

On holographic duals of certain isolated weighted homogeneous Gorenstein cDV singularities

Yuanyuan Fang ^a and Zekai Yu ^b

^a*Department of Mathematics, Tsinghua University,
Beijing, 10084, China*

^b*Qizhen College, Tsinghua University,
Beijing, 10084, China*

E-mail: fangyy21@mails.tsinghua.edu.cn, yuzk23@mails.tsinghua.edu.cn

ABSTRACT: We employ a novel approach, based on homological mirror symmetry for Landau-Ginzburg models, to demonstrate the non-existence of crepant resolutions for certain weighted homogeneous Gorenstein compound Du Val singularities. Physically, this implies that such singularities cannot serve as holographic backgrounds for four-dimensional $\mathcal{N} = 1$ superconformal quiver gauge theories realized on the worldvolume of a large number of D3-branes placed at the singular locus. This is confirmed by enumerating all consistent quiver gauge theories.

KEYWORDS: AdS-CFT Correspondence, Brane Dynamics in Gauge Theories, Differential and Algebraic Geometry, Supersymmetry and Duality

ARXIV EPRINT: [2412.10698](https://arxiv.org/abs/2412.10698)

Contents

1	Introduction	1
2	General strategies	4
2.1	K -stability of the singularities	4
2.2	Reduction of the problem	5
2.3	Symplectic cohomology	7
2.4	Hochschild cohomology: an introduction	9
2.5	Berglund-Hübsch-Krawitz mirror symmetry for Landau-Ginzburg models	10
3	Computation of symplectic cohomology of cE_6 and cE_7 singularities	16
3.1	cE_6 singularities	16
3.2	cE_7 singularities	21
4	Verification from the physics side	24
4.1	Consistency conditions for superconformal quiver gauge theory duals	25
4.2	Implementation of the search	30
5	Conclusions and future directions	31

1 Introduction

Compound Du Val (cDV) singularities form an important class of three-dimensional geometric singularities. They have a concrete description as hypersurfaces in \mathbb{C}^4 , which enables explicit computations. Moreover, they are closely related to the canonical ADE surface singularities, appearing as one-parameter deformations thereof. Among them, the *weighted homogeneous* cDV singularities are distinguished by the presence of an evident dilational \mathbb{C}^* -action. A class of these singularities is displayed in table 1.¹ The meaning of the rightmost column will be explained later. The additional symmetry makes them particularly useful in string-theoretic applications. For example, when the singularities are isolated, they can serve as input data in the type IIB geometric engineering program. The outcomes of geometric engineering are four-dimensional $\mathcal{N} = 2$ Argyres-Douglas superconformal field theories (SCFTs) [1, 2]. This provides an alternative geometric realization of certain Argyres-Douglas theories that are more conventionally obtained via compactification from six dimensions. Isolated cDV singularities also play an important role in M-theory geometric engineering [3–6], where they give rise to five-dimensional SCFTs of rank zero.

Three-fold singularities X furthermore appear in the context of AdS/CFT correspondence. In the large- N limit, four-dimensional $\mathcal{N}=1$ superconformal field theories on D3 branes is dual to type IIB string theory on $AdS_5 \times L_5$. [7–9], where L_5 denotes the *link*, a.k.a. the associated *Sasaki-Einstein* manifold, of the corresponding three-fold singularity X . In this holographic framework, one expects the coordinate ring of the singularity to correspond — at least heuristically — to the chiral ring of the dual $\mathcal{N}=1$ SCFT [10, 11].

¹More precisely, the hypersurface equations in the list are defined only up to what is called a *weight-one deformations*, i.e., deformations that preserve the original \mathbb{C}^* -actions.

J	singularity	K -stable range of parameters
A_{N-1}	$x_1^2 + x_2^2 + x_3^N + z^k = 0$	$\frac{N}{2} < k < 2N, N \geq 2$
	$x_1^2 + x_2^2 + x_3^N + x_3 z^k = 0$	$\frac{N^2-1}{2N-1} < k < 2N - 2, N \geq 2$
D_N	$x_1^2 + x_2^{N-1} + x_2 x_3^2 + z^k = 0$	$\frac{2N^2-8N+6}{2N-3} < k < 4N - 4, N \geq 4$
	$x_1^2 + x_2^{N-1} + x_2 x_3^2 + z^k x_3 = 0$	$(N = 4, 5, 1 < k < 2N)$ $(N \geq 6, \frac{N^2-4N}{2N-2} < k < 2N)$
E_6	$x_1^2 + x_2^3 + x_3^4 + z^k = 0$	$1 < k < 24$
	$x_1^2 + x_2^3 + x_3^4 + z^k x_3 = 0$	$1 < k < 18$
	$x_1^2 + x_2^3 + x_3^4 + z^k x_2 = 0$	$1 < k < 16$
E_7	$x_1^2 + x_2^3 + x_2 x_3^3 + z^k = 0$	$1 < k < 36$
	$x_1^2 + x_2^3 + x_2 x_3^3 + z^k x_3 = 0$	$1 < k < 28$
E_8	$x_1^2 + x_2^3 + x_3^5 + z^k = 0$	$1 < k < 60$
	$x_1^2 + x_2^3 + x_3^5 + z^k x_3 = 0$	$2 < k < 48$
	$x_1^2 + x_2^3 + x_3^5 + z^k x_2 = 0$	$1 < k < 40$

Table 1. Three-fold isolated weighted homogeneous compound Du Val singularities in [1, 2]. The rightmost row shows the K -stable range of parameters.

Compared to the classical geometric engineering of four-dimensional $\mathcal{N} = 2$ supersymmetric field theories, this set-up preserves a smaller number of Poincaré supercharges. Because of the reduced supersymmetry, the extent to which the singular geometry determines the field-theoretic data remains an open question. A guiding principle is that whenever the singularity admits a non-commutative crepant resolution (NCCR), the dual superconformal field theory is a quiver gauge theory with superpotential, completely determined by the underlying NCCR [12].

Later developments showed that for a consistent large- N dual $\mathcal{N}=1$ SCFT to exist, additional geometric conditions may be required — such as K -stability with respect to a certain conical \mathbb{C}^* -action [10, 11]. Concretely, for weighted homogeneous cDV singularities in table 1, K -stability with respect to the apparent \mathbb{C}^* -action imposes a constraint on the parameter k . Accordingly, in the rightmost column, k is restricted to its K -stable range for physical relevance. Nevertheless, K -stability is not technically required when N is small.

However, the following technical questions remain unresolved: under what conditions does a three-fold singularity X admit a four-dimensional $\mathcal{N} = 1$ superconformal theory dual in the holographic sense? Moreover, when is the dual field theory described by a scale-invariant quiver gauge theory? Answers to these questions are not obvious, and, as observed in [13] it is strongly possible that some singularities do not provide holographic backgrounds for a superconformal quiver gauge theory. From a mathematical perspective, this occurs when the singularity admits no NCCR. For isolated cDVs, the existence of an NCCR is known to be equivalent to the existence of a crepant resolution (CR)² [14], which possibly does not exist.

²By crepant resolution of a variety X we mean a birational morphism

$$f : Y \rightarrow X, \quad f^* K_X = K_Y$$

from a smooth variety Y to the singularity X . K_X and K_Y are the canonical divisors on X and Y , respectively. We emphasize that Y must be smooth. X is required to be a *Gorenstein* singularity for technical reasons. The intuition behind a birational morphism is that f contracts subvarieties of codimension at least one, and is generically an isomorphism.

The physical relevance of crepant resolutions (and their non-commutative counterparts) can be understood as follows. From the viewpoint of the low-energy worldvolume theory, the geometry is effectively smoothed. When a crepant resolution exists, the brane fractionates into components wrapping the irreducible exceptional loci. By standard string theory reasoning, each wrapped brane gives rise to a gauge group factor. This setup does not rely on a large- N limit and was extensively studied in the early 2000s under the name *fractional branes*, see e.g. [15–18]. We apologize for not being able to give a more complete list of references.

To the best of our knowledge, a systematic understanding of the existence of crepant resolutions of cDV singularities remains incomplete, particularly for those of compound E_n type (cE_n). However for quasi-homogeneous isolated cDV singularities, several physics-based results exist, notably including the examples listed in table 1. In [6], Valandro and collaborators studied M-theory geometric engineering on these singularities. They analyzed the Higgs branch structure of the resulting five-dimensional SCFTs using a *gauge-theoretic* strategy, i.e., treating the 5d theory as a reduction of a seven-dimensional $\mathcal{N} = 1$ gauge theory with a varying adjoint Higgs field Φ . The seven-dimensional theory itself can be viewed as geometrically engineered by putting M-theory on deformed *ADE* surface singularities. The geometry of the deformed *ADE* surfaces is encoded in the vacuum expectation values of Φ . Consequently, the resulting low energy field theory can be analyzed using representation-theoretic techniques. Through this approach, they established the answer for existence of crepant resolutions for the singularities in table 1, summarized in the corresponding tables of their works. Similar strategy appeared in other related studies, e.g. [19–22]. Morally, this method aligns with the classical algebro-geometric framework used to study the *simultaneous resolution* of families of deformed *ADE* singularities [23, 24]. For example, the existence of a small resolution of a cDV singularity is related to the possibility of lifting the classifying map from the base — denoted \mathbb{C}_w in [6] — from \mathfrak{h}/W to \mathfrak{h} . Since \mathfrak{h}/W retains only the invariant polynomials, there is no canonical lift to the Lie algebra \mathfrak{g} or its Cartan subalgebra \mathfrak{h} . But when additional choices are introduced, such a lift becomes possible. This is the abstract role played by Φ ; see for instance [25].³

In this paper, we propose an alternative approach — grounded in symplectic geometry and mirror symmetry — to address the question of existence of crepant resolutions of the cE_n singularities listed in table 1. Physically, this corresponds to understanding the existence of superconformal quiver gauge theory duals associated with these three-fold singularities. Our aim is to present a logically transparent framework that may offer new perspectives on this class of problems.

Specifically, our methods substantiate the following claim, which is somehow a corollary of the results in [6]

Claim 1.1 *A singularity with $J = E_n$ in the list above, within the K -stable range, admits a crepant resolution if and only if it is, up to weight-one deformations, one of the following four types:*

$$\begin{aligned} x^2 + y^3 + z^4 + w^{12} &= 0, \\ x^2 + y^3 + yz^3 + w^{18} &= 0, \end{aligned}$$

³We thank the anonymous referee for drawing our attention to this reference.

$$\begin{aligned} x^2 + y^3 + z^5 + w^{30} &= 0, \\ x^2 + y^3 + yz^3 + w^2z &= 0. \end{aligned}$$

Note that most of the singularities with $J = E_n$ will be genuinely of cE_n type. Their conjectured holographic duals appear in [25–27], where the associated quiver diagrams coincide with the affine Dynkin diagrams of ADE types.⁴ The superpotentials in [27] agrees with [25] as well after integrating out massive adjoints. In the holographic correspondence, weight-one deformations in the geometry translates to exactly marginal deformations in the field theory. So these deformations neither alter the quiver gauge description nor affect the underlying noncommutative crepant resolution.

Our methodology unfolds in three main stages:

1. **Reduction to two families.** By surveying existing mathematical results, we restrict our attention to only two families of singularities in table 1.
2. **Symplectic cohomology criterion.** Invoking an important conjecture established in [28], which links the negative-degree symplectic cohomology of the Milnor fiber to the existence of a crepant resolution for a cDV singularity, we reduce the question of “Does this singularity admit a crepant resolution?” to the symplectic cohomology computation of its Milnor fiber, in negative degrees.
3. **Computational implementation via mirror symmetry.** Homological mirror symmetry furnishes a concrete tool for these computations: one calculates the Hochschild cohomology of the category of (maximally-)equivariant matrix factorizations of the mirror singularity. The general setup and examples appear in section 2 where relevant background is explained, and the full computation is carried out in section 3.

In section 4, we then verify our mathematical conclusions from the physics perspective by enumerating all consistent $\mathcal{N} = 1$ superconformal quiver gauge theories, confirming that no additional cases arise beyond the four identified in Claim 1.1. Finally, section 5 summarizes our findings and offers concluding remarks.

2 General strategies

In this section, we briefly review the key mathematical concepts. For an introduction to K-stability, we refer the readers to [11], with its physical applications deferred to section 4. Essential results from birational geometry are collected in [29].

2.1 K-stability of the singularities

The first ingredient is a necessary condition for a three-fold singularity to admit a four-dimensional $\mathcal{N} = 1$ superconformal dual in the large N limit: the existence of a Ricci-flat conical metric. In the mathematical framework, this metric is guaranteed by K-stability of the singularity with respect to a specific \mathbb{C}^* -action [10]. Concretely, this \mathbb{C}^* -action splits into two commuting parts: a real scaling along the radial direction of the singularity

⁴One cannot rule out theories that are conformally dual to them.

X ,⁵ and the so-called *Reeb flow* on the associated Sasaki-Einstein manifold L . Under the AdS/CFT correspondence, the field-theoretic $U(1)_R$ symmetry is identified with the Reeb flow action, normalized so that the holomorphic top form acquires weight two.⁶ We adopt this normalization throughout this paper.

The Reeb flow lies within the isometry group of the associated Sasaki-Einstein manifold, and these isometries typically originate from the obvious $U(1)$ actions present in the singularity's defining equation. Each cDV singularity in table 1 carries one such manifest $U(1)$ symmetry; suspended singularities with form $uv + f(z, w) = 0$ (cA_n types) admit an additional $U(1)$ action, which we do not consider here. Accordingly, we take the Reeb vector to be generated by the manifest $U(1)$ symmetry. One then tests K-stability with respect to this action by the criteria of [11], yielding the inequalities in N and k listed in table 1. The physical significance of K-stability is discussed further in section 4.

2.2 Reduction of the problem

Our goal is to investigate the technical problem of existence of superconformal quiver gauge theory duals in large N of the weighted homogeneous singularities for cE_n singularities listed in table 1. It is equivalent to the problem of non-commutative crepant resolutions of the singularity (see e.g. [12, 18]).

Unlike cA_n and cD_n singularities, crepant resolutions for the cE_n families remain largely uncharted. Fortunately, several mathematical constraints enable our analysis, as summarized below:

Resolutions via semi-universal unfolding. For hypersurface singularities of the form

$$F(x, y, z, w) = f_{ADE}(x, y, z) + w^k = 0. \tag{2.1}$$

Brieskorn's criterion (see e.g. [24, 30]) asserts that a crepant resolution exists precisely when k is an integer multiple of the Coxeter number of the corresponding ADE Lie algebra. For instance, $k = 2n - 2$ for D_n , $k = 12$ for E_6 , $k = 18$ for E_7 and $k = 30$ for E_8 . This result, however, does not seem to immediately extend to more general deformations such as

$$F = f_{ADE}(x, y, z) + zw = 0, \tag{2.2}$$

for which the singularity structure differs substantially from the simple w^k suspension.

