domain of integration is no longer real but is complex as has been indicated. The question one would really like to know is just what influence causality, unitarity, and Lorentz invariance have upon the general structure of an N-fold representation of a multiple Feynman integral. In the case of the four-point function, the condition of causality contained in the $m^2-i\varepsilon$ implies that the spectral function is non-zero only in a real domain on the boundary of the physical sheet, and the interesting question is just what simple (or complicated) property of the higher functions contains the concept of causality.

A GENERALIZED UNITARITY RELATION

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Last year at the Ninth International Conference on High Energy Physics, Landau [1)] presented some new theorems on singularities of perturbation theory amplitudes. He showed that when we discuss a particular singularity, we only need to look at a " skeleton " of the Feynman graph, a " reduced graph." (See Fig. 1). The circles, which are the vertices of the reduced graphs, stand for any arbitrarily complicated subgraphs. Landau showed that a singularity is obtained when all the lines of a reduced graph correspond to particles which are simultaneously on the mass shell, and in addition satisfy certain geometrical relations. Each of the reduced graphs shown could arise from the same Feynman graph, and correspond to different singularities of the same amplitude.

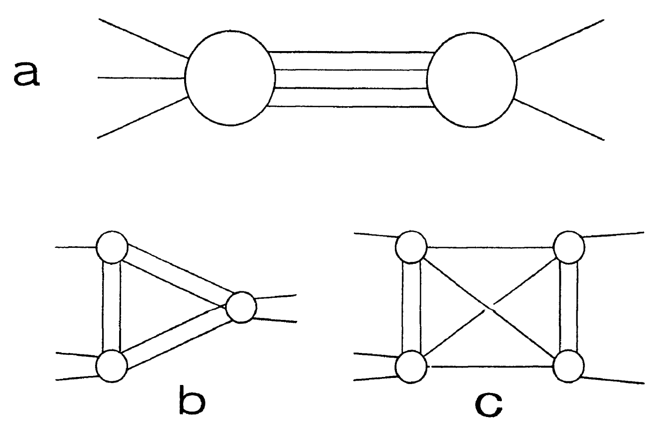

Fig. 1 Example of " reduced graph."

When we analyze these singularities further, we find that they are sometimes poles, and sometimes branch points. The residue of a pole is, of course, obtained by considering the subgraphs for the case that the lines leading into them represent particles which satisfy Landau's condition. When the singularity is a branch point, the discontinuity of the amplitude across the branch cut is obtained by an equally simple prescription. For each line of the Feynman graph which also appears explicitly in the reduced graph, the Feynman propagator is replaced by a delta function. In other words, the particles which correspond to the lines of the reduced graph are always taken to be on the mass shell. This prescription, when it is applied to a reduced graph like that on the top of Fig. 1, is equivalent to the familiar unitarity property of the S-matrix.

This theorem will perhaps be a little clearer after we outline a brief proof. The main idea of the proof is that we rewrite the Feynman integral in terms of the

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virtual masses of the intermediate particles, that is, in terms of the magnitudes of their four-momenta. The structure of the integral is then most evident [2].

We must first see whether the squares of the four-momenta which are associated with the lines of the reduced graph are all independent. If these squares do vary independently when the virtual momenta vary, then we can make a change of variables, using as variables these virtual masses and whatever additional variables we might need. Then we get an integral of the form :

$$F=\int\frac{dq_1^2}{M_1^2-q_1^2}\cdots\int\frac{dq_m^2}{M_m^2-q_m^2}G(q_1^2\dots q_m^2)\,, \qquad (1)$$

where $G(q_l^2\dots q_m^2)$ is the result of integrating over all the other variables. This function G, and the contours over which the q_i^2 are integrated, also depend on the external momenta.

