Modelling quantum phenomena in the de Sitter universe

Pethybridge, Ben

Awarding institution:
King's College London

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Benjamin J. Pethybridge

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Supervised by Dr. Dionysios Anninos

Department of Mathematics
King's College London
United Kingdom

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Dedicated to G. E. Troughton

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Abstract

The de Sitter manifold is a maximally symmetric cosmological spacetime which solves the vacuum Einstein field equation with a positive cosmological constant. Physically it is a highly relevant solution to these equations of motion; de Sitter describes the asymptotic behaviour of the current phase of our universe and finds an application as the leading order behaviour of an inflationary cosmology.

Seeking to understand the microscopic degrees of freedom of gravity and matter in this spacetime, in this thesis we develop tools for quantum field theory on a fixed de Sitter background and a toy model for gravity in dS_2.

In the initial part of the thesis we consider the generic-dimensional case, and build a theoretical toolkit using ambient space methods to calculate Wightman functions for fermions and symmetric traceless tensor fields in any dimension. In the process of this exposition we also consider the representations of the de Sitter isometry group, the Bunch-Davies vacuum and the "In-In" formalism for calculating cosmological correlation functions. This serves as an introduction to the spacetime and to current literature on observables relevant to inflation. Our exploration of a toy model arises from careful consideration of classical and quantum fields on two-dimensional de Sitter in the second part of the thesis. We consider the role of a particular series of representations of the de Sitter isometry group: the discrete series of unitary irreducible representations of $SL(2,\mathbb{R})$. Focusing on theories with states and operators transforming in these representations, we are led to a highly constrained gauge theory describing non-propagating gravity coupled to matter in dS_2. We show that this theory has distinctive, discrete-series operators and a highly constrained Hilbert space. In the final section of this thesis we argue that these qualities also appear in the context of the $q = 2$ SYK model. We make a conjecture that the microscopic degrees of freedom of the bulk theory may be described in terms of this solvable quantum-mechanical model and build on the body of evidence for this relationship. We also calculate the CFT data of the SYK model for future comparison to the bulk theory. We finally comment on applications and future developments of this model, placing the work in the context of current developments in theoretical cosmology.

Contents

1. Introduction

In this thesis we explore models of quantum behaviour in de Sitter spacetime.

The de Sitter (dS) manifold is one of three maximally symmetric, Lorentzian solutions to the Einstein field equation

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu} \;, \tag{1.0.1}$$

in the vacuum case where we set the energy momentum tensor, $T_{\mu\nu} = 0$. Here $G_{\mu\nu}$ is the Einstein tensor, $g_{\mu\nu}$ is the metric, Λ is the cosmological constant and $\kappa = 8\pi G_N$ is related to the gravitational constant. The other two vacuum solutions are flat Minkowski space, and anti-de Sitter (AdS) spacetime. These three are the solutions to the (1.0.1) with no matter and a positive, zero or negative cosmological constant respectively,

$$\begin{aligned} \Lambda > 0 &: \text{ de Sitter,} \\ \Lambda = 0 &: \text{ Minkowski,} \\ \Lambda < 0 &: \text{ Anti-de Sitter.} \end{aligned}$$

Solving (1.0.1) gives the global de Sitter metric in dimension $d + 1$,

$$\frac{ds^2}{\ell^2} = -d\tau^2 + \cosh^2\tau d\Omega_d^2 \;, \tag{1.0.2}$$

where $\tau \in \mathbb{R}$ and $d\Omega_d^2$ is the d-dimensional spherical metric; ℓ is the de Sitter length

$$\ell^2 = \frac{d(d-1)}{2\Lambda} \;. \tag{1.0.3}$$

This metric covers the entire Lorentzian de Sitter hyperbola, shown in figure 1.0.1.

Of the three vacuum solutions to (1.0.1), de Sitter is the only non-static manifold: global time (τ) parameterises a foliation of changing spacelike slices. In these coordinates, the spacelike slice is a contracting and expanding d-sphere with respect to Lorentzian time (see figure 1.0.1). The non-static nature of de Sitter is the source of our motivation, as dS furnishes a very simple model of an expanding spacetime.

In this introductory section we discuss the experimental observations which motivate the study of quantum phenomena in expanding gravitational backgrounds. We then introduce the observables that have dominated literature on de Sitter and consider open problems relevant to the de Sitter observer. We describe the challenges of building quantum field theoretic and gravitating models in this context and propose a group-theoretical strategy for making progress in this research.

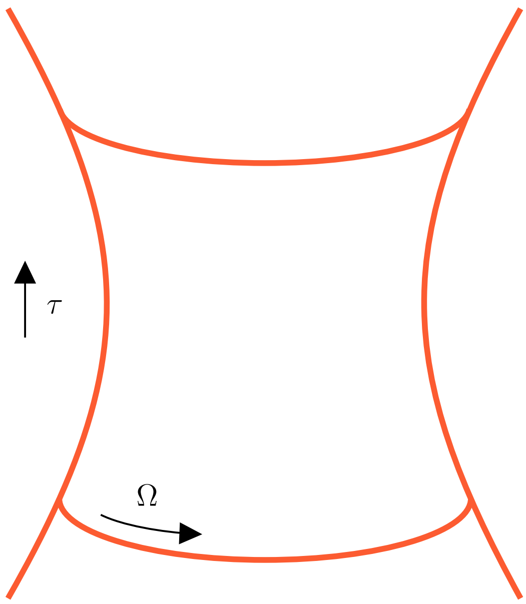

Figure 1.0.1: dS_{d+1} is a spatial d-sphere contracting and expanding with respect to Lorentzian global time τ. In this figure the dS manifold is embedded in a $d+2$-dimensional ambient space, a perspective we make use of in chapter 3 to derive propagators for tensorial and spinorial fields. The vertical direction is timelike and all but one of the spatial directions parameterized by Ω are suppressed, so that each point on the embedded manifold represents a $(d-1)$-sphere.

1.1 Experimental motivation

Quantum phenomena in de Sitter have been researched for many years [1], and the potential relevance of this spacetime to our universe has been explored since its definition [2]. But the motivation to study quantum field theory on a fixed de Sitter background and quantum-gravitational fluctuations around a de Sitter vacuum has accumulated more rapidly over the past three decades. For a large part, this is due to experimental evidence indicating the presence of a small positive cosmological constant in our universe.

The fact that the universe is currently expanding has been measured by the redshift of faraway galaxies since the time of Hubble [3]. As methods became more precise, new observations [4, 5] additionally showed that the rate of this expansion is accelerating exponentially. As the simplest model of a spacetime exhibiting accelerating expansion is realised when the effect of a positive cosmological constant dominates the effect of gravitating matter, it was concluded that de Sitter space described the leading-order behaviour of the universe. This effect is described as the "dark energy domination" of cosmic evolution.

These experiments, corroborated by further studies in the intervening years [6–11] measured the redshift of galaxies in which type 1a supernovae occurred. The studies made use of these objects' well-understood luminosity and spectra [12], to measure not only their distance but the velocity of their recession. Some of the most recent data from the DES Supernova survey [13], including ~ 1500 supernovae, is displayed in figure 1.1.1.

This information has been compared against various models of the evolution of the universe, of which the ΛCDM model is the simplest [14]. This model has contributions to the energy density of the matter in the universe (in terms of the matter energy-momentum tensor $T_{\mu\nu}$) from matter Ω_M and radiation Ω_R, as well as a contribution to the cosmological constant Λ with energy density Ω_Λ. Matching supernova data to the solutions of (1.0.1) under the simplest possible assumptions implies a negligible contribution from radiation $\Omega_R \sim 0$ and non-zero Ω_M and Ω_Λ. The value of Ω_M includes the effect of not only observable matter, but also implies the presence of "cold dark matter" (CDM) which has not been directly observed at the present time. The existence of dark matter is a fascinating puzzle about the make-up of our universe, however in this thesis we do not discuss it any further. Instead, we focus on the consequences of a small positive cosmological constant Λ. The de Sitter spacetime is the simplest version of such a spacetime, and approximates any solution for which the effect of the cosmological constant dominates that of the matter.

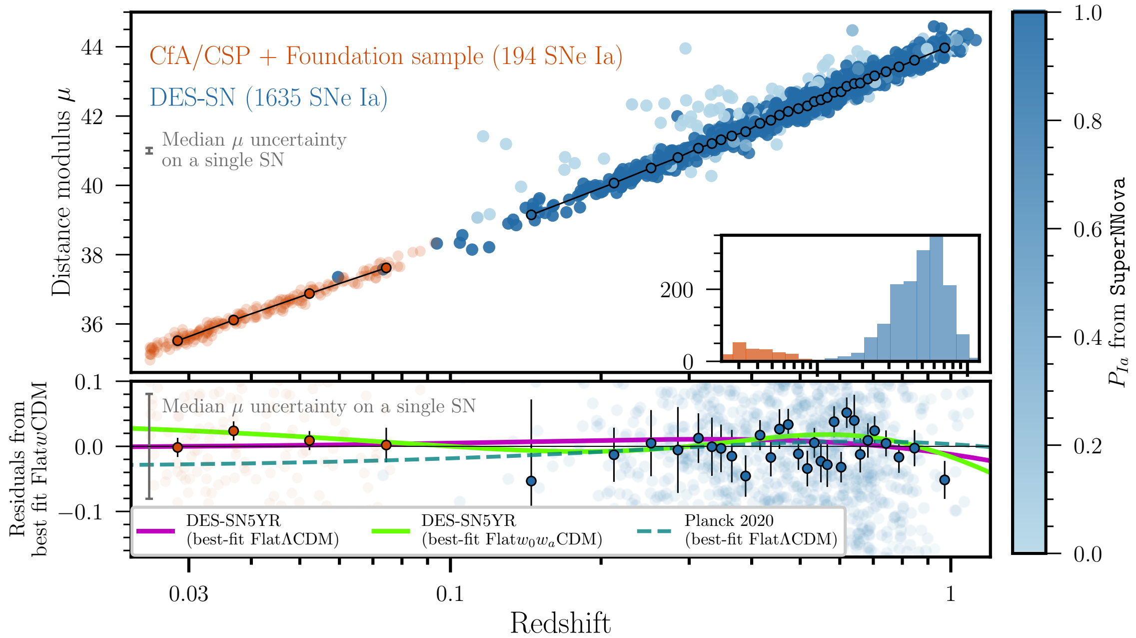

Figure 1.1.1: The Hubble diagram for the Dark Energy Survey (DES-SN5YR) [13], including data from type Ia supernovae. The upper panel measures the distance of supernovae (logarithmically via the distance modulus) with respect to redshift for the 1635 supernovae measured by DES, along with previous low-redshift data taken from [4, 5, 15–17]. The probability of them being type 1a is described by the shade of blue according to the right-hand scale. The inset is a histogram of the data with respect to redshift. The lower panel is a comparison of the data with the best-fit ΛCDM model in purple, and a related model in light green, as well as the line of best fit of ΛCDM according to data of the Planck Survey [18]. The redshift measured here is consistent with a small cosmological constant. This figure was reproduced from [13] with permission from the authors.

While measurements of supernovae remain the strongest evidence for the accelerating expansion of the universe, data has also accrued from other experimental sources. An important one, which we will explore in the next section, is cosmic microwave background (CMB) radiation. This radiation was emitted by energetic processes as the dense plasma in the early universe recombined, finally allowing transmission of electromagnetic radiation. These data include information about the accelerating expansion of the universe via baryon acoustic oscillations (BAO) [19, 20], comparisons between regions of over-dense baryon matter observed today and in the CMB. The cosmological parameters Ω_Λ an Ω_M are further constrained by the effect of the expansion on measurements of temperature fluctuations of the CMB [18] and weak gravitational lensing [21]. The latter makes use of the gravitational lensing of cosmic radiation by galaxies to map the density of matter between the source and observer. Taken together the constraints of these data on the cosmological parameters is shown in figure 1.1.2.

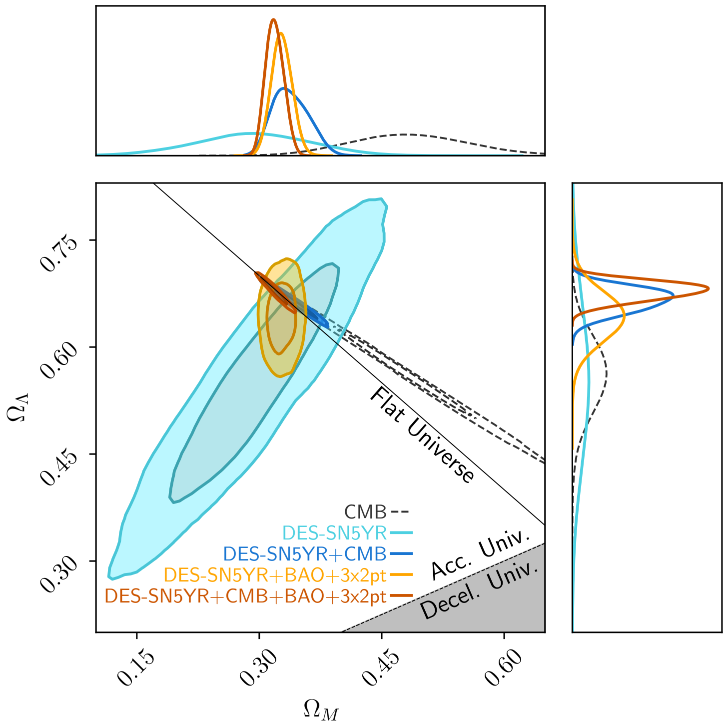

Figure 1.1.2: The intersection of supernova redshift data (light blue) [13] with that of baryon acoustic oscillations [19, 20] and weak gravitational lensing (orange and red) [21], and CMB temperature fluctuations (dark blue and red) [18]. These intersections further constrain the Ω_Λ and Ω_M parameters of the ΛCDM model. Here the possibility of non-flat spatial slicing is considered, and the areas of the domain which describe an accelerating or decelerating expansion are marked. This figure was reproduced from [13] with permission from the authors.

There is phenomenological tension between experimental results within the ΛCDM model. Both “late universe” techniques (such as supernova redshifts) and “early universe” methods (measurements of the CMB) agree on the accelerating nature of expansion. However, given the parameters of the model as derived from the supernova data [20], CMB observations [18] disagree on the *current* rate of expansion. The experiments used to generate these data are becoming more accurate and by now there is a significant statistical difference between the two values of the Hubble parameter.

A systematic error in this case would have to apply to a wide variety of data sources and protocols, consequently this has led to examination of the assumptions made in the ΛCDM model [22].

So while the assumptions and detail of this model remains the subject of scientific debate [23, 24], current experimental evidence suggests that de Sitter has relevance as a description of the accelerated expansion of our universe. We are deeply motivated to understand what the consequences of asymptotic de Sitter characteristics in a universe like ours might be. There are potentially far reaching consequences for physics at all scales, as we will discuss in the next section.

1.2 Observables in inflationary cosmology

The CMB signal fluctuates according to the thermal behaviour of the matter and radiation from which it was emitted. An important aspect of these fluctuations is that they are almost completely isotropic: in every direction, the CMB is black-body radiation with average temperature $\sim 2.725\text{K}$ [25]. This implies the universe was in thermal equilibrium when the radiation was created, placing constraints on the causality properties of the early universe. In order to explain the observed thermal equilibrium the ΛCDM model includes an "inflationary" epoch in the early development of the universe, occurring after the big bang and previous to recombination [26–29].

In this phase of its evolution the rate of expansion increased exponentially, much like in our current, dark energy dominated epoch. As a result the inflationary phase of the universe may also be approximated by de Sitter. In fact, the fluctuations of the CMB are independent of distance, rendering the CMB (almost) scale invariant. This is the hallmark of an approximately de Sitter phase in the evolution of the universe.

The models of inflation which predict these observations must include a mechanism to bring the inflationary phase to an end, as well as interacting matter and gravitational effects. As a result these models are more complicated than pure de Sitter. The simplest inflationary scenario breaks the de Sitter symmetries and includes the first order effect of gravity via the "slowly rolling" inflaton field [30]. However, a body of literature has developed around calculations of correlation functions on spacelike slices in inflationary spacetimes [31]. These functions relate to the power spectrum of the black body radiation, and the near scale invariant fluctuations of the CMB described above. Section 12 of [30] provides an interpretation of these observables in inflation, as well as a review of the calculations in Appendix C. The cosmological correlation function has also found an application as a probe of high energy processes [32]. Current developments use the large amount of symmetry in the problem to constrain inflationary observables in a similar way to the conformal bootstrap [33–38]. These efforts are described as the cosmological bootstrap, recent work has been summarised in [39]. Applications of this idea include constraining the types of particles and theories allowable in inflation [40].

There is a direct analogue of the inflationary observable in pure de Sitter spacetime, of the schematic form

$$\lim_{\eta_i \to 0} \langle \Omega | \phi(\eta_1, x_1), \phi(\eta_2, x_2)...\phi(\eta_n, x_n) | \Omega \rangle \ , \tag{1.2.1}$$

where ϕ is a generic field operator and $|\Omega\rangle$ is the Bunch Davies vacuum [41] we define below. Here, the de Sitter manifold is locally parameterised using the planar (or Poincaré) coordinate system with coordinates $-\infty > \eta > 0, x^a \in \mathbb{R}$,

$$\frac{ds^2}{\ell^2} = \frac{-d\eta^2 + dx^a dx_a}{\eta^2} \ , \tag{1.2.2}$$

such that $a = 1, .., d$ and contracted indices are summed over. This coordinate patch covers half of the de Sitter slice, and is displayed as the blue area in figure 1.2.1, the Penrose diagram for de Sitter space. The limit $\eta_i \to 0$ is that approaching the boundary in dS, a spacelike slice in the far future.

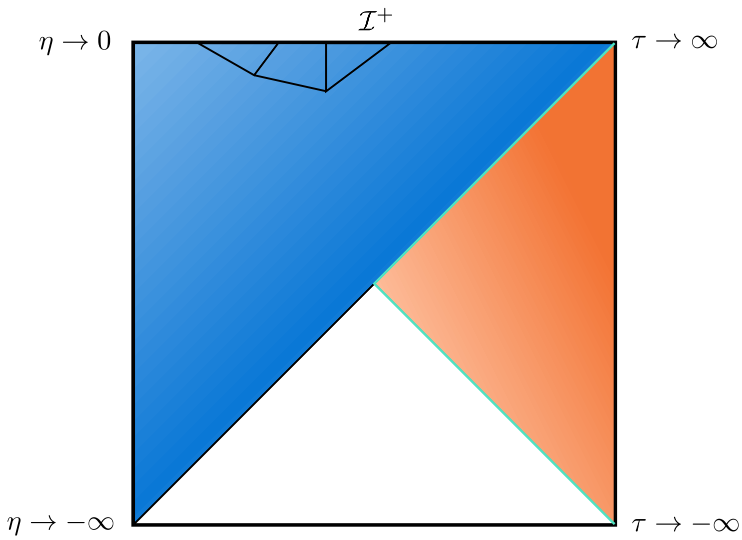

Figure 1.2.1: The Penrose diagram of conformally compactified de Sitter space in $d + 1$ dimensions. The salient features are depictions of two of the commonly used coordinate patches: Poincaré (or planar) coordinates in blue, and the static region accessible to the observer located on the right hand border in orange. The late-time slice is labelled $\mathcal{I}^+$, with a single diagram contributing to the cosmological four point function (1.2.1) displayed in analogy to Witten diagrams familiar from AdS. The cosmological horizon defining the area in causal contact with the static patch observer is displayed in light blue. All but a single spatial coordinate have been suppressed.

In a generic curved space the vacuum state will be non-unique [42], where equivalent vacua are related via the Bogoliubov transformation. In dS, there is a well defined state with respect to which late-time observables are calculated, the Bunch-Davies state, $|\Omega\rangle$. It is defined by constraining the state to the vacuum of Minkowski in the low-frequency limit. Although this is not the only choice [43, 44], it has the benefit of replicating the Euclidean vacuum on the sphere after Wick rotation [45, 46]. The de Sitter correlation function (1.2.1) is calculated with respect to this vacuum state using the in-in formalism, as reviewed in section 2.2.4, and [47]. Reviews of this formalism, as well as general quantum phenomena in de Sitter, include [48–51].

Approaching the problem of inflation from a theoretical viewpoint rather than a phenomenological one has led to a body of research concerned with calculating (1.2.1). Techniques to simplify the calculation have been developed, notably by writing the correlation functions in terms of boundary observables in Euclidean Anti-de Sitter [52–55] and utilising S-matrix methods in conjunction with the wavefunctional formalism for QFT [56–60].

The majority of this work has focused on the perturbative regime in dS, for example infra-red divergences in correlation functions have been analysed at the one-loop level [61, 62]. The next steps for boundary correlation functions in de Sitter and inflation include achieving higher loop expansions of these observables, as well as developing an understanding of the non-perturbative structures that control cosmological correlators at all loop orders. As a part of this ongoing work, unitarity constraints have been applied to the Källén-Lehman spectral decomposition [63,64] in dS, a non-perturbative decomposition of the two-point function [65, 66].

One of the applications of this thesis is the development of simple toy models in de Sitter, within which the calculation of correlation functions of the type (1.2.1) might be easier at higher orders in perturbation theory. Using this strategy, recent developments have also considered solvable models within which the non-perturbative late-time correlation functions may be attainable [67, 68].

1.2.1 Other observables in de Sitter spacetime

Correlation functions on the boundary are a relevant observable for inflation, but they cannot provide a complete description of dS for all observers [49]. In the context of inflation, the late-time behaviour of the approximately de Sitter phase is accessible as the observer looks back to information embedded on a Cauchy slice far in the past, long after the inflationary phase has passed [69]. However, if we consider the observer living on the geodesic on the right axis of figure 1.2.1, they are only in causal contact with the orange triangle of the static patch: no signal emitted in the blue area can reach them. As the late time slice, $\mathcal{I}^+$, and the correlation function signified by the diagram occur in the blue area, these observables are not measurable by the geodesic observer [70]. Instead, this part of the spacetime is measured only by the inflationary "meta-observer" looking back after the de Sitter phase has passed. The inaccessibility of $\mathcal{I}^+$ is a consequence of the cosmological horizon surrounding the observer [71]. In fact, any massive process, even those initially within the causal reach of the static patch observer will eventually pass the cosmological horizon in asymptotically de Sitter spacetime [72].

In addition, if we consider a spacetime with asymptotic de Sitter behaviour, correlation functions such as (1.2.1) are not stable against gravitational fluctuations [73]. In comparison, in the AdS case gravitational fluctuations are suppressed as they reach the boundary, leaving correlation functions of insertions of local observables invariant [74].

Alternative approaches to physical observables for cosmological spacetimes have been considered. Approaches for the static patch consider a correlation function at two points on the worldline of the observer [70,75], or the inclusion of a holographic screen between the geodesic and the horizon, described as the "stretched horizon" [76–78]. There is also the possibility that a piece of de Sitter might be embedded in a spacetime with a boundary at spatial infinity; however, for an embedding of dS in AdS this only seems possible in two dimensions [79–81].

In summary, the two regimes for which de Sitter is an approximate description of the universe provide the motivation to study quantum field theory and quantum gravity in this spacetime from a theoretical, as well as phenomenological, perspective. However, the observable relevant for inflation is dependent on the de Sitter phase coming to an end, while the dark energy domination of the universe requires a model closer to an eternal, asymptotic de Sitter state. In this sense, observations of the current epoch are closer to that of the observer on a timelike geodesic in the de Sitter static patch. As a response to the motivation presented in this section we make use of a very general approach to studying quantum phenomena in de Sitter in the following, with applications to both regimes.

1.3 Constructing models for de Sitter

In this thesis we develop concrete methods to calculate the Wightman function for spinning fields in de Sitter in any dimension [82]. We also build a model of quantum gravity in asymptotic dS_2 [83] and present the evidence for its microscopic description in terms of a quantum mechanical theory [84]. The methods we discuss, and the existence of such a toy model has applications for both the late-time and inflationary regimes discussed in the last section: answering questions on the conformal structure underlying bulk de Sitter phenomena, and providing a quantum mechanical description of a spacetime much like the one we currently inhabit.

Building models of quantum phenomena in de Sitter is complicated by the existence of a horizon, the non-static bulk, and the spacelike nature of the boundary. Here we describe a few of these intersecting puzzles, and provide the strategy used in this thesis to overcome them. From here we focus on quantum field theory in a fixed de Sitter background and gravitating, curved spacetimes with asymptotic de Sitter behaviour, rather an inflationary spacetime. Nevertheless, we will occasionally refer to these as "cosmological spacetimes".

1.3.1 Unitarity and de Sitter invariance

If we consider the global coordinate patch of (1.0.2), the shift in timelike coordinate

$$\tau \to \tau + \tau_0 \ , \tag{1.3.1}$$

is not a Killing isometry. There is no globally defined timelike Killing vector in de Sitter space: a fact which has far reaching consequences for a quantum field theory on a fixed de Sitter background. We consider coordinates, $t \geq 0$ and $0 \leq r \leq 1$ on the static patch

$$ds^2 = -(1-r^2)dt^2 + \frac{dr^2}{1-r^2} + r^2 d\Omega_{d-1}^2 \ , \tag{1.3.2}$$

where we have set the de Sitter length $\ell = 1$. This patch does not cover the entire spacetime, and the time-translation isometry of these coordinates is not timelike across the rest of the manifold.

On quantisation we choose a particular foliation of spacetime, such that the states live on spacelike slices and are evolved infinitesimally by the Hermitian Hamiltonian. The quantisation of a theory on a non-static spacetime is dependent on the coordinate system. In de Sitter the Hamiltonian can only be time-independent when supported solely in the static patch of a given observer. Quantizing instead in the planar (1.2.2) or global (1.0.2) patch (as required to examine late-time behaviour) renders the Hamiltonian time-dependent [41]. As a result the concept of energy is poorly defined in dS, and quantum mechanical processes are unconstrained by energy conservation [85, 86]. For this reason, it has been stated that quantum field theory in de Sitter is non-unitary. In this thesis, unitarity will be defined by the action of the de Sitter isometries on the states of the Hilbert space as described in [87, 88].

Concretely, the eigenbasis of the Hamiltonian ceases to be the convenient spanning set of the Hilbert space, due to its dependence on global time. However, while general time-translation invariance does not hold in de Sitter, dS_{d+1} has the maximum number of isometries (infinitesimally, killing vectors). Much like Minkowski space and AdS, in dimension $d+1$ there are

$$\frac{(d+1)(d+2)}{2}\,, \tag{1.3.3}$$

isometry generators. The isometry group for de Sitter is $SO(1, d+1)$. Imposing the invariance of a model with respect to these isometries is the analogy of imposing invariance under the Poincaré group in Minkowski quantum field theory. Much like constraints placed on QFT in flat space by Lorentz invariance and locality, invariance under the de Sitter group is a guiding principle for research on quantum phenomena in dS [89–92].