Topology of the link and number of exceptional curves. For hypersurface singularities, results are known relating its birational geometric properties and topology of its link structures.⁷ From [30–32] one summarizes as follows: if an isolated weighted homogeneous

⁵Recall that the general philosophy of AdS/CFT from D3-branes is that, in the near horizon geometry, the radial direction of the normal bundle combines with the world-volume of branes to make AdS_5 , while the angular direction remains internal. The radial direction in the current setup is just the radial direction of singular 3-folds.

⁶Such normalization is due to the fact that the coordinates x, y, z, w are identified as gauge invariant operators on the field theory side via holographic duality. To identify the Reeb vector with the generator of the $U(1)_R$ symmetry in the field theory, we adopt such a normalization.

⁷We thank Prof. Dan Xie for drawing our attention in this direction.

Singularity	$f(X)$	Constraints from K -stability
$x_1^2 + x_2^2 + x_3^N + z^k = 0$	$g.c.d.(N, k) - 1$	$\frac{N}{2} < k < 2N, N \geq 2$
$x_1^2 + x_2^2 + x_3^N + x_3 z^k = 0$	$g.c.d.(N - 1, k)$	$\frac{N^2 - 1}{2N - 1} < k < 2N - 2, N \geq 2$
$x_1^2 + x_2^{N-1} + x_2 x_3^2 + z^k = 0$	$\frac{g.c.d.(2N-2, k)+2}{2}$ for $\frac{2N-2}{g.c.d.(2N-2, k)}$ odd; 1 for k and $\frac{2N-2}{g.c.d.(2N-2, k)}$ even; 0 for k odd	$\frac{2N^2 - 8N + 6}{2N - 3} < k < 4N - 4, N \geq 4$
$x_1^2 + x_2^{N-1} + x_2 x_3^2 + z^k x_3 = 0$	$g.c.d.(N, k)$ for $\frac{N}{g.c.d.(N, k)}$ odd; 0 otherwise	$(N = 4, 5, 1 < k < 2N)$ $(N \geq 6, \frac{N^2 - 4N}{2N - 2} < k < 2N)$
$x_1^2 + x_2^3 + x_3^4 + z^k = 0$	6 for $k = 0(mod12)$; 2 for $k = 3, 6, 9(mod12)$; 0 for $k \neq 0(mod3)$	$1 < k < 24$
$x_1^2 + x_2^3 + x_3^4 + z^k x_3 = 0$	6 for $k = 0(mod9)$; 0 otherwise	$1 < k < 18$
$x_1^2 + x_2^3 + x_3^4 + z^k x_2 = 0$	6 for $k = 0(mod8)$; 2 for $k = 4(mod8)$; 1 for $k \neq 0(mod4)$	$1 < k < 16$
$x_1^2 + x_2^3 + x_2 x_3^3 + z^k = 0$	7 for $k = 0(mod18)$; 1 for k even and $k \neq 0(mod18)$; 0 for k odd	$1 < k < 36$
$x_1^2 + x_2^3 + x_2 x_3^3 + z^k x_3 = 0$	7 for $k = 0(mod14)$; 1 for k even and $k \neq 0(mod14)$; 0 for k odd	$1 < k < 28$
$x_1^2 + x_2^3 + x_3^5 + z^k = 0$	8 for $k = 0(mod30)$; 0 otherwise	$1 < k < 60$
$x_1^2 + x_2^3 + x_3^5 + z^k x_3 = 0$	8 for $k = 0(mod24)$; 0 otherwise	$1 < k < 48$
$x_1^2 + x_2^3 + x_3^5 + z^k x_2 = 0$	8 for $k = 0(mod20)$; 0 otherwise	$1 < k < 40$

Table 2. Mass parameters of the cDV singularities in questions, along with their range of K-stability.

cDV singularity X admits a small resolution (which is a crepant resolution) $\pi : Y \rightarrow X$, then the number of curves that is contracted is precisely given by the number $f(X)$ of mass deformations of the singularity

$$l = f(X),$$

regarded as defining a 4d $\mathcal{N}=2$ Argyres-Douglas SCFT. Namely, it is the number of deformations of mass dimension 1, where the mass dimensions Δ are normalized such that

$$\sum_{i=1}^4 \Delta_i - \Delta_F = 1.$$

In particular, the number of exceptional curves is the same for any such a small resolution. This result allows us to explicitly compute the number of curves contracted in such a resolution. In particular, the number of mass parameter for the singularities listed in table 1 were determined in the appendix of [33] and are recorded in table 2 below.

As we exclude smooth geometry, a necessary condition for existence of a crepant resolution is $l > 0$. As shown in table 2, one observes that $f(X) = N$ holds precisely for the following families of singularities

$$\begin{aligned} x_1^2 + x_2^2 + x_3^{N+1} + z^{N+1} &= 0, \\ x_1^2 + x_2^{N-1} + x_2 x_3^2 + z^{2N-2} &= 0, \end{aligned}$$

and $f(X) = 6, 7, 8$ respectively for the following singularities

$$\begin{aligned} x_1^2 + x_2^3 + x_3^4 + z^{12} &= 0, \\ x_1^2 + x_2^3 + x_2 x_3^3 + z^{18} &= 0, \\ x_1^2 + x_2^3 + x_3^5 + z^{30} &= 0, \end{aligned}$$

up to weight-one deformations. Moreover, $f(X) = 1$ for Morrison-Pinkham example $x^2 + y^3 + yz^3 + zt^2 = 0$. These are the correct values of exceptional curves of known singularities admitting crepant resolutions. Other singularities typically admit small values of $f(X)$, as expected.

In summary, the set of cE_n singularities in table 1 which

1. do not fit into the criteria of Brieskorn and
2. admit nontrivial mass deformations and
3. are not related to the known ones that can be resolved crepantly by weight-one deformations

are summarized in the following

Problem 2.1 *How to determine the existence of crepant resolutions of the following singularities?*

$$cE_6 : x^2 + y^3 + z^4 + yt^k : k = 2, 3, 5, 6, 7, 9, 10, 11, 13, 14, 15(f = 1); k = 4, 12(f = 2).$$

$$cE_7 : x^2 + y^3 + yz^3 + zt^k : k = 4, 6, 8, 10, 12, 16, 18, 20, 22, 24, 26(f = 1).$$

Here we record the corresponding values of f for reference.

The main result of the paper is to show via a novel method that none of the singularities in Problem 2.1 admit a crepant resolution. This means that they do not admit a 4d $\mathcal{N} = 1$ superconformal quiver gauge theory dual.

2.3 Symplectic cohomology

In this subsection, we briefly review the notion of symplectic cohomology and explain its implications on the existence of crepant resolutions. Our discussion follows the treatments in [28, 34, 35] and the overview in [30].

Consider a hypersurface singularity defined by a polynomial equation

$$f(x_1, x_2, x_3, x_4) = 0,$$

in \mathbb{C}^4 , with an isolated singular point at the origin, one studies its deformations via the family

$$F(x_1, \dots, x_4, t) := f(x_1, \dots, x_4) + t = 0,$$

parametrized by $t \in \mathbb{C}$. For sufficiently small and generic t , the set

$$M_X(t) := \{f^{-1}(t)\} \cap B(0, r),$$

inside a ball $B(0, r) \subset \mathbb{C}^4$ is smooth, called the *Milnor fiber*. This small deformation does not alter the behavior at infinity, so its boundary is the same as the boundary of the original singularity, namely the link

$$L_X := \{f^{-1}(0)\} \cap S_r^7,$$

which is a real 5-dimensional manifold. This is also called the Sasaki-Einstein manifold associated to X .

Each Milnor fiber $M_X(t)$ of the singularity X is contained in \mathbb{C}^4 and hence can be endowed with the canonical symplectic form. This makes the Milnor fiber a symplectic

manifold with contact type boundary, together with a complex structure. Standard reasoning in symplectic geometry requires completing the Milnor fiber by attaching a cylindrical end. This allows one to define its *symplectic cohomology* $SH^*(M_X(t))$, with integer grading.

More intuitively, symplectic cohomology counts critical trajectories of certain canonically defined action functional on M_X . These include two parts, the constant orbits, i.e. points, and Reeb orbits on the link manifold. The degrees of such orbits are determined by what is called the *Conley-Zehnder index*. A full definition uses a version of Hamiltonian Floer cohomology, which is beyond the scope of this note. It suffices to think of it as counting trajectories (or solitons) connected by gradient flows determined by Floer’s equations (or instantons), similar to what was done in Morse cohomology. Curious readers are referred to e.g. [36] for a review. As alluded to above, symplectic cohomology splits into two parts:

- **Positive symplectic cohomology** SH_+ coming from non-constant periodic Reeb orbits,
- **Negative symplectic cohomology** SH_- coming from constant orbits, i.e., points.

The constant-orbit contribution can be reduced to Morse cohomology of M_X , which coincides with the usual singular cohomology $H^*(M_X)$. These fit into a long exact sequence

$$\cdots \rightarrow H^{*-1}(M_X) \rightarrow SH_+^*(M_X) \rightarrow SH^*(M_X) \rightarrow H^*(M_X) \rightarrow \cdots .$$

It turns out that $SH^*(M_X(t))$ does not depend on the choice of small deformation t , so we denote it simply by $SH^*(M_X)$. Moreover, $SH^*(M_X)$ carries the structure of a graded cohomology ring, which is a symplectic invariant of the Milnor fiber [37].

For isolated Gorenstein cDV singularities, the symplectic cohomology of the Milnor fiber has a very simple form:

$$\begin{cases} \text{rank } SH^3(M_X) & = \mu , \\ SH^k(M_X) & = 0 \text{ for } k = 2 \text{ or } k \geq 4 . \end{cases} \tag{2.3}$$

where μ denotes the *Milnor number* of the singularity.

Building on explicit computations on weighted homogeneous isolated cDV singularities, an interesting conjecture was brought up in [28], which expects that a compound Du Val singularity has crepant resolution with l irreducible exceptional curves if and only if the symplectic cohomology of its Milnor fiber in all negative degrees has rank l . It has been proved for all cA_n singularities in [35].

In section 3, we will show that among all candidates, only the four cases highlighted in the introduction exhibit symplectic cohomology groups in negative degrees whose ranks match exactly the number of exceptional curves in a crepant resolution. In particular, the numbers are nonzero. Under the assumption that the “only if” direction of the above conjecture holds, this result leads to the main claim of our paper in the introduction.

Relying this argument on a conjecture outright may seem overly ambitious, nevertheless, in section 4 we will present physical arguments that lend it support.

Moreover, direct computation of symplectic cohomology is difficult. The technical tool for this turns out to be homological mirror symmetry: we do computations in *Hochschild cohomology* in the mirror side.

2.4 Hochschild cohomology: an introduction

Hochschild cohomology is widely applied in the study of mirror symmetry, as it is a bridge connecting closed string data to the open string ones. The mathematical essence of Hochschild cohomology is that it encodes the deformations of a structure. The following material is standard in the literature.

In its most elementary form, Hochschild cohomology is used to describe the infinitesimal deformation of an associative algebra up to equivalence. In other word it is the tangent space of the “moduli space” of an associative algebra at a specific point. More concretely, an associative algebra A consists of a \mathbb{C} vector space V and associative \mathbb{C} -bilinear multiplication $m_0 : V \times V \rightarrow V$. A deformation of the algebra A means a deformation of the multiplication from, say, $m_0(x, y)$ to $m(x, y)$. Being an infinitesimal deformation means that one can expand m near m_0 and looks at only the first order terms: $m(x, y) = m_0(x, y) + \epsilon(x, y)$. The constraint is associativity of multiplication

$$m(m(x, y), z) = m(x, m(y, z)).$$

The first order equation reads

$$m_0(\epsilon(x, y), z) + \epsilon(m_0(x, y), z) - m_0(x, \epsilon(y, z)) - \epsilon(x, m_0(y, z)) = 0.$$

Simplifying notation one may abbreviate $m_0(x, y)$ by $x \cdot y$. We obtain the cocycle condition

$$x \cdot \epsilon(y, z) - \epsilon(x \cdot y, z) + \epsilon(x, y \cdot z) - \epsilon(x, y) \cdot z = 0. \tag{2.4}$$

Two multiplications m and m' are regarded equivalent if they differ only by a reparametrization, i.e., there is a linear map $h : V \rightarrow V$ such that

$$h(m'(x, y)) = m(h(x), h(y)),$$

for all x, y . Expanding to first order $h(x) = x + \delta(x)$ this means that m' and m differ by a coboundary

$$m'(x, y) - m(x, y) = m(\delta(x), y) + m(x, \delta(y)) - \delta(m(x, y)),$$

specializing to m_0 this becomes

$$m(x, y) - x \cdot y = x \cdot \delta(y) - \delta(x \cdot y) + \delta(x) \cdot y. \tag{2.5}$$

The space of infinitesimal deformations of the product $m_0(-, -)$, or equivalently that of the associative algebra A , is given by the space of maps ϵ obeying (2.4) modulo the image of coboundaries of the form (2.5). This is precisely the second Hochschild cohomology of A as an A -bimodule, where A acts on itself both from the left and from the right, usually denoted by $HH^2(A, A)$.

Note that the space of maps ϵ and δ are respectively the space of homomorphisms from tensor product of A to A , by linearity. More formally and algorithmically, the Hochschild cohomology of A is computed as the cohomology of the reduced bar complex

$$0 \rightarrow A \rightarrow \text{hom}_{\text{Vect}_{\mathbb{C}}}(A, A) \rightarrow \text{hom}_{\text{Vect}_{\mathbb{C}}}(A \otimes A, A) \rightarrow \text{hom}_{\text{Vect}_{\mathbb{C}}}(A^{\otimes 3}, A) \rightarrow \dots,$$

where each term denotes the space of \mathbb{C} -multilinear maps from tensor product of A to itself, respectively. The differential

$$d : \text{hom}(A^{\otimes n}, A) \rightarrow \text{hom}(A^{\otimes n+1}, A),$$

is given by

$$(d\omega_n)(x_0, x_1, \dots, x_n) = x_0 \cdot \omega_n(x_1, \dots, x_n) + \sum_{i=1}^n (-1)^i \omega_n(x_1, \dots, x_{i-1} \cdot x_i, x_{i+1}, \dots, x_n) + (-1)^{n+1} \omega_n(x_0, \dots, x_{n-1}) \cdot x_n.$$

(2.5) and (2.4) are concrete examples of the differential with $n = 1, 2$ respectively. In fact the differential is the alternating sum of the most general way of producing an element via a map that requires one fewer argument, preserving the order of inputs.

