{R} \right] + \int_{\hat{r}+\epsilon}^{\infty} dr \left[\pi_L \dot{L} + \pi_R \dot{R} \right] \right] - \int dt M_+ \\
& + \int dt\, \dot{\hat{r}} \hat{L} \hat{R} \left[\sqrt{\left(\frac{R'_<}{\hat{L}} \right)^2 + \frac{2M}{\hat{R}} - 1} - \sqrt{\left(\frac{R'_>}{\hat{L}} \right)^2 + \frac{2M_+}{\hat{R}} - 1} \right] \\
& + \int dt\, \eta \hat{R} \dot{\hat{R}} \log \left| \frac{\frac{R'_<}{\hat{L}} - \eta \sqrt{\left(\frac{R'_<}{\hat{L}} \right)^2 + \frac{2M}{\hat{R}} - 1}}{\frac{R'_>}{\hat{L}} - \eta \sqrt{\left(\frac{R'_>}{\hat{L}} \right)^2 + \frac{2M_+}{\hat{R}} - 1}} \right|
\end{aligned}
\tag{3.6.10}
$$

With the L and R functions of the Schwarzschild-Painlevé metric,

$$
L(t, r) = 1, \quad R(t, r) = r \tag{3.6.11}
$$

the terms originating from outside the shell boundaries vanish and the rest of the expression becomes more compact

$$
\begin{aligned}
S &= \int dt \left[\dot{\hat{r}} \hat{r} \left[\sqrt{\frac{2M}{\hat{r}}} - \sqrt{\frac{2M_+}{\hat{r}}} \right] + \eta \hat{r} \dot{\hat{r}} \log \left| \frac{1 - \eta \sqrt{\frac{2M}{\hat{r}}}}{1 - \eta \sqrt{\frac{2M_+}{\hat{r}}}} \right| - M_+ \right] \\
&= \int dt \left[\dot{\hat{r}} \left[\sqrt{2Mr} - \sqrt{2M_+ r} \right] + \eta \hat{r} \dot{\hat{r}} \log \left| \frac{\sqrt{r} - \eta \sqrt{2M}}{\sqrt{r} - \eta \sqrt{2M_+}} \right| - M_+ \right]
\end{aligned}
\tag{3.6.12}
$$

Note that this gauge violates the boundary condition 3.5.17: the right hand side reduces to zero. The action obtained in this gauge can be regarded as an approximation valid away from the horizon. This should not pose a problem, as we will only take into account the dominant, late-time contributions.

Now that we have imposed a gauge, we need to define new canonical momenta in terms of the new coordinates. In the r direction, there is a radial momentum p_r, derived from the Lagrangian (i.e., the integrand in the previous equation),

$$
p_r := \frac{\partial \dot{S}}{\partial \dot{\hat{r}}} = \sqrt{2M\hat{r}} - \sqrt{2M_+ \hat{r}} + \eta \hat{r} \log \left| \frac{\sqrt{r} - \eta \sqrt{2M}}{\sqrt{r} - \eta \sqrt{2M_+}} \right| \tag{3.6.13}
$$

so the action in equation (3.6.12) becomes

$$
S = \int dt \left[p_r \dot{\hat{r}} - M_+ \right] \tag{3.6.14}
$$

From this equation, it is clear that the momentum in the t direction equals M_+ up to a constant. At $t = 0$, the shell is in its initial position and the total mass-energy in the system equals M. So the temporal momentum p_t equals

$$p_t := M - M_+ \tag{3.6.15}$$

However, being a constant, the mass term M does not alter the resulting equations of motion. We might as well discard it and use

$$S = \int dt \left[p_r \dot{\hat{r}} + p_t\right] \tag{3.6.16}$$

The radial momentum is then

$$p_r = \sqrt{2M\hat{r}} - \sqrt{2\left(M - p_t\right)\hat{r}} + \eta\hat{r}\log\left|\frac{\sqrt{\hat{r}} - \eta\sqrt{2M}}{\sqrt{\hat{r}} - \eta\sqrt{2\left(M - p_t\right)}}\right| \tag{3.6.17}$$

We have used up most of our gauge freedom. The action as expressed presently is not only pleasantly compact, but more importantly, it is expressed in terms of canonical momenta with respect to the Schwarzschild-Painlevé time and the radial coordinate of the shell.

3.7 The quantised action

In section 3.2 we derived that in the WKB regime, momenta can be quantised by replacing them with the derivative with respect to the corresponding coordinate of the action expressed as an explicit function. Fortunately, we know the trajectory $\hat{r}(t)$ of an emitted shell. It is the radial null geodesic derived it in section 2.6. Hence the quantised action is written as

$$S(t, r) = S(0, r) + \int_0^t d\bar{t}\, \left[p_r \dot{\hat{r}}(\bar{t}) + p_t\right] \tag{3.7.1}$$

At $t = 0$, the modes $v_k(0, r)$ are

$$v_k(0, r) \simeq e^{ikr} \tag{3.7.2}$$

so the initial term in the action function is

$$S(0, r) = kr \quad (k > 0) \tag{3.7.3}$$

The radial momentum is independent of time, so we might as well take $t = 0$ in our calculations.

$$p_r(0) = \frac{\partial S}{\partial \hat{r}} = \sqrt{2M\hat{r}(0)} - \sqrt{2M_+\hat{r}(0)} + \eta\hat{r}(0)\log\left|\frac{\sqrt{\hat{r}(0)} - \eta\sqrt{2M}}{\sqrt{\hat{r}(0)} - \eta\sqrt{2M_+}}\right| \tag{3.7.4}$$

The logarithmic term is singular at the horizon, so around it

$$\left|\sqrt{2M\hat{r}} - \sqrt{2M_+\hat{r}}\right| \ll \left|\eta\hat{r}\log\left|\frac{\sqrt{\hat{r}} - \eta\sqrt{2M}}{\sqrt{\hat{r}} - \eta\sqrt{2M_+}}\right|\right| \tag{3.7.5}$$

Keeping only the terms that contribute at infinity, the action function now equals

$$S(t,r) = S(0,r) + \int_{\hat{r}(0)}^{r(t)} dr\, p_r(r) + \int_0^t dt\, p_t(t)$$
$$\simeq k\hat{r}(0) + \eta \int_{\hat{r}(0)}^{r(t)} d\hat{r}\, \hat{r} \log \left| \frac{\sqrt{\hat{r}} - \eta\sqrt{2M}}{\sqrt{\hat{r}} - \eta\sqrt{2M_+}} \right| + [M - M_+]\, t \qquad (3.7.6)$$

Consistency then requires that

$$k = \eta \hat{r}(0) \log \left| \frac{\sqrt{\hat{r}(0)} - \eta\sqrt{2M}}{\sqrt{\hat{r}(0)} - \eta\sqrt{2M_+}} \right| \qquad (3.7.7)$$

or, in exponential form

$$e^{\frac{\eta k}{\hat{r}(0)}} = \frac{\sqrt{\hat{r}(0)} - \eta\sqrt{2M}}{\sqrt{\hat{r}(0)} - \eta\sqrt{2M_+}} \qquad (3.7.8)$$

In section 2.6 we determined the paths of radial null geodesics in the Schwarzschild-Painlevé metric. At $t = 0$, it follows from equation (2.6.18) that

$$\eta r_1(0) - 2\sqrt{2M_+ r_1(0)} + 4\eta M_+ \log\left(\sqrt{2M_+} - \eta\sqrt{r_1(0)}\right) = 0 \qquad (3.7.9)$$

Around the horizon, redshift is very large and non-singular terms do not contribute significantly. We can in fact drop the second term on the right hand side of equation (2.6.17) as well as the one on the left hand side of equation (3.7.9). For the trajectory of the shell, we merely need to pick $r(t) = \hat{r}(t)$ to obtain that

$$t - 4M_+ \log\left(\sqrt{r} - \eta\sqrt{2M_+}\right) = -4M_+ \log\left(\sqrt{\hat{r}(0)} - \eta\sqrt{2M_+}\right) \qquad (3.7.10)$$

or

$$e^{-\frac{t}{4M_+}} = \frac{\sqrt{\hat{r}(0)} - \eta\sqrt{2M_+}}{\sqrt{r} - \eta\sqrt{2M_+}} \qquad (3.7.11)$$

To cast the action function in its final form, as a function of only time t, the radial coordinate r, the mode wavelength k and the black hole mass M, we need to solve equations (3.7.8) and (3.7.11) to obtain explicit forms for the present unknowns $\hat{r}(0)$ and M_+. However, we have no way to solve equations of this form explicitly. Fortunately, we can obtain a very good approximation by replacing $\hat{r}(0)$ and M_+ in the exponential terms by their zeroth-order values of $2M$ and M, respectively.

$$\sqrt{\hat{r}(0)} - \eta\sqrt{2M} \simeq \left(\sqrt{\hat{r}(0)} - \eta\sqrt{2M_+}\right) e^{\frac{\eta k}{2M}} \qquad (3.7.12)$$

$$\sqrt{\hat{r}(0)} - \eta\sqrt{2M_+} \simeq \left(\sqrt{r} - \eta\sqrt{2M_+}\right) e^{-\frac{t}{4M}} \qquad (3.7.13)$$

In appendix B.6, we show that these are solved by

$$\sqrt{\hat{r}(0)} \simeq \eta\sqrt{2M} + \left(\sqrt{r} - \eta\sqrt{2M}\right) \frac{e^{\frac{\eta k}{2M} - \frac{t}{4M}}}{\left(e^{\frac{\eta k}{2M}} - 1\right) e^{-\frac{t}{4M}} + 1} \tag{3.7.14}$$

However, we can do better than this. Now that we have equations for M_+ and $\hat{r}(0)$ in terms of M k, t and r, we might as well use them to determine a correction to the zeroth-order approximation M we used in the exponential functions.

In outgoing trajectories, i.e., $\eta = 1$, the exponential term involving the relevant (large) wavelengths is much greater than one.

$$e^{\frac{k}{2M}} \gg 1 \tag{3.7.15}$$

Within this approximation, the values for $\sqrt{2M_+}$ and $\sqrt{\hat{r}(0)}$ are identical. This allows us to replace M by a single parameter M' in both equations.

$$M'(r) := M + \left(\sqrt{2Mr} - 2M\right) \frac{e^{\frac{k}{2M} - \frac{t}{4M}}}{e^{\frac{k}{2M} - \frac{t}{4M}} + 1} \tag{3.7.16}$$

The values for outgoing trajectories are then given by

$$\sqrt{2M_+} = \sqrt{2M} + \left(\sqrt{r} - \sqrt{2M}\right) \frac{\left(e^{\frac{k}{2M'}} - 1\right) e^{-\frac{t}{4M'}}}{\left(e^{\frac{k}{2M'}} - 1\right) e^{-\frac{t}{4M'}} + 1} \tag{3.7.17}$$

$$\sqrt{\hat{r}(0)} = \sqrt{2M} + \left(\sqrt{r} - \sqrt{2M}\right) \frac{e^{\frac{k}{2M'} - \frac{t}{4M'}}}{\left(e^{\frac{k}{2M'}} - 1\right) e^{-\frac{t}{4M'}} + 1} \tag{3.7.18}$$

Now M_+ depends on r, but it does not depend on the integration variable $\hat{r}$, so the integral in equation (3.7.6) is of the form

$$I := \int_a^b dx\, x \log\left|\frac{\sqrt{x} - \alpha}{\sqrt{x} - \beta}\right| \tag{3.7.19}$$

In equation (3.6.16), we are only interested in the dominant late-time contributions, so we can neglect all terms that are regular near the horizon. In the last result, only the logarithmic terms qualify. The result is (appendix B.7)

$$I \simeq \frac{b^2}{2}\log\left|\frac{\sqrt{b} - \alpha}{\sqrt{b} - \beta}\right| - \frac{a^2}{2}\log\left|\frac{\sqrt{a} - \alpha}{\sqrt{a} - \beta}\right| - \frac{\alpha^4}{2}\log\left|\frac{\sqrt{b} - \alpha}{\sqrt{a} - \alpha}\right| + \frac{\beta^4}{2}\log\left|\frac{\sqrt{b} - \beta}{\sqrt{a} - \beta}\right| \tag{3.7.20}$$

To obtain an effective action, let us use this result in equation (3.7.6), filling in the proper values of a, b, α and β and dropping regular terms that are not relevant for the particle paths.

$$\begin{aligned} S \simeq \frac{1}{2}r^2 \log\left|\frac{\sqrt{r}-\sqrt{2M}}{\sqrt{r}-\sqrt{2M_+}}\right| - \frac{1}{2}(\hat{r}(0))^2 \log\left|\frac{\sqrt{\hat{r}(0)}-\sqrt{2M}}{\sqrt{\hat{r}(0)}-\sqrt{2M_+}}\right| \\ - 2M^2 \log\left|\frac{\sqrt{r}-\sqrt{2M}}{\sqrt{\hat{r}(0)}-\sqrt{2M}}\right| + 2{M_+}^2 \log\left|\frac{\sqrt{r}-\sqrt{2M_+}}{\sqrt{\hat{r}(0)}-\sqrt{2M_+}}\right| \end{aligned} \tag{3.7.21}$$

Even in this expression, some irrelevant contributions remain. From equations (3.7.8) and (3.7.11), we have

$$\log\left|\frac{\sqrt{\hat{r}(0)}-\sqrt{2M}}{\sqrt{\hat{r}(0)}-\sqrt{2M_+}}\right| = \frac{k}{2M_+} \tag{3.7.22}$$

$$\log\left|\frac{\sqrt{r}-\sqrt{2M_+}}{\sqrt{\hat{r}(0)}-\sqrt{2M_+}}\right| = \frac{t}{4M_+} \tag{3.7.23}$$

so these contributions are not singular at the horizon. What remains is

$$\begin{aligned} S &\simeq \frac{1}{2}r^2 \log\left|\frac{\sqrt{r}-\sqrt{2M}}{\sqrt{r}-\sqrt{2M_+}}\right| - 2M^2 \log\left|\frac{\sqrt{r}-\sqrt{2M}}{\sqrt{\hat{r}(0)}-\sqrt{2M}}\right| \\ &= \left[2M^2 - \frac{1}{2}r^2\right] \log\left|\frac{\sqrt{\hat{r}(0)}-\sqrt{2M}}{\sqrt{r}-\sqrt{2M}}\right| + \frac{tr^2}{8M_+} + \frac{kr^2}{4M_+} \end{aligned} \tag{3.7.24}$$

where we have used equations (3.7.22) and (3.7.23) to join both remaining logarithmic terms into a more convenient form. The resulting terms are obviously regular around the horizon and can be neglected. Filling in the solution for $\sqrt{\hat{r}(0)}$ presented in equation (3.7.18) results in the action functional in terms of only time, the radial coordinate, the mode wavelength and the mass of the black hole.

$$S \simeq \left[2M^2 - \frac{1}{2}r^2\right] \log\left|\frac{e^{\frac{k}{2M'}-\frac{t}{4M'}}}{\left(e^{\frac{k}{2M'}}-1\right)e^{-\frac{t}{4M'}}+1}\right| \tag{3.7.25}$$

In the large-wavelength regime (3.7.15), this reduces to

$$S \simeq \left[2M^2 - \frac{1}{2}r^2\right] \log\left|\frac{e^{\frac{k}{2M'}-\frac{t}{4M'}}}{e^{\frac{k}{2M'}-\frac{t}{4M'}}+1}\right| \tag{3.7.26}$$

Finally, for the radiation as observed from a distance, late-time contribution is dominant. We might as well do away with the numerator in the logarithmic function altogether to obtain

$$S(t,r) \simeq \left[\frac{1}{2}r^2 - 2M^2\right] \log\left(e^{\frac{k}{2M'}-\frac{t}{4M'}}+1\right) \tag{3.7.27}$$

This is the final, quantised action for the dominant modes of a Schwarzschild black hole emitting a single shell at a time in the radial direction.

3.8 Results

Now that we have the action, we can use it to obtain the particle flux derived in 3.1. That requires some algebra.

Our solution only carries physical significance outside of the shell, i.e., for $r > 2(M + \omega_k)$. To determine the particle flux we can pick any reference radius. However, to obtain the best approximation in the WKB regime, we require that the wavelengths be as short as possible, which is when r is close to the horizon. Hence we shall choose

$$r = 2(M + \omega_k) \tag{3.8.1}$$

In WKB approximation

$$v_k(t, 2M + 2\omega_k) = e^{iS(t,2M+2\omega_k)} = e^{i\left[2\omega_k^2+4M\omega_k\right]\log\left(1+e^{\frac{k}{2M'}-\frac{t}{4M'}}\right)} \tag{3.8.2}$$

where M' equals

$$M'(2M + 2\omega_k) = M + \left[2\sqrt{M^2 + M\omega_k} - 2M\right] \frac{e^{\frac{k}{2M}-\frac{t}{4M}}}{1 + e^{\frac{k}{2M}-\frac{t}{4M}}} \tag{3.8.3}$$

Now the Bogoliubov coefficients (3.1.5) we need are

$$\alpha_{kk'} = \frac{1}{2\pi}\frac{1}{u_k(t,r)}\int_{-\infty}^{\infty} dt\, e^{i\omega_k t} e^{i\left[2\omega_{k'}^2+4M'\omega_{k'}\right]\log\left(1+e^{\frac{k'}{2M'}-\frac{t}{4M'}}\right)} \tag{3.8.4}$$

$$\beta_{kk'} = \frac{1}{2\pi}\frac{1}{u_k(t,r)}\int_{-\infty}^{\infty} dt\, e^{i\omega_k t} e^{-i\left[2\omega_{k'}^2+4M'\omega_{k'}\right]\log\left(1+e^{\frac{k'}{2M'}-\frac{t}{4M'}}\right)} \tag{3.8.5}$$

These integrals are not very pretty. In fact, they cannot be solved analytically. Fortunately, the saddle point approximation discussed in Appendix B.9 comes to the rescue. In the integrals that define the Bogoliubov coefficients, the pre-factor $2\omega_{k'}^2 + 4M'\omega_{k'}$ is very large compared to the logarithm, so this method should yield a very good approximation. The respective exponential argument functions are

$$g_\alpha(t) := \log\left(1 + e^{\frac{k'}{2M'}-\frac{t}{4M'}}\right) \tag{3.8.6}$$

$$g_\beta(t) := -\log\left(1 + e^{\frac{k'}{2M'}-\frac{t}{4M'}}\right) \tag{3.8.7}$$

To determine the saddle point, we need to take the derivative with respect to time.

$$\dot{g}_\alpha(t) = -\frac{1}{4M'}\frac{e^{\frac{k'}{2M'}-\frac{t}{4M'}}}{1+e^{\frac{k'}{2M'}-\frac{t}{4M'}}} \tag{3.8.8}$$

$$\dot{g}_\beta(t) = \frac{1}{4M'}\frac{e^{\frac{k'}{2M'}-\frac{t}{4M'}}}{1+e^{\frac{k'}{2M'}-\frac{t}{4M'}}} \tag{3.8.9}$$

Evidently, both derivatives equal zero when

$$e^{\frac{k'}{2M'}-\frac{t}{4M'}} \to 0 \tag{3.8.10}$$

However, that condition is only met at $t = \infty$. The time derivative of for $\alpha_{kk'}$ also approaches zero when

$$e^{\frac{k}{2M}-\frac{t}{4M}}\bigg|_{\alpha_{kk'}} \to \infty \tag{3.8.11}$$

Although in this limit

$$\frac{e^{\frac{k'}{2M'}-\frac{t}{4M'}}}{1+e^{\frac{k'}{2M'}-\frac{t}{4M'}}} \to 1 \tag{3.8.12}$$

M' is sufficiently large that we may consider the saddle point reached. The derivative for $\beta_{kk'}$ approaches zero when

$$e^{\frac{k}{2M'}-\frac{t}{4M'}}\bigg|_{\beta_{kk'}} \simeq -\frac{1}{2} \tag{3.8.13}$$

for then the left hand side (3.8.12) equals -1.

Since k is always real by definition, for the ratio of the Bogoliubov coefficients, only the imaginary part of t is relevant. From (3.8.11) it follows that

$$\operatorname{Im} t_\alpha = 0 \tag{3.8.14}$$

and from (3.8.13)

$$\operatorname{Im}\left(\frac{k}{2M'} - \frac{t_\beta}{4M'}\right) = \pi \quad \Rightarrow \quad \operatorname{Im} t_\beta = 4\pi M' \tag{3.8.15}$$

Apart from the sign of the exponent, only the pre-factor differs between $\alpha_{kk'}$ and $\beta_{kk'}$. From (3.8.4) we thus have

$$\left|\frac{\alpha_{kk'}}{\beta_{kk'}}\right| \simeq e^{4\pi M'\omega_k} \tag{3.8.16}$$

At the saddle point, the mass parameter (3.8.3) becomes

$$M' \simeq M - \left[2\sqrt{M^2 + M\omega_k} - 2M\right] \tag{3.8.17}$$

To first order in ω_k for $\omega_k \ll M$,

$$M'(\omega_k) \simeq M - \omega_k \tag{3.8.18}$$

Insertion in 3.8.16 yields our final expression for the ratio of Bogoliubov coefficients

$$\left|\frac{\alpha_{kk'}}{\beta_{kk'}}\right| \simeq e^{4\pi(M-\omega_k)\omega_k} \tag{3.8.19}$$

With this result, the particle flux measured at infinity, which we derived in section 3.1, equals

$$\Phi_\infty(\omega_k) \simeq \frac{d\omega_k}{2\pi}\frac{\kappa(\omega_k)}{e^{8\pi\omega_k(M-\omega_k)}-1} \tag{3.8.20}$$

which is the Hawking result [1] with a correction term. In engineering units,

$$\Phi_\infty(\omega) \simeq \frac{d\omega}{2\pi}\frac{\kappa(\omega)}{e^{8\pi\frac{G}{c^5}\omega(c^2M-\hbar\omega)}-1} \tag{3.8.21}$$

where $\hbar$ is the reduced Planck constant, c is the Einstein constant and G is the gravitational constant. This corresponds to a corrected Hawking temperature

$$T_{\mathrm{H}}(\omega) \simeq \frac{\hbar c^5}{8\pi k G\left(c^2M-\hbar\omega\right)} \tag{3.8.22}$$

As one would expect, in the small-energy limit the temperature derived without accounting for back-reaction is recovered.

3.9 Discussion

In order to calculate the particle flux coming from a black hole, we have made a number of assumptions. The first is of course is that the behaviour of the black hole is described by the Schwarzschild metric. The second assumption is that we only need to take into account the s-wave (i.e., radial) contribution because of the radial symmetry of the Schwarzschild black hole. Thirdly, we consider only the self-gravitation of one shell at a time. This might be a rather fundamental shortcoming of this model, as Per Kraus and Frank Wilczek mentioned that semi-classical methods such as the WKB approach might not be feasible when two or more shell discontinuities are present [2]. Finally, we also assumed that all radiated quanta are massless. This is not a fundamental limitation, but it does simplify the calculation. In general, one would expect even most massive particles to be highly relativistic.

We also made some approximations. The most important one is that we applied the WKB method: we assumed that high-frequency modes are dominant and that the wave function only depends on the classical action. This allowed us to go from a point-particle to a quantum field theory description in a mathematically and physically sound and non-ambiguous way.