The next step of the proof is to look at the first $m-l$ integrations in F, and identify the part of these integrals which is responsible for the existence of the Landau singularity. The idea can be explained by a simple example :

$$f=\int_p dq^2(M^2-q^2)^{-1}g(q^2)\,. \qquad (2)$$

The function f will become singular when the contour of integregation is "pinched" between the pole at $q^2=M^2$ and some singularity of $g(q^2)$, as in Fig. 2a. This pinched contour can be replaced by the two contours in Fig. 2b, in which the contour R gives a contribution to f which is regular when the singularities coalesce and therefore is of no interest in the present discussion. The singular part of f is obtained entirely from the small circle around the pole, so, in other words, we may replace the factor $(M^2-q^2)^{-1}$ by $2\pi i\delta(q^2-M^2)$.

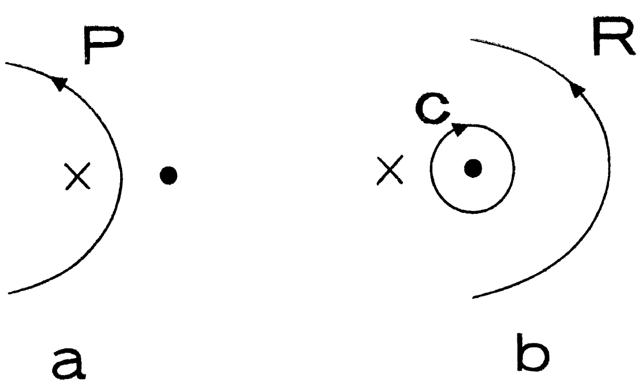

Fig. 2 Contours of integration.

We may apply this argument in turn to all but one of the q^2 integrations in Eq. (1) but the last integral is taken over a contour which is not pinched. Instead the end-point of the contour approaches the pole at $q_m^2=M_m^2$ when the amplitude F becomes singular. (It can be shown that this is equivalent to Landau's conditions.) When we analytically continue F past its singularity the end-point of the contour moves past the pole. Since the end-point can pass either above or below the pole, we may end up on either of two branches of F. The difference between the value of F on these two branches is just $2\pi i$ times the residue of the pole, so the rule of replacing the reciprocal of (M^2-q^2) by $2\pi i\delta(q^2-M^2)$ holds for each line of the reduced graph.

When the virtual masses of the particles in the reduced graph are not independent, that is when one of these masses is a function of the other virtual masses and of the external momenta, there is one less integration variable than there are poles. Then when we calculate the singular part of the amplitude we take the residue of all the poles for which there are integrations, but there remains at the end-pole one pole so we see explicitly that in this case the singularity of the amplitude is a pole. A simple example is given by a graph which consists of two parts connected by a single line. In this case the virtual momentum of the connecting line does not vary at all, when the momenta of the external lines are fixed. The importance of such poles in practical problems is already well understood [3].

The theorem we have just discussed applies to any finite order graph of the regularized, renormalized, perturbation theory. If we assume that the perturbation expansion can be treated as a formally convergent series, we can sum over all Feynman graphs which have a particular reduced graph, and which contribute to a particular discontinuity of an amplitude. When we do this, we automatically sum over all the subgraphs which contribute to the vertices of the reduced graph. When the vertex is a simple three-line vertex, the sum over all graphs gives the renormalized coupling constant, since the three entering momenta are all on the mass shell. The more complicated vertices become exact scattering amplitudes, or production amplitudes. In this way we obtain a generalization of the polology technique, because in the spectral representation of any amplitude, the contribution of all branch cut integrals and all poles can be represented in terms of other physical processes. We are assuming here, of course, that the sum does not have any singularities which do not appear in finite order terms.

One of the most interesting applications of the generalized unitary relation arises in Mandelstam's representation of scattering amplitudes [4]. The Mandelstam spectral functions are discontinuity functions which are associated with reduced graphs such as that in Fig. 3. The Mandelstam reduced graphs have at least four vertices, with each of the four external lines of the graph being connected to a separate vertex. For concreteness, let us talk about nucleon-nucleon scattering. We omit discussion of nucleon spin (although, as is well known, keeping track of the spin is most of the work in any practical calculation). Let us also just look at graphs in which there are only two mesons exchanged between the nucleons.