The notion of Hochschild cohomology can be generalized to more complicated algebraic structures, and to appropriate categories roughly by regarding the morphism spaces as an algebra. The category we will consider is the category of *equivariant matrix factorizations* of a 2d $\mathcal{N} = (2, 2)$ Landau-Ginzburg model. A pedagogical explanation of this is difficult; it suffices to know that it is the appropriate mathematical framework to describe the category of B-branes in this Landau-Ginzburg models with a weighted homogeneous superpotential.

2.5 Berglund-Hübsch-Krawitz mirror symmetry for Landau-Ginzburg models

Homological mirror symmetry (HMS) provides a powerful bridge between mirror Calabi-Yau geometries by identifying their categories of D-branes — namely, the derived Fukaya category of the A-model on one side and the derived category of coherent sheaves of the B-model on the mirror. For brevity, we will simply refer to them as the A-branes and B-branes.

When the varieties under consideration develop singularities, the classical formulation of HMS does not apply directly. Defining the appropriate categories in the presence of geometric singularities is highly nontrivial. For instance, B-branes associated with singularities of algebraic varieties have been studied in the framework of *singularity category* introduced by Orlov, see e.g. [38], which roughly encode the local geometric data near the singularity. In some cases where the singularity is defined by an affine hypersurface, this category coincides with that of B-branes in the Landau-Ginzburg (LG) model defined by the same potential. On the A-side, there is by now a well-established version of HMS within the symplectic geometry community, involving the category of A-branes in the corresponding LG models.

These developments suggest the following guiding idea: when studying the *local geometry* of singularities, the relevant categories should be replaced by those of the underlying LG models defined by the same potential. This perspective is consistent with the general philosophy of string theory, where a geometric singularity is replaced by a nonsingular superconformal field theory — either a worldvolume CFT or an LG model. This viewpoint extends beyond the traditional Landau-Ginzburg/Calabi-Yau (LG/CY) correspondence [39, 40] which concerns compact Calabi-Yau varieties.⁸

⁸We thank the referee for carefully pointing this out.

In the Landau-Ginzburg framework, B-branes are described by matrix factorizations, while A-branes correspond to vanishing cycles of a symplectic Lefschetz fibration. The only well-defined notion of mirror symmetry between LG models is that proposed by Berglund, Hübsch [41] and later clarified by Krawitz [42]. Originally, it was a combination of geometric mirror symmetry and LG/CY correspondence. Namely, starting from an LG model, one finds the corresponding compact Calabi-Yau 3-fold as the zero loci of superpotential in *weighted projective spaces*. In the same way one may find the Calabi-Yau 3-fold corresponding to the Berglund-Hübsch mirror LG model. The claim is that these two Calabi-Yau 3-folds are mirror to each other in a suitable sense. This engineers many new mirror pairs and generalizes the earlier work of Greene and Plesser [43]

Although the pioneering construction of Berglund and Hübsch allows one to perform the mirror operation starting from the LG model, its interpretation is primarily geometric through LG/CY correspondence. The work of Krawitz, however, provides a fully intrinsic definition of mirror symmetry at the level of LG models themselves. We will not explain this, but turn to the basic construction below. Altogether, this represents a highly algebraic incarnation of mirror symmetry.

Consider an *invertible* polynomial W of $n + 1$ variables, which means a weighted homogeneous polynomial consisting of $n + 1$ monomials

$$W(x_1, x_2, \dots, x_{n+1}) = \sum_{i=1}^{n+1} \prod_{j=1}^{n+1} x_j^{A_{ij}},$$

where A is a rank n integral matrix with nonvanishing determinant. Taking the transpose of A then yields the *Berglund-Hübsch mirror* polynomial

$$\check{W}(x_1, x_2, \dots, x_{n+1}) = \sum_{i=1}^{n+1} \prod_{j=1}^{n+1} x_j^{A_{ij}^T}.$$

By construction, both W and \check{W} admit a natural \mathbb{C}^* -action. One fixes a system of positive integer weights $(d_1, \dots, d_{n+1}; h)$ with $\text{g.c.d.}(d_1, d_2, \dots, d_{n+1}, h) = 1$, so that

$$W(\lambda^{d_1} x_1, \dots, \lambda^{d_{n+1}} x_{n+1}) = \lambda^h W(x_1, \dots, x_{n+1}) \quad (\forall \lambda \in \mathbb{C}^*).$$

The requirement that the corresponding singularity remains at finite distance in the moduli space [44] $\hat{c} = \sum_{i=1}^4 (1 - 2q_i) < 2$ is equivalent to the following condition

$$d_0 := h - \sum_{i=1}^{n+1} d_i < 0.$$

Beyond this \mathbb{C}^* -action, W often enjoys a larger, finite symmetry group. One convenient description introduces an auxiliary coordinate t_0 and defines

$$\Gamma_W := \left\{ (t_0, t_1, \dots, t_{n+1}) \in (\mathbb{C}^*)^{n+2} : \prod_{j=1}^{n+1} t_j^{A_{ij}} = t_0 t_1 t_2 \dots t_{n+1} \right\}.$$

This group acts on both \mathbb{C}^{n+1} via multiplication by (t_1, \dots, t_{n+1}) and \mathbb{C}^{n+2} via multiplication by (t_0, \dots, t_{n+1}) . The former action clearly preserves W , in the sense that the outcome is

a W up to a multiplicative factor. It preserves $W + x_0x_1 \cdots x_{n+1}$ as well, regarded as a polynomial in \mathbb{C}^{n+2} coordinatized by $(x_0, x_1, \dots, x_{n+1})$.

The homological mirror symmetry conjectures for Berglund-Hübsch mirror pairs (which is a special case of Berglund-Hübsch-Krawitz mirror) can be stated as follows (See e.g. [28])

Conjecture 2.2 *There is a quasi-equivalence of idempotent complete A_∞ -categories*

$$\mathcal{F}(\check{W}) \simeq \text{mf}(\mathbb{C}^{n+1}, \Gamma_W, W) \tag{2.6}$$

between the Fukaya-Seidel category of a Morsification of \check{W} and the dg-category of Γ_W -equivariant matrix factorizations of W .

Conjecture 2.3 *There is a quasi-equivalence of idempotent complete A_∞ -categories*

$$\mathcal{W}(\check{W}^{-1}(1)) \simeq \text{mf}(\mathbb{C}^{n+2}, \Gamma_W, W + x_0x_1 \cdots x_{n+1}) \tag{2.7}$$

between the wrapped Fukaya category of the Milnor fiber $\check{W}^{-1}(1)$ and the dg-category of Γ_W -equivariant matrix factorizations of $W + x_0x_1 \cdots x_{n+1}$.

Modulo technicality, this is a homological mirror symmetry statement establishing the equivalence of A-branes in the LG model (\mathbb{C}^{n+1}, W) and B-branes in the LG model $(\mathbb{C}^{n+1}, \Gamma_{\check{W}}, \check{W})$. It turns out that for computational convenience it is recommended to replace the latter by $(\mathbb{C}^{n+2}, \Gamma_{\check{W}}, \check{W} + x_0x_1 \cdots x_{n+1})$, introducing the extra x_0 -coordinate. Using either of these conjectures and the fact

$$SH^*(M_X) \cong HH^*(\mathcal{W}(M_X)), \tag{2.8}$$

it was proved in [28] that if

$$HH^2(\text{mf}(\mathbb{C}^{n+2}, \Gamma_W, W)) = 0,$$

then there is an isomorphism

$$SH^*(\check{W}^{-1}(1)) \cong HH^*(\text{mf}(\mathbb{C}^{n+2}, \Gamma_W, W)). \tag{2.9}$$

Note that (2.8) establish the symplectic-geometric invariant on the left to the more algebraic object on the right.

Therefore, once we can prove that $HH^2(\text{mf}(\mathbb{C}^{n+2}, \Gamma_W, W)) = 0$ for the singularity W , we can compute all their relevant symplectic cohomology groups via Hochschild cohomology at the mirror side. The latter is computable, albeit often very tedious. In particular, for isolated Gorenstein cDV singularities defined by invertible polynomials, one already knows that

$$SH^k(M_X) = 0, \text{ for } k = 2 \text{ or } k \geq 4.$$

So the isomorphism (2.9) is fully consistent with these vanishing results.

We will make use of the formula (2.11) to determine the Hochschild cohomology of the equivariant matrix factorizations [34]. From a physics perspective its role is computing the

Hilbert spaces of a Landau Ginzburg orbifold. Before we state the formula, let us introduce the necessary notations.

As we have mentioned, elements in Γ_W act canonically on the affine coordinates via coordinate-wise multiplications

$$(t_0, \dots, t_{n+1}) \cdot (x_0, x_1, x_2, \dots, x_{n+1}) = (t_0 x_0, t_1 x_1, \dots, t_{n+1} x_{n+1}).$$

The character χ of the group Γ_W is defined as

$$\Gamma_W \rightarrow \mathbb{C}^*, \quad \chi(t_0, t_1, \dots, t_{n+1}) = t_0 t_1 \cdots t_{n+1} = \prod_{j=1}^{n+1} t_j^{A_{ij}}. \quad (2.10)$$

The character records an overall factor of each monomial contained in W when acted on by $(t_0, \dots, t_{n+1}) \in \Gamma_W$. The kernel of χ is the finite subgroup

$$\ker \chi = \left\{ (t_0, \dots, t_{n+1}) \in (\mathbb{C}^*)^{n+2} : \prod_{j=1}^{n+1} t_j^{A_{ij}} = 1, t_0 = t_1^{-1} \cdots t_{n+1}^{-1} \right\}.$$

Under this action, each element $\gamma \in \ker \chi$ splits the coordinate space $V = \mathbb{C}^{n+2}$ (with basis x_0, \dots, x_{n+1}) into its fixed subspace V_γ and its complement subspace N_γ in V . Given a choice of $\gamma \in \ker \chi$, restricting W to the set V_γ , i.e., the set of γ -fixed variables, is denoted W_γ . Let Jac_{W_γ} be the associated Jacobian ring of W_γ . One picks a basis of the Jacobian ring as a vector space and label it by J_γ , this choice is eventually immaterial on the final constructions.

Introduce dual coordinates x_i^\vee , for $i = 0, 1, \dots, n+1$, on which Γ_W acts by

$$(t_0, \dots, t_{n+1}) \cdot x_i^\vee = t_i^{-1} x_i^\vee.$$

Equivalently, the character of x_i^\vee is inverse the character of x_i . For a general monomial $\underline{m} := \prod_i x_i^{b_i}$, where $b_i \geq -1$ and each factor with $b_i = -1$ represents x_i^\vee , one can compute its character $\chi_{\underline{m}}$ by multiplying together all factors: $\chi_{\underline{m}} = \prod_i t_i^{b_i}$.

Now we can state the formula, which reads [34]

$$\begin{aligned} HH^t(\text{mf}(\mathbb{C}^{n+2}, \Gamma_W, W)) \cong & \bigoplus_{\substack{\gamma \in \ker \chi, l \geq 0, \\ t - \dim N_\gamma = 2u}} \left(H^{-2l}(dW_\gamma) \otimes \Lambda^{\dim N_\gamma} N_\gamma^\vee \right)_{(u+l)\chi} \oplus \\ & \bigoplus_{\substack{\gamma \in \ker \chi, l \geq 0, \\ t - \dim N_\gamma = 2u+1}} \left(H^{-2l-1}(dW_\gamma) \otimes \Lambda^{\dim N_\gamma} N_\gamma^\vee \right)_{(u+l+1)\chi}, \end{aligned} \quad (2.11)$$

where $H^*(dW_\gamma)$ denotes the cohomology of the associated Koszul complex, $\Lambda^{\dim N_\gamma} N_\gamma^\vee$ is the top exterior power of N_γ^\vee . Although the Koszul complex looks horrible, its cohomology actually concentrates in at most two degrees when the singularity W_γ is isolated. More precisely, for each $\gamma \in \ker \chi$, there are two possible situations:

1. x_0 is not fixed by γ : in this case, W_γ has an isolated critical point at the origin. The cohomology of Koszul complex is concentrated in degree 0 which is isomorphic to Jac_{W_γ} , hence only the term $l = 0$ in the first direct sum contributes to the summand. Choosing a basis J_γ of Jacobian rings Jac_{W_γ} , each contributing element can be written as

$$\underline{m} = p x_0^\vee x_{j_1}^\vee \cdots x_{j_{n+1-k}}^\vee,$$

where k is the number of fixed coordinates among $\{x_1, \dots, x_{n+1}\}$, $x_{j_1}, \dots, x_{j_{n+1-k}}$ label the unfixed coordinates and $p \in J_\gamma$.⁹ One then imposes the character condition $\chi_{\underline{m}} = u\chi$ and sums over all integers u .

2. x_0 is fixed by γ : only the terms $l = 0$ contribute to the summand, but one for each direct sum. Monomials contributing to the first summand ($t - \dim N_\gamma = 2u$) are schematically $\underline{m} = x_0^{b_0} p x_{j_1}^\vee \cdots x_{j_{n+1-k}}^\vee$ such that $\chi_{\underline{m}} = u\chi$. Monomials contributing to the second summand ($t - \dim N_\gamma = 2u + 1$) are $\underline{m} = x_0^{b_0} p x_0^\vee x_{j_1}^\vee \cdots x_{j_{n+1-k}}^\vee$ such that $\chi_{\underline{m}} = u\chi$. Here b_0 is a non-negative integer and $p \in J_\gamma$.

Finally, one sum over all the elements $\gamma \in \ker \chi$ and all allowed integers u be derive $HH^t(\text{mf}(\mathbb{C}^{n+2}, \Gamma_W, W))$.

In summary, there are three types of monomials that may potentially contribute to the Hochschild cohomology for a given $\gamma \in \ker \chi$. These are called $A_\gamma, B_\gamma, C_\gamma$ in [28]

$$\begin{aligned}
 A_\gamma &= \begin{cases} \{x_0^\beta p x_{j_1}^\vee \cdots x_{j_{n+1-k}}^\vee : p \in J_\gamma, \beta = 0, 1, 2, \dots\} & \text{if } x_0 \text{ is fixed by } \gamma \\ \emptyset & \text{otherwise} \end{cases}, \\
 B_\gamma &= \begin{cases} \{x_0^\beta p x_0^\vee x_{j_1}^\vee \cdots x_{j_{n+1-k}}^\vee : p \in J_\gamma, \beta = 0, 1, 2, \dots\} & \text{if } x_0 \text{ is fixed by } \gamma \\ \emptyset & \text{otherwise} \end{cases}, \\
 C_\gamma &= \begin{cases} \emptyset & \text{if } x_0 \text{ is fixed by } \gamma \\ \{p x_0^\vee x_{j_1}^\vee \cdots x_{j_{n+1-k}}^\vee : p \in J_\gamma\} & \text{otherwise} \end{cases}.
 \end{aligned}$$

These monomials \underline{m} will be called “good” once their characters obey $\chi_{\underline{m}} = u\chi$. Each good A_γ monomial contributes rank one to $HH^{2u+n-k+1}$; each good B_γ and C_γ monomial contributes rank one to $HH^{2u+n-k+2}$, where k is the number of coordinates in $\{x_1, \dots, x_{n+1}\}$ that are fixed by γ .

