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Bohr Hamiltonian with deformation-dependent mass

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Abstract.

The Bohr Hamiltonian with a mass depending on the nuclear deformation is solved using the techniques of supersymmetric quantum mechanics (SUSYQM). Analytical expressions for spectra and wave functions are obtained. Spectra and B(E2) transition rates are calculated for more than 50 γ -unstable nuclei and more than 60 prolate deformed nuclei using the Davidson potential and the Kratzer potential. In addition to solving the long standing problem of the too rapid increase of the moment of inertia with deformation, the method reveals a conformal factor in the Bohr Hamiltonian, embedding the Bohr space in six dimensions.

1. Introduction

Bohr Hamiltonians with mass dependent on the nuclear deformation have been introduced in [1, 2, 3] and further considered in [4]. Here we are going to briefly review why these are needed (Section 2), how they can be solved (Section 3), why they are useful (Section 4), and how they are related to other approaches to nuclear collectivity (Section 5).

2. Why is deformation dependent mass (DDM) needed

In the framework of the Bohr Hamiltonian [5], nuclear collective motion is described in terms of the collective coordinates β (expressing the size of the deformation) and γ (related to the shape of the deformed nucleus), plus the three Euler angles (showing the orientation of the nucleus in space). The mass, m , is usually considered to be a constant.

Within this framework, the moments of inertia of atomic nuclei are found to increase as β^2 , i.e. too fast [6] in comparison to the experimental data. We are going to see that within the deformation dependent mass (DDM) framework, this increase is slowed down (see Fig. 1), while the one parameter entering in the dependence of the mass on the deformation can be related to the curvature of the 5D space, as well as to parameters of the Interacting Boson Model (IBM) [7], the complementary model describing nuclear collective motion in algebraic language.

Effective masses depending on the coordinates are not unheard of. They have been used for a long time in semiconductor theory [8, 9, 10, 11, 12, 13], in compositionally graded crystals [14], quantum dots [15], and liquid crystals [16], as well as in energy-density functional theory applied to nuclei [17], quantum liquids [18] and metal clusters [19].



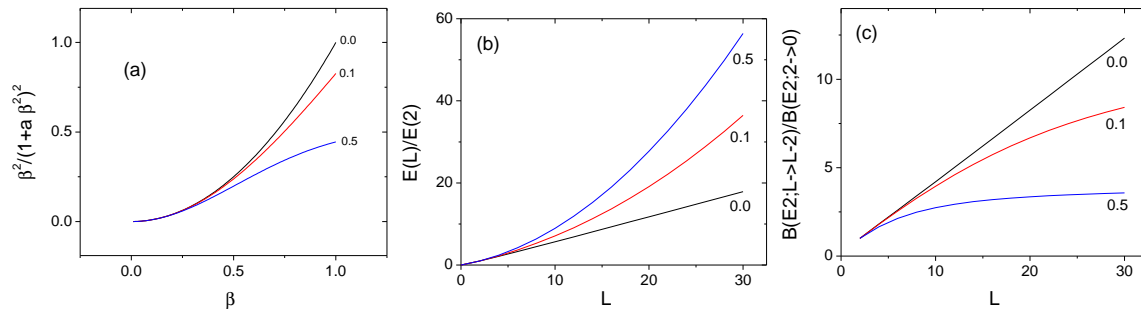


Figure 1. The function $\beta^2/f^2(\beta) = \beta^2/(1+a\beta^2)^2$, to which moments of inertia are proportional in the DDM Davidson model, plotted (a) as a function of the nuclear deformation β for different values of the parameter a [1]. Energy levels $E(L)$ [normalized to the energy of the first excited state, $E(2)$] (b) and $B(E2; L \rightarrow L - 2)$ transition rates [normalized to the transition from the first excited state to the ground state, $B(E2; 2 \rightarrow 0)$] (c) are shown for the ground state band as functions of the angular momentum L in the DDM Davidson model for $\beta_0 = 1$ and varying values of a . Figures taken from Ref. [1].

Additional motivation for introducing DDM in the Bohr Hamiltonian comes from the following points:

1) In the framework of the Interacting Boson Model (IBM) [7], it is known that in its geometrical limit [20, 21, 22], obtained through the use of coherent states [7], terms of the form $\beta^2\pi^2$ and/or more complicated terms appear [23], in addition to the usual term of the kinetic energy, π^2 . Thus it might be appropriate to search for a modified form of the Bohr Hamiltonian, in which the kinetic energy term will be modified by terms containing β^2 and/or more complicated terms.

2) Detailed comparisons to experimental data have recently pointed out [24, 25, 26] that the mass tensor of the collective Hamiltonian cannot be considered as a constant and should be taken as a function of the collective coordinates, with quadrupole and hexadecapole terms present in addition to the monopole one. This approach has been recently extended [27, 28] to odd nuclei as well.