If the barred line in Fig. 3 refers to a single nucleon, the vertices give the pion-nucleon coupling constant, and the spectral function is just that obtained in fourth order perturbation theory. Other reduced graphs are obtained if the barred line represents a continuum state, with a nucleon plus a meson, etc. Then the two upper circles represent, for example, the meson-nucleon scattering amplitude. Since this reduced graph arises in calculation of the discontinuity of the imaginary part of the nucleon-nucleon scattering amplitude we may give all the propagators on the left half of the diagram masses with small positive, imaginary parts, and conversely on the right. Therefore one of the upper circles actually represents the conjugate scattering amplitude, so the integral over the meson-nucleon scattering angle reduces to an application of the ordinary unitarity relation to the meson-nucleon scattering matrix, and the entire top part of the graph gives just the imaginary part

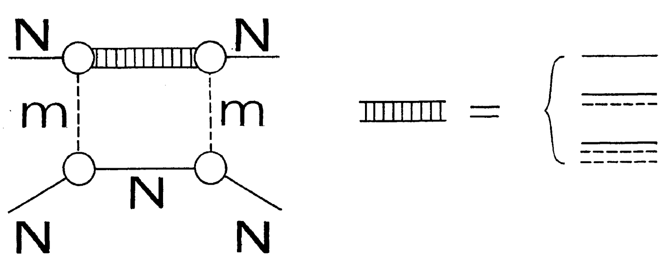

Fig. 3 Nucleon-nucleon scattering.

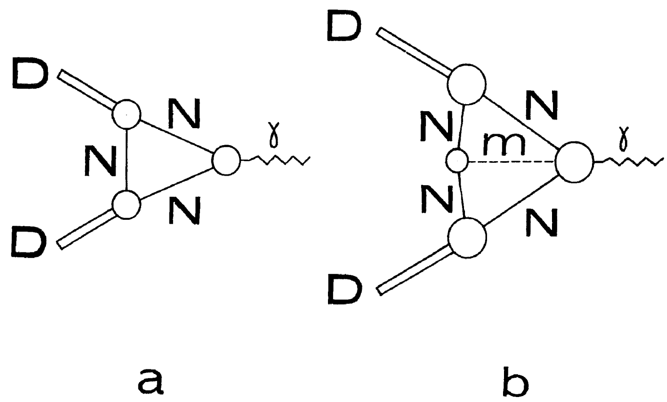

Fig. 4 Deuteron form factor.

of the meson-nucleon scattering amplitude. The final result has the form [5]

$$[T_{NN}(s,t)] = g^2 \int_{s_0}^{s_1} \frac{ds' \,\mathrm{Im}\, T_{\pi N}(s't)}{V(s,t,s')} \,. \qquad (3)$$

A result similar to this has been obtained independently by Cini and Fubini [6]. They used the formal apparatus of the more usual dispersion theory. The calculation based on generalized unitarity is more direct, and also provides a physical picture of Mandelstam's representation. The spectral functions arise rather mysteriously in the usual treatments, but have a simple physical interpretation in terms of reduced graphs.

Let us now turn to another problem, that of the structure of the deuteron, and in particular, the electromagnetic form factor of the deuteron. The discontinuity theorem, by enabling us to calculate the spectral functions associated with the so-called anomalous thresholds, allows us to develop a theory of the form factor which is independent of older methods involving wave functions. However, it is extremely important to keep the older approaches in mind simultaneously as a guide. We shall here treat only a simplified version of this problem, in which we again suppose that all of the particles which enter are spinless, even the photon. The form factor is the $(DD\gamma)$ vertex considered as a function of the momentum q imparted by a static external field. We write $t = q^2$; the physical values of t are negative. The form factor $F_D(t)$ has singularities along the positive real t axis. The singularity closest to the origin is associated with the reduced graph shown in Fig. 4a; the threshold is at $t_0^{\frac{1}{2}} = 4\alpha \frac{D}{2N}$. Our notation is that D, N, and m are the masses of the deuteron, nucleon, and meson, respectively, and $\alpha^2 = N^2 - \frac{1}{4}D^2$. Let us define the spectral function to be

