

Trajectory-based conservation laws for massive spin-zero relativistic quantum particles in $1+1$ spacetime

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Abstract. We present novel aspects of a trajectory-based theory of massive spin-zero relativistic quantum particles. In this approach, the quantum trajectory ensemble is the fundamental entity. It satisfies its own action principle, leading to a dynamical partial differential equation (via the Euler-Lagrange procedure), as well as to conservation laws (via Noether's theorem). In this paper, we focus on the derivation of the latter. In addition to the usual expected energy and momentum conservation laws, there is also a third law that emerges, associated with the conditions needed to maintain global simultaneity. We also show that the nonrelativistic limits of these conservation laws match those of the earlier, nonrelativistic quantum trajectory theory [J. Chem. Phys. **136**, 031102 (2012)].

1. Introduction

Wave-based quantum mechanics, and its relativistic generalizations in the form of quantum field theories, have been more or less fully developed (modulo renormalization) since the 1930s [1, 2, 3]. Moreover, perturbation calculations based on those theories (such as quantum electrodynamics) have achieved tremendous success, in terms of numerical agreement with experiment. And yet for all that, much disagreement still remains—e.g., as to the true significance of the wavefunction (ontic vs. epistemic), and whether or not it completely represents a quantum state. For example, there is the issue of whether or not the wavefunction need be supplemented with “hidden variables,” and also, exactly what happens when the wavefunction “collapses.” Perhaps not surprisingly, the various interpretations of quantum mechanics that have been developed over the years [4, 5, 6, 7, 8, 9, 10, 11] provide different answers to these questions.

On the other hand, what if there were no need of a wavefunction at all in a quantum theory? In recent years, attempts have been made to formulate a complete standalone theory of quantum mechanics without wavefunctions [12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23]. In particular, in 2010, one of the authors (B. Poirier) proposed a theoretical framework in which a quantum state is represented *solely* by an ensemble of real-valued probabilistic trajectories [14, 17]. With the notable exception of spin, this nonrelativistic version of the trajectory-based theory turns out to be formally mathematically equivalent to the standard wave-based Schrödinger equation [12, 13, 14, 17]—though it can be derived completely independently [14, 17]. More recently, a *discrete* version of the trajectory-based theory has also been proposed [20, 21, 22, 23], which is *not* consistent with the Schrödinger equation except in the continuous limit. In either form,



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i.e. discrete or continuous, the trajectory-based theory also suggests a new *interpretation* of quantum mechanics, as was also first proposed in 2010 [14]. This has come to be known as the “many interacting worlds” interpretation, in contradistinction to the many worlds interpretation of Everett [10, 11].

It should be emphasized that in the trajectory-based formulation (at least, the original continuous version as developed by the authors), the relevant dynamical partial differential equations (PDEs) emerge from an action extremization principle, applied to a trajectory-based Lagrangian density (reminiscent of classical field theory). The resultant solutions are therefore ensembles of true *trajectories*, as opposed to paths. In particular, there is no *integration* over paths, only an Euler-Lagrange *extremization*, as in classical (field) theory. As a consequence, it becomes a very straightforward matter to apply Noether’s theorem, to derive proper conservation laws, based on the continuous symmetries. It should be noted that this provides a huge advantage over say, Bohmian mechanics [5, 6], for which the quantum potential and force are treated as external fields, arising from the time- and space-dependent wavefunction. As a result, e.g., there are no satisfactory momentum conservation laws for Bohmian trajectories—at least none that emerge from the Bohmian theory itself, as has been lamented by at least one author [6].

In reality, the quantum trajectories of the trajectory-based theory are *identical* with Bohmian trajectories—meaning that both theories somehow manage to employ the same quantum potential and forces. The difference is that in the trajectory-based approach, these quantities are formulated solely from the trajectory ensemble itself, in a manner that exhibits both time and space translation invariance. In a sense, these symmetries were thus “there all along” in the Bohm theory, it is just that they are obscured by the theory’s reliance on a wavefunction. In the trajectory-based approach, in contrast, these symmetries are naturally revealed in the translation invariances of the trajectory-based Lagrangian density expression (Sec. 2.2).

With the success of the trajectory-based quantum theory in the nonrelativistic regime—and particularly given its wholly *trajectory-based* character—it is promising to generalize this theory for *relativistic* quantum particles. In spite of the success of quantum electrodynamics in terms of its agreement with experimental results, there is currently no satisfactory relativistic quantum theory for describing *individual* (or a fixed number of) spin-zero particles with a well-defined probabilistic interpretation. For instance, a promising candidate, the Klein-Gordon theory, does not achieve a probabilistic interpretation for a single quantum particle, due to the fact that the current density j^0 is not always positive semidefinite and that the four-current j^α can either be timelike or spacelike [6, 24, 25]. Although for spin-1/2 particles at least, these maladies are repaired by the Dirac equation, even the Dirac theory gives rise to unphysical negative energy states, when interpreted as a single-particle theory. Insofar as fixed, multi-particle Dirac equations are concerned, these do exist in practice, but all of them are *ad hoc* beyond two particles—a fact which, e.g., has thus far hampered the field of relativistic quantum chemistry [26].

These deficiencies suggest the value of exploring a trajectory-based approach for a single or fixed number of relativistic quantum particles, at least in contexts where QED pair creation/annihilation effects are of lesser importance. To this end, our group has developed a relativistic generalization of the trajectory-based quantum theory [27, 28]. Specifically, we have derived relativistic quantum trajectory dynamical PDEs, which reduce to *both* the nonrelativistic trajectory PDEs, *and* the classical relativistic dynamical equations, in the appropriate limits. To our eyes, a remarkable feature of the relativistic quantum trajectory PDEs is that they *are not* equivalent to Klein-Gordon (except in the above limits of course). The reason is that they are essentially nonlinear—in fact, they can be recast in wave form as the Klein-Gordon equation *plus* a nonlinear correction. What is more, the relativistic trajectory-based theory gives rise to a well-defined timelike probability current j^μ , with a temporal component j^0 that is positive-semidefinite over all spacetime (all trajectories are subluminal). Moreover, there are

no negative energy quantum trajectory ensemble solutions. Finally, the relativistic quantum trajectory theory gives rise to *global simultaneity submanifolds*—restoring the appealing, special relativistic notion of global simultaneity, that would ordinarily have to be discarded as soon as particle acceleration is introduced.

The above properties certainly distinguish the trajectory-based theory from the wave-based Klein-Gordon theory for massive spin-zero relativistic quantum particles. The trajectory approach may thus find some meaningful predictive value, again in contexts where QED pair creation/annihilation effects are of less importance (or can be simulated as external potentials). Although the dynamical PDEs for the trajectory-based relativistic quantum theory have already been derived [27], and even implemented numerically [28], it remains to work out the associated continuous symmetries and conservation laws. Such is the purpose of this paper, where for the moment we restrict ourselves to a $1+1$ flat Minkowski spacetime. In future work, we will consider the full $3+1$ case, for which angular momentum conservation laws emerge (and for non-spin-zero particles, intrinsic angular momentum)—as does a non-trivial constraint on the trajectory velocity field that turns out to be associated with global simultaneity (Sec. 4.4).

From a mathematical standpoint, even the restricted $1+1$ case considered here is of interest, for two reasons. First, unlike the usual wave-based quantum treatment, for which the spatial coordinate x is an *independent* variable, here, x is a *dependent* variable—indeed, it is no less than the very trajectory ensemble that we are solving for. The corresponding spacelike independent coordinate is C , which labels a particular trajectory within the ensemble [i.e., the ensemble itself can be denoted $x(C)$]. Note that with respect to continuous symmetries, one can examine the Lagrangian density both with respect to the *extrinsic* spatial coordinate x , and also independently with respect to the *intrinsic* spatial coordinate C . Second, being a relativistic theory, there are actually *two* dependent variables, (t, x) —i.e., the extrinsic spacetime coordinates—that depend on the two intrinsic independent variables, (λ, C) , where the “ensemble time” parameter λ describes the evolution of trajectories within the ensemble. All of this provides a rather rich structure, insofar as the continuous symmetries are concerned.

2. Formulation of the trajectory-based $3+1$ relativistic quantum theory

2.1. Preliminaries

In the trajectory-based relativistic quantum theory, the spacetime of a single relativistic spin-zero particle is represented by a 4d $3+1$ Riemannian manifold, which is presumed to be flat [27]. Because there is exactly one quantum trajectory passing through each point in spacetime, the ensemble of quantum trajectories can be denoted by inertial coordinates $x^\alpha = (ct, \mathbf{x})$, which depend on the intrinsic coordinates $X^\mu = (c\lambda, \mathbf{C})$. Thus, \mathbf{C} denotes a specific trajectory from the ensemble, and for a fixed choice of \mathbf{C} , the resultant $x^\alpha(\lambda)$ denotes the trajectory path through spacetime as a function of the “ensemble time” evolution parameter λ . The intrinsic or “natural” coordinates are defined in such a way that the contours of $\lambda(x^\alpha)$ are everywhere skew-orthogonal to the trajectories, but are otherwise arbitrary. In particular, the theory admits arbitrary and independent reparametrizations of the intrinsic time ($\lambda \rightarrow \lambda'$) and space ($\mathbf{C} \rightarrow \mathbf{C}'$) coordinates.

Due to the skew-orthogonality property described above, each contour of $\lambda(x^\alpha)$ can be interpreted as a global *simultaneity submanifold* [27]. In general, these will be curved, owing to the accelerated motion of some or all of the relativistic quantum trajectories that make up the ensemble. Even for free particles this is the case, owing to the quantum forces that arise from variations in particle density across spacetime. (The same is also true in the nonrelativistic case). Note that in general, it is not possible to take λ to be the proper time—at least not across the entire trajectory ensemble. Only when all trajectories are parallel and unaccelerated is this the case—a frame-defining situation that we refer to as *quantum inertial motion*.

In all versions of the trajectory-based quantum theory, all quantum effects arise from

the mutual interactions of nearby trajectories or “worlds.” The quantum potential Q , and (covariant) quantum force f_α^Q , thus arise as forms that involve the “spatial” derivatives (i.e., derivatives with respect to the \mathbf{C} variables) of the trajectory ensemble solution, $x^\alpha(X^\mu)$. A key feature is that these expressions do *not* involve any X^0 , or λ , derivatives, effectively restricting their evaluation to the simultaneity submanifolds. This is one of two postulates used to formulate the trajectory-based theory [27].

Throughout this paper, the Greek indices $\alpha, \beta, \gamma, \dots$ run over the spacetime inertial coordinate labels, i.e. $0, 1, \dots$, whereas μ, ν, \dots serve a similar function for the curvilinear natural coordinate system X^μ . Latin indices run over spatial (or spacelike) coordinate labels $1, \dots$, as per usual, with l, m, \dots used for inertial coordinates, and i, j, \dots for natural coordinates. We use the Einstein summation convention for repeated indices. When an index is repeated three or more times, the summation symbol is restored for clarity.

In inertial coordinates, the metric tensor $\eta^{\alpha\beta}$ is the usual Minkowski one, with $\tilde{\eta} = \text{diag}(-1, 1, 1, 1)$. In natural coordinates, the metric tensor is denoted $g^{\mu\nu}$. These two metric tensors are related by the transformation [27]:

$$g_{\mu\nu} = \eta_{\alpha\beta} \frac{\partial x^\alpha}{\partial X^\mu} \frac{\partial x^\beta}{\partial X^\nu}. \quad (1)$$

Note that the skew-orthogonality condition results in a $g_{\mu\nu}$ that is *block-diagonal* in its matrix form, *i.e.*,

$$\tilde{g} = \begin{pmatrix} g_{00} & 0 \\ 0 & \tilde{\gamma} \end{pmatrix}, \quad (2)$$

where the “spatial metric” $\tilde{\gamma}$ is the 3×3 spatial block of the full metric tensor \tilde{g} . We define the determinants of \tilde{g} and $\tilde{\gamma}$ to be:

$$g = \det \tilde{g} = g_{00}\gamma, \quad \gamma = \det \tilde{\gamma}. \quad (3)$$

The proper time τ for the quantum particle is defined via

$$d\tau^2 = -\frac{1}{c^2} \eta_{\alpha\beta} dx^\alpha dx^\beta. \quad (4)$$

Due to Eqs. (1), (2) and (4), we have

$$g_{00} = -\left(\frac{d\tau}{d\lambda}\right)^2. \quad (5)$$

2.2. Dynamical PDEs governing time evolution

We obtain the dynamics of the relativistic quantum particle by extremizing the action, following a generalized Euler-Lagrange procedure. For a classical free particle, the expression for the relativistic Lagrangian is well-described in the literature [29, 30, 31, 32]. For a relativistic quantum particle, the corresponding trajectory-based Lagrangian density is constructed in Ref. [27]. The action is expressed as

$$\begin{aligned} S &= \int d^4X \mathcal{L} \\ &= \int d^4X \frac{\rho(X^\mu)}{c} L, \end{aligned} \quad (6)$$

where \mathcal{L} is the Lagrangian density and L is a scalar invariant quantity referred to as the “Lagrangian.”