The other approximations were of a mathematical nature: we determined the ratio of the Bogoliubov coefficients using the saddle point approximation and we discarded terms in the action that are regular around the horizon. Furthermore, we solved the squares of the initial shell radius and the total enclosed mass in the equivalent of a second-order approximation and, finally, we only included terms linear in the mode frequency in the final particle flux.

However, we have successfully reproduced Hawking's result [1] to first order in radiated energy while taking account self-gravitation. Whereas Hawking's original derivation only depended on the initial mass M of the black hole, this result only depends on the final mass, $M - \omega$, after emission of a particle of energy ω. This also seems unlikely. However, our result is a first-order approximation in the small-energy limit. It might very well be that higher-order terms do incorporate the initial mass.

4 A tunnelling approach to Hawking radiation

There is another way to derive particle production at the horizon of a black hole. Both physically and mathematically, it is an entirely different approach. Again we will apply a WKB approximation, but we will do this in a different context: that of a single-particle wave function tunnelling through a potential wall. Intuitively, this is a nice approach, but one has to be careful to fully understand the procedure physically. In this section, we will reproduce the derivation by Maulik Parikh and Frank Wilczek [3].

4.1 Tunnelling through the Schwarzschild horizon

When a shell of positive energy ω is emitted from the horizon, the mass of the black hole should decrease by an amount ω. The total mass-energy sum of the system, however, should of course be conserved. Let us therefore consider a shell travelling radially in the Painlevé metric of a Schwarzschild black hole with mass $M - \omega$,

$$ds^2 = -\left(1 - \frac{2(M-\omega)}{r}\right) dt^2 + 2\sqrt{\frac{2(M-\omega)}{r}} dtdr + dr^2 + r^2 d\Omega_2{}^2 \quad (4.1.1)$$

Although we consider the back-reaction in the resulting black hole mass, we neglect changes in the metric during the emission of ω. We thus require that $\omega \ll M$. We also assume that the trajectory of the wave is purely radial.

Let us assume we can describe the shell of energy by a single-particle wave function $\psi(r)$. One can write this in the form of an exponential function with a suitable argument function $\phi(r)$ that we have yet to determine

$$\psi(r) = e^{\phi(r)} \quad (4.1.2)$$

The single-particle time-independent Schrödinger equation in this form is

$$\begin{aligned} Ee^{\phi(r)} &= -\frac{1}{2m}\partial_r^2 e^{\phi(r)} + V(r)e^{\phi(r)} \\ &= -\frac{1}{2m}\left[\partial_r^2\phi(r) + (\partial_r\phi(r))^2\right] e^{\phi(r)} + V(r)e^{\phi(r)} \end{aligned} \quad (4.1.3)$$

Hence

$$2m\,(V(r) - E) = \partial_r^2\phi(r) + (\partial_r\phi(r))^2 \quad (4.1.4)$$

The left hand side is manifestly real, but the right hand side contains derivatives of a complex function. Let us therefore split $\partial_r\phi(r)$ into a real and an imaginary part, corresponding to the amplitude and the phase of the wave function, respectively,

$$\partial_r\phi(r) = A(r) + iB(r) \quad (4.1.5)$$

Equation (4.1.4) now equals

$$2m\,(V(r)-E) = \partial_r A(r) + i\partial_r B(r) + A^2(r) - B^2(r) + 2iA(r)B(r) \quad (4.1.6)$$

We split both sides into separate equations for the real and imaginary parts

$$2m\,(V(r)-E) = \partial_r A(r) + A^2(r) - B^2(r) \quad (4.1.7)$$

$$0 = \partial_r B(r) + 2A(r)B(r) \quad (4.1.8)$$

The left hand side of (4.1.7) is the square of minus the classic radial momentum

$$E = \frac{p_r^2}{2m} + V(r) \Rightarrow p_r = \sqrt{2m\,(E - V(r))} \quad (4.1.9)$$

In the WKB approximation, the amplitude varies slowly

$$|\partial_r A| \ll A^2(r), \quad |\partial_r A| \ll B^2(r) \quad (4.1.10)$$

Equation (4.1.7) now reduces to

$$B^2(r) - A^2(r) = p_r^2 \quad (4.1.11)$$

Since we are in a tunnelling regime ($V(r) \gg E$), the momentum p_r is imaginary. Equation (4.1.11) implies that

$$A(r) = \pm \mathrm{Im}\, p_r, \quad B(r) = \pm \mathrm{Re}\, p_r \quad (4.1.12)$$

The emission rate Γ can be determined from the squared norm of the wave function within the trajectory.

$$\Gamma = \Gamma_0\, |\psi|^2 = \Gamma_0\, e^{2\int dr A(r)} = \Gamma_0\, e^{-2\,\mathrm{Im}\int_{r_{\mathrm{in}}}^{r_{\mathrm{out}}} dr\, p_r} \quad (4.1.13)$$

The integral in the exponential argument is in fact the action due to the radial momentum

$$S_p := \int_{r_{\mathrm{in}}}^{r_{\mathrm{out}}} dr\, p_r \quad (4.1.14)$$

The integration bounds r_{in} and r_{out} correspond to the radial positions of the shell before and after emission from the horizon, respectively. From

$$r_{\mathrm{in}} = 2M \quad \text{and} \quad r_{\mathrm{out}} = 2(M - \omega) \quad (4.1.15)$$

While following the path across the horizon, the energy wave obtains a radial momentum p_r and an energy ω

$$S_p = \int_{r_{\mathrm{in}}}^{r_{\mathrm{out}}} dr \int_{M}^{M-\omega} dH \frac{dp_r}{dH} \quad (4.1.16)$$

Using the second Hamilton equation

$$\frac{dr}{dt} = \frac{dH}{dp_r} \quad (4.1.17)$$

and filling in the trajectory of an outgoing massless particle (2.6.16) renders us with

$$S_p = \int_{r_{\text{in}}}^{r_{\text{out}}} dr \int_M^{M-\omega} dH \frac{1}{1-\sqrt{\frac{2H}{r}}} \tag{4.1.18}$$

With the substitution

$$u := \sqrt{\frac{2H}{r}} \Rightarrow H = \frac{1}{2} r u^2 \Rightarrow \frac{dH}{du} = ru \tag{4.1.19}$$

the radial momentum becomes

$$p_r = \int_M^{M-\omega} dH \frac{1}{1-\sqrt{\frac{2H}{r}}} = \int_{u(M)}^{u(M-\omega)} du \frac{ru}{1-u} \tag{4.1.20}$$

Now $u = 1$ is obviously a singular point of order one and it lies on the real axis within the path of integration. In fact, it is located exactly at the Schwarzschild horizon. We can deform the path around this singularity and use Cauchy's integral formula for real-valued singularities

$$p_r = -\frac{1}{2} \oint_C du \frac{ru}{u-1} = -\pi i r u|_{u=1} = -\pi i r \tag{4.1.21}$$

Putting this result into equation (4.1.18), we have

$$\begin{aligned} S_p &= -\int_{r_{\text{in}}}^{r_{\text{out}}} dr\, \pi i r = -\pi i \left[\frac{1}{2} r^2\right]_{r=2M}^{2(M-\omega)} = -2\pi i \left(\omega^2 - 2M\omega\right) \\ &= 2\pi i \omega \left(2M - \omega\right) \end{aligned} \tag{4.1.22}$$

Conservation of energy and momentum require that when the particle is emitted, an antiparticle with energy $-\omega$ be also created. This particle then travels into the black hole, following the $\eta = -1$ path of equation (2.6.16), thereby tunnelling from r_{out} to r_{in}. The integration bounds of the energy integral remain the same, since the energy content within the black hole is reduced by ω as well. The action of such an antiparticle thus equals

$$\begin{aligned} S_{p^\dagger} &= \int_{r_{\text{out}}}^{r_{\text{in}}} dr \int_M^{M-\omega} dH \frac{1}{-1-\sqrt{\frac{2H}{r}}} \\ &= \int_{r_{\text{in}}}^{r_{\text{out}}} dr \int_M^{M-\omega} dH \frac{1}{\sqrt{\frac{2H}{r}}+1} \end{aligned} \tag{4.1.23}$$

Under the substitution (4.1.19), the second integral yields a familiar result

$$\int_{r_{\text{in}}}^{r_{\text{out}}} dr \int_{u(M)}^{u(M-\omega)} du \frac{ru}{u+1} = \pi i r u|_{u=-1} = -\pi i r \tag{4.1.24}$$

Hence (4.1.23) is equal to (4.1.22) and the antiparticle is also described by an action with imaginary part $2\pi\omega\,(2M-\omega)$. Together, particle and antiparticle creation form a pair forming process and they must be considered complementary elements of a single process

$$S = S_p + S_{p^\dagger} = 4\pi i\omega\,(2M-\omega) \tag{4.1.25}$$

Finally, equation (4.1.13) can be filled in to obtain the emission rate

$$\Gamma = \Gamma_0\, e^{-8\pi\omega(M-\frac{1}{2}\omega)} \tag{4.1.26}$$

corresponding to a temperature

$$T_H = \frac{1}{4\pi(2M-\omega)} \tag{4.1.27}$$

As with the approach of section 3, the result reproduces Hawking's result to first order in ω [1] with a correction term. However, the second order term differs from the previous result. Interestingly, the exponent in the flux derived in this manner depends on equal contributions of the initial black hole mass M and the final black hole mass $M-\omega$.

4.2 Tunnelling through De Sitter horizons

Since the tunnelling approach turned out to be a simple and effective method to derive particle creation in Schwarzschild space-time, a logical next step is to apply this approach in other space-times in which particle creation is expected.

n-dimensional De Sitter space-time is the maximally symmetric space-time with the positive normalised Ricci curvature. For this reason, it plays an important role in cosmology. It lives in an n-dimensional submanifold of an $(n+1)$-dimensional manifold with a Minkowski space-time. With the Minkowski metric expressed in terms of the coordinates x^μ,

$$ds^2 = -dt^2 + dx_i dx^i \tag{4.2.1}$$

The submanifold is defined by the equation

$$\eta_{\mu\nu}x^\mu x^\nu = \alpha^2 \tag{4.2.2}$$

where α is a constant. In this sense, it is the Lorentzian equivalent of an n-sphere. It is not uncommon to pick the gauge $\alpha = 1$. De Sitter space-time has some properties that are very similar to that of the Schwarzschild metric. This is perhaps most manifest in static coordinates

$$ds^2 = -(1-r^2)dt_s^2 + (1-r^2)^{-1}dr^2 + r^2 d\Omega_{d-2}^2 \tag{4.2.3}$$

for $(d+1)$ dimensions, with the coordinates $(t_s, r, \{\theta^i\})$ defined such [5] that

$$x^0 = \sqrt{1-r^2}\sinh t_s \qquad x^i = r\theta^i \qquad x^n = \sqrt{1-r^2}\cosh t_s \tag{4.2.4}$$

for $r < -1 \vee r > 1$ and

$$x^0 = \sqrt{r^2-1}\cosh t_s \qquad x^i = r\theta^i \qquad x^n = \sqrt{r^2-1}\sinh t_s \tag{4.2.5}$$

for $-1 < r < 1$.

In static coordinates, the horizon appears at $r = \pm 1$. However, we can take a substitution analogous to (2.6.5)

$$dt = dt_s + r(1-r^2)^{-1} dr \tag{4.2.6}$$

For the static time element, this yields

$$\begin{aligned} dt_s^2 &= \left(dt - r(1-r^2)^{-1} dr\right)^2 \\ &= dt^2 + r^2(1-r^2)^{-2} dr^2 - 2r(1-r^2)^{-1} dt\, dr \end{aligned} \tag{4.2.7}$$

Applying this substitution to (4.2.6) renders us with the De Sitter-Painlevé metric

$$\begin{aligned} ds^2 &= -(1-r^2)dt^2 - r^2(1-r^2)^{-1}dr^2 + 2rdt\,dr + (1-r^2)^{-1}dr^2 + r^2 d\Omega_{d-2}^2 \\ &= -(1-r^2)dt^2 + 2rdt\,dr + (-r^2+1)(1-r^2)^{-1}dr^2 + r^2 d\Omega_{d-2}^2 \\ &= -(1-r^2)dt^2 + 2rdt\,dr + dr^2 + r^2 d\Omega_{d-2}^2 \end{aligned} \tag{4.2.8}$$

As with the Painlevé metric for Schwarzschild black holes, there is no longer a coördinate singularity at $r = \pm 1$. In ADM terminology, the metric functions of static De Sitter coordinates are

$$L_{\mathrm{dSP}}(t,r) = 1, \qquad R_{\mathrm{dSP}}(t,r) = r \tag{4.2.9}$$

$$N^t_{\mathrm{dSP}}(t,r) = 1, \qquad N^r_{\mathrm{dSP}}(t,r) = r \tag{4.2.10}$$

The differential substitution (4.2.6) corresponds to a coordinate change $t_s \to t$

$$\begin{aligned} t &= \int dt_s + \int dr\,\frac{r}{1-r^2} = t_s - \frac{1}{2}\int du\,\frac{1}{u-1} = t_s - \frac{1}{2}\ln|u-1| \\ &= t_s - \frac{1}{2}\ln\left|r^2-1\right| \end{aligned} \tag{4.2.11}$$

using the substitution $u := r^2$ and choosing the integration constant to equal 0. We can also identify radial null curves, i.e. curves for which $ds^2 = 0$ and $d\Omega_2^2 = 0$ in equation (4.2.8)

$$\begin{aligned} 0 &= -(1-r^2)dt^2 + 2rdt\,dr + dr^2 \\ &= (dr - (1-r)dt)(dr + (1+r)dt) \end{aligned} \tag{4.2.12}$$

This equation is solved by the curves

$$dr = (1 - r)\, dt \quad \vee \quad dr = -(1 + r)\, dt \tag{4.2.13}$$

In other words, null curves must obey

$$\frac{dr}{dt} = \pm 1 - r \tag{4.2.14}$$

An outgoing particle travelling across the static coordinate horizon at $r = +1$ must have a positive radial velocity even at $r < 1$, hence it must follow the 'plus' curve. Similarly, a particle crossing $r = -1$ must follow the 'minus' curve.

Let us attempt to use a strategy similar to the one we used in sections 4.1-4.7 to determine a temperature for De Sitter space. The analogue of equation (4.1.16) for a quantum of energy ω crossing the horizon of a De Sitter space-time at $r = +1$ with a total energy E is

$$\begin{aligned} S_p &= \int_{r_{\mathrm{in}}}^{r_{\mathrm{out}}} dr \int_E^{E-\omega} dH \frac{dt}{dr} = \int_{r_{\mathrm{in}}}^{r_{\mathrm{out}}} dr \int_E^{E-\omega} dH \frac{1}{1-r} \\ &= \int_{r_{\mathrm{in}}}^{r_{\mathrm{out}}} dr \frac{\omega}{r-1} \end{aligned} \tag{4.2.15}$$

where we again used the second Hamilton equation (4.1.17) to relate radial momentum to radial velocity. The complex-valued integral has a simple pole at $r = 1$. Since the integration curve $[r_{\mathrm{in}}, r_{\mathrm{out}}]$ lies on the real axis, we have to take a path that curves around the pole. However, only half of such a path lies within the singularity, so the action in equation (4.2.15) equals

$$S_p = \pi\omega i \tag{4.2.16}$$

The antiparticle that travels in the opposite direction has an action

$$S_{p^\dagger} = \int_{r_{\mathrm{out}}}^{r_{\mathrm{in}}} dr \int_E^{E-\omega} dH \left.\frac{dt}{dr}\right|_- = -\int_{r_{\mathrm{out}}}^{r_{\mathrm{in}}} dr \frac{\omega}{r+1} = \pi\omega i$$

Combining the contributions yields an action function

$$S = 2\pi\omega i \tag{4.2.17}$$

This corresponds to an emission rate

$$\Gamma = \Gamma_0\, e^{-2\pi\omega} \tag{4.2.18}$$

This is simply a Boltzmann distribution with a temperature T

$$T = \frac{1}{2\pi} \tag{4.2.19}$$

Note that back-reaction is not considered in this calculation. In section 4.1, the metric depended on the energy content within the horizon and the radiation trajectory in the action integral took the change in energy due to its emission into account. In a De Sitter metric, however, no such influence is present. Since we expect back-reaction to exist, we can assume that some shift of the location of the horizon takes place and therefore, tunnelling within the framework of section 4.1 to occur. The amplitude derived should therefore be interpreted as a small-energy limit.

4.3 Back-reaction in De Sitter space-time

In the preceding section, we derived particle emission at the horizon of De Sitter space-time. We did not account for back-reaction. However, as with Schwarzschild space-time, energy conservation mandates that any increase of mass-energy at one side of the horizon be paired with a corresponding decrease on the other side.

Whereas the Schwarzschild metric's raison d'être is the enclosed mass, usually denoted M, enclosed by one side of the horizon, De Sitter space is curved intrinsically and not by mass. Thus, we need to add curvature due to the mass-energy flow of back-reaction.

Birkhoff's theorem states that there is only one vacuum solution to the Einstein equation that observes spherical symmetry. This is the generalised Schwarzschild space-time, which can always be written as

$$ds^2 = -f(t,r)dt_s^2 + \frac{1}{f(t,r)}dr^2 + r^2 d\Omega_2^2 \tag{4.3.1}$$

in four dimensions, an ADM metric with $N^r = 0$ and the $L = 1, R = r$ gauge. An implication of Birkhoff's theorem is that if space-time is to be curved by mass, spherical symmetry can only be preserved if this mass resides in the centre of of curvature.

Back-reaction produces mass-energy beyond the horizon. This means that there is a separate, Schwarzschild contribution due to this mass-energy, in addition to the intrinsic curvature De Sitter space-time observes. This is usually called Schwarzschild-De Sitter space-time. With an energy E, its line element in four dimensions is

$$ds^2 = -\left(1 - r^2 - \frac{2E}{r}\right)dt_s^2 + \left(1 - r^2 - \frac{2E}{r}\right)^{-1}dr^2 + r^2 d\Omega_2^2 \tag{4.3.2}$$

As we would expect, in the limit $E \to 0$, this is just four-dimensional De Sitter space-time and the event horizon lies at $r = \pm 1$. For general E, we have to find the real roots of the depressed cubic equation

$$r^3 - r + 2E = 0 \tag{4.3.3}$$

This is most easily done using Cardamo's method. The roots are

$$r_S = e^{i\pi\frac{2k}{3}}\sqrt[3]{-\sqrt{E^2-\frac{1}{27}}-E}+e^{i\pi\frac{2(k+2)}{3}}\sqrt[3]{\sqrt{E^2-\frac{1}{27}}-E}, \quad k\in\mathbb{Z} \tag{4.3.4}$$

There are three solutions, whereas the horizon of De Sitter space-time horizon only lies at two radial coordinates. Let us explore each of these solutions in the low energy limit $E \to 0$, i.e., De Sitter space-time.

$$\begin{aligned} r_S(0) &= e^{i\pi\frac{2k}{3}}\sqrt[3]{-\sqrt{-\frac{1}{27}}}+e^{i\pi\frac{2(k+2)}{3}}\sqrt[3]{\sqrt{-\frac{1}{27}}} \\ &= \begin{cases} -1 & k=0 \pmod 3 \\ 0 & k=2 \pmod 3 \\ 1 & k=1 \pmod 3 \end{cases} \end{aligned} \tag{4.3.5}$$

as we would expect from inspection of the metric equation. The -1 and $+1$ solutions are evidently just the event horizon of De Sitter space-time; the $r = 0$ solution corresponds to the central singularity of the Schwarzschild metric.

If E is very small as compared to r, we might as well take a Taylor series around $E = 0$ to obtain a more manageable equation. To that end, we have

$$\begin{aligned} \frac{dr_S}{dE} &= \frac{e^{i\pi\frac{2k}{3}}}{3}\left(-\sqrt{E^2-\frac{1}{27}}-E\right)^{-\frac{2}{3}}\left(\frac{-E}{\sqrt{E^2-\frac{1}{27}}}-1\right) \\ &\quad+\frac{e^{i\pi\frac{2(k+2)}{3}}}{3}\left(\sqrt{E^2-\frac{1}{27}}-E\right)^{-\frac{2}{3}}\left(\frac{E}{\sqrt{E^2-\frac{1}{27}}}-1\right) \end{aligned} \tag{4.3.6}$$

Around $E = 0$ the coefficient of the linear term is then

$$\begin{aligned} \left.\frac{dr_S}{dE}\right|_{E=0} &= -\frac{e^{i\pi\frac{2k}{3}}}{3}\left(-\sqrt{-\frac{1}{27}}\right)^{-\frac{2}{3}}-\frac{e^{i\pi\frac{2(k+2)}{3}}}{3}\left(\sqrt{-\frac{1}{27}}\right)^{-\frac{2}{3}} \\ &= \begin{cases} 2 & (k=1 \pmod 3) \\ -1 & (\text{otherwise}) \end{cases} \end{aligned} \tag{4.3.7}$$

So to first order, the horizon lies at

$$r_S(E) \simeq \pm 1 - E \tag{4.3.8}$$

In fact, the second derivatives vanish at $E = 0$, so

$$r_S(E) = \pm 1 - E + \mathcal{O}(E^3) \tag{4.3.9}$$

Since radiated energy shells originate at the coordinate horizon, we need coordinates that are valid at the horizon. By a procedure similar to that of Schwarzschild and De Sitter space-times, we obtain the aptly called Schwarzschild-De-Sitter-Painlevé metric. Starting with the substitution corresponding to the new time coordinate t,

$$\begin{aligned} dt &:= dt_s - \sqrt{\left(1 - r^2 - \frac{2E}{r}\right)^{-2} - \left(1 - r^2 - \frac{2E}{r}\right)^{-1}} dr \\ &= dt_s - \sqrt{\frac{r^2 + \frac{2E}{r}}{\left(1 - r^2 - \frac{2E}{r}\right)^2}} dr = dt_s - \frac{\sqrt{r^2 + \frac{2E}{r}}}{1 - r^2 - \frac{2E}{r}} dr \end{aligned} \tag{4.3.10}$$

Squaring this yields

$$dt_s^2 = dt^2 - \frac{r^2 + \frac{2E}{r}}{\left(1 - r^2 - \frac{2E}{r}\right)^2} dr^2 + 2 \frac{\sqrt{r^2 + \frac{2E}{r}}}{1 - r^2 - \frac{2E}{r}} dt dr \tag{4.3.11}$$

leading to the Schwarzschild-De-Sitter-Painlevé metric

$$ds^2 = -\left(1 - r^2 - \frac{2E}{r}\right) dt^2 + dr^2 + 2\sqrt{r^2 + \frac{2E}{r}} dt dr + r^2 d\Omega_2^2 \tag{4.3.12}$$

We still expect radiation to travel in radial paths, so for a massless particle

$$\begin{aligned} 0 &= -\left(1 - r^2 - \frac{2E}{r}\right) dt^2 + dr^2 + 2\sqrt{r^2 + \frac{2E}{r}} dt dr \\ &= \left(dr + \left(\sqrt{r^2 + \frac{2E}{r}} - 1\right) dt\right) \left(dr + \left(\sqrt{r^2 + \frac{2E}{r}} + 1\right) dt\right) \end{aligned} \tag{4.3.13}$$

resulting in the path equation

$$\frac{dr}{dt} = \pm 1 - \sqrt{r^2 + \frac{2E}{r}} \tag{4.3.14}$$

Now that we know the path of radially propagating shell, we can derive an action function. Since we start with empty De Sitter space-time, the enclosed energy E equals zero before emission, leading to the De Sitter limit of the Schwarzschild-De Sitter metric, but upon emission of a shell of positive energy ω, it increases to ω. Again using the trick with the second Hamilton equation, the action of a positive-energy particle originating at the horizon at $r = +1$ is

$$S_p = \int_{r_{\text{in}}}^{r_{\text{out}}} dr \int_0^{\omega} dH \frac{dt}{dr} = \int_{r_s(0)}^{r_s(\omega)} dr \int_0^{\omega} dH \frac{1}{1 - \sqrt{r^2 + \frac{2H}{r}}} \tag{4.3.15}$$

With a substitution similar to the one in equation (4.1.19),

$$u := \sqrt{r^2 + \frac{2H}{r}} \Rightarrow H = \frac{1}{2} r \left[u^2 - r^2\right] \Rightarrow \frac{dH}{du} = ru \tag{4.3.16}$$

the second integral equals

$$\int_{u(0)}^{u(\omega)} du \frac{ru}{1-u} \tag{4.3.17}$$

This integral is of the same form as the one in equation (4.1.21), hence it yields the same result, $-\pi i r$. This renders us with the action integral

$$S_p = -\pi i \int_{r_s(0)}^{r_s(\omega)} dr\, r \simeq -\pi i \left[\frac{1}{2} r^2\right]_{r=1}^{1-\omega} = \pi i \left(\omega - \frac{\omega^2}{2}\right) \tag{4.3.18}$$

The antiparticle contributes towards the action as well,

$$S_{p^\dagger} = \int_{r_{\text{out}}}^{r_{\text{in}}} dr \int_{\omega}^{0} dH \left.\frac{dt}{dr}\right|_{-} = -\int_{r_s(\omega)}^{r_s(0)} dr \int_{\omega}^{0} dH \frac{1}{1 + \sqrt{r^2 + \frac{2H}{r}}} \tag{4.3.19}$$

With the same substitution (4.3.17) as for the positive energy particle, we obtain

$$S_{p^\dagger} = \pi i \int_{r_s(\omega)}^{r_s(0)} dr\, r \simeq \pi i \left[\frac{1}{2} r^2\right]_{r=1-\omega}^{1} = \pi i \left(\omega - \frac{\omega^2}{2}\right) \tag{4.3.20}$$

Again, the antiparticle contribution is identical to that produced by the particle. The combined action corresponds to an emission rate

$$\Gamma = \Gamma_0\, e^{-2\pi\omega\left(1 - \frac{1}{2}\omega\right)} \tag{4.3.21}$$

To first order in ω, this is again a Boltzmann distribution with a temperature T_{dS}. Apparently, we have derived De Sitter radiation with a correction term. Our result agrees with that obtained by Brian Greene, Maulik Parikh and Jan Pieter van der Schaar, who used a similar method [14].