$$f(t) = (2\pi i)^{-1}[F_D(t)],$$

or

$$F_D(t) = \int_{t_0}^{\infty} \frac{dt' f(t)}{t'-t}, \tag{5}$$

(if we may assume no subtractions are needed). The spectral function associated with the reduced graph in Fig. 4a is, according to the general discontinuity formula,

$$f_0(t) = F_N(t)\frac{C^2 D}{(2\pi)^2}\int d^4k\delta_1\delta_2\delta_3, \tag{6}$$

where $F_N(t)$ is the nucleon form factor. Since the integral in (6) involves nothing but delta functions it is very easy to evaluate, and leads to :

$$f_0(t) = \frac{C^2}{4\pi}F_N(t)\frac{1}{t^{\frac{1}{2}}}\frac{D}{(D^2-\frac{1}{4}t)^{\frac{1}{2}}} \tag{7}$$

The factor $D(D^2-\frac{1}{4}t)^{-\frac{1}{2}}$ in Eq. (7) can be pictured as a Fitzgerald contraction effect. The remaining factors are just the results obtained from the tail of the non-relativistic wave function.

The reduced graph with the next lowest threshold is shown in Fig. 4b. This graph contains a pion production vertex which we must discuss before we can consider the contribution of the graph to the form factor. For a fixed value of $t = q^2$, we may envisage using a Mandelstam representation in the two variables $s_1 = (k+q_1)^2$ and $s_2 = (k+q_2)^2$. There is a pole in each of these two variables, arising from connecting nucleon lines, and also the usual branch points. The poles lie much closer to the relevant region of integration than do the branch points, so as a first approximation we take just these two pole terms. The remainder of the vertex gives what is commonly known as a " non-additivity " correction, which would be very interesting to examine in a further study.

The reduced graph of Fig. 4b corresponds to the Landau scheme consisting of the solid lines in Fig. 5. The threshold calculated from the geometry of the diagram is :

$$t_1^{\frac{1}{2}} = 2m\left(\frac{D}{2N}\right)^2 + 4\alpha\frac{D}{2N}\left(1-\frac{m^2}{4N^2}\right)^{\frac{1}{2}}. \tag{8}$$

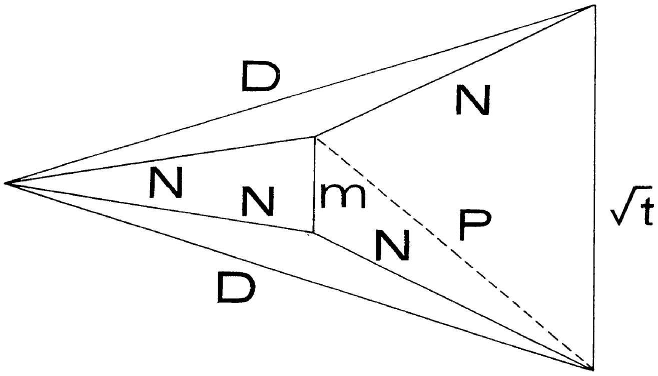

Fig. 5 Landau scheme for the deuteron form factor.

The discontinuity of $F_D(t)$ is calculated from an integral in which the two internal points of the diagram of Fig. 5 vary over four-dimensional space, subject to the restriction that the five internal solid lines have fixed lengths. It is useful to take as one of the variables the length of the dotted line shown. This is in fact equal to the virtual mass P of the intermediate nucleon which is responsible for the pole.