In Ref. [27], $\rho(X^\mu)$ is defined as the 4d scalar probability density of weight $W = -1$. This can be expressed as:

$$\rho(X^\mu) = \frac{d\tau}{d\lambda} f(\mathbf{C}), \quad (7)$$

where $f(\mathbf{C})$ denotes the 3d-probability density on \mathbf{C} -space. Note that $f(\mathbf{C})$ is independent of λ . This follows from the *second* postulate of the theory, i.e. that probability is conserved along individual trajectories [27]. In this paper, we find it convenient to work with the “reparametrized Lagrangian” $L^{(\lambda)}$, defined as

$$L^{(\lambda)} = \frac{d\tau}{d\lambda} L. \quad (8)$$

The net effect is to replace $\rho(X^\mu)$ with $f(\mathbf{C})$, which is more convenient in practice. Thus,

$$\mathcal{L} = \frac{1}{c} \frac{d\tau}{d\lambda} f(\mathbf{C}) L = \frac{1}{c} f(\mathbf{C}) L^{(\lambda)}. \quad (9)$$

We find it convenient to decompose the (reparametrized) Lagrangian (density) into three parts:

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I - \mathcal{L}_Q, \quad (10)$$

$$L = L_0 + L_I - L_Q, \quad (11)$$

$$L^{(\lambda)} = L_0^{(\lambda)} + L_I^{(\lambda)} - L_Q^{(\lambda)} \quad (12)$$

The 0, I , and Q terms refer to (classical) kinetic, (classical) external field, and quantum contributions, respectively. The three parts of $L^{(\lambda)}$ in Eq. (12) are defined as follows [27]:

$$L_0^{(\lambda)} = -mc^2 \sqrt{-\frac{1}{c^2} \eta_{\alpha\beta} \frac{\partial x^\alpha}{\partial \lambda} \frac{\partial x^\beta}{\partial \lambda}}, \quad (13)$$

$$L_I^{(\lambda)} = \frac{q}{c} \frac{\partial x^\alpha}{\partial \lambda} A_\alpha(x), \quad (14)$$

$$L_Q^{(\lambda)} = \frac{\hbar^2}{8m} \frac{d\tau}{d\lambda} \left(\frac{f(\mathbf{C})}{\gamma^{1/2}} \right)^{-2} \gamma^{ij} \frac{\partial}{\partial C^i} \left(\frac{f(\mathbf{C})}{\gamma^{1/2}} \right) \frac{\partial}{\partial C^j} \left(\frac{f(\mathbf{C})}{\gamma^{1/2}} \right). \quad (15)$$

We note that the external field $A^\alpha(x)$ in Eq. (14) is a function of x^α , in general.

Note that different forms are possible for the quantum contribution to the (reparametrized) Lagrangian, $L_Q^{(\lambda)}$, depending on the choice of gauge. In Eq. (15) above, we have chosen the “Holland gauge” (which is also utilized in Ref. [13, 27]), because it is simplest. On the other hand, the “Poirier gauge”, for which $L_Q^{(\lambda)} = Q$, is arguably more natural, because the resultant action is essentially equivalent to the phase of the wavefunction (at least in the nonrelativistic case).

Although the expression for $L_Q^{(\lambda)}$ depends on the choice of gauge, the resultant dynamical PDEs and Noether-theorem-derived conservation laws are independent of the gauge. In particular, all trajectories in the ensemble are found to evolve in accord with the classical relativistic force provided by the external vector potential $A^\alpha(x)$, together with a quantum force contribution, obtained from the relativistic quantum potential Q as follows:

$$Q = -\frac{\hbar^2}{2m} \gamma^{-1/4} f^{-1/2} \frac{\partial}{\partial C^i} \left[\gamma^{1/2} \gamma^{ij} \frac{\partial}{\partial C^j} f^{1/2} \gamma^{-1/4} \right], \quad (16)$$

$$f_Q^\alpha = -\frac{\partial x^\alpha}{\partial C^i} \gamma^{ij} \frac{\partial Q}{\partial C^j} \quad (17)$$

In Eq. (17) above, f_Q^α is the *contravariant* form of the covariant quantum force introduced in Sec. 2.1—i.e., $f_Q^\alpha = \eta^{\alpha\beta} f_\beta^Q$. Note that f_α^Q is *almost* equal to $-\partial Q/\partial x^\alpha$, except that the λ component of the four-gradient must be projected out due to the first postulate—thus effectively replacing $g^{\mu\nu}$ with γ^{ij} in Eq. (17) [27].

In addition to the choice of gauge, one also has a choice of natural coordinate reparametrization, as discussed above. The above forms are generic, in the sense that they apply for any choice of natural coordinates. Throughout most of this paper, we leave the choice of the natural timelike coordinate λ generic, although in Ref. [27], specific choices are explored. Regarding the natural spacelike coordinates, here, a very convenient choice—and one which is always available—is to define \mathbf{C} such that the density $f(\mathbf{C}) = 1$ everywhere. We call this a “uniformizing” choice for \mathbf{C} [17, 27], and it has the great advantage that $f(\mathbf{C})$ vanishes from the Lagrangian forms—which, in addition, further simplify, because $\partial f(\mathbf{C})/\partial C^i = 0$. We shall assume the use of uniformizing \mathbf{C} coordinates henceforth throughout this work, although explicit coordinate forms will not appear again until Sec. 4.

3. Action extremization, dynamical PDEs, and Noether’s theorem

3.1. Introduction

One of the primary advantages of the quantum theoretical framework employed here—as compared with both traditional wave-based theories, as well as the discrete trajectory-based theory [20]—is that it derives from a standalone trajectory-based Lagrangian. As such, it is a wholly straightforward matter to examine the form of the Lagrangian in order to determine its invariant symmetries. The corresponding conservation laws can then be obtained using the Noether procedure [32, 33, 34, 35, 36, 37, 38]. Additionally, since the formalism is inherently trajectory-based, the relativistic quantum, nonrelativistic quantum, and relativistic classical cases are all treated on an equal footing. It therefore becomes equally straightforward to consider the nonrelativistic and/or classical limits, and to verify agreement with previously established forms for these limiting cases.

From examination of Eqs. (13) to (15), it is clear that the *only* dependence of the (reparametrized) Lagrangian on the inertial coordinates x^α enters through the external vector potential contribution, $L_I^{(\lambda)}$. This is to be expected, and is exactly the same dependence as in the classical relativistic case. Thus for *free* particles (e.g., $A_\alpha(x) = 0$), the trajectory-based relativistic quantum Langrangian is independent of both inertial space and time variables, $x^\alpha = (ct, \mathbf{x})$. Consequently, by Noether’s theorem, there exist associated *inertial* momentum and energy conservation laws. This will be taken up again in Sec. 4.2.

On the other hand, as discussed in Sec. 1, the trajectory-based theory also has the *independent* or natural coordinates to contend with, i.e. $X^\mu = (c\lambda, \mathbf{C})$. By inspection, none of Eqs. (13) to (15) show an explicit dependence on the natural time-like coordinate λ , no matter how it is parametrized—a feature we will exploit in Sec. 5. Here, of course, the significance is that Noether’s theorem then implies a *second* energy conservation law, associated with the *natural* time evolution. As for the explicit dependence on the natural space-like coordinates \mathbf{C} , we see that Eqs. (13) and (14) are completely devoid of any explicit or implicit reference to \mathbf{C} (i.e. even with respect to C^i derivatives). Since these are wholly classical contributions to the Lagrangian, this is exactly as it should be—i.e., parallel classical “worlds” do not interact in any sense.

As for the quantum contribution of Eq. (15), here, we see that \mathbf{C} enters in, both in terms of derivatives (explicitly observed, and also implicit in the definition of γ^{ij}), and also through the $f(\mathbf{C})$ dependence. On the other hand, the latter, explicit coordinate dependence can be made to disappear through the use of uniformizing coordinates, as discussed. This means that the present theory also admits a *natural* momentum or “ \mathbf{C} -conservation” law. In principle, then, we have a total of eight energy and momentum conservation laws in 3 + 1 spacetime, which reduce

to four such laws in the $1 + 1$ restriction, as considered explicitly in the next section. Because trajectory ensembles are employed, these take the form of continuity equations for *density* and associated *flux* quantities. The physical meaning of these equations will become clear in the next section.

In this section, we derive the natural coordinate Noether conservation laws for the trajectory-based relativistic quantum theory in full $3 + 1$ spacetime. Due to subtleties associated with the generality of the natural time-like coordinate λ , we find it necessary to utilize a general approach as described, e.g., in Ref. [35]. In particular, in the following subsection, we use Noether's theorem to derive the Euler-Lagrange equations and the Noether current equations.

3.2. Generic derivation of dynamical PDEs

Consider an infinitesimal transformation of the natural coordinates X^μ and the inertial coordinates x^α ,

$$X^\mu \rightarrow Y^\mu = X^\mu + \delta X^\mu, \quad (18)$$

$$x^\alpha(X^\sigma) \rightarrow y^\alpha(Y^\sigma) = x^\alpha(X^\sigma) + \delta x^\alpha(X^\sigma), \quad (19)$$

where $\delta x^\alpha(X^\sigma)$ is called the “total variation.” In addition, there is the transformation of x^α restricted to the point X^σ , i.e.

$$x^\alpha(X^\sigma) \rightarrow y^\alpha(X^\sigma) = x^\alpha(X^\sigma) + \Delta x^\alpha(X^\sigma), \quad (20)$$

where $\Delta x^\alpha(X^\sigma)$ is called the “symmetry variation.” In Appendix A, it is shown that $\delta x^\alpha(X^\sigma)$ and $\Delta x^\alpha(X^\sigma)$ are related via

$$\Delta x^\alpha(X^\sigma) = \delta x^\alpha(X^\sigma) - (\partial_\mu x^\alpha) \delta X^\mu, \quad (21)$$

to first order in δX^μ , where

$$\partial_\mu \equiv \frac{\partial}{\partial X^\mu}. \quad (22)$$

It is also proved in Appendix A that

$$\partial_\mu(\Delta x^\alpha) = \Delta(\partial_\mu x^\alpha), \quad (23)$$

and that

$$\partial_\mu(\delta x^\alpha) = \delta(\partial_\mu x^\alpha) + (\partial_\nu x^\alpha) \partial_\mu(\delta X^\nu), \quad (24)$$

to first order in δX^ν .

Under the transformation of Eqs. (18) and (19), the variation of the action is

$$\delta S = 0, \quad (25)$$

where

$$\begin{aligned} \delta S &= \int d^4 Y \mathcal{L} \left[y^\alpha(Y^\sigma), \frac{\partial}{\partial Y^\mu} y^\alpha(Y^\sigma), \frac{\partial^2}{\partial Y^\mu \partial Y^\nu} y^\alpha(Y^\sigma); Y^\sigma \right] \\ &\quad - \int d^4 X \mathcal{L} \left[x^\alpha(X^\sigma), \frac{\partial}{\partial X^\mu} x^\alpha(X^\sigma), \frac{\partial^2}{\partial X^\mu \partial X^\nu} x^\alpha(X^\sigma); X^\sigma \right]. \end{aligned} \quad (26)$$

Here it is assumed that the functional form of the Lagrangian density \mathcal{L} is invariant. The form-invariance of the Lagrangian density is discussed in Ref. [32].

In the following proof, we use an abbreviated notation for the Lagrangian density, as evaluated at different combinations of the above coordinates:

$$\mathcal{L}[x^\alpha, X^\sigma] \equiv \mathcal{L} \left[x^\alpha(X^\sigma), \frac{\partial}{\partial X^\mu} x^\alpha(X^\sigma), \frac{\partial^2}{\partial X^\mu \partial X^\nu} x^\alpha(X^\sigma); X^\sigma \right], \quad (27)$$

$$\mathcal{L}[y^\alpha, Y^\sigma] \equiv \mathcal{L} \left[y^\alpha(Y^\sigma), \frac{\partial}{\partial Y^\mu} y^\alpha(Y^\sigma), \frac{\partial^2}{\partial Y^\mu \partial Y^\nu} y^\alpha(Y^\sigma); Y^\sigma \right], \quad (28)$$

$$\mathcal{L}[y^\alpha, X^\sigma] \equiv \mathcal{L} \left[y^\alpha(X^\sigma), \frac{\partial}{\partial X^\mu} y^\alpha(X^\sigma), \frac{\partial^2}{\partial X^\mu \partial X^\nu} y^\alpha(X^\sigma); X^\sigma \right]. \quad (29)$$

The dependence of \mathcal{L} on the inertial coordinates x^α and their derivatives with respect to X^μ is considered up to the second derivatives of x^α , which is due to our adoption of the Holland gauge for $L_Q^{(\lambda)}$ in Eq. (15).

