Numeric simulation of gravitational waves in Randall-Sundrum cosmology

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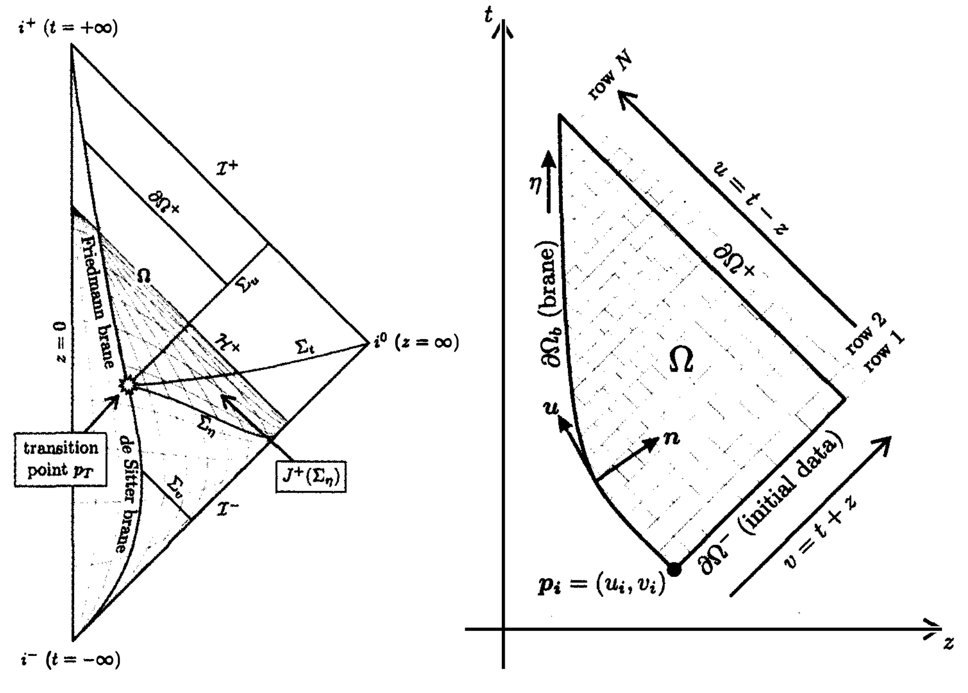
Motivated by the problem of the evolution of bulk gravitational waves in Randall-Sundrum cosmology, we have developed a characteristic numerical scheme to solve 1+1 dimensional wave equations in the presence of a moving timelike boundary. This code has been used to predict the spectral tilt of the stochastic gravitational wave background in brane cosmology for a variety of higher-dimensional (i.e. 'bulk') initial conditions. Here, we give a qualitative picture of how gravitational waves behave in the braneworld scenario, and summarize some of our main results.

1 Randall-Sundrum cosmology

It is well known that the Randall-Sundrum (RS) braneworld model[1] is in excellent agreement with general relativity at low energies. This is the principal appeal of the model; it is one of the only examples of a scenario involving a large extra dimension that entails no serious conflicts with general relativity. However, this means that one needs to consider high energy or strong gravity scenarios to properly test the model. One possibility is to examine the high energy epoch of braneworld cosmology, where exact solutions of the 5-dimensional field equations are known. Well-understood braneworld phenomena include a modified cosmic expansion and early times and 'dark radiation' effects, whereby the Weyl curvature of the bulk projected on the brane acts as an additional geometric source in the Friedmann equation.

But if one wants to move beyond the exact description of the background geometry in these cosmological models, there are significant technical difficulties. A cosmological brane is essentially a moving boundary in a static 5-dimensional background — anti-de Sitter space in the RS model (*cf.* Fig. 1) — so perturbations are described by bulk wave equations with boundary conditions enforced on a non-trivial timelike surface. While it is possible to make some analytic progress when the brane is moving 'slowly' [2,3,4], the more interesting case of a fast-moving, high-energy brane remains impervious to such treatment.



(a) Conformal diagram of RS bulk geometry (b) The computational domain Ω

Figure 1: Conformal diagram (a) illustrating the causal structure of a braneworld model of the early universe. In this plot, the three 'ordinary' spatial dimensions have been suppressed; hence, the brane is represented by a simple timelike trajectory. An initial purely-de Sitter inflationary phase is followed by a 'Friedmann', or high-energy radiation, phase. Our code assumes that the gravitational wave content of the model is known on the initial null surface Σ_u, and then calculates the field amplitude throughout the spacetime region Ω. The region is shown in a conventional spacetime diagram on the right (b). Superimposed on Ω is a (particularly coarse) example of the computational grid we use to discretize and solve the master wave equation.

The purpose of this work is to present a new numeric algorithm to solve wave equations in the presence of a moving boundary. For the sake of simplicity, we restrict ourselves to a class of wave equations and boundary conditions that correspond to tensor, or gravitational wave (GW), perturbations. This is not the first attempt to deal with these equations numerically: previous efforts include pseudo-spectral[5,6,7] and direct evolution[8,9,10,11,12] methods using various null and non-null coordinate systems in which the brane is stationary. Unfortunately, not all of these algorithms agree with one another. In particular, Hiramatsu et al.[6] predict a flat GW background spectrum at high frequencies, while Ichiki and Nakamura[8] predict a red spectrum. These two groups have very different prescriptions for setting initial conditions in the bulk, and it used to be unclear whether this was the source of tension between the two results. However, with our new code we have been able to definitively state that both initial conditions lead to a flat spectrum, provided that the energy scale of brane inflation is sufficiently high. Furthermore, we have shown have a much wider class of initial conditions can lead to the same result; implying that a flat GW background is a somewhat generic prediction of this class of braneworld models. For a comprehensive account of how these conclusions are obtained, the interested reader should consult Seahra.[13]

