Faraday Resonance in Dynamically Bar Unstable Stars

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Abstract

We investigate the nonlinear behaviour of the dynamically unstable rotating star for the bar mode by both three-dimensional hydrodynamics in Newtonian gravity and our simplified mathematical model. We find that an oscillation along the rotation axis is induced throughout the growth of the unstable bar mode, and that its characteristic frequency is twice as that of the bar mode, which oscillates mainly along the equatorial plane. We also find that, by examine several azimuthal modes, mode coupling to even modes, i.e., the bar mode and higher harmonics, significantly enhances the amplitudes of odd modes, unless they are exactly zero initially. Therefore, non-axisymmetric azimuthal modes cannot be neglected at late times in the growth of the unstable bar-mode even when starting from an almost axially symmetric state.

Dynamical bar instability in a rotating equilibrium star takes place when the ratio β ($\equiv T/W$) between rotational kinetic energy T and the gravitational binding energy W exceeds the critical value $\beta_{\rm dyn}$ (≈ 0.27 for an uniformly rotating incompressible body in Newtonian gravity [1]). Determining the onset of the dynamical bar-mode instability, as well as the subsequent evolution of an unstable star, requires a fully nonlinear hydrodynamic simulation. Recent numerical simulations also show that dynamical bar instability can occur at significantly lower β than the threshold $\beta_{\rm crt} \approx 0.27$ in some cases.

Our main concern in this paper is not to determine the onset of the instability, but to study the dynamical features of the bar. For this purpose, we numerically study the growing behaviour of the azimuthal modes in the nonlinear regime for a longer timescale. One interesting issue of nonlinear evolution is the possibility of resonant growth of other azimuthal modes triggered by the dynamical bar-mode instability. One candidate for such resonance is Faraday resonance, which is excited by the external periodic force. The dynamically unstable bar mode may work for other azimuthal oscillation modes as an external periodic force. Although the oscillation is not exactly periodic, but rather quasi-periodic, it may trigger a parametric resonance.

The other interesting issue of nonlinear evolution is to study the physical mechanism of the growth of odd modes. Are there unstable modes with odd numbers (e.g., $m = 1$ or 3) in addition to the unstable bar mode? Are the amplitudes of the odd modes enhanced by mode coupling? In order to understand the growth of odd modes, we investigate the evolution of a simplified model. The model's description of mode coupling, unstable growth, and decay mimics the realistic system very well. Moreover, the number and growth rates of the unstable modes are easily controlled. The model, therefore, deepens our understanding of the nonlinear behavior of unstable bar-mode growth in rotating stars. The physical mechanism is confirmed by comparing the model problem with a more realistic calculation of a dynamically unstable star simulated using three-dimensional hydrodynamics in Newtonian gravity.

A more detailed discussion is presented in Refs. [2, 3]. Throughout this paper, we use the geometrized units with $G = 1$ and adopt Cartesian coordinates (x, y, z) with the coordinate time t.

We study four different differentially rotating stars, which are detailed in Table 1 of Ref. [2] to investigate the nonlinear behaviour of the non-axisymmetric dynamical bar instabilities using three dimensional hydrodynamics in Newtonian gravity. We disturbed 1% of the equilibrium density by a non-axisymmetric perturbation to enhance any dynamically unstable mode.

We show the diagnostics of the model III (the weakest dynamically bar unstable system among model I – III) here, which contain both amplitude and phase in Fig. 1. The behaviours in the diagnostics

