PERTURBED NONLINEAR EQUATIONS: APPLICATION TO THE KORTEWEG-DEVRIES EQUATION CONSIDERED AS A PERTURBED EULER EQUATION.

M. IRAC-ASTAUD
Laboratoire de Physique Théorique et Mathématique -E.R. CNRS 177 - Université Paris VII, 2, Place Jussieu, 75251 PARIS, France.

ABSTRACT. When two dynamical systems of partial nonlinear equations differ by a term considered as a perturbation, one is called free ,the other one perturbed. Their solutions equal at the initial time, are related by an integral equation that allows to write the perturbed solution as an expansion, the terms of which are completely explicit expressions of the free solution. This generalises the usual perturbation theories around free solutions satisfying linear equations. This result is applied to the KDV equation.

In the first section of this paper, we establish two integral equations used in the second section to relate the solutions of two systems differing by a perturbation term. One of them, affine with respect to the perturbed solution, furnishes an iterative process to obtain this solution in terms of the free one. The other one let appear a generalisation of the Green function for nonlinear equations. In the third part, we apply the previous result to the KDV equation considered as a perturbed Euler equation.

I INTEGRAL EQUATIONS RELATING FREE AND PERTURBED SUBSTITUTION OPERATORS

Let $\mathscr{F}$ be any vector space and $\mathscr{G}$ the space of the mappings from $\mathscr{F}$ into $\mathscr{F}$. The substitution operator $\Phi^{\lambda *}_{ts} \in \mathscr{L}(\mathscr{G},\mathscr{G})$ is related to the flow $\Phi^{\lambda}_{ts} \in \mathscr{G}$ by the formula

$$\Phi^{\lambda *}_{ts}(F) = F \circ \Phi^{\lambda}_{ts} \qquad , \forall\, F \in \mathscr{G}. \tag{1}$$

It fulfils the semigroup properties

$$\Phi^{\lambda *}_{t\tau}\ \Phi^{\lambda *}_{st} = \Phi^{\lambda *}_{s\tau}, \tag{2}$$

and

$$\Phi^{\lambda *}_{ss} = \bar{I} \quad , \tag{3}$$

where $\bar{I}$ is the identity in $\mathcal{L}(\mathcal{B},\mathcal{B})$. In the following we will call free the quantities when $\lambda=0$, perturbed the other ones. The <u>field</u> $S^{\lambda *}(t)$ associated with the operator $\Phi^{\lambda *}$ is defined by

$$\left\{\frac{\partial}{\partial\tau}\ \Phi^{\lambda *}_{\tau t}\ \right\}_{\tau=t} = S^{\lambda *}(t) \tag{4}$$

Using the equations (2) and (4), we deduce

$$\frac{\partial}{\partial t}\ \Phi^{\lambda *}_{ts} = \Phi^{\lambda *}_{ts}\ S^{\lambda *}(t) \quad ; \tag{5}$$

The property (3) acts as an initial condition. From (2) and (3), it results that $\Phi^{\lambda *}_{ts}$ and $\Phi^{\lambda *}_{st}$ are mutually inverse;the equation (5) gives then

$$\frac{\partial}{\partial t}\ \Phi^{\lambda *}_{st} = -\ S^{\lambda *}(t)\ \Phi^{\lambda *}_{st}\ . \tag{6}$$

Assuming that the field $S^{\lambda *}(t)$ takes the form $S^{o *}(t)+\lambda\ N^{*}(t)$, we have

$$\frac{\partial}{\partial\tau}\ \Phi^{\lambda *}_{\tau s}\ \Phi^{o *}_{t\tau} = \Phi^{\lambda *}_{\tau s}\ \lambda N^{*}(\tau)\Phi^{o *}_{t\tau} \quad \text{and} \quad \frac{\partial}{\partial\tau}\ \Phi^{o *}_{\tau s}\ \Phi^{\lambda *}_{t\tau} = -\ \Phi^{o *}_{\tau s}\ \lambda N^{*}(\tau)\Phi^{\lambda *}_{t\tau} \tag{7}$$

where $\Phi^{o *}$ is equal to $\Phi^{\lambda *}$ for $\lambda=0$. The integrations of these equalities on the interval [s,t] lead to

$$\Phi^{\lambda *}_{ts} = \Phi^{o *}_{ts} + \lambda\int_{s}^{t} d\tau\ \Phi^{o *}_{\tau s}\ N^{*}(\tau)\Phi^{\lambda *}_{t\tau} \quad \text{and} \quad \Phi^{\lambda *}_{ts} = \Phi^{o *}_{ts} + \lambda\int_{s}^{t} d\tau\ \Phi^{\lambda *}_{\tau s}\ N^{*}(\tau)\Phi^{o *}_{t\tau} \tag{8}$$