3. How can the DDM Bohr Hamiltonian be solved

If the mass depends on the coordinates, then mass and momentum do not commute. One then has to modify the kinetic energy operator, however keeping it Hermitian. The most general solution of this problem, involving two free parameters, has been given by von Roos [12].

In order to derive a DDM Bohr Hamiltonian, one has to work in the 5-dimensional (5D) space of the Bohr Hamiltonian, involving the Bohr–Wheeler coordinates β, γ and the three Euler angles, and allowing the mass to depend on the deformation β . Using the usual Pauli–Podolsky prescription [29], one arrives at the DDM Bohr Hamiltonian given explicitly in Refs. [1, 2, 3].

This Hamiltonian can be solved using Supersymmetric Quantum Mechanics (SUSYQM) [30, 31], as pointed out in Ref. [32]. The basic steps of the solution, described for simplicity in 1D, are:

1) Factorize [33] the Hamiltonian H_1 in the form $A^\dagger A$. Its supersymmetric partner, H_2 , is AA^\dagger . The two Hamiltonians share the same kinetic energy, but their potentials are $V_1 = W^2 - W'$ and $V_2 = W^2 + W'$, where W is called the superpotential and W' is its first derivative with respect to the relevant variable (β in the present case). The basic feature of these two Hamiltonians is that they share the same spectrum, the only difference being that the ground state of H_1 has

no partner in the spectrum of H_2 . Using the operator A one can jump from the eigenfunctions of H_1 to the eigenfunctions of H_2 with one node less, while using the operator A^\dagger one can jump from the eigenfunctions of H_2 to the eigenfunctions of H_1 with one node more [30, 31].

2) Write a hierarchy of Hamiltonians [30, 31]. Each Hamiltonian H_{i+1} is defined as $A_i A_i^\dagger$, in terms of the operators A_i, A_i^\dagger of the previous Hamiltonian $H_i = A_i^\dagger A_i$, and is subsequently factorized into $H_{i+1} = A_{i+1}^\dagger A_{i+1}$, where A_{i+1}, A_{i+1}^\dagger are new operators.

3) Impose shape invariance conditions [34] (integrability conditions guaranteeing exact solvability). Two potentials, V_1 and V_2 , which are supersymmetric partners, are shape invariant if they have the same functional dependence on the relevant variable (β in this case), while the values of the parameter are different for each one, and the relative displacement does not depend on the variable [34]. Symbolically, $V_2(\beta; a_1) = V_1(\beta; a_2) + R(a_1)$, where a_1, a_2 are the parameter values and R the relative displacement, or $A(a_1)A^\dagger(a_1) = A^\dagger(a_2)A(a_2) + R(a_1)$.

Analytical expressions for the wave functions are obtained, being square integrable and ensuring the hermiticity of the Hamiltonian [2, 3]. $B(E2)$ transition rates can then be calculated numerically [2, 3].

Only some specific potentials are shape invariant and can be treated by the above method. These include [30, 31]:

- 1) the shifted harmonic oscillator,
- 2) the 3D oscillator and the Coulomb potential, the only ones which are soluble by this method in 3D,
- 3) the Morse, Rosen–Morse I (trigonometric), Rosen–Morse II (hyperbolic), Scarf I (trigonometric), Scarf II (hyperbolic), Eckart, and Poeschl–Teller potentials.

In what follows, we are going to consider (see Fig. 2) the Davidson potential [35] (a special case of the 3D oscillator) and the Kratzer potential [36] (a special case of the Coulomb problem).

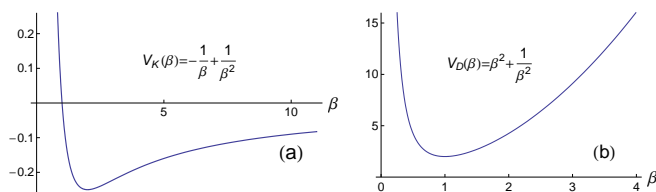


Figure 2. The Kratzer (a) and Davidson (b) potentials. The quantities shown are dimensionless, while all free parameters have been set equal to unity, for the sake of simplicity. Figure taken from Ref. [3].

4. What is the DDM Bohr Hamiltonian useful for

In both of the DDM Davidson [35] and DDM Kratzer [36] models, the mass depends on the deformation through the form $m(\beta) = m_0/f^2(\beta)$, where m_0 is a constant. In the Davidson case, one has $f(\beta) = 1 + a\beta^2$, while in the case of Kratzer one has $f(\beta) = 1 + a\beta$. In both cases, the parameter a , expressing the degree of dependence of the mass on the deformation, has to be much less than unity.