The problem of determining ranks of corresponding Hochschild cohomology groups is then reduced to the problem of counting solutions to a set of integral linear (congruence) equations. In the next section, we will apply this tool to compute the symplectic cohomology of the candidate singularities in Problem 2.1. As we are considering singularities in \mathbb{C}^4 , n is specialized to be 3.

The example of cA_1 . To illustrate how the machinery works, let us recall a simple cA_1 example discussed in section 8.4 of [30]. This singularity is defined by the invertible polynomial $W = x_1^2 + x_2^2 + x_3^2 + x_4^2$. On the physics side, it admits a superconformal quiver gauge theory dual — namely, the Klebanov-Witten quiver with its standard superpotential [8]. Mathematically, the singularity has a crepant resolution with a single irreducible exceptional curve. The Berglund-Hübsch mirror is given by the same potential $\check{W} = W$, orbifolded by

$$\Gamma_{\check{W}} = \{(t_0, \dots, t_4) | t_1^2 = t_2^2 = t_3^2 = t_4^2 = t_0 t_1 \cdots t_4\} \subset (\mathbb{C}^*)^5.$$

⁹Note that we do not care about the overall sign since eventually only the rank of HH^t counts.

The character χ sends an element to $t_0 t_1 \cdots t_4$. The group $\ker \chi$ as above

$$G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2,$$

where each \mathbb{Z}_2 factor contains two choices of t_i , $i = 1, 2, 3, 4$. t_0 can be eliminated. This gives a total of 16 elements in G . Let us see what subspaces they fix in \mathbb{C}^5 . For example, $(-1, 1, 1, 1)$ sends $(x_0, x_1, x_2, x_3, x_4)$ to $(-x_0, -x_1, x_2, x_3, x_4)$, hence it fixes \mathbb{C}^3 spanned by the last three coordinate lines, and $N_\gamma = \text{span}\{x_0, x_1\}$. We have

$$\check{W}_\gamma = x_2^2 + x_3^2 + x_4^2,$$

so $\text{Jac}_{W_\gamma} \cong \mathbb{C}$. One can take $J_\gamma = \{1\}$. As x_0 is not fixed by γ , the above rules mean that there is only one element $x_0^\vee x_1^\vee$ with character $t_0^{-1} t_1^{-1}$. This is a C_γ monomial but not a good one, since its character is not in general a multiple of χ . We conclude that this element γ does not contribute to the Hochschild cohomology. Similar analysis revealed that there is no good C_γ monomial. In fact if the good condition forces such a monomial to be $x_0^\vee x_1^\vee x_2^\vee x_3^\vee x_4^\vee$, which can never appear as a C_γ monomial. The only possible choice of γ that fixes none of x_1, \dots, x_4 is $(-1, -1, -1, -1)$, which fixes x_0 . However this is actually a good B_γ monomial, and it contributes to $HH^3(\mathbb{C}^5, \Gamma_{\check{W}}, \check{W})$. This is in fact the only contribution to HH^3 . This agrees with the fact that SH^3 of the Milnor fiber of the singularity defined by W is of rank 1, the Milnor number of the singularity.

Now we turn to A_γ and B_γ monomials. This requires considering γ that fixes x_0 . Observe that $\text{Jac}_\gamma \cong \mathbb{C}$ in all situations, where we remove x_0 in the space V_γ where W_γ is defined. Hence we always take $J_\gamma = \{1\}$. One finds that the only good A_γ monomials are the following:

- $x_0^\beta x_1^\vee x_2^\vee x_3^\vee x_4^\vee$ for $\gamma = (-1, -1, -1, -1)$, with $\beta = 1, 3, 5, \dots$. In fact when $\beta = 2k - 1$ this monomial can be rearranged to be $x_0^{-2k-1} x_1^{4k} x_2^\vee x_3^\vee x_4^\vee$, which has character $\chi_{\underline{m}} = \chi^{\otimes(-2k-1)}$, or simply denoted as $\chi_{\underline{m}} = (-2k - 1)\chi$. Hence it contributes rank one to the Hochschild cohomology at degree $2 - 4k$.
- x_0^β for $\gamma = (1, 1, 1, 1)$, with $\beta = 0, 2, \dots$. When $\beta = 2k$ this has character $\chi_{\underline{m}} = -2k\chi$. Hence it contributes rank one to the Hochschild cohomology at degree $-4k$.

Altogether these contribute to the Hochschild cohomology group once in each nonpositive even degree. Now the good B_γ monomials other than the one discussed above can be obtained in a similar way, i.e.,

- $x_0^\beta x_0^\vee x_1^\vee x_2^\vee x_3^\vee x_4^\vee$ for $\gamma = (-1, -1, -1, -1)$, with β a non-negative even number. When $\beta = 2k$. This has character $\chi_{\underline{m}} = (-2k - 1)\chi$. Hence it contributes rank one to the Hochschild cohomology at degree $3 - 4k$.
- $x_0^\beta x_0^\vee$ for $\gamma = (1, 1, 1, 1)$, with β a positive odd number. When $\beta = 2k + 1$ this has character $\chi_{\underline{m}} = -2k\chi$. Hence it contributes rank one to the Hochschild cohomology at degree $1 - 4k$.

Altogether these contribute to the Hochschild cohomology group once at each odd degree not greater than 1. As a result we obtain

$$\begin{cases} HH^3(\text{mf}(\mathbb{C}^5, \Gamma, \check{W})) = 1, \\ HH^{d \leq 1}(\text{mf}(\mathbb{C}^5, \Gamma, \check{W})) = 1, \\ HH^{d=2 \text{ or } d \geq 4}(\text{mf}(\mathbb{C}^5, \Gamma, \check{W})) = 0, \end{cases} \quad (2.12)$$

which is in agreement with (2.3). It is important to bear in mind that a given monomial may appear as either a A_γ, B_γ or C_γ monomial.

3 Computation of symplectic cohomology of cE_6 and cE_7 singularities

Our goal in this section is to compute symplectic cohomology of (Milnor fiber of) the singularities $W = 0$ in Problem 2.1. Once the defining polynomials are all invertible, one can apply the techniques from the mirror side developed above. As a first step, one must verify that $HH^2(\text{mf}(\mathbb{C}^5, \Gamma_{\check{W}}, \check{W}))$ vanish.

3.1 cE_6 singularities

These singularities are

$$W = x^2 + y^3 + z^4 + yw^k : 1 < k < 16.$$

The Berlund-Hübsch mirrors are (after relabeling the coordinates)

$$\check{W} = x^2 + y^4 + z^3w + w^k : 1 < k < 16.$$

Our goal here is to show that for these mirror singularities the Hochschild cohomology satisfies

$$HH^2(\text{mf}(\mathbb{C}^5, \Gamma_{\check{W}}, \check{W})) = 0$$

and moreover HH^t stabilizes for $t < 0$ if and only if $k = 8$.

Example $k = 8$. This case is a bit more difficult than the computation of Brieskorn-Pham singularities considered in [28], due to the mixing of \mathbb{C}^* -actions on z and w . We follow and slightly generalize a method proposed in [35]. We illustrate the method for $k = 8$ as follows.

First, one can compute the normalized weights: $(d_0, \dots, d_4) = (-4, 12, 6, 7, 3)$ and $h = 24$. The basis of Jacobian algebra is taken to be

$$J = \left\{ \begin{array}{ll} 1, z, z^2, w, \dots, w^7, zw, \dots, zw^7 & \text{if both } z, w \text{ are fixed.} \\ 1, w, \dots, w^6 & \text{if } w \text{ is fixed but } z \text{ is not.} \\ 1 & \text{if } w \text{ is not fixed.} \end{array} \right\} \otimes \left\{ \begin{array}{ll} 1 & \text{if } y \text{ is not fixed.} \\ 1, y, y^2 & \text{if } y \text{ is fixed.} \end{array} \right\}. \quad (3.1)$$

In the presence of the mixing of \mathbb{C}^* -actions, one can simplify the problem by lifting the group $\Gamma_{\check{W}}$ to its covering. To simplify notation, we write Γ for $\Gamma_{\check{W}}$ in the following. We define a map Ψ

$$\Psi: (\mathbb{C}^*)^4 \rightarrow \Gamma, \quad (u_1, u_2, u_3, \tau) \mapsto (\tau^{d_0} u_1^{-1} u_2^{-1} u_3^2, \tau^{d_1} u_1, \tau^{d_2} u_2, \tau^{d_3} u_3, \tau^{d_4} u_3^{-3}) = (t_0, t_1, \dots, t_4). \quad (3.2)$$

This map is certainly surjective. Demanding that the image is contained in Γ imposes the following

$$u_1^2 = u_2^4 = u_3^{24} = 1.$$

With this new parametrization, the character $\chi \circ \Psi = \tau^h$. The kernel of map Ψ is precisely \mathbb{Z}_h . Now consider the $\ker(\chi \circ \Psi) = \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_{24} \times \mathbb{Z}_h$, hence $\ker\chi = \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_{24}$ with generators the roots of unity. An element $\gamma \in \ker\chi$ acts on coordinates via $(x_0, \dots, x_4) \mapsto (u_1 u_2^3 u_3^2 x_0, u_1 x, u_2 y, u_3 z, u_3^{-3} w)$ where we have made an identification $x_1 = x, x_2 = y, x_3 = z, x_4 = w$.

Now one can compute the character of any monomial $\underline{m} = x_0^{b_0} x^{b_1} \dots w^{b_4}$ to be $\chi_{\underline{m}} = \tau^{n_0} u_1^{n_1} u_2^{n_2} u_3^{n_3}$ where

$$\begin{cases} n_0 &= -4b_0 + 12b_1 + 6b_2 + 7b_3 + 3b_4, \\ n_1 &= b_1 - b_0, \\ n_2 &= b_2 - b_0, \\ n_3 &= -3b_4 + b_3 + 2b_0. \end{cases} \quad (3.3)$$

The existence of a corresponding $\gamma \in \ker\chi$ then means: $b_i = 0$ if x_i is fixed by γ ; $\prod_{j \in I \subset \{1,2,3,4\}} x_j \in J_\gamma$ if $x_j \in I$ are coordinates fixed by γ . Recall the order of u_i , we see that the condition for \underline{m} to be “good” is

$$2|n_1, \quad 4|n_2, \quad 24|n_3, \quad 24|n_0. \quad (3.4)$$

Then one concludes that $u = n_0/h$. This monomial will hence contribute to the Hochschild cohomology at a certain degree, depending on u and on which type it belongs to.

In principle, the algorithm for computing the rank at a certain value of u is the following: one has to enumerate all γ , first identifying whether it fixes x_0 or not to cast the corresponding monomials into one of the three types. Then one enumerates on all monomials in the Jacobian algebras of the fixed variables, and apply constraints (3.3), (3.4) to find number of integral solutions (b_0, b_1, \dots, b_4) .

In the case at hand, there is a short-cut. Observe that $b_1 \equiv b_2 \pmod{2}$. The only possibilities for (b_1, b_2) are thus $(-1, -1), (-1, 1), (0, 0)$ and $(0, 2)$. First we discuss the A -type monomials. There are only few choices of γ . Let ζ be an 24–th root of unity.

1. $\gamma = (1, 1, 1)$: in this case, z, w are fixed, so they appear as monomials in the Jacobian.

- If $b_1 = b_2 = 0$, the constraints imply that $4|b_0$ and $24|(b_0 + b_3 - 3b_4)$. A computation then shows that $b_0 \equiv 0, 4 \pmod{12}$. There are two possible monomials, x_0^{12k} and $x_0^{12k+4} z w^3$, which contribute two generators in HH^{-4k} ($k \geq 0$);
- If $b_1 = 0, b_2 = 2$, the constraints imply that $4|(b_0 - 2)$ and $24|(b_0 + b_3 - 3b_4)$. One concludes that $b_0 \equiv 6, 10 \pmod{12}$. The monomials $x_0^{12k+6} y^2 w^4$ and $x_0^{12k+10} y^2 z w^7$ contribute two generators in HH^{-4k} .

2. $\gamma = (1, 1, -1)$: in this case, z, w are not fixed, so they appear as $z^\vee w^\vee$.

- If $b_1 = b_2 = 0$, the constraints imply that $4|b_0$ and $24|(b_0 + 2)$. There is no solution;
- If $b_1 = 0, b_2 = 2$, the constraints imply that $4|(b_0 - 2)$ and $24|(b_0 + 2)$. Again, no solution.

3. $\gamma = (-1, 1, \pm i)$: $b_1 = -1, b_2 = 1$. The constraints impose that $4|(b_0 - 1)$ and $24|(2b_0 + 2)$. There is no solution.
4. $\gamma = (-1, -1, 1)$: $b_1 = b_2 = -1, 4|(b_0 + 1), 24|(2b_0 + b_3 - 3b_4)$. One concludes that $b_0 \equiv -1, 3$ or $7 \pmod{12}$. The case $b_0 \equiv -1 \pmod{12}$ requires extra care. When $b_0 = -1$ there is a good monomial $x_0^\vee x^\vee y^\vee z^2$. It contributes to HH^3 . Otherwise, there are three monomials $x_0^{3+12k} x^\vee y^\vee w^2, x_0^{7+12k} x^\vee y^\vee z w^5, x_0^{11+12k} x^\vee y^\vee z^2$ where $k \geq 0$. Two of them contribute to HH^{-4k} and the remaining one contributes to HH^{-4k-2} .
5. $\gamma = -1, -1, -1$: $b_1 = b_2 = -1, 4|(b_0 + 1), 24|(2b_0 + 2)$. One concludes that $b_0 \equiv -1 \pmod{12}$. The case $b_0 = -1$ yields a good monomial $x_0^\vee x^\vee y^\vee z^\vee w^\vee$. It contributes to HH^3 . Otherwise, $x_0^{11+12k} x^\vee y^\vee z^\vee w^\vee$ contributes to HH^{-4k-2} .
6. $\gamma = (-1, i, \pm \zeta^9)$: $4|(b_0 + 1), 24|(2 + 2b_0)$. One concludes that $b_0 \equiv -1 \pmod{12}$. For the same reason, both HH^3 and HH^{-4k-2} acquire two generators, since there are two possible choices of γ .
7. $\gamma = (-1, -i, \pm \zeta^3)$: the same as in case 6.

To summarize, one obtains Hochschild cohomology groups of rank 6 at all nonpositive even degrees, and also at degree 3. The contributions to HH^3 are in fact due to B -type monomials. Nevertheless, one performs the following trick: write $1 = x_0 x_0^\vee$ and insert it to the A -type monomials obtained above. This produces all the remaining B -type monomials. One finds that they contribute exactly to Hochschild cohomology groups at one degree higher than the corresponding A -monomials. In this way, one obtains HH^t of rank 6 at all odd degree less than or equal to one.