4.4 The Unruh effect by tunnelling

Horizons may even appear in flat Minkowski space-time. Suppose an observer in Minkowski space-time extending in the y and z directions experiences a constant positive proper acceleration a in the x direction. Then

$$\frac{d^2x}{d\tau^2} = a, \quad \frac{d^2t}{d\tau^2} = \frac{d^2y}{d\tau^2} = \frac{d^2z}{d\tau^2} = 0 \tag{4.4.1}$$

so for any set of coordinates, the scalar product of proper velocities equals

$$\eta_{\mu\nu} \frac{d^2x^\mu}{d\tau^2} \frac{d^2x^\nu}{d\tau^2} = a^2 \tag{4.4.2}$$

If we want to re-express the Minkowski coordinates t and x in terms of the proper time τ of the observer, we need to solve

$$\left(\frac{d^2x}{d\tau^2}\right)^2 - \left(\frac{d^2t}{d\tau^2}\right)^2 = a^2 \tag{4.4.3}$$

and

$$\left(\frac{dx}{d\tau}\right)^2 - \left(\frac{dt}{d\tau}\right)^2 = -1 \tag{4.4.4}$$

These equations are solved by

$$\frac{dt}{d\tau} = \cosh(a\tau), \quad \frac{dx}{d\tau} = \sinh(a\tau) \tag{4.4.5}$$

Integration results in the parametrised equations

$$t(\tau) = a^{-1}\sinh(a\tau) + t_0 \tag{4.4.6}$$

$$x(\tau) = a^{-1}\cosh(a\tau) + x_0 \tag{4.4.7}$$

If we place the observer at the Minkowski origin, $t_0 = x_0 = 0$.

In order for a set of coordinates $\{y^\mu\}$ to be comoving with respect to the observer, the components of the proper velocity must obviously be

$$\frac{dy^0}{d\tau} = 1, \quad \frac{dy^j}{d\tau} = 0 \tag{4.4.8}$$

The natural choice for y^0 would be τ, while y and z are already independent of τ, since the observer's motion is restricted to the x direction, so it makes sense to pick $y^2 = y$ and $y^3 = z$. In order to describe points on the (t, x) plane that do not lie on the curves (4.4.6) and (4.4.7), we can scale by the third coordinate, ξ $(0 < \xi < \infty)$ and, for the sake of convenience, a factor a. Then we have the Rindler coordinates $y^\mu = (\tau, \xi, y, z)$. In these coordinates, the observer lives at $\xi = a^{-1}$.

The Minkowski coordinates are expressed in terms of Rindler coordinates as

$$t = \xi\sinh(a\tau), \quad x = \xi\cosh(a\tau) \tag{4.4.9}$$

The corresponding differential coordinates equations are

$$dt = a\xi\cosh(a\tau)\,d\tau - \sinh(a\tau)\,d\xi \tag{4.4.10}$$

$$dx = a\xi\sinh(a\tau)\,d\tau - \cosh(a\tau)\,d\xi \tag{4.4.11}$$

leading to the Rindler line element

$$\begin{aligned} ds^2 &= -dt^2 + dx^2 + dy^2 + dz^2 \\ &= -a^2\xi^2\cosh^2(a\tau)\,d\tau^2 - \sinh^2(a\tau)\,d\xi^2 + a^2\xi^2\sinh^2(a\tau)\,d\tau^2 \\ &\quad + \cosh^2(a\tau)\,d\xi^2 + dy^2 + dz^2 \\ &= -a^2\xi^2 d\tau^2 + d\xi^2 + dy^2 + dz^2 \end{aligned} \tag{4.4.12}$$

Hence the metric tensor's components are

$$g_{\tau\tau} = -a^2\xi^2, \quad g_{\xi\xi} = g_{yy} = g_{zz} = 1, \quad g_{\mu\nu} = 0 \quad (\mu \neq \nu) \tag{4.4.13}$$

so the metric determinant equals

$$\det g = -a^2\xi^2 \tag{4.4.14}$$

The determinant equals zero when ξ does. Since $\xi = 0$ is the only solution to the equation $x = 0$, there is a coordinate horizon at $x = 0$ in Minkowski coordinates. The observer cannot receive any particle originating from beyond this horizon and particle number he measures needs not match the particle number on the other side of the horizon, observed from a distance.

In fact, the observer experiences thermal radiation with a temperature that depends linearly on acceleration,

$$T_U = \frac{a}{2\pi} \tag{4.4.15}$$

This was first derived by Steven Fulling, Paul Davies and Bill Unruh and it is called the Unruh effect [20, 21].

Locally, any metric resembles the Minkowski metric. One can approximate the natural coordinates of an observer close to the horizon by the Rindler coordinates. In other words, any phenomenon where an event horizon leads to particle creation being observed by an observer at a distance can be derived by taking the appropriate Unruh temperature for an observer infinitely close to the horizon.

An excellent illustration is provided by the Schwarzschild black hole and the Hawking radiation it produces. An observer infinitely close to the horizon is subject to a constant proper acceleration due the curvature caused by the black hole mass. Yet the horizon cannot be reached in a finite amount of his own time. The point of view of the observer can be described in the Rindler coordinates defined above.

We start by postulating that the Rindler time is the Schwarzschild time

$$\tau = t_S \tag{4.4.16}$$

Then we must have

$$a^2\xi^2 = 1 - \frac{2M}{r} \Rightarrow a\xi = \sqrt{1 - \frac{2M}{r}} \tag{4.4.17}$$

Furthermore, close to the horizon we must have that

$$d\xi^2 = \left(1 - \frac{2M}{r}\right)^{-1} dr^2 \Rightarrow \frac{d\xi}{dr} = \sqrt{1 - \frac{2M}{r}} \tag{4.4.18}$$

where we have used that ξ is always positive. The simplest way to determine the acceleration a in terms of these coordinates is by taking the derivative with respect to ξ on both sides of the result in the first equation and then using the result of the second equation

$$a = \frac{1}{2\sqrt{1-\frac{2M}{r}}} \frac{2M}{r^2} \frac{d\xi}{dr} = \frac{M}{r^2} \tag{4.4.19}$$

which corresponds to acceleration due to a gravitational attraction by a mass M located at a distance r, as we would expect. Hence an observer infinitely close to the Schwarzschild horizon at $r = 2M$ observes a Rindler coordinate metric with the line element

$$ds^2 = -\left(1 - \frac{2M}{r}\right) d\tau^2 + d\xi^2 + r^2 d\Omega_2^2 \tag{4.4.20}$$

By equation (4.4.15), the associated Unruh temperature is

$$T_{U,S} = \frac{M}{8\pi M^2} = \frac{1}{8\pi M} \tag{4.4.21}$$

which is, as we would expect, the Hawking temperature of a Schwarzschild black hole when back-reaction is not taken into account.

Let us now try and derive the Unruh temperature (4.4.15) using the tunnelling approach from in the previous sections. τ is a timelike coordinate and the Rindler metric is stationary with respect to τ. The second Hamilton equation in Rindler coordinates is

$$\frac{d\xi}{d\tau} = \frac{dH}{dp_\xi}, \tag{4.4.22}$$

From equation (4.4.12), it follows that null geodesics in the ξ direction obey

$$0 = -a^2\xi^2 d\tau^2 + d\xi^2 \tag{4.4.23}$$

which is solved by

$$\frac{d\xi}{d\tau} = a\xi \tag{4.4.24}$$

since a and ξ are positive by definition.

The action due to the momentum of a particle of energy ω travelling in the ξ direction from ξ_{in} to ξ_{out} is

$$\begin{aligned} S &= \int_{\xi_{\text{in}}}^{\xi_{\text{out}}} d\xi \int_E^{E-\omega} dH \frac{d\tau}{d\xi} = \int_{\xi_{\text{in}}}^{\xi_{\text{out}}} d\xi \int_E^{E-\omega} dH \frac{1}{a\xi} \\ &= \int_{\xi_{\text{in}}}^{\xi_{\text{out}}} d\xi\, \omega \frac{1}{a\xi} = \pi i \frac{\omega}{a} \end{aligned} \tag{4.4.25}$$

where we have used Cauchy's integration method at $\xi = 0$. The WKB tunnelling amplitude equals

$$\Gamma = \Gamma_0 e^{-2\,\mathrm{Im}S} = e^{-2\pi\frac{\omega}{a}} \tag{4.4.26}$$

which is a Boltzmann distribution for a temperature T_U

$$T_U = \frac{a}{2\pi} \tag{4.4.27}$$

Hence we have derived the famed Unruh effect. In a sense, this is the most general form of any particle production phenomenon due to an event horizon in space-time.

4.5 The instanton approach

In order to understand the calculation in section 4.1 better, let us repeat the essential part of the calculation in a slightly more rigorous way. We will do this by writing a path integral for an instanton, a particle-like phenomenon in quantum mechanics. Part of the following analysis closely follows Coleman [10].

Given the wave function $\psi(r_a; t_a)$ at spatial coordinate r_a and time t_a, causality requires that one can write the wave function at time t_b and spatial coordinate r_b in terms of a propagator $K(r_b, t_b; r_a, t_a)$

$$\psi(r_b, t_b) = \int dr_a\, K(r_b, t_b; r_a, t_a)\psi(r_a, t_a) \tag{4.5.1}$$

By the completeness relation

$$\int dr_s\, |r_s, t_s\rangle\langle r_s, t_s| = 1 \tag{4.5.2}$$

we can write the wave function as

$$\begin{aligned} \psi(r_b, t_b) = \langle r_b, t_b|\psi\rangle &= \int dr_a \langle r_b, t_b|r_a, t_a\rangle\langle r_a, t_a|\psi\rangle \\ &= \int dr_a \langle r_b, t_b|r_a, t_a\rangle\psi(r_a, t_a) \end{aligned} \tag{4.5.3}$$

Hence the propagator in equation (4.5.1) equals

$$K(r_b, t_b; r_a, t_a) = \langle r_b, t_b|r_a, t_a\rangle \tag{4.5.4}$$

Since we can incorporate any intermediate point r_c in equation (4.5.1) to obtain

$$\psi(r_b, t_b) = \int dr_a dr_c\, K(r_b, t_b; r_c, t_c)K(r_c, t_c; r_a, t_a)\psi(r_a, t_a), \tag{4.5.5}$$

Richard Feynman proposed that to determine this propagator, one has to integrate over *all* possible midpoints, thus accounting for all possible paths that would lead from state $|a\rangle$ to state $|b\rangle$, by means of a path integral. If we divide the time between t_a and t_b in n equal pieces of length Δt each, the appropriate path integral is then

$$\langle r_b, t_b | r_a, t_a\rangle = \lim_{n\to\infty} \int dr_1 \dots dr_n \langle r_b, t_b | r_n, t_n\rangle \left[\prod_{j=1}^{n-1} \langle r_{j+1}, t_{j+1} | r_j, t_j\rangle\right] \langle r_1, t_1 | r_a, t_a\rangle \tag{4.5.6}$$

Now

$$|r, t\rangle = e^{-iHt} |r, 0\rangle \tag{4.5.7}$$

and since $\Delta t = t_{j+1} - t_j$, we may rewrite the elements of the path integral in the Heisenberg picture

$$\langle r_{j+1}, t_{j+1} | r_j, t_j\rangle = \left\langle r_{j+1} \middle| e^{-i\Delta t H} \middle| r_j \right\rangle \tag{4.5.8}$$

If the Hamiltonian follows the Ansatz

$$H(r, p) = h_p(p) + h_r(r), \tag{4.5.9}$$

such as the free particle Hamiltonian

$$H_f(r, p) = \frac{p^2}{2m} + V(r), \tag{4.5.10}$$

the element can be written as

$$\langle r_{j+1}, t_{j+1} | r_j, t_j\rangle = \left\langle r_{j+1} \middle| e^{-i\Delta t h_p} e^{-i\Delta t h_r} \middle| r_j \right\rangle \tag{4.5.11}$$

For the free particle Hamiltonian H_f, the element is

$$\begin{aligned}\langle r_{j+1}, t_{j+1} | r_j, t_j\rangle &= e^{-i\Delta t\, V(\frac{1}{2}[r_{j+1}+r_j])} \left\langle r_{j+1} \middle| e^{-i\Delta t \frac{p^2}{2m}} \middle| r_j \right\rangle \\ &= \frac{1}{2\pi} e^{-i\Delta t\, V(\frac{1}{2}[r_{j+1}+r_j])} \int dp \left\langle r_{j+1} \middle| e^{-i\Delta t \frac{p^2}{2m}} \middle| p \right\rangle \langle p | r_j\rangle\end{aligned} \tag{4.5.12}$$

where we have used completeness of states. Now

$$\begin{aligned}\left\langle r_{j+1} \middle| e^{-i\Delta t \frac{p^2}{2m}} \middle| p \right\rangle &= \sum_{k=0}^{\infty} \frac{1}{k!} \left\langle r_{j+1} \middle| \left(-i\Delta t \frac{p^2}{2m}\right)^k \middle| p \right\rangle \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{-i\Delta t}{2m}\right)^k \left\langle r_{j+1} \middle| p^{2k} \middle| p \right\rangle \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} \left(-i\Delta t \frac{p^2}{2m}\right)^k \langle r_{j+1} | p\rangle \\ &= e^{-i\Delta t \frac{p^2}{2m}} \langle r_{j+1} | p\rangle\end{aligned} \tag{4.5.13}$$

and $\langle p|r\rangle$ is just the free particle wave function e^{ipr}, while

$$\langle r|p\rangle = \langle p|r\rangle^\dagger = e^{-ipr} \tag{4.5.14}$$

so

$$\begin{aligned}
\langle r_{j+1}, t_{j+1}|r_j, t_j\rangle &= \frac{1}{2\pi} e^{-i\Delta t\, V(\frac{1}{2}[r_{j+1}+r_j])} \int dp\, e^{ip[r_{j+1}-r_j]-i\Delta t \frac{p^2}{2m}} \\
&= \sqrt{\frac{-im}{2\pi\Delta t}} e^{\frac{i}{2}\frac{m}{\Delta t}(r_{j+1}-r_j)^2 - i\Delta t\, V(\frac{1}{2}[r_{j+1}+r_j])} \\
&= \sqrt{\frac{-im}{2\pi\Delta t}} e^{i\Delta t\left[\frac{m}{2}\left(\frac{r_{j+1}-r_j}{\Delta t}\right)^2 - V(\frac{1}{2}[r_{j+1}+r_j])\right]}
\end{aligned} \tag{4.5.15}$$

where we have used the Gaussian integral

$$\int_{-\infty}^{\infty} dx\, e^{-ax^2-2bx} = \sqrt{\frac{\pi}{a}} e^{\frac{b^2}{a}} \tag{4.5.16}$$

The free particle path integral is then

$$\langle r_b, t_b|r_a, t_a\rangle = \lim_{n\to\infty} \left(\frac{-im}{2\pi\Delta t}\right)^{\frac{n+1}{2}} \int dr_1 \ldots dr_n e^{i\sum_{j=1}^{n}\Delta t\left[\frac{m}{2}\left(\frac{r_{j+1}-r_j}{\Delta t}\right)^2 - V(\frac{1}{2}[r_{j+1}+r_j])\right]} \tag{4.5.17}$$

The limit $n \to \infty$ implies $\Delta t \to 0$, hence the sum in the exponent becomes an integral and the change of r per time step becomes the derivative of r with respect to time.

$$\langle r_b, t_b|r_a, t_a\rangle = \lim_{n\to\infty} \left(\frac{-im}{2\pi\Delta t}\right)^{\frac{n+1}{2}} \int dr_1 \ldots dr_n e^{i\int_{t_a}^{t_b} dt\left[\frac{m}{2}\dot{r}^2 - V(\frac{1}{2}[r_{j+1}+r_j])\right]} \tag{4.5.18}$$

The argument of the exponent is just the classical action times i. Defining the integration measure $[Dr(t)]$

$$[Dr(t)] := \left(\frac{-im}{2\pi\Delta t}\right)^{\frac{n+1}{2}} dr_1 \ldots dr_n \tag{4.5.19}$$

we can write the propagator as

$$K(r_b, t_b; r_a, t_a) = \int [Dr(t)]\, e^{iS[r(t)]} \tag{4.5.20}$$

where S is the classical action for a single particle of mass m with a potential $V(r)$. If we consider a particle's path from $t = t_a$ to t_b, the action is

$$S = \int_{t_a}^{t_b} dt \left[\frac{1}{2} m\dot{r}^2(t) - V(r(t))\right] \tag{4.5.21}$$

Up to this point, this analysis has assumed space-time is Euclidean. For the analysis to hold in Minkowski space-time, one needs to perform a Wick rotation: to replace the purely real time t by the purely imaginary time $\tau = -it$, i.e., time is rotated by $-\frac{\pi}{2}$ in the complex plane. The Minkowski metric is then replaced by a Euclidean metric for the new coordinates $y^\mu = (\tau, \{r^i\})$.

$$ds^2 = -dt^2 + dr_i dr^i = d\tau^2 + dr_i dr^i = dy_\mu dy^\mu \tag{4.5.22}$$

The transition amplitude in (4.5.20) is then replaced by its Euclidean counterpart

$$K_E(r_b, \tau_b; r_a, \tau_a) = \langle r_a | e^{-H\tau} | r_b \rangle = \int [Dr(\tau)] \, e^{-S_E[r(\tau)]} \tag{4.5.23}$$

with a Euclidean action

$$S_E\left[r(\tau)\right] = \int_{\tau_a}^{\tau_b} d\tau \left[\frac{1}{2} m\dot{r}^2(\tau) + V(r(\tau))\right] \tag{4.5.24}$$

where a dot denotes a derivative with respect to τ. However, the problem at hand lives in a curved space-time expressed by the Gullstrand-Painlevé metric. We need to go one step further and multiply the integrand by the square root of the metric determinant. Inserting the Gullstrand-Painlevé ADM functions (2.6.8) in the result of equation (4.5.25) yields

$$\sqrt{-g} = r^2 \sin^2\theta \tag{4.5.25}$$

However, for the flat Minkowski metric $\eta_{\mu\nu}$

$$\eta_{tt} = -1, \quad \eta_{rr} = 1, \quad \eta_{\theta\theta} = r^2, \quad \eta_{\phi\phi} = r^2 \sin^2\theta, \quad \eta_{\mu\nu} = 0 \ (\mu \neq \nu) \tag{4.5.26}$$

the integration weight is

$$\sqrt{-\eta} = r^2 \sin^2\theta \tag{4.5.27}$$

Apparently, our coordinates have the nice property that one can integrate over them as though they were Minkowski coordinates.