With the above abbreviated notations, the variation of the action is expressed as:

$$\delta S = \int d^4 Y \mathcal{L}[y^\alpha, Y^\sigma] - \int d^4 X \mathcal{L}[x^\alpha, X^\sigma]. \quad (30)$$

Since

$$\frac{\partial Y^\mu}{\partial X^\nu} = \delta_\nu^\mu + \frac{\partial \delta X^\mu}{\partial X^\nu}, \quad (31)$$

we obtain

$$\begin{aligned} d^4 Y &= \left| \det \left(\frac{\partial Y^\mu}{\partial X^\nu} \right) \right| d^4 X \\ &\approx \left(1 + \frac{\partial \delta X^\mu}{\partial X^\mu} \right) d^4 X. \end{aligned} \quad (32)$$

Under the transformation Eq. (19), the resulting transformation of the Lagrangian density is:

$$\mathcal{L}[x^\alpha, X^\sigma] \rightarrow \mathcal{L}[y^\alpha, Y^\sigma] = \mathcal{L}[x^\alpha, X^\sigma] + \delta \mathcal{L}[x^\alpha, X^\sigma], \quad (33)$$

Likewise, the restricted transformation of Eq. (20) leads to the following transformed Lagrangian density:

$$\mathcal{L}[x^\alpha, X^\sigma] \rightarrow \mathcal{L}[y^\alpha, X^\sigma] = \mathcal{L}[x^\alpha, X^\sigma] + \Delta \mathcal{L}[x^\alpha, X^\sigma]. \quad (34)$$

where $\mathcal{L}[y(X)]$ is defined in Eq. (29). Similar to the derivation of Eq. (21), it is straightforward to show that $\delta \mathcal{L}[x^\alpha, X^\sigma]$ and $\Delta \mathcal{L}[x^\alpha, X^\sigma]$ are related via

$$\Delta \mathcal{L}[x^\alpha, X^\sigma] = \delta \mathcal{L}[x^\alpha, X^\sigma] - \frac{\partial \mathcal{L}[x^\alpha, X^\sigma]}{\partial X^\mu} \delta X^\mu, \quad (35)$$

up to $\mathcal{O}(\delta X)$.

Using Eqs. (30), (32), (33) and (35) and keeping only the terms that are linear in the variations, we have

$$\begin{aligned}
\delta S &= \int d^4Y \mathcal{L}[y^\alpha, Y^\sigma] - \int d^4X \mathcal{L}[x^\alpha, X^\sigma] \\
&= \int d^4X \left(1 + \frac{\partial \delta X^\mu}{\partial X^\mu} \right) (\mathcal{L}[x^\alpha, X^\sigma] + \delta \mathcal{L}[x^\alpha, X^\sigma]) - \int d^4X \mathcal{L}[x^\alpha, X^\sigma] \\
&= \int d^4X \left[\frac{\partial \delta X^\mu}{\partial X^\mu} \mathcal{L}[x^\alpha, X^\sigma] + \delta \mathcal{L}[x^\alpha, X^\sigma] \right] \\
&= \int d^4X \left[\frac{\partial \delta X^\mu}{\partial X^\mu} \mathcal{L}[x^\alpha, X^\sigma] + \Delta \mathcal{L}[x^\alpha, X^\sigma] + \frac{\partial \mathcal{L}[x^\alpha, X^\sigma]}{\partial X^\mu} \delta X^\mu \right] \\
&= \int d^4X \left[\Delta \mathcal{L}[x^\alpha, X^\sigma] + \frac{\partial}{\partial X^\mu} (\mathcal{L}[x^\alpha, X^\sigma] \delta X^\mu) \right]. \tag{36}
\end{aligned}$$

From Eq. (34), we note that

$$\Delta \mathcal{L}[x^\alpha, X^\sigma] = \mathcal{L}[y^\alpha, X^\sigma] - \mathcal{L}[x^\alpha, X^\sigma], \tag{37}$$

which is obtained at the fixed point X^σ . Similarly, Δx^α , $\Delta(\partial_\mu x^\alpha)$ and $\Delta(\partial_\mu \partial_\nu x^\alpha)$ are defined at the fixed point X^σ . Thus we have:

$$\begin{aligned}
\Delta \mathcal{L} &= \frac{\partial \mathcal{L}}{\partial x^\alpha} \Delta x^\alpha + \frac{\partial \mathcal{L}}{\partial(\partial_\mu x^\alpha)} \Delta(\partial_\mu x^\alpha) + \sum_{\mu, \nu} \frac{1}{2} (1 + \delta_{\mu\nu}) \frac{\partial \mathcal{L}}{\partial(\partial_\mu \partial_\nu x^\alpha)} \Delta(\partial_\mu \partial_\nu x^\alpha) \\
&= \left(\frac{\partial \mathcal{L}}{\partial x^\alpha} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu x^\alpha)} + \sum_{\mu, \nu} \frac{1}{2} (1 + \delta_{\mu\nu}) \partial_\mu \partial_\nu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \partial_\nu x^\alpha)} \right) \Delta x^\alpha \\
&\quad + \left[\left(\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu x^\alpha)} \right) \Delta x^\alpha + \frac{\partial \mathcal{L}}{\partial(\partial_\mu x^\alpha)} \partial_\mu (\Delta x^\alpha) \right] \\
&\quad + \left[- \sum_{\mu, \nu} \frac{1}{2} (1 + \delta_{\mu\nu}) \left(\partial_\mu \partial_\nu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \partial_\nu x^\alpha)} \right) \Delta x^\alpha \right. \\
&\quad \left. + \sum_{\mu, \nu} \frac{1}{2} (1 + \delta_{\mu\nu}) \frac{\partial \mathcal{L}}{\partial(\partial_\mu \partial_\nu x^\alpha)} \partial_\mu \partial_\nu (\Delta x^\alpha) \right]. \tag{38}
\end{aligned}$$

After some algebra as shown in Appendix B, it is revealed that

$$\begin{aligned}
\Delta \mathcal{L} &= \left(\frac{\partial \mathcal{L}}{\partial x^\alpha} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu x^\alpha)} + \sum_{\mu, \nu} \frac{1}{2} (1 + \delta_{\mu\nu}) \partial_\mu \partial_\nu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \partial_\nu x^\alpha)} \right) \Delta x^\alpha \\
&\quad + \partial_\mu \left\{ \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu x^\alpha)} - \sum_{\rho, \nu} \frac{1}{2} \delta^{\mu\rho} (1 + \delta_{\rho\nu}) \partial_\nu \frac{\partial \mathcal{L}}{\partial(\partial_\rho \partial_\nu x^\alpha)} \right] \Delta x^\alpha \right. \\
&\quad \left. + \sum_{\rho, \nu} \frac{1}{2} \delta^{\mu\rho} (1 + \delta_{\rho\nu}) \frac{\partial \mathcal{L}}{\partial(\partial_\rho \partial_\nu x^\alpha)} \partial_\nu (\Delta x^\alpha) \right\}. \tag{39}
\end{aligned}$$

Substituting Eq. (39) into Eq. (36), we have:

$$\begin{aligned}
 \delta S = & \int d^4X \left\{ \left(\frac{\partial \mathcal{L}}{\partial x^\alpha} - \partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu x^\alpha)} \right. \right. \\
 & + \sum_{\mu,\nu} \frac{1}{2} (1 + \delta_{\mu\nu}) \partial_\mu \partial_\nu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \partial_\nu x^\alpha)} \left. \right) \Delta x^\alpha \\
 & + \partial_\mu \left[\left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu x^\alpha)} - \sum_{\rho,\nu} \frac{1}{2} \delta^{\mu\rho} (1 + \delta_{\rho\nu}) \partial_\nu \frac{\partial \mathcal{L}}{\partial (\partial_\rho \partial_\nu x^\alpha)} \right) \Delta x^\alpha \right. \\
 & \left. \left. + \sum_{\rho,\nu} \frac{1}{2} \delta^{\mu\rho} (1 + \delta_{\rho\nu}) \frac{\partial \mathcal{L}}{\partial (\partial_\rho \partial_\nu x^\alpha)} \partial_\nu (\Delta x^\alpha) + \mathcal{L} \delta X^\mu \right] \right\} \\
 = & 0. \tag{40}
 \end{aligned}$$

The terms in the first set of parentheses in the first line of Eq. (40) pertain to the variations Δx^α , which are independent. These give rise to the Euler-Lagrange equations, i.e. the dynamical PDEs:

$$\frac{\partial \mathcal{L}}{\partial x^\alpha} - \partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu x^\alpha)} + \sum_{\mu,\nu} \frac{1}{2} (1 + \delta_{\mu\nu}) \partial_\mu \partial_\nu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \partial_\nu x^\alpha)} = 0, \tag{41}$$

for $\alpha = 0, 1, 2, 3$.

3.3. Generic derivation of (infinitesimal) Noether current and conservation laws

The remaining terms in Eq. (40) take the form of a vanishing four-divergence or continuity equation. These are the terms that give rise to the conservation laws,

$$\partial_\mu \mathcal{J}^\mu = 0, \tag{42}$$

where the (infinitesimal) Noether current is

$$\begin{aligned}
 \mathcal{J}^\mu = & \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu x^\alpha)} - \sum_{\rho,\nu} \frac{1}{2} \delta^{\mu\rho} (1 + \delta_{\rho\nu}) \partial_\nu \frac{\partial \mathcal{L}}{\partial (\partial_\rho \partial_\nu x^\alpha)} \right) \Delta x^\alpha \\
 & + \sum_{\rho,\nu} \frac{1}{2} \delta^{\mu\rho} (1 + \delta_{\rho\nu}) \frac{\partial \mathcal{L}}{\partial (\partial_\rho \partial_\nu x^\alpha)} \partial_\nu (\Delta x^\alpha) + \mathcal{L} \delta X^\mu. \tag{43}
 \end{aligned}$$

Due to Eq. (21), we have:

$$\begin{aligned}
 \mathcal{J}^\mu = & \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu x^\alpha)} - \sum_{\rho,\nu} \frac{1}{2} \delta^{\mu\rho} (1 + \delta_{\rho\nu}) \partial_\nu \frac{\partial \mathcal{L}}{\partial (\partial_\rho \partial_\nu x^\alpha)} \right) (\delta x^\alpha - (\partial_\sigma x^\alpha) \delta X^\sigma) \\
 & + \sum_{\rho,\nu} \frac{1}{2} \delta^{\mu\rho} (1 + \delta_{\rho\nu}) \frac{\partial \mathcal{L}}{\partial (\partial_\rho \partial_\nu x^\alpha)} \partial_\nu (\delta x^\alpha - (\partial_\sigma x^\alpha) \delta X^\sigma) + \delta_\sigma^\mu \mathcal{L} \delta X^\sigma. \tag{44}
 \end{aligned}$$

Due to Eq. (24), we have

$$\begin{aligned}
 \partial_\nu (\delta x^\alpha - (\partial_\sigma x^\alpha) \delta X^\sigma) &= \partial_\nu (\delta x^\alpha) - (\partial_\nu \partial_\sigma x^\alpha) \delta X^\sigma - \partial_\sigma x^\alpha \partial_\nu (\delta X^\sigma) \\
 &= \left[\delta (\partial_\nu x^\alpha) + \partial_\lambda x^\alpha \partial_\nu (\delta X^\lambda) \right] - (\partial_\nu \partial_\sigma x^\alpha) \delta X^\sigma \\
 &\quad - \partial_\sigma x^\alpha \partial_\nu (\delta X^\sigma) \\
 &= \delta (\partial_\nu x^\alpha) - (\partial_\nu \partial_\sigma x^\alpha) \delta X^\sigma. \tag{45}
 \end{aligned}$$

Substituting Eq. (45) into Eq. (44), we obtain

$$\begin{aligned}
 \mathcal{J}^\mu = & \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu x^\alpha)} - \sum_{\rho, \nu} \frac{1}{2} \delta^{\mu\rho} (1 + \delta_{\rho\nu}) \partial_\nu \frac{\partial \mathcal{L}}{\partial(\partial_\rho \partial_\nu x^\alpha)} \right] \delta x^\alpha \\
 & + \sum_{\rho, \nu} \frac{1}{2} \delta^{\mu\rho} (1 + \delta_{\rho\nu}) \frac{\partial \mathcal{L}}{\partial(\partial_\rho \partial_\nu x^\alpha)} \delta(\partial_\nu x^\alpha) \\
 & - \left\{ \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu x^\alpha)} - \sum_{\rho, \nu} \frac{1}{2} \delta^{\mu\rho} (1 + \delta_{\rho\nu}) \partial_\nu \frac{\partial \mathcal{L}}{\partial(\partial_\rho \partial_\nu x^\alpha)} \right] \partial_\sigma x^\alpha \right. \\
 & \left. + \sum_{\rho, \nu} \frac{1}{2} \delta^{\mu\rho} (1 + \delta_{\rho\nu}) \frac{\partial \mathcal{L}}{\partial(\partial_\rho \partial_\nu x^\alpha)} \partial_\nu \partial_\sigma x^\alpha - \delta_\sigma^\mu \mathcal{L} \right\} \delta X^\sigma. \tag{46}
 \end{aligned}$$

3.4. Generic derivation of conservation laws based on natural coordinate invariance

At this stage, we exploit the fact that the solution trajectory ensemble $x^\alpha(X^\mu)$ is invariant with respect to all infinitesimal translations in the *natural* coordinates, X^μ . Note that this natural coordinate symmetry holds (in uniformizing coordinates) *even* when there is an external vector potential present. As shown below, consideration of all such infinitesimal translations gives rise to the energy-momentum tensor.