In this work, we begin by using the principle due to Wigner [93, 94], that single particle states in the Hilbert space of a quantum field theory are organised into unitary irreducible representations (UIRs) of the isometry group of the fixed background spacetime. In Minkowski this organises the particles of the standard model into massive and massless single particle excitations, with fixed (half) integer spin depending on the quantum numbers associated to the Lorentz subalgebra of the Poincaré algebra. In AdS, states are organised into single particle excitations depending on which UIR of the universal cover of $SO(2, d)$ they realise. Similarly, single particle states in dS are organised into the UIRs fo the de Sitter group $SO(1, d+1)$. This is a well understood subject within the mathematical literature. Since initial work [95–97], all of the unitary irreducible representations of the Lie algebra $\mathfrak{so}(1, d+1)$ have been classified [98–100]. A recent review of the subject, written with the application to physics in mind, is available here [101]. In order to understand representations of half-integer spin, as we do in sections chapter 2 and chapter 5, it is also necessary to include in our analysis UIRs of the double cover of the de Sitter group. The physical properties of these fields in de Sitter is a subject of ongoing study [102–105].

In this thesis, the starting point for chapter 2 and chapter 3 are two understudied representations of the de Sitter group. Firstly, we consider fermionic representations in any dimension, in order to better understand the classical, free and perturbative analysis that has been well studied for scalar fields, in the spin-half case. In chapter 3 we build models which are characterised by states that transform in the discrete series of scalar representations in dS_2. In this sense, representations of the de Sitter group are the starting point for the entirety of the research that was conducted while writing this thesis.

1.3.2 Low-dimensional semiclassical gravity

By coupling a quantum field theory to a background de Sitter spacetime it is possible to show that the static patch observer experiences a thermal bath of cosmic radiation, much like the Rindler observer in Minkowski space [42, 48, 71]. Cosmic radiation is the result of tracing over the degrees of freedom inaccessible to the observer, leaving a thermal density matrix of states in the static patch at temperature

$$T_{dS} = \frac{1}{2\pi\ell} \ . \tag{1.3.4}$$

This observation would seem to be analogous to the temperature of Hawking radiation for the black hole [106], and suggests a thermodynamic description for the cosmological horizon. Naively extrapolating the Bekenstein-Hawking formula for the entropy of the event horizon of a black hole [107, 108], the first suggestion for an entropy to associate to the cosmological horizon is

$$S_{dS} = \frac{l^{d-1}\mathrm{Vol}(S^{d-1})}{4G_N} \ . \tag{1.3.5}$$

In fact, this is justified by expanding the gravitational path integral semiclassically around the Euclideanised de Sitter saddle point solution [45, 71], the $d+1$ sphere,

$$\log \mathcal{Z}_{grav} = S_{dS} + \mathcal{O}\left(\log S_{dS}\right) \ , \tag{1.3.6}$$

where

$$\mathcal{Z}_{grav} = \sum_{\mathcal{M}} \int [Dg_{\mu\nu}] e^{-S_E[g]} \ , \tag{1.3.7}$$

$\mathcal{M}$ are compact manifolds, and $S_E[g]$ is the Einstein-Hilbert action evaluated at tree level on the Euclidean $(d+1)$-sphere. This statement picks up corrections away from the semiclassical approximation, and will be further corrected by the addition of interacting matter [77, 109, 110]. A generic model of gravity and matter in dS should include a quantum mechanical Hilbert space over which it is possible to take the trace to derive such an entropy [111].

For black holes, the question of the quantum mechanical origin of the entropy has been studied in light of the AdS/CFT correspondence for many years [112–114], and in particular it has been useful to move to lower dimensions [115–117]. Here gravity is not dynamical, simplifying the path integral calculation and allowing a clear exposition of the Hilbert space. As an example, topological gravity in AdS_2 has a pair of timelike boundaries and the entropy of the black hole in this dimension is described by the entanglement entropy of the thermofield double state [118, 119].

In spite of the topological nature of pure gravity in two dimensions, the horizon is still a feature of the spacetime for black holes [117, 120] and for de Sitter [77, 81, 121]. Interesting dynamical models which include gravity can be constructed in a number of ways: for example by coupling the theory of gravity to matter [122–126]. It is also possible to consider theories in two dimensions which are projections of dynamical four-dimensional gravity, as is the case for JT gravity [127–129]. In this way calculations performed in 2D can have consequences for the fully dynamical case of gravity in $(A)dS_4$.

Similarly, in the work presented here, we consider the low dimensional case of gravity in dS_2. We approach the topic from the Hilbert space perspective [92], as we start from the representation theory of the de Sitter group, and discover a gravitating theory while looking for realisations of the discrete series of particle representations of $SL(2, \mathbb{R})$. In this work the simpler structure of the isometry group of de Sitter in two dimensions is the first tool that we make use of in attempting to build models of quantum phenomena in de Sitter from a representation theoretical perspective.

1.3.3 Microscopic toy models

Given the success of AdS/CFT, it has been hypothesised that a generic gravitating system will have a holographic description [130]. The best understood examples of holography are dualities between gravitational theories in a bulk spacetime with asymptotic AdS_{d+1} behaviour, and conformal field theories (CFTs) on the d-dimensional boundary Minkowski space [131, 132]. Away from this context we lack perturbative or solvable examples of holographic descriptions of gravity in terms of quantum theories in one fewer dimension. de Sitter exhibits a similar structure of isometries to AdS, but a very different asymptotic structure [133]. In this sense we regard the study of holography in de Sitter as an opportunity to explore phenomena that are not easily included in the AdS/CFT paradigm, and to test the understanding that we have gained from this static laboratory.

In AdS/CFT, gravitational states in the bulk are dual to the states of the conformal field theory on the Lorentzian boundary cylinder which are evolved in the standard way by the boundary Hamiltonian of the CFT [74]. Such a CFT on a cylinder is radially quantised under a conformal transformation to the plane. Unitarity of the Lorentzian conformal generators is expressed in the Euclideanisation of a Lorentzian CFT as reflection positivity. These two properties, reflection positivity and radial quantisation, are assumed for the proof of the convergence of the OPE and the state-operator correspondence [134]. This conformal structure constrains the observables of the bulk via the correspondence with the boundary [135, 136].

On the other hand, the totally spacelike asymptotic boundary of de Sitter is accessible in the limit of the bulk timelike coordinate, $\eta \to 0$ or $\tau \to \infty$ in planar or global coordinates respectively [133]. The late-time boundary cannot host states and local operators in the same way as the AdS boundary, as there is no timelike cooordinate for which to define a Hamiltonian [51, 137]. In addition, the Euclidean conformal behaviour of de Sitter will not necessarily be described by the Euclideanisation of a Lorentzian CFT. In the dS case, the Hermiticity properties of the generators of the Euclidean conformal symmetry are derived from the bulk, rather than being wick rotated from the Lorentzian conformal unitarity conditions. As the proofs of the state operator correspondence and convergence of the OPE in CFT depend on radial quantisation and the Lorentzian unitarity conditions, the de Sitter theory is not expected to exhibit this structure [138].

The late-time de Sitter boundary is a flat, spacelike slice, and the projection of the de

Sitter isomorphisms onto this surface are the Euclidean conformal group, $SO(1, d+1)$, where d is also the dimension of the boundary. As described in section 1.2, Euclidean conformal symmetry is widely used to constrain correlation functions of local operators such as (1.2.1) in the cosmological bootstrap [39]. This inspires the simplest conjecture for de Sitter holography, which suggests a Euclidean CFT lives on the late-time de Sitter boundary, in a direct analogy to AdS/CFT [139]. In particular, the path integral of the conformal system on the boundary has been conjectured to be dual to the wavefunctional of the bulk theory in dS [31,140–142], the cosmological "wavefunctional of the universe." The interpretation of this suggestion is still an open question, and other holographic descriptions of dS, such as the stretched horizon described in section 1.3.2 are also being developed.

Moreover, the de Sitter boundary is gravitating. As a result we do not expect a holographic de Sitter theory to be defined only in terms of local observables on the late-time slice. This mirrors recent developments on the information problem in AdS/CFT which use non-local observables to tell a more complete story of bulk reconstruction, as reviewed in [143, 144]. A fully gravitating theory in de Sitter will treat the isometry group $SO(1, d+1)$ as a gauge redundancy [145]. Many approaches to achieve this gauging, with pure de Sitter as the vacuum state, have been attempted including averaging the states of the gravitational theory over the group [146–150].

In the context of these puzzles, concrete examples of correspondences between de Sitter models with gravity and quantum theories in one fewer dimension are invaluable tools for building a better physical intuition. In spite of the extant possibility that cosmological spacetimes may be included in string-related gravitational models [111, 151]. The fact that asymptotic de Sitter is yet to be found as a vacuum solution to a string theory [152, 153] has hampered progress in building effective models of quantum gravity in dS. In the AdS case, models such as type IIB string theory on $\text{AdS}_5 \times S^5$, dual to $N = 4$ super Yang-Mills in 4D [131], give concrete predictions, upon which generalisations can be formed and tested. The aim of this research is to explore a potential example of such a model of dS.

A model which does describe a de Sitter correspondence [139, 142, 154] exists in four dimensions. As expected, the conformal theory is very exotic from the perspective of the Euclidean CFTs that are familiar from the literature. Nevertheless, it was shown in [155] that the conformal theory shared a Hilbert space and observables with higher-spin gravity in dS_4. However, this bulk theory is non-Lagrangian Vasiliev higher-spin gravity [156], complicating analysis. Inspired by two-dimensional toy models of AdS/CFT discussed in section 1.3.2, we consider the two-dimensional case in order to find a similar model in de Sitter.

1.4 Structure of the thesis

This thesis was compiled as a "thesis incorporating publications" under the Kings' College London guidelines. The thesis includes large portions of the published manuscripts [82, 83], as well as the preprint [84]. The papers are edited to provide a single narrative and to enable their consistent integration into this thesis.

The thesis is split into three chapters. In chapter 2 we introduce quantum field theory in a pure de Sitter background, concentrating on the slightly underserved case of fermionic fields in dS_{d+1}. In this chapter we present [82], where we provide ambient space tools to derive propagators for massive spin-half fields in all dimensions in section 2.3, as well as clarifying the methods for bosonic symmetric traceless tensor fields in section 2.2. We also make reference to the representations of the de Sitter group in dimension $d+1$ described in section 2.1.

In chapter 3 we focus on the case of dS_2 and use representation theory as our starting point, considering the special case of representations of the isometry group $SL(2,\mathbb{R})$, which are fully described in appendix B. We present content published in [83]. We then undertake a study of the discrete series of unitary irreducible representations. We compare these representations to the well understood case of the principal series in section 3.2 and provide a model which exhibits one of the UIRs in its single particle Hilbert space. We then consider the tensor products of single particle states and comment on the appearance and importance of the discrete series in an interacting scalar theory in section 3.3. This leads us to a theory incorporating gravity in dS_2, which we show includes operators transforming in the discrete series in section 3.4. As these models have a gravitational character we make some comments about the types of observables we are able to consider in asymptotically de Sitter theories of gravity in section 3.5.

In chapter 4 we describe the $q = 2$ SYK model and make a holographic proposal for the discrete series as first published in [83]. The proposal is outlined first in section 4.1. We then develop the quantum mechanical model of interest, presenting [84]. Here the complex $q = 2$ SYK model is explored in the latter part of section 4.2 and the evidence is presented that this model shares an operator algebra and Hilbert space with that of section 3.4.2. We then calculate the three point function of the integer-weight operators in the semiclassical limit in section 4.2.2.

We make some comments about the various future directions for development of the model, as well as the next steps to build evidence for the holographic relationship in chapter 5. We note that the conventions for the thesis are stored in appendix A.

2. dS_{d+1}

In this chapter we study efficient computational methods for spinorial and tensorial quantum fields in a pure de Sitter background. In particular, we develop the ambient space formalism for these fields in any dimension, with positive m^2. This places the fields in the complementary or principal series of representations of the de Sitter algebra, as discussed in section 2.1. In this chapter we calculate the Wightman function for scalars, symmetric traceless tensors (STTs) and fermions and briefly discuss the application to the in-in formalism.

It was recognised to be computationally convenient to treat field theory from the perspective of an ambient space with an embedded dS slice since the earliest work on the subject by Dirac [1, 157]. Lifting fields to a higher dimensional space, to linearise the action of the isometry group and manifest the maximal symmetry, has been developed thoroughly as a technique in the context of flat-space with conformal symmetry, as well as in Euclidean AdS [158–162]. This formulation fostered new results in holography and greatly improved our conceptual understanding of QFT in curved spacetime [74]. Well-honed methods have appeared to deal with arbitrary tensor representations [136, 163], as well as spinors in odd dimension [164]. Most importantly, it has become a commonplace tool to evaluate Witten diagrams in AdS_{d+1}.

The ambient space picture arises naturally, given the realisation that the isometry group of dS_{d+1} is $SO(1, d+1)$. In $\mathbb{R}^{1,d+1}$ there are three classes of Lorentz-invariant submanifold, defined by points $P^A \in \mathbb{R}^{1,d+1}$ with $P^A P_A = -1, 0, +1$: respectively corresponding to EAdS_{d+1}, the projective lightcone (i.e. the embedding of CFT_d), and dS_{d+1}. Each class of submanifold inherits a group action from the higher dimensional space, defined by the push forward of the isometry generators of $SO(1, d+1)$. The benefits are numerous, for the most part one can avoid specific choices of a coordinate chart and differential operations become easy to write down, manipulate, and compute. In addition, the notion of the boundary limit appears naturally by identifying points at the boundary of these spaces with points on the projective lightcone. One considers points on the submanifold $X^A X_A = 1$, parameterised in

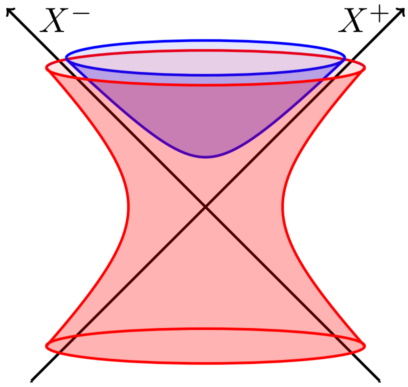

Figure 2.0.1: $EAdS_2$ (blue) and dS_2 (red) realised as Lorentz-invariant subset of $\mathbb{R}^{1,2}$, here represented with Lightcone axes $X^\pm = X^0 \pm X^2$.

terms of null vectors, $X^A = \lambda P^A + \frac{1}{\lambda} \dots$ with $P^A P_A = 0$. The limit $\lambda \to \infty$ gives insertions on the lightcone, which are identified with operators in a CFT_d. These operators are naturally classified by their representation of $SO(1, d+1)$, and therefore specified by a complex weight Δ, and a representation of $SO(d)$. As discussed in section 1.3, the representation theory of the de Sitter group is the natural starting place for a study of de Sitter quantum field theory. We list the possible representations in section 2.1.

The embedding picture for dS has received less attention than that of AdS. However, it has surfaced in calculations for bosonic quantities [53, 165–167], for both integer and half-integer spin in the specific context of dS_4 [168], and appeared as part of a wider effort to understand massless fields of general spin in maximally symmetric spacetimes [169, 170]. It has also been used to consider the general properties of interacting two-point functions in [171, 172]. The goal of this chapter is foremost to give a self-contained, dS oriented, entry point to these methods, with a view toward future applications.

We also develop the ambient space approach for massive spinors in general dimensions and derive their Wightman function. The equivalent construction in (E)AdS has only been performed in odd dimension in the embedding [164], or in a specific coordinate system [173]. We provide a convenient calculational toolkit for any integer or half-integer spin, while also contributing to the literature on fermionic fields in de Sitter.

The topic of the propagator of a spin $\frac{1}{2}$ field in dS has received some attention previously. After an initial attempt within the ambient space formalism by Dirac [1], the calculation of the propagator for Dirac fermions in de Sitter has mostly been attempted using mode decomposition in the tetrad formalism. Work on the subject has been reported in [174–177], with which our results might be compared. Relevant calculations for spinors on Euclidean, maximally symmetric surfaces have been considered in [178–180], as well as in (E)AdS [173, 181] and CFT [160, 182]. The main result of this chapter is a concise expression for the Wightman function of Dirac spinors in dS_{d+1} in terms of the de Sitter invariant distance. We also give coordinate expressions for this two-point function in planar and global coordinate systems.

Outline

The plan of the chapter is the following. We first list the possible representations of the de sitter group relevant to the case of dS_{d+1} in section 2.1. We then review the construction of the bosonic two-point function using the ambient space formalism in section 2.2.3, analogously to that of scalars [48] which is available in (2.2.16). This allows us to discuss the different charts one can use to cover the slice in (2.2.2) and (2.2.3), as well as the analytical properties of the Wightman functions, their interpretation in perturbation theory in section 2.2.4, and the late-time limit of operator insertions in section 2.2.5. We transpose the work of [161] for spinning operators to the dS slice in appendix D, including the Wightman function for the symmetric traceless case.

We then consider the uplift of Dirac fermions in section 2.2.3. We construct these using a method inspired by [158, 183], matching the transformation of the spinorial fields in the ambient space and on the slice, written explicitly in (2.3.21) to (2.3.24). We constrain ambient spinors to obtain irreducible dS spinors, and showcase a formalism unifying this analysis in both even and odd dimensions with the final constraint given by (2.3.25). We then compute the Wightman function of Dirac spinors, by uplifting the Dirac equation and solving it in (2.3.52) and (2.3.53). Explicit expressions are given in this section for planar coordinates, and in appendix E for global coordinates. We close with a discussion of our results in section 2.4 and future directions.

2.1 Representation theory of $\mathfrak{so}(1, d+1)$

In this section we collect some results regarding the representation theory of the de Sitter algebra , $\mathfrak{so}(1, d+1)$ in the case $d \geq 2$. For a more thorough treatment, one should refer to the canonical work [99], or to the recent review of the subject [101]. Spinorial representations are only lightly touched in the literature, we present some interesting elementary statements regarding them from [184, 185], a more recent review can be found in [105].

The representation theory of de Sitter has some similarities to the familiar examples of conformal field theory and (E)AdS. The operators we are concerned with transform in finite dimensional representations of the de Sitter group, note that these cannot be unitary representations due to their finite dimensionality. In this chapter we consider symmetric traceless tensors of generic integer spin s, and spin $s = \frac{1}{2}$ representations of the universal cover of the ambient space Lorentz group, which acts as the de Sitter group once pulled to the de Sitter slice. The action of this group is described in the ambient space formalism in [165].

As we discuss in section 1.3.1, the spectrum of single particle excitations of the fields should transform in unitary irreducible representations of the de Sitter group. To induce the irreducible unitary representations we consider the maximal subgroup $SO(1, 1) \times SO(d) \in SO(1, d + 1)$, and label representation using label (Δ, ρ) with ρ a representation of $\mathfrak{so}(d)$. Unitarity imposes a complex interplay between the spin representation and the allowed values of Δ. These organise into two continuous series and two discrete [186]. For clarity we treat the symmetric traceless tensors and the spinorial cases separately.

2.1.1 Symmetric traceless tensors

Irreducible representations of $\mathfrak{so}(d)$ are specified by a weight-vector $\vec{s} = (s_1, s_2, ...s_r)$, of dimension $r = \lfloor \frac{d}{2} \rfloor$, with half-integer entries $s_i \in \frac{1}{2}\mathbb{N}$ [187]. In the bosonic case $\vec{s}$ defines a Young tableau with rows of length $s_1 \geq s_2 \geq .... \geq s_r$. Symmetric Traceless Tensor (STT) representations correspond to Young tableaux with one row and $s \in \mathbb{N}$ boxes, i.e. $\vec{s} = (s, 0, ..., 0)$. Let p be the number of non zero entries in $\vec{s}$, so that only the spin 0 case is distinguished. The induced unitary irreducible representations (UIRs) are entirely specified given the spin representation and the value of Δ. The allowed values of Δ for a given STT representation decompose into the following series :

- Principal series : $\Delta = \frac{d}{2} + i\nu$, $\nu \in \mathbb{R}^+$,
- Complementary series : $\Delta \in (\frac{d}{2}, d-p)$,
- Exceptional series I : for $d+1 = 2n$, $s = 0$ we have $\Delta = d + k$ such that $k = \mathbb{N}$.
- Exceptional series II: $\Delta = t + d - 1$ with $\Delta(d-\Delta) + s(d+s-2) = (s-1-t)(d+s+t-3)$ for $t \in 0, 1, ..., s-1$,

For the principal and complementary series, we additionally identify representations with weight Δ to those with weight $d - \Delta$. These are unitary in de Sitter as well as equivalent to those of weight Δ, under the intertwining isomorphism given by the 'shadow transformation', which is discussed in [87, 101, 188]. For the exceptional series the shadow transformation is not an isomorphism and so there is an additional UIR for each of the values of Δ listed above for these cases.

The conformal dimension Δ can be related to the mass of the fields and states in de Sitter. The most natural choice of the mass parameter for various representations in de Sitter is the subject of [189], and leads to a description of unique representations in the case of de Sitter for higher spin fields. In particular there exist 'massless' and 'partially massless' representations, associated with the exceptional series. These are discussed at length in [46, 155, 187, 190–192]. These are unitary, in the above sense, and correspond to discrete values of the mass between the lower bound of the complementary series and $m = 0$. The partially massless representations have an intermediate amount of gauge freedom, interpolating the massless and massive cases. In this chapter we do not consider either of the exceptional series, and therefore focus on strictly massive STTs such that

$$m^2\ell^2 \equiv \Delta(d-\Delta) + s(d+s-2) > 0 \ , \tag{2.1.1}$$

where ℓ is the de Sitter length.

2.1.2 Spinors

In dimension $d+1 \geq 3$, local spinor fields transform in the familiar spin $\frac{1}{2}$ representations of the local Lorentz group [193]. They may be constructed by demanding they solve the Dirac equation with the appropriate spin covariant derivative [189].

The Dirac equation imposes a specific form of the mass term in the Laplacian eigenproblem [174, 183, 191]

$$(\nabla\!\!\!/ - m)\psi = 0 \implies \left(\Box_{dS} - \left(m^2 + \frac{R}{2}\right)\right)\psi = 0\,. \tag{2.1.2}$$

From which it follows that, in terms of the UIRs, we can only find

- Principal series : $\Delta = \frac{d}{2} - im\ell$
- Exceptional series : $\Delta = \frac{d}{2}$.

Complementary series single particle UIRs of the de Sitter group are excluded from the spectra of half-integer spin representations of the fields [105, 184, 185]. This nontrivial point follows from the observation that the fermionic UIRs, induced by those of the double cover of the compact subgroup, $Spin(d)$, are incompatible with the positivity of the intertwining operator, however this positivity is required to construct a unitary inner product [87]. This follows from the general statement made in [185], that positivity requires the inducing representation of $Spin(d)$ to be equivalent to its Weyl conjugate. In the case $d = 2p$, this excludes the half-integer spin representations of $Spin(d)$, which are necessarily chiral. In the case, $d = 2p + 1$, faithful (injective) representations of $Spin(d)$ are also excluded from the complementary series, by the requirement that the highest weight state must have a final entry with integer values. The injective representations of $Spin(d)$ are precisely those of half-integer spin, as the bosonic representations are double valued.

The works previously cited include an analysis of higher rank spinor tensors and introduce the possibility of partially massless tunings for m for these fields, we leave this topic for later analysis. Representation theory and no-go theorems for these cases is the subject of [102–105].

2.1.3 A note on the dimension

The results of this chapter, in particular the Wightman function, of the STT and the spinor are valid in all dimensions for massive fields. However the representation theory presented in this section is valid only for $d \geq 2$. In the two dimensional case, which we describe in appendix B, there is a single discrete series instead of the two exceptional series listed for bosonic fields, this is the subject of study in chapter 3. For purely fermionic fields ($s = \frac{1}{2}$) the low-dimensional cases $d = 1, 2$ are exceptional [101, 105], in particular for $d = 1$ there exists a fermionic analogue to the discrete series in dS_2. However, the results of this chapter are for the strictly massive, principal series only. It is possible to show they can be extended to principle series fermions in $d = 1, 2$.

2.2 Embedding methods for tensor fields

In this section we review the basic tools of the ambient space, developed thoroughly for EAdS_{d+1} symmetric traceless tensors by [161], adapted to dS_{d+1}. Our aim is to set the logic and notation, and give a survey of the existing literature. The discussion of local coordinates and the scalar propagator is a classic topic, see for example [48, 49]. The analytic structure of the Wightman function and its link to the in-in formalism is explained in its full generality and details in [47].

2.2.1 Local coordinates on the dS slice

Consider dS_{d+1} as a submanifold of $\mathbb{R}^{1,d+1}$ Setting the de Sitter length (1.0.3) $\ell = 1$, it corresponds to points X^A satisfying[1]

$$X^2 = -X^+X^- + X^aX_a = X_{d+1}^2 + X^\mu X_\mu = -X_0^2 + X^iX_i = 1\,. \tag{2.2.1}$$

All our conventions can be found in appendix A. In what follows X will generically denote an ambient space vector field satisfying such a constraint P will be a generic ambient space vector. We obtain a dS-foliation of the spacelike region of the ambient space by multiplying X^A by a real number $\ell > 0$. From here we work exclusively in the regime with fixed unit de Sitter length, $\ell = 1$. An example of a parametrisation of the de Sitter slice is given by conformally flat (planar) coordinates, the analogue of Poincaré coordinates in AdS

$$X^A = (X^+, X^-, X^a) = \frac{1}{\eta}(1, x^2 - \eta^2, x^a)\,. \tag{2.2.2}$$

Note $x^a \in \mathbb{R}$ and we choose $\eta < 0$, where η increases from $-\infty$ towards 0^+. This patch covers the causal future of an observer sitting at the origin in the far past, and therefore includes the late-time slice. These coordinates are usually most convenient for explicit calculations. Another interesting choice are global coordinates

$$X^A = (X^0, X^i) = (\sinh\tau, \omega^i \cosh\tau)\,, \tag{2.2.3}$$

[1] We use indices $A, B, ... = 0, 1, ...d+1$ or $A, B, ... = +, -, 1, ...d$ for light-cone coordinates. Lower case letters specify the ranges $a, b, .. = 1, 2, ...d$ for latin characters at the beginning of the alphabet, $i, j, ... = 1, 2, ...d+1$ for those in the middle and, $\mu, \nu, ... = 0, 1, ...d$ as usual for greek indices. Contractions between indices will be performed with the Minkowski metric ($\eta_{\mu,\nu}$, η_{AB}) or the Euclidean metric (δ_{ab},δ_{ij}).