The functional derivative of the Wick-rotated action with respect to the coordinate r is

$$\frac{\delta S_E}{\delta r} = \frac{dV}{dr}(\tau) - \frac{d}{d\tau}\left(m\frac{dr}{d\tau}(\tau)\right) = V'(r(\tau)) - m\ddot{r}(\tau) \tag{4.5.28}$$

or, written as an integral over τ,

$$\frac{\delta S_E}{\delta r} = \int d\overline{\tau} \left[V'(r(\overline{\tau})) - m\ddot{r}(\overline{\tau})\right] \delta\left(\overline{\tau} - \tau\right) \tag{4.5.29}$$

where δ denotes the Dirac delta function. Then the second functional derivative of S_E with respect to r is

$$\frac{\delta^2 S_E}{\delta r^2} = V''(r(\tau)) - m\frac{d^2}{d\tau^2} \tag{4.5.30}$$

We may express the paths we are to integrate over as a complete set $\{r_j(t)\}$ of orthonormal functions, i.e., for any pair r_j, r_k,

$$\int_{\tau_b}^{\tau_a} d\tau\, r_j(\tau) r_k(\tau) = \delta_{jk} \tag{4.5.31}$$

that obey the boundary conditions

$$r(\tau_b) = r(\tau_a) = 0 \tag{4.5.32}$$

If we know any particular solution $\overline{r}(\tau)$ to the boundary conditions, then the most general function obeying the boundary conditions is

$$r(\tau) = \overline{r}(\tau) + \sum_j a_j r_j(\tau) \tag{4.5.33}$$

The new integration measure then becomes

$$[Dr(\tau)] = \left(\frac{-im}{2\pi\Delta t}\right)^{\frac{n+1}{2}} da_1 \dots da_n \tag{4.5.34}$$

For a stationary point $\overline{r}$, we have

$$0 = \frac{\delta S_E}{\delta r}(\overline{r}) = V'(\overline{r}) - m\ddot{\overline{r}} \tag{4.5.35}$$

This is identical to the equation of motion for a free particle of mass m with a potential $-V(\overline{r})$. The corresponding energy is the constant

$$E = \frac{1}{2}m\dot{\overline{r}}^2 - V(\overline{r}) \tag{4.5.36}$$

One can pick the r_js such that every r_j is an eigenfunction of the second partial derivative of the action (4.5.30) at the stationary point, with an eigenvalue λ_j

$$V''(\overline{r})r_j(\tau) - m\frac{d^2 r_j}{d\tau^2}(\tau) = \lambda_j r_j(\tau) \tag{4.5.37}$$

In the semiclassical limit, $\hbar \ll 1$, the propagator (4.5.23) can now approximated using the saddle point approximation. Dimension analysis quickly reveals that in a unit system where $\hbar \neq 1$, the transition amplitude equals

$$K_E(r_b, \tau_b; r_a, \tau_a) = \int [Dr(\tau)]\, e^{-\frac{1}{\hbar}S_E[r(\tau)]} \tag{4.5.38}$$

Using equation (B.9.3),

$$\begin{aligned}
K_E(r_b, \tau_b; r_a, \tau_a) &= \left(\frac{-im}{2\pi\Delta t}\right)^{\frac{n+1}{2}} \int da_1 \ldots da_n \, e^{-\frac{1}{\hbar}S_E[r(\tau)]} \\
&\approx \left(\frac{-im}{2\pi\Delta t}\right)^{\frac{n+1}{2}} \prod_{j=1}^{n} \sqrt{\frac{-2\pi\hbar}{\frac{d^2 S_E}{dr_j^2}[\overline{r}]}} e^{-\frac{1}{\hbar}S_E[\overline{r}]} \\
&= \frac{1}{\sqrt{2\pi\hbar}} \left(\frac{\hbar m}{\Delta t}\right)^{\frac{n+1}{2}} e^{-\frac{1}{\hbar}S_E[\overline{r}]} \prod_j \frac{1}{\sqrt{\lambda_j}} \qquad (4.5.39)
\end{aligned}$$

Now the product of eigenvalues equals the determinant of the linear transformation. If we define

$$\omega := \sqrt{|V''(r(\tau))|} \qquad (4.5.40)$$

the previous result equals

$$K_E(r_b, \tau_b; r_a, \tau_a) = \frac{1}{\sqrt{2\pi\hbar}} \left(\frac{\hbar m}{\Delta t}\right)^{\frac{n+1}{2}} e^{-\frac{1}{\hbar}S_E[\overline{r}]} \frac{1}{\sqrt{\det\left(\omega^2 - m\frac{d}{d\tau^2}\right)}} \qquad (4.5.41)$$

$$= \sqrt{\frac{\omega}{2\pi\hbar}} \left(\frac{\hbar m}{\Delta t}\right)^{\frac{n+1}{2}} e^{-\frac{1}{\hbar}S_E[\overline{r}]} e^{-\frac{\omega}{2}(\tau_b - \tau_a)} \qquad (4.5.42)$$

$$= N\sqrt{\frac{\omega}{2\pi\hbar}} e^{-\frac{1}{\hbar}S_E[\overline{r}]} e^{-\frac{\omega}{2}(\tau_b - \tau_a)} \qquad (4.5.43)$$

where N is the scaling constant

$$N := \left(\frac{\hbar m}{\Delta t}\right)^{\frac{n+1}{2}} \qquad (4.5.44)$$

This method is readily applied to a rounded V-shaped potential. However, it can also be applied to a (symmetric) rounded W-shaped potential. If the centre of the potential is at $r = 0$ and the minima are located at $-a$ and a, the transition amplitude

$$K_E(-a, -\tau_a; a, \tau_a) = \langle -a | e^{-\frac{1}{\hbar}H\tau_a} | a \rangle \qquad (4.5.45)$$

is the tunnelling probability for the rounded Λ-shaped middle section of the potential. Wick rotation transforms the rounded W into a rounded M. In the limit $t_a \to \infty$, only the rounded V-shaped middle of the potential remains relevant for this propagator. The remaining domain of the potential cannot be reached within any finite time. At $r = \pm a$, the second derivative of r equals zero, but so does the potential. Since the energy E defined in equation (4.5.36) is a constant, it must also be equal to zero. Then

$$V(r) = \frac{1}{2}m\dot{r}^2 \qquad (4.5.46)$$

so the Euclidean action is

$$S_E = m \int_{-\frac{1}{2}\tau_1}^{\frac{1}{2}\tau_1} d\tau\, \dot{r}^2(\tau) = m \int_{-a}^{a} dr\, \dot{r}(\tau) = \int_{-a}^{a} dr\, \sqrt{2mV(r)} \qquad (4.5.47)$$

which, not coincidentally, corresponds to the WKB transmission amplitude. The pseudo-particle defined in this way is called an instanton, or an anti-instanton if time is inverted (i.e., $t \to -t$), which leads to opposite signs in the integration boundaries.

In section 4.1 we performed a WKB tunnelling calculation with a 'potential' due to the crossing of a space-time horizon. A shell of positive energy was created at the horizon, starting an outward journey while another shell with negative energy moved back into the black hole. To develop a path integral formulation in curved space-time, we need to re-write the classical action in a covariant fashion. However, then it no longer makes sense to write the action as an integral over time. We need to integrate a Lagrangian density over the four dimensions of the space-time manifold.

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{2m} \dot{\rho}^2(t) - U(r(t)) \right] \qquad (4.5.48)$$

where ρ is the mass density and U is the potential density.

For ADM space-time, $\sqrt{-g}$ is given by equation (2.1.8). More specifically, for the Schwarzschild-Painlevé coordinates we used,

$$\sqrt{-g} = r^2 \sin^2\theta \qquad (4.5.49)$$

This is the same result one would have for flat Minkowski space-time in spherical coordinates. For the Minkowski metric $\eta_{\mu\nu}$

$$\eta_{tt} = -1, \quad \eta_{rr} = 1, \quad \eta_{\theta\theta} = r^2, \quad \eta_{\phi\phi} = r^2 \sin^2\theta, \quad \eta_{\mu\nu} = 0 \ (\mu \neq \nu) \qquad (4.5.50)$$

the integration weight is

$$\sqrt{-\eta} = r^2 \sin^2\theta \qquad (4.5.51)$$

Hence integration of a spherically-symmetric function over the hypervolume in Schwarzschild space-time can be written as

$$\int d^4x \sqrt{-g} f(t,r) = 4\pi \int dt\, dr\, r^2 f(t,r) \qquad (4.5.52)$$

We might interpret the positive energy shells in section 4.1 as instantons, and the negative energy shells as anti-instantons, that are separated by the Schwarzschild horizon.

4.6 On the symmetry of the horizon

If the analysis in section 4.1 is physically sound, one would expect a similar analysis in different coordinates to yield the same result. However, the dependence on the Hamilton equation suggests a dependence on the choice of a time coordinate. Borun Chowdhury argues [15] that when space-time is divided in a part within a thin shell and a part outside of it, the mass in general differs on both sides and so should the time coordinate.

Another point made by Chowdhury is that the barrier at the Schwarzschild horizon is asymmetric in the sense that an observer travelling inward across the horizon follows a path that differs from that of an observer travelling outward. More specifically, an incoming observer encounters a barrier whereas an outgoing observer does not. Hence the emission coefficient for incoming shells cannot be expected to equal that of outgoing shells.

$$\Gamma_{\text{in}} \neq \Gamma_{\text{out}} \tag{4.6.1}$$

Chowdhury argues that equation (4.1.13) is only correct for tunnelling through symmetric barriers and we should in fact use the generalisation

$$\Gamma = \Gamma_0 \, e^{-\text{Im} \oint dr \, p_r(r)} = \Gamma_0 \, e^{-\text{Im} \int_{r_+}^{r_-} dr \, p_{r,\text{in}}(r) - \text{Im} \int_{r_-}^{r_+} dr \, p_{r,\text{out}}(r)} \tag{4.6.2}$$

where r_+ and r_- are the radii of the Schwarzschild horizon before and after emission of the shell.

One might try and deal with this problem by using Eddington-Finkelstein coordinates for the Schwarzschild metric. They are introduced in section A.7. This coordinate system provides a different set of coordinates for incoming and outgoing paths, but it preserves the radial coordinate of the Schwarzschild metric and it is continuous and well-defined around the Schwarzschild horizon.

Let us derive the action for a particle tunnelling through a Schwarzschild black hole again using Eddington-Finkelstein coordinates. Specifically, let us choose u as the time coordinate for the outgoing particle and v for the incoming antiparticle. The line element for outgoing paths in coordinates $\{u, r, \theta, \phi\}$ is (equation (A.7.13))

$$ds^2 = -\left(1 - \frac{2M}{r}\right) du^2 - 2du dr + r^2 d\Omega_2^2 \tag{4.6.3}$$

A massless outgoing particle in radial curve (i.e., $d\Omega_2 = 0$) obeys

$$0 = -\left(1 - \frac{2M}{r}\right) du^2 - 2du dr \Rightarrow \frac{dr}{du} = -\frac{1}{2}\left(1 - \frac{2M}{r}\right) \tag{4.6.4}$$

as we would expect. If we accept u as the time coordinate, the tunnelling action is

$$\begin{aligned}\int_{r_-}^{r_+} dr\, p_{r,\text{out}} &= \int_{2M}^{2M-\omega} dr \int_{M}^{M-\omega} dH \frac{dp_r}{dH} = \int_{M}^{M-\omega} dH \int_{2M}^{2M-\omega} dr \frac{du}{dr} \\ &= -2 \int_{M}^{M-\omega} dH \int_{2M}^{2M-\omega} dr \left(1 - \frac{2H}{r}\right)^{-1} \\ &= -2 \int_{M}^{M-\omega} dH \int_{2M}^{2M-\omega} dr \frac{r}{r-2H} = -2\pi i \int_{2M}^{2M-\omega} 2H \\ &= -2\pi i \left((2M-\omega)^2 - 4M^2\right) = 2\pi i\, \omega(2M-\omega) \qquad (4.6.5)\end{aligned}$$

where we have reversed the order of integration in the second equality for the sake of convenience. Since the corresponding antiparticle does not encounter a barrier, the imaginary component of its momentum must equal zero and hence it does not contribute to the particle flux.

$$\Gamma = \Gamma_0\, e^{-4\pi\omega(M-\frac{1}{2}\omega)} \qquad (4.6.6)$$

This is the square root of the result (4.1.26) and the corresponding temperature is half the value associated with that result. Without the antiparticle contribution we assumed to exist in section 4.1, the radiation derived is different from that of the method of section 3 even to first order in energy and from Hawking's result [1]. Evidently, at least one method appears to be incorrect or incomplete.

Our derivations of De Sitter radiations are burdened by the same problem: only the positive energy particle faces the barrier, so the anti-particle contribution incorporated does not contribute to the tunnelling amplitude.

4.7 A contribution due to time

In the preceding section we discussed a problem with the calculation in section 4.1. If the asymmetry of the barrier posed by the Schwarzschild horizon is accounted for, the exponent in the emission rate is half of what we expect it to be. Could it be that we are missing a contribution?

In stationary metrics, one can split the action into a spatial part (due to momentum) and a temporal part (due to energy)

$$S(x^\mu) = S_t(t) + S_s(\vec{x}) \qquad (4.7.1)$$

We used this property in section (3.2). In the preceding sections, however, we only considered the spatial term, as only the imaginary part of the action contributes to tunnelling.

Valeria Akhmedova, Terry Pilling, Andrea de Gill and Douglas Singleton argue that a temporal contribution to the action should be accounted for in a tunnelling calculation when timelike and spacelike coordinates are exchanged at the horizon [16]. As discussed in section 4.2, in De Sitter space-time, only the section of space-time at one side of the horizon is covered by a set of real static coordinates. One might argue that upon emission, a particle goes from the region within the horizon to the region outside of the horizon.

The Schwarzschild space-time metric is similar in this regard, which was also pointed out by Emil T. Akhmedov, Terry Pilling and Douglas Singleton [17]. This is most evident in Kruskal-Szekeres coordinates $\{T, R, \theta, \phi\}$, which are described in appendix A.8. They are defined separately for region I (outside of the event horizon) and region II (within the event horizon). For region I,

$$T = \sqrt{\frac{r}{2M} - 1}\, e^{\frac{r}{4M}} \sinh \frac{t_S}{4M} \tag{4.7.2}$$

$$R = \sqrt{\frac{r}{2M} - 1}\, e^{\frac{r}{4M}} \cosh \frac{t_S}{4M} \tag{4.7.3}$$

For region II, the terms under the square root change sign and the hyperbolic sine and cosine switch places. Alternatively, we might choose to describe region II by the coordinates of region I, but with

$$t_S \to t_S - 2M\pi i \tag{4.7.4}$$

Performing this substitution in the region I Kruskal-Szekeres coordinates yields the region II Kruskal-Szekeres coordinates

$$T = \sqrt{1 - \frac{r}{2M}}\, e^{\frac{r}{4M}} \cosh \frac{t_S}{4M} \tag{4.7.5}$$

$$R = \sqrt{1 - \frac{r}{2M}}\, e^{\frac{r}{4M}} \sinh \frac{t_S}{4M} \tag{4.7.6}$$

Before emission, our shells are within the horizon (nominally described by the region II coordinates), but afterwards, they are soundly in region I. If we are to describe them in any continuous set of coordinates, we need to account for a step sideways on the complex plane at the horizon in Schwarzschild time. Apart from moving in the radial direction with an imaginary momentum, our shell moves in the imaginary time. The momentum associated with time is just the difference in mass-energy in the system, which is $-\omega$ There is an additional contribution in the action

$$S_{t,p} = -\int_{t_e}^{t_e - 2M\pi i} dt\, \omega = 2\pi i M \omega \tag{4.7.7}$$

where t_e is the time of emission, which, as one would expect, turns out to be irrelevant.

In the preceding section, we concluded that the antiparticle does not contribute towards the imaginary part of the radial momentum, since it does not cross the barrier. However, the antiparticle does make the shift from section I to section II. The antiparticle has minus the energy of the particle and it makes a shift from $t_S \to t_S + 2M\pi i$, so it yields the same contribution to the action as the particle. We must therefore add to the action in equation (4.1.22) a term

$$S_t = 4\pi i M\omega \tag{4.7.8}$$

yielding the emission rate

$$\Gamma = \Gamma_0 \, e^{-8\pi\omega(M-\frac{1}{4}\omega)} \tag{4.7.9}$$

In this way we recover Hawking's result to first order in ω. However, the second order result differs from that derived in section 3. Interestingly, it depends on both the initial mass M and the final mass $M - \omega$, but not in equal amounts.

In De Sitter and Schwarzschild-De Sitter space-time, a similar coordinate discontinuity exists. To describe both sides of the horizon by the same set of coordinates, time needs to be shifted by

$$t_s \to t_s - \frac{1}{2}\pi i \tag{4.7.10}$$

at the horizon. This leads to a contribution towards the action equal in magnitude to that obtained from the radial momentum.

$$S_{p,t} = \int_{t_e}^{t_e - \frac{1}{2}\pi i} dt\, \omega = \frac{1}{2}\pi\omega i \tag{4.7.11}$$

An antiparticle created simultaneously, but travelling in the opposite direction does not face a barrier, but it does yield a temporal contribution. It experiences the inverse of the time shift the particle experiences,

$$t_s \to t_s + \frac{1}{2}\pi i \tag{4.7.12}$$

since its energy is $-\omega$, the contribution to the action is identical to that of the particle. Including the temporal contributions, the emission rate becomes

$$\Gamma = \Gamma_0 \, e^{-2\pi\omega\left(1-\frac{1}{4}\omega\right)} \tag{4.7.13}$$

With the additional terms proposed by Akhmedova et al., the first-order results obtained using other methods are recovered while accounting for the asymmetry of the horizon.

4.8 Discussion

The tunnelling method designed by Maulik Parikh and Frank Wilczek [3] proved to be a simple and intuitive way to derive Hawking radiation and, more generally, particle creation due to event horizons. Like the method used in section 3, it takes into account back-reaction while reproducing Stephen Hawking's result [1] to first order in wavelength. However, it does so in an entirely different manner and the second-order term in the result differs. For a black hole of mass M emitting a particle of energy ω, the amplitude is

$$\Gamma(\omega) \propto e^{-8\pi G\omega(M-\frac{1}{2}\omega)} \tag{4.8.1}$$

Whereas Hawking's original result depends only on the initial mass M and the back-reaction method by Per Kraus and Frank Wilczek only depends on the final mass $M - \omega$, this particle flux depends on the final and initial masses in equal amounts. Due to the nature of an emission process, this seems likely.

If we are to take a closer look at the symmetry of the horizon, we see that only the particle that moves outward contributes to the tunnelling flux whereas the anti-particle that travels inward does not. This was pointed out by Borun Chowdhury [15]. However, the shift from Kruskal region I to region II made by a radiated shell appears to introduce an additional contribution [17, 16]. If both are taken into account, we end up with

$$\Gamma(\omega) \propto e^{-8\pi G\omega(M-\frac{1}{4}\omega)} \tag{4.8.2}$$

We were also able to reproduce the well-known result for the De Sitter temperature using the tunnelling approach. We had to take into account the asymmetry of the event horizon [15] as well as a temporal contribution due to a shift in imaginary time at the horizon that both the particle and the antiparticle experience [17]. To account for back-reaction in De Sitter space-time, we looked at a 3 + 1-dimensional Schwarzschild-De Sitter space-time that radiates away enclosed energy due to the anti-particles produced in tandem with the particles that radiate towards observers at a distance. We determined the horizons to second order in the emitted shell energy. This introduced a first-order correction term equal to that derived by Brian Greene, Maulik Parikh and Jan Pieter van der Schaar in a similar derivation [14].

$$\Gamma_{\text{dS}} \propto e^{-2\pi\left(\omega-\frac{1}{2}\omega\right)} \tag{4.8.3}$$

When the asymmetry of the horizon and the proposed additional contributions towards the imaginary part of the action are taken into account, the second-order term is halved,

$$\Gamma_{\text{dS}} \propto e^{-2\pi\left(\omega-\frac{1}{4}\omega\right)} \tag{4.8.4}$$

The Unruh radiation experienced by an observer moving with a constant acceleration through Minkowski space-time can also be described using the tunnelling framework in the Rindler metric. The observer sees a flux coming from the Rindler horizon

$$\Gamma_{\mathrm{U}}(\omega) \propto e^{-2\pi \frac{\omega}{a}} \tag{4.8.5}$$

which is the same result as obtained by Paul Davies [20] and Bill Unruh [21].

5 Conclusion

There are several methods to derive Bekenstein-Hawking radiation. In this paper, we reviewed two methods that account for back-reaction. That is, in the methods used, the energy radiated at the event horizon is compensated for by a reduction of the total amount of mass-energy enclosed in the the black hole, as required by mass-energy conservation.

We looked at two approaches: a quantum field approach in WKB approximation with self-gravitation first derived by Per Kraus and Frank Wilczek [2] and a WKB tunnelling approach developed by Maulik Parikh and Frank Wilczek [3].

The former method is a more formal approach that relies on the action for a system consisting of a black hole of mass $M - \omega$ and a shell of energy ω. Using the Hamilton-Jacobi formalism, the action is quantised to first order in the reduced Planck constant. Finally, the particle flux outside of the horizon is determined by a Bogoliubov transform of the creation and annihilation operators for the modes at the horizon. The amplitude for an observer at a distance looking at a Schwarzschild black hole of mass M is that obtained by Stephen Hawking [1] with a correction term,

$$\Gamma \propto e^{-8\pi G\omega(M-\omega)} \tag{5.1}$$

where ω is the energy and κ is the fraction of particles that reach an observer at infinity. This was derived in section 3.

The other method is a tunnelling-like approach. A shell of energy at the event horizon is produced in tandem with an antiparticle of negative but otherwise equal energy. Whereas the particle travels outward radially, the antiparticle travels in the opposite direction. Semi-classically, the imaginary part of the radial momentum causes the shell to tunnel through the horizon while it shrinks, thereby effectively moving to a position at a finite distance from the horizon. This can be regarded as an instanton-anti-instanton pair production. In section 4.1, we showed that with this method

$$\Gamma \propto e^{-8\pi G\omega(M-\frac{1}{2}\omega)} \tag{5.2}$$

Some concerns over the validity of the tunnelling method have been raised [15]. The Parikh and Wilczek article relies on two identical contributions from the positive and negative energy shells, but since only the outgoing (positive energy) shell needs to cross a barrier, the resulting particle flux should properly be the square root of this result. However, the shift from the Kruskal region I to region II adds a contribution [17]. Taking into account both observations, the result for a Schwarzschild black hole of mass M is

$$\Gamma \propto e^{-8\pi G\omega(M-\frac{1}{4}\omega)} \tag{5.3}$$

as shown in section 4.7.

All three derivations produced the familiar Hawking radiation flux to first order in ω, but to second order in ω, the tunnelling method yields results different from the more formal derivation. Without back-reaction, final and initial mass are indiscriminate and hence the Hawking particle flux only depends on the initial mass M of the black hole. The result obtained by the quantum field calculation is expressed in terms of the final mass $M - \omega$. The 'symmetric' tunnelling method yields a flux that depends on on the initial and final masses in equal amounts, whereas the result of the 'corrected' counterpart does so in the proportion three to one.

Given the nature of the emission process, it seems likely that the particle flux depends on both the initial and the final mass. A priori, there is no reason why these dependencies would need to be equal in magnitude. However, a back-of-the-envelope calculation suggested [23] by Jan Pieter van der Schaar shows that equal contributions are indicated by thermodynamics. Suppose a black hole of mass M radiates a quantum of energy $\delta\omega$ in a thermodynamic distribution of inverse temperature β

$$\Gamma_{\delta\omega} = \Gamma_0 \, e^{-\beta\delta\omega} \tag{5.4}$$

where Γ_0 is a constant. If the total radiation is ω,

$$\omega = \sum_j \delta\omega_j, \tag{5.5}$$

the resulting distribution Γ_ω is

$$\Gamma_\omega = \prod_j \Gamma_{\delta\omega_j} = \Gamma_0 \, e^{-\sum_j \beta_j \delta\omega_j} \tag{5.6}$$

Mass-energy conservation requires that the amount of energy radiated equals minus the change of energy in the black hole

$$d\omega = -dE \tag{5.7}$$

In the small-energy limit ($\delta\omega \to 0$), the distribution reduces to

$$\Gamma_\omega = \Gamma_0 \, e^{\int_M^{M-\omega} dE \, \beta} \tag{5.8}$$

Hawking's original calculation [1], as well as the first-order in radiated energy of the preceding derivations, produces a thermal spectrum with an inverse temperature

$$\beta(E) = 8\pi E \tag{5.9}$$

With this result, the integral in equation 5.8 results in

$$\int_M^{M-\omega} dE \, \beta = 4\pi\omega \, (2M - \omega) \tag{5.10}$$

yielding

$$\Gamma_\omega = \Gamma_0 \, e^{-8\pi\omega\left(M-\frac{1}{2}\omega\right)} \tag{5.11}$$

Equation (5.8) may also be expressed in terms of entropy,

$$\Gamma_\omega = \Gamma_0 \, e^{\int dS} = \Gamma_0 \, e^{S_f - S_i} \tag{5.12}$$

where S_i and S_f are the initial and final entropies, respectively.