It remains to analyze the C -type monomials. There are 66 possible choices of γ that does not fix x_0 ; each of them may lead to one C -type monomial. 2 of them fix z, w . 6 of them fix w without fixing z , while the others fix neither of z and w . One finds that, except for these 6 elements, all 66 elements contribute one generator respectively to HH^3 . In total, we obtain

$$\begin{cases} HH^3(\text{mf}(\mathbb{C}^5, \Gamma, \check{W})) = 66, \\ HH^{d \leq 1}(\text{mf}(\mathbb{C}^5, \Gamma, \check{W})) = 6, \\ HH^{d=2 \text{ or } d \geq 4}(\text{mf}(\mathbb{C}^5, \Gamma, \check{W})) = 0. \end{cases} \tag{3.5}$$

At positive degrees, this coincide with the known symplectic cohomology of the original singularities $W = 0$. The second Hochschild cohomology of $\check{W} = 0$ vanishes as expected, and so (2.9) can be applied. As a sanity check, we see that the Milnor number of $W = 0$ is precisely 66. We then conclude that the symplectic cohomology groups of the Milnor fiber of $W^{-1}(0)$ have rank 6 at all negative degrees.

General values of k . With the experience of dealing with $k = 8$, it is now straightforward to compute the Hochschild cohomology for general values of k .

We do not have to evaluate d_i in general, since only the proportion $\omega_i = \frac{d_i}{h}$ matters. We have $(\omega_0, \dots, \omega_4) = (\frac{1}{4} - \frac{k+2}{3k}, \frac{1}{2}, \frac{1}{4}, \frac{k-1}{3k}, \frac{1}{k})$. The $h : 1$ covering homomorphism is given by (3.2) again, but the kernel of χ is now

$$u_1^2 = u_2^4 = u_3^{3k} = 1.$$

It acts via $(x_0, \dots, x_4) \mapsto (u_1 u_2^3 u_3^2 x_0, u_1 x, u_2 y, u_3 z, u_3^{-3} w)$. The choice of monomial basis is the following

$$J = \left\{ \begin{array}{ll} 1, z, z^2, w, \dots, w^{k-1}, zw, \dots, zw^{k-1} & \text{if both } z, w \text{ are fixed.} \\ 1, w, \dots, w^{k-2} & \text{if } w \text{ is fixed but } z \text{ is not.} \\ \text{1if } w & \text{is not fixed.} \end{array} \right\} \otimes \left\{ \begin{array}{ll} 1 & \text{if } y \text{ is not fixed.} \\ 1, y, y^2 & \text{if } y \text{ is fixed.} \end{array} \right\}. \quad (3.6)$$

Good monomials obey the following conditions

$$2|(b_1 - b_0), \quad 4|(b_2 - b_0), \quad 3k|(b_3 - 3b_4 + 2b_0).$$

And it is required that $u = \left(\frac{1}{4} - \frac{k+2}{3k}\right)b_0 + \frac{b_1}{2} + \frac{b_2}{4} + \frac{(k-1)b_3}{3k} + \frac{b_4}{k}$ is integral. Again, there are four choices of the tuple (b_1, b_2) . The elements γ that fix x_0 can be case into one of the five classes

1. $\gamma = (1, 1, 1)$: in this case, z, w are fixed, so they appear as monomials in the Jacobian.

- $b_1 = b_2 = 0$. In all, number of generators contributed to HH^{2u} is the number of integral tuples (b_0, b_3, b_4) obeying the following constraints

$$\begin{cases} u = \left(\frac{1}{4} - \frac{k+2}{3k}\right)b_0 + \frac{(k-1)b_3}{3k} + \frac{b_4}{k}, \\ b_0 \geq 0, \quad 4|b_0, \quad 3k|(2b_0 + b_3 - 3b_4), \\ z^{b_3} w^{b_4} \in J. \end{cases}$$

- $b_1 = 0, b_2 = 2$. In all, number of generators contributed to HH^{2u} is the number of integral tuples (b_0, b_3, b_4) obeying the following constraints

$$\begin{cases} u = \left(\frac{1}{4} - \frac{k+2}{3k}\right)b_0 + \frac{1}{2} + \frac{(k-1)b_3}{3k} + \frac{b_4}{k}, \\ b_0 \geq 0, \quad 4|(b_0 - 2), \quad 3k|(2b_0 + b_3 - 3b_4), \\ z^{b_3} w^{b_4} \in J. \end{cases}$$

2. $\gamma = (1, 1, -1)$: in this case, z, w are not fixed, so they appear as $z^\vee w^\vee$.

- Number of generators contributed to HH^{2u+2} is the number of integral tuples (b_0, b_3, b_4) obeying the following constraints

$$\begin{cases} u = \left(\frac{1}{4} - \frac{k+2}{3k}\right)b_0 - \frac{(k+2)}{3k}, \\ b_0 \geq 0, \quad 4|b_0, \quad 3k|(2b_0 + 2), \\ z^{b_3} w^{b_4} \in J. \end{cases}$$

- Number of generators contributed to HH^{2u+2} is the number of integral tuples (b_0, b_3, b_4) obeying the following constraints

$$\begin{cases} u = \left(\frac{1}{4} - \frac{k+2}{3k}\right)b_0 - \frac{(k+2)}{3k} + \frac{1}{2}, \\ b_0 \geq 0, \quad 4|b_0, \quad 3k|(2b_0 + 2), \\ z^{b_3} w^{b_4} \in J. \end{cases}$$

3. $\gamma = (-1, -1, 1)$: number of generators contributed to HH^{2u+2} is the number of integral tuples (b_0, b_3, b_4) obeying the following constraints

$$\begin{cases} u = \left(\frac{1}{4} - \frac{k+2}{3k}\right) b_0 - \frac{3}{4} + \frac{(k-1)b_3}{3k} + \frac{b_4}{k}, \\ b_0 \geq 0, \quad 4|(b_0 + 1), \quad 3k|(2b_0 - b_3 + 3b_4), \\ z^{b_3} w^{b_4} \in J. \end{cases}$$

Note that for $b_0 = -1$ there is an extra generator of HH^3 , at $u = 0$.

4. $\gamma = (-1, -1, -1)$: contributions to HH^{2u+4} are from integral tuples (b_0, b_3, b_4) obeying the following constraints

$$\begin{cases} u = \left(\frac{1}{4} - \frac{k+2}{3k}\right) b_0 - \frac{3}{4} - \frac{k+2}{3k}, \\ b_0 \geq 0, \quad 4|(b_0 + 1), \quad 3k|(2b_0 + 2), \\ z^{b_3} w^{b_4} \in J. \end{cases}$$

Note that for $b_0 = -1$ there is an extra generator of HH^3 , at $u = -1$.

5. $\gamma = (-1, 1, \pm i)$: this is only possible when $4|k$. Contributions to HH^{2u+3} are from integral tuples (b_0, b_3, b_4) obeying the following constraints

$$\begin{cases} u = \left(\frac{1}{4} - \frac{k+2}{3k}\right) b_0 - \frac{1}{4} - \frac{k+2}{3k}, \\ b_0 \geq 0, \quad 4|(b_0 - 1), \quad 3k|(2b_0 + 2), \\ z^{b_3} w^{b_4} \in J. \end{cases}$$

6. $\gamma = (-1, i, \pm\sqrt{-i})$ or $(-1, -i, \pm\sqrt{i})$: contributions to HH^{2u+4} are from integral tuples (b_0, b_3, b_4) obeying the following constraints

$$\begin{cases} u = \left(\frac{1}{4} - \frac{k+2}{3k}\right) b_0 - \frac{1}{4} - \frac{k+2}{3k}, \\ b_0 \geq 0, \quad 4|(b_0 + 1), \quad 3k|(2b_0 + 2), \\ z^{b_3} w^{b_4} \in J. \end{cases}$$

Note that there are in total four extra tuples contributing to HH^3 for this class.

As in the example where $k = 8$, B -monomials contribute the same amount to cohomology groups at one degree higher (but not to HH^4). C -monomials contribute in total $9k - 12$ generators to $HH^3(\text{mf}(\mathbb{C}^5, \Gamma, \check{W}))$. Hence one finds that the rank at degree 3 is $9k - 6$, which reproduces the Milnor number of W as expected.

We implement the above computations using **Mathematica**. The results at degree $-10 \leq d \leq 2$ are listed in table 3. In particular, we find that $HH^2(\text{mf}(\mathbb{C}^5, \Gamma, \check{W})) = 0$ for all k , and $\text{rank}(HH^{d < 0}(\text{mf}(\mathbb{C}^5, \Gamma, \check{W})))$ stabilizes only if $k = 8$, which is the case shown in detail in the previous example. Hence we assert that only in that case there exists a crepant resolution.

k	rank $HH^d, d = -10, -9, \dots, 1, 2$
2	3, 3, 2, 2, 3, 3, 2, 2, 2, 2, 3, 3, 0
3	2, 2, 2, 2, 1, 1, 3, 3, 3, 3, 2, 2, 0
4	2, 2, 3, 5, 5, 5, 2, 2, 3, 5, 5, 5, 0
5	3, 3, 1, 1, 4, 4, 3, 3, 2, 2, 4, 4, 0
6	5, 5, 3, 3, 3, 3, 3, 3, 3, 3, 5, 5, 0
7	2, 2, 4, 4, 3, 3, 5, 5, 1, 1, 5, 5, 0
8	6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 0
9	2, 2, 5, 5, 3, 3, 5, 5, 1, 1, 6, 6, 0
10	5, 5, 3, 3, 3, 3, 5, 5, 3, 3, 6, 6, 0
11	6, 6, 3, 3, 5, 5, 4, 4, 2, 2, 6, 6, 0
12	6, 6, 2, 2, 6, 6, 3, 5, 5, 5, 6, 6, 0
13	5, 5, 3, 3, 6, 6, 2, 2, 4, 4, 6, 6, 0
14	3, 3, 3, 3, 6, 6, 3, 3, 5, 5, 6, 6, 0
15	3, 3, 5, 5, 6, 6, 1, 1, 5, 5, 6, 6, 0

Table 3. $HH^d(\text{mf}(\mathbb{C}^5, \Gamma, \check{W}))$ for $-10 \leq d \leq 2$.

3.2 cE_7 singularities

The singularities are

$$W = x^2 + y^3 + yz^3 + zw^k : 2 \leq k \leq 26, 2|k.$$

The corresponding Berglund-Hübsch mirrors are

$$\check{W} = x^2 + y^3z + z^3w + w^k : 2 \leq k \leq 26, 2|k.$$

Our goal is to show that the rank of Hochschild cohomology groups of these mirror singularities is zero at degree 2, and stabilize at all negative degrees if and only if $k = 2$ or $k = 14$.

Hochschild cohomologies of these singularities are much more difficult to compute than the E_6 cases, due to the mixing of characters. We follow and slightly generalize the trick in [28]. Define the group $G \subset \mathbb{Z}_2 \times \mathbb{Z}_3 \times (\mathbb{C}^*)^2$ which contains elements (s, μ, ρ, τ) with relation $\rho^3 = \mu^2\tau^{4k+2}$. This can be mapped surjectively to Γ via

$$\Psi : G \rightarrow \Gamma, \quad (s, \mu, \rho, \tau) \mapsto (s\rho^{-1}\mu^2\tau^{k-4}, s\tau^{3k}, \rho, \mu\tau^{2k-2}, \tau^6) = (t_0, t_1, \dots, t_4). \quad (3.7)$$

Now $\chi \circ \Psi = \tau^{6k}$. One can see that $\ker\Psi$ is the following subgroup

$$\left\{ (s = 1, \mu = \tau^{2-2k}, \rho = 1, \tau) \mid \tau^6 = 1 \right\}.$$

This is an order 6 subgroup, so one concludes that the map Ψ is 6 to 1.

One picks the monomial basis

$$\begin{aligned}
 J_\gamma \text{ fixes } y,z,w &= \begin{cases} y^l z^i w^j, & l = 0, 1; i = 0, 1, 2; j = 0, 1, \dots, k-2, \\ y^2 w^j, & j = 0, 1, \dots, k-2, \\ y^l z^i w^{k-1}, & l = 0, 1; i = 0, 1, 2, \end{cases} \\
 J_\gamma \text{ fixes } z,w &= \begin{cases} z^i w^j, & i = 0, 1; j = 0, 1, \dots, k-1, \\ z^2, & \end{cases} \\
 J_\gamma \text{ fixes } z,y &= \begin{cases} y^i z^j, & i = 0, 1; j \geq 0, \\ y^2, & \end{cases} \\
 J_\gamma \text{ fixes } w &= \{w^j, \quad j = 0, 1, \dots, k-1.
 \end{aligned}$$

We note that if γ fixes z, y but not w , then the singularity $\check{W}_\gamma = 0$ is not isolated, and the simplification below (2.11) may break down. However, as we will see, it is expected that the caveat is immaterial in the current context.

As before, consider first the case that γ fixes x_0 . This means the following

$$\begin{cases} s\tau^{k-4} = \rho\mu, \\ \rho^3 = \mu^2\tau^{4k+2}, \\ \tau^{6k} = 1. \end{cases} \tag{3.8}$$

Eliminating ρ yields $s\mu = \tau^{k+14}$ and in particular $\tau^{84} = 1$. Hence the details depend on the g.c.d. of $6k$ and 84 . Since $\text{g.c.d.}(6k, 84) = 12$ for all $k \neq 14$, we find that there are two possible γ , given by $\tau^6 = 1$ and $\tau^6 = -1$,¹⁰ respectively, that fix x_0 for each such k . One such γ fixes all coordinates, while the other fixes x_0 only. When $k = 14$, τ can only be determined up to $\tau^{84} = 1$, hence there are fourteen elements γ . One of them fixes all, six of them fix x_0, x and the remaining seven fix x_0 only. In that case, one finds the following

$$(s, \mu, \rho, \tau) = (1, \zeta^{4m}\tau^4, \zeta^{11m}, \tau | \tau^6 = \zeta^m, m = 0, 1, \dots, 13), \tag{3.9}$$

where one picks a 14-th root of unity and let it be ζ .

Hence, denote a monomial $\underline{m} = x_0^{b_0} x^{b_1} \dots w^{b_4}$ as before, we can compute its character and look at the following

1. γ fixes all variables: $\chi_{\underline{m}} = \chi^{\otimes u}$ means that, s, μ and ρ must be eliminated by only applying the defining relations $s^2 = \mu^3 = 1$ and $\rho^3 = \mu^2\tau^{4k+2}$. Hence there are contributions to HH^{2u} from integral tuples (b_0, b_2, b_3, b_4) obeying the following constraints

$$\begin{cases} b_0 \geq 0, & 2|b_0, & 3|(b_2 - b_0), & 3 \mid \left(2b_0 + b_3 + 2\frac{b_2 - b_0}{3}\right), \\ 6ku = (4k + 2)\frac{b_2 - b_0}{3} + b_0(k - 4) + b_3(2k - 2) + 6b_4, \\ x_0^{b_0} y^{b_2} z^{b_3} w^{b_4} \in J. \end{cases}$$

¹⁰Recall that Ψ is 6 to 1, so each possibility yields only one element γ .