Consider δX^μ to be a fixed infinitesimal translation, *i.e.*, $\delta X^\mu = \epsilon^\mu$, where ϵ^μ are constants. Since the trajectory ensemble is invariant under any such infinitesimal translation, we have

$$X^\mu \rightarrow Y^\mu = X^\mu + \epsilon^\mu, \tag{47}$$

$$x^\alpha(X^\sigma) \rightarrow y^\alpha(Y^\sigma) = x^\alpha(X^\sigma), \tag{48}$$

which leads to $\delta x^\alpha(X^\sigma) = 0$. Thus, the Noether current in Eq. (46) becomes

$$\begin{aligned}
 \mathcal{J}^\mu = & \sum_{\rho, \nu} \frac{1}{2} \delta^{\mu\rho} (1 + \delta_{\rho\nu}) \frac{\partial \mathcal{L}}{\partial(\partial_\mu \partial_\nu x^\alpha)} \delta(\partial_\nu x^\alpha) \\
 & - \left\{ \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu x^\alpha)} - \sum_{\rho, \nu} \frac{1}{2} \delta^{\mu\rho} (1 + \delta_{\rho\nu}) \partial_\nu \frac{\partial \mathcal{L}}{\partial(\partial_\rho \partial_\nu x^\alpha)} \right] \partial_\sigma x^\alpha \right. \\
 & \left. + \sum_{\rho, \nu} \frac{1}{2} \delta^{\mu\rho} (1 + \delta_{\rho\nu}) \frac{\partial \mathcal{L}}{\partial(\partial_\rho \partial_\nu x^\alpha)} \partial_\nu \partial_\sigma x^\alpha - \delta_\sigma^\mu \mathcal{L} \right\} \epsilon^\sigma. \tag{49}
 \end{aligned}$$

In addition to Eqs. (47) and (48), we use the following property:

$$\frac{\partial}{\partial X^\nu} x^\alpha(X^\sigma) \rightarrow \frac{\partial}{\partial Y^\nu} y^\alpha(Y^\sigma) = \frac{\partial}{\partial X^\nu} x^\alpha(X^\sigma). \tag{50}$$

We note that $\partial_\nu \epsilon^\sigma = 0$ since ϵ^σ is a constant four-vector. Equation (50) is valid because we have set $\delta X^\sigma = \epsilon^\sigma$ and taken into account $\partial_\nu \epsilon^\sigma = 0$. Thus, we have

$$\delta(\partial_\nu x^\alpha) = 0. \tag{51}$$

Because of Eq. (51), the current in Eq. (46) becomes

$$\begin{aligned}
 \mathcal{J}^\mu = & - \left\{ \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu x^\alpha)} - \sum_{\rho, \nu} \frac{1}{2} \delta^{\mu\rho} (1 + \delta_{\rho\nu}) \partial_\nu \frac{\partial \mathcal{L}}{\partial(\partial_\rho \partial_\nu x^\alpha)} \right] \partial_\sigma x^\alpha \right. \\
 & \left. + \sum_{\rho, \nu} \frac{1}{2} \delta^{\mu\rho} (1 + \delta_{\rho\nu}) \frac{\partial \mathcal{L}}{\partial(\partial_\rho \partial_\nu x^\alpha)} \partial_\nu \partial_\sigma x^\alpha - \delta_\sigma^\mu \mathcal{L} \right\} \epsilon^\sigma. \tag{52}
 \end{aligned}$$

From the above expression, we obtain

$$\mathcal{J}^\mu = -\Theta^\mu{}_\sigma \epsilon^\sigma, \quad (53)$$

where $\Theta^\mu{}_\sigma$ is the (canonical) energy-momentum tensor, defined by

$$\begin{aligned} \Theta^\mu{}_\sigma &= \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu x^\alpha)} - \sum_{\rho, \nu} \frac{1}{2} \delta^{\mu\rho} (1 + \delta_{\rho\nu}) \partial_\nu \frac{\partial \mathcal{L}}{\partial(\partial_\rho \partial_\nu x^\alpha)} \right] \partial_\sigma x^\alpha \\ &+ \sum_{\rho, \nu} \frac{1}{2} \delta^{\mu\rho} (1 + \delta_{\rho\nu}) \frac{\partial \mathcal{L}}{\partial(\partial_\rho \partial_\nu x^\alpha)} \partial_\nu \partial_\sigma x^\alpha - \delta_\sigma^\mu \mathcal{L}. \end{aligned} \quad (54)$$

From Eqs. (42) and (53), it is straightforward to show that

$$\partial_\mu \Theta^\mu{}_\sigma = 0, \quad (55)$$

for $\sigma = 0, 1, 2, 3$. Clearly, $\sigma = 0$ corresponds to the energy conservation law (continuity equation), and $\sigma = i$ to the momentum conservation law, associated with natural coordinate translation symmetry. Likewise, the $\mu = 0$ component corresponds to energy or momentum density, whereas $\mu = j$ corresponds to the corresponding flux quantities. The above natural coordinate conservation laws have been derived here for the full $3 + 1$ spacetime, but have straightforward simplifications for the $1 + 1$ case considered in Sec. 4.

4. Specific conservation laws in $1 + 1$ spacetime

4.1. Introduction

In Sec. 3, generic forms of the conservation laws were derived, for $3 + 1$ (or really arbitrary-dimensional) spacetime. Here, we restrict consideration to just the $1 + 1$ spacetime case, and also replace the generic Lagrangians with the explicit forms from Sec. 2.2. The restriction to a single spatial dimension helps to simplify the resultant expressions, as does the use of uniformizing coordinates in terms of which $f(C) = 1$.

Instead of eight separate conservation laws, there are now just four, associated with the two natural variables $(c\lambda, C)$ and the two inertial variables, (ct, x) . Note that the latter two depend on the functional form of the Lagrangian density \mathcal{L} being invariant with respect to translations in inertial coordinates x^α . Thus, we are henceforth *ignoring* the external vector potential contribution \mathcal{L}_I —i.e., only the free particle case is considered.

In this section, Noether's theorem is applied to obtain the four conservation laws in the form of two-term $1 + 1$ continuity equations (the flux “vectors” now include just a single component). Three of the four continuity equations will be seen to give rise to independent dynamical constraints, whereas the fourth is a trivial expression that does not correspond to anything new. The three nontrivial conservation laws derived here correspond to conservation of energy, momentum, and the C -conservation law, respectively. All three of these conservation laws have been previously identified in the context of the nonrelativistic trajectory-based theory [17, 19]. Accordingly, it becomes possible to compare the nonrelativistic limits of the present relativistic results with the previously derived non-relativistic trajectory-based expressions. This is done in Sec. 5.

In this section, we use the generic Noether current results from Sec. 3 to derive specific $1 + 1$ forms for the natural coordinate conservation laws associated with λ and C . We find that the natural coordinate “momentum” continuity equation becomes the C -conservation law, as discussed. In contrast, the λ or natural coordinate “energy” conservation law is seen to become a trivial equation, with vanishing density. Thus, it is the *inertial coordinate* continuity equations, for ct and x , that are seen to give rise to the usual energy and momentum conservation laws, respectively. These are derived first—conveniently, from the Euler-Lagrange equations themselves.

4.2. General inertial coordinate conservation laws: energy and momentum

From Eq. (41), the $1 + 1$ Euler-Lagrange equations become:

$$\frac{\partial \mathcal{L}}{\partial x^\alpha} - \frac{\partial}{\partial \lambda} \left(\frac{\partial \mathcal{L}}{\partial \left(\frac{\partial x^\alpha}{\partial \lambda} \right)} \right) - \frac{\partial}{\partial C} \left(\frac{\partial \mathcal{L}}{\partial \left(\frac{\partial x^\alpha}{\partial C} \right)} \right) + \frac{\partial^2}{\partial C^2} \left(\frac{\partial \mathcal{L}}{\partial \left(\frac{\partial^2 x^\alpha}{\partial C^2} \right)} \right) = 0, \quad (56)$$

for each component $\alpha = 0, 1$. Recall that in the free-particle case considered here, $\mathcal{L}_I = 0$, i.e., so that the contribution from Eq. (14) vanishes. Under these conditions, \mathcal{L} has no explicit dependence on x^α explicitly, meaning that

$$\frac{\partial \mathcal{L}}{\partial x^\alpha} = 0, \quad (57)$$

so that the first term of Eq. (56) vanishes, resulting in

$$\frac{\partial}{\partial \lambda} \left(\frac{\partial \mathcal{L}}{\partial \left(\frac{\partial x^\alpha}{\partial \lambda} \right)} \right) + \frac{\partial}{\partial C} \left(\frac{\partial \mathcal{L}}{\partial \left(\frac{\partial x^\alpha}{\partial C} \right)} \right) - \frac{\partial^2}{\partial C^2} \left(\frac{\partial \mathcal{L}}{\partial \left(\frac{\partial^2 x^\alpha}{\partial C^2} \right)} \right) = 0. \quad (58)$$

Next, we define the following quantities:

$$\Pi^\alpha \equiv \eta^{\alpha\beta} \frac{\partial \mathcal{L}}{\partial \left(\frac{\partial x^\beta}{\partial \lambda} \right)}, \quad (59)$$

$$\Phi^\alpha \equiv \eta^{\alpha\beta} \left[\frac{\partial \mathcal{L}}{\partial \left(\frac{\partial x^\beta}{\partial C} \right)} - \partial_C \left(\frac{\partial \mathcal{L}}{\partial \left(\frac{\partial^2 x^\beta}{\partial C^2} \right)} \right) \right]. \quad (60)$$

Substituting into Eq. (58), we find:

$$\frac{\partial}{\partial \lambda} \Pi^\alpha + \frac{\partial}{\partial C} \Phi^\alpha = 0, \quad (61)$$

where $\alpha = 0, 1$. The physical interpretation of this equation becomes a bit clearer with the following minor redefinitions:

$$\rho_E = c\Pi^0 \quad ; \quad J_E = c\Phi^0; \quad (62)$$

$$\rho_P = \Pi^1 \quad ; \quad J_P = \Phi^1 \quad (63)$$

As suggested by the new quantity units, ρ_E and ρ_P represent energy and momentum densities, respectively, whereas J_E and J_P are the corresponding energy and momentum fluxes. Equation (61) thus leads to two continuity equations, corresponding to energy and momentum conservation, respectively:

$$\frac{\partial \rho_E}{\partial \lambda} + \frac{\partial J_E}{\partial C} = 0, \quad (64)$$

$$\frac{\partial \rho_P}{\partial \lambda} + \frac{\partial J_P}{\partial C} = 0, \quad (65)$$

It is interesting that these two conservation laws, Equations (64) and (65), can be derived directly from the Euler-Lagrange equation of Eq. (58)—i.e., from the dynamical PDEs themselves. Also, we note that the individual expressions for Π^α and Φ^α depend in general on the choice of gauge—with the explicit forms that will be provided in Sec. 4.3 corresponding to the Holland gauge (i.e., \mathcal{L}_Q chosen as per Eq. (15)). However, the continuity conditions themselves are gauge-invariant.

4.3. Specific inertial coordinate conservation laws: energy and momentum

Before substituting the specific forms of Eqs. (13) and (15) into the expressions of Sec. 4.2, it is convenient to introduce the following notation:

$$t_\lambda \equiv \left. \frac{\partial t}{\partial \lambda} \right|_C, \quad t' \equiv \left. \frac{\partial t}{\partial C} \right|_\lambda, \quad (66)$$

$$x_\lambda \equiv \left. \frac{\partial x}{\partial \lambda} \right|_C, \quad x' \equiv \left. \frac{\partial x}{\partial C} \right|_\lambda. \quad (67)$$

In terms of these forms, the generic inertial density and flux expressions become

$$\rho_E = -\frac{\partial \mathcal{L}}{\partial t_\lambda}, \quad J_E = -\left(\frac{\partial \mathcal{L}}{\partial t'} - \frac{\partial}{\partial C} \left(\frac{\partial \mathcal{L}}{\partial t''} \right) \right), \quad (68)$$

$$\rho_P = \frac{\partial \mathcal{L}}{\partial x_\lambda}, \quad J_P = \frac{\partial \mathcal{L}}{\partial x'} - \frac{\partial}{\partial C} \left(\frac{\partial \mathcal{L}}{\partial x''} \right). \quad (69)$$

In uniformizing coordinates, the $1+1$ reparametrized free-particle Lagrangian expressions become:

$$L_Q^{(\lambda)} = \frac{\hbar^2}{8m} \sqrt{t_\lambda^2 - x_\lambda^2/c^2} \frac{(x'x'' - c^2t't'')^2}{(x'^2 - c^2t'^2)^3}, \quad (70)$$

$$L^{(\lambda)} = -mc^2 \sqrt{t_\lambda^2 - x_\lambda^2/c^2} - \frac{\hbar^2}{8m} \sqrt{t_\lambda^2 - x_\lambda^2/c^2} \frac{(x'x'' - c^2t't'')^2}{(x'^2 - c^2t'^2)^3}. \quad (71)$$