This is only a back-of-the-envelope calculation that is merely based on thermodynamics and mass-energy conservation, but it shows rather nicely that one would expect a quadratic correction of half magnitude and opposite sign to the first-order term. In other words, the symmetric tunnelling result is the most likely result, even though its physical basis appears to be the least solid of the three results discussed.

It is difficult to judge which result is most likely to be correct within the relevant regime. The tunnelling approach is a nice intuitive way to look at particle production, but it is limited by the fact that fundamentally, it is a quantum mechanical approach to a problem of quantum fields. The more formal approach connects better to our physical understanding of the problem, but it too involves several approximations and the second-order term in the resulting flux does not seem as credible as that obtained by tunnelling. Since our results are only to second order in emitted shell energy, unknown higher order correction terms might also play a role.

Any correction to Hawking's result produces a flux that is not completely thermal. This means that fundamentally, information may be contained by the particles, which might be a step towards a possible solution to the black hole information paradox. However, that such a correction term can be obtained from thermodynamics is quite likely an indication that there may be little room for any physical information in the particle flux. Since thermodynamics is a property of macrostates, one would expect that microstates that we do not understand as of yet play a role. Further understanding of these is needed to draw conclusions about the implications for the information paradox.

We also applied the tunnelling method to $(3+1)$-dimensional De Sitter space-time. Accounting for back-reaction results in a particle flux distribution from the De Sitter horizons that equals the well-known standard result to first order in the emitted particle energy with a correction term

$$\Gamma_{\text{dS}} \propto e^{-2\pi\omega\left(1-\frac{1}{2}\omega\right)} \tag{5.13}$$

The second order term differs by a factor two from the one obtained by Brian Greene, Maulik Parikh and Jan Pieter van der Schaar in a derivation that does not take into account the concerns over the asymmetry of the horizon and the imaginary time contribution [14]. This is in line with the result for Hawking radiation from Schwarzschild black holes.

Finally, we used the tunnelling framework to derive the distribution of Unruh radiation for the Rindler metric, which is the perspective of a linearly accelerated observer in flat Minkowski space-time. For an observer with a proper acceleration a, the amplitude is

$$\Gamma_{\mathrm{U}} \propto e^{-2\pi \frac{\omega}{a}} \tag{5.14}$$

This is identical to the results obtained by Paul Davies [20] and Bill Unruh [21].

The fundamental laws of physics require us to account for back-reaction if we are to derive the particle flux in any system that is dependent on the mass contained on one side of the horizon. Terms of second and higher order in wavelength should thus be expected to exist in the particle flux from a Schwarzschild black hole as well as that of a De Sitter horizon. The tunnelling approach is a nice and intuitive way to derive the particle production while accounting for back-reaction and it produces a credible result. However, as long as we do not completely understand the microstructure of the Schwarzschild black hole, the most reliable result can only be the one built on the most solid physical foundations. This is the Kraus and Wilczek method used in section 3.

A Space-time technicalities

In section 2, we introduced the ADM form metric. We discussed its properties and we determined some quantities, but we left some laborious derivations to this appendix.

A.1 Another expression for the extrinsic curvature tensor

The extrinsic curvature tensor (equation (2.4.6)) can also be written in a more explicit form. To obtain it, we start with the Lie derivative in the form with covariant derivatives (equation (2.4.4)).

$$\begin{aligned} K_{\mu\nu} &= \frac{1}{2}\left[n^\lambda P_{\mu\nu;\lambda} + n^\lambda{}_{;\mu} P_{\lambda\nu} + n^\lambda{}_{;\nu} P_{\mu\lambda}\right] \\ &= \frac{1}{2}\Big[n^\lambda g_{\mu\nu;\lambda} - \sigma n^\lambda n_{\mu;\lambda} n_\nu - \sigma n^\lambda n_\mu n_{\nu;\lambda} + n^\lambda{}_{;\mu} g_{\lambda\nu} - \sigma n^\lambda{}_{;\mu} n_\lambda n_\nu \\ &\qquad + n^\lambda{}_{;\nu} g_{\mu\lambda} - \sigma n_{\mu;\nu} n_\mu n_\lambda\Big] \\ &= \frac{1}{2}\left[n_{\mu;\nu} + n_{\nu;\mu} - \sigma n^\lambda n_{\mu;\lambda} n_\nu - \sigma n^\lambda n_\mu n_{\nu;\lambda} - \sigma n^\lambda{}_{;\mu} n_\lambda n_\nu - \sigma n^\lambda{}_{;\nu} n_\mu n_\lambda\right] \end{aligned} \tag{A.1.1}$$

However, $K_{\mu\nu}$ is symmetric, so

$$n^\lambda{}_{;\mu} n_\lambda n_\nu + n^\lambda{}_{;\nu} n_\mu n_\lambda = \frac{1}{2}\left[\sigma_{;\mu} n_\nu + \sigma_{;\nu} n_\mu\right] = 0 \tag{A.1.2}$$

and we might as well write

$$K_{\mu\nu} = n_{\nu;\mu} - \sigma n_\mu n^\lambda n_{\nu;\lambda} \tag{A.1.3}$$

This expression is used in appendix A.2 to derive Gauss' equation.

A.2 Deriving Gauss' equation

Gauss' equation relates the Riemann curvature tensor on a manifold M with metric $g_{\mu\nu}$ to its counterpart on a submanifold Σ with induced metric h_{ij}. In this appendix, we derive Gauss' equation and from it an equation for the Ricci curvature scalar on M. This analysis is based upon the treatment of hypersurfaces in Carroll [8] and Wald [7].

Any vector can be projected along a hypersurface by applying the projection tensor. The hypersurface covariant derivative operator $\widehat{D}_\rho$ of an (m, n) tensor T is therefore given by

$$\widehat{D}_\rho T^{\mu_1\ldots\mu_m}{}_{\nu_1\ldots\nu_n} = P^\alpha{}_\rho P^{\mu_1}{}_{\beta_1} \cdots P^{\mu_m}{}_{\beta_m} P^{\gamma_1}{}_{\nu_n} \cdots P^{\gamma_n}{}_{\nu_n} T^{\beta_1\ldots\beta_m}{}_{\gamma_1\ldots\gamma_n;\alpha} \tag{A.2.1}$$

The commutator of the covariant derivative of a vector field V^μ effectively defines the Riemann tensor on the hypersurface.

$$\left[\widehat{D}_\mu, \widehat{D}_\nu\right] V^\rho = \widehat{\mathcal{R}}^\rho{}_{\sigma\mu\nu} V^\sigma \tag{A.2.2}$$

For a one-form ω_μ, we then have

$$\left[\widehat{D}_\mu, \widehat{D}_\nu\right] \omega_\rho = P_{\rho\alpha} \widehat{\mathcal{R}}^\alpha{}_{\sigma\mu\nu} P^{\sigma\beta} \omega_\beta = \widehat{\mathcal{R}}_\rho{}^\beta{}_{\mu\nu} \omega_\beta \tag{A.2.3}$$

while from equation (A.2.1) it follows that

$$\begin{aligned}
\widehat{D}_\mu \widehat{D}_\nu \omega_\rho &= \widehat{D}_\mu \left[P^\alpha{}_\nu P^\beta{}_\rho \omega_{\beta;\alpha}\right] = P^\gamma{}_\mu P^\epsilon{}_\rho P^\delta{}_\nu \left[P^\alpha{}_\delta P^\beta{}_\epsilon \omega_{\beta;\alpha}\right]_{;\gamma} \\
&= P^\gamma{}_\mu P^\alpha{}_\nu P^\beta{}_\rho \omega_{\beta;\alpha;\gamma} + P^\gamma{}_\mu P^\delta{}_\nu P^\alpha{}_{\delta;\gamma} P^\beta{}_\rho \omega_{\beta;\alpha} + P^\gamma{}_\mu P^\epsilon{}_\rho P^\alpha{}_\nu P^\beta{}_{\epsilon;\gamma} \omega_{\beta;\alpha} \\
&= P^\gamma{}_\mu P^\alpha{}_\nu P^\beta{}_\rho \omega_{\beta;\alpha;\gamma} + P^\gamma{}_\mu P^\delta{}_\nu [g^\alpha{}_\delta - \sigma n^\alpha n_\delta]_{;\gamma} P^\beta{}_\rho \omega_{\beta;\alpha} \\
&\quad + P^\gamma{}_\mu P^\epsilon{}_\rho P^\alpha{}_\nu \left[g^\beta{}_\epsilon - \sigma n^\beta n_\epsilon\right]_{;\gamma} \omega_{\beta;\alpha} \\
&= P^\gamma{}_\mu P^\alpha{}_\nu P^\beta{}_\rho \omega_{\beta;\alpha;\gamma} - \sigma P^\gamma{}_\mu P^\delta{}_\nu \left[n^\alpha n_{\delta;\gamma} + n^\alpha{}_{;\gamma} n_\delta\right] P^\beta{}_\rho \omega_{\beta;\alpha} \\
&\quad - \sigma P^\gamma{}_\mu P^\epsilon{}_\rho P^\alpha{}_\nu \left[n^\beta n_{\epsilon;\gamma} + n^\beta{}_{;\gamma} n_\epsilon\right] \omega_{\beta;\alpha}
\end{aligned} \tag{A.2.4}$$

where we have used metric compatibility in the last identity. The projection tensor will project any vector V^μ orthogonal to the normal vector n^μ, so with equation (2.4.2)

$$P^\lambda{}_\mu V^\mu n_\lambda = P_{\lambda\mu} V^\mu n^\lambda = 0 \tag{A.2.5}$$

Hence the second term in both of the square brackets of the result (A.2.4) vanishes. Furthermore, from equation (A.1.3) we obtain

$$P^\lambda{}_\mu n_{\nu;\lambda} = \left[g^\lambda{}_\mu - \sigma n^\lambda n_\mu\right] n_{\nu;\lambda} = n_{\nu;\mu} - \sigma n^\lambda n_\mu n_{\nu;\lambda} = K_{\mu\nu} \tag{A.2.6}$$

For the covariant derivative of the vector counterpart, we have

$$P^\lambda{}_\mu n^\nu{}_{;\lambda} = P^\lambda{}_\mu \left(g^{\nu\tau} n_\tau\right)_{;\lambda} = g^{\nu\tau} P^\lambda{}_\mu n_{\tau;\lambda} = g^{\nu\tau} K_{\tau\mu} = K^\nu{}_\mu \tag{A.2.7}$$

Finally, we can simplify somewhat by noting that

$$\begin{aligned}
P^\delta{}_\nu K_{\mu\delta} &= \left[g^\delta{}_\nu - \sigma n^\delta n_\nu\right] K_{\mu\delta} = \left[g^\delta{}_\nu - \sigma n^\delta n_\nu\right] K_{\delta\mu} \\
&= K_{\nu\mu} - \sigma n^\delta n_\nu n_{\mu;\delta} + n^\delta n_\nu n_\delta n^\lambda n_{\mu;\lambda} \\
&= K_{\mu\nu} - \sigma n^\delta n_\nu n_{\mu;\delta} + \sigma n_\nu n^\lambda n_{\mu;\lambda} = K_{\mu\nu}
\end{aligned} \tag{A.2.8}$$

where we have used the symmetry of the extrinsic curvature tensor in the second and fourth equalities. When combined, these results allows us to rewrite the expression for $\widehat{D}_\mu \widehat{D}_\nu \omega_\rho$

$$\begin{aligned}
\widehat{D}_\mu \widehat{D}_\nu \omega_\rho &= P^\gamma{}_\mu P^\alpha{}_\nu P^\beta{}_\rho \omega_{\beta;\alpha;\gamma} - \sigma P^\beta{}_\rho K_{\mu\nu} n^\alpha \omega_{\beta;\alpha} \\
&\quad - \sigma P^\alpha{}_\nu K_{\mu\rho} n^\beta \omega_{\beta;\alpha}
\end{aligned} \tag{A.2.9}$$

The third term contains the inner product of ω with a covariant derivative. If we rewrite using the Leibniz product rule, one of the terms turns out to equal the mixed form of the extrinsic curvature tensor as derived in equation (A.2.7) while the other term equals zero for it is an inner product between orthogonal tensors.

$$P^{\alpha}{}_{\nu} n^{\beta} \omega_{\beta;\alpha} = P^{\alpha}{}_{\nu} \left[\left(n^{\beta} \omega_{\beta} \right)_{;\alpha} - n^{\beta}{}_{;\alpha} \omega_{\beta} \right] = -K^{\beta}{}_{\nu} \omega_{\beta} \tag{A.2.10}$$

With the previous results, the hypersurface Riemann tensor applied on a one-form as written down in (A.2.3) becomes

$$\begin{aligned} \widehat{\mathcal{R}}_{\rho}{}^{\beta}{}_{\mu\nu} \omega_{\beta} &= \left[P^{\gamma}{}_{\mu} P^{\alpha}{}_{\nu} - P^{\gamma}{}_{\nu} P^{\alpha}{}_{\mu} \right] P^{\beta}{}_{\rho} \omega_{\beta;\alpha;\gamma} - \sigma P^{\beta}{}_{\rho} \left[K_{\mu\nu} - K_{\nu\mu} \right] n^{\alpha} \omega_{\beta;\alpha} \\ &\quad - \sigma \left[K_{\nu\rho} K^{\beta}{}_{\mu} - K_{\mu\rho} K^{\beta}{}_{\nu} \right] \omega_{\beta} \\ &= P^{\gamma}{}_{\mu} P^{\alpha}{}_{\nu} P^{\beta}{}_{\rho} \mathcal{R}_{\beta}{}^{\delta}{}_{\gamma\alpha} \omega_{\delta} + \sigma \left[K_{\mu\rho} K^{\beta}{}_{\nu} - K_{\nu\rho} K^{\beta}{}_{\mu} \right] \omega_{\beta} \end{aligned} \tag{A.2.11}$$

Hence the hypersurface Riemann tensor in this form can be written as

$$\widehat{\mathcal{R}}_{\rho}{}^{\sigma}{}_{\mu\nu} = P^{\gamma}{}_{\mu} P^{\alpha}{}_{\nu} P^{\beta}{}_{\rho} P^{\sigma}{}_{\delta} \mathcal{R}_{\beta}{}^{\delta}{}_{\gamma\alpha} + \sigma \left[K_{\mu\rho} K^{\sigma}{}_{\nu} - K_{\nu\rho} K^{\sigma}{}_{\mu} \right]$$

If we now raise the first index, lower the second index and relabel the dummy indices, we obtain Gauss' equation

$$\widehat{\mathcal{R}}^{\rho}{}_{\sigma\mu\nu} = P^{\rho}{}_{\alpha} P^{\beta}{}_{\sigma} P^{\gamma}{}_{\mu} P^{\delta}{}_{\nu} \mathcal{R}^{\alpha}{}_{\beta\gamma\delta} + \sigma \left[K^{\rho}{}_{\mu} K_{\sigma\nu} - K^{\rho}{}_{\nu} K_{\sigma\mu} \right] \tag{A.2.12}$$

which in turn provides us with a relation between the Ricci tensor of ADM space-time and that of a constant-time slice

$$\begin{aligned} \widehat{\mathcal{R}} &= P^{\nu\sigma} \widehat{\mathcal{R}}^{\rho}{}_{\sigma\rho\nu} = \left[g^{\nu\sigma} - \sigma n^{\nu} n^{\sigma} \right] \widehat{\mathcal{R}}^{\rho}{}_{\sigma\rho\nu} \\ &= g^{\nu\sigma} P^{\rho}{}_{\alpha} P^{\beta}{}_{\sigma} P^{\gamma}{}_{\rho} P^{\delta}{}_{\nu} \mathcal{R}^{\alpha}{}_{\beta\gamma\delta} + \sigma g^{\nu\sigma} \left[K^{\rho}{}_{\rho} K_{\sigma\nu} - K^{\rho}{}_{\nu} K_{\sigma\rho} \right] \\ &\quad - \sigma P^{\rho}{}_{\alpha} P^{\beta}{}_{\sigma} P^{\gamma}{}_{\rho} P^{\delta}{}_{\nu} \mathcal{R}^{\alpha}{}_{\beta\gamma\delta} n^{\nu} n^{\sigma} - n^{\nu} n^{\sigma} \left[K^{\rho}{}_{\mu} K_{\sigma\nu} - K^{\rho}{}_{\nu} K_{\sigma\mu} \right] \end{aligned} \tag{A.2.13}$$

The normalised normal vector is orthogonal to the extrinsic curvature tensor

$$n^{\sigma} K_{\sigma\mu} = n^{\sigma} n_{\mu;\sigma} - \sigma n^{\sigma} n_{\sigma} n^{\rho} n_{\mu;\rho} = 0 \tag{A.2.14}$$

so the last two terms of equation (A.2.13) are in fact equal to zero. The second Riemann tensor is projected orthogonally to the normal tensors it is contracted with, hence that term also equals zero. Furthermore, in the first term two projection tensors can be contracted directly and the metric tensor results in another contraction.

$$\begin{aligned}
\widehat{\mathcal{R}} &= P^{\gamma}{}_{\alpha} P^{\beta\nu} P^{\delta}{}_{\nu} \mathcal{R}^{\alpha}{}_{\beta\gamma\delta} + \sigma g^{\nu\sigma} \left[K^{\rho}{}_{\rho} K_{\sigma\nu} - K^{\rho}{}_{\nu} K_{\sigma\rho}\right] \\
&= \left[g^{\gamma}{}_{\alpha} - \sigma n^{\gamma} n_{\alpha}\right] \left[g^{\beta\delta} - \sigma n^{\beta} n^{\delta}\right] \mathcal{R}^{\alpha}{}_{\beta\gamma\delta} + \sigma \left[K^{\rho}{}_{\rho} K^{\nu}{}_{\nu} - K^{\rho\sigma} K_{\sigma\rho}\right] \\
&= \mathcal{R}^{\alpha\beta}{}_{\alpha\beta} - \sigma \mathcal{R}^{\alpha}{}_{\beta\alpha\delta} n^{\beta} n^{\delta} - \sigma g^{\beta\delta} \mathcal{R}_{\alpha\beta\gamma\delta} n^{\gamma} n^{\alpha} \\
&\quad + \sigma \left[K^2 - K^{\rho\sigma} K_{\rho\sigma}\right] \\
&= \mathcal{R} - \sigma \mathcal{R}_{\beta\delta} n^{\beta} n^{\delta} - \sigma g^{\beta\delta} \mathcal{R}_{\beta\alpha\delta\gamma} n^{\gamma} n^{\alpha} + \sigma \left[K^2 - K^{\rho\sigma} K_{\sigma\rho}\right] \\
&= \mathcal{R} - 2\sigma \mathcal{R}_{\mu\nu} n^{\mu} n^{\nu} + \sigma \left[K^2 - K^{\mu\nu} K_{\mu\nu}\right] \qquad \text{(A.2.15)}
\end{aligned}$$

where K denotes the scalar quantity $K^{\mu}{}_{\mu}$. Let us determine $\mathcal{R}_{\mu\nu} n^{\mu} n^{\nu}$ by exploiting the way the Riemann tensor acts on a vector

$$\begin{aligned}
\mathcal{R}_{\mu\nu} n^{\mu} n^{\nu} &= \mathcal{R}^{\rho}{}_{\mu\rho\nu} n^{\mu} n^{\nu} = \left[n^{\rho}{}_{;\rho;\nu} - n^{\rho}{}_{;\nu;\rho}\right] n^{\nu} \\
&= \left(n^{\rho}{}_{;\rho} n^{\nu}\right)_{;\nu} - n^{\rho}{}_{;\rho} n^{\nu}{}_{;\nu} - \left(n^{\rho}{}_{;\nu} n^{\nu}\right)_{;\rho} + n^{\rho}{}_{;\nu} n^{\nu}{}_{;\rho} \qquad \text{(A.2.16)}
\end{aligned}$$

However,

$$\begin{aligned}
K^{\mu\nu} K_{\mu\nu} &= g^{\mu\rho} g^{\nu\sigma} \left(n_{\sigma;\rho} - \sigma n_{\rho} n^{\kappa} n_{\sigma;\kappa}\right) \left(n_{\mu;\nu} - \sigma n_{\nu} n^{\lambda} n_{\mu;\lambda}\right) \\
&= n^{\rho}{}_{;\nu} n^{\nu}{}_{;\rho} + n^{\mu} n^{\kappa} n^{\nu}{}_{;\kappa} n_{\nu} n^{\lambda} n_{\mu;\lambda} - \sigma n_{\mu;\nu} n^{\mu} n^{\kappa} n^{\nu}{}_{;\kappa} - \sigma n_{\sigma;\rho} n^{\sigma} n^{\lambda} n^{\rho}{}_{;\lambda} \\
&= n^{\rho}{}_{;\nu} n^{\nu}{}_{;\rho} + n^{\mu} n^{\kappa} n^{\nu}{}_{;\kappa} n_{\nu} n^{\lambda} n_{\mu;\lambda} - 2\sigma n_{\mu;\nu} n^{\mu} n^{\kappa} n^{\nu}{}_{;\kappa} \qquad \text{(A.2.17)}
\end{aligned}$$

and

$$\begin{aligned}
K^2 &= \left(K^{\mu}{}_{\mu}\right)^2 = \left(n^{\mu}{}_{;\mu} - \sigma n^{\lambda} n_{\mu} n^{\mu}{}_{;\lambda}\right)^2 \\
&= n^{\mu}{}_{;\mu} n^{\nu}{}_{;\nu} + n^{\kappa} n_{\mu} n^{\mu}{}_{;\kappa} n^{\lambda} n_{\nu} n^{\nu}{}_{;\lambda} - 2\sigma n^{\mu}{}_{;\mu} n^{\lambda} n_{\nu} n^{\nu}{}_{;\lambda} \qquad \text{(A.2.18)}
\end{aligned}$$

so the Ricci tensor applied on two normalised normal factors equals

$$\mathcal{R}_{\mu\nu} n^{\mu} n^{\nu} = K^{\mu\nu} K_{\mu\nu} - K^2 + \left(n^{\rho}{}_{;\rho} n^{\mu} - n^{\mu}{}_{;\nu} n^{\nu}\right)_{;\mu}$$

Inserting this result in equation (A.2.15) finally yields the scalar curvature of the entire space-time in terms of the scalar curvature of the submanifold, scalar contractions of the extrinsic curvature tensor and covariant derivatives of the normalised normal vector.

$$\mathcal{R} = \widehat{\mathcal{R}} + \sigma \left[K^{\mu\nu} K_{\mu\nu} - K^2\right] + 2\sigma \left(n^{\mu}{}_{;\nu} n^{\nu} - n^{\nu}{}_{;\nu} n^{\mu}\right)_{;\mu} \qquad \text{(A.2.19)}$$

This result is used in appendix A.5 to determine the Ricci scalar curvature for a general ADM form metric.