When P is fixed we have separate integrals which are both similar to that in Eq. (6), so we obtain directly a simple form for the spectral function :

$$f_1(t) = -f_0(t)\frac{g^2}{4\pi}\int_{P_o{}^2}^{P^2(t)} \frac{dP^2}{P^2-N^2}\times$$

$$\times\frac{N^2}{[P^2-(D-N)^2]^{\frac{1}{2}}[(D+N)^2-P^2]^{\frac{1}{2}}} \tag{9}$$

The limits of integration as obtained from the geometry of Fig. 5 are :

$$P_0^2 = N^2 + 2m^2\left(\frac{D}{2N}\right)^2 + 4\alpha m\frac{D}{2N}\left(1-\frac{m^2}{4N^2}\right)^{\frac{1}{2}},$$

$$P^2(t) = N^2 + \tfrac{1}{2}t - 2\alpha t^{\frac{1}{2}}\left(1-\frac{t}{4D^2}\right)^{\frac{1}{2}}.$$

Note that $f_1(t)$ contains $f_0(t)$ as a factor. The term (P^2-N^2) in the denominator is the pole of the pion production vertex. We can see explicitly in Eq. (9) how close to the pole the main part of the integral is located. If we let $N\to\infty$ in Eq. (9), keeping α and m fixed, we obtain the non-relativistic limit. This limit is exactly equal to the result obtained from the non-relativistic Schrodinger equation with a Yukawa potential, when one calculates the effect of the potential on the wave function which is of lowest order, and longest range [7].

The two contributions to the form factor which we have calculated have the form

$$f_D(t) = f_b(t) F_N(t) ,$$

where $F_N(t)$ is the nucleon form factor, and f_b we may call the "bare" part of the deuteron form factor. We should like to derive such a formula not only for the discontinuity of the form factor, but also for the form factor itself. The elementary picture of the deuteron tells us that the charge density of the deuteron is obtained by folding the density of "bare" protons into the proton's charge density, or, in other words, that the form factor is the product of the two corresponding form factors. We therefore consider the quantity

$$F'_D(t) = F_b(t) F_N(t)$$

and ask, is $F'_D = F_D$? This question is answered by showing that F_D and F'_D have the same analytic structure [8].

Both F_D and F'_D have singularities at t_0 and t_1, and F'_D is so constructed that the discontinuities at these branch points are identical, apart from the "non-additive" term. However, F_D and F_N both have additional singularities such as represented by the reduced graphs (a) and (b) in Fig. 6, which obviously occur at the same value of t for both F_D and F_N. Let us call these "t" singularities, since their location involves only t and not other quantities such as the external masses D or N, and denote the discontinuities by $[F_D]_t$ and $[F'_D]_t = F_b[F_N]_t$. Note that $[F'_D]_t$ is singular at t_0; the discontinuity is $[[F'_D]_t]_0 = [F_b]_0 \times [F_N]_t$. The function $[F_D]_t$ is also singular at t_0; the reduced graph is shown in Fig. 6c. This is a rather special graph; it consists of two parts connected by a single vertex. While such a graph does not occur when we discuss the singularities of the form factor itself, we must take it into account when we consider the singularities of the discontinuity function [9]. We find that when we apply the discontinuity theorem to this graph the discontinuity of the discontinuity is the product of two factors, one associated with each of the two parts, and in fact $[[F_D]_t]_0 = [F_b]_0 \times [F_N]_t = [[F'_D]_t]_0$. From this it can be seen to follow that $[F_D]_t = [F'_D]_t$. (If a given Feynman graph has additional singularities we proceed inductively; also we are still ignoring the non-additive term). From this we deduce

$$F_D(t) = F_b(t) F_N(t) + F_{\text{non-add.}}(t) \tag{10}$$

This result has great practical significance, because it means that in calculating a theoretical deuteron form factor for physical values of t we do not need to know the nucleon form factor in the unphysical region; we can use the experimental data directly. Moreover the definition of the non-additive term is unambiguous, and the method of calculating it is straightforward (at least in principle).