In the Davidson case, moments of inertia turn out to be proportional to $\beta^2/(1 + a\beta^2)^2$, while in the Kratzer case they increase as $\beta^2/(1 + a\beta)^2$. In both cases, the increase is not following β^2 , as in the plain Bohr case, but is moderated by the denominator, the moderation increasing with increasing a . One can easily see that energy ratios $E(L)/E(2)$ increase with increasing a , while transition ratios $B(E2; L \rightarrow L - 2)/B(E2; 2 \rightarrow 0)$ decrease with increasing a [2] (see Fig. 1).

Separation of variables can be easily achieved in the following cases.

1) Potentials independent of γ , referred to as γ -unstable potentials. They can describe nuclei lying relatively near to closed shells, which exhibit nearly vibrational behaviour. More than 50 nuclei of this kind have been fitted using both the Davidson and the Kratzer potentials, the results given in Refs. [2, 3] (see Fig. 3).

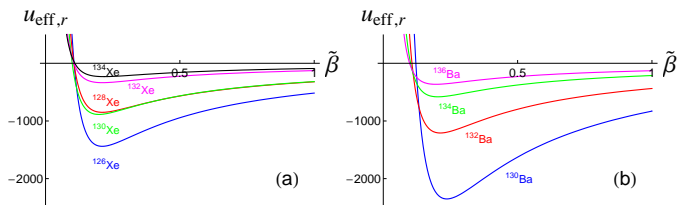


Figure 3. Effective DDM Kratzer potentials for $L = 4$ for some Xe (a) and Ba (b) isotopes. Figure taken from Ref. [3].

2) Potentials of the form $v(\beta, \gamma) = u(\beta) + w(\gamma)/\beta^2$, with $w(\gamma) = c\gamma^2$, i.e. a sharp potential centered at $\gamma = 0$. The term β^2 guarantees exact solvability in this case, since it allows the $w(\gamma)$ term to be absorbed into the centrifugal term. These potentials can describe well deformed nuclei, lying near the middle of the proton and neutron shells. More than 60 nuclei of this kind have been fitted using both the Davidson and the Kratzer potentials, the results given in Refs. [2, 3] (see Fig. 4).

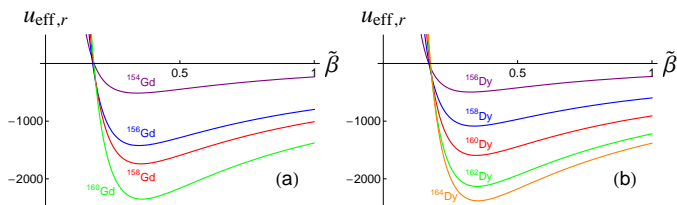


Figure 4. Effective DDM Kratzer potentials for $L = 4$ for some Gd (a) and Dy (b) isotopes. Figure taken from Ref. [3].

3) Potentials of the form $v(\beta, \gamma) = u(\beta) + w(\gamma)/\beta^2$, with $w(\gamma) = c(\gamma - 30^\circ)^2$, i.e. a sharp potential centered at $\gamma = 30^\circ$. Again the term β^2 guarantees exact solvability in this case. These potentials can describe nuclei exhibiting features of triaxiality.

It turns out that the Kratzer potential [3] can fit the $N = 90$ nuclei ^{150}Nd , ^{152}Sm , ^{154}Gd , while the Davidson potential [2] cannot fit them. These nuclei are the best experimental manifestations [37] of the X(5) critical point symmetry [38], occurring at the point of the shape/phase transition [39] between spherical and prolate deformed shapes. It is known that the potential at the transition point should be flat [38]. The above result can then be explained by the fact that the Kratzer potential for appropriate parameter values becomes wide and relatively flat, while the Davidson potential always possesses a deep minimum.

5. Relation of DDM to other approaches to nuclear collectivity

The extra parameter a introduced by the DDM approach can be connected to quantities appearing in other collective models.

5.1. Connection of the DDM parameter to the space curvature

It is known that the Position Dependent Mass (PDM) approach is equivalent to the introduction of curved spaces, or to the employment of deformed canonical commutation relations [40]. Here we briefly describe the way in which the DDM parameter a can be connected to the curvature of space [4].

In the usual Bohr Hamiltonian (where the 5 variables can be enumerated as q_i with $i = 1, \dots, 5$), it is known that the kinetic energy can be written in the form $T = (m/2)(ds/dt)^2$, where the length element is $ds^2 = g_{ij}dq_i dq_j$, where g_{ij} is a 5×5 matrix [41].