A.3 The curvature scalar of a constant-time submanifold

To determine the Ricci curvature scalar for a constant-time submanifold, we rewrite equation (2.2.4)

$$\widehat{\mathcal{R}} = h^{ij}\left[\widehat{\Gamma}^k_{ij,k} - \widehat{\Gamma}^k_{ki,j} + \widehat{\Gamma}^k_{ks}\widehat{\Gamma}^s_{ji} - \widehat{\Gamma}^k_{js}\widehat{\Gamma}^s_{ki}\right] \tag{A.3.1}$$

Since the inverse induced metric is diagonal, only terms with $j = i$ contribute to the curvature scalar of the induced metric. With the induced metric in equations (2.3.7) and the Christoffel connection in equations (2.3.9), we have

$$\begin{aligned}
\widehat{\mathcal{R}} &= h^{ii}\left[\widehat{\Gamma}^k_{ii,k} - \widehat{\Gamma}^k_{ki,i} + \widehat{\Gamma}^k_{ks}\widehat{\Gamma}^s_{ii} - \widehat{\Gamma}^k_{is}\widehat{\Gamma}^s_{ki}\right] \\
&= h^{rr}\Big[\widehat{\Gamma}^r_{rr,r} - \widehat{\Gamma}^r_{rr,r} - \widehat{\Gamma}^\theta_{\theta r,r} - \widehat{\Gamma}^\phi_{\phi r,r} + \left(\widehat{\Gamma}^r_{rr} + \widehat{\Gamma}^\theta_{\theta r} + \widehat{\Gamma}^\phi_{\phi r}\right)\widehat{\Gamma}^r_{rr} \\
&\quad - \left(\widehat{\Gamma}^r_{rr}\right)^2 - \left(\widehat{\Gamma}^\theta_{\theta r}\right)^2 - \left(\widehat{\Gamma}^\phi_{\phi r}\right)^2\Big] \\
&\quad + h^{\theta\theta}\left[\widehat{\Gamma}^r_{\theta\theta,r} - \widehat{\Gamma}^\phi_{\phi\theta,\theta} + \left(\widehat{\Gamma}^r_{rr} + \widehat{\Gamma}^\theta_{\theta r} + \widehat{\Gamma}^\phi_{\phi r}\right)\widehat{\Gamma}^r_{\theta\theta} - 2\widehat{\Gamma}^\theta_{\theta r}\widehat{\Gamma}^r_{\theta\theta} - \left(\widehat{\Gamma}^\phi_{\phi\theta}\right)^2\right] \\
&\quad + h^{\phi\phi}\Big[\widehat{\Gamma}^r_{\phi\phi,r} + \widehat{\Gamma}^\theta_{\phi\phi,\theta} + \left(\widehat{\Gamma}^r_{rr} + \widehat{\Gamma}^\theta_{\theta r} + \widehat{\Gamma}^\phi_{\phi r}\right)\widehat{\Gamma}^r_{\phi\phi} + \widehat{\Gamma}^\phi_{\phi\theta}\widehat{\Gamma}^\theta_{\phi\phi} - 2\widehat{\Gamma}^\phi_{\phi r}\widehat{\Gamma}^r_{\phi\phi} \\
&\quad - 2\widehat{\Gamma}^\phi_{\phi\theta}\widehat{\Gamma}^\theta_{\phi\phi}\Big] \\
&= L^{-2}\left[-2\left(\frac{R'}{R}\right)' + \frac{L'}{L}\left(\frac{L'}{L} + 2\frac{R'}{R}\right) - \left(\frac{L'}{L}\right)^2 - 2\left(\frac{R'}{R}\right)^2\right] \\
&\quad + R^{-2}\left[-\left(\frac{RR'}{L^2}\right)' + \frac{1}{\sin^2\theta} - \frac{RR'}{L^2}\left(\frac{L'}{L} + 2\frac{R'}{R}\right) + 2\frac{R'^2}{L^2} - \frac{\cos^2\theta}{\sin^2\theta}\right] \\
&\quad + R^{-2}\sin^{-2}\theta\Big[-\left(\frac{RR'}{L^2}\right)'\sin^2\theta + \sin^2\theta - \cos^2\theta \\
&\quad - \frac{RR'}{L^2}\sin^2\theta\left(\frac{L'}{L} + 2\frac{R'}{R}\right) - \cos^2\theta + 2\frac{R'^2}{L^2}\sin^2\theta + 2\cos^2\theta\Big] \\
&= L^{-2}\left[-2\frac{R''}{R} + 2\frac{R'^2}{R^2} + 2\frac{L'R'}{LR} - 2\frac{R'^2}{R^2}\right] \\
&\quad + R^{-2}\left[-2\left(\frac{RR'}{L^2}\right)' - 2\frac{RR'}{L^2}\left(\frac{L'}{L} + 2\frac{R'}{R}\right) + 4\frac{R'^2}{L^2} + 2\right] \\
&= -2\frac{R''}{L^2R} + 2\frac{L'R'}{L^3R} - 2\frac{R''}{L^2R} - 2\frac{R'^2}{L^2R^2} + 4\frac{L'R'}{L^3R} \\
&\quad - 2\frac{L'R'}{L^3R} - 4\frac{R'^2}{L^2R^2} + 4\frac{R'^2}{L^2R^2} + \frac{2}{R^2} \\
&= -4\frac{R''}{L^2R} + 4\frac{L'R'}{L^3R} - 2\frac{R'^2}{L^2R^2} + \frac{2}{R^2}
\end{aligned} \tag{A.3.2}$$

A.4 The extrinsic curvature tensor for ADM space-time

To obtain the scalar curvature of the ADM metric, one needs the components of the extrinsic curvature tensor as defined by equation (2.4.6). The first term equals zero for all components of the projection tensor that equal zero; the other terms then equal zero because $P_{\lambda\theta}$ and $P_{\lambda\phi}$ are only non-zero when λ equals θ and ϕ, respectively, but then n^λ is equal to zero.

$$\begin{aligned}
K_{tt} &= \frac{1}{2}\left[n^t P_{tt,t} + n^r P_{tt,r} + n^t{}_{,t} P_{tt} + n^r{}_{,t} P_{rt} + n^t{}_{,t} P_{tt} + n^r{}_{,t} P_{tr}\right] \\
&= \frac{1}{2} n^t P_{tt,t} + \frac{1}{2} n^r P_{tt,r} + n^t{}_{,t} P_{tt} + n^r{}_{,t} P_{rt} \\
&= -\frac{1}{N^t}\left(L^2 N^r \dot{N}^r + L\dot{L} N^{r2}\right) + \frac{N^r}{N^t}\left(L^2 N^r N^{r\prime} + LL' N^{r2}\right) \\
&\quad + \frac{\dot{N}^t}{N^{t2}} L^2 N^{r2} + \left(\frac{\dot{N}^r}{N^t} - \frac{N^r \dot{N}^t}{N^{t2}}\right) L^2 N^r \\
&= \frac{N^{r2}}{N^t} N^{r\prime} L^2 + \frac{N^{r3}}{N^t} LL' - \frac{N^{r2}}{N^t} L\dot{L} \qquad \text{(A.4.1)} \\
K_{rr} &= \frac{1}{2}\left[n^t P_{rr,t} + n^r P_{rr,r} + n^t{}_{,r} P_{tr} + n^r{}_{,r} P_{rr} + n^t{}_{,r} P_{tr} + n^r{}_{,r} P_{rr}\right] \\
&= \frac{1}{2} n^t P_{rr,t} + \frac{1}{2} n^r P_{rr,r} + n^r{}_{,r} P_{rr} + n^t{}_{,r} P_{tr} \\
&= -\frac{L\dot{L}}{N^t} + \frac{N^r}{N^t} LL' + \left(\frac{N^{r\prime}}{N^t} - \frac{N^r N^{t\prime}}{N^{t2}}\right) L^2 + \frac{N^{t\prime}}{N^{t2}} L^2 N^r \\
&= \frac{N^r}{N^t} LL' - \frac{1}{N^t} L\dot{L} + \frac{N^{r\prime}}{N^t} L^2 \qquad \text{(A.4.2)} \\
K_{rt} &= K_{tr} \\
&= \frac{1}{2}\left[n^t P_{tr,t} + n^r P_{tr,r} + n^t{}_{,t} P_{tr} + n^r{}_{,t} P_{rr} + n^t{}_{,r} P_{tt} + n^r{}_{,r} P_{tr}\right] \\
&= -\frac{1}{N^t}\left(\frac{1}{2} L^2 \dot{N}^r + L\dot{L} N^r\right) + \frac{N^r}{N^t}\left(\frac{1}{2} L^2 N^{r\prime} + LL' N^r\right) \\
&\quad + \frac{1}{2}\left(\frac{\dot{N}^t}{N^{t2}} + \left(\frac{N^{r\prime}}{N^t} - \frac{N^r N^{t\prime}}{N^{t2}}\right)\right) L^2 N^r + \frac{1}{2}\left(\frac{\dot{N}^r}{N^t} - \frac{N r \dot{N}^t}{N^{t2}}\right) L^2 \\
&\quad + \frac{1}{2}\frac{N^{t\prime}}{N^{t2}} L^2 N^{r2} \\
&= \frac{N^r N^{r\prime}}{N^t} L^2 - \frac{N^r}{N^t} L\dot{L} + \frac{N^{r2}}{N^t} LL' \\
K_{\theta\theta} &= \frac{1}{2}\left[n^t P_{\theta\theta,t} + n^r P_{\theta\theta,r}\right] = -\frac{R\dot{R}}{N^t} + \frac{N^r}{N^t} RR' \qquad \text{(A.4.3)} \\
K_{\phi\phi} &= \frac{1}{2}\left[n^t P_{\phi\phi,t} + n^r P_{\phi\phi,r}\right] = -\frac{R\dot{R}}{N^t}\sin^2\theta + \frac{N^r}{N^t} RR' \sin^2\theta \qquad \text{(A.4.4)}
\end{aligned}$$

A.5 The curvature scalar of ADM space-time

Equation (A.2.19) provides a recipe for the Ricci curvature scalar in terms of the Ricci scalar of a submanifold, the extrinsic curvature tensor and a total covariant derivative wee need not worry about for the moment. In appendices (A.3) and (A.4) we determined the ingredients, so let us now perform the algebra.

The trace of the extrinsic curvature tensor contains many terms, but fortunately, most either equal zero or equal another term due to symmetry. Inspection of equations (2.5.5) - (2.5.9) reveals that for ADM space-time, the trace contains the following non-zero terms.

$$\begin{aligned} K^{\mu\nu}K_{\mu\nu} &= g^{\mu\rho}g^{\nu\sigma}K_{\rho\sigma}K_{\mu\nu} \\ &= \left(g^{tt}\right)^2(K_{tt})^2 + (g^{rr})^2(K_{rr})^2 + 2\left(g^{tr}\right)^2 K_{tt}K_{rr} + 4g^{tt}g^{tr}K_{tt}K_{tr} \\ &\quad + 4g^{rr}g^{rt}K_{rr}K_{rt} + 2g^{tt}g^{rr}(K_{tr})^2 + 2\left(g^{tr}\right)^2(K_{tr})^2 \\ &\quad + \left(g^{\theta\theta}\right)^2(K_{\theta\theta})^2 + \left(g^{\phi\phi}\right)^2(K_{\phi\phi})^2 \end{aligned} \tag{A.5.1}$$

We will also need the square of the extrinsic curvature scalar. It consists of the following non-zero terms.

$$\begin{aligned} K^2 &= (g^{\mu\nu}K_{\mu\nu})^2 \\ &= \left(g^{tt}K_{tt}\right)^2 + (g^{rr}K_{rr})^2 + 4\left(g^{tr}K_{tr}\right)^2 + \left(g^{\theta\theta}K_{\theta\theta}\right)^2 + \left(g^{\phi\phi}K_{\phi\phi}\right)^2 \\ &\quad + 2g^{tt}g^{rr}K_{tt}K_{rr} + 4g^{tt}g^{tr}K_{tt}K_{tr} + 4g^{rr}g^{rt}K_{rr}K_{tr} \\ &\quad + 2\left(g^{tt}K_{tt} + g^{rr}K_{rr} + 2g^{tr}K_{tr}\right)\left(g^{\theta\theta}K_{\theta\theta} + g^{\phi\phi}K_{\phi\phi}\right) \\ &\quad + 2g^{\theta\theta}g^{\phi\phi}K_{\theta\theta}K_{\phi\phi} \end{aligned} \tag{A.5.2}$$

where some terms remain factorised for the sake of clarity as well as further computational convenience. Fortunately, many of the terms in the above equations cancel.

$$\begin{aligned} K^{\mu\nu}K_{\mu\nu} - K^2 &= 2\left(g^{tr}\right)^2 K_{tt}K_{rr} + 2g^{tt}g^{rr}(K_{tr})^2 - 2\left(g^{tr}\right)^2(K_{tr})^2 \\ &\quad - 2g^{tt}g^{rr}K_{tt}K_{rr} - 2g^{\theta\theta}g^{\phi\phi}K_{\theta\theta}K_{\phi\phi} \\ &\quad - 2\left(g^{tt}K_{tt} + g^{rr}K_{rr} + 2g^{tr}K_{tr}\right)\left(g^{\theta\theta}K_{\theta\theta} + g^{\phi\phi}K_{\phi\phi}\right) \end{aligned} \tag{A.5.3}$$

This is still a formidable expression, so let us start by determining some parts. We will need to fill in the components of the inverse metric tensor and the extrinsic curvature tensor from equations (2.1.6) and (2.5.5) - (2.5.9), respectively.

$$
\begin{aligned}
&2\left[(g^{tr})^2 - g^{tt}g^{rr}\right]K_{tt}K_{rr} \\
&\quad = 2\frac{1}{L^2N^{t2}}\left[\frac{N^{r2}}{N^t}N^{r\prime}L^2 + \frac{N^{r3}}{N^t}LL' - \frac{N^{r2}}{N^t}L\dot{L}\right]\left[\frac{N^r}{N^t}LL' - \frac{1}{N^t}L\dot{L} + \frac{N^{r\prime}}{N^t}L^2\right] \\
&\quad = 2\frac{1}{L^2N^{t2}}\left(\frac{N^r}{N^t}N^{r\prime}L^2 + \frac{N^{r2}}{N^t}LL' - \frac{N^r}{N^t}L\dot{L}\right)^2 \qquad \text{(A.5.4)}
\end{aligned}
$$

while

$$
2\left[g^{tt}g^{rr} - (g^{tr})^2\right](K_{tr})^2 = -2\frac{1}{L^2N^{t2}}\left(\frac{N^rN^{r\prime}}{N^t}L^2 - \frac{N^r}{N^t}L\dot{L} + \frac{N^{r2}}{N^t}LL'\right)^2 \qquad \text{(A.5.5)}
$$

Apparently, the first four terms in equation (A.5.3) cancel. The angular components term, however, does contribute.

$$
\begin{aligned}
2g^{\theta\theta}g^{\phi\phi}K_{\theta\theta}K_{\phi\phi} &= 2\left(-\frac{1}{N^t}\frac{\dot{R}}{R} + \frac{N^r}{N^t}\frac{R'}{R}\right)^2 \\
&= \frac{2}{N^{t2}}\frac{\dot{R}^2}{R^2} + 2\frac{N^{r2}}{N^{t2}}\frac{R'^2}{R^2} - 4\frac{N^r}{N^{t2}}\frac{\dot{R}R'}{R^2} \qquad \text{(A.5.6)}
\end{aligned}
$$

Finally,

$$
\begin{aligned}
&g^{tt}K_{tt} + g^{rr}K_{rr} + 2g^{tr}K_{tr} \\
&\quad = -\frac{N^{r2}}{N^{t3}}N^{r\prime}L^2 - \frac{N^{r3}}{N^{t3}}LL' + \frac{N^{r2}}{N^{t3}}L\dot{L} + \frac{N^r}{N^t}\frac{L'}{L} - \frac{1}{N^t}\frac{\dot{L}}{L} + \frac{N^{r\prime}}{N^t} - \frac{N^{r3}}{N^{t3}}LL' \\
&\quad + \frac{N^{r2}}{N^{t3}}L\dot{L} - \frac{N^{r2}N^{r\prime}}{N^{t3}}L^2 + 2\frac{N^{r2}N^{r\prime}}{N^{t3}}L^2 - 2\frac{N^{r2}}{N^{t3}}L\dot{L} + 2\frac{N^{r3}}{N^{t3}}LL' \\
&\quad = \frac{N^r}{N^t}\frac{L'}{L} - \frac{1}{N^t}\frac{\dot{L}}{L} + \frac{N^{r\prime}}{N^t} \qquad \text{(A.5.7)}
\end{aligned}
$$

and

$$
\begin{aligned}
\left(g^{\theta\theta}K_{\theta\theta} + g^{\phi\phi}K_{\phi\phi}\right) &= -\frac{1}{N^t}\frac{\dot{R}}{R} + \frac{N^r}{N^t}\frac{R'}{R} - \frac{1}{N^t}\frac{\dot{R}}{R} + \frac{N^r}{N^t}\frac{R'}{R} \\
&= -\frac{2}{N^t}\frac{\dot{R}}{R} + 2\frac{N^r}{N^t}\frac{R'}{R} \qquad \text{(A.5.8)}
\end{aligned}
$$

Now that we have gathered all the bits, the Ricci scalar of ADM spacetime can be obtained from equations (A.2.19), (A.5.3), (A.3.2) and (A.5.4) -

(A.5.8).

$$\begin{aligned}\mathcal{R} = &-4\frac{R''}{L^2R} + 4\frac{L'R'}{L^3R} - 2\frac{R'^2}{L^2R^2} + \frac{2}{R^2} - \frac{2}{N^{t2}}\frac{\dot{R}^2}{R^2} - 2\frac{N^{r2}}{N^{t2}}\frac{R'^2}{R^2} \\ &+ 4\frac{N^r}{N^{t2}}\frac{\dot{R}R'}{R^2} + 4\frac{N^r}{N^{t2}}\frac{L'}{L}\frac{\dot{R}}{R} - \frac{4}{N^{t2}}\frac{\dot{L}}{L}\frac{\dot{R}}{R} + 4\frac{N^{r\prime}}{N^{t2}}\frac{\dot{R}}{R} - 4\frac{N^{r2}}{N^{t2}}\frac{L'}{L}\frac{R'}{R} \\ &+ 4\frac{N^r}{N^{t2}}\frac{\dot{L}}{L}\frac{R'}{R} - 4\frac{N^rN^{r\prime}}{N^{t2}}\frac{R'}{R} + 2\left(n^{\nu}{}_{;\nu}n^{\mu} - n^{\mu}{}_{;\nu}n^{\nu}\right)_{;\mu} \qquad \text{(A.5.9)}\end{aligned}$$

This result is used in section 3.3 to derive the Einstein-Hilbert action for an ADM-form metric.

A.6 Parametrisation of the Schwarzschild-Painlevé null geodesics

For incoming particles ($\eta = -1$), the result of (2.6.17) is easily simplified to three terms

$$\begin{aligned}t_-(r) &= -r + 2\sqrt{2Mr} - 2M\log\frac{\left(\sqrt{2M}+\sqrt{r}\right)^2}{2M-r} - 2M\log(r-2M) \\ &= -r + 2\sqrt{2Mr} - 2M\log\left(\sqrt{2M}+\sqrt{r}\right)^2 \\ &= -r + 2\sqrt{2Mr} - 4M\log\left(\sqrt{2M}+\sqrt{r}\right) \qquad \text{(A.6.1)}\end{aligned}$$

For the outgoing particle ($\eta = 1$) path, we first need to invert the first logarithmic term

$$\begin{aligned}t_+(r) &= r + 2\sqrt{2Mr} + 2M\log\left(\frac{\sqrt{2M}-\sqrt{r}}{\sqrt{2M}+\sqrt{r}}\right) + 2M\log(r-2M) \\ &= r + 2\sqrt{2Mr} + 2M\log\frac{\left(\sqrt{2M}-\sqrt{r}\right)^2}{2M-r} + 2M\log(r-2M) \\ &= r + 2\sqrt{2Mr} + 2M\log\left(\sqrt{2M}-\sqrt{r}\right)^2 \\ &= r + 2\sqrt{2Mr} + 4M\log\left(\sqrt{2M}-\sqrt{r}\right) \qquad \text{(A.6.2)}\end{aligned}$$

Hence

$$t_\eta(r) = \eta r + 2\sqrt{2Mr} + 4\eta M\log\left(\sqrt{2M}-\eta\sqrt{r}\right) \qquad \text{(A.6.3)}$$

A.7 Eddington-Finkelstein coordinates

The Eddington-Finkelstein coordinates provide another way to describe a Schwarzschild black hole. Put more precisely, they are *two* sets of coordinates; one for paths going that start outside of the black hole and end inside and one for paths that go the other way around. A nice and, for our purposes, useful property is that there is no coordinate singularity at the Schwarzschild radius.