2. γ fixes x_0 only: $b_1 = b_2 = b_3 = b_4 = -1$. The character condition says that there are contributions to HH^{2u+4} from the integer b_0 obeying the following constraints

$$\begin{cases} b_0 \geq 0, & 2|(b_0 - 1), & 3|(1 + b_0), & 3 \left| \left(2b_0 + 2 - 2\frac{b_0+1}{3} \right) \right., \\ 6ku = -(4k + 2)\frac{1+b_0}{3} + b_0(k - 4) - 5k - 4. \end{cases}$$

3. γ fixes x_0 and x only: this is possible only when $k = 14$. $b_2 = b_3 = b_4 = -1$; b_1 can only be 0 in the Jacobian algebra. There are contributions to HH^{2u+3} from integer b_0 obeying the following constraints

$$\begin{cases} b_0 \geq 0, & 2|b_0, & 3|(1 + b_0), & 3 \left| \left(2b_0 + 2 - 2\frac{b_0+1}{3} \right) \right., \\ 6ku = -(4k + 2)\frac{1+b_0}{3} + b_0(k - 4) - 2k - 4, & u \text{ is integral.} \end{cases}$$

These exhaust all possible A -monomials. Some remarks are in order. First of all, one can still perform the substitution to obtain B -monomials which contribute to one degree higher, but as in the E_6 case there can be extra B -monomials with $b_0 = -1$. To find them, note that as the character of x_0 contains s , $\chi_{\underline{m}}$ depends merely on τ only if b_0 is even or b_0 is odd with $b_1 = -1$. If one further demands that γ fixes x_0 , then the only source for those extra B -monomials are from class 2 of the above, with $\underline{m} = x_0^\vee x^\vee y^\vee z^\vee w^\vee$. Each such γ contributes one generator to HH^3 . Therefore we exhaust all possible A - and B -monomials from this procedure.

A second remark is that, by implementing the above computations in `Mathematica`, we found that no A - and B -monomial obtained in the procedure above contributes to $HH^2(\text{mf}(\mathbb{C}^5, \Gamma, \check{W}))$, as one may expect. On the other hand, stabilization at negative degrees happens precisely when $k = 2$ and $k = 14$. Class 3 in fact contributes nothing; this is desired since it yields generators of cohomology at odd degrees.

It remains to analyze C -monomials. We will show that they contribute only to $HH^3(\text{mf}(\mathbb{C}^5, \Gamma, \check{W}))$. We do not aim to count the precise rank of HH^3 ; in fact, it suffices to ensure that they do not contribute to HH^2 in order for (2.9) to hold.

Again, we work by enumeration. To obtain a good monomial, γ must not fix x once it does not fix x_0 . Observe from (3.7) that once γ fixes y and w it must then fix z . The constraints read

$$\begin{cases} b_0 = -1, & 3|(b_2 + 1), & 3 \left| \left(1 + b_3 + 2\frac{b_2+1}{3} \right) \right., \\ 6ku = (4k + 2)\frac{b_2+1}{3} + 4 - 4k + b_3(2k - 2) + 6b_4, & u \text{ is integral.} \end{cases}$$

There are the following cases

1. γ fixes y only. One concludes that $\rho = 1$. The only possible \underline{m} appears $x_0^\vee x^\vee z^\vee w^\vee$. $b_0 = 0$ violates the constraints.
2. γ fixes y, z . $\underline{m} = x_0^\vee x^\vee y^2 w^\vee$. This contributes to HH^3 .
3. γ fixes y, z, z . $\underline{m} = x_0^\vee x^\vee y^2 w^i$ where $i = 0, 1, \dots, k - 2$. Then $u = (b_4 + 1)/k$ cannot be integral.
4. γ fixes z only. $\underline{m} = x_0^\vee x^\vee y^\vee w^\vee$. Constraints violated by $b_3 = 0$.

k	$\text{rank } HH^d, d = -10, -9, \dots, 1, 2$
2	1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 0
4	2, 2, 1, 1, 1, 1, 2, 2, 2, 2, 2, 0
6	2, 2, 3, 3, 2, 2, 1, 1, 3, 3, 3, 0
8	2, 2, 2, 2, 4, 4, 1, 1, 3, 3, 4, 0
10	5, 5, 1, 1, 4, 4, 3, 3, 2, 2, 5, 0
12	3, 3, 4, 4, 2, 2, 5, 5, 1, 1, 6, 0
14	7, 7, 7, 7, 7, 7, 7, 7, 7, 7, 7, 0
16	3, 3, 5, 5, 2, 2, 6, 6, 1, 1, 7, 0
18	6, 6, 3, 3, 4, 4, 5, 5, 2, 2, 7, 0
20	7, 7, 1, 1, 6, 6, 4, 4, 3, 3, 7, 0
22	6, 6, 1, 1, 7, 7, 3, 3, 4, 4, 7, 0
24	4, 4, 3, 3, 7, 7, 2, 2, 5, 5, 7, 0
26	2, 2, 5, 5, 7, 7, 1, 1, 6, 6, 7, 0

Table 4. $HH^d(\text{mf}(\mathbb{C}^5, \Gamma, \check{W}))$ for $-10 \leq d \leq 2$.

5. γ fixes z, w . It is only possible that $\underline{m} = x_0^\vee x^\vee y^\vee z^2$. This contributes to HH^3 .
6. γ fixes w only. $\underline{m} = x_0^\vee x^\vee y^\vee z^\vee w^i$ where $i = 0, 1, \dots, k - 2$. No integral u exists.
7. γ fixes nothing. $\underline{m} = x_0^\vee x^\vee y^\vee z^\vee w^\vee$. One finds that $u = 1$; this contributes to HH^3 .

Combined with the remarks in the previous paragraph, we assert that all the cE_7 singularities in Problem 2.1 admit no crepant resolutions. The results at degree $-10 \leq d \leq 2$ are listed in table 4.

4 Verification from the physics side

In this section, we substantiate our results from the physics perspective by invoking the AdS/CFT correspondence:

$$\text{large } N \text{ 4d } \mathcal{N}=1 \text{ quiver SCFT} \quad \longleftrightarrow \quad K\text{-stable 3-fold singularity with NCCR.}$$

Crucially, the structure of the quiver and its superpotential is determined by the underlying NCCR. Throughout this work, we have confined ourselves to isolated terminal singularities, for which the existence of an NCCR is equivalent to that of a crepant resolution — thus yielding a transparent physical interpretation of the duality. Strong evidence for this AdS/CFT correspondence comes from matching invariant data on both sides, particularly the leading order central charge a and the Hilbert series of the affine ring of X [10, 27].¹¹

¹¹The chiral ring of the gauge theory splits into mesonic and baryonic sectors. The mesonic sector is captured by the coordinate ring of the symmetric product $M_{vac} = X^N/S_N$. Moreover, the mesonic operators splits into single-trace and multi-trace sectors. The single-trace mesonic operators are precisely those arising from a single copy of the affine coordinate ring $\mathbb{C}[X]$ of X . The full mesonic ring corresponds to the coordinate ring $\mathbb{C}[M_{vac}] = (\mathbb{C}[X]^{\otimes N})^{S_N}$ of the symmetric product M_{vac} , which is the algebra of symmetric functions on $\mathbb{C}[X]$. For further details, the readers can refer to [10].

On the field theory side, central charge a of the SCFT can be computed from the quiver Hilbert series [45]. On the geometric side, a is inversely proportional to the volume of the associated Sasaki-Einstein manifold, which itself can be computed from the singularity’s Hilbert series. Thus, a necessary condition for a consistent holographic duality is the equality of the two Hilbert series:

$$H_{\text{sing}}(t) = H_{00}(t),$$

where $H_{\text{sing}}(t)$ denotes the Hilbert series of the singularity, and $H_{00}(t)$ the quiver Hilbert series at the distinguished node 0.

Assuming the AdS/CFT correspondence, the existence of a crepant resolution can be tested by classifying all candidate quiver SCFTs — without prior knowledge of their superpotentials — whose quiver Hilbert series at a chosen node 0 coincides with the Hilbert series of the singularity. Such gauge theories must also satisfy additional consistency requirements, which we detail in subsection 4.1. Our objective is to demonstrate that, for each singularity listed in Problem 2.1, no admissible quiver SCFT exists. By AdS/CFT correspondence, this absence of field-theoretic candidates implies the nonexistence of crepant resolutions for these singularities. The explicit search procedure is described in subsection 4.2.

It is important to emphasize that this correspondence remains conjectural: matching Hilbert series provides only a necessary condition for holographic duality. Even if the series coincide, one must still compare additional data — such as the operator spectrum in the field theory versus those in the gravity theory — to confirm a genuine duality. However, for our “no-go” argument, verifying the mismatch of Hilbert series alone is sufficient.

4.1 Consistency conditions for superconformal quiver gauge theory duals

Scale invariance, unitarity and central charges. A key requirement for consistency is the vanishing of the NSVZ β -functions for each gauge coupling [46]. Equivalently, the ABJ anomaly for $U(1)_R$ should vanish

$$\text{Tr}(RG_iG_i) = 0. \tag{4.1}$$

Here the trace runs over all Weyl fermions charged under the simple gauge group G_i . In terms of group-theoretic data, this condition becomes

$$C_2(G_i) + \sum_{k \text{ chiral}} T(\text{Rep}_k)(R_k - 1) = 0, \tag{4.2}$$

where $C_2(G_i)$ is the quadratic Casimir of the adjoint representation of G_i and $T(\text{Rep}_k)$ denotes the Dynkin index of the irreducible representation Rep_k , where the k -th chiral superfield transforms. The R_k denotes the R charge of the k -th chiral under Rep_k representation of G_i . In our quiver theories, only adjoint and bifundamental chiral multiplets appear. For later reference, we summarize the Dynkin indices for the adjoint and fundamental representations of the $SU(N)$ gauge group:

$$T(\text{adj}) = N = C_2(SU(N)), \quad T(\text{fund}) = \frac{1}{2}.$$

Another important constraint is unitarity, which requires every gauge-invariant chiral operator to have scaling dimension $\Delta \geq 1$. Using the 4d $\mathcal{N} = 1$ shortening condition for chiral operators

$$\Delta = \frac{3}{2}R,$$

the unitarity requires that $R \geq \frac{2}{3}$. Any gauge invariant chiral operator saturating this bound is free. Moreover, this shortening condition also ensures that R charge of a composite chiral operator is simply the sum of the R charges of its constituent chiral fields.

The central charges a and c depend on the $U(1)_R$ symmetry via the 't Hooft anomalies $\text{Tr } R$ and $\text{Tr } R^3$ [47]:

$$a = \frac{3}{32}(3 \text{Tr } R^3 - \text{Tr } R), \quad c = \frac{1}{32}(9 \text{Tr } R^3 - 5 \text{Tr } R).$$

All such anomalies can be computed directly from the quiver data and the assumed R charge assignments. On the other hand, one may extract the leading large N central charge from the asymptotic behavior of the quiver Hilbert series $H_{00}(t)$ [48]. Writing $t = e^{-s}$, one finds

$$H_{00}(\exp(-s)) = \frac{a_0}{s^3} + \frac{a_1}{s^2} + \dots$$

Accordingly, the leading coefficient a_0 in the small- s expansion of the Hilbert series directly fixes the central charge a and c of the dual SCFT.¹²

$$a = c = \frac{27}{32} \frac{1}{a_0} N^2. \tag{4.3}$$

Stability of chiral ring. Under the AdS/CFT dictionary, the requirement that a polarized chiral ring (chiral ring with specialized R symmetry) is “stable” in the sense of four-dimensional $\mathcal{N}=1$ SCFTs is expectedly equivalent to the K-stability of the dual singularity [11]. Concretely, one implements this stability by a generalized a-maximization procedure over trial $U(1)$ mixings of the R-symmetry [11, 13]. If the chiral ring fails this test — because a superpotential deformation becomes irrelevant or an operator’s R-charge falls below the unitarity bound — the putative SCFT does not exist, even when the geometry admits a noncommutative crepant resolution. Hence, imposing K-stability on the singularity provides a simple and powerful field-theoretic criterion to rule out such inconsistent quiver candidates.

We therefore impose K-stability on the singularity, which restricts us to the region of parameter space listed in table 1 [11, 49]. Concretely, this entails two conditions:

1. **Positivity of Futaki invariants** Futaki invariants $F(X, \zeta, \eta)$ of some test configurations generated by η are positive. Let ζ be the Reeb vector field encoding the R charges w_1, w_2, w_3, w_4 of coordinates x, y, z, w normalized s.t. the $(3, 0)$ form $\Omega = \frac{dx \wedge dy \wedge dz \wedge dw}{df}$ has R charge two, i.e., $w_1 + w_2 + w_3 + w_4 - d = 2$. For each test configuration generated by a vector field η with weight (v_1, v_2, v_3, v_4) , one computes the Futaki invariant $F(X, \zeta, \eta)$. Theorem 3.1 of [49] states that K-stability requires

$$F(X, \zeta, \eta) > 0, \text{ for each nontrivial } \eta.$$

¹²In the large N limit, for quivers consisting of only adjoints and bifundamentals, $\text{Tr } RGG = 0$ ensures that $\text{Tr } R = 0$, therefore $a = c$.

In our coordinate basis,

$$\begin{aligned}
 F(X, \zeta, \eta) = & -[v_4 w_1 w_2 w_3 (w_1 + w_2 + w_3 - 2w_4 - d) \\
 & + v_3 w_1 w_2 w_4 (w_1 + w_2 + w_4 - 2w_3 - d) \\
 & + v_2 w_1 w_3 w_4 (w_1 + w_3 + w_4 - 2w_2 - d) \\
 & + v_1 w_2 w_3 w_4 (w_2 + w_3 + w_4 - 2w_1 - d)].
 \end{aligned}
 \tag{4.4}$$

2. Unitarity bound on R charges Since each coordinate x, y, z, w correspond to a gauge-invariant operator under holographic dual. Unitarity requires the R charges of them should be greater than or equal to $\frac{2}{3}$.

These two criteria together carve out the K-stable region of (N, k) , as recorded in table 1. We now illustrate the application of these two criteria with a specific example.