The form factor of the deuteron also enters, but in a different way and only as an approximation, in the impulse approximation to scattering by a deuteron. The impulse approximation to the meson-deuteron scattering amplitude is :

$$T_{\pi D}(s, t) = \int \psi_D(k) T_{\pi N}(s'(k), t; k) \psi_D\left(k - \frac{q}{2}\right) d^3k , \tag{11}$$

where q is the momentum transfer. The meson-nucleon scattering amplitude is extrapolated off the mass shell by an amount depending on k. It is usual to make a further approximation, in which $T_{\pi N}$ is replaced by an average value, which is on the mass shell; then the integral over k gives the form factor, and we obtain :

$$T_{\pi D}(s, t) = T_{\pi N}(s', t) F(t) , \tag{12}$$

where $s' = s'(s, t)$.

Now let us investigate the spectral representation of $T_{\pi D}$, using t as the variable. There are thresholds at the same points t_0 and t_1 which occur in the electro-

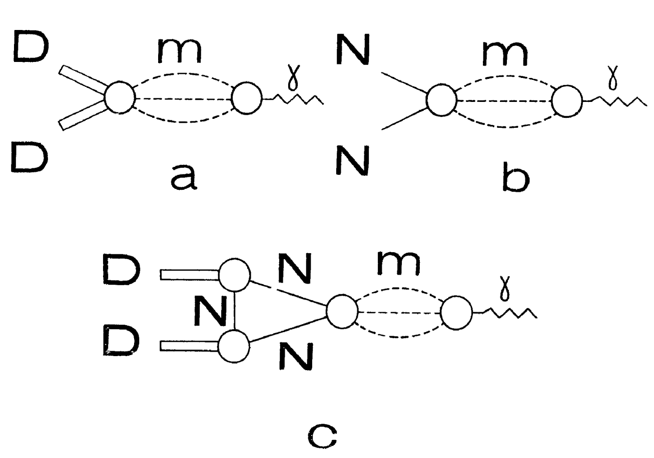

Fig. 6 "t" singularities in form factors.

magnetic form factor, and some new ones which we will discuss shortly. Let us confine our attention to the tail of the deuteron wave function, that is, to the branch point t_0; the next term is only slightly more complicated. The reduced graph is shown in Fig. 7a. The associated spectral function can be written in the form

$$f_{\pi D}(st) = f_b(t) \oint \frac{ds' T_{\pi N}(s', t)}{2\pi i[(s'-s_+)^{\frac{1}{2}}(s'-s_-)^{\frac{1}{2}}]}, \qquad (13)$$

where $f_b(t)$ is again the bare spectral function. The form of Eq. (13) is derived by a very simple argument based on examination of the geometry of the Landau scheme associated with the reduced graph in Fig. 7a. The limits of variation of s' are also obtained from the geometry:

$$s_\pm = (4D^2-t)^{-1}\{(s-m^2)^2 - \Big[[s^2+st+(D^2-m^2)^2 - 2s(D^2+m^2)]^{\frac{1}{2}} \pm (N^2t-4\alpha^2D^2)^{\frac{1}{2}}\Big]^2\}.$$

The form of Eq. (13) suggests that we write:

$$T_{\pi D}(s, t) = F_b(t) \oint \frac{ds' T_{\pi N}(s', t)}{2\pi i[(s'-s_+)(s'-s_-)]^{\frac{1}{2}}} + \widetilde{T}_{\pi D}(s, t). \qquad (14)$$

It is not hard to show that the first term is equal to Eq. (12), apart from relativistic corrections—terms which involve the square of the nucleon velocity.

The extra term $\widetilde{T}_{\pi D}$ in Eq. (14) is a result of the greater complexity of the pion-nucleon scattering vertex, as compared to the $(NN\gamma)$ vertex, which shows up in the fact that not all the additional singularities are "t" singularities. The pion-deuteron scattering amplitude has a singularity corresponding to the reduced graph of Fig. 7, where the barred line represents any intermediate state. The first term of Eq. (14) also is singular, at exactly the same value of t. This singularity arises from a pinching of the s' contour between one of the singularities of the denominator and the singularity of $T_{\pi N}(s', t)$ associated with graph c of Fig. 7. It occurs on both sheets of the Riemann surface (with respect to the cut from t_0), while the singularity of $T_{\pi D}$ exists on only one branch.