These two equations relate the operators $\Phi^{\lambda *}_{ts}$ and $\Phi^{o *}_{ts}$. Their iterations can be performed and give

$$\Phi^{\lambda *}_{ts} = \Phi^{o *}_{ts} + \sum_{k=1}^{\infty} \lambda^{k}\int_{s}^{t} d\tau_{k}\int_{\tau_{k}}^{t} d\tau_{k-1}\cdot\cdot\int_{\tau_{2}}^{t} d\tau_{1}\,\Phi^{o *}_{\tau_{k}s}N^{*}(\tau_{k})\Phi^{o *}_{\tau_{k-1}\tau_{k}}\cdot\cdot\Phi^{o *}_{\tau_{1}\tau_{2}}N^{*}(\tau_{1})\Phi^{o *}_{t\tau_{1}} \tag{9}$$

In this relation the perturbed substitution operator $\Phi^{\lambda *}$ is expressed in terms of the free one $\Phi^{o *}$. Let us summarise this result:

Let a substitution operator $\Phi^{\lambda *}$ belonging to $\mathcal{L}(\mathcal{G},\mathcal{G})$, $\mathcal{G}$ being the space of the mappings from a vector space $\mathcal{F}$ into itself, satisfy a differential equation: $\frac{\partial}{\partial t}\Phi^{\lambda *}_{ts} = \Phi^{\lambda *}_{ts}(S^{o *}(t)+\lambda N^{*}(t))$ with the condition that $\Phi^{\lambda *}_{ts}$ is the identity when t=s. The operator $\Phi^{\lambda *}$ is related to $\Phi^{o *}$ by the integral equations (8) that can be solved by iteration giving the expression (9) of $\Phi^{\lambda *}$ in terms of $\Phi^{o *}$.

II INTEGRAL EQUATIONS FOR NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

We now apply this result to two interesting cases (general results are given in [I]), the space $\mathcal{F}$ is that of the smooth mappings of $\mathbb{R}^n$ in $\mathbb{R}$ (resp.$\mathbb{R}^2$). The value at x of $\Phi^{\lambda}_{ts}[y]$, $u[\lambda,t,s,x;y]$, is solution of the nonlinear partial differential equation (resp.equations) obtained by applying (5) to the identity I in $\mathcal{G}$, the result being itself applied to a function $y \in \mathcal{F}$,

$$\frac{\partial}{\partial t}u[\lambda,t,s,x;y]= S^{\lambda}[t,x;u] \quad ,x\in \mathbb{R}^n,\; u[\lambda,t,s,x;y]\in \mathbb{R} \;(resp.\mathbb{R}^2) \qquad (10)$$

where $S^{\lambda}[t,x;u]$ belongs to $\mathbb{R}$ (resp.$\mathbb{R}^2$)and is defined by

$$S^{\lambda}[t,x;y]= \left(\left[\left[S^{\lambda *}(t)\right](I)\right](y)\right](x) \qquad (11)$$

The initial condition resulting from (3) reads

$$u[\lambda,s,s,x;y]= y(x),\; y(x)\in \mathbb{R} \;(resp.\mathbb{R}^2). \qquad (12)$$

The flow Φ_{ts} and the field S^{λ} are now the usual ones associated with the equation (10).

When $u[\lambda,t,s,x;y]$ belongs to $\mathbb{R}$, the first relation (8) leads to

$$u[\lambda,t,s,x;y]= u[0,t,s,x;y]+\lambda\int_s^t d\tau\int d\xi \left\{N[\tau,\xi;z]\frac{\delta}{\delta z(\xi)}u[\lambda,t,\tau,x;z]\right\}_{z(x)=u[0,\tau,s,x;} \qquad (13)$$

In this equation the functional λN is the difference between S^{λ} and S^{o}. This integral equation (13) relates the perturbed solution $u[\lambda,t,s,x;y]$ to the free one $u[0,t,s,x;y]$ of the equation (10), satisfying the same initial condition (12). This equation is <u>linear with respect to the perturbed solution</u> and therefore <u>easy to iterate</u>. Analogously the second relation (8) furnishes an integral equation linear with respect to $u[0,t,s,x;y]$, that is an explicit functional equation for $u[\lambda,t,s,x;y]$ when the free solution is known.

$$u[\lambda,t,s,x;y]=u[0,t,s,x;y]+\lambda\int_s^t d\tau\int d\xi\ \left\{N[\tau,\xi;z]\frac{\delta}{\delta z(\xi)}u[0,t,\tau,x;z]\right\}_{z(x)=u[\lambda,\tau,s,x;y} \quad (14)$$

When the free equation is linear ,its solution is linear with respect to y and the coefficient of N under the integral is a kernel that is independent of u and equal to the Green function associated with the linear equation. In the case of nonlinear equation this coefficient can therefore be viewed as a generalisation of the Green function.