Where ω^i parameterise a spacelike S_d and $\omega^i \omega_i = 1$. These coordinates cover the entire slice as described in (1.0.2).

The formalism of this chapter avoids the necessity of choosing a coordinate patch. However, we make use of the planar patch to make our construction explicit and prove some coordinate independent results in section 2.3, appendix E contains the specialisation of our methods to global coordinates.

More generally, local coordinates x^μ chart coordinate patches on the slice, providing a map $X^A(x^\mu)$. From this map, we can apply the machinery of differential geometry of a submanifold, greatly simplified since the ambient space is flat [194]. One can define frame-fields ${e_\mu}^A \equiv \frac{\partial X^A}{\partial x^\mu}$, which define the push-forward of ambient tensors to the slice. For instance, we recover the planar metric of (1.2.2)

$$ds^2 = \frac{-d\eta^2 + dx_a dx^a}{\eta^2} = \frac{dx_\mu dx^\mu}{x_0^2} \,. \tag{2.2.4}$$

Note that, to simplify notation, the lower case indices of the slice are all $SO(1,d)$ tangent space indices, and are therefore contracted with the Minkowski metric. To recover the spacetime quantities these should be contracted with the ordinary tetrad (as opposed to the "frame field" we have just defined). Though the use of specific coordinates is practically necessary to perform computations intrinsically, the main advantage of the embedding picture is that for almost all computations, we do not need to choose a specific parameterisation. One can work only with embedding objects, which are in one-to-one correspondence with local objects on the slice.

2.2.2 Tensor fields and differential operators

We illustrate the construction using tensor fields. First, notice that ${e_\mu}^A X_A = 0$, hence all longitudinal components of a tensor have zero projection on the dS slice. This redundancy can be fixed by considering only transverse tensors. From this, we define the uplift of tensorial operators on dS as transverse tensor fields in the ambient space.

For example, a Symmetric Traceless Tensor (STT) operator $T_{A_1 \dots A_l}(X)$ in the ambient space is a dS tensor of same rank and symmetry properties, provided

$$X^{A_i} T_{A_1 \dots A_l}(X) = 0 \,.$$

The uplift of the induced metric

$$G_{AB} = \eta_{AB} - \frac{X_A X_B}{X^2} = \eta_{AB} - X_A X_B\,, \tag{2.2.5}$$

defines a projector to the slice. dS objects are formed by contracting indices with G_{AB}. Note the change of sign in (2.2.5) with respect to the result in EAdS [161]. Contracting all indices with G_{AB} defines differential operators on the slice. For example the covariant derivative

$$\nabla_A = G_A{}^B \frac{\partial}{\partial X^B} = \partial_A - X_A X^B \partial_B\,, \tag{2.2.6}$$

transforms only between objects defined in dS. It acts on a generic tensor field $T_{A\ldots}$ as

$$\nabla_A T_{B\ldots} = G_A{}^{A'} G_B{}^{B'} \ldots \frac{\partial T_{B'\ldots}}{\partial X^{A'}}\,. \tag{2.2.7}$$

In practice, indices should be avoided when possible, and an index-free formalism is favoured here. This is done by considering STT operators as scalar, homogeneous polynomials in a dummy transverse and null vector variable W^A. i.e. given $T_{A_1\ldots A_l}(X)$, we take $T(X,W) \equiv W^{A_1}\ldots W^{A_l} T_{A_1\ldots A_l}(X)$, with polarisation vector W^A such that $W \cdot X = W^2 = 0$.[2] The first condition ensures that W^A is a dS vector, the second simplifies all traces. One can check that together the conditions on W^A imply that for any choice of parameterisation x^μ, $W^A \equiv w^\mu e_\mu{}^A$, with $w^\mu w^\nu g_{\mu\nu} = 0$. In a sense, polarisation takes care of the projection, if $t_{\mu_1\ldots\mu_l}$ is the push forward of $T_{A_1\ldots A_l}$, then $T(X(x), W(x,w)) = t(x,w)$. Having contracted all indices, it can be necessary to free them once again. This is done by using a vector differential operator of homogeneous degree -1 in W, whose image only contains STT tensors, and preserves the constraint $X \cdot W = W^2 = 0$. These choices uniquely fix it to be the uplift of the Todorov operator, K_A defined by

$$K_A \equiv \left(\frac{d-1}{2} + W \cdot \frac{\partial}{\partial W}\right)\left(\frac{\partial}{\partial W^A} - X_A X \cdot \frac{\partial}{\partial W}\right) - \frac{W_A}{2}\left(\frac{\partial^2}{\partial W \cdot \partial W} - \left(X \cdot \frac{\partial}{\partial W}\right)^2\right)\,. \tag{2.2.8}$$

The Todorov operator has been used in the CFT literature [160], and also appears in conformal geometry as the Thomas operator [196].[3] In the index-free notation, the

[2] One can extend the index-free formalism to mixed-symmetry tensor in a straightforward way following [195], by using multiple polarisation vectors and imposing the symmetry of the Young tableau, by hand or using grassmanian variables.

[3] One should note that subsequent freeing and contraction of indices returns the same object up

covariant derivative is slightly modified, as an explicit computation shows

$$\nabla_A T(X,W) = \left(\frac{\partial}{\partial X^A} - X_A X \cdot \frac{\partial}{\partial X} - W_A X \cdot \frac{\partial}{\partial W}\right) T(X,W)\,. \tag{2.2.9}$$

2.2.3 Propagators

We now consider the two-point function of free bulk scalar fields ϕ, $\Pi(X,Y) = \langle\phi(X)\phi(Y)\rangle$ with respect to a de Sitter invariant vacuum state. We choose the Bunch-Davies vacuum through appropriate boundary conditions. We derive the Wightman function, which obeys the homogeneous equations of motion. Lorentz invariance of the ambient space forces this function to depend only on the geodesic distance between points

$$u(X,Y) = \frac{(X-Y)^2}{2} = 1 - X\cdot Y. \tag{2.2.10}$$

In this chapter we use the slightly more convenient form

$$z(X,Y) = 1 - \frac{u}{2} = \frac{1}{2}(1 + X\cdot Y). \tag{2.2.11}$$

Therefore $z = 1$, $z > 1$, $z < 1$ implies the points are seperated by null, timelike or spacelike geodesics respectively, $z = 0$ implies X is null with respect to the antipodal point of Y [48, 49]. We rewrite the Casimir eigenvalue problem as an ODE for $g(z) = \Pi(X,Y)$, where $\mathcal{C}$ defines the Casimir operator of the de Sitter group with the appropriate differential action on the field

$$(\mathcal{C} - \boldsymbol{\Lambda})\Pi(X,Y) = 0\,. \tag{2.2.12}$$

The action of $\mathcal{C}$ and the eigenvalue $\boldsymbol{\Lambda}$ is fixed by the representation of $SO(1,d+1)$. The uniqueness of the Laplacian as a second order differential operator commuting with the isometries of the de Sitter slice, implies the Klein-Gordon and Casimir equation are interchangeable. This is true up to a constant change in the eigenvalue for spinning fields [165], as we discuss in appendix D.

to a spin-dependent constant. This can be checked in an example

$$W\cdot K(W\cdot V)^l = l\left(\frac{d+1}{2} + l - 2\right)(W\cdot V)^l\,.$$

The de Sitter isometries are linearly expressed in the ambient space, where the Casimir operator takes the form

$$\widehat{\mathcal{C}} = -\frac{1}{2}\widehat{L}_{AB}\widehat{L}^{AB} \,. \tag{2.2.13}$$

The operators $\widehat{L}_{AB}$ generate the Lorentz group in the ambient space, through a differential realisation $\mathcal{L}_{AB}$ and possible spin parts on the various fields. Throughout the thesis we will use a hat $\hat{}$ to denote abstract operators, all operators without the hat will be defined with respect to the representations on which they act. For the scalar case the eigenvalue is fixed by the conformal weight and dimension of the space $\mathbf{\Lambda} = \Delta(\Delta - d)$ and we only need

$$\mathcal{L}_{AB} = X_A \partial_B - X_B \partial_A \,. \tag{2.2.14}$$

The Wightman function must solve the equation of motion given in (2.2.12). The hypergeometric differential equation [197] is found by changing variable to z,

$$(1-z)zg''(z) + \frac{d+1}{2}\,(2z-1)\,g'(z) + \Delta(\Delta - d)g(z) = 0\,; \tag{2.2.15}$$

Generic solutions are given by linear combinations

$$g(z) = \kappa \, {}_2F_1\left(\Delta, d-\Delta; \frac{d+1}{2}; z\right) + \tilde{\kappa} \, {}_2F_1\left(\Delta, d-\Delta; \frac{d+1}{2}; 1-z\right).$$

We must consider the analytic structure of this function. First of all, note that both functions diverge at the coincident limit, while the second one diverges also at antipodal points $X = -Y$. This latter singularity is screened behind the dS horizon. We consider Wightman functions with singularities only at coincident points, thus setting $\tilde{\kappa} = 0$. This choice of boundary condition selects the Bunch-Davies vacuum from the continuum of possible vacua, often referred to as α vacua in the literature [43, 44], by design this choice also allows causality to be preserved.

Finally, we must fix the normalisation. Noting that in the coincident limit, the Wightman function is blind to the curvature of space, we match the normalisation of the flat-space result [139]

$$g(z) = \frac{\Gamma(\Delta)\Gamma(d-\Delta)}{(4\pi)^{d+1/2}\Gamma\left(\frac{d+1}{2}\right)} \, {}_2F_1\left(\Delta, d-\Delta; \frac{d+1}{2}; z\right). \tag{2.2.16}$$

We refer to the Wightman function obtained from this choice of boundary condition as the Hadamard form of the two-point function.

In Euclidean signature, this would be the final answer. However, rotating to Lorentzian time, the hypergeometric function develops a branch-cut along time-like separations. We must specify how to evaluate this, in a way that specifies the time-ordering of the different points. This subtlety is slightly obscured in this formalism for which $\Pi(X,Y) = \Pi(Y,X)$. The $i\epsilon$ prescription must be brought back by hand when needed, usually at the coordinate level. We describe this process in section 2.2.4, where we briefly recapitulate the in-in formalism for perturbation theory in de Sitter [46, 47].

For the purposes of review we reserve the case of the STT propagator for appendix D, the equivalent calculation in EAdS is performed in [161]. For tensorial correlation functions, one must sum over all allowed structures, which involves a choice of basis. For example in the case of the spin-1 Wightman function, we have equivalent possible bases

$$\begin{aligned}\Pi(X,W_1;Y,W_2) &= (W_1\cdot W_2)g_0(u) + ((W_1\cdot Y)(W_2\cdot X))g_1(u)\\ &= (W_1\cdot W_2)f_0(u) + ((W_1\cdot\nabla_X)(W_2\cdot\nabla_Y))f_1(u).\end{aligned} \tag{2.2.17}$$

Which can be generalised to higher spin. When we perform the computation for spinors, we will similarly find a sum of allowed structures multiplied by scalar functions.

2.2.4 The perturbative prescription

The observables most regularly calculated in de Sitter are insertions of operators on a single time slice with respect to the Bunch-Davies vacuum as we considered in (1.2.1). We assume the theory approaches the free theory far in the past, where the fields act on the Bunch-Davies vacuum as in Minkowski spacetime. Here we use t to represent either Planar, η, or global, τ, time. Splitting the Hamiltonian between the quadratic part, H_0, and the interacting Hamiltonian, H_{int}, we seek time evolution operators $U(t,t_0)$ which satisfy

$$\mathcal{Q}(t) = U^\dagger(t,t_0)\mathcal{Q}_I(t)U(t,t_0)\,. \tag{2.2.18}$$

Where $\mathcal{Q}_I(t)$ is an operator including only the field insertions which evolve with respect to H_0 and therefore propagate as free fields. The operator $U(t,t_0)$ solves the same ODE as in the Minkowski case, and performing the similar calculation for the inverse leaves us with

$$U(t,t_0) = T\left\{\exp\left[-i\int_{-\infty^+}^{t} dt_r H_I(t_r)\right]\right\} \tag{2.2.19}$$

$$U^\dagger(t,t_0) = \bar{T}\left\{\exp\left[i\int_{-\infty^-}^{t} dt_l H_I(t_l)\right]\right\}. \tag{2.2.20}$$

H_I is the evolution, with respect to H_0 of H_{int} and the $\bar{T}$ implies the expansion should be anti-time ordered, with the insertions of operators placed in order of increasing t from left to right. We have now taken the limit and prescription $t_0 \to -\infty(1 \pm i\epsilon) = -\infty^\pm$, for the time evolution operator and inverse respectively. This implies the following $i\epsilon$ prescription for t

$$t_l \to t_l(1+i\epsilon)\,, \qquad\qquad t_r \to t_r(1-i\epsilon)\,. \tag{2.2.21}$$

From the expressions above, Wick contraction will require three different propagators, all of which may be derived from the Wightman function. In planar coordinates, we may rewrite the prescription in terms of the invariant length z and $\alpha = \mathrm{sgn}(\eta_l - \eta_r)$. Labelling contractions of fields with l/r for inclusion in T and $\bar{T}$ respectively, we require the propagators

$$\Pi_{ll}(z) = g(z + i\epsilon) \tag{2.2.22}$$

$$\Pi_{rr}(z) = g(z - i\epsilon) \tag{2.2.23}$$

$$\Pi_{lr}(z) = g(z - i\alpha\epsilon) \tag{2.2.24}$$

In general the prescription for z will be parameterisation dependent. A coordinate independent formulation seems slightly out of reach, although we note that the prescription for z is the same for global and planar coordinates. This formalism is reviewed and used in many places, among which notable recent examples include [53, 54, 62].

2.2.5 Boundary limit

From the embedding picture, the discussion of the boundary limit of operator insertion is greatly streamlined. The starting point is to choose a specific limiting geometry, by fixing a choice for the projective null vector P^A corresponding to boundary insertions. One can parameterise a generic point on the dS slice X using a pair of null vectors,

P and Q such that $P^2 = Q^2 = 1 - 2P \cdot Q = 0$ and a single variable λ

$$X^A = \lambda P^A + \frac{1}{\lambda} Q^A \tag{2.2.25}$$

The boundary limit is achieved by taking $\lambda \to \infty$, which maps projectively the insertions of fields in the dS slice to those at points P^A on the lightcone. Given a specific coordinate system, there is usually an obvious parameterisation of this type. For example, for planar and global coordinates we can write

$$\left(X^+, X^-, X^a\right) = \left(\frac{1}{\eta}, \frac{x^2 - \eta^2}{\eta}, \frac{x^a}{\eta}\right) = \frac{1}{\eta}\left(1, x^2, x^a\right) + \ldots \,, \tag{2.2.26}$$

$$\left(X^0, X^i\right) = \left(\sinh(t), \cosh(t)\omega^i\right) = \frac{e^t}{2}\left(1, w^i\right) + \ldots \,. \tag{2.2.27}$$

Order by order in the expansion of the correlator, one can identify the correlator of the bulk field and some boundary insertions. At leading order, one obtains the bulk-to-boundary correlator for the boundary primary operator. For example, consider the Wightman function found previously,

$$\begin{aligned}\lim_{\lambda\to\infty} \Pi(X = \lambda P + \ldots, Y) = \lambda^{-\Delta} \frac{1}{(-2P \cdot Y)^{\Delta}} \frac{4^{\Delta}\Gamma(\Delta)\Gamma(d - 2\Delta)}{(4\pi)^{\frac{d+1}{2}}\Gamma\left(\frac{d+1}{2} - \Delta\right)} \\ + \lambda^{-(d-\Delta)} \frac{1}{(-2P \cdot Y)^{d-\Delta}} \frac{4^{d-\Delta}\Gamma(d - \Delta)\Gamma(2\Delta - d)}{(4\pi)^{\frac{d+1}{2}}\Gamma\left(\frac{1-d}{2} + \Delta\right)}\end{aligned} \tag{2.2.28}$$

The pair of leading terms can be identified with the bulk to boundary correlator a conformal primary field $\mathcal{O}_\Delta(\lambda P) = \lambda^{-\Delta}\mathcal{O}_\Delta(P)$ and its shadow dual, $\mathcal{O}_{d-\Delta}(P)$, as expected [87].

2.3 Spinors in dS_{d+1}

We turn to a systematic treatment of spinors. We begin with ambient spinor fields, and constructively show, using the methods of [158, 183], how to constrain them to obtain irreducible spinors of the dS slice in (2.3.25). As a by-product of this analysis, we write the action of symmetry generators with fields of generic spin in the bulk of dS in (2.3.21) to (2.3.24). We introduce an index-free notation for spinors, and showcase these tools by computing the propagator of Dirac spinors in dS_{d+1} given by (2.3.52) and (2.3.53). We finally discuss the late-time limit of the propagator given in (2.3.57). Odd and even dimension are initially treated separately, though the final result for Dirac spinors is shown to be equivalent. The case of spinors in odd EAdS was covered in [164], our added value lies in giving an explicit construction which generalises both to other coordinates and to even dimensions.

2.3.1 Constraint and transformation law

Embedding the dS slice leads to a straightforward realisation of its isometries, mapping them to Lorentz transformations of the ambient space. Using ambient-space gamma matrices Γ_A, we define $\Sigma_{AB} \equiv \frac{1}{4}[\Gamma_A, \Gamma_B]$ and set $\mathcal{L}_{AB} \equiv P_A \frac{\partial}{\partial P^B} - P_B \frac{\partial}{\partial P^A}$. The transformations act linearly on the ambient spinor field Ψ via

$$[\widehat{L}_{AB}, \Psi(P)] = -(\mathcal{L}_{AB} + \Sigma_{AB})\Psi(P) \ , \tag{2.3.1}$$

$$[\widehat{L}_{AB}, \widehat{L}^{CD}] = -\eta_A{}^C \widehat{L}_B{}^D + \eta_B{}^C \widehat{L}_A{}^D + \eta_A{}^D \widehat{L}_B{}^C - \eta_B{}^D \widehat{L}_A{}^C \ , \tag{2.3.2}$$

$$= -4\eta_{[A}{}^{[C} \widehat{L}_{B]}{}^{D]} \ , \tag{2.3.3}$$

We note that $\widehat{L}_{AB}, \mathcal{L}_{AB}$ and Σ_{AB} obey the same commutation relations.

In what follows we consider Ψ and $\overline{\Psi}$: respectively a Dirac spinor representation of $SO(1, d+1)$ and its conjugate. As we did previously for tensors, we will *define* a dS spinor as a constrained object which lives in the ambient space. The Lorentz invariant constraint will make this object irreducible, in the same way that transversality did, and ensure that it contains degrees of freedom corresponding exactly to those of a dS spinor. We perform our construction in a specific parameterisation of the dS slicing $X^2 = 1$, but the results and constraint are coordinate independent. The transformation law (2.3.1) maps to a local realisation of the isometries acting on components of Ψ on the slice; matching this local realisation to the intrinsic calculation implies a constraint on Ψ which corresponds to an irreducible field ψ in dS. For the planar parameterisation the algebra has a natural basis of generators

$$\widehat{D} = 2\widehat{L}_{+-}\,, \tag{2.3.4}$$

$$\widehat{P}_a = 2\widehat{L}_{-a}\,, \tag{2.3.5}$$

$$\widehat{K}_a = 2\widehat{L}_{a+}\,, \tag{2.3.6}$$

$$\widehat{M}_{ab} = \widehat{L}_{ab}\,. \tag{2.3.7}$$

When written explicitly in planar coordinates, these operators act on a scalar field ϕ as Killing vectors specified by

$$D\phi = x^\mu \frac{\partial}{\partial x^\mu}\phi\,, \tag{2.3.8}$$

$$P_a\phi = \frac{\partial}{\partial x^a}\phi\,, \tag{2.3.9}$$

$$K_a\phi = \left(2x_a x^\mu \frac{\partial}{\partial x^\mu} - x^\mu x_\mu \frac{\partial}{\partial x^a}\right)\phi\,, \tag{2.3.10}$$

$$M_{ab}\phi = \left(x_b \frac{\partial}{\partial x^a} - x_a \frac{\partial}{\partial x^b}\right)\phi\,. \tag{2.3.11}$$

The intrinsic action on other dS representations is fixed by the addition of a spin part to the action of the operators. For a field transforming in a given representation of $SO(1,d)$, we add a contribution from the spin matrix to the action of M_{ab}, for a spinor this is $\tilde{\Sigma}_{\mu\nu} = \frac{1}{4}[\gamma_\mu, \gamma_\nu]$. We then use the Jacobi identity to fix the spin-part of all the remaining generators in terms of the spin matrix $\tilde{\Sigma}_{\mu\nu}$. Note that the indices used here are the $SO(1,d)$ tangent space indices as described in appendix A, so that $\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}$. This analysis provides the general form of the isometry generators acting on spinning dS bulk fields ψ in planar coordinates,

$$D\psi(x^\mu) = x^\mu \frac{\partial}{\partial x^\mu}\psi\,, \tag{2.3.12}$$

$$P_a\psi(x^\mu) = \frac{\partial}{\partial x^a}\psi\,, \tag{2.3.13}$$

$$K_a\psi(x^\mu) = \left(2x_a x^\mu \frac{\partial}{\partial x^\mu} - x^\mu x_\mu \frac{\partial}{\partial x^a} + 2\tilde{\Sigma}_{a\mu}x^\mu\right)\psi\,, \tag{2.3.14}$$

$$M_{ab}\psi(x^\mu) = -\left(x_a \frac{\partial}{\partial x^b} - x_b \frac{\partial}{\partial x^a} + \tilde{\Sigma}_{ab}\right)\psi\,. \tag{2.3.15}$$

Note that $\tilde{\Sigma}_{ab}$ needs not be the ab component of Σ_{AB}, though it is often the case these are defined independently until stated otherwise. We show the analogous calculation for the related example of global coordinates in appendix E. Though this appears superficially similar to the transformation law of a field of weight 0 under the conformal group, one should be mindful of the index range. Only in the late-time limit $\lim_{\eta\to 0^+}\psi(\eta,x) = (-\eta)^\Delta\psi(x)$, do we recover the usual form of the Conformal algebra acting on primary fields of weight Δ.

We can now compare how the elements of Ψ transform in the ambient space, (2.3.1) with the transformation in the planar patch, (2.3.12) to (2.3.15), to isolate a quantity transforming like a spinor of dS. Though an explicit choice is made, we stress that the results are not unique to the planar parameterisation, precisely because of the homogeneity of dS. As further proof of this, in the appendix E we perform the analogous analysis for global coordinates.

Uplifting spinors of dS$_{2n+1}$

When the ambient space is now even dimensional, we decompose the ambient space spinor as a direct sum of two spinors in $Spin(1,2n)$,

$$\Psi = \begin{pmatrix}\chi\\ \rho\end{pmatrix}\,. \tag{2.3.16}$$

Associated to the dS slice we have matrices γ_μ, from which we may construct those of the ambient space

$$\Gamma_0 = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\otimes \mathbb{1}\,, \tag{2.3.17}$$

$$\Gamma_a = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\otimes \gamma_0\gamma_a\,, \tag{2.3.18}$$

$$\Gamma_{d+1} = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\otimes \mathbb{1}\,, \tag{2.3.19}$$

$$\Gamma_\star = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\otimes -i\gamma_0\,. \tag{2.3.20}$$

From these choices, one may compare $[\widehat{L}_{AB},\chi]$ in the ambient space, (2.3.1), with that defined by (2.3.12) to (2.3.15). Since the $-+$ terms give an inhomogeneous contribution to the dilation, we consider a rescaled field $\sqrt{-\eta}\chi$, simplifying the action on the fields to

$$D\left(\sqrt{-\eta}\chi(x^\mu)\right) = x^\mu \frac{\partial}{\partial x^\mu}\left(\sqrt{-\eta}\chi\right) , \tag{2.3.21}$$

$$P_a\left(\sqrt{-\eta}\chi(x^\mu)\right) = \frac{\partial}{\partial x^a}\left(\sqrt{-\eta}\chi\right) , \tag{2.3.22}$$

$$K_a\left(\sqrt{-\eta}\chi(x^\mu)\right) = \left(2x_a x^\rho \frac{\partial}{\partial x^\rho} - x^\rho x_\rho \frac{\partial}{\partial x^a}\right)\left(\sqrt{-\eta}\chi\right) - \sqrt{-\eta}\left(x_a\chi - \gamma_a\gamma_0\rho\right) , \tag{2.3.23}$$

$$M_{ab}\left(\sqrt{-\eta}\chi(x^\mu)\right) = -\left(x_a\frac{\partial}{\partial x^b} - x_b\frac{\partial}{\partial x^a} + \frac{1}{4}[\gamma_a, \gamma_b]\right)\left(\sqrt{-\eta}\chi\right) . \tag{2.3.24}$$

Clearly $\sqrt{-\eta}\chi$ has the correct transformation law under translations, rotations, and the dilation. However, the special conformal transformations are more problematic, with an inhomogeneous contribution from the second spinor ρ in (2.3.23). We regain the coordinate transformation given in (2.3.14) by setting $\rho = -\gamma_0\gamma_\mu x^\mu \chi$. This can be rewritten as a Lorentz-covariant constraint on the ambient spinor Ψ,

$$\Gamma_A X^A \Psi = \Psi . \tag{2.3.25}$$

At this point, we stress the analogy with the tensorial case. We showed that generic ambient spinors do not define dS spinors. However, there is a specific class of (Lorentz-covariant) constrained spinors in the embedding which are in one-to-one correspondence with spinors on the slice. Hence, we can define the uplift of dS spinors as precisely those constrained spinors, in total correspondence with what we saw for tensors. One can also note that this constraint is the only possibility, since $(X \cdot \Gamma)^2 = 1$, and the spinor with eigenvalue -1 is related to the one we consider by multiplication with $\Gamma_\star$.