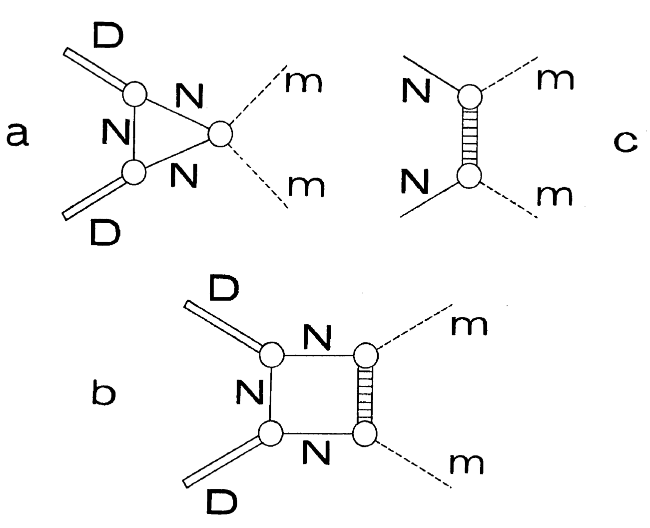

Fig. 7 Pion-deuteron scattering.

The nucleon pole term of $T_N(s' = N^2)$ gives a contribution to the spectral representation of $\widetilde{T}_{\pi D}$ which is obtained from

$$[\widetilde{T}_{\pi D}(s, t)]_{N^2} = g^2 F_b(t)[(N^2-s_+)(N^2-s_-)]^{-\frac{1}{2}}. \qquad (15)$$

The continuum states ($\sqrt{s'} > N+m$) give

$$[\widetilde{T}_{\pi D}(s, t)]_{s'} = \pi^{-1} \operatorname{Im} T_{\pi N}(s', t) F_b(t)[(s'-s_+) \times (s'-s_-)]^{-\frac{1}{2}}. \qquad (16)$$

In Eq. (15) and Eq. (16), $F_b(t)$ must be taken to be on the sheet which *does not* contain the singularity of $T_{\pi D}(s, t)$.

The term $\widetilde{T}_{\pi D}(s, t)$ is obviously to be interpreted as reproducing the effects which in the older theory arise from extrapolation of the scattering matrix off of the mass shell, but we are able to express these effects entirely in terms of the ordinary scattering matrix. The treatment of the threshold at $t = t_1$ is only slightly more complicated than that given here, as long as the non-additive effects are ignored. By continuing the spectral analysis of $T_{\pi D}(s, t)$ other reduced graphs can be brought in, and in particular, the effects of multiple scattering can be included.

In summary, we have tried to indicate here the utility of the discontinuity formula through brief discussions of the Mandelstam representation and of the anomalous thresholds in the deuteron problem. It should be pointed out that much further study must be given to the remaining unsettled questions about the theoretical foundation of the techniques presented here, and also about their application to more complicated problems. Nevertheless, it appears that Landau's discussion of singularities, as supplemented by the general unitarity relation, incorporates the maximum information obtainable from convariance and causality, and provides the basis for a complete graphical calculus. The examples are intended to show that the application of this graphical calculus to physical problems is straightforward, and has the particular advantage that the relation to older techniques and intuitive concepts is always directly evident.

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DISCUSSION

OPPENHEIMER: Would you amplify your remarks that the non-additive terms are easy to calculate?

CUTKOSKY: I do not say that they are easy to calculate; what I say is that the way in which one should attack the problem is straightforward. For example, in the electromagnetic problem, we consider values of the momentum transfer which are not 0. The photon is not a free photon. In fact, one has to consider photoproduction for various values of t in order to put it into the spectral representation of the form factor. I presume the same techniques that have been used very successfully in these problems can be used here as well, although there is the added complication that t is not zero. One can use a Mandelstam representation for appropriate values of t.

