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The equivalence principle is NOT a Noether symmetry

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Abstract The connection between the equivalence principle and Noether's theorem was discussed in Capozziello and Ferrara (Int J Geom Methods Mod Phys 21:2440014, 2024). However, it is known that the Noether symmetry condition is independent of the equations of motions as follows from Hamilton's principle. In this paper, we critically examine the analysis presented in the aforementioned work, highlighting its flaws, and provide a detailed demonstration that there is no connection between Noether's theorem and the equivalence principle. Furthermore, we offer various insights on symmetry analysis to prevent the perpetuation of inaccuracies regarding Noether's work.

1 Introduction

Symmetry analysis plays a critical role in the analytical treatment of nonlinear differential equations. Symmetries are essential in many aspects of physical science, particularly in gravitational physics, and are thoroughly explored in the literature. However, despite the extensive number of studies in mathematical science and physics on the treatment of symmetries, many inaccuracies persist in the literature, leading to incorrect conclusions and results.

In a recent study [1], they investigated the relation of Noether's theorem with that of the Einstein's equivalence principle. The authors concluded that Einstein's equivalence principle is a Noether symmetry for General Relativity and for metric-affine theories. However, these conclusions were reached due to a flawed application of symmetry analysis and a misinterpretation of Noether's theorems. In this study, we critically examine the methodology used in [1] and demonstrate that their results are incorrect. We firmly state that Einstein's equivalence principle is not a Noether symmetry. The structure of the paper is organized as follows.

In Sect. 2 we discuss the general form of Noether's theorems. In Sect. 3 we present the application of Noether's symmetry conditions for the geodesic Lagrangian, where we point the flawed results of [1]. Finally, in Sect. 4 we draw our conclusions.

2 Noether's theorems

Symmetry plays a fundamental role in numerous aspects of the physical world. The systematic treatment of symmetry analysis was introduced in the late nineteenth century by Sophus Lie, whose work revolutionized the approach to differential equations. Lie's contributions introduced the concept of considering the infinitesimal representations of finite transformations of continuous groups. This methodology facilitated the transition from group theory to a local algebraic representation, enabling a deeper exploration of invariance properties under these transformations.

The primary goal of determining the invariant transformations that leave a given differential equation unchanged is to simplify the solution of the equation under study. The presence of symmetries enables one to solve differential equations through repeated reduction of order, often using a reverse series of quadratures or by determining a sufficient number of first integrals.

A few decades later based on the spirit of Lie's approach, Emmy Noether presented her pioneer work which include two main theorems [2]. Noether's first theorem treats the invariance of the functional of the Calculus of Variations – the Action Integral in Mechanics – under an infinitesimal transformation. On the other hand, Noether's second theorem relates variational symmetries to conserved quantities. Notably, Noether's work allows the coefficient functions of the infinitesimal transformations to depend on the derivatives of the dependent variables, extending the applicability of symmetry analysis.

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Moreover, the boundary function on the Action Integral can include higher-order derivatives of the dependent variables. Thus, a series of studies which discuss generalizations of Noether's work are all included on Emmy Noether's original work, for more details we refer the reader to the recent discussion in [3]. Some prior studies of Noether's work which investigate the symmetries are the independent studies of Hamel [4,5], of Herglotz [6], of Knesner [7] and of Klein [8]. An English translation of Noether's work and more historical details on the foundations of the two theorems are presented in [9].

In the following lines we present the basic elements of Noether's theorems.

Let us introduce the infinitesimal transformation

$$\bar{t} = t + \varepsilon\xi, \qquad \bar{q} = q + \varepsilon\eta, \tag{1}$$

where the variable dependences of the functions ξ and η is arbitrary and ε is an infinitesimal parameter, that is, $\varepsilon^2 \to 0$.