When this argument is reversed, we see that any spinor of the form $\left(\Gamma_A X^A + 1\right)\Psi$, with Ψ unconstrained, has a top component which transforms like $\frac{1}{\sqrt{-\eta}}$ times a dS spinor. In analogy to the tensorial case, we can work with unconstrained spinors like (2.3.16), and contract them with a constrained dummy polarisation spinor $\overline{S}$, which incorporates the projection prefactor. This allows us to work with scalar fields $\mathbf{\Psi}(X, \overline{S}) \equiv \overline{S}\Psi(X)$, instead of spinors. On the slice, we also work with the scalar $\overline{s}\psi(\eta, x)$, allowing us to avoid the use of spinor indices. The constraint is then transferred to, $\overline{S}$, which also takes care of the projection, as in the tensorial case. The dummy spinor in the ambient space and in the slice are related directly,

$$\overline{S}\Psi = \overline{s}\psi\,. \tag{2.3.26}$$

We can use the constraint to derive the ambient space $\overline{S}$ in terms of the intrisic $\overline{s}$, taking the complex conjugate we recover the projection to conjugate spinors. The spinor polarisations are given by

$$X^A\Gamma_A S = -S \qquad \Longrightarrow \qquad S = \begin{pmatrix} \gamma_0 \\ -\not{x} \end{pmatrix} \frac{s}{\sqrt{-\eta}}\,, \tag{2.3.27}$$

$$\overline{S}X^A\Gamma_A = \overline{S} \qquad \Longrightarrow \qquad \overline{S} = \frac{\overline{s}}{\sqrt{-\eta}} \begin{pmatrix} -\not{x}\gamma_0 & 1 \end{pmatrix}\,. \tag{2.3.28}$$

We define dS spinors, in analogy to the tensor case, as homogeneous scalar polynomials in the (commuting) polarisation variables

$$\boldsymbol{\Psi}(X, \lambda\bar{S}) = \lambda\boldsymbol{\Psi}(X, \bar{S})\,, \tag{2.3.29}$$

with obvious generalisation to multiple spinor indices. It is of course possible to work directly with constrained spinors, with free indices. Then, one must write out spinor-structures which manifestly obey the constraint, these match those constructed in [178]. Switching to the index free notation only makes manipulation simpler, one can simply write all non-vanishing scalar objects satisfying the constraint on S and the homogeneity requirement.

The explicit expressions we gave for S are specific to the choice of parameterisation of dS, one can translate these back into other choices, though this can be a lengthy endeavour. We showcase the computation for global coordinates in appendix E. In this work we preferentially compute results directly in the embedding, and only evaluate in a specific parameterisation, if necessary, as a final step.

The case of dS$_{2n}$

In even dimensions the spin group admits irreducible Weyl representations, this more intricate case requires a slightly more subtle approach. We start by discussing the transformation induced by the Lorentz group of the ambient space, and derive the uplift of the dS Weyl-spinors. We then use this result to reconstruct the full Dirac spinor in even dimensions.

The ambient space is odd dimensional. So naively, the Dirac spinor representations are the same on the slice and in the embedding, and we could be tempted to try to use unconstrained ambient objects. However, from the previous computation it is clear that such an object does not transform as it should on the slice. One may expect from the case of even ambient space that the constrained Dirac spinor maps to local irreducible spinors of the slice, which are the Weyl spinors in even dS. Inspired by these remarks, we consider the chiral representation of the gamma matrices on the slice

$$\gamma_\mu = \begin{pmatrix} 0 & \sigma_\mu \\ \overline{\sigma}_\mu & 0 \end{pmatrix} , \tag{2.3.30}$$

$$\sigma_\mu = (\mathbb{1}, \sigma_a) = \sigma_\mu^\dagger , \tag{2.3.31}$$

$$\overline{\sigma}_\mu = (-\mathbb{1}, \sigma_a) = \overline{\sigma}_\mu^\dagger . \tag{2.3.32}$$

Consider the basis of ambient gamma matrices in (2.3.17) to (2.3.20), with the lower slice γ-matrices replaced by chiral σ-matrices

$$\mathit{\Gamma}_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes \mathbb{1} , \tag{2.3.33}$$

$$\mathit{\Gamma}_a = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \sigma_a , \tag{2.3.34}$$

$$\mathit{\Gamma}_{d+1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \mathbb{1} . \tag{2.3.35}$$

Many of the results of the previous section can be reused. For example, the adjustment of the action of D and P are identical. Rotations indeed involve $\Sigma_{ab} = \frac{1}{4}[\sigma_a, \sigma_b]$ which is identical for left and right handed Weyl spinors. For the boosts we encounter $\Sigma_{a\mu}x^\mu = \frac{1}{2}\left(\sigma_a\left(-\alpha\eta\mathbb{1}a + \sigma_b x^b\right) - x_a\right)$, with $\alpha = \pm 1$ corresponding to LH-spinors and RH-spinors respectively. It follows that an ambient spinor of the form

$$\sqrt{-\eta}\begin{pmatrix} \chi \\ (\alpha\eta\mathbb{1} - x^a\sigma_a)\,\chi \end{pmatrix} , \tag{2.3.36}$$

encodes a chiral spinor in its top component, with $\alpha = \pm 1$ defining the handedness. Such a spinor solves an eigenvalue equation of the form $X^A \mathit{\Gamma}_A \Psi_\alpha = \alpha\Psi_\alpha$. The key difference is the absence of $\mathit{\Gamma}_\star$ to exchange α, implying the sign is now a meaningful distinction between inequivalent representations. Since any such spinor can be written as $\left(X^A\varGamma_A+\alpha\right)\Psi$ with Ψ unconstrained, we can proceed as in the odd case to define a polarisation spinor S, with eigenvalue equation $\overline{S}X\cdot\varGamma=\alpha\overline{S}$.[4]

To discuss the uplift of the Dirac spinor, it is convenient to adopt the terminology of [198]. In even dimensions, we have two Weyl spinors which together form a Dirac spinor. However, in odd dimension, we call the irreducible Dirac representation, a Pauli spinor, and an $SU(2)$ doublet of Pauli spinors a Cartan spinor. We showed in the previous section, that in even dimensional ambient space, a constrained Dirac spinor encodes a Pauli spinor on the slice. On the other hand, when the ambient space has odd dimension, the constrained ambient Pauli spinor encodes the Weyl spinors. It follows naturally that we can build a constrained Cartan spinor which encodes a Dirac spinor. Effectively, we recompose the uplift of the Dirac spinor as a sum of its Weyl parts $\Psi'=\Psi_+\oplus\Psi_-$. We pick Gamma matrices given by $\Gamma'_A=\varGamma_A\otimes\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)$, such that the constrained Cartan spinor obeys the eigenvalue equation $\Gamma'_AX^A\Psi'=\Psi'$, as in odd dimension. The symplectic structure of the $SU(2)$ doublet implies the existence of two special invariant matrices: the identity, and the symplectic form $J=\epsilon_{ij}$, which exchanges the two Pauli spinors. One is free to perform a similarity transformation U to a more convenient basis. To make contact with the odd dimensional case, we reorder the components of Ψ using

$$U=\frac{1}{\sqrt{2}}\begin{pmatrix}1&0&0&0\\0&0&-1&0\\0&1&0&0\\0&0&0&1\end{pmatrix}. \tag{2.3.37}$$

We then relabel $\Gamma_A=U\Gamma'_AU^\dagger$, as in (2.3.17) to (2.3.20). We also obtain a supplementary matrix through $UJU^\dagger\equiv-i\Gamma_\star$. Hence, although we are in odd ambient dimension, we have a coordinate independent construction of $\Gamma_\star$, the uplift of $\gamma_\star$ in the even dimensional slice. The output of this discussion is that the formalism defined for dS_{2n+1} can be reused without modification for dS_{2n}, as long as one considers Dirac fields, while chiral fields may be considered using the more intricate chiral picture outlined above.

[4]This chiral picture will be useful to those interested in supersymmetry in de Sitter and chiral interactions, but for many purposes it is easier to work with the reducible Dirac spinor.

2.3.2 Propagator and boundary limit

From our kinematic discussion, we have a dimension independent formalism to uplift Dirac spinors to the ambient space. The most general two-point function, compatible with the constraints on the polarisations and the homogeneity requirements, may only include the following structures

$$\left\langle \mathbf{\Psi}(X, \overline{S}_1)\overline{\mathbf{\Psi}}(Y, S_2)\right\rangle = \overline{S}_1 S_2 g_+(z) + \overline{S}_1 \Gamma_\star S_2 g_-(z)\,, \tag{2.3.38}$$

We parameterise our functions as before in terms of the dS invariant variable, $z = \frac{1}{2}(1 + X \cdot Y)$. We can also use the explicit expressions we found to project these structures down to a dS slice expression if required. For example, in planar coordinates ((2.2.2)) we find

$$\overline{S}_1 S_2 = \frac{\overline{s}_1 \gamma_\mu (x-y)^\mu s_2}{\sqrt{x^0 y^0}}\,, \tag{2.3.39}$$

$$\overline{S}_1 \Gamma_\star S_2 = \frac{\overline{s}_1 i\gamma_0 \gamma_\mu (\tilde{x}-y)^\mu s_2}{\sqrt{x^0 y^0}}\,, \tag{2.3.40}$$

where $\tilde{x}^\mu = (-x^0, x^a)$ is the time-reversal of x^μ. In order to uplift the Dirac equation to the ambient space, we present an intuitive argument which reaches the same conclusion as the intrinsic argument of [198, 199]. The operator $\not{\nabla}$ should be a first order differential operator, longitudinal to the dS slice, whose image acting on a dS spinor is again, a dS spinor. The first requirement tells us that ∇_A includes only transverse objects, for example $G_{AB}\partial^B$ and $\Sigma_{AB}X^B$. The second requirement tells us that $\{\not{\nabla}, \not{X}\} = 0$. A minimal ansatz coming from the first constraint is easily fixed using the second one, to give us the final form of the Dirac operator in the embedding

$$\not{\nabla} = \Gamma^A \left(G_{AB}\partial^A - \Sigma_{AB}X^B \frac{1}{X \cdot X} \right) \tag{2.3.41}$$

$$= \not{\partial} - \not{X} X \cdot \partial - \frac{d+1}{2}\not{X}\,. \tag{2.3.42}$$

This offers a convenient derivation of the result stated in [164, 198]. One can explicitly check in the planar parameterisation that this operator reproduces the action of the covariant derivative on spinor as can be computed from the tetrad formalism [173]

$$\Gamma^A\nabla_A\Psi = \begin{pmatrix}\gamma_0 \\ -\not{x}\end{pmatrix}\frac{\gamma^\mu\nabla_\mu\psi}{\sqrt{-\eta}}\,, \tag{2.3.43}$$

$$\gamma^\mu\nabla_\mu\psi = \eta\gamma^\mu\partial_\mu\psi + \frac{d}{2}\gamma_0\psi\,. \tag{2.3.44}$$

It follows that $\bar{\Psi}\not{\nabla}\Psi = -2\bar{\psi}\not{\nabla}\psi$. Similarly $\bar{\Psi}\Psi = 0$, hence the mass term must be uplifted with a factor of $\Gamma_\star$, transforming Ψ into a spinor with eigenvalue $\alpha = -1$. Explicitly, we find $\bar{\Psi}\Gamma_\star\Psi = 2i\bar{\psi}\psi$. All together

$$\bar{\psi}\left(\not{\nabla} + m\right)\psi \equiv -\frac{1}{2}\bar{\Psi}\left(\not{\nabla} + im\Gamma_\star\right)\Psi\,, \tag{2.3.45}$$

from which we can rewrite the Dirac equation in the ambient space as

$$\left(i\Gamma_\star\not{\nabla} - m\right)\Psi = 0\,. \tag{2.3.46}$$

This form is used so that $\bar{S}\left(i\Gamma_\star\not{\nabla} - m\right)\Psi$ is of the same type as $\bar{S}\Psi$, and we may write $\bar{S}i\Gamma_\star\not{\nabla}\frac{\partial}{\partial\bar{S}} - m$. For the case $m \neq 0$, in odd ambient space, the explicit factor of $\Gamma_\star$ prevents us from writing the Dirac equation of a single constrained Pauli spinor. This is expected due to the lack of a chiral representation in odd dimensions.

One can also make use of the Casimir equation previously discussed. From the CFT literature, for example [183], the Casimir of the d dimensional conformal group associated to fields transforming in a spin $\frac{1}{2}$ representation, is known and given by

$$C_{\Delta,\frac{1}{2}} = \Delta(d-\Delta) + \frac{d(d-1)}{8}\,^{5}. \tag{2.3.47}$$

The Casimir is built starting from the differential realisation of $\widehat{L}_{AB}$ on the fermionic fields,

$$X_A\frac{\partial}{\partial X^B} - X_B\frac{\partial}{\partial X^A} + \overline{S}\Sigma_{AB}\frac{\partial}{\partial\overline{S}} = \mathcal{L}_{AB} + \Sigma_{AB}\,.$$

where $\Sigma_{AB} = \frac{1}{4}[\Gamma_A, \Gamma_B]$. We can now evaluate $\mathcal{C} = -\frac{1}{2}\left(\mathcal{L}+\Sigma\right)_{AB}\left(\mathcal{L}+\Sigma\right)^{AB}$,

[5] This can be rederived by using the explicit form of the Casimir operator and considering the action of the Casimir on primary states in a CFT_d

$$\mathcal{C} = \underbrace{-\partial^2 + X_A X_B \partial^A \partial^B + (d+1) X \cdot \partial}_{-\Box_{dS}} + \underbrace{\frac{(d+2)(d+1)}{8}}_{-\frac{1}{2}\Sigma^2} + \underbrace{X \cdot \partial - \bar{S}\left(\Gamma^A X_A\right)\left(\Gamma^B \partial_B\right)\frac{\partial}{\partial \bar{S}}}_{-\Sigma\cdot\mathcal{L}} \,. \tag{2.3.48}$$

Almost all of the pieces act trivially on the structures we defined in (2.3.39). In fact, only the last element of the third piece changes between the two structures appearing in the propagator, by picking up a sign. More importantly, they are completely decoupled.

It is now a straightforward exercise to act with the Casimir equation on the propagator ansatz and to collect terms. The differential equation we obtain is simple, owing largely to the choice of variable

$$\left(z(z-1)\partial_z^2 - \left(\frac{d+2-\alpha}{2} - (d+2)z\right)\partial_z - \left(\Delta + \frac{1}{2}\right)\left(d + \frac{1}{2} - \Delta\right)\right) g_\alpha(z) = 0 \,. \tag{2.3.49}$$

These hypergeometric differential equations are equivalent to the set of coupled, first order equations one finds using the Dirac equation

$$(z-1)\partial_z g_+(z) + \frac{d+1}{2} g_+(z) + im g_-(z) = 0 \,, \tag{2.3.50}$$

$$z\partial_z g_-(z) + \frac{d+1}{2} g_-(z) + im g_+(z) = 0 \,, \tag{2.3.51}$$

provided we identify $\Delta = \frac{d}{2} + im$. These differential equations are solved by hypergeometric functions, just as in the case of the scalar field in section 2.2.3. Solving the Dirac equation while requiring that all singularities lie at coincident point, fixes both functions in terms of one constant κ_ψ

$$g_+(z) = \kappa_\psi \, {}_2F_1\left(d - \Delta + \frac{1}{2}, \Delta + \frac{1}{2}; \frac{d+1}{2}; z\right) \,, \tag{2.3.52}$$

$$g_-(z) = \kappa_\psi \frac{(d - 2\Delta)}{d+1} \, {}_2F_1\left(d - \Delta + \frac{1}{2}, \Delta + \frac{1}{2}; \frac{d+1}{2} + 1; z\right) \,. \tag{2.3.53}$$

The Hadamard condition fixes the leading singularity of the Wightman function to the same normalisation as in flat-space, where it diverges as $-\frac{1}{4}\Gamma\left(\frac{d+1}{2}\right)\pi^{-\frac{d+1}{2}} u^{-\frac{d+1}{2}} \gamma_\mu \partial^\mu u$. One can check that all singularities contained in $g_-(z)$ are subleading, hence only $g_+(z)$ contributes, as we expect from the explicit coordinate form of the spinorial structures $\overline{S}_1 S_2$ and $\overline{S}_1 \Gamma_\star S_2$. Matching the overall constant, we find that the normalisation is given by

$$\kappa_\psi = -\frac{1}{4} \frac{\Gamma\left(\Delta + \frac{1}{2}\right) \Gamma\left(d - \Delta + \frac{1}{2}\right)}{(4\pi)^{\frac{d+1}{2}} \Gamma\left(\frac{d+1}{2}\right)} . \tag{2.3.54}$$

Note that, in the massless limit $\Delta \to \frac{d}{2}$, $g_-(z) \to 0$, while $g_+(z)$ is finite. From this expression for the Wightman function, one can reconstruct the different propagators to be used in perturbation theory as previously explained for the scalar case and given in (2.2.22) to (2.2.24).

For the boundary limit of the two-point function we proceed as in the scalar case, by parameterising the bulk in term of boundary vectors $X = \lambda P + \ldots$. We note that the spinor polarisations have a nontrivial power-law divergence as one approaches the boundary

$$\lim_{\lambda \to \infty} \frac{1}{\sqrt{\lambda}} \overline{S} = \overline{S}_\partial \, , \tag{2.3.55}$$

$$\overline{S}_\partial \not{P} = 0 \, . \tag{2.3.56}$$

This can be checked explicitly for the case of conformally flat coordinates, and by homogeneity this must be true for any other limiting procedure toward the boundary. The boundary limit of the two-point function at late time is given by

$$\begin{aligned} \lim_{\lambda \to \infty} \Pi(X = \lambda P + \ldots, Y, \overline{S}_1, S_2) = \lambda^{-\Delta} & \left(K_\Delta \frac{\overline{S}_{1,\partial}(1 + \Gamma_\star) S_2}{(-2P \cdot Y)^{\Delta + \frac{1}{2}}} + \mathcal{O}(\lambda) \right) \\ & + \lambda^{\Delta - d} \left(K_{d-\Delta} \frac{\overline{S}_{1,\partial}(1 - \Gamma_\star) S_2}{(-2P \cdot Y)^{d - \Delta + \frac{1}{2}}} + \mathcal{O}(\lambda) \right) , \end{aligned} \tag{2.3.57}$$

with $K_\Delta \equiv -\frac{1}{8\sqrt{\pi}^{d+2}} \Gamma\left(\Delta + \frac{1}{2}\right) \Gamma\left(\frac{d+1}{2} - \Delta\right)$. These again match the form expected for the bulk-to-boundary correlation function of spinors with weight Δ and $d - \Delta$.

2.4 Future Directions

We conclude this chapter with a short description of the potential extensions and applications of the above formalism and results.

- The most obvious extension of the techniques discussed above is the study of a greater range of fields of mixed integer and half-integer spin, including those with gauge redundancy. In particular, it would be interesting to apply these techniques for fields describing the massless and partially massless UIRs described in [155, 189–191], which are particular to dS. Another potential extension of our work would be to generalise to dS the study of weight shifting operators which simplify calculations in higher spin CFT and AdS [163, 200]. In [201] they touch on this subject. Extending this discussion to spinors and away from the late-time limit should prove worthwhile. Finally, while we worked with pure dS above, our explicit results should be useful to consider spinors in asymptotically dS spacetime, an important area for further analysis [73, 133].

- We also have a significant interest in the use of the ambient formalism to study the relation between QFT in fixed dS_{d+1} and S_{d+1} [202–205]. The time-like coordinate relates the respectively Lorentzian and Euclidean field theoretic properties of these spaces for both global and static coordinates described in [70, 111]. In addition, there is a relation which encodes the thermodynamics of the cosmological horizon in the Euclideanisation of the static patch [77, 109, 110, 121, 124, 125]. Fermionic Green's functions for euclidean maximally symmetric spaces have been derived in [178]. The consequences of Wick rotation for propagators on these spaces should be relevant to this effort, as would a better understanding in general of the Euclidean theory of S_d, perhaps building on [204].

- The findings of this chapter further an expanding body of research on the development of a rigorous, analytic and group theoretic treatment of QFT in dS. It is possible to apply our spinor construction to the work of [87, 88] on the unitarity and existence of scalar operators at the late time limit. We have included statements on the limiting behaviour of the Wightman function in the late-time regime for spinors in (2.3.57), which may be directly compared with allegorical two-point functions of spin-half primaries constructed using these methods.

- The structures found in planar coordinates for the spinor correlator are akin to the ones encountered in BCFT [206]. Pushing this analogy further would be an interesting pursuit. One could also, in the Euclidean picture, consider the construction of isometry generators through topological surface operators, as in BCFT [207], with the hope of new insights on Ward identities and asymptotic symmetries in dS [73, 133]. Another possible extension is to make contact between our formalism and the one developed for massless fields [208].

- We contextualised our work in the ongoing effort on the perturbative front, using the in-in formalism [31,53,54,62]. The generalisation of these works to fermionic fields and tensors is a natural objective, for example, in the construction of the effective AdS action studied in detail for scalar fields in [54]. Simplification of perturbative calculations in dS is achieved there by constructing a non-unitary Lagrangian in an AdS background which reproduces the dS results at each order in perturbation theory. Additionally, the cosmological bootstrap effort offers hope for another application of the ideas included in our work [35, 36, 38, 65] perhaps enlarging our knowledge of CFT like structures in dS beyond the previous work on the dS/CFT correspondence [139, 140, 142, 154]. In particular, the development of the Källén-Lehman spectral representation and the expansion of the four-point function of boundary operators in terms of conformal partial waves will have relevance for the study of spinor theories in analogy to the scalar case.

3. dS_2

In Minkowski space, Wigner's classification [93] links the very tangible concept of a particle—a tiny streaming packet of energy and momentum—to the abstract notion of a unitary irreducible representation (UIR) of the Poincaré group. This principle is so powerful at constraining the observable physics [209] that one is led to apply it in other maximally symmetric spacetimes. In de Sitter space, this has been advocated either directly or indirectly in several papers [46, 87–90, 101, 147, 156, 172, 187, 210–216]. Relatedly, dS/CFT considerations [31, 139, 140, 154, 155] and the bootstrap and S-matrix methods [32–35, 37, 52, 56–60, 65, 92, 141, 201] exploit the de Sitter isometries to constrain physical phenomena. In this chapter, we explore the logic of Wigner's classification on particle dynamics in the setting of quantum fields propagating on a two-dimensional de Sitter spacetime (dS_2).

We choose to work in dS_2, so as to have a simplified playground in which to explore the relevant physics. Indeed, the dS_2 group of isometries, $SO(1,2)$ (or its double cover $SL(2,\mathbb{R})$) shares many features in common with the $SO(1,4)$ isometry group of dS_4. Of particular interest is that both groups contain discrete series UIRs, in addition to the more generic principal and complementary series representations, associated with heavy fields in de Sitter, as was first established in the works of Bargmann and Harish-Chandra [95–97]. In dS_4, the discrete series UIRs appear in the single-particle Hilbert space of higher-spin gauge fields, both massless and partially massless [189, 191, 217, 218], including, as a special case, the linearized graviton. However, in dS_2, healthy models with discrete series UIRs have been more elusive. For example, they have been shown to arise in the single-particle Hilbert space in free tachyonic scalar models [219, 220].

One reason to concern ourselves with discrete series UIRs is that they generically appear in the multi-particle (tensor-product) Hilbert space of heavy fields [99, 100]. This statement deserves scrutiny. Since de Sitter is a time-dependent spacetime, energy is not conserved, allowing for interesting phenomenology. For example [85, 86, 212], a scalar particle of mass m_0 in de Sitter space, coupled via a cubic interaction to particles with masses m_1 and m_2, can decay into these particles even if $m_0 < m_1 + m_2$;

a process forbidden in flat space by energy conservation. A similarly counter-intuitive fact is that the two-particle Hilbert space of a single heavy field in de Sitter carries the discrete series UIR, which, as we just mentioned, is generally constructed as a scalar tachyon in dS_2. We are thus motivated to explore how these discrete series UIRs can arise in different QFT constructions on a rigid dS_2 background. We also investigate how the discrete series contribute in a Källén-Lehmann [63,64] spectral decomposition of the two-point function for general interacting scalar fields in dS_2 (see [54,65,66,92,172,212,221] for related discussions).

We find that we can also construct operators furnishing the discrete series representations in BF-type gauge theories on dS_2. However, such BF gauge theories are topological and as a result these UIRs eventually are projected out of the gauge-invariant Hilbert space. In a sense, they only exist in the pre-Hilbert space of the theory, meaning they may come alive if we break the gauge invariance or do something to alter the structure of the spacetime. At least semiclassically, BF gauge theories can be thought of as higher-spin fields in two-dimensions [222,223], thus providing a link between the two- and four-dimensional constructions of these UIRs.

The simplest example is a BF theory with $SL(2,\mathbb{R})$ gauge group, which can be mapped to de Sitter JT gravity [224,225]. The discrete series operators are built from the Weyl mode of the two-dimensional metric. But since one must further impose the diffeomorphism constraints of the theory, the Hilbert space is severely reduced. This is reminiscent of the need to gauge the dS_4 isometry group in semiclassical quantum gravity near a dS_4 vacuum, which restricts the Hilbert space to just the de Sitter invariant states [145–148]. This naturally leads us to ask how to construct gravitational observables. In particular, we discuss how contact terms in n-point functions of local operators in dS_2 [88] propagate into the appropriate gauge invariant observables.

Outline

The chapter is structured as follows. In section 3.1 we discuss the basic geometric aspects of dS_2 and introduce various relevant coordinate systems. In section 3.2 we consider the unitary irreducible representations of $SL(2,\mathbb{R})$, the free Green's function for particles in the principal series UIR and a simple model furnishing the $\Delta = 1$ discrete series UIR. In section 3.3 we elaborate on the Källén-Lehman decomposition of a de Sitter invariant two-point function of interacting scalar fields.

In section 3.4 we describe how the operator content of BF-type gauge theories furnish the discrete series UIR, at the level of the pre-Hilbert space. In section 3.5 we discuss how the constraints from gauging the dS_2 isometry group in a theory of gravity affect observables and correlation functions, bearing in mind the role of potential contact terms. In appendix B we review the unitary irreducible representations of the $SO(1,2)$ isometry group of dS_2. In appendix F we construct a discrete series UIR via a Clebsch-Gordan analysis at the level of a quantum mechanics of two degrees of freedom each furnishing a principal series UIR.

3.1 Geometry of dS_2

In this section we discuss the geometry of a two-dimensional de Sitter spacetime. We also discuss some group theoretic properties of its isometry group $SO(1,2) \cong SL(2,\mathbb{R})/\mathbb{Z}_2$.