Utilizing Eqs. (9), (71) and (68), we obtain specific forms for the energy density and flux:

$$\rho_E = \frac{mc^2 t_\lambda}{\sqrt{t_\lambda^2 - x_\lambda^2/c^2}} + \frac{\hbar^2}{8m} \frac{t_\lambda}{\sqrt{t_\lambda^2 - x_\lambda^2/c^2}} \frac{(x'x'' - c^2t't'')^2}{(x'^2 - c^2t'^2)^3}, \quad (72)$$

$$J_E = \frac{\hbar^2 c^2 t'}{4m} \left[\sqrt{t_\lambda^2 - x_\lambda^2/c^2} \left(\frac{(x''^2 - c^2t'^2) + (x'x''' - c^2t't''')}{(x'^2 - c^2t'^2)^3} \right. \right. \\ \left. \left. - \frac{3(x'x'' - c^2t't'')^2}{(x'^2 - c^2t'^2)^4} \right) + \frac{(t_\lambda t'_\lambda - x_\lambda x'_\lambda/c^2)}{\sqrt{t_\lambda^2 - x_\lambda^2/c^2}} \frac{(x'x'' - c^2t't'')^2}{(x'^2 - c^2t'^2)^3} \right]. \quad (73)$$

Utilizing Eqs. (9), (71) and (69), we obtain the momentum density and flux in similar fashion:

$$\rho_P = \frac{mx_\lambda}{\sqrt{t_\lambda^2 - x_\lambda^2/c^2}} + \frac{\hbar^2}{8mc^2} \frac{x_\lambda}{\sqrt{t_\lambda^2 - x_\lambda^2/c^2}} \frac{(x'x'' - c^2t't'')^2}{(x'^2 - c^2t'^2)^3}, \quad (74)$$

$$J_P = \frac{\hbar^2 x'}{4m} \left[\sqrt{t_\lambda^2 - x_\lambda^2/c^2} \left(\frac{(x''^2 - c^2t'^2) + (x'x''' - c^2t't''')}{(x'^2 - c^2t'^2)^3} - \frac{3(x'x'' - c^2t't'')^2}{(x'^2 - c^2t'^2)^4} \right) \right. \\ \left. + \frac{(t_\lambda t'_\lambda - x_\lambda x'_\lambda/c^2)}{\sqrt{t_\lambda^2 - x_\lambda^2/c^2}} \frac{(x'x'' - c^2t't'')^2}{(x'^2 - c^2t'^2)^3} \right]. \quad (75)$$

These equations represent the trajectory-based energy and momentum conservation laws for a massive spin-zero relativistic quantum particle in $1+1$ spacetime.

4.4. Natural coordinate conservation laws: C -conservation

In analogy with the inertial quantities of Eqs. (62) and (63), we find it useful to introduce the corresponding natural coordinate quantities,

$$\tilde{\rho}_E = \Theta^0_0, \quad \tilde{J}_E = c\Theta^0_1, \quad (76)$$

$$\tilde{\rho}_P = \frac{1}{c}\Theta^0_1, \quad \tilde{J}_P = \Theta^1_1, \quad (77)$$

designated with a ‘~’. This leads to natural coordinate ‘‘energy’’ and ‘‘momentum’’ continuity equations:

$$\frac{\partial \tilde{\rho}_E}{\partial \lambda} + \frac{\partial \tilde{J}_E}{\partial C} = 0 \quad (78)$$

$$\frac{\partial \tilde{\rho}_P}{\partial \lambda} + \frac{\partial \tilde{J}_P}{\partial C} = 0 \quad (79)$$

Equation (79), being directly associated with the C coordinate, corresponds to what has been called the ‘‘ C -conservation law’’ [17, 19]. Equation (78) does not correspond to any independent conservation law, for reasons to be discussed.

We first examine Eq. (79), for which the generic forms for the $1+1$ quantities are given by

$$\tilde{\rho}_P = \frac{\partial \mathcal{L}}{\partial t_\lambda} t' + \frac{\partial \mathcal{L}}{\partial x_\lambda} x', \quad (80)$$

$$\begin{aligned} \tilde{J}_P = & \left[\frac{\partial \mathcal{L}}{\partial t'} - \frac{\partial}{\partial C} \left(\frac{\partial \mathcal{L}}{\partial t''} \right) \right] t' + \frac{\partial \mathcal{L}}{\partial t''} t'' \\ & + \left[\frac{\partial \mathcal{L}}{\partial x'} - \frac{\partial}{\partial C} \left(\frac{\partial \mathcal{L}}{\partial x''} \right) \right] x' + \frac{\partial \mathcal{L}}{\partial x''} x'' - \mathcal{L}. \end{aligned} \quad (81)$$

To obtain specific forms for these quantities, we substitute Eqs. (9) and (71) into Eqs. (80) and (81), which yields

$$\begin{aligned} \tilde{\rho}_P = & \frac{-mc^2 (t_\lambda t' - x_\lambda x'/c^2)}{\sqrt{t_\lambda^2 - x_\lambda^2/c^2}} - qt' \Phi(t, x) + \frac{q}{c} x' A(t, x) \\ & - \frac{\hbar^2}{8m} \frac{(t_\lambda t' - x_\lambda x'/c^2)}{\sqrt{t_\lambda^2 - x_\lambda^2/c^2}} \frac{(x' x'' - c^2 t' t'')^2}{(x'^2 - c^2 t'^2)^3}, \end{aligned} \quad (82)$$

$$\begin{aligned} \tilde{J}_P = & \frac{\hbar^2}{4m} \left\{ \sqrt{t_\lambda^2 - x_\lambda^2/c^2} \left[\frac{(x''^2 - c^2 t''^2) + (x' x''' - c^2 t' t''')}{(x'^2 - c^2 t'^2)^2} - \frac{7}{2} \frac{(x' x'' - c^2 t' t'')^2}{(x'^2 - c^2 t'^2)^3} \right] \right. \\ & \left. + \frac{(t_\lambda t' - x_\lambda x'/c^2)}{\sqrt{t_\lambda^2 - x_\lambda^2/c^2}} \frac{(x' x'' - c^2 t' t'')}{(x'^2 - c^2 t'^2)^2} \right\} \\ & + mc^2 \sqrt{t_\lambda^2 - x_\lambda^2/c^2} + qt_\lambda \Phi(t, x) - \frac{q}{c} x_\lambda A(t, x). \end{aligned} \quad (83)$$

Note that for completeness, we have restored the external vector potential contribution—since natural coordinate translation symmetry does not require that this vanish.

Equation (79) is denoted as the C -conservation law, with $\tilde{\rho}_P$ and \tilde{J}_P given by Eqs. (82) and (83), respectively. The C -conservation law has an interesting place in the context of trajectory-based quantum theory [17, 19]. In the 1d non-relativistic theory, it simply states that the

velocity field is single-valued, or that quantum trajectories do not cross—or rather, it ensures that this property is maintained for all times, if it is true at the initial time. Likewise, in the many-dimensional non-relativistic case, it ensures that vortices do not arise, if they are not present initially. One might argue that these are undesirable or *ad hoc* constraints to impose on an otherwise arbitrary velocity vector field. In the relativistic context, however, the C -conservation law takes on a more substantial meaning. This is because it is *directly* linked to the conditions that must be imposed on the velocity vector field in order to ensure the consistent construction of global simultaneity submanifolds—on which the whole of the relativistic theory depends [27]. The required (Frobenius) condition can be specified using a kind of parallel-transport argument—which as it happens, simply does not manifest in the $1+1$ special case. Consequently, further discussion will be reserved for future publications.

We now turn our attention to Eq. (78), whose generic $1+1$ quantities become

$$\tilde{\rho}_E = \left(\frac{\partial \mathcal{L}}{\partial t_\lambda} \right) t_\lambda + \left(\frac{\partial \mathcal{L}}{\partial x_\lambda} \right) x_\lambda - \mathcal{L}, \quad (84)$$

$$\begin{aligned} \tilde{J}_E = & \left(\frac{\partial \mathcal{L}}{\partial t'} - \frac{\partial}{\partial C} \left(\frac{\partial \mathcal{L}}{\partial t''} \right) \right) t_\lambda + \left(\frac{\partial \mathcal{L}}{\partial t''} \right) t'_\lambda \\ & + \left(\frac{\partial \mathcal{L}}{\partial x'} - \frac{\partial}{\partial C} \left(\frac{\partial \mathcal{L}}{\partial x''} \right) \right) x_\lambda + \left(\frac{\partial \mathcal{L}}{\partial x''} \right) x'_\lambda. \end{aligned} \quad (85)$$

By substituting Eqs. (9) and (71) into the above expressions, we obtain the following specific forms:

$$\tilde{\rho}_E = 0, \quad (86)$$

$$\begin{aligned} \tilde{J}_E = & \frac{\hbar^2}{4m} \left\{ (x' x_\lambda - c^2 t' t_\lambda) \left[\sqrt{t_\lambda^2 - x_\lambda^2/c^2} \left(\frac{(x''^2 - c^2 t''^2) + (x' x''' - c^2 t' t''')}{(x'^2 - c^2 t'^2)^3} \right. \right. \right. \\ & \left. \left. \left. - \frac{3(x' x'' - c^2 t' t'')^2}{(x'^2 - c^2 t'^2)^4} \right) + \frac{(t_\lambda t'_\lambda - x_\lambda x'_\lambda/c^2)}{\sqrt{t_\lambda^2 - x_\lambda^2/c^2}} \frac{(x' x'' - c^2 t' t'')}{(x'^2 - c^2 t'^2)^3} \right] \right. \\ & \left. - (x' x'_\lambda - c^2 t' t'_\lambda) \sqrt{t_\lambda^2 - x_\lambda^2/c^2} \frac{(x' x'' - c^2 t' t'')}{(x'^2 - c^2 t'^2)^3} \right\}. \end{aligned} \quad (87)$$

We note that $L^{(\lambda)}$ in Eq. (71) is positive-homogeneous of degree 1 in $\partial_\mu x^\alpha$, and so is \mathcal{L} . Thus, the vanishing $\tilde{\rho}_E$ at the coordinate level, obtained in Eq. (86), results from Eq. (84) together with Euler's theorem of homogeneous functions [39]. In any event, the corresponding continuity equation, Eq. (78), simply states that the C space integral of $\tilde{\rho}_E$ must be zero for all time—which is already implied by the fact that $\tilde{\rho}_E$ itself is zero. As a conservation law, the result is therefore trivial.

On the other hand, the vanishing of $\tilde{\rho}_E$ in Eq. (86) is itself a useful result, because it implies a constraint between the *inertial* energy and momentum densities, ρ_E and ρ_P . Specifically, we have

$$\tilde{\rho}_E = -\rho_E t_\lambda + \rho_P x_\lambda - \mathcal{L} = 0, \quad (88)$$

which is easily obtained from Eqs. (84), (86), (68) and (69). The Eq. (88) constraint can be useful, particularly when deriving the nonrelativistic limits of ρ_E and J_E .

5. Nonrelativistic limits of the conservation laws

In Sec. 4, we have obtained three trajectory-based conservation laws for the relativistic quantum mechanics of a massive spin-zero particle. In this section, we verify that the nonrelativistic limits of these energy, momentum and C -conservation continuity equations agree with those obtained from the nonrelativistic trajectory-based theory. In order to ensure a self-consistent comparison, we begin by evaluating the nonrelativistic limit of the action S , and show its agreement with the nonrelativistic action $S^{(NR)}$. Then, we proceed to obtain the nonrelativistic limits of the corresponding continuity equation density and flux quantities. Note that all of these quantities (as opposed to the continuity conditions themselves) are gauge-dependent. Accordingly, we work in the Holland gauge throughout.

5.1. Nonrelativistic limit of the action

Due to Eqs. (6) and (12), the action can be decomposed into three parts,

$$S = S_0 + S_I + S_Q, \quad (89)$$

where

$$\begin{aligned} S_0 &= \int d\lambda dC f(C) L_0^{(\lambda)}, & S_I &= \int d\lambda dC f(C) L_I^{(\lambda)}, \\ S_Q &= \int d\lambda dC f(C) \left(-L_Q^{(\lambda)} \right), \end{aligned} \quad (90)$$

and $L_0^{(\lambda)}$, $L_I^{(\lambda)}$ and $L_Q^{(\lambda)}$ are presented in Eqs. (13), (14) and (15), respectively. Since $L_0^{(\lambda)}$, $L_I^{(\lambda)}$ and $L_Q^{(\lambda)}$ are homogeneous functions of degree 1 in $\partial_\lambda x^\alpha$, the corresponding actions S_0 , S_I and S_Q are parameter-invariant [39]. In other words, the action S remains invariant under arbitrary reparameterization, $\lambda \rightarrow \lambda'$.

For obtaining the nonrelativistic limit of the action S , the most convenient choice is to set $\lambda = t$. With this choice, we have:

$$x_\lambda = \dot{x} \equiv \frac{\partial x}{\partial t} \Big|_C \quad ; \quad \beta \equiv \frac{1}{c} \frac{\partial x}{\partial t} \Big|_C = \frac{\dot{x}}{c} \quad ; \quad t_\lambda = 1 \quad (91)$$

In addition, recall that we are now using a uniformizing choice of C , so that $f(C) = 1$ in Eq. (90) above. Thus we have

$$\begin{aligned} S_0 &= \int d\lambda dC (-mc^2) \sqrt{-\frac{1}{c^2} \eta_{\alpha\beta} \frac{\partial x^\alpha}{\partial \lambda} \frac{\partial x^\beta}{\partial \lambda}} \\ &\approx \int dt dC (-mc^2) \sqrt{1 - \frac{1}{c^2} \dot{x}^2}. \end{aligned} \quad (92)$$

Since the relativistic quantum trajectories are all subluminal, in general, we have $0 \leq |\beta| < 1$.