We consider the Action Integral

$$S = \int_{t_0}^{t_1} L\left(t, q, \dot{q}\right) dt. \tag{2}$$

where $L\left(t, q, \dot{q}\right)$ is the Lagrange function. Under the Action of the infinitesimal transformation (1) reads

$$\bar{S} = \int_{\bar{t}_0}^{\bar{t}_1} L\left(\bar{t}, \bar{q}, \dot{\bar{q}}\right) d\bar{t}. \tag{3}$$

Thus, expanding in first order of the parameter ε we find

$$\bar{A} = \int_{t_0}^{t_1} \left[L + \varepsilon \left(\xi \frac{\partial L}{\partial t} + \eta \frac{\partial L}{\partial q} + \zeta \frac{\partial L}{\partial \dot{q}} + \dot{\xi} L \right) \right] dt + \varepsilon \left[\xi t_1 L(t_1, q_1, \dot{q}_1) - \xi t_0 L(t_0, q_0, \dot{q}_0) \right], \tag{4}$$

or equivalently

$$\bar{A} = A + \varepsilon \int_{t_0}^{t_1} \left(\xi \frac{\partial L}{\partial t} + \eta \frac{\partial L}{\partial q} + \zeta \frac{\partial L}{\partial \dot{q}} + \dot{\xi} L \right) dt + \varepsilon F, \tag{5}$$

where

$$F = \xi t_1 L(t_1, q_1, \dot{q}_1) - \xi t_0 L(t_0, q_0, \dot{q}_0), \tag{6}$$

$L(t_0, q_0, \dot{q}_0)$ and $L(t_1, q_1, \dot{q}_1)$ are the values of L at the endpoints t_0 and t_1 respectively, and

$$\zeta = \dot{\eta} - \dot{q}\dot{\tau}. \tag{7}$$

Since function F depends only upon the endpoints it can be written as $F = -\int_{t_0}^{t_1} \dot{f} dt$. Therefore, the variation of the Action Integral (2) remain invariant under the application of the infinitesimal transformation (1) if and only if

$$\xi \frac{\partial L}{\partial t} + \eta \frac{\partial L}{\partial q} + \zeta \frac{\partial L}{\partial \dot{q}} + \dot{\xi} L - \dot{f} = 0 \tag{8}$$

that is

$$X^{[1]} L + \dot{\xi} L - \dot{f} = 0, \tag{9}$$

in which $X^{[1]} = X + \zeta \partial_{\dot{q}}$ is the first extension of the vector field X in the jet space, and

$$X = \xi \partial_t + \eta \partial_q. \tag{10}$$

is the generator of the infinitesimal transformation (1). When condition (9) the vector field is called Noether symmetry. Without loss of generality the coefficients of the vector field X, i.e. ξ and η is arbitrary which means that they can depend on the derivatives of the dependent variables or on more general forms.

Therefore, the classification of Noether's symmetries as presented in [1] and [10] into generalized Noether symmetries, when $\dot{\xi} \neq 0$, Noether symmetries for canonical Lagrangians $\dot{\xi} = 0$, $\dot{f} = const$, and internal Noether symmetries $\dot{\xi} = 0$, $\dot{f} = 0$, it is at least unnecessary. It can lead to non mathematically acceptable conclusions and makes the issue of symmetries problematic. It shows lack on the mathematical literature of the subject of symmetry.

If we impose Hamilton's principle for the Action Integral (2) we end up with the Euler–Lagrange equations of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0. \tag{11}$$

Hence, with the use of the equations of motion (11) the Noether symmetry condition (9) can be written in the equivalent form

$$\frac{d}{dt}\left(f + \xi \mathcal{H} - \eta p\right) = 0, \tag{12}$$

in which $\mathcal{H} = p\dot{q} - L$ is the Hamiltonian function and $p = \frac{\partial L}{\partial \dot{q}}$ is the momentum. The latter expression states that the function

$$\Phi\left(t, q, p\right) = \xi \mathcal{H} - \eta p + f, \tag{13}$$

is a conserved quantity for the equations of motion (11). This is known as Noether's second theorem.

We emphasize that symmetry is the generator of an infinitesimal transformation that leaves the Action Integral invariant, and the existence of such symmetry is independent of the Euler–Lagrange equation from the calculus of variations. The Euler–Lagrange equation arises from the application of Hamilton's principle, where the variable q is subjected to a zero-endpoint variation. However, no such restriction applies to the infinitesimal transformations introduced by Noether. Indeed the equation of motion can exist either if the Action Integral does not possess any symmetries.