Let us begin by reviewing the geometry of dS_2. The usual starting point is to view this spacetime as a Lorentzian hypersurface embedded in a three-dimensional ambient Minkowski spacetime, satisfying the equation

$$-\left(X^0\right)^2+\left(X^1\right)^2+\left(X^2\right)^2=\ell^2 , \qquad X^A \equiv \left(X^0, X^1, X^2\right) \in \mathbb{R}^3 . \tag{3.1.1}$$

As we did for the generic case in section 2.2, where we did set $\ell = 1$: In this chapter we allow the de Sitter length to take a generic value. The metric of dS_2 is induced from the flat metric of the ambient spacetime

$$ds^2 = \eta_{AB}\, dX^A dX^B = -\left(dX^0\right)^2+\left(dX^1\right)^2+\left(dX^2\right)^2 , \tag{3.1.2}$$

by solving (3.1.1). It follows from the above construction that the isometry group of dS_2 is the Lorentz group in three-dimensions, $SO(1,2)$. We will also permit fermionic fields on this spacetime, meaning we should actually consider the double cover of $SO(1,2)$, namely $G \equiv SL(2,\mathbb{R})$. The generators of G are constructed analogously to section 2.2; however, in this chapter we work with explicitly Hermitian generators

$$\mathcal{L}_{AB} = -i\left(X_A\partial_B - X_B\partial_A\right) , \tag{3.1.3}$$

which under commutation satisfy

$$[\mathcal{L}_{AB}, \mathcal{L}_{CD}] = -i\left(\eta_{AD}\mathcal{L}_{BC} - \eta_{AC}\mathcal{L}_{BD} + \eta_{BC}\mathcal{L}_{AD} - \eta_{BD}\mathcal{L}_{AC}\right) . \tag{3.1.4}$$

In this chapter, we will again use a hat $\hat{}$ to denote abstract operators. Whenever an operator appears without a hat, we mean the representation as a differential operator. If we define

$$L_0 \equiv \mathcal{L}_{12} , \qquad L_{\pm1} \equiv -\left(\mathcal{L}_{02} \pm i\mathcal{L}_{01}\right) , \tag{3.1.5}$$

we observe that the above algebra is isomorphic to that of $SL(2,\mathbb{R})$,

$$[L_n, L_m] = (n-m)L_{n+m} . \tag{3.1.6}$$

The quadratic Casimir on a scalar field is given by

$$\mathcal{C} \equiv \frac{1}{2}\mathcal{L}_{AB}\mathcal{L}^{AB} = L_0^2 - \frac{1}{2}\left(L_{-1}L_1 + L_1 L_{-1}\right) . \qquad (3.1.7)$$

Note this is the same as defined in (2.2.13) given the additional factors of i in (3.1.3). From here, we will use the shorthand $L_\pm \equiv L_{\pm 1}$. The maximal compact subgroup of G is $K \equiv SO(2)$, generated by L_0. It is worth noting that dS_2 shares its isometry group with the Poincaré disk, also known as Euclidean AdS_2.

Just like the propagators of chapter 2, correlation functions on the spacetime will be given as functions of de Sitter invariant quantities. The two-point function will depend on the following de Sitter invariant distance, defined in terms of coordinates on the hyperboloid as in (2.2.10):

$$u(X,Y) \equiv \frac{\eta_{AB}\left(X^A - Y^A\right)\left(X^B - Y^B\right)}{2\ell^2} = 1 - \frac{\eta_{AB}X^A Y^B}{\ell^2} , \qquad (3.1.8)$$

where contractions are made using the ambient Minkowski metric (3.1.2). From this formula we conclude that $u(X,X) = 0$, as expected, whereas $u(X,-X) = 2$ for antipodally separated points. Thus points are spacelike separated for $u > 0$, null separated when $u = 0$, and timelike separated for $u < 0$. Lastly, for $u > 2$, there exist no spacelike geodesic paths connecting the two points.

We can select from a variety of parameterisations of the induced metric on the hypersurface (3.1.1). Here we list a few.

Global coordinates on $\mathbf{dS}_2$. The global chart:

$$X^A = \ell\left(\sinh\tau,\, \cos\vartheta\cosh\tau,\, \sin\vartheta\cosh\tau\right), \qquad (3.1.9)$$

covers the entire manifold, resulting in the two-dimensional line element

$$\frac{ds^2}{\ell^2} = -d\tau^2 + \cosh^2\tau\, d\vartheta^2 , \qquad \tau \in \mathbb{R} , \quad \vartheta \sim \vartheta + 2\pi . \qquad (3.1.10)$$

In this coordinate system, the invariant distance $u(X,Y)$ defined in (3.1.8) can be expressed simply as

$$u = 1 + \sinh\tau\sinh\tau' - \cos(\vartheta - \vartheta')\cosh\tau\cosh\tau' . \qquad (3.1.11)$$

The Killing vector fields of dS_2 in the global chart are given explicitly by

$$\mathcal{L}_{12} = -i\partial_\vartheta \ , \tag{3.1.12}$$

$$\mathcal{L}_{01} = i\left(\cos\vartheta\,\partial_\tau - \sin\vartheta\,\tanh\tau\,\partial_\vartheta\right) \ , \tag{3.1.13}$$

$$\mathcal{L}_{02} = i\left(\sin\vartheta\,\partial_\tau + \cos\vartheta\,\tanh\tau\,\partial_\vartheta\right) \ , \tag{3.1.14}$$

which respectively generate the rotational and two boost symmetries of dS_2. Similarly, we have

$$L_0 = -i\partial_\vartheta \ , \qquad L_\pm = e^{\mp i\vartheta}\left(-i\tanh\tau\partial_\vartheta \pm \partial_\tau\right) \ . \tag{3.1.15}$$

The quadratic Casimir on scalar fields, is a differential operator given by

$$\mathcal{C} = -\ell^2\Box_{\text{dS}} \,, \tag{3.1.16}$$

where $\Box_{\text{dS}}$ is the scalar Laplacian on dS_2.

One can unwrap the spatial S^1 by taking $\vartheta \in \mathbb{R}$ in (3.1.10), as considered in [226,227]. The resulting spacetime is smooth and has an isometry group given by the universal cover of G, namely $\widetilde{SL}(2,\mathbb{R})$, which is also the isometry group of Lorentzian AdS_2.

Conformal compactification of dS_2. It will occasionally be convenient to work in a global coordinate system with the infinite coordinate time compactified to a finite interval, given by

$$\frac{ds^2}{\ell^2} = \frac{-dT^2 + d\vartheta^2}{\sin^2 T} \ , \qquad T \in (-\pi, 0) \ , \tag{3.1.17}$$

and obtained from (3.1.9) by the identification

$$\cosh\tau = -\frac{1}{\sin T} \ . \tag{3.1.18}$$

In this coordinate system, the invariant distance $u(X,Y)$ takes the following form

$$u = \frac{\cos(T-T') - \cos(\vartheta-\vartheta')}{\sin T \sin T'} = \frac{2\sin\left(\vartheta^- - \vartheta'^-\right)\sin\left(\vartheta^+ - \vartheta'^+\right)}{\sin\left(\vartheta^+ - \vartheta^-\right)\sin\left(\vartheta'^+ - \vartheta'^-\right)} \ , \tag{3.1.19}$$

where

$$\vartheta^\pm \equiv \frac{\vartheta \pm T}{2} \tag{3.1.20}$$

are, respectively, the instantaneous left and right moving coordinates on global dS_2. The metric (3.1.17) is conformally equivalent to the Lorentzian cylinder over a finite time-interval. In this case we have Killing vectors given by

$$\mathcal{L}_{12} = -i\partial_\vartheta \ , \tag{3.1.21}$$

$$\mathcal{L}_{01} = -i\left(\cos\vartheta\ \sin T\,\partial_T + \sin\vartheta\ \cos T\,\partial_\vartheta\right)\ , \tag{3.1.22}$$

$$\mathcal{L}_{02} = -i\left(\sin\vartheta\ \sin T\,\partial_T - \cos\vartheta\ \cos T\,\partial_\vartheta\right)\ . \tag{3.1.23}$$

These give us:

$$L_0 = -i\partial_\vartheta\ , \qquad L_\pm = -e^{\mp i\vartheta}\left(i\cos T\,\partial_\vartheta \pm \sin T\,\partial_T\right)\ . \tag{3.1.24}$$

Planar coordinates on dS_2. We also make use of the planar coordinate system when discussing phenomena that are localised near a boundary point. This coordinate patch is analogous to the Poincaré patch in AdS. The metric is

$$\frac{ds^2}{\ell^2} = \frac{-d\eta^2 + dx^2}{\eta^2}\ , \qquad \eta \in (-\infty, 0)\ , \qquad x \in \mathbb{R}\ , \tag{3.1.25}$$

and covers half of the global dS manifold. It contains the late-time slice with a single point removed. It is obtained from the embedding coordinates

$$X^A = \frac{\ell}{2\eta}\left(1 + x^2 - \eta^2,\, 2x,\, 1 - x^2 + \eta^2\right)\ . \tag{3.1.26}$$

Euclidean continuation to S^2. The global patch of dS_2 can be Wick rotated to the standard metric on the two-sphere. In the coordinates (3.1.10) one takes $\tau \to i\left(\psi - \frac{\pi}{2}\right)$ and restricts the range of $\psi \in (0, \pi)$ to obtain the smooth geometry. In the coordinates (3.1.17) one takes $T \to -iX - \frac{\pi}{2}$ to obtain the metric

$$\frac{ds^2}{\ell^2} = \frac{dX^2 + d\vartheta^2}{\cosh^2 X}\ , \tag{3.1.27}$$

with $X \in \mathbb{R}$. Again this metric is that of the two-sphere. Similarly to (3.1.18), the relationship between X and the polar angle ψ is

$$\cosh X = \frac{1}{\sin\psi}\ . \tag{3.1.28}$$

At least in the absence of gravity, quantum field theory on the Euclidean sphere plays an important role in the axiomatic formulation of quantum field theory on de Sitter space (see for instance [46]). In particular, correlation functions of local operators on the Euclidean sphere can be continued to dS_2 correlation functions in the Euclidean vacuum.

Discrete symmetries

In addition to the continuous isometries described above, the dS_2 spacetime also enjoys discrete antipodal (A), parity (P), and time reversal (T) symmetries [228].

	P	T	A
Global	$\vartheta \to 2\pi - \vartheta$	$\tau \to -\tau$	$\vartheta \to \vartheta + \pi\ ,\ \tau \to -\tau$
Planar	$x \to -x$	$\times$	$\eta \to -\eta$
Conformal	$\vartheta \to 2\pi - \vartheta$	$T \to -\pi - T$	$\vartheta \to \vartheta + \pi\ ,\ T \to -\pi - T$

The above table summarises the action of these symmetries in terms of the corresponding coordinates of each coordinate patch, these are derived for the fermion in appendix C.

3.2 Unitary irreducible representations

3.2.1 General theory

The unitary irreducible representations (UIRs) of $SO(1,2)$ are well known [99] (see e.g. [65, 87, 91, 92, 101, 109, 229] for recent literature). Here we will be brief. Let us label the eigenvalue of the quadratic Casimir (3.1.7) as $\Delta(\Delta-1)$. The quantity Δ is known as the *conformal weight* and labels a particular representation. Only a few possible choices of Δ lead to unitary irreducible representations, which we review in appendix B.

To set the stage, we remind the reader that a state $|n,\Delta\rangle$ in a UIR is labelled by its eigenvalues under the maximal commuting subgroup of $SL(2,\mathbb{R})$:

$$\widehat{\mathcal{C}}|\Delta,n\rangle = \Delta(\Delta-1)|\Delta,n\rangle \ , \qquad \widehat{L}_0|\Delta,n\rangle = -n|\Delta,n\rangle \ , \quad \widehat{L}_\pm|\Delta,n\rangle = -(n\pm\Delta)|\Delta,n\pm1\rangle \ . \tag{3.2.1}$$

Recall that operators denoted with a $\widehat{\ }$ signify abstract matrix (not necessarily differential) operators. Because $\widehat{L}_0$ is a compact generator, which acts by rotating the de Sitter hyperboloid, its eigenvalues must be integers (or half integers for the double cover).

To have a UIR means we have a positive, semi-definite inner product on the space of states defined above. The various distinct UIRs for which this is possible are:

- The *principal series*, π_ν, for which $\Delta = \frac{1}{2}(1+i\nu)$ with $\nu\in\mathbb{R}$.
- The *complementary series*, γ_Δ, for which $0<\Delta<1$.
- The *discrete series*, $D^\pm_\Delta$, for which Δ is either a positive integer or half integer. The $(+)$ refers to the highest-weight module, which has an element annihilated by $\widehat{L}_+$. The $(-)$ refers to a lowest weight module which contains an element annihilated by $\widehat{L}_-$.

Moreover, as suggested by the Casimir eigenvalue, there is an isomorphism between π_ν and $\pi_{-\nu}$ and γ_Δ and $\gamma_{1-\Delta}$. The isomorphism does not hold for the $D^\pm_\Delta$, as unitarity restricts $\Delta\geq1$ for $D^\pm_\Delta$. The ranges of n differ across the various representations, so we provide a handy summary in table 3.2.1 for the scalar representations. For both scalar and fermionic representations, see appendix B.

Much like states can be organised into UIRs, we can also discuss the transformation

Rep.	**Range of Δ**	**Range of n**	**Scalar $m^2\ell^2$**
π_ν	$\Delta = \frac{1}{2}(1+i\mathbb{R})$	$n \in \mathbb{Z}$	$m^2\ell^2 > \frac{1}{4}$
γ_Δ	$0 < \Delta < 1$	$n \in \mathbb{Z}$	$0 < m^2\ell^2 < \frac{1}{4}$
$D^\pm_\Delta$	$\Delta \in \mathbb{Z}^+$	$n = \mp\Delta, \mp(\Delta+1), \ldots$	$m^2\ell^2 = -t(t+1)$ with $t \in \mathbb{N}_0$

Table 3.2.1: Summary of the various scalar representations and their eigenvalues under the $SL(2,\mathbb{R})$ algebra. For more details see appendix B.

properties of certain operators under $SL(2,\mathbb{R})$. In particular, an operator $\mathcal{O}_{\Delta,n}$ satisfying

$$\left[\widehat{L}_0, \mathcal{O}_{\Delta,n}\right] = n\mathcal{O}_{\Delta,n} \ , \qquad \left[\widehat{L}_\pm, \mathcal{O}_{\Delta,n}\right] = (n \pm \Delta)\, \mathcal{O}_{\Delta,n\pm1} \ , \tag{3.2.2}$$

is said to be a conformal operator of weight Δ. Although we have states and operators furnishing UIRs of the de Sitter group, there is no state-operator correspondence as is usual for conformal field theory [54]. Acting with a local conformal operator at the Euclidean de Sitter boundary creates a non-normalizable state due to coincident point singularities. Relatedly, the future boundary of de Sitter space, though Euclidean, arises as the end point of the bulk Lorentzian spacetime's time evolution. As such, the imprints of a Lorentzian structure such as a non-commuting operator algebra and standard Hermiticity conditions must be obeyed at $\mathcal{I}^+$.

Recall from (3.1.16) that the Casimir operator $\mathcal{C}$ can be represented by the Laplacian on dS_2 as $\mathcal{C} = -\ell^2\Box_{\mathrm{dS}}$. The equation of motion for a massive scalar field is

$$\Box_{\mathrm{dS}}\, \phi = m^2\phi \ , \tag{3.2.3}$$

implying the relationship

$$\Delta(\Delta-1) = -m^2\ell^2 \quad \Longrightarrow \quad \Delta = \frac{1}{2}\left(1 \pm \sqrt{1-4m^2\ell^2}\right) \ . \tag{3.2.4}$$

From here we determine that states in the complementary series are faithfully represented by scalars whose mass squared satisfies: $0 < m^2\ell^2 < \frac{1}{4}$, whereas states in the principal series are given by scalars with $m^2\ell^2 > \frac{1}{4}$. In order to obtain a state in the discrete series, for $\Delta = 1 + t$ with $t \in \mathbb{N}_0$, we would need

$$m^2\ell^2 = -t(t+1) \ . \tag{3.2.5}$$

Namely, for $t = 0$, the scalar is massless, otherwise the scalar must be tachyonic. Perhaps, then, discrete series states are unphysical in dS_2, and can be rightfully ignored. However, recalling [100] the tensor product of scalar representations:

$$\pi_\nu \otimes \pi_{\nu'} = \left(\bigoplus_\Delta D^\pm_\Delta\right) \oplus \left(2\int_{\mathbb{R}^+} d\omega\, \pi_\omega\right) \ , \tag{3.2.6}$$

one notes that $D^\pm_\Delta$ invariably makes an appearance in the two-particle Hilbert space of massive scalar fields on dS_2.[1] Given Wigner's interpretation of UIRs as single particle states in quantum field theory, we are led to ask if these discrete series bound states can be interpreted as asymptotic free particle states under some suitable definition. In fact, much of this chapter will concern itself with how the discrete series appears in different guises in the context of dS_2. In section 3.3 we will explore the consequences of $D^\pm_\Delta$ appearing in the multi-particle Hilbert space by studying the spectral decomposition (or Källén-Lehmann representation) of the two-point function of a heavy interacting scalar on dS_2.

An interesting consequence of our analysis is that a QFT whose classical equations of motion on dS_2 are given by (3.2.3) is not guaranteed to have a state of dimension Δ (related to the mass via the relation (3.2.4)) in its Hilbert space. This is most obvious, whenever the the metric degrees of freedom are dynamical. Suitably gauging the ambient $SL(2,\mathbb{R})$ isometry of dS_2 will result in a reduction of the physical Hilbert space—a fact we will explore in a few examples in section 3.4.

3.2.2 Free Fock space: principal series

So far our exposition has been quite abstract, so as an instructive aid to the reader, we will show how to build the principal series UIR, as in (3.2.1), using single-particle excitations of a free quantum field theory on dS_2. We start with the action:

$$S = -\frac{1}{2}\int d^2x\sqrt{-g}\left[g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + m^2\phi^2\right] \ , \tag{3.2.7}$$

from which we can derive the equation of motion (3.2.3):

$$\Box_{\mathrm{dS}}\, \phi = m^2\phi \ . \tag{3.2.8}$$

[1]We elaborate on the two-particle Hilbert space from the perspective of a Hilbert space built from wavefunctions on S^1 in appendix F.

In this section, we will insist that $m^2\ell^2 > \frac{1}{4}$ such that $\Delta = \frac{1}{2}(1+i\nu)$. There are several ways to decompose our classical field into modes. Working in global coordinates (3.1.9), we choose to expand our field in modes that are regular near the pole of the lower half-sphere $\left(\tau \to -\frac{i\pi}{2}\right)$ in the Euclidean continuation of the global coordinates defined above (3.1.27). Being regular on the south pole, these modes define a Hadamard state, as we will come to see. That is, we write

$$\phi(\tau,\vartheta) = \sum_{n=-\infty}^{\infty} a_n^{\Delta}\,\phi_n^{E,\Delta}(\tau,\vartheta) + a_n^{\Delta\dagger}\phi_n^{*E,\Delta}(\tau,\vartheta) \tag{3.2.9}$$

where [44]:

$$\phi_n^{E,\Delta}(\tau,\vartheta) = f_n^{E,\Delta}(\tau)\frac{e^{-in\vartheta}}{\sqrt{2\pi}} \ , \tag{3.2.10}$$

and the time dependent factor is:

$$f_n^{E,\Delta}(\tau) \equiv e^{i\alpha_n}\sqrt{\frac{\Gamma(\Delta-|n|)\Gamma(1-\Delta-|n|)}{2}}\,P_{-\Delta}^{|n|}\left(i\sinh\tau\right) \ , \tag{3.2.11}$$

where $P_a^b(x)$ is an associated Legendre function. We have chosen a particular phase factor

$$e^{2i\alpha_n} = e^{i\pi\left(|n|+\frac{1}{2}\right)}\frac{\Gamma(|n|+1-\Delta)\Gamma(1+\Delta)}{\Gamma(|n|+\Delta)\Gamma(2-\Delta)} \ , \tag{3.2.12}$$

which will play an important role in what follows. These modes are normalized such that

$$(\phi_n^{E,\Delta},\phi_m^{E,\Delta}) = \delta_{nm} \ , \qquad (\phi_n^{E,\Delta},\phi_m^{*E,\Delta}) = 0 \ , \tag{3.2.13}$$

where the bracket $(\cdot,\cdot)$ denotes Klein-Gordon inner-product:

$$(\phi_1,\phi_2) = -i\int_\Sigma d\Sigma^\mu\left(\phi_1\partial_\mu\phi_2^* - \phi_2^*\partial_\mu\phi_1\right) = -i\cosh\tau\int_0^{2\pi} d\vartheta\left(\phi_1\partial_\tau\phi_2^* - \phi_2^*\partial_\tau\phi_1\right) \ . \tag{3.2.14}$$

The phase factor (3.2.12) ensures that these modes transform nicely under the generators of $SL(2,\mathbb{R})$ given in (3.1.15):

$$L_0\phi_n^{E,\Delta} = -n\phi_n^{E,\Delta} \ , \qquad L_\pm\phi_n^{E,\Delta} = -(n\pm\Delta)\phi_{n\pm1}^{E,\Delta} \ . \tag{3.2.15}$$

One may find the above equation surprising, given that the Euclidean modes have an admixture of falloffs ($e^{-\Delta\tau}$ and $e^{-(1-\Delta)\tau}$) at late times, but it is nevertheless possible to express them as transforming properly under the conformal algebra.

Canonical quantisation proceeds by promoting $\phi(\tau,\vartheta)$ and its canonical conjugate

$\pi(\tau, \vartheta)$ to operators, where

$$\pi(\tau, \vartheta) \equiv \frac{\delta \mathcal{L}}{\delta(\partial_\tau \phi(\tau, \vartheta))} = \cosh \tau \, \partial_\tau \phi \ , \tag{3.2.16}$$

and demanding

$$[\phi(\tau, \vartheta), \pi(\tau, \vartheta')] = i\delta(\vartheta - \vartheta') \ . \tag{3.2.17}$$

This can be achieved by promoting $(a_n^\Delta, a_n^{\Delta\dagger})$ to operators that satisfy:

$$[a_n^\Delta, a_m^{\Delta\dagger}] = \delta_{nm} \ , \qquad [a_n^\Delta, a_m^\Delta] = [a_n^{\Delta\dagger}, a_m^{\Delta\dagger}] = 0 \ . \tag{3.2.18}$$

We must also choose a state on top of which we build our Fock space. The Euclidean vacuum $|\Omega\rangle$, is defined such that

$$a_n^\Delta |\Omega\rangle = 0 \ , \qquad \forall n \ . \tag{3.2.19}$$

What remains is to identify the basis states of the principal series UIR: $|\Delta, n\rangle$. A natural expectation is the following:

$$|\Delta, n\rangle \equiv a_n^{\Delta\dagger} \, |\Omega\rangle \tag{3.2.20}$$

at least at the single particle level [230]. To check that this is indeed correct, we must write down the conformal generators in the basis of creation and annihilation operators:

$$\widehat{L}_n = - \sum_{k=-\infty}^{\infty} (k + n\Delta) \, a_{k+n}^{\Delta\dagger} a_k^\Delta \ , \tag{3.2.21}$$

for $n = \{-1, 0, 1\}$. Using the canonical commutation relations (3.2.18), we find

$$\left[\widehat{L}_n, \widehat{L}_m\right] = (n - m)\widehat{L}_{n+m} \ , \tag{3.2.22}$$

as required. It is also straightforward to check that

$$\left[\widehat{L}_n, \phi(\tau, \vartheta)\right] = -L_n \phi(\tau, \vartheta) \ , \tag{3.2.23}$$

where the operators on the right hand side are the differential representation of the algebra given in (3.1.15). Finally given the definitions, a short computation yields:

$$\widehat{L}_0 \, a_n^{\Delta\dagger}|\Omega\rangle = -n \, a_n^{\Delta\dagger}|\Omega\rangle \ , \quad \widehat{L}_\pm \, a_n^{\Delta\dagger}|\Omega\rangle = -(n \pm \Delta) a_{n\pm 1}^{\Delta\dagger}|\Omega\rangle \ . \tag{3.2.24}$$

Hence we see that we have correctly identified $|\Delta, n\rangle \equiv a_n^{\Delta\dagger}|\Omega\rangle$. Given these definitions, the mode functions may be expressed as overlaps of the field operator ϕ and the state $|\Delta, n\rangle$:

$$\phi_n^{E,\Delta}(\tau,\vartheta) = \langle\Omega|\,\phi(\tau,\vartheta)\,|\Delta,n\rangle\ , \qquad\qquad \phi_n^{*E,\Delta}(\tau,\vartheta) = \langle\Delta,n|\,\phi(\tau,\vartheta)\,|\Omega\rangle\ . \quad (3.2.25)$$

3.2.3 Free two-point function

We now turn to the two-point function, or propagator, of a free, minimally-coupled, massive scalar field ϕ on dS_2

$$G_{\mathrm{f}}(X,Y) \equiv \langle\Omega|\phi(X)\phi(Y)|\Omega\rangle\ . \quad (3.2.26)$$

The subscript "f" refers to the fact that it is free. This satisfies

$$\left(\Box_{\mathrm{dS}} - m^2\right) G_{\mathrm{f}}(X,Y) = 0\ , \quad (3.2.27)$$

with Δ related to m through (3.2.4). Here we are choosing to study the Wightman function, but we could just as well study the retarded, advanced or Feynman propagator by replacing the right hand side of (3.2.27) with $\delta(X,Y)/\sqrt{-g}$ and choosing suitable boundary conditions. The above differential equation can be expressed as an ODE of the de Sitter invariant distance $u(X,Y)$ defined in (3.1.8)

$$u(2-u)G_{\mathrm{f}}''(u) + 2(1-u)G_{\mathrm{f}}'(u) + \Delta(\Delta-1)G_{\mathrm{f}}(u) = 0\ . \quad (3.2.28)$$

Being a second order differential equation, there are two linearly independent solutions

$$G_{\mathrm{f}}(u) = c_1\,{}_2F_1\left(\Delta, 1-\Delta, 1, 1-\tfrac{u}{2}\right) + c_2\,{}_2F_1\left(\Delta, 1-\Delta, 1, \tfrac{u}{2}\right)\ . \quad (3.2.29)$$

For Δ in the principal or complementary series, the term proportional to c_1 has the appropriate lightcone singularity in the limit $u \to 0$, while the term proportional to c_2 has an antipodal singularity in the limit $u \to 2$. Both behaviours are allowed in a de Sitter invariant state, however, we will further demand that the state $|\Omega\rangle$, upon which we build our Fock space, be Hadamard, which disallows any field singularities at spacelike-separated points. This fixes:

$$G_{\mathrm{f}}^{\Delta}(u) \equiv \frac{\Gamma(\Delta)\Gamma(1-\Delta)}{4\pi}\,{}_2F_1\left(\Delta, 1-\Delta, 1, 1-\tfrac{u}{2}\right)\ , \qquad \Delta \in \pi_\nu \quad \text{or} \quad \gamma_\Delta\ , \quad (3.2.30)$$

where the coefficient is set by demanding that we match onto the flat space answer in the limit $u \to 0$, which, in this case, is

$$G_{\mathrm{f}}(u) \underset{u\to 0}{\approx} -\frac{1}{4\pi}\log\frac{u}{2} \ . \tag{3.2.31}$$

Moreover, for π_ν, we can show the Hadamard two-point function admits a Fourier decomposition in terms of the Euclidean modes constructed in section 3.2.2:

$$G_{\mathrm{f}}^{\Delta}(u) = \sum_{n=-\infty}^{\infty} \phi_n^{E,\Delta}(\tau,\vartheta)\phi_n^{*E,\Delta}(\tau',\vartheta') = \frac{\Gamma(\Delta)\Gamma(1-\Delta)}{4\pi} {}_2F_1\left(\Delta, 1-\Delta, 1, 1-\tfrac{u}{2}\right) \ , \quad \Delta \in \pi_\nu \ . \tag{3.2.32}$$

The case of Δ in the discrete series is subtle and requires some care. Let us parametrize $\Delta = 1+t$ for $t = 0, 1, 2, \ldots$. Note that for these values of Δ, the coefficient of (3.2.30) diverges as $\Gamma(-t)$. This divergence has a physical origin [231], as we will discuss shortly. For now, let us repeat the exercise and try and solve (3.2.28) for these values of Δ. The two independent solutions are:

$$G_{\mathrm{f}}(u) = c_1\, P_t(1-u) + c_2\, Q_t(1-u) \ , \tag{3.2.33}$$

where P_t and Q_t are Legendre functions of order t. A peculiarity: The term proportional to c_1 is a polynomial of order t in $(1-u)$ and thus has no lightcone divergence in the limit $u \to 0$. Alternatively, the term proportional to c_2 has both lightcone and antipodal divergences. If we demand that $|\Omega\rangle$ be Hadamard, we are required to set $c_2 = 0$, throwing away both the coincident-point and antipodal singularities, *together*. This leaves a correlator free of divergences, or branch cuts—which is certainly not expectated for a local quantum field. It would seem, then, that there is no room for the discrete series to contribute to the two-point function, at least if we are to have a standard coincident point singularity.