When v is solution of an equation of the type $\frac{\partial^2}{\partial t^2}v = \mathscr{S}^0[t,x;v]+ \lambda\mathscr{N}[t,x;v]$, and fulfils as its first t derivative initial conditions at t= s, we introduce $u= (v,\frac{\partial}{\partial t}v)\in \mathbb{R}^2$, so that u satisfies the equations (10) and (12), we then get

$$v[\lambda,t,s,x;y] = v[0,t,s,x;y]+\lambda\int_s^t d\tau\int d\xi\left\{\mathscr{N}[\tau,\xi;z]\frac{\delta}{\delta z_2(\xi)}v[\lambda,t,\tau,x;z_1,z_2]\right\}_{z_1(x)= v[0,\tau,s,x\ y],z_2(x)= \frac{\partial}{\partial\tau}v[0,\tau,s,x;y} \quad (15)$$

When $\mathscr{S}^0[t,x;v]= \frac{\partial^2}{\partial x^2} v+ e^v$, this expression relates the solutions of the perturbed and of the free Liouville equations.

III KDV EQUATION CONSIDERED AS A PERTURBED EULER EQUATION.

The Korteveg-DeVries equation corresponds to equation (10) where $S^\lambda[t,x;u]= -\left[u.\frac{\partial}{\partial x}u + \lambda.\frac{\partial^3}{\partial x^3} u\right]$,n= 1,$u\in \mathbb{R}$.When λ is a small dispersion parameter ,the last term can be considered as a perturbation term added to the Euler hydrodynamic equation [II] The solution of this last equation fulfilling the initial condition (12) is implicitly given by

$$u[0,t,s,x;y] = y(x-(t-s)u[0,t,s,x;y]). \quad (16)$$

The perturbation theory assumes that the perturbed solution is analytic with respect to λ at least in a neighbourhood of λ = 0 and t = s and can be written $u[\lambda,t,s,x;y]= \sum_{n\geq 0}\lambda^n u_n(t,x)$ [III]. The coefficient $u_0(t,x)$ is equal to $u[0,t,s,x;y]$. The direct substitution of this expression in (10) gives a linear partial differential equation for the u_n in terms of the preceding ones that can be solved by tedious calculations. On the contrary the method lying on the result of the preceding section leads to an easy and quite systematic process. Let us calculate the first term. Using the equation (13) we have

$$u_1(t,x) = \int_s^t d\tau \int d\xi \left\{ N[\tau,\xi;z] \frac{\delta}{\delta z(\xi)} u[0,t,\tau,x;z] \right\}_{z(x)=u[0,\tau,s,x;y]} \quad (17)$$

Taking the functional derivative with respect to $z(\xi)$ of (16) gives

$$\frac{\delta}{\delta z(\xi)} u[0,t,\tau,x;z] = \delta(x-(t-\tau)u[0,t,\tau,x;z]-\xi).(1+(t-\tau)z'(\xi))^{-1} \quad (18)$$

In this expression we have to replace $z(\xi)$ by $u_o(\tau,\xi)$ and ξ by its value resulting from the Dirac distribution. Putting $\zeta= x-(t-s)u_o(t,x)$ in the equation (18) we have

$$u_o(t,x) = u_o(t,\zeta+(t-s)y(\zeta)) = y(\zeta) \quad (19)$$

From the expression of ζ and (19), we deduce that $x = \zeta+(t-s)y(\zeta)$. Formula (19) implies that $u_o(t,\zeta+(t-s)y(\zeta)) = u_o(\tau,\zeta+(\tau-s)y(\zeta))$. Replacing in this equality ξ and ζ by their expressions we obtain

$$u_o(t,x) = u_o(\tau,\xi) \quad , \text{ for } \xi = x-(t-\tau)u_o(t,x). \quad (20)$$

The differentiation of (20) with respect to x gives

$$1+(t-\tau)u_o'(\tau,\xi) = (1-(t-\tau)u_o'(t,x))^{-1} = f(\tau) \quad (21)$$

where $u_o'(t,x)$ is the derivative of u_o with respect to x. When N in (17) is a function of u, $u_1(t,x) = N(u_o(t,x)).[(t-s)-(t-s)^2u_o'(t,x)]$.

In the case of the KDV equation, the integral (17) involves the third derivative of $u_o(\tau,\xi)$ with respect to ξ. A calculation analogous to that giving (21) leads to

$$u_o'''(\tau,\xi)= f^4(\tau) \; [u_o'''(t,x)+ 3(t-\tau)f(\tau) \; u_o''(t,x)]. \quad (22)$$

By putting this expression in (17) finally we get

$$u_1(t,x)=- \frac{(t-s)f(s)}{2}\left[u_o'''(t,x) \; (1+f(s))+(t-s).u_o''(t,x)^2 f(s) \; (1+2f(s))\right] \quad (23)$$

The following u_n are obtained as functionals of u_o by the same type of calculation straightforwardly performed.

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