On the other hand, in the DDM Davidson case it can be proved [2] that the DDM Bohr Hamiltonian can obtain the same form, with matrix elements $g'_{ij} = g_{ij}/f^2$. Therefore, f is a conformal factor, i.e. a factor not affecting the angles (i.e., γ and the three Euler angles). One can further see that for the volume element one has $dv' = dv/f^5$, while for the wave functions $\Psi' = \Psi f^{5/2}$, where the primed quantities refer to the DDM Bohr Hamiltonian, while the unprimed ones regard the original Bohr Hamiltonian.

Conformal factors are known to occur during the embedding from N into $N + 1$ dimensions [4]. As an example, let us consider the embedding from 2D to 3D. The surface of a sphere represents a 2D space with constant curvature, on which the Pythagorean theorem is not valid. The Pythagorean theorem becomes valid if we embed the 2D sphere into a 3D space, in which $x_1^2 + x_2^2 + x_3^2 = R^2$. The length element in cartesian coordinates reads $dl^2 = dx_1^2 + dx_2^2 + dx_3^2$, while in polar coordinates it reads $dl^2 = dr^2/Q + r^2 d\theta^2$, where $Q = 1 - (r/R)^2$. Performing the conformal transformation $r = r_1/F$, where $F = 1 + (r_1/(2R))^2$, the length element takes the form $dl^2 = (dr_1^2 + r_1^2 d\theta^2)/F^2$, which is identical to the euclidean form up to the conformal factor F .

Exactly the same picture appears when embedding the 5D Bohr Hamiltonian into a 6D space [2]. The line element in the 5D euclidean space of the original Bohr Hamiltonian reads $ds^2 = d\beta^2 + \beta^2 d\Omega_4^2$, where $d\Omega_4^2$ is a complicated function of γ and the three Euler angles. After the conformal transformation, the line element for the DDM Davidson Hamiltonian [2] reads $(ds')^2 = (d\beta^2 + \beta^2 d\Omega_4^2)/f^2$, with $f = 1 + a\beta^2$. The conformal transformation from the embedding of 5D into 6D reads $\beta' = \beta/F$, where $F = 1 + (\beta/(2R))^2$. Comparing f and F we see that $a = 1/(4R^2)$, i.e. the parameter contained in the dependence of the mass on the deformation is related to the radius of curvature R of a hypersphere in the 5D space [4]. It is also clear that the case of mass independent from the deformation ($a = 0$, as in the original Bohr Hamiltonian) corresponds to a hyperplane in the 5D space [4].

5.2. Connection of the DDM parameter to IBM parameters

The procedure inverse to the embedding from N into $N + 1$ dimensions is the stereographic projection from $N + 1$ to N dimensions. As an example, let us consider the stereographic projection from 3D to 2D. In this procedure, a point (x_1, x_2, x_3) on a sphere is projected onto a point (X_1, X_2) on a plane passing through the south pole of the sphere and being orthogonal to the axis connecting the north and south poles. The projection is carried out through a line starting from the north pole of the sphere, passing through the point (x_1, x_2, x_3) and ending at the point (X_1, X_2) . Using cylindrical coordinates (r, θ, x_3) on the sphere and polar coordinates (r_1, Θ) on the plane, one can easily prove that $\theta = \Theta$ and $r = r_1/F$, where $F = 1 + (r_1/(2R))^2$. We remark that the conformal factor F appearing in the stereographic projection from 3D to 2D is the same as the conformal factor appearing in the embedding from 2D into 3D. The same conformal factor appears in general in the stereographic projection from $N + 1$ to N dimensions and in the embedding from N into $N + 1$ dimensions.

The Hamiltonian of the Interacting Boson Model (IBM) [7], possessing an overall U(6) symmetry, is known to live in a 6D space. It is known to possess a classical limit [20, 21, 22, 23], obtained [22] through a stereographic projection from 6D to 5D. It is therefore legitimate

to compare the classical limit of the IBM Hamiltonian to the DDM Davidson Hamiltonian. Preliminary results [42] indicate that the DDM parameter a is related to

- a) the strength of the pairing interaction in the U(5) (vibrational) limit,
- b) the strength of the quadrupole-quadrupole interaction in the SU(3) (deformed) limit,
- c) the difference between the strength of the (sd)-pairing and the strength of the (dd)-pairing in the O(6) (γ -unstable) limit.

Thus it turns out that the IBM Hamiltonian has built in the curvature of the 5D space, while the original Bohr Hamiltonian corresponds to a flat 5D space (a 5D hyperplane). To allow agreement with the IBM predictions, the Bohr Hamiltonian has to be enriched by allowing the nuclear mass to depend on the deformation, thus making the DDM Bohr Hamiltonian to correspond to a curved 5D space.

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