As we will discuss in the next section, this conclusion is not quite correct. Following the work of [219, 231–233], we will show that a more delicate treatment indeed leads to a contribution from the discrete series UIR.

Revisiting the discrete series two-point function

We now proceed to explain the physical origin behind the divergence in (3.2.30) when $\Delta = 1+t$, following [231]. For this, let us recall that the Hadamard Wightman function on dS_2 can be obtained via analytic continuation of the two-point function on an S^2 of radius ℓ [46, 204] . Thus, we should compute the Euclidean path integral

$$G_{\rm E}^{\Delta}(\Omega,\Omega') = \frac{\int \mathcal{D}\phi\, \phi(\Omega)\phi(\Omega')\, e^{-S_{\rm E}[\phi]}}{\int \mathcal{D}\phi\, e^{-S_{\rm E}[\phi]}} \ , \tag{3.2.34}$$

where the Euclidean action is given by

$$S_{\rm E}[\phi] = \frac{1}{2}\int_{S^2} d^2x\sqrt{g}\,\phi(\Omega)\left[-\Box_{S^2}+m^2\right]\phi(\Omega) \ . \tag{3.2.35}$$

As usual, to evaluate the path integral it is convenient to expand the field $\phi(\Omega)$ in a basis of eigenfunctions of the two-sphere Laplacian, as

$$\phi(\Omega) = \sum_{L=0}^{\infty}\sum_{M=-L}^{L} c_{LM}\, Y_L^M(\Omega) \ , \qquad\qquad \mathcal{D}\phi = \prod_{L=0}^{\infty}\prod_{M=-L}^{L} dc_{LM} \ . \tag{3.2.36}$$

We have chosen the $Y_L^M(\Omega)$ to be real-valued, such that the c_{LM} are themselves real-valued. The $Y_L^M(\Omega)$ satisfy the standard orthonormality conditions

$$\Box_{S^2} Y_L^M = -\frac{L(L+1)}{\ell^2} Y_L^M \ , \qquad\qquad \int_{S^2} d^2x\sqrt{g}\, Y_L^M Y_{L'}^{M'} = \ell^2\,\delta_{LL'}\delta_{MM'} \ , \tag{3.2.37}$$

as well as the completeness relation

$$\sum_{L=0}^{\infty}\sum_{M=-L}^{L} Y_L^M(\Omega) Y_L^M(\Omega') = \delta(\Omega,\Omega') \ . \tag{3.2.38}$$

Performing the remaining Gaussian integrals over the c_{LM} leads to the Euclidean Green's function in momentum space

$$G_{\rm E}^{\Delta}(\Omega,\Omega') = \sum_{L=0}^{\infty}\sum_{M=-L}^{L} \frac{Y_L^M(\Omega)Y_L^M(\Omega')}{L(L+1)+m^2\ell^2} \ . \tag{3.2.39}$$

Upon employing the addition theorem

$$\sum_{M=-L}^{L} Y_L^M(\Omega)Y_L^M(\Omega') = \frac{2L+1}{4\pi} P_L(1-u(\Omega,\Omega')) \ , \tag{3.2.40}$$

we can further express the Euclidean Green's function in the following form

$$G_{\rm E}^{\Delta}(\Omega,\Omega') = \frac{1}{4\pi}\sum_{L=0}^{\infty} \frac{2L+1}{L(L+1)+m^2\ell^2} P_L(1-u(\Omega,\Omega')) \ . \tag{3.2.41}$$

In turn, for generic $m^2\ell^2$, the above sum can be performed explicitly, resulting in an expression involving the hypergeometric function. Namely,

$$G_{\mathrm{E}}^{\Delta}(\Omega,\Omega') = \frac{\Gamma(\Delta)\Gamma(1-\Delta)}{4\pi} {}_2F_1\left(\Delta, 1-\Delta, 1, 1-\tfrac{u(\Omega,\Omega')}{2}\right) \ , \tag{3.2.42}$$

as expected from (3.2.30), and where we have used (3.2.4), which relates $m^2\ell^2$ with Δ. Here $u(\Omega,\Omega')$ is the appropriate geodesic distance on the S^2, equivalent to the analytic continuation of (3.1.8) to Euclidean signature. In this case, it is straightforward to verify that

$$1 - \frac{u(\Omega,\Omega')}{2} = \cos^2\left(\frac{\theta(\Omega,\Omega')}{2}\right) \implies u(\Omega,\Omega') = 2\sin^2\left(\frac{\theta(\Omega,\Omega')}{2}\right) \ , \tag{3.2.43}$$

where $\theta(\Omega,\Omega')$ is the angle subtended by a geodesic arc connecting Ω to Ω'. Notice, however, that when $m^2\ell^2 = -t(t+1)$, with $t \in \mathbb{N}_0$, there are a collection of $(2t+1)$ modes, precisely those with $L = t$, with vanishing Euclidean action. The integrals over these modes necessarily lead to divergences which must be dealt with.[2] Thus, the Lorentzian discrete-series divergence originates from the fact that this theory suffers from a Euclidean vacuum state which is non-normalizable, precisely due to these problematic modes [43].

One can now proceed to try and define an appropriate Euclidean two-point function for the discrete series UIR [231]. The idea is to eliminate the problematic modes from the sum altogether:

$$H_{\mathrm{f}}^{\Delta=1+t}(\Omega,\Omega') = \sum_{\substack{L=0\\L\neq t}}^{\infty} \sum_{M=-L}^{L} \frac{Y_L^M(\Omega) Y_L^M(\Omega')}{L(L+1) - t(t+1)} \ , \tag{3.2.44}$$

and try to give this function a Lorentzian Hilbert space interpretation, as in [219,220]. From here on, we will label free propagators on the discrete series as H_{f}, the "f" again referring to the fact that it is free, so as to distinguish it from the typical two-point function of the principal and complementary series. This procedure is inherently ad-hoc, and the final answer will necessarily be ambiguous, moreover, it is difficult to reconcile with local quantum field theory, although in the next section we will give an example of how to proceed when $\Delta = 1$.

[2]The modes with $L < t$ have negative Euclidean action, and may seem even more problematic. One way to deal with these is by analytically continuing the contour of integration for the offending c_{LM}'s.

Note that the completeness relation (3.2.38) implies that

$$\left[-\ell^2\Box_{S^2} - t(t+1)\right] H_{\rm f}^{\Delta=1+t}(\Omega,\Omega') = \delta(\Omega,\Omega') - \frac{2t+1}{4\pi}P_t(1-u(\Omega,\Omega')) \ , \quad (3.2.45)$$

where we have used the addition theorem (3.2.40). We see that the Klein-Gordon operator acting on the two-point function (with the problematic zero-modes removed) isn't sourced by a local δ-function disturbance, but rather, by a function supported on the entire S^2—evidence of some tension with locality. Moreover, this right hand side implies that the notion of the identity operator on the Hilbert space needs modification whenever the discrete series is concerned. Since all we've done is remove an entire $SO(3)$ representation from the sum, the final answer remains $SO(3)$-invariant, and the analytically continued result will therefore be de Sitter invariant.

It is possible to solve this inhomogeneous Klein-Gordon equation outright, giving:

$$H_{\rm f}^{\Delta=1+t}(u;\alpha) = -\frac{1}{4\pi}P_t(1-u)\left(\log\frac{u}{2}+\alpha\right) - \frac{1}{2\pi}\sum_{s=0}^{t-1}\frac{2s+1}{t(t+1)-s(s+1)}P_s(1-u) \ , \quad (3.2.46)$$

where the constant α is an ambiguity proportional to a homogeneous solution to (3.2.33). The Green's function defined in (3.2.44) is equivalent to (3.2.46) with $\alpha = 0$, but we have included the $\alpha \neq 0$ term in order to provide the general solution to (3.2.45), which reflects the ambiguity in defining the procedure for removing the zero-modes. Indeed, in [231], the parameter α is related to a BRST gauge-fixing procedure. The formula (3.2.46) has the appropriate short-distance singularity as in (3.2.31), as expected for a two-point function in a Hadamard state, but the coefficient α can't be fixed by any local requirement. We will also make use of the following definition

$$H_{\rm f}^{\Delta=1+t}(u) \equiv H_{\rm f}^{\Delta=1+t}(u;0) \ . \quad (3.2.47)$$

Suffice it to say: if we ever encounter the Klein-Gordon operator with $m^2\ell^2 = -t(t+1)$ under any circumstance in dS_2, we should exercise care. In what follows, we will provide some examples where such equations arise.

Comment on positivity of $H_{\rm f}$ for $0 < u \leq 2$: In the spacelike separated regime ($0 < u \leq 2$), Euclidean and Lorentzian correlators agree. On the sphere, we typically interpret the two-point function at antipodally-separated points as the norm of a state. Based on this intuition, we expect the two-point function in this regime to be positive definite. However (3.2.46) is oscillatory in this regime, and is not sign definite at antipodal points for every t. This is unlike the principal and complementary series correlators (3.2.42), which are positive definite for $0 < u \leq 2$.

Comment on late-time behaviour of $H_{\rm f}$: We now turn to the late-time behaviour of these correlation functions. Choosing the global coordinate system (3.1.10), and taking $\tau = \tau' \to \infty$ we find the following late-time behaviour for a discrete series two-point function:

$$\lim_{\tau\to\infty} H_{\rm f}^{\Delta=1+k}(u;\alpha) = e^{2k\tau}\frac{(-1)^{k+1}}{4\pi}\begin{pmatrix} k-\frac{1}{2} \\ k \end{pmatrix}\sin^{2k}\left(\frac{\vartheta-\vartheta'}{2}\right)\left\{2\tau+\alpha+\log\left[\frac{1}{4}\sin^2\left(\frac{\vartheta-\vartheta'}{2}\right)\right]\right\} \tag{3.2.48}$$

where we have used k in place of t to disambiguate it from the time coordinate τ. This piece of the correlation function grows at late times. Note that the dependence on the ambiguous parameter α is subleading at the future boundary, albeit only polynomially in the global time τ. On the other hand, the late-time behaviour of the free principal series correlator is

$$\lim_{\tau\to\infty} G_{\rm f}^{\Delta}(u) = e^{-2\Delta\tau}\frac{\Gamma(\Delta)\Gamma\left(\frac{1}{2}-\Delta\right)}{4\pi^{3/2}}\left|\sin\left(\frac{\vartheta-\vartheta'}{2}\right)\right|^{-2\Delta} + \text{c.c.}\ , \qquad \Delta\in\pi_\nu\ , \tag{3.2.49}$$

meaning that, at late times, the discrete series contributions, if present, will wash out the imprint of the principal series on the conformal boundary.

Before moving on, let us briefly comment on a familiar example of a discrete series theory: the case of the free, massless scalar. The massless free boson has an action invariant under constant shifts of the field $\phi(x) \to \phi(x) + c$, and integrating over the constant mode of ϕ leads to a divergence since this mode is not Gaussian suppressed in Euclidean signature, again rendering the Laplacian operator non-invertible. There are two familiar remedies: we can either compactify the zero-mode by identifying $\phi \sim \phi + R$ in which case the field ϕ is no longer well-defined as a local scalar operator on Hilbert space and we must instead consider operators such as $:e^{2\pi i\phi/R}:$,[3] or we can gauge the shift symmetry, in which case ϕ is not a gauge-invariant operator on Hilbert space. In both cases, the bare field $\phi(x)$ loses its status as a well-defined quantum field acting on Hilbert space.

[3] Even in the case of the compact free scalar, global constraints arise when the zero-mode is treated carefully, see exercise 9.2 of [234]. For example, shifts of ϕ act as $U(1)$ phase rotations of the vertex operator $:e^{2\pi i\phi/R}:$, and the only non-vanishing vacuum correlators are those of charge-neutral strings of vertex operators.

It is of crucial importance that we acknowledge that the discrete series can not be relegated as an easy-to-ignore curiosity. In appendix F, we solve a quantum mechanical model that may be thought of as the late-time single-particle Hilbert space of two principal series fields propagating in dS_2. We show that the discrete series arises in the the two-particle Hilbert space of this quantum mechanical example, via a simple Clebsch-Gordan analysis. The projecting-out of the problematic modes happens simply by demanding normalizability of the two-body wavefunctions.

In the following section, we will examine the case of a free massless scalar with a gauged shift symmetry and show that the correlators of this theory can be derived starting from (3.2.46).

3.2.4 Scalar with a gauged shift symmetry and the $\Delta = 1$ discrete series

With the general discussion of the previous section now behind us, let us provide a simple example where the Green's function (3.2.46) (for $\Delta = 1$) makes an indirect appearance. Consider a massless scalar coupled to a gauge field:

$$S = -\frac{1}{2}\int d^2x\sqrt{-g}g^{\mu\nu}\left(\partial_\mu\phi - A_\mu\right)\left(\partial_\nu\phi - A_\nu\right) + k\int d^2x\sqrt{-g}\,B\,\epsilon^{\mu\nu}F_{\mu\nu}\ . \qquad (3.2.50)$$

Here, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, B is a real-valued scalar and $\epsilon^{\mu\nu}$ is the antisymmetric Levi-Civita tensor with $\epsilon^{T\vartheta} = +1/\sqrt{-g}$.[4] The model's global shift symmetry $\phi(x) \to \phi(x) + c$, absent the gauge field, is promoted to a local symmetry. We take the shift symmetry to be non-compact such that $k \in \mathbb{R}$. Explicitly, the model's Abelian gauge invariance is $\phi(x) \to \phi(x) + \omega(x)$ and $A_\mu \to A_\mu + \partial_\mu\omega$ with ω a smooth real-valued function.

Had we not gauged the shift symmetry, the model would suffer from a pathological zero mode, as explained in the previous section. In [231], the constant shift mode is gauged via a non-local condition on the S^2. The model (3.2.50) follows the spirit of [231], but the advantage of this setup is that we are always in the realm of local quantum field theory.

It is convenient to consider the model in the global coordinate system (3.1.17), for

[4] We will distinguish between the Levi-Civita tensor and symbol by denoting the latter as $\tilde{\epsilon}^{\mu\nu}$ such that $\tilde{\epsilon}^{T\vartheta} = +1$.

which the Weyl factor drops out altogether. Gauge invariant operators are given by

$$B \ , \qquad F_{\mu\nu} \ , \qquad \mathcal{O}_\mu(T,\vartheta) = \partial_\mu\phi(T,\vartheta) - A_\mu(T,\vartheta) \ , \qquad \chi = \oint_{\mathcal{C}} A_\mu dx^\mu \ , \quad (3.2.51)$$

and combinations thereof. The curve $\mathcal{C}$ is taken to be a closed spacelike curve. In addition, the dressed operators

$$\mathcal{O}_q(T,\vartheta) = e^{-iq\phi(T,\vartheta)} \exp iq\int_{\mathcal{L}} A_\mu dx^\mu \ , \qquad (3.2.52)$$

where $\mathcal{L}$ is a curve beginning at some reference point and ending at (T,ϑ), can be arranged into gauge-invariant combinations by taking products for which the sum of the q vanishes.

We can construct a Hilbert space by acting on the vacuum state $|\Omega\rangle$ with suitable combinations or distributions of the gauge-invariant operators. Working in the $A_T = 0$ gauge, the ensuing constraint is given by

$$2k\,\partial_\vartheta B - \partial_T\phi = 0 \ . \qquad (3.2.53)$$

This fixes the non-constant spatial modes of B, leaving only the constant mode $b \equiv \frac{1}{2\pi}\oint d\vartheta B$ as an independent gauge-invariant operator. We must further ensure invariance under residual gauge transformations given when ω is purely a function of ϑ. This can be used to gauge away the spatial non-zero modes of A_ϑ, again leaving χ as the gauge-invariant operator.

Thus, we land on the non-gauge invariant operator algebra

$$[\phi(T,\vartheta), \partial_T\phi(T,\vartheta')] = i\delta(\vartheta-\vartheta') \ , \qquad [b,\chi] = -\frac{i}{2k} \ . \qquad (3.2.54)$$

To create single-particle states, we can build a creation operator out of the gauge-invariant operators. Creation and annihilation operators which stem from gauge invariant operators, expressed in the $A_T = 0$ gauge, read as follows

$$a_n - a_{-n}^\dagger \equiv i\sqrt{\frac{2}{|n|}}\int d\vartheta e^{-in\vartheta}\partial_T\phi(T,\vartheta)|_{T=0} \ , \qquad (3.2.55)$$

$$a_n + a_{-n}^\dagger \equiv -i\,\mathrm{sgn}\,n\sqrt{\frac{2}{|n|}}\int d\vartheta e^{-in\vartheta}\left(\partial_\vartheta\phi(0,\vartheta) - A_\vartheta(0,\vartheta)\right) \ , \qquad (3.2.56)$$

where $n \in \mathbb{Z}/\{0\}$. We thus define the vacuum $|\Omega\rangle$ as the state annihilated by the a_n, while acting with the $a_n^\dagger$ for either $n > 0$ (or $n < 0$) furnishes the lowest (highest) weight $\Delta = 1$ UIR. Additional states are created by acting with the operators χ and $\mathcal{O}_q(T, \vartheta)$.

One can also consider the model in Euclidean signature, on the two-sphere, whose metric is given by (3.1.27). The path-integral of interest is now

$$\mathcal{Z}_{\rm BF} = \int \frac{\mathcal{D}\phi \mathcal{D}A_\mu \mathcal{D}B}{\mathrm{vol}\,\mathcal{G}} \, e^{-S_E[\phi, A_\mu]} \, e^{ik\int \sqrt{g} B \epsilon^{\mu\nu} F_{\mu\nu}} \, , \tag{3.2.57}$$

where

$$S_E[\phi, A_\mu] = \frac{1}{2}\int d^2x \sqrt{g} g^{\mu\nu} \left(\partial_\mu \phi - A_\mu\right) \left(\partial_\nu \phi - A_\nu\right) \, , \tag{3.2.58}$$

and $\mathrm{vol}\,\mathcal{G}$ is the volume of the gauge group. Path-integrating over B imposes that $A_\mu = \partial_\mu \xi$ is locally pure gauge, and ξ is a non-constant function which we use to parameterize the entire field configuration space of flat-curvature connections. Up to a Jacobian, this imposes $\mathcal{D}A_\mu \to \mathcal{D}\xi$. One can subsequently eliminate any ξ dependence from the action by a shift in $\phi \to \phi + \xi$. The path-integral over ξ then cancels against $\mathrm{vol}\,\mathcal{G}$, save for the zero-mode corresponding to the constant part of the gauge group. This remaining zero-mode is cancelled by the integral over the constant mode of ϕ. One subsequently computes expectation values of gauge-invariant operators. Employing the results in section 3.2.3 we have the Euclidean two-point function

$$\langle \mathcal{O}_\mu(\Omega) \mathcal{O}_\mu(\Omega') \rangle = -\frac{1}{4\pi} \partial_\mu \partial_{\mu'} \left(\log \frac{u(\Omega, \Omega')}{2} + \alpha \right) \, , \tag{3.2.59}$$

where Ω and Ω' are points on the two-sphere, and

$$u(\Omega, \Omega') = \frac{\cosh(X - X') - \cos(\vartheta - \vartheta')}{\cosh X \cosh X'} \, , \tag{3.2.60}$$

is the invariant length on the two-sphere, in analogy with (3.1.8). The result can be Wick rotated back to dS_2, producing an $SL(2, \mathbb{R})$ covariant result. Importantly (3.2.59) is obtained by taking derivatives of $H_{\rm f}^{\Delta=1}(u; \alpha)$ written in (3.2.46), and the necessity to compute a correlation function of gauge-invariant operators kills any dependence on the ambiguous coefficient α.

It may be possible to repeat this exercise and gauge the non-constant global shift-symmetries of the scalar in the case of the $t \neq 0$ discrete series.

The $\Delta = 2$ discrete series equation

Now we briefly comment on an example where the $\Delta = 2$ UIR makes an appearance. We will consider a different example in section 3.4.2, which shares some features with this one. Recall, following (3.2.3), that the $\Delta = 2$ Casimir equation is:

$$\ell^2 \Box_{\text{dS}} \phi = -2\phi \; . \tag{3.2.61}$$

This equation can be derived in a setting where we couple 2d quantum gravity to a two-dimensional conformal-matter field theory with large positive central charge. Upon integrating out the matter-CFT, the fluctuations of the Weyl factor ω of the physical metric in the Weyl gauge take the form of a tachyonic scalar in de Sitter (as noted in footnote 6 of [124], see also [235]). To be explicit, parameterize the metric $g_{\mu\nu} = e^{2\omega(T,\vartheta)}\tilde{g}_{\mu\nu}$ with $\tilde{g}_{\mu\nu}$ given by (3.1.17). The constant-curvature equation of motion $R[g] = 2/\ell^2$ can be written as:

$$\ell^2 \Box_{\text{dS}} \omega(T,\vartheta) \approx -2\omega(T,\vartheta) \; , \tag{3.2.62}$$

for small $\omega(T,\vartheta)$. As we discuss in section 3.4.2, in a somewhat different example, the conformal factor $\omega(T,\vartheta)$ is subject to the residual diffeomorphism constraints, which, in turn, remove the three Euclidean zero-modes of (3.2.62).

3.2.5 Discrete series analogues in dS_4

Let us now discuss the analogues of the discrete series in four-dimensional de Sitter space. The isometry group of dS_4 is $SO(1,4)$, so in addition to conformal dimension Δ, UIRs in this setting are also labeled by a spin s quantum number associated to the $SO(3)$-rotation subgroup of $SO(1,4)$. A spin s field on dS_4 has Casimir eigenvalue, directly generalizing the $SL(2,\mathbb{R})$ case:

$$\mathcal{C} = \Delta(\Delta - 3) + s(s+1) \; . \tag{3.2.63}$$

There are two possible discrete series analogs in dS_4 (see section 4 of [101]):

- **Exceptional Type I:** In the $s = 0$ sector, one finds a collection of discrete scalar UIRs, known as the Exceptional Type I representations, labeled by a discrete conformal dimension $\Delta = 3+k$ where k, again, is a non-negative integer. One proposal for a free-field-theoretic construction of these representations is the following: Consider the action

$$S = -\frac{1}{2}\int d^4x\,\sqrt{-g}\left[g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + m_k^2\,\phi^2\right] \tag{3.2.64}$$

where the mass is tachyonic and satisfies:

$$m_k^2\ell^2 = -k(k+3)\ , \tag{3.2.65}$$

analogous to (3.2.5). This field is massless for $k=0$, but is otherwise tachyonic for larger values of k. As noted in [231], this theory suffers from a set of unsuppressed Euclidean zero-modes associated to the following symmetry of the action (3.2.64):

$$\phi \to \phi + \lambda_k\ , \qquad \lambda_k = s_{A_1A_2\ldots A_k}X^{A_1}X^{A_2}\ldots X^{A_k}\ , \tag{3.2.66}$$

where $s_{A_1A_2\ldots A_k}$ is a real traceless and symmetric constant tensor and the X^A are coordinates on the ambient hyperboloid, as in (3.1.1). For $k=0$, this is the familiar shift symmetry of the free, massless scalar. If this symmetry is gauged [219, 231], as in the $\Delta = 1$ example of section 3.2.4, then the theory might be amenable to quantisation. So far, no one has yet attempted the exercise.

- **Exceptional Type II:** For $s \neq 0$, one finds an additional family of discrete UIRs realised as free spin-s (partially) massless gauge fields in dS_4. These fields have:

$$\Delta = 2 + t\ , \qquad\qquad t = 0, 1, \ldots s-1\ . \tag{3.2.67}$$

The quantity t is called the *depth*. To realize these exceptional series on de Sitter, consider the following field theory of a fully-symmetrized, transverse, traceless, spin-s field which satisfies the following equations of motion [236]:

$$\left[\Box_{\text{dS}} - m^2 + \frac{s(s-2)-2}{\ell^2}\right]\phi_{\mu_1\ldots\mu_s} = 0\ , \quad \nabla^\nu\phi_{\nu\mu_2\ldots\mu_s} = 0\ , \quad \phi^\nu{}_{\nu\mu_3\ldots\mu_s} = 0\ . \tag{3.2.68}$$

At generic values of the mass, this equation propagates $2s+1$ degrees of freedom. In dS_4, the Higuchi bound for such a spin-s field is given by:

$$m^2\ell^2 \geq s(s-1)\ . \tag{3.2.69}$$

Below this value of the mass, one of the Stückelberg fields that implement the transverse-tracelessness conditions obtains a ghost-like kinetic term, rendering the theory non-unitary. However, there are a set of special masses, all at or below the Higuchi bound where the theory develops a gauge symmetry that removes the ghosts. These masses are:

$$m_{s,t}^2 \ell^2 = (s-1-t)(s+t) \ , \qquad\qquad t = 0, 1, \ldots, s-1 \ . \tag{3.2.70}$$

At these special points, known as the *partially massless* points, the equations of motion (3.2.68) develop a symmetry under $\phi \to \phi + \delta\phi$, where

$$\delta\phi_{\mu_1 \ldots \mu_s} = \nabla_{(\mu_{t+1}} \nabla_{\mu_{t+2}} \ldots \nabla_{\mu_s} \lambda_{\mu_1 \ldots \mu_t)} + \ldots \tag{3.2.71}$$

and the additional dots indicate terms with fewer derivatives.[5] The gauge parameter must itself satisfy:

$$\left[\Box_{\rm dS} + \frac{(s-1)(s+2)-t}{\ell^2} \right] \lambda_{\mu_1 \ldots \mu_t} = 0 \ , \quad \nabla^\nu \lambda_{\nu\mu_2 \ldots \mu_t} = 0 \ , \quad \lambda^\nu{}_{\nu\mu_3 \ldots \mu_t} = 0 \ . \tag{3.2.72}$$

In total, this gauge symmetry amounts to removing a massive spin-t field's worth of propagating degrees of freedom. Thus the partially massless spin-s field propagates a total of $2(s-t)$ degrees of freedom. Note that the maximal-depth field with $t = s-1$, the field is massless and propagates two polarisations, just like the graviton. At the time of writing, besides Vasiliev theory on dS$_4$ (which has an infinite tower of massless fields), no consistent interacting theory with partially massless fields is known. Vasiliev theory possesses an enormous higher-spin gauge symmetry [156], one which encompasses the underlying $SO(1,4)$ global symmetry of the dS$_4$ spacetime. So while we might naively think the Vasiliev fields carry conformal dimensions labeled by $\Delta = 1+s$, the gauge-invariant Hilbert space only consists of $\Delta = 0$ states.