Starting with the Schwarzschild metric 2.6.1, let us define the so-called tortoise coordinate r^*

$$r^* := r + 2M \log\left|\frac{r}{2M} - 1\right| \tag{A.7.1}$$

The derivative with respect to r equals

$$\frac{dr^*}{dr} := 1 + \frac{1}{\frac{r}{2M} - 1} = 1 + \frac{\frac{r}{2M}}{\frac{r}{2M} - 1} = \left(1 - \frac{2M}{r}\right)^{-1} \tag{A.7.2}$$

If we replace the radial coordinate of Schwarzschild metric with the tortoise coordinate, the line element becomes

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt_S^2 + \left(1 - \frac{2M}{r}\right) dr^{*2} + r^2 d\Omega_2^2 \tag{A.7.3}$$

where we have kept the radial coordinate in the 2-sphere contribution for the sake of clarity. The first two diagonal components of the metric are now equal, except for the sign. The path of a radial null geodesic in these coordinates is

$$dr^* = \pm dt_S \tag{A.7.4}$$

Hence we might express the metric in light cone coordinates u and v

$$u := t_S - r^*, \quad v := t_S + r^* \tag{A.7.5}$$

such that radial null geodesics are described by lines of constant u or v. In inverted differential form,

$$dt_S = \frac{1}{2}(dv + du), \quad dr^* = \frac{1}{2}(dv - du) \tag{A.7.6}$$

The line element in these coordinates is given by

$$\begin{aligned} ds^2 &= \frac{1}{4}\left(1 - \frac{2M}{r}\right)\left(-(dv + du)^2 + (dv - du)^2\right) + r^2 d\Omega_2^2 \\ &= -\left(1 - \frac{2M}{r}\right) du\, dv + r^2 d\Omega_2^2 \end{aligned} \tag{A.7.7}$$

However, an observer obeying the radial null curves (A.7.4) would naturally express his world in the Schwarzschild radial coordinate and the coordinate along his curve, v for the ingoing curve and u for the outgoing counterpart. For in-falling Eddington-Finkelstein coordinates $\{v, r, \theta, \phi\}$ we have

$$2dt_S = dv + du = dv + dt_S - dr^* \tag{A.7.8}$$

so the Schwarzschild time differential is

$$dt_S = dv - dr^* = dv - \left(1 - \frac{2M}{r}\right)^{-1} dr \tag{A.7.9}$$

yielding the line element

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dv^2 + 2dvdr + r^2 d\Omega_2^2 \tag{A.7.10}$$

where the contributions to g_{rr} from the Schwarzschild metric and the square on the right hand side of the previous relation have cancelled. Similarly, for out-falling Eddington-Finkelstein coordinates $\{u, r, \theta, \phi\}$

$$2dt_S = dv + du = dt_S + dr^* + du \tag{A.7.11}$$

so the Schwarzschild time differential is

$$dt_S = dr^* + du = \left(1 - \frac{2M}{r}\right)^{-1} dr + du \tag{A.7.12}$$

yielding the line element

$$ds^2 = -\left(1 - \frac{2M}{r}\right) du^2 - 2dudr + r^2 d\Omega_2^2 \tag{A.7.13}$$

A.8 Kruskal-Szekeres coordinates

Another coordinate system that can be used to describe a Schwarzschild black hole is the Kruskal-Szekeres set of coordinates (U, V, θ, ϕ)

$$T = \sqrt{\frac{r}{2M} - 1}\, e^{\frac{r}{4M}} \sinh \frac{t_S}{4M} \tag{A.8.1}$$

$$R = \sqrt{\frac{r}{2M} - 1}\, e^{\frac{r}{4M}} \cosh \frac{t_S}{4M} \tag{A.8.2}$$

while θ and ϕ remain unchanged. A relation for t can be obtained by dividing T by R

$$\frac{T}{R} = \tanh \frac{t_S}{4M} \tag{A.8.3}$$

Similarly, by using that $\cosh^2 x - \sinh^2 x = 1\, \forall\, x \in \mathbb{R}$, we have

$$T^2 - R^2 = \left(1 - \frac{r}{2M}\right) e^{\frac{r}{2M}} \tag{A.8.4}$$

The t coordinate is easily derived in explicit form,

$$t_S = 4M \tanh^{-1} \frac{T}{R} \tag{A.8.5}$$

leading to the differential

$$dt_S = \frac{4M}{1-\frac{T^2}{R^2}}\left(\frac{1}{R}dT - \frac{T}{R^2}dR\right) = \frac{4M}{R^2-T^2}\left(R\,dT - T\,dR\right) \tag{A.8.6}$$

To derive the r coordinate from equation (A.8.4), we need the Lambert W function, which has the property that

$$W(z) := w : we^w = z \tag{A.8.7}$$

whence

$$\frac{dz}{dw} = (1+w)\,e^w \tag{A.8.8}$$

By inversion, the derivative of W with respect to z equals

$$\frac{dW}{dz}(z) = \frac{dw}{dz} = \frac{1}{(1+w)\,e^w} = \frac{1}{(w^{-1}+1)\,z} = \frac{1}{z}\frac{W(z)}{1+W(z)} \tag{A.8.9}$$

unless $z = 0$ or $z = -e^{-1}$.

By defining

$$u := \frac{r}{2M} - 1 \tag{A.8.10}$$

we can rewrite equation (A.8.4) as

$$T^2 - R^2 = -u\,e^{u+1} = -ue^1e^u \tag{A.8.11}$$

which is solved by

$$u = W(e^{-1}\left(R^2-T^2\right)) \tag{A.8.12}$$

Then

$$r = 2M\left[W(e^{-1}\left(R^2-T^2\right)) + 1\right] \tag{A.8.13}$$

The differential dr can be derived using equation (A.8.9)

$$\begin{aligned} xdr &= 2M\frac{e^1}{R^2-T^2}\frac{W(e^{-1}\left(R^2-T^2\right))}{1+W(e^{-1}\left(R^2-T^2\right))}e^{-1}\left(2RdR - 2TdT\right) \\ &= 4M\frac{W(e^{-1}\left(R^2-T^2\right))}{1+W(e^{-1}\left(R^2-T^2\right))}\frac{RdR-TdT}{R^2-T^2} \end{aligned} \tag{A.8.14}$$

If we use a mixed-coordinate notation, we can fill in the right hand side of equation (A.8.10) again

$$dr = 4M\frac{\left(\frac{r}{2M}-1\right)}{\frac{r}{2M}}\frac{RdR-TdT}{R^2-T^2} = 4M\left(1-\frac{2M}{r}\right)\frac{RdR-TdT}{R^2-T^2} \tag{A.8.15}$$

Now we can fill in equations (A.8.6) and (A.8.15) in the Schwarzschild line element to obtain

$$\begin{aligned}
ds^2 &= -\left(1-\frac{2M}{r}\right)\frac{16M^2}{(R^2-T^2)^2}(R\,dT - T\,dR)^2 \\
&\quad + \left(1-\frac{2M}{r}\right)\frac{16M^2}{(R^2-T^2)^2}(R\,dR - T\,dT)^2 + r^2 d\Omega_2^2 \\
&= \frac{16M^2}{(R^2-T^2)^2}\left(1-\frac{2M}{r}\right)\left[(R\,dR - T\,dT)^2 - (R\,dT - T\,dR)^2\right] + r^2 d\Omega_2^2 \\
&= \frac{16M^2}{R^2-T^2}\left(1-\frac{2M}{r}\right)\left(dR^2 - dT^2\right) + r^2 d\Omega_2^2 \qquad \text{(A.8.16)}
\end{aligned}$$

The cross terms cancelled in the final equality. Continuing the mixed-coordinate system, let us use equation (A.8.4) to simplify this result somewhat further. Then we have the the Kruskal-Szekeres line element

$$ds^2 = \frac{32M^3}{r} e^{-\frac{r}{2M}}\left(dR^2 - dT^2\right) + r^2 d\Omega_2^2 \qquad \text{(A.8.17)}$$

A radial (i.e., $d\Omega_2 = 0$) null geodesic obeys

$$dR^2 = dT^2 \Rightarrow \frac{dR}{dT} = \pm 1 \qquad \text{(A.8.18)}$$

The Schwarzschild metric is only valid outside of the horizon ($r > 2M$). Similarly, the Kruskal-Szekeres metric in the above coordinates is only valid for $T^2 - R^2 < 0$ and $R > 0$. However, we can extend the coordinates to the inner region ($r < 2M$) by defining

$$T_{\mathrm{II}} = \sqrt{1-\frac{r}{2M}}\, e^{\frac{r}{4M}} \cosh\frac{t_S}{4M} \qquad \text{(A.8.19)}$$

$$R_{\mathrm{II}} = \sqrt{1-\frac{r}{2M}}\, e^{\frac{r}{4M}} \sinh\frac{t_S}{4M} \qquad \text{(A.8.20)}$$

such that

$$\frac{R_{\mathrm{II}}}{T_{\mathrm{II}}} = \tanh\frac{t_S}{4M} \qquad \text{(A.8.21)}$$

The region outside of the black hole horizon is commonly denoted as region I, whereas the interior is denoted region II. Physical paths exist from region I to region II, but not the other way around. What is in region II stays there.

B Particle production technicalities

In section 3, we derived particle production at the horizon of a Schwarzschild black hole using the method used Per Kraus and Frank Wilczek [2]. However, we left a few of the more involved calculations for this appendix.

B.1 The Einstein-Hilbert Action

Let us motivate the Einstein-Hilbert action (3.3.2) physically by deriving the vacuum Einstein equations from it, in a manner similar to the one in Carroll [8] and Wald [7]. Upon variation of the inverse metric by $\delta g^{\mu\nu}$, the action changes by

$$\delta S_G = \frac{1}{16\pi}\int d^4x \left[\left(\delta\sqrt{-g}\right)\mathcal{R} + \sqrt{-g}\left(\delta g^{\mu\nu}\right)\mathcal{R}_{\mu\nu} + \sqrt{-g}\, g^{\mu\nu}\delta\mathcal{R}_{\mu\nu}\right] \quad \text{(B.1.1)}$$

Now

$$\delta\sqrt{-g} = -\frac{\delta g}{2\sqrt{-g}} = \frac{g\, g^{\mu\nu}\delta g_{\mu\nu}}{2\sqrt{-g}} = -\frac{1}{2}\sqrt{-g}\, g_{\mu\nu}\delta g^{\mu\nu} \quad \text{(B.1.2)}$$

where we have used Jacobi's formula for the derivative of the determinant in the second equality and from the definition of the Riemann tensor it follows that

$$\delta\mathcal{R}^{\rho}{}_{\sigma\mu\nu} = \delta\Gamma^{\rho}_{\nu\sigma,\mu} - \delta\Gamma^{\rho}_{\mu\sigma,\nu} + \Gamma^{\rho}_{\mu\lambda}\delta\Gamma^{\lambda}_{\nu\sigma} + \left(\delta\Gamma^{\rho}_{\mu\lambda}\right)\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\delta\Gamma^{\lambda}_{\mu\sigma} - \left(\delta\Gamma^{\rho}_{\nu\lambda}\right)\Gamma^{\lambda}_{\mu\sigma} \quad \text{(B.1.3)}$$

The difference between two Christoffel symbols is a proper (1,2) tensor and it clear that the terms on the right hand side are in fact those of two covariant derivatives

$$\delta\mathcal{R}^{\rho}{}_{\sigma\mu\nu} = \delta\Gamma^{\rho}_{\nu\sigma;\mu} - \delta\Gamma^{\rho}_{\mu\sigma;\nu}$$

Note that this the covariant derivative of a varied quantity, not the variation of a derivative. From this, it follows that the variation in the Ricci tensor yields

$$\delta\mathcal{R}_{\mu\nu} = \delta\mathcal{R}^{\rho}{}_{\mu\rho\nu} = \delta\Gamma^{\rho}_{\nu\mu;\rho} - \delta\Gamma^{\rho}_{\rho\mu;\nu} \quad \text{(B.1.4)}$$

In $g^{\mu\nu}\delta\mathcal{R}_{\mu\nu}$, both ρ and ν are dummy indices that we can relabel. We can also use inverse metric compatibility (i.e., $g_{\mu\nu;\sigma} = 0$ and $g^{\mu\nu}{}_{;\sigma} = 0$) to write the expression as a single total derivative

$$\begin{aligned} g^{\mu\nu}\delta\mathcal{R}_{\mu\nu} &= \left[g^{\mu\nu}\delta\Gamma^{\sigma}_{\nu\mu} - g^{\mu\sigma}\delta\Gamma^{\rho}_{\rho\mu}\right]_{;\sigma} - g^{\mu\nu}{}_{;\sigma}\delta\Gamma^{\rho}_{\nu\mu} + g^{\mu\sigma}{}_{;\sigma}\delta\Gamma^{\rho}_{\rho\mu} \\ &= \left[g^{\mu\nu}\delta\Gamma^{\sigma}_{\nu\mu} - g^{\mu\sigma}\delta\Gamma^{\rho}_{\rho\mu}\right]_{;\sigma} \end{aligned} \quad \text{(B.1.5)}$$

By applying Stokes' theorem, the last term in equation (B.1.1) now only contributes at the boundary. If we assume the variation to vanish at infinity, the integral equals zero.

$$\int d^4x\sqrt{-g}\left[g^{\mu\nu}\delta\Gamma^{\sigma}_{\nu\mu} - g^{\mu\sigma}\delta\Gamma^{\rho}_{\rho\mu}\right]_{;\sigma} = 0 \quad \text{(B.1.6)}$$

Filling in the other terms leaves us with the action variation

$$\delta S_G = \frac{1}{16\pi}\int d^4x\sqrt{-g}\left[-\frac{1}{2}\mathcal{R}g_{\mu\nu} + \mathcal{R}_{\mu\nu}\right]\delta g^{\mu\nu} \tag{B.1.7}$$

Finally, for δS to always equal zero as $\delta g^{\mu\nu}$ goes to zero, the quantity enclosed within the brackets must equal zero

$$\mathcal{R}_{\mu\nu} - \frac{1}{2}\mathcal{R}g_{\mu\nu} = 0 \tag{B.1.8}$$

which is in fact the vacuum Einstein equation.

B.2 The canonical action for a metric in ADM form

The relations (3.4.3) and (3.4.4) can be inverted to obtain equations for $\dot{L}$ and $\dot{R}$ in terms of the canonical momenta. From the first equation

$$\dot{R} = -\frac{N^t}{R}\left[\pi_L - \frac{N^r}{N^t}RR'\right] = -\frac{N^t}{R}\pi_L + N^r R' \tag{B.2.1}$$

and from the second equation

$$\begin{aligned}\dot{L} &= -\frac{N^t}{R}\left[\pi_R - \frac{1}{N^t}\left[-L\dot{R} + N^{r\prime}LR\right] - \frac{N^r}{N^t}\left[LR' + L'R\right]\right]\\ &= -\frac{N^t}{R}\pi_R - \frac{L\dot{R}}{R} + N^{r\prime}L + N^r\frac{LR'}{R} + N^r L'\\ &= -\frac{N^t}{R}\pi_R + N^t\frac{L}{R^2}\pi_L + N^{r\prime}L + N^r L' \end{aligned} \tag{B.2.2}$$

The Einstein-Hilbert action for the ADM metric as expressed in equation (3.3.7) can now be expressed in terms of the canonical momenta. To do so, we first need to compute the square of π_L

$${\pi_L}^2 = \frac{1}{N^{t2}}R^2\dot{R}^2 + \frac{N^{r2}}{N^{t2}}R^2R'^2 - 2\frac{N^r}{N^{t2}}R^2\dot{R}R' \tag{B.2.3}$$

and the product of π_L and π_R

$$\begin{aligned}\pi_L\pi_R &= \frac{1}{N^{t2}}LR\dot{R}^2 + \frac{1}{N^{t2}}\dot{L}R^2\dot{R} - \frac{N^{r\prime}}{N^{t2}}LR^2\dot{R} - \frac{N^r}{N^{t2}}LR\dot{R}R'\\ &\quad - \frac{N^r}{N^{t2}}\dot{L}R^2R' - \frac{N^rN^{r\prime}}{N^{t2}}LR^2R' - \frac{N^r}{N^{t2}}LR\dot{R}R' + \frac{N^r}{N^{t2}}L'R^2\dot{R}\\ &\quad + \frac{N^{r2}}{N^{t2}}LRR'^2 + \frac{N^{r2}}{N^{t2}}L'R^2R'\\ &= \frac{1}{N^{t2}}LR\dot{R}^2 + \frac{1}{N^{t2}}\dot{L}R^2\dot{R} - \frac{N^{r\prime}}{N^{t2}}LR^2\dot{R} - 2\frac{N^r}{N^{t2}}LR\dot{R}R'\\ &\quad - \frac{N^r}{N^{t2}}\dot{L}R^2R' - \frac{N^rN^{r\prime}}{N^{t2}}LR^2R' + \frac{N^r}{N^{t2}}L'R^2\dot{R}\\ &\quad + \frac{N^{r2}}{N^{t2}}LRR'^2 + \frac{N^{r2}}{N^{t2}}L'R^2R' \end{aligned} \tag{B.2.4}$$

Evidently, equation (3.3.7) for the gravitational contribution in the action equals

$$S_G = \int dt\, dr\, N^t \left[-\frac{RR''}{L} + \frac{L'RR'}{L^2} - \frac{R'^2}{2L} + \frac{L}{2} - \frac{\pi_L \pi_R}{R} - \frac{L{\pi_L}^2}{2R^2} \right] \tag{B.2.5}$$

B.3 The static mass in Schwarzschild space-time

Filling in equations (3.4.15) and (3.4.16) into equation (3.5.8) results in the following expression for $\mathcal{M}'$

$$\begin{aligned} \mathcal{M}' &= -\frac{R'\pi_L^2}{2R^2} + \frac{R'\pi_L\pi_R}{LR} - \frac{R'}{L}\left(\frac{RR'}{L}\right)' + \frac{(R')^3}{2L^2} + \frac{R'}{2} - \frac{R'\pi_L\pi_R}{LR} + \frac{\pi_L\pi_L'}{R} \\ &= -\frac{R'\pi_L^2}{2R^2} + \frac{\pi_L\pi_L'}{R} - \frac{(R')^3}{2L^2} - \frac{RR'R''}{L^2} + \frac{L'R(R')^2}{L^3} + \frac{R'}{2} \end{aligned} \tag{B.3.1}$$

To obtain the static mass, we need to integrate this relation

$$\begin{aligned} \mathcal{M}(r) &= \int_0^r d\bar{r} \left[-\frac{R'\pi_L^2}{2R^2} + \frac{\pi_L\pi_L'}{R} - \frac{(R')^3}{2L^2} - \frac{RR'R''}{L^2} + \frac{L'R(R')^2}{L^3} + \frac{R'}{2} \right]_{r=\bar{r}} \\ &= \int_0^r d\bar{r} \left[\left(\frac{\pi_L^2}{2R}\right)' - \left(\frac{RR'^2}{2L^2}\right)' + \frac{R'}{2} \right]_{r=\bar{r}} \\ &= \frac{\pi_L^2}{2R} - \frac{RR'^2}{2L^2} + \frac{R}{2} \end{aligned} \tag{B.3.2}$$

where we have used that $R(t, 0)$ equals zero for any suitable set of coordinates.

B.4 Variational of the action

Integration of the time derivative of the contribution to the action due to momentum in the L direction yields equation (3.6.2). To compute the integral I over L, let us perform a substitution

$$u := \frac{R'}{L} \Rightarrow \frac{dL}{du} = -\frac{R'}{u^2} \tag{B.4.1}$$

The constants R and R' can also be moved out of the integrand. Now we can integrate by parts and insert the boundaries

$$\begin{aligned}
I &= -RR' \int_{\frac{R'}{L}}^{0} du \frac{\sqrt{u^2 + \frac{2\mathcal{M}}{R} - 1}}{u^2} \\
&= RR' \left. \frac{\sqrt{u^2 + \frac{2\mathcal{M}}{R} - 1}}{u} \right|_{\frac{R'}{L}}^{0} - RR' \int_{\frac{R'}{L}}^{0} \frac{du}{\sqrt{u^2 + \frac{2\mathcal{M}}{R} - 1}} \\
&= RR' \left[\frac{\sqrt{u^2 + \frac{2\mathcal{M}}{R} - 1}}{u} - \log \left| u + \sqrt{u^2 + \frac{2\mathcal{M}}{R} - 1} \right| \right]_{u=\frac{R'}{L}}^{u\to 0} \\
&= RR' \left[\frac{L\sqrt{\left(\frac{R'}{L}\right)^2 + \frac{2\mathcal{M}}{R} - 1}}{R'} - \log \left| \sqrt{\frac{\mathcal{M}}{R} - 1} \right| \right. \\
&\qquad \left. + \log \left| \frac{R'}{L} + \sqrt{\left(\frac{R'}{L}\right)^2 + \frac{2\mathcal{M}}{R} - 1} \right| \right] \\
&= LR\sqrt{\left(\frac{R'}{L}\right)^2 + \frac{2\mathcal{M}}{R} - 1} + RR' \log \left| \frac{\frac{R'}{L} - \sqrt{\left(\frac{R'}{L}\right)^2 + \frac{2\mathcal{M}}{R} - 1}}{\sqrt{\frac{2\mathcal{M}}{R} - 1}} \right| \qquad \text{(B.4.2)}
\end{aligned}$$

In the second equality, the derivative of u^2 and the antiderivative of $\frac{1}{u^2}$ neatly cancel. In the fourth equality, the (infinite) boundary term from $L \to 0$ is discarded, for it does not carry physical significance. With this result, equation (3.6.2) equals

$$\begin{aligned}
S_L = &\int_{r_0}^{\hat{r}-\epsilon} drR \left[R' \log \left| \frac{\frac{R'}{L} - \sqrt{\left(\frac{R'}{L}\right)^2 + \frac{2M}{R} - 1}}{\sqrt{\frac{2M}{R} - 1}} \right| + L\sqrt{\left(\frac{R'}{L}\right)^2 + \frac{2M}{R} - 1} \right] \\
&+ \int_{\hat{r}+\epsilon}^{\infty} drR \left[R' \log \left| \frac{\frac{R'}{L} - \sqrt{\left(\frac{R'}{L}\right)^2 + \frac{2M_+}{R} - 1}}{\sqrt{\frac{2M_+}{R} - 1}} \right| + L\sqrt{\left(\frac{R'}{L}\right)^2 + \frac{2M_+}{R} - 1} \right] \qquad \text{(B.4.3)}
\end{aligned}$$

To determine the result of varying the above action with respect to R, we start with equation (3.6.4)

$$
\begin{aligned}
\delta_R S_L &= \int_{r_0}^{\infty} dr \int dR\, \pi_R - \frac{\partial S'_L}{\partial R'} \delta \hat{R} \Big|_{r-\epsilon}^{r+\epsilon} \\
&= \int_{r_0}^{\infty} dr \int dR\, \pi_R - \left[R \log \left| \frac{\frac{R'}{L} - \sqrt{\left(\frac{R'}{L}\right)^2 + \frac{2\mathcal{M}}{R} - 1}}{\sqrt{\frac{2\mathcal{M}}{R} - 1}} \right| \right. \\
&\quad + \frac{R\sqrt{\frac{2\mathcal{M}}{R} - 1}}{\frac{R'}{L} - \sqrt{\left(\frac{R'}{L}\right)^2 + \frac{2\mathcal{M}}{R} - 1}} \frac{\frac{1}{L} - \frac{R'}{L^2\sqrt{\left(\frac{R'}{L}\right)^2 + \frac{2\mathcal{M}}{R} - 1}}}{\sqrt{\frac{2\mathcal{M}}{R} - 1}} \\
&\quad \left. + \frac{RR'}{L\sqrt{\left(\frac{R'}{L}\right)^2 + \frac{2\mathcal{M}}{R} - 1}} \right]_{r=\hat{r}-\epsilon}^{\hat{r}+\epsilon} \\
&= \int_{r_0}^{\infty} dr \int dR\, \pi_R - \left[R \log \left| \frac{\frac{R'}{L} - \sqrt{\left(\frac{R'}{L}\right)^2 + \frac{2\mathcal{M}}{R} - 1}}{\sqrt{\frac{2\mathcal{M}}{R} - 1}} \right| \right. \\
&\quad \left. + R \frac{1 - \frac{R'}{L\sqrt{\left(\frac{R'}{L}\right)^2 + \frac{2\mathcal{M}}{R} - 1}}}{R' - L\sqrt{\left(\frac{R'}{L}\right)^2 + \frac{2\mathcal{M}}{R} - 1}} + \frac{RR'}{L\sqrt{\left(\frac{R'}{L}\right)^2 + \frac{2\mathcal{M}}{R} - 1}} \right]_{r=\hat{r}-\epsilon}^{\hat{r}+\epsilon} \\
&= \int_{r_0}^{\infty} dr \int dR\, \pi_R - \left[R \log \left| \frac{\frac{R'}{L} - \sqrt{\left(\frac{R'}{L}\right)^2 + \frac{2\mathcal{M}}{R} - 1}}{\sqrt{\frac{2\mathcal{M}}{R} - 1}} \right| \right]_{r=\hat{r}-\epsilon}^{\hat{r}+\epsilon} \\
&= \int_{r_0}^{\infty} dr \int dR\, \pi_R - \hat{R} \log \left| \frac{\frac{R'(r+\epsilon)}{L} - \sqrt{\left(\frac{R'(r+\epsilon)}{L}\right)^2 + \frac{2M_+}{\hat{R}} - 1}}{\sqrt{\frac{2M_+}{\hat{R}} - 1}} \right| \delta \hat{R} \\
&\quad + \hat{R} \log \left| \frac{\frac{R'(r-\epsilon)}{L} - \sqrt{\left(\frac{R'(r-\epsilon)}{L}\right)^2 + \frac{2M}{\hat{R}} - 1}}{\sqrt{\frac{2M}{\hat{R}} - 1}} \right| \delta \hat{R} \qquad \text{(B.4.4)}
\end{aligned}
$$