In the nonrelativistic limit, $|\beta| \ll 1$, and thus β can be used as an expansion parameter. Expanding $L_0^{(\lambda)}/(mc^2)$ up to $\mathcal{O}(\beta^2)$, we obtain S_0 in the nonrelativistic limit:

$$S_0 \approx \int dt dC \left(-mc^2 + \frac{1}{2} m \dot{x}^2 \right). \quad (93)$$

Upon evaluating the C integral, we obtain the usual classical result, in the nonrelativistic limit. Likewise, S_I is also a purely classical contribution—from the external vector potential. It is

clear that this reduces to the usual scalar potential in the nonrelativistic limit, although to first order in β , we have

$$\begin{aligned} S_I &= \int d\lambda dC \frac{q}{c} (\partial_\lambda x^\alpha) A_\alpha(t, x) \\ &\approx \int dt dC \left(-q\Phi(t, x) + \frac{q}{c} \dot{x} A(t, x) \right). \end{aligned} \quad (94)$$

Of course, the interesting case is S_Q . To evaluate the nonrelativistic limit of this quantity, we utilize the following procedure. First, setting $\lambda = t$, we have

$$\begin{aligned} S_Q &= \int d\lambda dC \left[-\frac{\hbar^2}{8m} \sqrt{t_\lambda^2 - \frac{1}{c^2} x_\lambda^2} \frac{(x' x'' - c^2 t' t'')^2}{(x'^2 - c^2 t'^2)^3} \right] \\ &\approx \int dt dC \left[-\frac{\hbar^2}{8m} \sqrt{1 - \frac{1}{c^2} \dot{x}^2} \frac{(x' x'' - c^2 t' t'')^2}{(x'^2 - c^2 t'^2)^3} \right]. \end{aligned} \quad (95)$$

In the nonrelativistic limit, it is straightforward to show that

$$\left(\frac{\partial x}{\partial \lambda} \Big|_C \right) \left(\frac{\partial (ct)}{\partial \lambda} \Big|_C \right)^{-1} \approx \left(\frac{\partial (ct)}{\partial C} \Big|_\lambda \right) \left(\frac{\partial x}{\partial C} \Big|_\lambda \right)^{-1}. \quad (96)$$

Replacing λ by t in Eq. (96), we obtain

$$\frac{\dot{x}}{c} = \beta \approx \frac{ct'}{x'}. \quad (97)$$

From Eq. (97), it can be shown that

$$ct'' \approx x''\beta + x'\beta', \quad (98)$$

$$ct''' \approx x'''\beta + 2x''\beta' + x'\beta''. \quad (99)$$

If we assume that $x''\beta \gg x'\beta'$, $x'''\beta \gg 2x''\beta'$ and $x'''\beta \gg x'\beta''$, then we obtain $ct'' \approx x''\beta$ and $ct''' \approx x'''\beta$. Therefore,

$$\beta \approx \frac{ct''}{x''} \approx \frac{ct'''}{x'''} \quad (100)$$

Due to Eqs. (91), (97) and (100), Equation (95) can be approximated as follows:

$$S_Q \approx \int dt dC \left(-\frac{\hbar^2}{8m} \frac{x''^2}{x'^4} \right) (1 - \beta^2)^{-1/2}. \quad (101)$$

In Ref. [27], it has been discussed that L_Q , defined in Eq. (11), is of the same order as the quantum potential Q , and that $Q/(mc^2) \approx \mathcal{O}(\beta^2)$, where $\mathcal{O}(\beta^n)$ denotes order n in β . Therefore, we have

$$\frac{L_Q}{mc^2} \approx \mathcal{O}(\beta^2), \quad (102)$$

$$\frac{L_Q^{(\lambda)}}{mc^2} = \frac{d\tau}{d\lambda} \frac{L_Q}{mc^2}. \quad (103)$$

In addition, for the nonrelativistic quantum trajectories, $L_Q^{(\text{NR})}$ can be expressed as:

$$L_Q^{(\text{NR})} = \frac{\hbar^2}{8m} \frac{x''^2}{x'^4}. \quad (104)$$

It is straightforward to show that $L_Q^{(\text{NR})}$ is of the same order as the quantum potential $Q^{(\text{NR})}$ for the nonrelativistic quantum trajectories, and that $Q^{(\text{NR})}/(mc^2) \approx \mathcal{O}(\beta^2)$. Therefore,

$$\frac{L_Q^{(\text{NR})}}{mc^2} = \frac{1}{8} \left(\frac{\hbar}{mc} \right)^2 \frac{x''^2}{x'^4} \approx \mathcal{O}(\beta^2). \quad (105)$$

We note that the nonrelativistic limits of L_Q and $L_Q^{(\lambda)}$ are the same up to $\mathcal{O}(\beta^2)$, which is $L_Q^{(\text{NR})}$.

Using Eqs. (102), (103) and (105), we expand $L_Q^{(\lambda)}/(mc^2)$ up to $\mathcal{O}(\beta^2)$. Equation (101) then becomes

$$S_Q \approx \int dt dC \left(-\frac{\hbar^2}{8m} \frac{x''^2}{x'^4} \right). \quad (106)$$

From Eqs. (93), (94) and (106), we obtain the nonrelativistic limit of the action:

$$S \approx \int dt dC \left(-mc^2 + \frac{1}{2} m \dot{x}^2 - q\Phi + \frac{q}{c} \dot{x} A - \frac{\hbar^2}{8m} \frac{x''^2}{x'^4} \right). \quad (107)$$

Up to the expected additive constant $(-mc^2)$, this action S agrees with $S^{(\text{NR})}$, which leads to the nonrelativistic trajectory-based dynamical PDE.

5.2. Nonrelativistic limit of the momentum conservation law quantities

A similar procedure can be applied to obtain nonrelativistic limits for the density and flux quantities in the energy, momentum and C -conservation continuity equations. We start by obtaining the nonrelativistic limit of Eq. (65). Due to Eqs. (91), (97) and (100), we obtain

$$\begin{aligned} \rho_P &\approx \frac{m\dot{x}}{\sqrt{1-\dot{x}^2/c^2}} + \frac{\hbar^2}{8mc^2} \frac{\dot{x}}{\sqrt{1-\dot{x}^2/c^2}} \frac{(x'x'' - c^2t't'')^2}{(x'^2 - c^2t'^2)^3} \\ &\approx mc^2 \left[\frac{\dot{x}}{c^2} (1-\beta^2)^{-1/2} + \frac{1}{8} \left(\frac{\hbar}{mc} \right)^2 \frac{x''^2}{x'^4} \frac{\beta}{c} (1-\beta^2)^{-3/2} \right], \end{aligned} \quad (108)$$

where \dot{x} and β are defined in Eq. (91). Due to Eq. (102), the second term in the bracket of Eq. (108) is of order β^3 . Expanding $\rho_P/(mc^2)$ up to $\mathcal{O}(\beta^2)$, we have

$$\rho_P \approx mc^2 \left[\frac{1}{c} \cdot \frac{\dot{x}}{c} + \mathcal{O}(\beta^3) \right] \approx m\dot{x}. \quad (109)$$

To obtain the nonrelativistic limit of J_P , we start from Eq. (75) and set $\lambda = t$. By using Eqs. (91), (97), (100) and (102) and making the substitution $ct' \approx \beta x'$, we obtain

$$J_P \approx mc^2 (1-\beta^2)^{-5/2} \left\{ \frac{1}{4} \left(\frac{\hbar}{mc} \right)^2 \left[\left(\frac{x'''}{x'^4} - 2 \frac{x''^2}{x'^5} \right) - \frac{\beta}{c} \frac{x''\dot{x}'}{x'^4} \right] \right\}. \quad (110)$$

After similar algebraic substitutions using the limiting forms from Sec. 5.1, and subsequent reduction to second order in β , Equation (110) becomes:

$$\begin{aligned} J_P &\approx mc^2 \left\{ \frac{1}{4} \left(\frac{\hbar}{mc} \right)^2 \left[\left(\frac{x'''}{x'^4} - 2 \frac{x''^2}{x'^5} \right) + \frac{1}{c} \mathcal{O}(\beta^3) \right] \right\} \\ &\approx \frac{\hbar^2}{4m} \left(\frac{x'''}{x'^4} - 2 \frac{x''^2}{x'^5} \right). \end{aligned} \quad (111)$$

Equations (109) and (111) match exactly the nonrelativistic forms as presented in Ref. [17]. Interestingly, in the nonrelativistic limit, the continuity equation for ρ_P and J_P , Eq. (65), yields the nonrelativistic trajectory-based dynamical PDE itself.

5.3. Nonrelativistic limit of the energy conservation law quantities

Next, we obtain the nonrelativistic limit of Eq. (72). Due to the constraint in Eq. (88), we can rewrite:

$$\rho_E = \frac{1}{t_\lambda} \left[\frac{\partial \mathcal{L}}{\partial x_\lambda} x_\lambda - \mathcal{L} \right], \quad (112)$$

$$J_E = \frac{1}{t_\lambda} \left[-\tilde{J}_E + \frac{\partial \mathcal{L}}{\partial t''} t'_\lambda + \left(\frac{\partial \mathcal{L}}{\partial x'} - \frac{\partial}{\partial C} \left(\frac{\partial \mathcal{L}}{\partial x''} \right) \right) x_\lambda + \frac{\partial \mathcal{L}}{\partial x''} x'_\lambda \right]. \quad (113)$$

We note that Eqs. (112) and (113) are equivalent to the expressions of ρ_E and J_E in Eq. (68). It is straightforward to utilize Eqs. (112) and (113) to obtain the nonrelativistic limits of ρ_E and J_E , respectively. From Eqs. (78) and (86), we find that the *natural* coordinate flux \tilde{J}_E is divergence-free, i.e.,

$$\frac{\partial \tilde{J}_E}{\partial C} = 0. \quad (114)$$

Defining

$$\phi_E = \left(\frac{\partial \mathcal{L}}{\partial x'} - \frac{\partial}{\partial C} \left(\frac{\partial \mathcal{L}}{\partial x''} \right) \right) x_\lambda + \frac{\partial \mathcal{L}}{\partial x''} x'_\lambda, \quad (115)$$

we find that Eq. (113) can be expressed as:

$$J_E = \frac{1}{t_\lambda} \left[-\tilde{J}_E + \frac{\partial \mathcal{L}}{\partial t''} t'_\lambda + \phi_E \right]. \quad (116)$$

From Eqs. (114), (115) and (116), we have

$$\begin{aligned} \frac{\partial J_E}{\partial C} &= \frac{\partial}{\partial C} \left\{ \frac{1}{t_\lambda} \left[-\tilde{J}_E + \frac{\partial \mathcal{L}}{\partial t''} t'_\lambda + \phi_E \right] \right\} \\ &= \frac{1}{t_\lambda} \left[-\frac{\partial \tilde{J}_E}{\partial C} + \frac{\partial}{\partial C} \left(\frac{\partial \mathcal{L}}{\partial t''} t'_\lambda + \phi_E \right) \right] - \frac{t'_\lambda}{t_\lambda^2} \left[-\tilde{J}_E + \frac{\partial \mathcal{L}}{\partial t''} t'_\lambda + \phi_E \right] \\ &= \frac{\partial}{\partial C} \left(\frac{\partial \mathcal{L}}{\partial t''} t'_\lambda \right) + \frac{1}{t_\lambda} \frac{\partial \phi_E}{\partial C} - \frac{t'_\lambda}{t_\lambda^2} \tilde{J}_E. \end{aligned} \quad (117)$$

In the nonrelativistic limit, we set $\lambda = t$, and then we have $t_\lambda = 1$, $t'_\lambda = 0$ and $x_\lambda = \dot{x}$. Therefore, Eq. (117) becomes

$$\frac{\partial J_E}{\partial C} \approx \frac{\partial \phi_E}{\partial C}. \quad (118)$$

Thus, Eq. (64) can be replaced by

$$\frac{\partial \rho_E}{\partial t} + \frac{\partial \phi_E}{\partial C} = 0, \quad (119)$$

Specific forms are obtained as follows. Utilizing Eqs. (9) and (71), we obtain:

$$\rho_E \approx \frac{mc^2}{\sqrt{1 - \dot{x}^2/c^2}} + q\Phi(x) + \frac{\hbar^2}{8m} \frac{1}{\sqrt{1 - \dot{x}^2/c^2}} \frac{(x'x'' - c^2t't'')^2}{(x'^2 - c^2t'^2)^3}, \quad (120)$$

From Eqs. (91), (97) and (100), we have:

$$\rho_E \approx mc^2 (1 - \beta^2)^{-1/2} + \frac{\hbar^2}{8m} \frac{x''^2}{x'^4} (1 - \beta^2)^{-3/2}, \quad (121)$$

where the external potential is temporarily ignored. We expand $\rho_E/(mc^2)$ up to $\mathcal{O}(\beta^2)$. From Eq. (102), we obtain:

$$\begin{aligned} \rho_E &\approx mc^2 \left[1 + \frac{1}{2} \left(\frac{\dot{x}}{c} \right)^2 + \frac{1}{8} \left(\frac{\hbar}{mc} \right)^2 \frac{x''^2}{x'^4} + \mathcal{O}(\beta^4) \right] \\ &\approx mc^2 + \frac{1}{2} m\dot{x}^2 + \frac{\hbar^2}{8m} \frac{x''^2}{x'^4} + q\Phi(x), \end{aligned} \quad (122)$$

where the external potential has been restored in the last form above.