In dS$_2$ there is no intrinsic spin: the $SO(3)$ rotation group is replaced by $SO(1) \cong \mathbb{Z}_2$ with respect to which states can be graded. We may therefore wonder if the discrete series states in dS$_2$ admit realisations that share features with either the Exceptional Series I or II of dS$_4$, as described above. Although the tachyonic scalar theory seems the

[5] As we will not need it, we do not provide the full expression for the gauge invariance of (3.2.68). The interested reader can find it in equation (2.5) of [237].

most natural to consider – due to the absence of spin in two-dimensions – in section 3.4 we will describe a scenario more closely related to the $s \neq 0$ partially-massless fields of $SO(1,4)$.

3.3 Spectral decomposition

In this section, we discuss the Källén-Lehmann [63, 64] spectral decomposition of the two-point function for general interacting scalar fields in dS_2. This has been discussed in many works before us [54, 65, 66, 172, 212, 221, 238, 239]. Our goal here is to focus on the contributions from discrete series states $D^{\pm}_{\Delta}$.

To set the stage, let us quickly review the spectral decomposition of the two-point function of an interacting scalar field ϕ in d-dimensional flat space:

$$G_{\rm i}(x-y) = \langle\Omega|\,\phi(x)\phi(y)\,|\Omega\rangle\ . \tag{3.3.1}$$

The subscript "i" denotes that it is "interacting." Under very general assumptions, this correlation function can be expressed as [63, 64, 209]

$$G_{\rm i}(x-y) = \int_0^\infty d\mu^2 \rho_\phi(\mu^2) \int \frac{d^dp}{(2\pi)^d} \frac{e^{ip\cdot(x-y)}}{p^2+\mu^2}\ . \tag{3.3.2}$$

In this form, unitarity of the two-point function demands $\rho_\phi(\mu^2) > 0$ and,

$$\int_0^\infty d\mu^2 \rho_\phi(\mu^2) = 1\ . \tag{3.3.3}$$

The usefulness of this representation stems from the fact that the interacting correlation function $G_{\rm i}(x-y)$ can be expressed, up to an undetermined function, out of free-field propagators, under any circumstance; allowing us to anchor our intuition on free-field theory, and free-field excitations. The quantity $\rho_\phi(\mu^2)$ is called the *spectral density* of the interacting field ϕ, and is chock-full of information about the underlying theory—namely it has within it data about the overlap between $\phi(x)\,|\Omega\rangle$ and any state in the Hilbert space. Hence, for example, it can tell us if the field $\phi(x)$ creates single-particle excitations, or if it is composite.

To see this, note that spectral function governs the analytic properties of the two-point amplitude $G_{\rm i}(p) = \int_0^\infty d\mu^2 \frac{\rho_\phi(\mu^2)}{p^2+\mu^2}$, which will have poles at the single-particle states created by $\phi(x)$ and a branch cut starting at the first multi-particle state. Hence if $G_{\rm i}(p)$ only has a single pole, we have an invariant notion of a 'free-field.' Alternatively we define $\phi(x)$ as composite if $G_{\rm i}(p)$ only has branch cuts and no poles.

On general curved backgrounds, we can't use analyticity arguments, as we do in flat space, to give an invariant notion to the meaning of single-particle (poles) and multi-particle (branch-cuts) states. But hope is not lost. de Sitter is a maximally-symmetric spacetime, and we have the constraining power of group theory to guide us—so while we may not have a notion of multi-*particles*, we do have tensor product representations, as we discuss now.

One might then think, following (3.2.41), that the natural generalisation of (3.3.2) to Euclidean dS_2 is:

$$G_{\mathrm{i}}(u) = \frac{1}{4\pi}\int_{\frac{1}{4\ell^2}}^{\infty} dm^2 \rho_\phi(m^2) \sum_{L=0}^{\infty} \frac{2L+1}{L(L+1)+m^2\ell^2} P_L(1-u) \; , \tag{3.3.4}$$

but, this formula, for the subtleties described in section 3.2.3, misses contributions from states in the discrete series $D^{\pm}_{\Delta}$, which are required to appear by group theoretic considerations [99].

3.3.1 A proposal

To derive the spectral decomposition, we start with the identity operator on the field-theoretic Hilbert space of a general interacting scalar field theory:

$$\mathbb{1} = |\Omega\rangle\langle\Omega| + \sum_{\Delta,n} \frac{|\Delta,n\rangle\langle\Delta,n|}{\langle\Delta,n|\Delta,n\rangle} \; . \tag{3.3.5}$$

The state $|\Omega\rangle$ is the (Hadamard) Bunch-Davies vacuum, and the states $|\Delta,n\rangle$ carry quantum numbers, respectively, under the quadratic Casismir $\mathcal{C}$ and the action of rotation L_0, as in (3.2.1).[6] The normalisation factor must be included for a generic UIR, since only the principal series can simultaneously be made unit normalized while also faithfully transforming under the action of the ladder operators (3.2.1), see appendix B.

Let us now consider the two-point function of a general interacting scalar field in dS_2:

$$G_{\mathrm{i}}(X,Y) \equiv \langle\Omega|\phi(X)\phi(Y)|\Omega\rangle \; . \tag{3.3.6}$$

[6]More generally, there is also a sum over states transforming with either positive (even) or negative (odd) action under the operator $e^{2\pi i L_0}$. The scalar case we are studying leaves only states with even action under this transformation. The odd states are relevant to the case of fermionic fields as considered in [55], to which our analysis can be extended. We note that the complementary series are non-unitary in the odd case, and so would not appear in the decomposition of the identity.

Inserting the identity operator, this implies:

$$G_{\rm i}(X,Y) = \langle\phi\rangle^2 + \sum_{\Delta,n} \frac{\langle\Omega|\,\phi(X)\,|\Delta,n\rangle\,\langle\Delta,n|\,\phi(Y)\,|\Omega\rangle}{\langle\Delta,n|\Delta,n\rangle}\,. \tag{3.3.7}$$

Although we have included the possibility of a vacuum expectation value for the field ϕ, from now on, we will assume that $\langle\phi\rangle \equiv \langle\Omega|\,\phi(X)\,|\Omega\rangle = 0$ in the Bunch-Davies state. We will use an additional fact about $SL(2,R)$ representation theory, namely that the operator $\mathbb{1}$ *excludes the complementary series* [99]. Using this:

$$\begin{aligned} G_{\rm i}(X,Y) = \sum_{t=0}^{\infty}&\left\{\sum_{n\neq\{-t,..,t\}} \frac{\Gamma(|n|+1+t)}{\Gamma(|n|-t)}\,\langle\Omega|\,\phi(X)\,|1+t,n\rangle\,\langle 1+t,n|\,\phi(Y)\,|\Omega\rangle\right\} \\ &+\int_{-\infty}^{+\infty} d\nu\left\{\sum_{n\in\mathbb{Z}} \langle\Omega|\,\phi(X)\left|\tfrac{1}{2}\left(1+i\nu\right),n\right\rangle\left\langle\tfrac{1}{2}\left(1+i\nu\right),n\right|\phi(Y)\,|\Omega\rangle\right\}\,. \end{aligned} \tag{3.3.8}$$

In the above equation, the first line includes the contributions from discrete series $D^{\pm}_{1+t}$ states (and we have combined the highest and lowest weight UIRs into a single sum), and the term out front is the normalisation factor (see appendix B). The second line contains the contributions coming from principal series states π_ν.[7]

Principal series contribution: We now proceed to write down the principal series contribution to (3.3.8). We have already done most of the work in section 3.2.2. This argument is similar to one that appeared in [65]. Given that our Fock space is built atop the Bunch-Davies state $|\Omega\rangle$ the symmetries of the problem require:

$$\langle\Omega|\,\phi(X)\left|\tfrac{1}{2}\left(1+i\nu\right),n\right\rangle = c(\nu)\phi_n^{E,\Delta=\frac{1}{2}(1+i\nu)}(X)\,, \tag{3.3.9}$$

$$\left\langle\tfrac{1}{2}\left(1+i\nu\right),n\right|\phi(Y)\,|\Omega\rangle = c(\nu)^*\phi_n^{*E,\Delta=\frac{1}{2}(1+i\nu)}(Y)\,, \tag{3.3.10}$$

where the mode functions are given in (3.2.10) and subsequent equations. The undetermined coefficient $c(\nu)$ contains information about the field ϕ and its interactions. Now, we can use (3.2.32) to express:

$$\sum_{n\in\mathbb{Z}} \langle\Omega|\,\phi(X)\left|\tfrac{1}{2}\left(1+i\nu\right),n\right\rangle\left\langle\tfrac{1}{2}\left(1+i\nu\right),n\right|\phi(Y)\,|\Omega\rangle = \rho_\phi(\nu)\,G_{\rm f}^{\Delta=\frac{1}{2}(1+i\nu)}\left(u(X,Y)\right) \tag{3.3.11}$$

[7]One can incorporate the complementary series γ_Δ by appropriately shifting the principal series contour.

where $G_{\rm f}^{\Delta}(u)$ is given in (3.2.30) and $\rho_\phi(\nu) \equiv |c(\nu)|^2 \geq 0$.

Discrete series contribution: Identifying the discrete series mode functions is a subtle problem. As described in section 3.2.3, this is because, on this representation and in stark contrast to the principal series, the corresponding Euclidean Laplacian operator for the discrete series is non-invertible. The non-invertibility of the Euclidean differential operator indicates a gauge redundant structure in the discrete series sector, and suggests [219] we must modify the Klein-Gordon representation of the Casimir operator. A potential modification is described at length in [219], but we will not review it here. We simply quote the following:

$$\sum_{n\neq\{-t,..,t\}} \frac{\Gamma(|n|+1+t)}{\Gamma(|n|-t)} \langle\Omega|\,\phi(X)\,|1+t,n\rangle\,\langle 1+t,n|\,\phi(Y)\,|\Omega\rangle = \sigma_\phi(t;\alpha_t)\, H_{\rm f}^{\Delta=1+t}\,(u(X,Y);\alpha_t)\ . \tag{3.3.12}$$

The left hand side of this equation suggests that we may be able to prove that $\sigma_\phi(t;\alpha_t) \geq 0$, but since the functions $H_{\rm f}^{\Delta=1+t}(u;\alpha)$ are not positive on the Euclidean section, at this stage we can not impose any positive conditions on the density σ_ϕ. Moreover, we have left in the inherent ambiguity with respect to the choice of α_t.

Final answer: We are now ready to write down the full Källén-Lehmann representation for an interacting scalar field ϕ in two dimensions:

$$G_{\rm i}(X,Y) = \sum_{t=0}^{\infty} \sigma_\phi(t;\alpha_t)\, H_{\rm f}^{\Delta=1+t}\,(u(X,Y);\alpha_t) + \int_{-\infty}^{\infty} d\nu\, \rho_\phi(\nu)\, G_{\rm f}^{\Delta=\frac{1}{2}(1+i\nu)}\,(u(X,Y))\ . \tag{3.3.13}$$

A remark is in order. While we were not able to constrain the sign of σ_ϕ, nor its dependence on the ambiguous parameters α_t, it must be that the right hand side leads to a unitary two-point function for an interacting scalar theory in de Sitter. This means that the total sum of discrete and principal series contributions must result in a two-point function with appropriate properties: positivity on the Euclidean section as well as a positive coincident point limit that grows at most logarithmically in u. While the density on the discrete series may or may not be sign definite, this certainly constrains how they may contribute to the final answer.

Given the discussion around (3.2.48), it is interesting to note that, if present, the discrete series contributions will dominate over the principal series contributions at the late-time boundary. While this seems bizarre, it tells us that we need to either search for a general principle that would exclude the contribution of the discrete series in the two-point function of scalars in dS_2, or understand how and why they may arise in interacting scalar field theories. We have no way to discount the discrete series representations based on group theoretic arguments alone, unless the de Sitter group is gauged—as is the case when gravity is turned on. In quantum field theory on a rigid dS_2 background, the discrete series states are, without a doubt, present in the tensor product Hilbert space of two species of particles, even in the absence of interactions. We take this as an invitation to try and understand the physical nature of these vexing representations.

3.4 Discrete series operators in BF theory

This section is concerned with BF gauge theories on dS_2. Our interest in these theories stems from the fact that, in a suitable gauge, the equation of motion

$$\Box_{\mathrm{dS}}\phi = -t(t+1)\phi \,, \qquad t \in \mathbb{Z} \tag{3.4.1}$$

arises quite naturally. One may use this to conclude that the discrete series UIRs have a role to play in BF theories on dS_2, but this is too quick. After all, BF theories are topological, meaning they are insensitive to the background on which they live, and there should be no imprint of the dS_2 background once we have appropriately quantised the theory. However, there is a sense in which the discrete series UIRs are realized at the level of the pre-Hilbert space of the field operators, in the sense of (3.2.2). That is, the discrete series UIRs are realized by the linearized field equations of the $SL(N,\mathbb{R})$ BF theory, with $N \geq 2$, as we will demonstrate below. These field operators do not survive the imposition of the gauge constraints, but if, for example, we add a boundary to dS_2, for example along a worldline in the static patch, we can imagine breathing life into these modes.

Moreover, semiclassically, the $SL(N,\mathbb{R})$ gauge theory is a two-dimensional version of higher-spin theory [222,240] whose gravitational subsector is governed by the $SL(2,\mathbb{R})$ embedding inside of $SL(N,\mathbb{R})$. The field operators play a crucial role in formulating the gauge theory, but they are subject to the gauge constraints. Imposing these gauge constraints, for the theory quantised on a spatial circle, cuts down the size of the pre-Hilbert space and leads to a physical (gauge-invariant) Hilbert space absent of any non-trivial $SL(2,\mathbb{R})$ representations. Nonetheless, the presence of the discrete series UIR, at the level of the pre-gauged operator algebra, plays an important role in characterizing the entanglement structure of the theory [241], as well as providing a convenient basis for expressing the wavefunctionals [224,225]. These pre-Hilbert space states may offer a guiding principle for a microphysical completion of the theory, as in [154,155].

The setup in this section may be compared with an analogous one in dS_4. At the free level, the discrete series UIR of $SO(1,4)$ may be realized as the single-particle Hilbert space of (partially) massless gauge fields of spin s, with $s = 1, 2, \ldots$ (along with their fermionic counterparts [102]) (see section 3.2.5). Turning on interactions among these fields, which include the linearized graviton for $s = 2$, requires that we gauge any residual symmetries, *including* the $SO(1,4)$ isometry group of dS_4, which is a subgroup of the diffeomorphism group [145, 146]. As such, non-trivial $SO(1,4)$ UIRs are projected out of the physical Hilbert space of the interacting theory. On its own, the free theory leads to a somewhat misleading picture. For example: A model that encapsulates these ideas is the Vasiliev theory with $\Lambda > 0$ [156], which has an infinite tower of interacting higher-spin gauge fields in dS_4. The gauge-invariant Hilbert space of the Vasiliev theory was argued in [155] to be dramatically reduced at the microscopic level. Related remarks for the theories at hand will be given in section 4.1.

3.4.1 Abelian BF theory

The Abelian BF theory with compact $U(1)$ gauge group is governed by the action

$$S_{\mathrm{BF}} = \frac{k}{4\pi}\int d^2x\sqrt{-g}\, B\,\epsilon^{\mu\nu}F_{\mu\nu}\ , \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\ . \tag{3.4.2}$$

Here, B is a real compact scalar field $B \cong B + 2\pi$, and $\epsilon^{\mu\nu}$ is the antisymmetric tensor with $\epsilon^{T\vartheta} = +1/\sqrt{-g}$. The theory is invariant under gauge transformations $A_\mu \to A_\mu + \partial_\mu\omega$ with ω a compact scalar of radius 2π. Although we will consider the theory on the spacetime (3.1.17), the action (3.4.2) is independent of the metric. Also note that similar to the case of Chern-Simons theory, the parameter k in front of the action is quantised.

The classical equations of motion are given by

$$\partial_\mu B = 0\ , \qquad \Box_{\mathrm{dS}}\phi = 0\ , \tag{3.4.3}$$

where we have picked the Lorenz gauge: $A^\mu \equiv \epsilon^{\mu\nu}\partial_\nu\phi$. We note that the constant part of ϕ is absent from the field configuration space, as it does not affect the physical field A_μ. The second equation in (3.4.3) is equivalent to that of a free massless scalar field in dS_2, i.e. (3.2.3) with $m^2 = 0$, but now with the zero-mode removed by construction. The solutions are

$$\phi(T,\vartheta) = \chi\frac{T}{2\pi} + \sum_{n\in\mathbb{Z}/\{0\}}\frac{e^{in\vartheta}}{2\pi}\left(\alpha_n\frac{\sin nT}{n} + \beta_n\cos nT\right) \underset{\lim_{T\to 0^-}}{\approx} \chi\frac{T}{2\pi} + \sum_{n\in\mathbb{Z}/\{0\}}\frac{e^{in\vartheta}}{2\pi}\left(T\alpha_n + \beta_n\right)\ , \tag{3.4.4}$$

subject to reality conditions $\alpha_n^* = \alpha_{-n}$ and $\beta_n^* = \beta_{-n}$, and we have singled out $\alpha_0 \equiv \chi$. Recall that β_0 is not included in the above sum because the gauge field A_μ

is insensitive to the constant mode of ϕ. Upon quantisation, B and ϕ are promoted to quantum operators, and (3.4.3) become operator equations. The operator χ is associated to the Wilson loop and forms a canonical pair with the constant mode of B. Under the decomposition (3.4.4), one might conclude that the modes α_n and β_n can be organised in terms of the $D^{\pm}_{\Delta}$ UIRs with $\Delta = 1$. χ and the constant mode of B, which are the only operators that survive the gauge constraints, furnish a singlet representation of $SL(2,\mathbb{R})$.

Having discussed the transformation properties of the operators B and ϕ, we note that it is *not* the case that the physical state space furnishes the $\Delta = 1$ UIRs. Starting from the above-mentioned pre-Hilbert space, properly imposing the gauge constraints will result in an enormous reduction of the physical Hilbert space.

Hilbert space of abelian BF theory. There are many ways to quantise this theory, but we will begin with a way that quickly identifies the Hilbert space [242]. This will be done in temporal gauge, where we set $A_T = 0$. This means we must also impose the constraint generated by A_T at the level of the action, namely:

$$\partial_\vartheta B = 0 \ . \tag{3.4.5}$$

Let us define $\chi \equiv \oint d\vartheta A_\vartheta$, which is the piece of A_ϑ invariant under the residual gauge freedom $A_\vartheta \to A_\vartheta + \partial_\vartheta \omega(\vartheta)$, and $b = \frac{1}{2\pi}\oint d\vartheta B$. Thus our gauge-fixed action is a simple quantum mechanical model:

$$S^{\text{g.f.}}_{\text{BF}} = \frac{k}{2\pi}\int dT\, b\, \dot{\chi} \ . \tag{3.4.6}$$

Recall that by flux quantisation $b \cong b + 2\pi$ and moreover, by compactness of the gauge group $\chi \cong \chi + 2\pi$. Thus the fields b and χ are canonically conjugate pairs, with commutation relation:

$$[\chi, b] = \frac{2\pi i}{k} \ . \tag{3.4.7}$$

Because of the compactness of the fields, the well-defined observables are e^{ib} and $e^{i\chi}$. We can work in an eigenspace of the operator e^{ib}, which is spanned by square-integrable wavefunctions $\Psi_n(\chi) = e^{in\chi}/\sqrt{2\pi}$ with $n \in \mathbb{Z}$. Recalling that the operator $b = -\frac{2\pi i}{k}\partial_\chi$, conjugate to χ, is also compact, such that $e^{ib} = e^{i(b+2\pi q)}$ with $q \in \mathbb{Z}$. We require that

$$-\tfrac{2\pi i}{k}\partial_\chi \Psi_n(\chi) \cong \left(-\tfrac{2\pi i}{k}\partial_\chi + 2\pi\right)\Psi_n(\chi) \ . \tag{3.4.8}$$

The above expression informs us that n should in fact be defined modulo k such that the correct range is $n = 0, 1, \ldots, k-1$. Thus the gauge-invariance significantly reduces the Hilbert space, rendering a naively infinite Hilbert space into one that is k-dimensional, and therefore incapable of carrying the $\Delta = 1$ UIR. Going back to our original gauge-theoretic variables, the Hilbert space is spanned by the wavefunctionals

$$\Psi_n\left[A_\mu\right] = \frac{1}{\sqrt{2\pi}}\, e^{in\oint d\vartheta A_\vartheta}\ , \qquad n = 0, 1, \ldots, k-1\ , \tag{3.4.9}$$

where $B = -i\delta/\delta A_\vartheta$.

3.4.2 $SL(2,\mathbb{R})$ BF theory as JT gravity

We now proceed to $SL(2,\mathbb{R})$ BF theory. The theory is semi-classically equivalent to a JT gravity [224, 225] built from gravitational degrees of freedom including a two-dimensional metric $g_{\mu\nu}$ and a scalar ϕ, and it is in these variables that we will analyze the model. The theory is governed by the following action:

$$S_{\text{JT}} = \frac{1}{16\pi G}\int_{\mathcal{M}} d^2x\sqrt{-g}\phi\left(R - \frac{2}{\ell^2}\right) \tag{3.4.10}$$

where we take ℓ^2 to be positive such that the theory admits dS_2 solutions. As described in [243], we may recast the above action as an $SL(2,\mathbb{R})$ gauge theory (in the first order formalism) by combining the zweibein e^a_μ and the spin-connection ω_{01} into an $SL(2,\mathbb{R})$ gauge field, while collecting the field ϕ and additional auxiliary fields (imposing the torsionless condition) into the adjoint-valued scalar B. Though related, neither the $SL(2,\mathbb{R})$ gauge symmetry, nor any subgroup thereof, of the BF theory (3.4.10) is to be identified with the $SL(2,\mathbb{R})$ isometry of the dS_2 vacuum which is in fact slightly broken by the general classical solution.

The equations of motion stemming from (3.4.10) are

$$\delta\phi: \qquad R - \frac{2}{\ell^2} = 0\,, \tag{3.4.11}$$

$$\delta g_{\mu\nu}: \qquad \left(\ell^2\nabla_\mu\nabla_\nu + g_{\mu\nu}\right)\phi = 0\,. \tag{3.4.12}$$

In global coordinates, the solution is

$$\frac{ds^2}{\ell^2} = \frac{-dT^2 + d\vartheta^2}{\sin^2 T}\ , \qquad \phi = \gamma_0\cot T + \left(\gamma_1 e^{i\vartheta} + \gamma_{-1}e^{-i\vartheta}\right)\csc T\ . \tag{3.4.13}$$

The breaking of de Sitter invariance is evidenced by a non-trivial profile for the dilaton ϕ.[8]

We will parallel the previous section's discussion on the Abelian BF theory. First we will pick a gauge where the $\Delta = 2$ discrete series equation of motion appears, allowing for a description of this UIR, at least at the level of the pre-Hilbert space. Afterwards we will explain how imposing the gauge constraints significantly cuts down the size of the Hilbert space. To this end, we parameterize the global dS_2 geometry in Weyl gauge as:

$$\frac{ds^2}{\ell^2} = e^{2\omega(\vartheta^+,\,\vartheta^-)}\,\frac{4\,d\vartheta^+ d\vartheta^-}{\sin^2(\vartheta^+ - \vartheta^-)}\ , \tag{3.4.15}$$

where the solution in (3.1.17) corresponds to $\omega(\vartheta^+,\,\vartheta^-) = 0$ with $T = \vartheta^+ - \vartheta^-$ and $\vartheta = \vartheta^+ + \vartheta^-$. Equation (3.4.11) governing the Weyl factor is then

$$\ell^2 \Box_{\mathrm{dS}}\omega(\vartheta^+,\,\vartheta^-) = 1 - e^{2\omega(\vartheta^+,\,\vartheta^-)} \approx -2\omega(\vartheta^+,\,\vartheta^-)\ , \tag{3.4.16}$$

where in the second expression we have expanded for small $\omega(\vartheta^+,\,\vartheta^-)$ and $\Box_{\mathrm{dS}}$ is the Laplacian with respect to the metric (3.4.13), which is the same as (3.4.15) with $\omega(\vartheta^+,\,\vartheta^-) = 0$. As noted in [121], this linearized equation governing $\omega(\vartheta^+,\,\vartheta^-)$ is the equation of a free scalar on a fixed dS_2 background with a tachyonic mass $m^2\ell^2 = -2$, corresponding to the discrete series equation for $\Delta = 2$. The non-linear nature of equation (3.4.16) can be thought of as self-interactions for the field ω. Therefore, the Weyl gauge here should be thought of as paralleling the Lorenz gauge in the Abelian BF theory.