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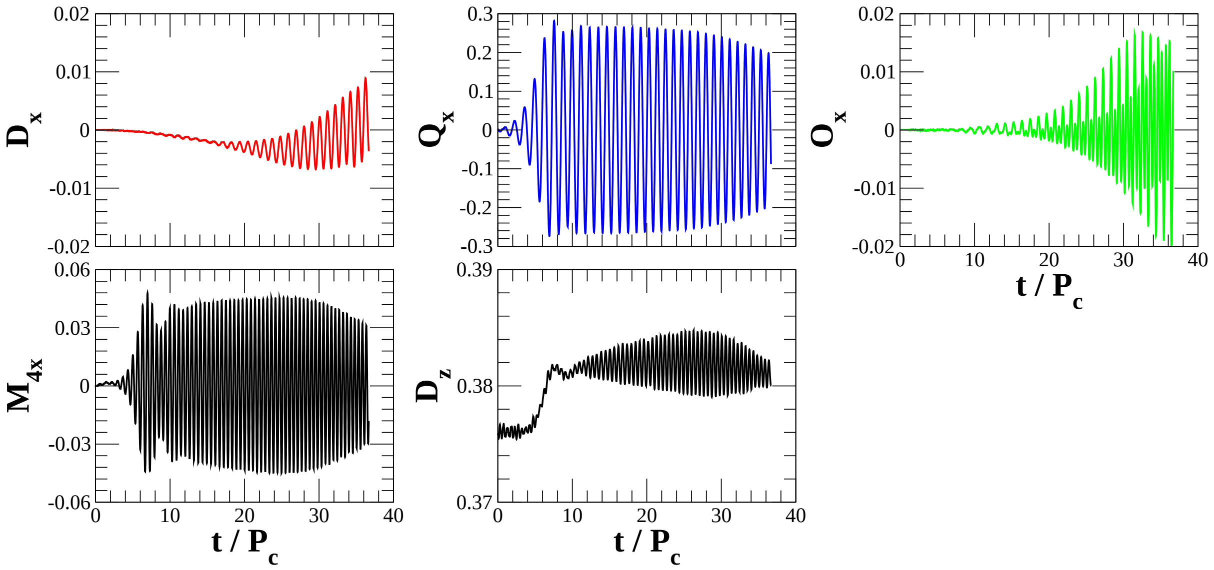

Figure 1: Diagnostics $\Re[D]$, $\Re[Q]$, $\Re[O]$, $\Re[M_4]$, and D_z as a function of t/P_c for the differentially rotating star of model III (see Table 1 of Ref. [2]). Note that the five diagnostics are defined in Ref. [2]. Hereafter, P_c is the central rotation period of the equilibrium star.

are clearly understood once we compute the spectra of the diagnostics (Fig. 2). From the spectra we find the following two remarkable features. One is that the spectra $|F_1|^2$, $|F_2|^2$, $|F_3|^2$ take a peak around $\omega_{\rm bar} \approx 5 \sim 6P_c^{-1}$ for model III, and the other is that $|F_3|^2$, $|F_4|^2$, $|F_z|^2$ take a peak around $\omega_{\rm quad} \approx 2\omega_{\rm bar} \approx 10 - 12P_c^{-1}$for bar unstable stars. Combining the present feature with the behaviour of the amplitude of the diagnostics (Fig. 1) [2], the dynamically unstable bar acts as follows.

Firstly the $m = 2$ mode grows and acts as a dominant mode of all because of the dynamical bar instability. Next the $m = 4$ mode grows because of the secondary harmonic of the $m = 2$ mode. In fact the saturation amplitude of the $m = 4$ is approximately ≈ 0.04, which is the order of the square of the saturation amplitude of the $m = 2$ ($\approx 0.2^2$). After that Faraday resonance occurs, which is clearly found in both D_z and $|F_z|^2$ from the fact $\omega_{\rm quad} \approx 2\omega_{\rm bar}$.

Note that Faraday resonance occurs in the fluid mechanics when the oscillation of the vertical direction is twice (2ω) as much as the one in the horizontal direction (ω) in the weakly nonlinear interaction [4]. Then, there is a resonance between $m = 1$ and $m = 2$, $m = 3$ and $m = 4$. The possibility of such resonances is three wave interaction: either $m = 1$ ($\omega_{\rm bar}$) and $m = 2$ ($\omega_{\rm bar}$) generates $m = 3$ ($\omega_{\rm bar} + \omega_{\rm bar}$)

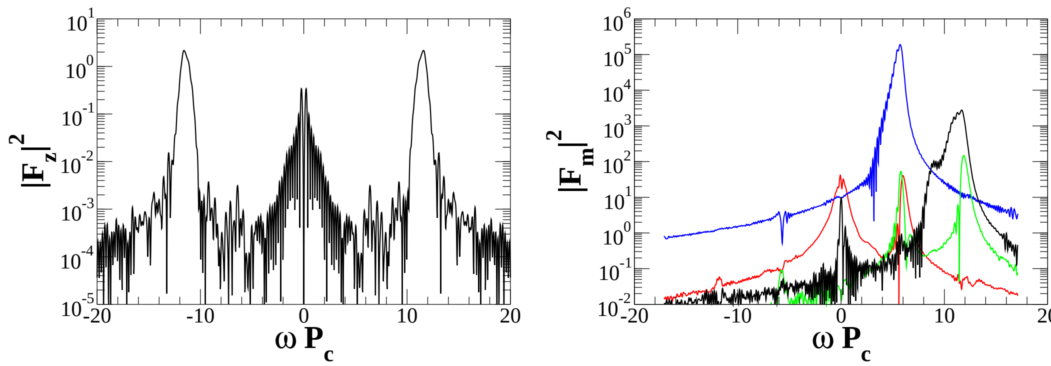

Figure 2: Spectra $|F_m|^2$ and $|F_z|^2$ as a function of ωP_c for the differentially rotating star of model III (see Table 1 of Ref. [2]). Red, blue, green, and black line of $|F_m|^2$ denote the values of $m = 1$, 2, 3, and 4, respectively. Note that the spectra $|F_m|^2$ and $|F_z|^2$ are defined in Ref. [2].