This observation clearly states that there can be no connection between the existence of variational symmetries and the existence for the equations of motion, or the equivalence principle. Nevertheless, in the following lines we will show in a pedagogical way how the authors of [1] were driven to a faulty conclusion.

3 Noether symmetries for geodesic Lagrangian

Consider a second-rank tensor $g_{\mu\nu}\left(q^k\right)$ which is a metric tensor, and the Lagrangian function

$$L\left(t,q,\dot{q}\right)=\frac{1}{2}g_{\mu\nu}\dot{q}^{\mu}\dot{q}^{\nu}. \tag{14}$$

We follow [1] and we study the Noether symmetry condition (9) for the Lagrangian function (14). As in [1] we focus in the case where the symmetry vector is $X=\eta\partial_q$ and we investigate for transformations where the boundary term is zero, i.e. $\dot{f}=0$.

Indeed, for this consideration the symmetry condition (9) reads

$$X^{[1]}L=0. \tag{15}$$

where now $X^{[1]}=\eta\partial_q+\dot{\eta}\partial_{\dot{q}}$.

In [1] it has been considered that the symmetry condition is $X\nabla L=0$, where ∇ describes the covariant derivative. However, the correct expansion of condition (15) which can be found in any standard textbook of symmetry analysis

$$X^{[1]}L\equiv\mathcal{L}_{X^{[1]}}L=L_{,q}\eta+L_{,\dot{q}}\dot{\eta}. \tag{16}$$

This indeed is a (guided) derivative in the jet space $\{q,\dot{q}\}$, but it should not be confused with the covariant derivative ∇ of the background geometry. This observation is sufficient to conclude that the main results of the analysis in [1] are at least inaccurate, and with the discussion we made before we can conclude that the equivalent principle is NOT a Noether symmetry.

Another point which deserve discussion is that Noether symmetries for a given Lagrangian leads to conservation laws for the Euler–Lagrange equations of the studied Lagrangian. Specifically, the application of Hamilton's principle for the Lagrangian (14) leads to the equations of motion

$$\ddot{q}^{\mu}+\mathring{\Gamma}^{\mu}_{\kappa\nu}\dot{q}^{\kappa}\dot{q}^{\nu}=0, \tag{17}$$

where $\mathring{\Gamma}^{\mu}_{\kappa\nu}$ is the Levi-Civita connection which defines the covariant derivative $\mathring{\nabla}$ with the property $\mathring{\nabla}_{\kappa}g_{\mu\nu}=0$. However, in [1] the authors considered the autoparallels which depend on a more general connection, and it is not clear how the geodesic Lagrangian (14) is related to the autoparallels for an arbitrary connection.

Furthermore, in [1] the authors explore the relation between the Noether symmetries for the geodesic Lagrangian with the symmetries of the metric tensor $g_{\mu\nu}$. Before concluding this work, for the sake of clarity, we note that in [11] the geometric nature of the Noether symmetries was investigated when the coefficients ξ and η are functions of t and q. Specifically, the Noether symmetries for the geodesic Lagrangian (14) are generated by the elements of the Homothetic group of the metric tensor $g_{\mu\nu}$ and form a subgroup which belongs to the special projective group for the $n+1$ decomposable spacetime as discussed in details in [12]. On the hand, when ξ and η are linear on the derivative $\dot{q}$, the symmetry conditions indicate that the generator of the symmetry vectors are the Killing tensors of the metric tensor $g_{\mu\nu}$ [13]. See also the discussion in [14] and references therein.

4 Conclusions

In this study, we provide a detailed discussion of Noether's theorem and its relationship with the equations of motion. Specifically, with regard to the geodesic Lagrangian, we demonstrate that there is no connection between the equivalence principle and Noether's symmetries.

Finally, we made various clarifications on what is called as a Noether symmetry, and we mentioned the complete connection of the Noether symmetries with the collineations of the metric tensor $g_{\mu\nu}$. Hopefully this work it helps to avoid perpetuating various inaccuracies on Noether's work.

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