To obtain the nonrelativistic limit of J_E , we start from Eq. (115). By using Eqs. (9), (71) and (115), we obtain

$$\begin{aligned} \phi_E &\approx \frac{\hbar^2}{4m} \left\{ \sqrt{1 - \dot{x}^2/c^2} \left[\frac{x'\dot{x} [(x''^2 - c^2t''^2) + (x'x''' - c^2t't''')]}{(x'^2 - c^2t'^2)^3} \right. \right. \\ &\quad \left. \left. - x'\dot{x}' \frac{(x'x'' - c^2t't'')}{(x'^2 - c^2t'^2)^3} - 3x'\dot{x} \frac{(x'x'' - c^2t't'')^2}{(x'^2 - c^2t'^2)^4} \right] \right. \\ &\quad \left. + \frac{(-\dot{x}\dot{x}'/c^2)}{\sqrt{1 - \dot{x}^2/c^2}} \frac{x'\dot{x} (x'x'' - c^2t't'')}{(x'^2 - c^2t'^2)^3} \right\}, \end{aligned} \quad (123)$$

in the nonrelativistic limit. By utilizing Eqs. (91), (97), (100) and (123) and making the substitution $ct' \approx \beta x'$, we obtain:

$$\begin{aligned} \phi_E &\approx mc^2 (1 - \beta^2)^{-3/2} \left\{ \frac{1}{4} \left(\frac{\hbar}{mc} \right)^2 \left[\left(\frac{x'''}{x'^4} - 2 \frac{x''^2}{x'^5} \right) \dot{x} - \frac{x''\dot{x}'}{x'^4} \right] \right. \\ &\quad \left. + \frac{1}{4} \left(\frac{\hbar}{mc} \right)^2 \frac{1}{c} \beta (1 - \beta^2) \dot{x} \left(1 - \frac{\beta\dot{x}'}{\dot{\beta}x' + \beta\dot{x}'} \right) \right\}. \end{aligned} \quad (124)$$

Similar manipulations as in Sec. 5.2 then yields

$$\begin{aligned} \phi_E &\approx mc^2 \left\{ \frac{1}{4} \left(\frac{\hbar}{mc} \right)^2 \left[\left(\frac{x'''}{x'^4} - 2 \frac{x''^2}{x'^5} \right) \dot{x} - \frac{x''\dot{x}'}{x'^4} + \mathcal{O}(\beta^4) \right] \right\} \\ &\approx \frac{\hbar^2}{4m} \left[\left(\frac{x'''}{x'^4} - 2 \frac{x''^2}{x'^5} \right) \dot{x} - \frac{x''\dot{x}'}{x'^4} \right]. \end{aligned} \quad (125)$$

Equations (122) and (124) match exactly the nonrelativistic forms as derived previously in the Holland gauge.

5.4. Nonrelativistic limit of the C -conservation law quantities

Finally, we turn to the nonrelativistic limit for the C -conservation law. We start from Eqs. (79), (82) and (83) and take the nonrelativistic limit using a similar procedure as in the previous subsections. Using Eqs. (91), (97), (100), Eqs (82) and (83) become:

$$\tilde{\rho}_P \approx (mc^2) \frac{x'}{c} \left[\frac{\dot{x}}{c} (1 - \beta^2)^{-1/2} + \frac{q}{mc^2} A(t, x) - \frac{1}{8} \left(\frac{\hbar}{mc} \right)^2 \frac{x''^2}{x'^4} \beta (1 - \beta^2)^{-1/2} \right], \quad (126)$$

$$\begin{aligned} \tilde{J}_P \approx (mc^2) & \left\{ \frac{1}{4} \left(\frac{\hbar}{mc} \right)^2 \left[\left(\frac{x'''}{x'^3} - \frac{5}{2} \frac{x''^2}{x'^4} \right) (1 - \beta^2)^{-1/2} - \frac{x'' \dot{x}'}{x'^3} \frac{\beta}{c} (1 - \beta^2)^{-1/2} \right] \right. \\ & \left. + \frac{q}{mc^2} \Phi(t, x) - \frac{q}{mc^2} \frac{\dot{x}}{c} A(t, x) + (1 - \beta^2)^{1/2} \right\}. \end{aligned} \quad (127)$$

Expanding $\tilde{\rho}_P/(mc^2)$ to order $\mathcal{O}(\beta^2)$ yields

$$\tilde{\rho}_P \approx m \dot{x} x' + \frac{q}{c} x' A(t, x). \quad (128)$$

By balancing β orders in Eq. (79), and using Eq. (127), we obtain

$$\tilde{J}_P \approx mc^2 - \frac{1}{2} m \dot{x}^2 + q \Phi(t, x) - \frac{q}{c} \dot{x} A(t, x) + \frac{\hbar^2}{4m} \left(\frac{x'''}{x'^3} - \frac{5}{2} \frac{x''^2}{x'^4} \right). \quad (129)$$

To summarize this section: in the nonrelativistic limit, the energy and momentum conservation laws are obtained from Eqs. (64) and (65), respectively, by setting $\lambda = t$ and restricting all expressions to second order in β . We have shown that, in the nonrelativistic limit, ρ_P , J_P , ρ_E and ϕ_E are given by Eqs. (109), (111), (122) and (125), respectively. Likewise, the C -conservation law in the nonrelativistic limit is obtained from Eq. (79), where the nonrelativistic limits of $\tilde{\rho}_P$ and \tilde{J}_P are given by Eqs. (128) and (129), respectively. Finally, we compare the nonrelativistic limits of the quantities above with the corresponding quantities derived previously for the nonrelativistic theory. Provided all comparisons are made in the Holland gauge, we find the results to be *identical* in every case—apart from the expected presence of the usual constant rest energy term mc^2 , which is immaterial for nonrelativistic dynamics. We thus conclude that the relativistic energy, momentum and C -conservation laws as derived here all give rise to the correct results in the nonrelativistic limit.

6. Summary and Conclusions

As a standalone theory, the trajectory-based formulation [12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23] presents a new and vastly different perspective on quantum mechanics, in terms of which the wavefunction itself becomes utterly extraneous. Historically speaking, valid alternative perspectives have often proven quite useful in physics. A good example would be Hamiltonian vs. Newtonian mechanics, in classical physics. In the case of the trajectory-based theory, this approach offers not only a new *interpretation* of quantum mechanics, but also a new *mathematical formulation*. There are evidently also numerical benefits, as well.

On the mathematical/theoretical front, there are obvious advantages to a standalone trajectory-based quantum theory—not least of which would be the natural ability to address the *classical limit*. Comparing waves to trajectories is not so straightforward—especially since x is an independent variable in a wave theory, but a *dependent* variable in a trajectory theory. A trajectory-based formulation enables an apples-to-apples comparison, which is quite valuable. Moreover, those trajectory theories that derive from an action extremization principle are *especially* valuable, in that they provide (via Noether's theorem) a natural understanding of

symmetry and conservation laws. It should be stressed that both Bohmian mechanics [5] and the discrete version of the trajectory-based theory [20], *fail* to deliver on this score. In particular, neither of these other trajectory approaches manifests a Lagrangian with all of the proper invariance symmetries—and hence, they cannot give rise to the full set of conservation laws obtained here.

The other key feature of this work is that it specifically addresses the relativistic generalization of quantum mechanics—at least, for massive particles of spin zero. In this context, it is perhaps surprising that the trajectory-based theory differs fundamentally from Klein-Gordon—but in ways that “repair” all of the deficiencies of the latter. Note that here, too, the fact that the formulation derives from an underlying Lagrangian and action principle—rather than from an existing wave equation that is merely “translated” into trajectory form—is of vital importance. In particular, past efforts to build a quantum trajectory theory “on top of” the already-flawed Klein-Gordon foundation have been doomed to failure—resulting, e.g., in superluminal trajectories [6].

In this article, we have analyzed the invariance of the trajectory-based relativistic quantum Lagrangian of Ref. [27], with respect to its continuous translation symmetries, in order to derive the corresponding conservation laws via Noether’s theorem, in the restricted context of $1+1$ spacetime. Despite being a “1d” application, there are in fact *four* such continuous symmetries to emerge—associated with both the two independent, natural, curvilinear spacetime coordinates $(c\lambda, C)$, and also with the two dependent inertial spacetime coordinates, (ct, x) . It is the latter two that give rise to the familiar energy and momentum conservation law continuity equations. As for the natural coordinates, λ does not lead to anything new, whereas C gives rise to the less familiar “ C -conservation law.” The natural coordinate results are quite interesting, given that the non-relativistic case does not even have an analog of λ , whereas the C law—though present, also, in the non-relativistic case—is not so easy to interpret there. In the relativistic context, in contrast, we find that the C -conservation law is intimately connected with global simultaneity.

Using Noether’s theorem, we have derived four continuity equations, Eqs. (64), (65), (78) and (79). Generic expressions for the requisite density and flux quantitites, i.e. $\rho_E, J_E, \rho_P, J_P, \tilde{\rho}_P, \tilde{J}_P, \tilde{\rho}_E$ and \tilde{J}_E , are provided in Eqs. (68), (69), (80), (81), (84) and (85). Specific forms for $\rho_E, J_E, \rho_P, J_P, \tilde{\rho}_P, \tilde{J}_P, \tilde{\rho}_E$ and \tilde{J}_E , are provided in Eqs. (72), (73), (74), (75), (82), (83), (86) and (87). Equation (78) does not correspond to a *bona fide* conservation law, due to the fact that $\tilde{\rho}_E$ vanishes, as indicated in Eq. (86). The three remaining continuity equations, Eqs. (64), (65) and (79), represent the energy-, momentum- and C -conservation laws, respectively. Finally, through some effort, we were able to obtain expressions for the nonrelativistic limit for each of the relevant quantities above (i.e., $\rho_E, J_E, \rho_P, J_P, \tilde{\rho}_P$ and \tilde{J}_P) and to compare these with the results derived from the nonrelativistic theory. Working consistently within the Holland gauge, perfect agreement was obtained in every case.

Despite progress, much work remains for the future. The most obvious direction will be to consider the full $3+1$ spacetime case. Generalizations of the present conservation laws for $3+1$ spacetime are straightforward enough—indeed, many of the generic results derived here, e.g. in Sec. 3, have already been worked out for the $3+1$ case. Of the results presented here, it is the C -conservation law whose $3+1$ generalization will be most interesting—owing to the connection with global simultaneity, as discussed. In addition to the conservation laws presented here, in the $3+1$ case, there is also rotational symmetry and angular momentum conservation to contend with. Lorentz invariance symmetry will also play a nontrivial role.

On the more technical side, natural generalizations of the present theory for arbitrary C coordinates (i.e., not just the uniformizing case), and also for other gauges, remain to be worked out. In particular, the so-called “Poirier” gauge [14, 17], has the property that the corresponding action S is very closely related to the phase of the wavefunction. This property is useful for comparing trajectory- and wave-based approaches, although it has the disadvantage that

$L_Q^{(\lambda)} = Q$ is somewhat more complicated than for the Holland gauge considered here, particularly for multiple spatial dimensions. In particular, the Poirier gauge involves one higher-order spatial (C) derivative than the Holland gauge [i.e., $\mathcal{L}_Q [x^\alpha(X), \partial_\mu x^\alpha(X), \partial_\mu \partial_\nu x^\alpha(X), \partial_\mu \partial_\nu \partial_\rho x^\alpha(X); X]$], which results in higher-order expressions for the relevant Euler-Lagrange and Noether quantities.

Finally, the most important generalization of the present work—and also the riskiest—will no doubt be for spin 1/2 particles. The “risk” comes from the fact that Lagrangian-based theories are more naturally suited for continuous, rather than discrete, domains—as has already been discussed in the context of discrete “many interacting worlds” theories. On the other hand, the great advantage of such a trajectory-based “Dirac equation” is that negative energy solutions are likely to disappear entirely. For one-electron systems, negative energy solutions are a nuisance; for two or more electrons, they can be devastating [26]. A trajectory-based version that avoids negative-energy solutions entirely, then, might be of tremendous value, in the context of many-electron quantum chemistry.

Appendix A. Proof of Equations (21), (23) and (24).