The derivative of the Lagrangian density with respect to M_+ is

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial M_+} &= \frac{RR'\sqrt{\frac{2M_+}{R}-1}}{\frac{R'}{L}-\sqrt{\left(\frac{R'}{L}\right)^2+\frac{2M_+}{R}-1}}\left(\frac{\frac{-1}{R\sqrt{\left(\frac{R'}{L}\right)^2+\frac{2M_+}{R}-1}}}{\sqrt{\frac{2M_+}{R}-1}}+\frac{\frac{R'}{L}-\sqrt{\left(\frac{R'}{L}\right)^2+\frac{2M_+}{R}-1}}{R\left(\frac{2M_+}{R}-1\right)^{\frac{3}{2}}}\right) \\
&\quad+\frac{L}{\sqrt{\left(\frac{R'}{L}\right)^2+\frac{2M_+}{R}-1}} \\
&= \frac{-R'}{\frac{R'}{L}-\sqrt{\left(\frac{R'}{L}\right)^2+\frac{2M_+}{R}-1}}\frac{1}{\sqrt{\left(\frac{R'}{L}\right)^2+\frac{2M_+}{R}-1}}+\frac{R'}{\frac{2M_+}{R}-1} \\
&\quad+\frac{L}{\sqrt{\left(\frac{R'}{L}\right)^2+\frac{2M_+}{R}-1}} \\
&= \frac{L\sqrt{\left(\frac{R'}{L}\right)^2+\frac{2M_+}{R}-1}}{\frac{R'}{L}-\sqrt{\left(\frac{R'}{L}\right)^2+\frac{2M_+}{R}-1}}\frac{1}{\sqrt{\left(\frac{R'}{L}\right)^2+\frac{2M_+}{R}-1}}+\frac{R'}{\frac{2M_+}{R}-1} \\
&= \frac{R'+L\sqrt{\left(\frac{R'}{L}\right)^2+\frac{2M_+}{R}-1}}{\frac{R'}{L}-\sqrt{\left(\frac{R'}{L}\right)^2+\frac{2M_+}{R}-1}}\frac{1}{\frac{R'}{L}+\sqrt{\left(\frac{R'}{L}\right)^2+\frac{2M_+}{R}-1}}+\frac{R'}{\frac{2M_+}{R}-1} \\
&= L\frac{\sqrt{\left(\frac{R'}{L}\right)^2+\frac{2M_+}{R}-1}}{\frac{2M_+}{R}-1}
\end{aligned} \tag{B.4.5}$$

Variation of M_+ by δM_+ then leads to a an action modified by $\delta_{M_+} S$

$$\delta_{M_+} S = \int_{\hat{r}+\epsilon}^{\infty} L\frac{\sqrt{\left(\frac{R'}{L}\right)^2+\frac{2M_+}{R}-1}}{\frac{2M_+}{R}-1}\delta M_+ \tag{B.4.6}$$

These results are used in section 3.6 to derive an explicit expression for the action that meets the constrains of section 3.5 and upholds equation (3.6.1) for the total derivative.

B.5 The action as a function of time

The total action as an integral over time is then

$$
\begin{aligned}
S =& \int dt \left[\int_{r_0}^{\hat{r}-\epsilon} dr \left[\pi_L \dot{L} + \pi_R \dot{R}\right] + \int_{\hat{r}+\epsilon}^{\infty} dr \left[\pi_L \dot{L} + \pi_R \dot{R}\right]\right] - \int dt M_+ \\
&+ \int dt\, \hat{R} \left[\dot{\hat{R}} \log \left| \frac{\frac{R'_<}{L} - \sqrt{\left(\frac{R'_<}{L}\right)^2 + \frac{2M}{\hat{R}} - 1}}{\sqrt{\frac{2M}{R} - 1}} \right| + \dot{\hat{r}} L \sqrt{\left(\frac{R'_<}{L}\right)^2 + \frac{2M}{\hat{R}} - 1} \right] \\
&- \int dt\, \hat{R}\dot{\hat{r}}\hat{L} \sqrt{\left(\frac{R'_>}{\hat{L}}\right)^2 + \frac{2M_+}{R} - 1} \\
&- \int dt\, \hat{R}\dot{\hat{R}} \log \left| \frac{\frac{R'(r-\epsilon)}{\hat{L}} - \sqrt{\left(\frac{R'(r-\epsilon)}{\hat{L}}\right)^2 + \frac{2M}{\hat{R}} - 1}}{\sqrt{\frac{2M}{\hat{R}} - 1}} \right| \qquad \text{(B.5.1)}
\end{aligned}
$$

$$
\begin{aligned}
=& \int dt \left[\int_{r_0}^{\hat{r}+\epsilon} dr \left[\pi_L \dot{L} + \pi_R \dot{R}\right] + \int_{\hat{r}+\epsilon}^{\infty} dr \left[\pi_L \dot{L} + \pi_R \dot{R}\right]\right] - \int dt M_+ \\
&+ \int dt\, \dot{\hat{r}}\hat{L}\hat{R} \left[\sqrt{\left(\frac{R'_<}{\hat{L}}\right)^2 + \frac{2M}{\hat{R}} - 1} - \sqrt{\left(\frac{R'_>}{\hat{L}}\right)^2 + \frac{2M_+}{R} - 1} \right] \\
&+ \int dt\, \hat{R}\dot{\hat{R}} \log \left| \frac{\frac{R'_<}{\hat{L}} - \sqrt{\left(\frac{R'_<}{\hat{L}}\right)^2 + \frac{2M}{\hat{R}} - 1}}{\frac{R'(r-\epsilon)}{\hat{L}} - \sqrt{\left(\frac{R'(r-\epsilon)}{\hat{L}}\right)^2 + \frac{2M}{\hat{R}} - 1}} \right| \qquad \text{(B.5.2)}
\end{aligned}
$$

Now we can fill in the constraints (3.5.14) and (3.5.16) in the last line

$$
\begin{aligned}
S =& \int dt \left[\int_{r_0}^{\hat{r}-\epsilon} dr \left[\pi_L \dot{L} + \pi_R \dot{R}\right] + \int_{\hat{r}+\epsilon}^{\infty} dr \left[\pi_L \dot{L} + \pi_R \dot{R}\right]\right] - \int dt M_+ \\
&+ \int dt\, \dot{\hat{r}}\hat{L}\hat{R} \left[\sqrt{\left(\frac{R'_<}{\hat{L}}\right)^2 + \frac{2M}{\hat{R}} - 1} - \sqrt{\left(\frac{R'_>}{\hat{L}}\right)^2 + \frac{2M_+}{R} - 1} \right] \\
&+ \int dt\, \hat{R}\dot{\hat{R}} \log \left| \frac{\frac{R'_<}{\hat{L}} - \sqrt{\left(\frac{R'_<}{\hat{L}}\right)^2 + \frac{2M}{\hat{R}} - 1}}{\frac{R'(r+\epsilon)}{\hat{L}} + \frac{\eta p}{\hat{L}\hat{R}} - \sqrt{\left(\frac{R'(r+\epsilon)}{\hat{L}}\right)^2 + \frac{2M}{\hat{R}} - 1} - \frac{p}{\hat{L}\hat{R}}} \right| \qquad \text{(B.5.3)}
\end{aligned}
$$

With the right hand side of equation (3.6.8)

$$
\begin{aligned}
S = & \int dt \left[\int_{r_0}^{\hat{r}-\epsilon} dr \left[\pi_L \dot{L} + \pi_R \dot{R} \right] + \int_{\hat{r}+\epsilon}^{\infty} dr \left[\pi_L \dot{L} + \pi_R \dot{R} \right] \right] - \int dt M_+ \\
& + \int dt \, \dot{\hat{r}} \hat{L} \hat{R} \left[\sqrt{\left(\frac{R'_<}{\hat{L}} \right)^2 + \frac{2M}{\hat{R}} - 1} - \sqrt{\left(\frac{R'_>}{\hat{L}} \right)^2 + \frac{2M_+}{R} - 1} \right] \\
& + \int dt \, \hat{R} \dot{\hat{R}} \log \left| \frac{\frac{R'_<}{\hat{L}} - \sqrt{\left(\frac{R'_<}{\hat{L}} \right)^2 + \frac{2M}{\hat{R}} - 1}}{\frac{R'_>}{\hat{L}} - \sqrt{\left(\frac{R'_>}{\hat{L}} \right)^2 + \frac{2M_+}{\hat{R}} - 1} + \frac{(\eta-1)p}{\hat{L}\hat{R}}} \right| \\
= & \int dt \left[\int_{r_0}^{\hat{r}-\epsilon} dr \left[\pi_L \dot{L} + \pi_R \dot{R} \right] + \int_{\hat{r}+\epsilon}^{\infty} dr \left[\pi_L \dot{L} + \pi_R \dot{R} \right] \right] - \int dt M_+ \\
& + \int dt \, \dot{\hat{r}} \hat{L} \hat{R} \left[\sqrt{\left(\frac{R'_<}{\hat{L}} \right)^2 + \frac{2M}{\hat{R}} - 1} - \sqrt{\left(\frac{R'_>}{\hat{L}} \right)^2 + \frac{2M_+}{\hat{R}} - 1} \right] \\
& + \int dt \, \eta \hat{R} \dot{\hat{R}} \log \left| \frac{\frac{R'_<}{\hat{L}} - \eta \sqrt{\left(\frac{R'_<}{\hat{L}} \right)^2 + \frac{2M}{\hat{R}} - 1}}{\frac{R'_>}{\hat{L}} - \eta \sqrt{\left(\frac{R'_>}{\hat{L}} \right)^2 + \frac{2M_+}{\hat{R}} - 1}} \right|
\end{aligned}
\tag{B.5.4}
$$

B.6 Solving the path equations

In equations (3.7.12) and (3.7.13), $\sqrt{\hat{r}(0)}$ and $\eta\sqrt{2M_+}$ can be separated by moving around a few terms.

$$\sqrt{\hat{r}(0)} \left(1 - e^{\frac{\eta k}{2M}}\right) \simeq \eta\sqrt{2M} - \eta\sqrt{2M_+} e^{\frac{\eta k}{2M}} \tag{B.6.1}$$

$$\eta\sqrt{2M_+} \left(1 - e^{-\frac{t}{4M}}\right) \simeq \sqrt{\hat{r}(0)} - \sqrt{r} e^{-\frac{t}{4M}} \tag{B.6.2}$$

The first equation is easily inserted into the second to obtain an equation for $\eta\sqrt{2M_+}$ independent of $\sqrt{\hat{r}(0)}$.

$$\eta\sqrt{2M_+} \left(1 - e^{-\frac{t}{4M}}\right) \simeq \frac{\eta\sqrt{2M} - \eta\sqrt{2M_+} e^{\frac{\eta k}{2M}}}{1 - e^{\frac{\eta k}{2M}}} - \sqrt{r} e^{-\frac{t}{4M}} \tag{B.6.3}$$

whence

$$\eta\sqrt{2M_+} \left(1 - e^{-\frac{t}{4M}} + \frac{e^{\frac{\eta k}{2M}}}{1 - e^{\frac{\eta k}{2M}}}\right) \simeq \frac{\eta\sqrt{2M}}{1 - e^{\frac{\eta k}{2M}}} - \sqrt{r} e^{-\frac{t}{4M}} \tag{B.6.4}$$

which finally results in an explicit expression for $\eta\sqrt{2M_+}$

$$\eta\sqrt{2M_+} \simeq \frac{\frac{\eta\sqrt{2M}}{1 - e^{\frac{\eta k}{2M}}} - \sqrt{r} e^{-\frac{t}{4M}}}{1 - e^{-\frac{t}{4M}} + \frac{e^{\frac{\eta k}{2M}}}{1 - e^{\frac{\eta k}{2M}}}} \tag{B.6.5}$$

To zeroth order, $\sqrt{2M_+}$ equals $\sqrt{2M}$, so it makes sense to try and simplify the equation by moving a term $\eta\sqrt{2M}$ out of the fraction.

$$\begin{aligned}
\eta\sqrt{2M_+} &\simeq \eta\sqrt{2M} + \frac{\eta\sqrt{2M}\left(\frac{1}{1-e^{\frac{\eta k}{2M}}} - 1 + e^{-\frac{t}{4M}} - \frac{e^{\frac{\eta k}{2M}}}{1-e^{\frac{\eta k}{2M}}}\right) - \sqrt{r}e^{-\frac{t}{4M}}}{1 - e^{-\frac{t}{4M}} + \frac{e^{\frac{\eta k}{2M}}}{1-e^{\frac{\eta k}{2M}}}} \\
&= \eta\sqrt{2M} + \frac{\eta\sqrt{2M}e^{-\frac{t}{4M}} - \sqrt{r}e^{-\frac{t}{4M}}}{1 - e^{-\frac{t}{4M}} + \frac{e^{\frac{\eta k}{2M}}}{1-e^{\frac{\eta k}{2M}}}} \\
&= \eta\sqrt{2M} + \left(\eta\sqrt{2M} - \sqrt{r}\right)\frac{\left(e^{\frac{\eta k}{2M}} - 1\right)e^{-\frac{t}{4M}}}{\left(e^{\frac{\eta k}{2M}} - 1\right)\left(1 - e^{-\frac{t}{4M}}\right) + 1} \\
&\simeq \eta\sqrt{2M} + \left(\sqrt{r} - \eta\sqrt{2M}\right)\frac{\left(e^{\frac{\eta k}{2M}} - 1\right)e^{-\frac{t}{4M}}}{\left(e^{\frac{\eta k}{2M}} - 1\right)e^{-\frac{t}{4M}} + 1}
\end{aligned} \tag{B.6.6}$$

where we have assumed that $e^{-\frac{t}{4M}} \gg 1$ in the last step.

To obtain $\sqrt{\hat{r}(0)}$, we insert this result into equation (B.6.1).

$$\begin{aligned}
\sqrt{\hat{r}(0)}\left(1 - e^{\frac{\eta k}{2M}}\right) \simeq& \left(1 - e^{\frac{\eta k}{2M}}\right)\eta\sqrt{2M} \\
&- \left(\sqrt{r} - \eta\sqrt{2M}\right)\frac{\left(e^{\frac{\eta k}{2M}} - 1\right)e^{\frac{\eta k}{2M} - \frac{t}{4M}}}{\left(e^{\frac{\eta k}{2M}} - 1\right)e^{-\frac{t}{4M}} + 1}
\end{aligned} \tag{B.6.7}$$

Evidently,

$$\sqrt{\hat{r}(0)} \simeq \eta\sqrt{2M} + \left(\sqrt{r} - \eta\sqrt{2M}\right)\frac{e^{\frac{\eta k}{2M} - \frac{t}{4M}}}{\left(e^{\frac{\eta k}{2M}} - 1\right)e^{-\frac{t}{4M}} + 1} \tag{B.6.8}$$

B.7 The radial momentum integral

In section 3.7 we came across an integral of the form

$$I := \int_a^b dx\, x \log\left|\frac{\sqrt{x} - \alpha}{\sqrt{x} - \beta}\right| \tag{B.7.1}$$

The logarithm can be expressed as two separate terms. Additionally, we can apply the substitution $x \to u^2$ if we multiply the integrand by $\frac{dx}{du} = 2u$.

$$I = 2\int_{\sqrt{a}}^{\sqrt{b}} du\, u^3 \left[\log|u - \alpha| - \log|u - \beta|\right] \tag{B.7.2}$$

This is the sum of two integrals, each of which can be simplified again by substitution. For the first term, we use $u \to w + \alpha$

$$
\begin{aligned}
I_1 &= 2 \int_{\sqrt{a}-\alpha}^{\sqrt{b}-\alpha} dw\, (w+\alpha)^3 \log|w| \\
&= \frac{1}{2}\left[(w+\alpha)^4 \log|w|\right]_{w=\sqrt{a}-\alpha}^{\sqrt{b}-\alpha} - \frac{1}{2}\int_{\sqrt{a}+\alpha}^{\sqrt{b}-\alpha} dw\, \frac{(w-\alpha)^4}{w} \\
&= \frac{b^2}{2}\log\left|\sqrt{b}-\alpha\right| - \frac{a^2}{2}\log\left|\sqrt{a}-\alpha\right| \\
&\quad - \frac{1}{2}\int_{\sqrt{a}+\alpha}^{\sqrt{b}-\alpha} dw\, \left[w^3 - 4\alpha w^2 + 6\alpha^2 w - 4\alpha^3 w + \frac{\alpha^4}{w}\right] \\
&= \frac{b^2}{2}\log\left|\sqrt{b}-\alpha\right| - \frac{a^2}{2}\log\left|\sqrt{a}-\alpha\right| \\
&\quad - \frac{1}{2}\left[\frac{1}{4}w^4 - \frac{4}{3}\alpha w^3 + 3\alpha^2 w^2 - 4\alpha^3 w + \alpha^4 \log w\right]_{w=\sqrt{a}-\alpha}^{\sqrt{b}-\alpha}
\end{aligned} \tag{B.7.3}
$$

The second integral term yields an identical result, except α being replaced by β and the overall sign being reversed. In equation (3.6.16), we are only interested in the dominant late-time contributions, so we can neglect all terms that are regular near the horizon. In the last result, only the logarithmic terms qualify. For I_1, this means

$$
I_1 \simeq \frac{b^2}{2}\log\left|\sqrt{b}-\alpha\right| - \frac{a^2}{2}\log\left|\sqrt{a}-\alpha\right| - \frac{\alpha^4}{2}\log\left|\frac{\sqrt{b}-\alpha}{\sqrt{a}-\alpha}\right| \tag{B.7.4}
$$

With the proper substitutions for I_2, the relevant parts of equation (B.7.1) are

$$
\begin{aligned}
I \simeq\ & \frac{b^2}{2}\log\left|\frac{\sqrt{b}-\alpha}{\sqrt{b}-\beta}\right| - \frac{a^2}{2}\log\left|\frac{\sqrt{a}-\alpha}{\sqrt{a}-\beta}\right| - \frac{\alpha^4}{2}\log\left|\frac{\sqrt{b}-\alpha}{\sqrt{a}-\alpha}\right| \\
& + \frac{\beta^4}{2}\log\left|\frac{\sqrt{b}-\beta}{\sqrt{a}-\beta}\right|
\end{aligned} \tag{B.7.5}
$$

B.8 Tunnelling momentum in explicit form

Integrating the radial momentum in the tunnelling calculation, given by equation (4.1.20), explicitly over the real axis yields

$$\begin{aligned}
p_r^T &= \int_M^{M-\omega} dH \frac{1}{\eta - \sqrt{\frac{2H}{r}}} = \int_{\sqrt{\frac{2M}{r}}}^{\sqrt{\frac{2(M-\omega)}{r}}} du \frac{ru}{1-\eta u} \\
&= r \int_{1-\sqrt{\frac{2M}{r}}}^{1-\sqrt{\frac{2(M-\omega)}{r}}} dw \left[1 - \frac{1}{w}\right] = r\left[w - \log|w|\right]_{w=1-\sqrt{\frac{2M}{r}}}^{1-\sqrt{\frac{2(M-\omega)}{r}}} \\
&= r\sqrt{\frac{2M}{r}} - r\sqrt{\frac{2(M-\omega)}{r}} + r\log\left|\frac{1-\sqrt{\frac{2M}{r}}}{1-\sqrt{\frac{2(M-\omega)}{r}}}\right| \\
&= \sqrt{2Mr} - \sqrt{2(M-\omega)r} + r\log\left|\frac{\sqrt{r}-\sqrt{2M}}{\sqrt{r}-\sqrt{2(M-\omega)}}\right|
\end{aligned} \tag{B.8.1}$$

where we have used the substitution

$$w := 1 - \eta u \Rightarrow u = \frac{1-w}{\eta} = \eta\,(1-w) \Rightarrow \frac{du}{dw} = -\eta \tag{B.8.2}$$

B.9 The saddle point approximation

In general, if an integral I over t has the form

$$I = \int_{-\infty}^{\infty} dt f(t) e^{\lambda g(t)} \quad \lambda \gg g(t)\,\forall\, t \tag{B.9.1}$$

and $g(t)$ has a single global maximum at t_0, the integrand has a much greater value around $t = t_0$, so the dominant contribution to the integral lies in that area. One would therefore expect a second-order Taylor series in $g(t)$ around the saddle point $t = t_0$ to yield a good approximation

$$I \simeq I_s := \int_{-\infty}^{\infty} dt f(t) e^{\lambda\left[g(t_0) + \dot{g}(t_0)(t-t_0) + \frac{1}{2}\ddot{g}(t_0)(t-t_0)^2\right]} \tag{B.9.2}$$

Of course, to find t_0 one looks for the t for which $\dot{g}(t) = 0$ so by definition the second term equals zero. The third term is just a Gaussian integral and the first term does not depend on t at all.

$$I_s = \sqrt{\frac{2\pi}{\lambda \ddot{g}(t_0)}} e^{\lambda g(t_0)} \int_{-\infty}^{\infty} dt f(t) \tag{B.9.3}$$

this is the saddle point approximation.

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Summary in Dutch – Samenvatting in het Nederlands

Als een zware ster aan het einde van zijn levensduur is, stort de materie in de ster ineen en wordt de ster een zwart gat. Dat is een object dat zo een sterke zwaartekracht uitoefent op zijn omgeving, dat elk deeltje dat binnen een bepaalde straal van het centrum van het zwarte gat komt daar onherroepelijk binnen blijft. Op deze straal bevindt zich de Schwarzschild-horizon. Omdat elke vorm van straling of materie die zich binnen de horizon bevindt daar altijd zal blijven komt er geen licht uit het object. Het is dus een zwart gat.

In 1974 berekende Stephen Hawking dat er *op* de horizon wel deeltjes naar buiten worden uitgezonden. Dit fenomeen noemt men Hawkingstraling en het vindt zijn oorsprong in het feit dat de vacua aan beide zijden van de horizon verschillen. In de berekening van Hawking is de straling *thermisch*: de distributie van deeltjes komt overeen met de straling van een voorwerp met een bepaalde temperatuur. Dat betekent onder andere dat er deeltjes kunnen worden uitgestraald die zwaarder zijn dan het zwarte gat. Bovendien gaat de berekening er vanuit dat het zwarte gat niet lichter wordt door straling. Daarbij wordt de wet van massa-energiebehoud geschonden.

In deze scriptie worden twee alternatieve berekeningen vergeleken. Deze nemen de afname van energie binnen het zwarte gat wel mee, en resulteren dan ook in een distributie van deeltjes die niet helemaal thermisch is.