Example: consider the family of singularities

$$X : x_1^2 + x_2^2 + x_3^4 + z^k = 0.$$

The R charges of $(x_1, x_2, x_3, z; d)$ is

$$(w_1, w_2, w_3, w_4; d) = \left(\frac{12k}{k+12}, \frac{8k}{k+12}, \frac{6k}{k+12}, \frac{24}{k+12}; \frac{24k}{k+12} \right).$$

To test K-stability, take the symmetry $\eta = (0, 0, 0, 1)$. A straightforward computation gives the Futaki invariant

$$F(X, \zeta, \eta) = -\frac{1152(-24+k)k^3}{(12+k)^4}.$$

$F > 0$ gives the constraints $0 < k < 24$. The k should be greater than 1 since it is an isolated singularity. Within the range $1 < k < 24$, the R charges of (x_1, x_2, x_3, z) are all larger than $\frac{2}{3}$, which satisfies the unitarity requirements.

NCCR and shape of the quiver. Because the gauge-theory quiver is encoded by NCCR of the singularity, we must examine all admissible NCCRs. In fact, the quiver underlying any NCCR is directly related to the dual graph of a corresponding crepant resolution. Morrison [50] proved that any small crepant resolution of an isolated Gorenstein threefold singularity yields a collection of exceptional curves whose intersection graphs are those shown in figure 1.

More precisely, the diagrams in figure 1 depict the quivers associated to NCCR with the distinguished node deleted.¹³ Each remaining node (labelled by its multiplicity $1, 2, \dots, n$) represents an exceptional \mathbb{P}^1 in the small resolution, and each pair of opposite arrows encodes transverse intersection between two such curves.

To reconstruct the 4d $\mathcal{N} = 1$ gauge theory quiver [52], one proceeds as follows:

- 1. Gauge node** Associate to each node as an SU type gauge group whose rank is proportional to N , the number of $D3$ branes probing the singularity.

¹³The deleted distinguished node corresponds to the coordinate ring itself as its module. It appears as a summand in any tilting module whose endomorphism algebra produces an NCCR.

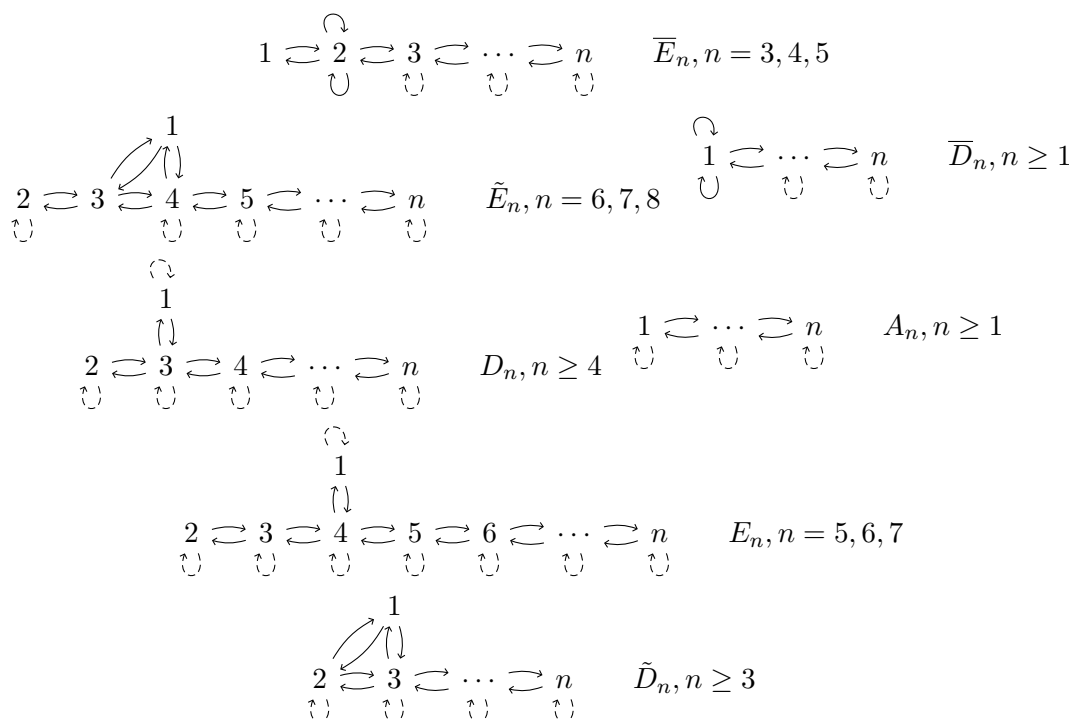
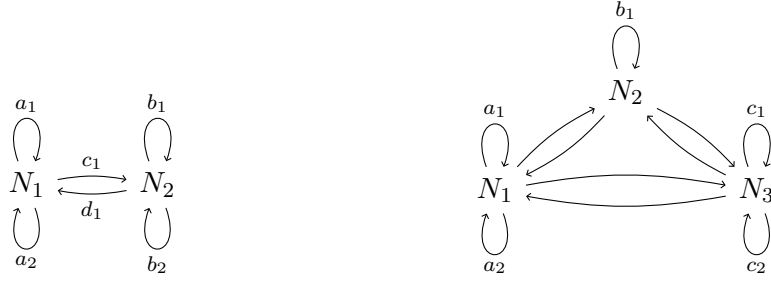


Figure 1. The possible shape of the quivers corresponding to the (one-node-deleted) NCCR of the cDV singularities. [51] The dotted arc represents that there may be an adjoint chiral or may be not.

2. **Distinguished node** Add a node corresponding to the trivial module.
3. **Bifundamental fields** Associate a pair of bifundamental chiral multiplets for each pair of opposite arrows between two nodes.
4. **Adjoint fields** A solid loop at a node indicates the presence of an adjoint chiral multiplet transforming in the adjoint of that gauge group; dashed loops may or may not correspond to additional adjoint fields.

When a singularity is known to admit crepant resolution, one can often construct the corresponding NCCR explicitly — for example, by employing matrix factorization methods. In contrast, for singularities whose crepant resolvability remains undecided (as in our case), a brute-force search for all candidate quiver SCFTs requires enumerations. Morrison’s classification of small crepant resolutions and their associated dual graphs enables us to restrict to a finite set of quiver topologies, avoiding unnecessary enumeration.

Since we only focus on the $f(X) = 1, 2$ case, i.e., the associated quiver have two-node or three-node. Among the quivers shown in figure 1, only the A_n and \bar{D}_n families admit two and three-node quivers. Therefore, most general two-node and three-node quivers are those shown in figure 2. We emphasize that these general quivers also includes quivers with fewer adjoint fields: setting the R -charge of an adjoint chiral to 1 effectively removes its contribution in the infrared, recovering the cases with less adjoint at that node.



(a) quiver for one mass deformation. (b) quiver for two mass deformations.

Figure 2. There are at most two adjoints on the distinguished node N_1 . There may be a pair of bifundamental chirals between all the other nodes and the distinguished node.

For the case $f = 1$, the quiver consists of two nodes associated to gauge groups $G = \text{SU}(N_1), \text{SU}(N_2)$. Imposing the NSVZ beta functions [46] for each gauge group give equations

$$\begin{cases} N_1(1 + R_{a_1} - 1 + R_{a_2} - 1) + \frac{1}{2}(R_{c_1} - 1 + R_{d_1} - 1)N_2 = 0, \\ N_2(1 + R_{b_1} - 1 + R_{b_2} - 1) + \frac{1}{2}(R_{c_1} - 1 + R_{d_1} - 1)N_1 = 0. \end{cases} \quad (4.5)$$

Here R_X denote R charges of the corresponding chiral field $X = a_1, a_2, b_1, b_2, c_1, d_1$. Physical consistency further requires both ranks to be positive: $N_1 > 0, N_2 > 0$. One can solve the ratio of ranks N_1, N_2 of the gauge groups from the beta functions in terms of R charges of the quiver.

For a given quiver gauge theory Q , one computes its matrix Hilbert series [27, 53] via the formula

$$H(Q, t) = \frac{1}{1 - M_Q(t) + t^2 M_Q^T(t^{-1}) - t^2}, \quad (4.6)$$

where adjacent matrix M_Q can be read from the quiver and R charges:

1. If $i \neq j$, the off-diagonal element M_{ij} of M_Q is

$$M_{ij} = \sum_{\text{bifund chirals in } (\mathbb{N}_i, \bar{\mathbb{N}}_j)} t^{R_{ij}}.$$

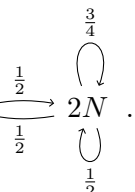
2. If $i = j$, the diagonal element M_{ii} is

$$M_{ii} = \sum_{\text{adjoint chiral fields}} t^{R_{ii}}.$$

Here R_{ij} and R_{ii} denote the R charge of bifundamental and adjoint chiral fields, respectively. The (i, j) entry of $H(Q, t)$ counts the oriented path from the node i to node j with the R charge grading. In particular, the quiver Hilbert series $H_{00}(Q, t)$ with respect to the distinguished node 0 is the $(0, 0)$ entry of the matrix $H(Q, t)$, which enumerates closed loops based at node, and therefore counts the gauge invariant scalar operators. Under the holographic duality, the H_{00} is believed to be identified with the Hilbert series of the affine coordinate ring of the dual singularity. Note that each node can be chosen as the node 0 and the corresponding quiver Hilbert series may be different. Therefore, in our search procedure, we compute and compare H_{00} for every possible node acting as node 0.

Now let us give an example of K-stable cDV singularity which is known to have CRs, along with the field theory dual.

Morrison-Pinkham example

Consider the two-node quiver Q : $N \begin{array}{c} \xrightarrow{\frac{1}{2}} \\ \xleftarrow{\frac{1}{2}} \end{array} 2N$. 

The R charges of bifundamental chiral fields c and d are $\frac{1}{2}$, while the two adjoint of the right node have $R_{b_1} = \frac{3}{4}$, $R_{b_2} = \frac{1}{2}$. Therefore, the matrix M_Q is

$$M_Q = \begin{pmatrix} t^{R_{a_1}} + t^{R_{a_2}} & t^{R_{c_1}} \\ t^{R_{d_1}} & t^{R_{b_1}} + t^{R_{b_2}} \end{pmatrix} = \begin{pmatrix} 0 & t^{\frac{1}{2}} \\ t^{\frac{1}{2}} & t^{\frac{1}{2} + \frac{3}{4}} \end{pmatrix}.$$

With the left node as the distinguished node, the quiver Hilbert series can be derived from M_Q using formula (4.6)

$$H_{00} = \frac{1 - t^{\frac{18}{4}}}{\left(1 - t^{\frac{9}{4}}\right) \left(1 - t^{\frac{6}{4}}\right) \left(1 - t^{\frac{4}{4}}\right) \left(1 - t^{\frac{7}{4}}\right)}.$$

This indicates that the corresponding geometry is generated by four fields of weights

$$(x, y, z, w) = \left(\frac{9}{4}, \frac{6}{4}, \frac{4}{4}, \frac{7}{4}\right),$$

and that the degree $\frac{18}{4}$ relation is

$$x^2 + y^3 + yz^3 + w^2z = 0.$$

This is precisely the cD_4 Morrison-Pinkham example, known to admit an NCCR [12].

4.2 Implementation of the search

We now outline our algorithm for identifying candidate quiver theories whose Hilbert series $H_{00}(t)$ matches that of each singularity in Problem 2.1. Although a large number of quivers might a priori satisfy the Hilbert series condition, scale invariance, unitarity, and the allowed quiver topologies (determined by $f(X) + 1$ nodes) reduce the search space dramatically. For simplicity, we impose three additional constraints:

1. All R charges of the quiver are in the range $[0, 2]$.¹⁴
2. Each pair of bifundamental chiral fields is assigned the same R charge due to symmetry.
3. If the common denominator R charges of the variable x, y, z, w is m , then we restrict all R charges of quiver to lie in the discrete set

$$\left\{0, \frac{1}{m}, \frac{2}{m}, \dots, \frac{2m}{m}\right\},$$

since x, y, z, w are composites of the fundamental chiral fields.

¹⁴There are two reasons of the simplification: 1) outside this range, there are some terms of the Hilbert series has negative powers of t . Therefore, the Hilbert series is hard to coincide those from the singularity side. 2) the quiver with R charges outside this range is much more difficult to find a sensible superpotential. [54].

With these constraints, we can exhaustively search all possible quivers with one, two or three nodes. As a consistency check, our program reproduces Morrison-Pinckham example described above.¹⁵ We then verify that no further $N = 1$ superconformal quiver exists, with the prescribed number of nodes, for the remaining cases:

$$\begin{array}{l}
 J = E_6 : \quad x^2 + y^3 + z^4 + yt^k : \begin{cases} k \in \{2, 3, 5, 6, 7, 9, 10, 11, 13, 14, 15\} \text{ (2-node quivers)} \\ k \in \{4, 12\} \text{ (3-node quivers)} \end{cases} \\
 J = E_7 : \quad x^2 + y^3 + yz^3 + zt^k : k \in \{4, 6, 8, 10, 12, 16, 18, 20, 22, 24, 26\} \text{ (2-node quivers)}
 \end{array}$$

In each instance, no admissible quiver with a correct Hilbert series is found, confirming the absence of a superconformal dual. This absence verifies our main claim.

5 Conclusions and future directions

In this paper we established our main claim through the novel mirror-symmetry based approach. It gives a new way to understand the existence of 4d $\mathcal{N} = 1$ superconformal quiver gauge theory duals of all singularities in table 1, with $J = E_6, E_7$ and E_8 . Most of these singularities fall into the cE_n types, with the notable exception of the classical cD_4 Morrison-Pinkham example. We verify this implication from the physics side, by searching for all possible field theory candidates. Although the search is subject to additional constraints, it provides compelling evidence for the claim. Finally, we conclude with a list of open questions which worth further investigation.

More general singularities. One may consider more general classes of singularities. The most immediate extensions are canonical singularities, and beyond them, Kawamata log-terminal (klt) singularities. Unlike cDV singularities, divisors may appear in a resolution of a canonical singularity. From a physics viewpoint one might imagine engineering gauge theories by wrapping D7-branes on such resolved geometries. However, to the best of our knowledge, no systematic study in the literature carries out this generalization. In particular, the relation between NCCRs and crepant resolutions is no longer transparent, and it is unclear whether the quivers with potentials produced from NCCRs should still be interpreted as the genuine quiver gauge theories and their superpotentials. For klt singularities, the situation is even less understood.

Within this broader framework, it is also natural to include non-isolated singularities, whose links are in general not smooth. Examples of this type already appear in [13].¹⁶ On the algebraic side, one may still construct noncommutative crepant resolutions via matrix factorizations. However, the non-isolated nature of singular locus means that any crepant resolution must be contract divisors, and the connection between such divisorial resolutions and the associated NCCR quivers is once again unclear. If the NCCRs do correspond to physical theories, the resulting quiver gauge models may involve chiral multiplets with irrational R charges (and hence irrational scaling dimensions).

¹⁵One can see the attached `Mathematica` notebook `Find dual.nb`.

¹⁶The non-isolated singularities considered in this paper admit at least two \mathbb{C}^* actions, which in particular allows for irrational R charges.