2 Numeric Method

The problem of predicting the propagation of a GW mode with 3-dimensional wavenumber k in this model can be reduced to solving the wave equation

$$\left[\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial z^2} + \epsilon_*(2+\epsilon_*) + \frac{15}{4z^2}\right]\psi(t,z) = 0, \quad (1)$$

on Ω. Here, ϵ_* is a dimensionless parameter that represents the density of brane matter normalized by the brane tension when the mode re-enters the Hubble horizon, ψ is a master variable governing tensor perturbations, and $\ell \lesssim 0.1\,\text{mm}$ is the curvature scale of the bulk. Assuming radiation-domination, the brane's trajectory is given by the Friedmann equation

$$\dot{z}^2 = (H\ell)^2 = \epsilon_* z^4(2+\epsilon_* z^4), \quad \dot{t} = \sqrt{1+\dot{z}^2}, \quad (2)$$

where an overdot indicates a derivative with respect to conformal time. The wavefunction ψ satisfies the following boundary condition on the brane:

$$\left[\mathbf{n}\cdot\nabla\psi + \frac{3}{2}\frac{\sqrt{1+(H\ell)^2}}{z}\psi\right]_b = 0, \quad (3)$$

where $\mathbf{n}$ is the brane normal. To complete the specification of the problem, we need to set initial data on the $\partial\Omega^-$ hypersurface. This surface is located in spacetime by demanding that the perturbation wavelength be s_0 times the horizon size at the epoch when $\partial\Omega^-$ crosses the brane.

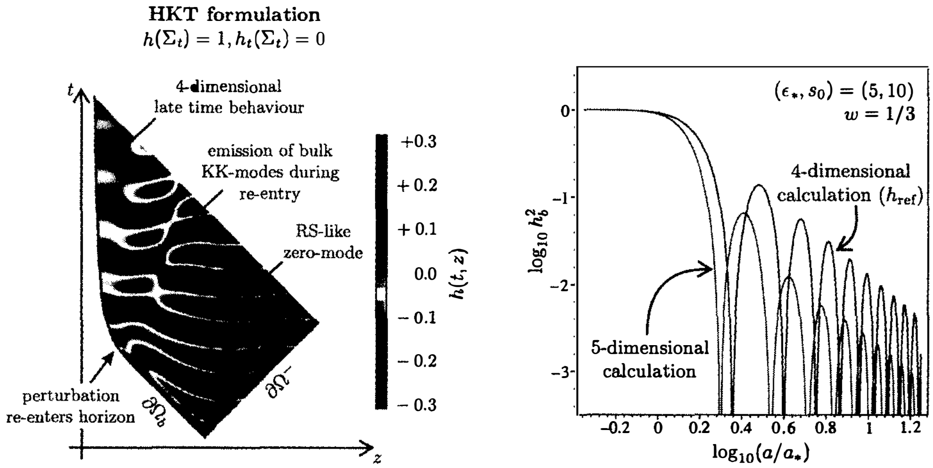
To solve for ψ numerically, we take inspiration from well established techniques in black hole perturbation theory. We discretize the computational domain as shown in Fig. 1(b); the evolution of ψ over a given cell is obtained by integrating the wave equation over the 'finite element' and applying the divergence theorem. Because the individual elements are based on the characteristics of the wave equation (1), we obtain a fast, accurate, and stable numeric algorithm.

3 Results

In Fig. 2, we show the result of a typical simulation of the GW amplitude. One can see how the value of the perturbation is frozen on the brane until it re-enters the horizon, as in 4-dimensional theory. After horizon re-entry, some of the GW energy is radiated away into the bulk, and at late times the perturbation on the brane decays as $1/a$ as usual. In Fig. 2(b), we show what the brane signal would be if one ignored the bulk and evolved the GWs as in ordinary general relativity. We see that the '5-dimensional' simulation result shows a suppressed late-time amplitude compared to the reference curve, reflecting the GW energy loss into the bulk that occurs at horizon crossing. Knowledge of the ratio between the simulation and reference amplitudes $\mathcal{R}$ as a function of the observed mode frequency f can be directly translated into a prediction for the spectral energy density Ω_{GW} of the GW background today[13]:

$$\Omega_{\text{GW}} \propto \mathcal{R}^2(f)\begin{cases} 54.9, & f \lesssim f_c, \\ 36.4(f/f_c)^{4/3}, & f \gtrsim f_c, \end{cases} \quad f_c \sim 3.3\times 10^{-5}\left(\frac{0.1\,\text{mm}}{\ell}\right)^{1/2}\text{Hz}. \quad (4)$$

Our simulations show that as long as $s_0 \gtrsim 100$, $\mathcal{R} \propto (f/f_c)^{-2/3}$ for $f \gtrsim f_c$ and 'reasonable' initial data; i.e., field configurations that do not vary too quickly along Σ_u. On the other hand, we have found $\mathcal{R} \sim 1$ for $f \lesssim f_c$ in all cases. Hence, a flat GW spectrum is recovered for all frequencies.



(a) Bulk profile

(b) Brane profile

Figure 2: Results of a typical numeric simulation using the initial conditions favoured by Hiramatsu et al. On the right, we have drawn what the brane GW signal $h_{\rm ref}$ would be if the bulk were neglected; i.e., if one solved the 4-dimensional master equation with a modified expansion rate given by (2).

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