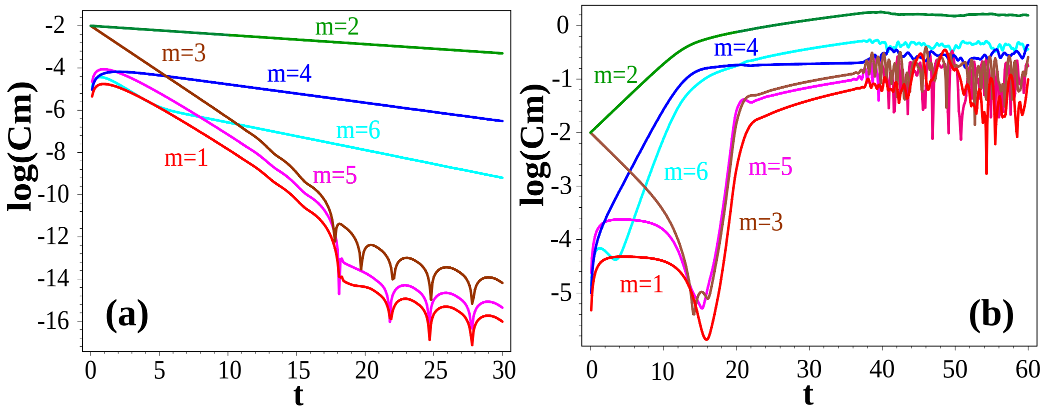

Figure 3: The time-evolution of the amplitudes of the lowest six Fourier modes ($m = 1, \cdots, 6$). Panel (a) is the result for $\nu = 0.15$ ($m = 2$ stable), (b) for $\nu = 0.05$ ($m = 2$ unstable). The initial coefficients (a_m, b_m) of the Fourier series expansion of the flow velocity $u(t, x)$ are $a_2 = a_3 = 10^{-2}$ for both cases. The behaviour of very small amplitude, say, $\log(C_m)$ ($C_m = \sqrt{{a_m}^2 + {b_m}^2}$) approximately less than -10 in panel (a), mostly comes from numerical truncation errors, and is therefore unimportant.

or $m = 3$ ($2\omega_{\rm bar}$) and $m = 2$ ($\omega_{\rm bar}$) generates $m = 1$ ($2\omega_{\rm bar} - \omega_{\rm bar}$) in the dominant part. It is the fact found in the nonlinear behaviour of the dynamically unstable bar system.

We introduce a simplified model [3] to examine the nonlinear evolution of the unstable modes, especially taking into account the nonlinearity and instability caused by an external force. Our model is Burgers' equation for a flow velocity $u(t, x)$ coupled to a scalar field $\phi(t, x)$:

$$\partial_t u + u \partial_x u = \nu \partial_x^2 u + \phi, \tag{1}$$

$$\partial_x^2 \phi + 2\phi = -u + 1, \tag{2}$$

where ν is a diffusion constant. We regard $u = 1$ and $\phi = 0$ as the background state and consider linear stability and nonlinear growth using Fourier series expansion from this uniform state.

The amplitudes of Fourier series expansion of the flow velocity $u(t, x)$ are shown in Fig. 3. Note that case (a) is stable to $m = 2$ mode, while case (b) unstable. In Fig. 3(b), the $m = 2$ mode grows exponentially until $t \sim 15$, where the amplitude of the $m = 2$ mode reaches the nonlinear regime: $10^{-2} \times \exp(0.3 \times 15) \sim 1$. All other even modes, originating from the bilinear coupling term $u \partial_x u$, also grow. The $m = 6$ mode is produced from the coupling between $m = 2$ and $m = 4$ and also from the quadric coupling of $m = 3$. Therefore, the amplitude of the $m = 6$ mode is not always smaller than that of $m = 4$. The growth of all even modes is slightly suppressed after the turning time $t \sim 15$. The turning time is also important for the odd modes. The odd modes decay for $t \lesssim 15$, but grow after that. Therefore, the nonlinearity of the amplitude of the $m = 2$ mode cannot be ignored even for the odd modes. The turning time corresponds to shock formation as will be discussed later. For Eqs. (1) and (2) all odd modes are always zero, if they are exactly zero initially. When there is at least one odd mode with a finite amplitude, the nonlinearity of the $m = 2$ mode enhances all odd modes.