Figure 3. Two quiver gauge theories in which every gauge node is an unitary gauge group. Both theories have $TrR = 0$. TrR^3 are all $\frac{2560N^2}{81}$. Furthermore, we have checked that their superconformal indices coincide in the large N limit.

It is also natural to investigate quotient singularities. Most cyclic quotients have been analyzed in the literature. As shown in [55], except for product of cyclic quotients, the singularities are typically non-isolated. In particular, quotients of \mathbb{C}^3 by finite subgroups of $SO(3)$, as studied in [27, 52], are generically non-isolated. Nevertheless, the proposed dual field-theory candidates in these works remain highly compelling.

Another direction is to search for new dual pairs of cDV singularities beyond the list of Wang and Xie. Once the Gorenstein condition is relaxed, three-fold terminal singularities are not necessarily hypersurfaces, and may instead arise as complete intersections or through other constructions. Exploring such cases may lead to previously unknown gauge/gravity duals.

Dualities between superconformal quiver gauge theories. In carrying out the search described in section 4, we discovered that the same algorithm also identifies candidate dualities among superconformal quiver gauge theories. Specifically, different quiver realizations sometimes produce the same Hilbert series. To further test whether these theories are truly dual, one can compare their large N superconformal indices (or single-trace indices) [56] and central charges. However, because matching these invariants is only a necessary — not a sufficient — condition for duality, each pair remains a provisional candidate that merits additional checks. Unfortunately, these theories are not holographic dual to any singularity considered in this paper.

In fact, additional duality candidates can be uncovered by first matching quiver Hilbert series and then verifying other physical data — such as central charges and large N superconformal indices. Note that the quiver Hilbert series need not be tied to the particular singularities studied here, this approach therefore provides a general algorithm for discovering new duality candidates.

Acknowledgments

We thank Dan Xie for providing the foundational insights and background research that inspired this work. ZY would like to thank Fulin Xu for various informative and inspiring conversations on birational geometry. We would also like to thank Michael Wemyss and Ban Lin for discussions. Finally, we express our gratitude to Robert Valandro for correspondences.

Data Availability Statement. This article has no associated data or the data will not be deposited.

Code Availability Statement. This article has code included as electronic supplementary material.

Supplementary material and data. The following files are provided as supplementary material/data and are accessible through the article's webpage.

- `Find_dual.nb` (Notebook for numerically implementing the search). Description: In our section 4 we enumerate all possible candidates for the physical quiver of the scale invariant $\mathcal{N} = 1$ gauge theories, under certain assumptions. As in footnote 15 it is implemented by computer through this mathematica notebook.

Open Access. This article is distributed under the terms of the Creative Commons Attribution License ([CC-BY4.0](https://creativecommons.org/licenses/by/4.0/)), which permits any use, distribution and reproduction in any medium, provided the original author(s) and source are credited.

References

- [1] Y. Wang and D. Xie, *Classification of Argyres-Douglas theories from M5 branes*, *Phys. Rev. D* **94** (2016) 065012 [[arXiv:1509.00847](https://arxiv.org/abs/1509.00847)] [[INSPIRE](#)].
- [2] D. Xie and S.-T. Yau, *4d $N = 2$ SCFT and singularity theory Part I: Classification*, [arXiv:1510.01324](https://arxiv.org/abs/1510.01324) [[INSPIRE](#)].
- [3] K.A. Intriligator, D.R. Morrison and N. Seiberg, *Five-dimensional supersymmetric gauge theories and degenerations of Calabi-Yau spaces*, *Nucl. Phys. B* **497** (1997) 56 [[hep-th/9702198](https://arxiv.org/abs/hep-th/9702198)] [[INSPIRE](#)].
- [4] F. Apruzzi et al., *5d Superconformal Field Theories and Graphs*, *Phys. Lett. B* **800** (2020) 135077 [[arXiv:1906.11820](https://arxiv.org/abs/1906.11820)] [[INSPIRE](#)].
- [5] C. Closset, S. Schäfer-Nameki and Y.-N. Wang, *Coulomb and Higgs branches from canonical singularities. Part I. Hypersurfaces with smooth Calabi-Yau resolutions*, *JHEP* **04** (2022) 061 [[arXiv:2111.13564](https://arxiv.org/abs/2111.13564)] [[INSPIRE](#)].
- [6] M. De Marco, A. Sangiovanni and R. Valandro, *5d Higgs branches from M-theory on quasi-homogeneous cDV threefold singularities*, *JHEP* **10** (2022) 124 [[arXiv:2205.01125](https://arxiv.org/abs/2205.01125)] [[INSPIRE](#)].
- [7] J.M. Maldacena, *The large N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [[hep-th/9711200](https://arxiv.org/abs/hep-th/9711200)] [[INSPIRE](#)].
- [8] I.R. Klebanov and E. Witten, *Superconformal field theory on three-branes at a Calabi-Yau singularity*, *Nucl. Phys. B* **536** (1998) 199 [[hep-th/9807080](https://arxiv.org/abs/hep-th/9807080)] [[INSPIRE](#)].
- [9] R. Eager, J. Schmude and Y. Tachikawa, *Superconformal Indices, Sasaki-Einstein Manifolds, and Cyclic Homologies*, *Adv. Theor. Math. Phys.* **18** (2014) 129 [[arXiv:1207.0573](https://arxiv.org/abs/1207.0573)] [[INSPIRE](#)].
- [10] D. Xie and S.-T. Yau, *Singularity, Sasaki-Einstein manifold, Log del Pezzo surface and $\mathcal{N} = 1$ AdS/CFT correspondence: Part I*, [arXiv:1903.00150](https://arxiv.org/abs/1903.00150) [[INSPIRE](#)].
- [11] T.C. Collins, D. Xie and S.-T. Yau, *K stability and stability of chiral ring*, [arXiv:1606.09260](https://arxiv.org/abs/1606.09260) [[INSPIRE](#)].
- [12] P.S. Aspinwall and D.R. Morrison, *Quivers from Matrix Factorizations*, *Commun. Math. Phys.* **313** (2012) 607 [[arXiv:1005.1042](https://arxiv.org/abs/1005.1042)] [[INSPIRE](#)].

- [13] M. Fazzi and A. Tomasiello, *Holography, Matrix Factorizations and K-stability*, *JHEP* **05** (2020) 119 [[arXiv:1906.08272](#)] [[INSPIRE](#)].
- [14] M. van den Bergh, *Non-commutative Crepant Resolutions*, in *The Legacy of Niels Henrik Abel*, O.A. Laudal, R. Piene eds., Springer Berlin Heidelberg (2004), p. 749–770 [[DOI:10.1007/978-3-642-18908-1_26](#)].
- [15] S. Franco et al., *Brane dimers and quiver gauge theories*, *JHEP* **01** (2006) 096 [[hep-th/0504110](#)] [[INSPIRE](#)].
- [16] S. Franco et al., *Gauge theories from toric geometry and brane tilings*, *JHEP* **01** (2006) 128 [[hep-th/0505211](#)] [[INSPIRE](#)].
- [17] S. Benvenuti et al., *An infinite family of superconformal quiver gauge theories with Sasaki-Einstein duals*, *JHEP* **06** (2005) 064 [[hep-th/0411264](#)] [[INSPIRE](#)].
- [18] D. Berenstein and M.R. Douglas, *Seiberg duality for quiver gauge theories*, [hep-th/0207027](#) [[INSPIRE](#)].
- [19] A. Collinucci, M. De Marco, A. Sangiovanni and R. Valandro, *Higgs branches of 5d rank-zero theories from geometry*, *JHEP* **10** (2021) 018 [[arXiv:2105.12177](#)] [[INSPIRE](#)].
- [20] A. Collinucci, A. Sangiovanni and R. Valandro, *Genus zero Gopakumar-Vafa invariants from open strings*, *JHEP* **09** (2021) 059 [[arXiv:2104.14493](#)] [[INSPIRE](#)].
- [21] A. Collinucci, M. De Marco, A. Sangiovanni and R. Valandro, *Flops of any length, Gopakumar-Vafa invariants and 5d Higgs branches*, *JHEP* **08** (2022) 292 [[arXiv:2204.10366](#)] [[INSPIRE](#)].
- [22] M. De Marco and A. Sangiovanni, *Higgs Branches of rank-0 5d theories from M-theory on (A_j, A_l) and (A_k, D_n) singularities*, *JHEP* **03** (2022) 099 [[arXiv:2111.05875](#)] [[INSPIRE](#)].
- [23] E. Brieskorn, *Über die auflösung gewisser singularitäten von holomorphen abbildungen*, *Math. Ann.* **166** (1966) 76.
- [24] E. Brieskorn, *Die auflösung der rationalen singularitäten holomorpher abbildungen*, *Math. Ann.* **178** (1968) 255.
- [25] J. Karmazyn, *The length classification of threefold flops via noncommutative algebras*, [arXiv:1709.02720](#).
- [26] S. Gubser, N. Nekrasov and S. Shatashvili, *Generalized conifolds and 4-Dimensional $N = 1$ SuperConformal Field Theory*, *JHEP* **05** (1999) 003 [[hep-th/9811230](#)] [[INSPIRE](#)].
- [27] Y. Fang, J. Feng and D. Xie, *Three dimensional quotient singularity and 4d $\mathcal{N} = 1$ AdS/CFT correspondence*, [arXiv:2310.15792](#) [[INSPIRE](#)].
- [28] J.D. Evans and Y. Lekili, *Symplectic cohomology of compound Du Val singularities*, [arXiv:2104.11713](#).
- [29] A. Grassi and T. Weigand, *On topological invariants of algebraic threefolds with $(\mathbb{Q}$ -factorial) singularities*, [arXiv:1804.02424](#) [[INSPIRE](#)].
- [30] C. Peters, *On isolated hypersurface singularities: algebra-geometric and symplectic aspects*, [arXiv:2405.03475](#).
- [31] H. Flenner, *Divisorenklassengruppen quasihomogener singularitäten*, *J. Reine Angew. Math.* **328** (1981) 128.
- [32] M. Caibar, *Minimal models of canonical singularities and their cohomology*, Ph.D. thesis, University of Warwick, Coventry, U.K (1999).

- [33] S. Giacomelli, *RG flows with supersymmetry enhancement and geometric engineering*, *JHEP* **06** (2018) 156 [[arXiv:1710.06469](#)] [[INSPIRE](#)].
- [34] Y. Lekili and K. Ueda, *Homological mirror symmetry for Milnor fibers of simple singularities*, [arXiv:2004.07374](#).
- [35] N. Adaloglou, F. Pasquotto and A. Zanardini, *Symplectic cohomology of quasihomogeneous cA_n singularities*, [arXiv:2404.17301](#).
- [36] P. Seidel, *A biased view of symplectic cohomology*, [arXiv:0704.2055](#).
- [37] P. Seidel, *Disjoinable Lagrangian spheres and dilations*, *Invent. Math.* **197** (2013) 299.
- [38] D. Orlov, *Triangulated categories of singularities and D-branes in Landau-Ginzburg models*, *Trudy Steklov Mat. Inst.* **246** (2004) 240 [[math/0302304](#)] [[INSPIRE](#)].
- [39] E. Witten, *Phases of $N = 2$ theories in two-dimensions*, *Nucl. Phys. B* **403** (1993) 159 [[hep-th/9301042](#)] [[INSPIRE](#)].
- [40] C. Vafa and N.P. Warner, *Catastrophes and the Classification of Conformal Theories*, *Phys. Lett. B* **218** (1989) 51 [[INSPIRE](#)].
- [41] P. Berglund and T. Hübsch, *A generalized construction of mirror manifolds*, *Nucl. Phys. B* **393** (1993) 377 [[hep-th/9201014](#)] [[INSPIRE](#)].
- [42] M. Krawitz, *FJRW rings and Landau-Ginzburg Mirror Symmetry*, [arXiv:0906.0796](#).
- [43] B.R. Greene and M.R. Plesser, *Duality in Calabi-Yau Moduli Space*, *Nucl. Phys. B* **338** (1990) 15 [[INSPIRE](#)].
- [44] S. Gukov, C. Vafa and E. Witten, *CFT's from Calabi-Yau four folds*, *Nucl. Phys. B* **584** (2000) 69 [[hep-th/9906070](#)] [[INSPIRE](#)].
- [45] R. Eager, *Equivalence of A-Maximization and Volume Minimization*, *JHEP* **01** (2014) 089 [[arXiv:1011.1809](#)] [[INSPIRE](#)].
- [46] V.A. Novikov, M.A. Shifman, A.I. Vainshtein and V.I. Zakharov, *Exact Gell-Mann-Low Function of Supersymmetric Yang-Mills Theories from Instanton Calculus*, *Nucl. Phys. B* **229** (1983) 381 [[INSPIRE](#)].
- [47] D. Anselmi, D.Z. Freedman, M.T. Grisaru and A.A. Johansen, *Nonperturbative formulas for central functions of supersymmetric gauge theories*, *Nucl. Phys. B* **526** (1998) 543 [[hep-th/9708042](#)] [[INSPIRE](#)].
- [48] D. Martelli, J. Sparks and S.-T. Yau, *The geometric dual of a-maximisation for Toric Sasaki-Einstein manifolds*, *Commun. Math. Phys.* **268** (2006) 39 [[hep-th/0503183](#)] [[INSPIRE](#)].
- [49] T.C. Collins and G. Székelyhidi, *Sasaki-Einstein metrics and K-stability*, *Geom. Topol.* **23** (2019) 1339 [[arXiv:1512.07213](#)] [[INSPIRE](#)].
- [50] D.R. Morrison, *The birational geometry of surfaces with rational double points*, *Math. Ann.* **271** (1985) 415.
- [51] J. August, *The Tilting Theory of Contraction Algebras*, [arXiv:1802.10366](#).
- [52] A.N. de Celis and Y. Sekiya, *Flops and mutations for crepant resolutions of polyhedral singularities*, [arXiv:1108.2352](#).
- [53] R. Bocklandt, *Graded Calabi Yau Algebras of dimension 3*, *J. Pure Appl. Algebra* **212** (2008) 14 [[math/0603558](#)].

- [54] B. Bajc, *Kutasov-Seiberg dualities and cyclotomic polynomials*, *JHEP* **06** (2019) 083 [[arXiv:1901.02846](#)] [[INSPIRE](#)].
- [55] B. Chen et al., *4d $\mathcal{N} = 2$ SCFT and singularity theory Part III: Rigid singularity*, *Adv. Theor. Math. Phys.* **22** (2018) 1885 [[arXiv:1712.00464](#)] [[INSPIRE](#)].
- [56] A. Gadde, L. Rastelli, S.S. Razamat and W. Yan, *On the Superconformal Index of $N = 1$ IR Fixed Points: A Holographic Check*, *JHEP* **03** (2011) 041 [[arXiv:1011.5278](#)] [[INSPIRE](#)].