CHEW: Am I correct in the impression that you are not afraid now of calculating the five-lined diagrams and more complicated things? Do you see your way clear to techniques for doing these?

CUTKOSKY: I would say that in principle I can see that there is a way to do it, but in practice there are still many things that have to be settled, such as the question of which are the most convenient variables in which to make the analytic continuations. Also, one has to find out just where the singularities are. As has been pointed out already, this morning, their positions will, in general, be in the complex region, but this is not a great difficulty as long as one knows where they are and one has a technique for studying the spectral function associated with these singularities. One can treat them in the same way as one treats the real singularities in the Mandelstam representation.

BJORKEN: When you replace your propagators by delta functions, some of the momenta become complex. Is there a question of analytic continuation?

CUTKOSKY: There is a question of analytic continuation which comes up in this way. In the ordinary unitarity condition, one has a relation in which one has just the positive frequency part of the delta function, and this comes up here as well. There will, in general, be different roots for the q^2 and you have to be careful to pick the right root. This is a question of being careful about which sheet of the Riemann surface one is on.

BJORKEN: Do you have a general rule for choosing these roots?

CUTKOSKY: In some simple cases I know just how to do it. In general, I do not have a criterion except that I know that there is just one root which contributes.

EDEN : I would like to comment on this point in connection with the five-legged diagram. If the five-legged diagram consists of one closed loop, then Cutkosky has shown that this corresponds to a pole and it is thought that poles might be dangerous for some of the experiments on pion production in the investigation of the pion-pion cross section. Recently, at Berkeley, Cook and Tarski have been investigating this point with particular regard to the location of the singularities. The location can be worked out tediously and the difficulty is to find which sheet the singularity is in. They have done this by tracing from a place where they know whether or not a curve is singular and simply following it through to the position of the pole. They are explicitly following the surface of singularities which Polkinghorne described this morning. Their indications up to now are that the pole in the $\pi+N\rightarrow\pi+\pi+N$ situation is probably not dangerous. The pole is, in fact, in the physical scattering region and therefore must be on another sheet. The same appears to be true regarding the pion production by protons on deuterons to give He^3 which is also underway in Berkeley. On the other hand, for the deuteron form-factor which was mentioned earlier, one has three pions going into a deuteron and an anti-deuteron. It seems that the five-legged closed loop pole might well be dangerous and contribute to the form factor in this case.

CUTKOSKY : I would not say that was dangerous. I would say it was interesting.

THE COMPLEX SINGULARITIES OF PARTIAL WAVE AMPLITUDES IN PERTURBATION THEORY

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(presented by J. G. Taylor)

Recently, a general method has been developed for locating the complex singularities of contributions from Feynman diagrams [1]. In particular, Tarski [2] has studied in detail the fourth order square diagram, obtaining, in a simple manner, the condition on the masses under which the two-dimensional representation of Mandelstam [3] is no longer valid. Using Tarski's analysis, we show that the complex singularities which invalidate Mandelstam's representation do not cause complex singularities of the partial-wave amplitudes, $C_l(q^2)$. This result is of importance in that the most useful analyticity property for scattering amplitudes is, in practice, a cut plane of analyticity for the partial-wave amplitude. Of course, if the outgoing particles differ from the incoming particles, there will be the expected "kinematic" complex branch points, arising from the branch points of the total energy s regarded as a function of q^2, the squared centre-of-mass momentum of an incoming particle.

Neglecting irrelevant factors, the contribution from the diagram

$$F(s,t) = \int_0^1 \prod_{i=1}^{4} dx_i \delta(1-\Sigma x_i)/D^2 \tag{1}$$

while

$$C_l(g^2) = \int_{-1}^{+1} d(\cos\theta) F(q^2, \cos\theta) P_l(\cos\theta) \tag{2}$$