The similarity can be seen in the time evolution of the Fourier components both in mathematical and three-dimensional numerical models [3]. The time evolution of the shape $u(t, x)$ is shown in Fig. 4. The $m = 2$ mode initially grows and the shape is enhanced before the turning time $t \sim 15$. The curve at $t = 4\pi$ clearly shows symmetric features due to the $m = 2$ mode. That is, the shape is the symmetry under translations $x \to x + \pi$, a "π-symmetry". The nonlinearity causes a shock as in the original Burgers' equation. After shock formation, the Gibbs phenomenon associated with Fourier series is seen at $t = 8\pi, 16\pi$. The overshoot is a numerical artifact and such behaviour always appears when a function having a sharp discontinuity is expressed as a Fourier series. Neglecting the Gibbs phenomenon, the symmetry due to the $m = 2$ mode can still be seen in the shape at $t = 8\pi$, whereas it is partially broken at $t = 16\pi$. The time $t = 16\pi$ in the mathematical model is much longer than that of nonlinear

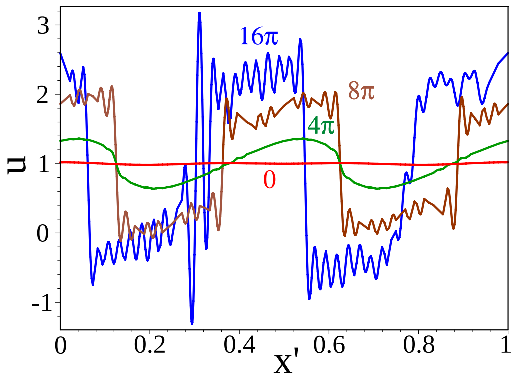

Figure 4: Snapshots of the flow velocity $u(t, x)$ are shown as a function of $x' = x/2\pi$ for $t = 0, 4\pi, 8\pi, 16\pi$ for case (b) in Fig. 3. The attached labels denote the time t.

saturation and that of growth of odd modes. Therefore, there is no counterpart in three-dimensional numerical simulations. The mathematical model suggests that a "π-symmetry" (i.e., symmetry under a 180° rotation around the z-axis) in the shape is broken in a longer timescale.

We investigate the nonlinear effects of dynamically bar unstable stars by means of both three dimensional hydrodynamic simulations in Newtonian gravity and our simplified mathematical model.

We find interesting mode coupling in the dynamically unstable system in the nonlinear regime, and that only before the destruction of the bar. The quasi-periodic oscillation mainly along the rotational axis is induced. The characteristic frequency is twice as big as that of the dynamically unstable bar mode. This feature is quite analogous to the Faraday resonance. Although our finding is only supported by the weakly nonlinear theory of fluid mechanics, we have also found the same feature of parametric resonance even in the strongly nonlinear regime [2].

We also find that our mathematically simplified model provides a concrete example showing the importance of mode coupling. The amplitudes of odd modes increase without unstable odd modes being present in the axially symmetric state; instead, they are enhanced by the bar instability with $m = 2$. We also confirmed that this physical picture is consistent with the results from a three-dimensional hydrodynamics simulation. Generally, the odd modes grow only after the bar instability reaches the nonlinear regime. The timescales of the mode coupling and the growth of unstable modes may depend on the rotation law and the strength of the initial instabilities. It is very rare that the initial perturbations in the hydrodynamics simulation should consist of purely even or odd modes only. Therefore, the unstable bar mode enhances the amplitudes of the all other modes at late times, no matter whether they are even or odd.

A similar mode coupling can be seen in numerical simulations for the one-armed spiral instability and the elliptical instability of rotating stars in Newtonian gravity. The initial models and the growth mechanism are different, but the turbulent-like behaviour appears in diagnostics of the azimuthal Fourier components at late times of nonlinear growth. The behaviour is also important for the nonlinear saturation of the unstable mode. Further study is necessary to explore the origin of the similarity seen in the development of different unstable modes. It is reasonable to assume that the nonlinearity in hydrodynamics is the source of this similarity.

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