In this Appendix, we prove Eqs (21), (23) and (24), starting with Eq. (21). From Eqs. (19) and (20), we have

$$\begin{aligned}\Delta x^\alpha(X^\sigma) &= y^\alpha(X^\sigma) - x^\alpha(X^\sigma) \\ &= y^\alpha(X^\sigma) - y^\alpha(Y^\sigma) + y^\alpha(Y^\sigma) - x^\alpha(X^\sigma) \\ &= -[y^\alpha(Y^\sigma) - y^\alpha(X^\sigma)] + \delta x^\alpha(X^\sigma).\end{aligned}\tag{A.1}$$

We note that

$$\begin{aligned}y^\alpha(Y^\sigma) - y^\alpha(X^\sigma) &= y^\alpha(X^\sigma + \delta X^\sigma) - y^\alpha(X^\sigma) \\ &= \left(y^\alpha(X^\sigma) + \frac{\partial y^\alpha}{\partial X^\mu} \delta X^\mu + \mathcal{O}((\delta X)^2) \right) - y^\alpha(X^\sigma) \\ &= \frac{\partial y^\alpha}{\partial X^\mu} \delta X^\mu + \mathcal{O}((\delta X)^2) \\ &= \frac{\partial x^\alpha}{\partial X^\mu} \delta X^\mu + \mathcal{O}((\delta X)^2),\end{aligned}\tag{A.2}$$

where the last step is obtained by keeping only the zeroth-order variation—i.e., $y^\alpha(X^\sigma) \approx x^\alpha(X^\sigma)$. Thus Eq. (A.1) becomes:

$$\begin{aligned}\Delta x^\alpha(X^\sigma) &= -[y^\alpha(Y^\sigma) - y^\alpha(X^\sigma)] + \delta x^\alpha(X^\sigma) \\ &= \delta x^\alpha(X^\sigma) - \frac{\partial x^\alpha}{\partial X^\mu} \delta X^\mu + \mathcal{O}((\delta X)^2).\end{aligned}\tag{A.3}$$

Therefore, up to the first order in δX^μ , Equation (21) is proved.

To prove Eq. (23), we start from

$$\Delta x^\alpha(X^\sigma) = y^\alpha(X^\sigma) - x^\alpha(X^\sigma).\tag{A.4}$$

Then

$$\begin{aligned}\partial_\mu (\Delta x^\alpha) &= \frac{\partial}{\partial X^\mu} (y^\alpha(X^\sigma) - x^\alpha(X^\sigma)) \\ &= \left(\frac{\partial y^\alpha(X^\sigma)}{\partial X^\mu} - \frac{\partial x^\alpha(X^\sigma)}{\partial X^\mu} \right) \\ &= \Delta (\partial_\mu x^\alpha).\end{aligned}\tag{A.5}$$

Therefore, Eq. (23) is proved.

To prove Eq. (24), we start from

$$\delta x^\alpha(X^\sigma) = y^\alpha(Y^\sigma) - x^\alpha(X^\sigma). \quad (\text{A.6})$$

Because

$$\begin{aligned} \frac{\partial Y^\nu}{\partial X^\mu} &= \frac{\partial (X^\nu + \delta X^\nu)}{\partial X^\mu} \\ &= \delta_\mu^\nu + \frac{\partial (\delta X^\nu)}{\partial X^\mu}, \end{aligned} \quad (\text{A.7})$$

we have

$$\begin{aligned} \partial_\mu (\delta x^\alpha) &= \frac{\partial}{\partial X^\mu} (y^\alpha(Y^\sigma) - x^\alpha(X^\sigma)) \\ &= \left(\frac{\partial y^\alpha(Y^\sigma)}{\partial X^\mu} - \frac{\partial y^\alpha(Y^\sigma)}{\partial Y^\mu} \right) + \left(\frac{\partial y^\alpha(Y^\sigma)}{\partial Y^\mu} - \frac{\partial x^\alpha(X^\sigma)}{\partial X^\mu} \right) \\ &= \left(\frac{\partial Y^\nu}{\partial X^\mu} \frac{\partial y^\alpha(Y^\sigma)}{\partial Y^\nu} - \frac{\partial y^\alpha(Y^\sigma)}{\partial Y^\mu} \right) + \delta \left(\frac{\partial x^\alpha(X^\sigma)}{\partial X^\mu} \right) \\ &= \left(\left(\delta_\mu^\nu + \frac{\partial (\delta X^\nu)}{\partial X^\mu} \right) \frac{\partial y^\alpha(Y^\sigma)}{\partial Y^\nu} - \frac{\partial y^\alpha(Y^\sigma)}{\partial Y^\mu} \right) + \delta \left(\frac{\partial x^\alpha(X^\sigma)}{\partial X^\mu} \right) \\ &= \left(\left(\frac{\partial y^\alpha(Y^\sigma)}{\partial Y^\mu} + \frac{\partial (\delta X^\nu)}{\partial X^\mu} \frac{\partial y^\alpha(Y^\sigma)}{\partial Y^\nu} \right) - \frac{\partial y^\alpha(Y^\sigma)}{\partial Y^\mu} \right) + \delta \left(\frac{\partial x^\alpha(X^\sigma)}{\partial X^\mu} \right) \\ &= \frac{\partial (\delta X^\nu)}{\partial X^\mu} \frac{\partial y^\alpha(Y^\sigma)}{\partial Y^\nu} + \delta \left(\frac{\partial x^\alpha(X^\sigma)}{\partial X^\mu} \right). \end{aligned} \quad (\text{A.8})$$

We note that

$$\begin{aligned} \frac{\partial y^\alpha(Y^\sigma)}{\partial Y^\nu} &= \frac{\partial}{\partial Y^\nu} (x^\alpha(X^\sigma) + \delta x^\alpha(X^\sigma)) \\ &= \frac{\partial X^\rho}{\partial Y^\nu} \frac{\partial}{\partial X^\rho} (x^\alpha(X^\sigma) + \delta x^\alpha(X^\sigma)). \end{aligned} \quad (\text{A.9})$$

Because

$$\begin{aligned} \frac{\partial X^\rho}{\partial Y^\nu} &= \frac{\partial (Y^\lambda - \delta X^\rho)}{\partial Y^\nu} \\ &= \delta_\nu^\rho - \frac{\partial (\delta X^\rho)}{\partial Y^\nu}, \end{aligned} \quad (\text{A.10})$$

Equation (A.9) becomes

$$\begin{aligned} \frac{\partial y^\alpha(Y^\sigma)}{\partial Y^\nu} &= \left(\delta_\nu^\rho - \frac{\partial (\delta X^\rho)}{\partial Y^\nu} \right) \frac{\partial}{\partial X^\rho} (x^\alpha(X^\sigma) + \delta x^\alpha(X^\sigma)) \\ &= \delta_\nu^\rho \frac{\partial x^\alpha(X^\sigma)}{\partial X^\rho} - \frac{\partial (\delta X^\rho)}{\partial Y^\nu} \frac{\partial x^\alpha(X^\sigma)}{\partial X^\rho} \\ &\quad + \delta_\nu^\rho \frac{\partial (\delta x^\alpha(X^\sigma))}{\partial X^\rho} - \frac{\partial (\delta X^\rho)}{\partial Y^\nu} \frac{\partial (\delta x^\alpha(X^\sigma))}{\partial X^\rho}. \end{aligned} \quad (\text{A.11})$$

By substituting (A.11) into (A.8) and keeping only the terms which are first-order in the variations, we obtain:

$$\begin{aligned}
 \partial_\mu (\delta x^\alpha) &= \frac{\partial(\delta X^\nu)}{\partial X^\mu} \frac{\partial y^\alpha(Y^\sigma)}{\partial Y^\nu} + \delta \left(\frac{\partial x^\alpha(X^\sigma)}{\partial X^\mu} \right) \\
 &= \frac{\partial(\delta X^\nu)}{\partial X^\mu} \left(\delta_\nu^\lambda \frac{\partial x^\alpha(X^\sigma)}{\partial X^\lambda} \right) + \delta \left(\frac{\partial x^\alpha(X^\sigma)}{\partial X^\mu} \right) \\
 &= \frac{\partial(\delta X^\nu)}{\partial X^\mu} \left(\frac{\partial x^\alpha(X^\sigma)}{\partial X^\nu} \right) + \delta \left(\frac{\partial x^\alpha(X^\sigma)}{\partial X^\mu} \right) \\
 &= \delta (\partial_\mu x^\alpha) + (\partial_\nu x^\alpha) \partial_\mu (\delta X^\nu).
 \end{aligned} \tag{A.12}$$

Therefore Eq. (24) is proved.

Appendix B. Proof of Eq. (39)

We start from the expression of $\Delta\mathcal{L}$ in Eq. (38). We define

$$\Lambda_\alpha^{\mu\nu} \equiv \frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_\nu x^\alpha)}. \tag{B.1}$$

And we define the terms in the bracket in the last line of Eq. (38) to be:

$$\Delta\mathcal{L}_2 \equiv - \sum_{\mu,\nu} \frac{1}{2} (1 + \delta_{\mu\nu}) \partial_\mu \partial_\nu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_\nu x^\alpha)} \Delta x^\alpha + \sum_{\mu,\nu} \frac{1}{2} (1 + \delta_{\mu\nu}) \frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_\nu x^\alpha)} \partial_\mu \partial_\nu (\Delta x^\alpha). \tag{B.2}$$

Thus,

$$\begin{aligned}
 \int d^4X \Delta\mathcal{L}_2 &= \int d^4X \left[- \sum_{\mu,\nu} \frac{1}{2} (1 + \delta_{\mu\nu}) \partial_\mu \partial_\nu \Lambda_\alpha^{\mu\nu} \Delta x^\alpha + \sum_{\mu,\nu} \frac{1}{2} (1 + \delta_{\mu\nu}) \Lambda_\alpha^{\mu\nu} \partial_\mu \partial_\nu (\Delta x^\alpha) \right] \\
 &= \int d^4X \left[- \sum_{\mu,\nu} \frac{1}{2} (1 + \delta_{\mu\nu}) \partial_\nu ((\partial_\mu \Lambda_\alpha^{\mu\nu}) \Delta x^\alpha) + \sum_{\mu,\nu} \frac{1}{2} (1 + \delta_{\mu\nu}) (\partial_\mu \Lambda_\alpha^{\mu\nu}) \partial_\nu (\Delta x^\alpha) \right. \\
 &\quad \left. + \sum_{\mu,\nu} \frac{1}{2} (1 + \delta_{\mu\nu}) \partial_\mu (\Lambda_\alpha^{\mu\nu} \partial_\nu (\Delta x^\alpha)) - \sum_{\mu,\nu} \frac{1}{2} (1 + \delta_{\mu\nu}) (\partial_\mu \Lambda_\alpha^{\mu\nu}) \partial_\nu (\Delta x^\alpha) \right] \\
 &= \int d^4X \left[- \sum_{\mu,\nu} \frac{1}{2} (1 + \delta_{\mu\nu}) \partial_\nu ((\partial_\mu \Lambda_\alpha^{\mu\nu}) \Delta x^\alpha) + \sum_{\mu,\nu} \frac{1}{2} (1 + \delta_{\mu\nu}) \partial_\mu (\Lambda_\alpha^{\mu\nu} \partial_\nu (\Delta x^\alpha)) \right] \\
 &= \int d^4X \left[- \sum_{\mu,\nu} \frac{1}{2} (1 + \delta_{\nu\mu}) \partial_\mu ((\partial_\nu \Lambda_\alpha^{\nu\mu}) \Delta x^\alpha) + \sum_{\mu,\nu} \frac{1}{2} (1 + \delta_{\mu\nu}) \partial_\mu (\Lambda_\alpha^{\mu\nu} \partial_\nu (\Delta x^\alpha)) \right].
 \end{aligned} \tag{B.3}$$

Because $\Lambda_\alpha^{\nu\mu} = \Lambda_\alpha^{\mu\nu}$ and $\delta_{\nu\mu} = \delta_{\mu\nu}$, we have

$$\begin{aligned}
 \int d^4X \Delta\mathcal{L}_2 &= \int d^4X \sum_{\mu,\nu} \frac{1}{2} (1 + \delta_{\mu\nu}) \partial_\mu [- (\partial_\nu \Lambda_\alpha^{\mu\nu}) \Delta x^\alpha + \Lambda_\alpha^{\mu\nu} \partial_\nu (\Delta x^\alpha)] \\
 &= \int d^4X \sum_{\mu,\nu} \partial_\mu \left[\frac{1}{2} \delta^{\mu\rho} (1 + \delta_{\rho\nu}) (- (\partial_\nu \Lambda_\alpha^{\rho\nu}) \Delta x^\alpha + \Lambda_\alpha^{\rho\nu} \partial_\nu (\Delta x^\alpha)) \right].
 \end{aligned} \tag{B.4}$$

Thus

$$\Delta\mathcal{L}_2 = \sum_{\mu,\nu} \partial_\mu \left[\frac{1}{2} \delta^{\mu\rho} (1 + \delta_{\rho\nu}) \left(- \left(\partial_\nu \frac{\partial\mathcal{L}}{\partial(\partial_\rho \partial_\nu x^\alpha)} \right) \Delta x^\alpha + \frac{\partial\mathcal{L}}{\partial(\partial_\rho \partial_\nu x^\alpha)} \partial_\nu (\Delta x^\alpha) \right) \right]. \quad (\text{B.5})$$

By substituting Eq. (B.5) into Eq. (38), we obtain Eq. (39). Therefore, Eq. (39) is proved.

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