In two-dimensions, a local region of the geometry is determined entirely by the Ricci scalar, so up to global effects, the general solution space to (3.4.16) must be the set of diffeomorphisms preserving the Weyl gauge (3.4.15)

$$e^{2\omega(\vartheta^+,\,\vartheta^-)} = \sin^2(\vartheta^+ - \vartheta^-)\frac{\partial_+ f(\vartheta^+)\partial_- g(\vartheta^-)}{\sin^2\left(f(\vartheta^+) - g(\vartheta^-)\right)}\ , \tag{3.4.17}$$

[8]Note that taking the trace of (3.4.12) leads to

$$\ell^2\Box_{\mathrm{dS}}\phi(\vartheta^+,\,\vartheta^-) = -2\phi(\vartheta^+,\,\vartheta^-)\,, \tag{3.4.14}$$

which is a tachyonic free scalar equation with $m^2\ell^2 = -2$. However, since ϕ is subject to a symmetric tensor's worth of equations, there are only three linearly independent solutions. We will make further comments on this observation below, but take note that this is precisely the number of Euclidean zero-modes admitted by the Euclidean continuation of (3.4.14).

and

$$\phi = \gamma_0 \cot\Big(f(\vartheta^+) - g(\vartheta^-)\Big) + \Big\{\gamma_1 e^{i\left[f(\vartheta^+)+g(\vartheta^-)\right]} + \gamma_{-1} e^{-i\left[f(\vartheta^+)+g(\vartheta^-)\right]}\Big\} \csc\Big(f(\vartheta^+) - g(\vartheta^-)\Big) \ . \tag{3.4.18}$$

with $f(\vartheta^+)$ and $g(\vartheta^-)$ smooth functions. The classical solution space is therefore labeled by the modes of the functions f and g, as well as the three parameters γ_i. In the Abelian BF theory, the zero-mode of the scalar ϕ in Lorenz gauge was projected out, allowing us to identify candidate operators that might furnish a $\Delta = 1$ UIR—at least at the level of the pre-Hilbert space. Our task now is to identify a similar mechanism in this setting, but for $\Delta = 2$, which requires projecting out three zero-modes.

Which are the zero-modes that must be projected out? The natural candidates are either the three Killing symmetries of the spacetime, or the three non-isometric conformal-Killing transformations. Of these the former are physical, since the de Sitter invariance of the background is broken by the dilaton profile, whereas the latter correspond to field redefinitions of Weyl factor ω, and therefore do not describe physically inequivalent data. We will now show precisely how this plays out. Starting from the linearized equation (3.4.16) and using the variables $T = \vartheta^+ - \vartheta^-$ and $\vartheta = \vartheta^+ + \vartheta^-$, we must solve:

$$\sin^2 T\left(-\partial_T^2 + \partial_\vartheta^2\right)\omega(T,\vartheta) = -2\omega(T,\vartheta) \ . \tag{3.4.19}$$

The solutions may be expressed in terms of angular modes around the spatial S^1, labelled by $n \in \mathbb{Z}$. For $n \in \mathbb{Z}/\{-1,0,1\}$, the solutions are

$$\omega(T,\vartheta) = \sqrt{|\sin T|}\sum_{n\in\mathbb{Z}/\{-1,0,1\}} e^{in\vartheta}\left(\alpha_n\sqrt{\frac{2}{\pi}}Q^{3/2}_{|n|-1/2}(\cos T) + \beta_n\sqrt{\frac{\pi}{2}}P^{3/2}_{|n|-1/2}(\cos T)\right) \ , \tag{3.4.20}$$

with $\alpha_n^* = \alpha_{-n}$ and $\beta_n^* = \beta_{-n}$. Near the future boundary $T \to 0^-$, we have two characteristic behaviours given by $\omega(T,\vartheta) \sim T^{-1}$ and $\omega(T,\vartheta) \sim T^2$. Concretely,

$$\omega(T,\vartheta) \approx \sum_{n\in\mathbb{Z}/\{-1,0,1\}} e^{in\vartheta}\left(\alpha_n\frac{|n|(n^2-1)}{3}T^2 + \frac{\beta_n}{T}\right) \ . \tag{3.4.21}$$

We must treat the $n \in \{-1,0,1\}$ modes separately. These are given by

$$\omega_{\{-1,0,1\}}(T,\vartheta) = \Big\{\alpha_0\left(1 - T\cot T\right) + \tfrac{1}{2}\left(\alpha_{-1}e^{-i\vartheta} + \alpha_1 e^{i\vartheta}\right)\left(T\csc T - \cos T\right) + \beta_0 \cot T + \left(\beta_{-1}e^{-i\vartheta} + \beta_1 e^{i\vartheta}\right)\csc T\Big\} . \quad (3.4.22)$$

Near $T \sim 0^-$, these modes behave as

$$\omega_{\{-1,0,1\}}(T,\vartheta) \underset{T\to 0^-}{\approx} \sum_{n\in\{-1,0,1\}} e^{in\vartheta}\left(\alpha_n \frac{T^2}{3} + \frac{\beta_n}{T}\right) . \quad (3.4.23)$$

Due to the reality conditions on $\omega(T,\vartheta)$, β_0 and α_0 are real valued. Which of these modes correspond to Killing symmetries, and which correspond to conformal Killing transformations? To determine this, let us expand the nonlinear solution (3.4.17) near the future boundary, where $T \to 0^-$, and we will furthermore choose the slice $g(x) = f(x) \equiv h(2x)$. On this slice one finds :

$$e^{2\omega(\vartheta^+,\vartheta^-)}|_{bdy} = 1 + \frac{T^2}{3}\left(2\,\text{Sch}\left(h\left(\vartheta\right),\vartheta\right) + 4h'\left(\vartheta\right)^2 - 1\right)\ldots . \quad (3.4.24)$$

This equation is invariant under the transformation

$$\tan h(\vartheta) \to \frac{a\tan h(\vartheta) + b}{c\tan h(\vartheta) + d} , \qquad ad - bc = 1 , \quad (3.4.25)$$

therefore, we see that there are three modes that leave the late-time form of the metric invariant. Moreover, these modes are proportional to T^2 at late times, suggesting that the α_n for $n = \{-1,0,1\}$ are associated to the (broken) Killing symmetries, and therefore should not be discarded.

A clearer exposition, making contact with the discussion around the tachyonic equation of motion for the $\Delta = 2$ discrete series UIR, can be achieved by repeating the above analysis at the linearized level. To linear order, the solutions (3.4.17) are

$$2\omega(\vartheta^+,\vartheta^-) = \epsilon\left\{\delta g'(\vartheta^-) + \delta f'(\vartheta^+) - 2\delta f(\vartheta^+)\cot\left(\vartheta^+ - \vartheta^-\right) + 2\delta g(\vartheta^-)\cot\left(\vartheta^+ - \vartheta^-\right)\right\} , \quad (3.4.26)$$

where we have taken $f(\vartheta^+) = \vartheta^+ + \epsilon\delta f(\vartheta^+)$ and $g(\vartheta^-) = \vartheta^- + \epsilon\delta g(\vartheta^-)$. Taking the late-time limit $T \to 0$ along the slice $\delta g(x) = \delta f(x) \equiv \delta h(2x)$, we gain an equation for the Weyl mode in terms of the linearized Schwarzian derivative:

$$\omega \approx \frac{2}{3}T^2\epsilon\left(\delta h'(\vartheta) + \delta h'''(\vartheta)\right) \ . \tag{3.4.27}$$

Expanding δh in modes $\delta h(\vartheta) = \sum_n c_n e^{in\vartheta}$, we see that the modes with $n = \{-1, 0, 1\}$ do not change the Weyl factor at late times.

Conformal Killing vectors. We now expect the β_n modes for $n = \{-1, 0, 1\}$ to be associated with conformal Killing transformations. These are field redefinitions of ω, and therefore are not in the physical phase space, leading us to discard them. To see how this works out, we study the conformal Killing equation:

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = g_{\mu\nu}\nabla_\rho \xi^\rho \ . \tag{3.4.28}$$

For the case of dS_2 in global coordinates (3.1.17) there are six globally well-defined conformal Killing vectors, smooth with respect to the Euclidean continuation to the two-sphere. Of these, three are divergence-free Killing vector fields that do not alter the original metric, and are essentially the three $\alpha_{-1,0,1}$ zero-modes identified earlier. The remaining three are conformal Killing vector fields that non-trivially transform the metric Weyl factor. Contracting the conformal Killing equation (3.4.28) with a covariant derivative yields

$$\ell^2 \Box_{\rm dS} \xi_\nu = -\xi_\nu \ , \tag{3.4.29}$$

from which it follows that

$$\ell^2 \Box_{\rm dS} \nabla_\mu \xi^\mu = -2\nabla_\mu \xi^\mu \ . \tag{3.4.30}$$

Upon identifying $\omega = \nabla^\mu \xi_\mu$, this is precisely the linearized form of (3.4.19). The three globally well-defined conformal Killing vector fields are given by (labelling vector fields as $V \equiv V^\mu \partial_\mu$)

$$\xi_1 = \partial_T \ , \quad \xi_2 = \cos T \sin\vartheta \partial_T + \cos\vartheta \sin T \partial_\vartheta \ , \quad \xi_3 = \cos T \cos\vartheta \partial_T - \sin\vartheta \sin T \partial_\vartheta \ . \tag{3.4.31}$$

Their corresponding, non-vanishing, divergence is

$$\omega = \nabla_\mu \left(a_1\, \xi_1^\mu + a_2\, \xi_2^\mu + a_3\, \xi_3^\mu\right) = \beta_0 \cot T + \left(\beta_{-1} e^{-i\vartheta} + \beta_1 e^{i\vartheta}\right) \csc T \ , \tag{3.4.32}$$

where $\beta_0 = -2a_1$, $\beta_{\pm 1} = -a_3 \mp ia_2$. These are the $\beta_{\pm 1}$ and β_0 modes in (3.4.22), which we can now interpret as the subset of globally well-defined conformal Killing vector fields. We thus interpret the three β_n modes of (3.4.22) in terms of a residual redundancy in the Weyl parameterisation (3.4.15).

Thus, just as removing the constant part of ϕ (β_0 in (3.4.4)) when working in the Lorenz gauge $A^\mu = \epsilon^{\mu\nu}\partial_\nu\phi$ in the Abelian $U(1)$ BF theory was necessary so as to not overcount field configurations; failure to remove the $\beta_{-1,0,1}$ modes from the configuration space of Weyl factors, would lead to an overcounting of field configurations.[9]

The three remaining α_n modes are then naturally paired to the three modes γ_n of the dilaton solution space (3.4.13). This again mirrors the story in the $U(1)$ BF example. There, the canonical pairing was between the constant part of B and the Wilson Loop operator $\alpha_0 \equiv \chi$. Indeed, a canonical analysis reveals that $-4\pi\partial_T\omega$ is momentum conjugate to ϕ [244], and the α_n modes have an appropriate Poisson bracket algebra with the γ_n modes of (3.4.13). Under the action of the Killing isometries, the triplets furnish a three-dimensional, and hence non-unitary, irreducible representation of $SL(2,\mathbb{R})$.

Hilbert space of $\mathbf{dS_2}$ JT gravity. Upon quantising the JT gravity theory (3.4.10), the fields $\phi(T,\vartheta)$ and $g_{\mu\nu}(T,\vartheta)$ are promoted to operators. The wave equation (3.4.19) becomes an operator equation governing $\omega(T,\vartheta)$. As we have just shown, the operator modes in the decomposition (3.4.20) furnish a $\Delta = 2$ irreducible representation of $SL(2,\mathbb{R})$. The operators α_n and β_n, for $n \in \mathbb{Z}/\{-1,0,1\}$, appearing in (3.4.20) constitute a basis for the representation. The highest- and lowest-weight towers are labelled by quantum numbers $n = 2,3,\dots$ and $n = -2,-3,\dots$ as expected. The operators α_n with $n \in \{-1,0,1\}$ are naturally paired up with the three modes γ_n associated to ϕ in (3.4.13). As mentioned, these furnish a three-dimensional non-unitary irreducible representation of $SL(2,\mathbb{R})$.

But as in the Abelian BF theory, although the operator algebra furnishes the $\Delta = 2$ representation, the physical Hilbert space is much smaller. This reduction arises when the gauge constraints of the BF theory are properly imposed. Instead, the Hilbert

[9]It follows that the configuration space of the boundary values of $\omega(T,\vartheta)$ should be understood in terms of a quotient space of the set of all boundary metrics modulo the $SL(2,\mathbb{R})$ redundancy, a feature that has prominently appeared in the AdS_2 considerations of the model [129]. This echoes naturally with the group theoretic necessity of removing the three modes $n = \{-1,0,1\}$ from the $\Delta = 2$ discrete series UIR.

space is spanned by a family of states parameterised by a single real parameter, indicating a Hilbert space of a quantum mechanical, rather than quantum field theoretic, nature. Concretely, one can consider the problem in the Schrödinger picture, as we did for the $U(1)$ BF theory in (3.4.9). Here, one studies wavefunctionals $\Psi_\Sigma[h(u), \phi(u)]$ of the induced metric $h(u)$ and dilaton field $\phi(u)$ on a Cauchy surface whose points are labelled by u. The wavefunctions are subject to the constraints of the JT theory. As shown in [245], one can solve the constraints exactly. The spatial diffeomorphism redundancy allows us to pick a gauge of constant $\phi(u) = \phi_0$. In addition one must impose the associated momentum constraint on the wavefunctions. Prior to doing so, the wavefunctions are functionals of the boundary metric $h(u)$. (In the AdS$_2$ context, this would be the functional of the Schwarzian mode [129].) The momentum constraint further enforces that the $\Psi_\Sigma[h(u), \phi_0]$ are functions of the constant mode of $h(u)$ only (see appendix C of [244] for example). Thus, the physical state-space of the $SL(2,\mathbb{R})$ BF theory on a global spatial slice is insensitive to the $\Delta = 2$ conformal operators stemming from (3.4.20).

3.4.3 $SL(N,\mathbb{R})$ BF theory, briefly

Our discussion permits a direct generalisation to the $SL(N,\mathbb{R})$ BF theory. We comment on this case briefly here, and leave a general analysis to future work. The $SL(N,\mathbb{R})$ BF theory has been studied in [222, 223, 240, 246] as a higher-spin extension of JT gravity. Provided the $SL(2,\mathbb{R})$ is principally embedded in the $SL(N,\mathbb{R})$ gauge group, the theory contains a spectrum of fields of spin $s = 2, 3, \ldots, N$. These can be viewed as decomposing the adjoint representation of $\mathfrak{sl}(N,\mathbb{R})$ into traceless and symmetric $\mathfrak{sl}(2,\mathbb{R})$ tensors as follows:

$$t^a = \oplus_{k=1}^{N-1} t^{A_1 \ldots A_k} \ , \qquad a = 1, \ldots, N^2 - 1 \ , \tag{3.4.33}$$

where t^a is a generator of $\mathfrak{sl}(N,\mathbb{R})$ in the adjoint, and the $t^{A_1 \ldots A_k}$ are rank-k traceless symmetric tensors of $\mathfrak{sl}(2,\mathbb{R})$, each with $(2k+1)$ components, such that the sum gives a total of $N^2 - 1$ components. This is indeed the dimension of the adjoint representation of $\mathfrak{sl}(N,\mathbb{R})$.

The gravitational subsector of the theory is captured by the $SL(2,\mathbb{R}) \subset SL(N,\mathbb{R})$ embedding, and the models admit a (near) dS$_2$ vacuum solution. The equation governing fluctuations (in a suitably chosen gauge) about the dS$_2$ vacuum generalizes

(3.4.16) to the following collection of fluctuation equations [246]

$$\ell^2\Box_{\text{dS}}\omega_s(T,\vartheta) = s(1-s)\omega_s(T,\vartheta)\ , \quad s=2,3,\ldots,N\ . \tag{3.4.34}$$

The Schwarzian boundary mode is extended to an $SL(N,\mathbb{R})$ version. The three-redundant modes observed in the $SL(2,\mathbb{R})$ case are replaced by $(2s+1)$ redundant modes for each s. As for the previous cases, although there are operators furnishing the discrete series UIRs for $\Delta = 2,3,\ldots,N$ the state-space of the theory does not. The details of this will be presented in future work. At $N\to\infty$ the gauge-symmetry is generated by an infinite dimensional higher-spin algebra [223], which is an extension of the $N\to\infty$ limit of $SL(N,\mathbb{R})$, reminiscent of the algebra governing four-dimensional Vasiliev theory [156, 247].

3.5 Structures at $\mathcal{I}^+$, contact terms, and gravitational constraints

To conclude this chapter we present certain important structures that arise when considering matter-field observables coupled to gravity. Specifically, we will show how to implement the gauge constraints on the late-time matter-field observables, and the implications that arise as a result of imposing these constraints. Throughout the section, we work in the semiclassical limit, where the fluctuations of the metric field are suppressed. Nevertheless, quantum gravity will play a role inasmuch as it requires us to impose the gauge constraints on the matter. These constraints are of particular importance for the theories discussed in the previous section 3.4, which realise the discrete series UIR in the pre-Hilbert space of a gauge theory. The importance of gravitational constraints were pointed out in early work [145–147, 248, 249], and explored more recently in [75, 149, 155].

To construct gauge-invariant observables at the late-time boundary, we will consider expectation values of conformal operators (made out of matter fields) integrated over the late-time spatial slice. Thus, the implementation of the diffeomorphism constraints is performed at the late-time conformal boundary, in a similar way to how vertex operators are integrated over the string worldsheet in order to construct operators invariant under the Virasoro constraints. Because we advocate integrating over the spatial slice, it will be crucial to keep track of any and all contact terms that arise in late-time (pre-integrated) dS_2 correlation functions. We thus start with a discussion on the allowed structures present in late-time correlation functions.

3.5.1 Allowed structures for correlators on $\mathcal{I}^+$

In dS_2, the future boundary is conformal to an S^1, and conformal operators on the conformal circle transform as described in (3.2.2):

$$\left[\widehat{L}_0, \mathcal{O}_{\Delta,n}\right] = n\mathcal{O}_{\Delta,n} \ , \qquad \left[\widehat{L}_\pm, \mathcal{O}_{\Delta,n}\right] = (n \pm \Delta)\, \mathcal{O}_{\Delta,n\pm 1} \ . \tag{3.5.1}$$

This is the type of setup imagined in dS/CFT where it is posited that interacting quantum fields on a de Sitter background reorganize themselves into a Euclidean CFT on the future boundary at $\mathcal{I}^+$. We can combine this collection of operators into a single object called a local quasi-primary field

$$\mathcal{O}_\Delta(\vartheta) \equiv \sum_n \mathcal{O}_{\Delta,n} \frac{e^{-in\vartheta}}{\sqrt{2\pi}} \ , \tag{3.5.2}$$

which transforms covariantly under $SL(2,\mathbb{R})$ coordinate transformations

$$\mathcal{O}_\Delta(\vartheta) \to \left(\frac{\partial\vartheta'}{\partial\vartheta}\right)^\Delta \mathcal{O}_\Delta(\vartheta') \ , \qquad \tan\frac{\vartheta'}{2} = \frac{a\tan\frac{\vartheta}{2}+b}{c\tan\frac{\vartheta}{2}+d} \ , \tag{3.5.3}$$

with $(a,b,c,d) \in \mathbb{R}$ and $\det\begin{pmatrix} a & b \\ c & d\end{pmatrix} = 1$. The three generators that exponentiate to form the group elements are:

$$e^{i\lambda H} \longleftrightarrow \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \ , \qquad e^{i\lambda K} \longleftrightarrow \begin{pmatrix} 1 & 0 \\ -\lambda & 1 \end{pmatrix} \ , \qquad e^{i\lambda D} \longleftrightarrow \begin{pmatrix} 1+\frac{\lambda}{2} & 0 \\ 0 & \frac{1}{1+\frac{\lambda}{2}} \end{pmatrix} \ , \tag{3.5.4}$$

and the action of these generators on a quasi-primary field of dimension Δ is

$$H \equiv \ 2i\cos\frac{\vartheta}{2}\left[\Delta\ \sin\frac{\vartheta}{2} - \cos\frac{\vartheta}{2}\partial_\vartheta\right] \ , \tag{3.5.5}$$

$$K \equiv -2i\sin\frac{\vartheta}{2}\left[\Delta\ \cos\frac{\vartheta}{2} + \sin\ \frac{\vartheta}{2}\partial_\vartheta\right] \ , \tag{3.5.6}$$

$$D \equiv -i\left[\Delta\ \cos\vartheta + \sin\vartheta\,\partial_\vartheta\right] \ . \tag{3.5.7}$$

These Hermitian generators are related to the complexified ones described previously via the relationship

$$H = L_0 - \frac{1}{2}(L_+ + L_-) \ , \quad K = L_0 + \frac{1}{2}(L_+ + L_-) \ , \quad D = -\frac{i}{2}(L_+ - L_-) \ , \tag{3.5.8}$$

where the operators

$$L_0 = -i\partial_\vartheta \ , \qquad L_\pm = e^{\mp i\vartheta}\left(\mp\Delta - i\partial_\vartheta\right) \ , \tag{3.5.9}$$

generate the algebra of $SL(2,\mathbb{R})$ as given in (3.1.6).

Under this general structure correlation functions of quasi-primary operators transform as expected, namely

$$\langle \mathcal{O}_{\Delta_1}(\vartheta_1)\mathcal{O}_{\Delta_2}(\vartheta_2)\ldots\mathcal{O}_{\Delta_n}(\vartheta_n)\rangle = \left[\prod_{i=1}^{n}\left(\frac{\partial\vartheta'}{\partial\vartheta}\right)^{\Delta_i}\Bigg|_{\vartheta=\vartheta_i}\right]\langle \mathcal{O}_{\Delta_1}(\vartheta'_1)\mathcal{O}_{\Delta_2}(\vartheta'_2)\ldots\mathcal{O}_{\Delta_n}(\vartheta'_n)\rangle\,. \tag{3.5.10}$$

and if the underlying theory is invariant under $SL(2,\mathbb{R})$ transformations, then the correlation functions must be invariant as well. We will now raise some points which, to our knowledge, have not previously been highlighted in the literature.

Two-point function: In general treatments on conformal field theory, one determines that invariance under (3.5.10) entirely fixes the two-point function of quasi-primary operators:

$$\langle \mathcal{O}_{\Delta_1}(\vartheta_1)\mathcal{O}_{\Delta_2}(\vartheta_2)\rangle = \begin{cases} \frac{c_{12}}{\left[\sin^2\left(\frac{\vartheta_1-\vartheta_2}{2}\right)\right]^{\frac{\Delta_1+\Delta_2}{2}}}\,, & \Delta_1 = \Delta_2 \\ 0\,, & \Delta_1 \neq \Delta_2 \end{cases} \qquad \text{(typical)}\,. \tag{3.5.11}$$

This above statement presupposes that no local contact terms contribute to the correlation function of two operators. The logic behind this reasoning stems from the formulation of CFT in Euclidean signature. In this setting, contact terms in correlation functions correspond to ultraviolet ambiguities in the definition of the local operator $\mathcal{O}_\Delta(\vartheta)$.

The setting for dS/CFT is different. Correlation functions at the future boundary encode the entire bulk history. As we will soon demonstrate in an example, local contact terms in correlation functions can now arise naturally from the bulk Heisenberg algebra of quantum fields. Therefore they are not UV ambiguities, but rather, are fixed by the canonical structure of the bulk Hilbert space. With this in mind, we present the most general structure allowed for a two-point function which is invariant under (3.5.10):

$$\langle \mathcal{O}_{\Delta_1}(\vartheta_1)\mathcal{O}_{\Delta_2}(\vartheta_2)\rangle = \begin{cases} a_{12}\,\delta(\vartheta_1-\vartheta_2)\,, & \Delta_1 = 1-\Delta_2 \\ b_{12}\,g_{\Delta_1+\Delta_2}(\vartheta_1-\vartheta_2) + \frac{c_{12}}{\left[\sin^2\left(\frac{\vartheta_1-\vartheta_2}{2}\right)\right]^{\frac{\Delta_1+\Delta_2}{2}}}\,, & \Delta_1 = \Delta_2 \\ 0\,, & \text{otherwise} \end{cases}\,. \tag{3.5.12}$$

The function g_Δ only contains contact terms

$$g_\Delta(\vartheta) = \begin{cases} \delta(\vartheta) \ , & \Delta = 1 \\ \delta'(\vartheta) \ , & \Delta = 2 \\ \delta''(\vartheta) + \frac{\delta(\vartheta)}{4} \ , & \Delta = 3 \\ \delta'''(\vartheta) + \delta'(\vartheta) \ , & \Delta = 4 \\ \delta^{(4)}(\vartheta) + \frac{5}{2}\delta''(\vartheta) + \frac{9}{16}\delta(\vartheta) \ , & \Delta = 5 \\ \delta^{(5)}(\vartheta) + 5\delta'''(\vartheta) + 4\delta'(\vartheta) \ , & \Delta = 6 \\ \vdots \end{cases} . \tag{3.5.13}$$

Crucially, the contact term proportional to a_{12} in (3.5.12) occurs when an operator is paired with an operator in a shadow representation. For principal series operators, this means we have paired an operator with its complex conjugate. It is worth noting that the late-time behaviour of the discrete series two-point function (3.2.48) is in tension with the structures present in (3.5.12).

Three-point function: In a similar vein to the discussion above, standard CFT treatments state that invariance under (3.5.10) also completely fixes the three point function:

$$\langle \mathcal{O}_{\Delta_1}(\vartheta_1)\mathcal{O}_{\Delta_2}(\vartheta_2)\mathcal{O}_{\Delta_3}(\vartheta_2)\rangle = \frac{c_{123}}{\left[\sin^2\left(\frac{\vartheta_1-\vartheta_2}{2}\right)\right]^{\frac{\Delta_1+\Delta_2-\Delta_3}{2}} \left[\sin^2\left(\frac{\vartheta_1-\vartheta_3}{2}\right)\right]^{\frac{\Delta_1+\Delta_3-\Delta_2}{2}} \left[\sin^2\left(\frac{\vartheta_2-\vartheta_3}{2}\right)\right]^{\frac{\Delta_2+\Delta_3-\Delta_1}{2}}} \quad \text{(typical)} \ . \tag{3.5.14}$$

Again, this presupposes that none of the points are coincident. Allowing for contact terms, we find

$$\langle \mathcal{O}_{\Delta_1}(\vartheta_1)\mathcal{O}_{\Delta_2}(\vartheta_2)\mathcal{O}_{\Delta_3}(\vartheta_2)\rangle = \\ c_{123}\left(b_{12}\, g_{\Delta_1+\Delta_2-\Delta_3}(\vartheta_1-\vartheta_2) + \left[\sin^2\left(\frac{\vartheta_1-\vartheta_2}{2}\right)\right]^{-\frac{\Delta_1+\Delta_2-\Delta_3}{2}}\right) \\ \times\left(b_{13}\, g_{\Delta_1+\Delta_3-\Delta_2}(\vartheta_1-\vartheta_3) + \left[\sin^2\left(\frac{\vartheta_1-\vartheta_3}{2}\right)\right]^{-\frac{\Delta_1+\Delta_3-\Delta_2}{2}}\right) \\ \times\left(b_{23}\, g_{\Delta_2+\Delta_3-\Delta_1}(\vartheta_2-\vartheta_3) + \left[\sin^2\left(\frac{\vartheta_2-\vartheta_3}{2}\right)\right]^{-\frac{\Delta_2+\Delta_3-\Delta_1}{2}}\right) , \tag{3.5.15}$$

and the function g_Δ is given in (3.5.13). One must exercise care when using this expression, as we must tune the coefficients such that the correlation function make sense as in a distributional sense.

As we will now demonstrate, these contact terms can unambiguously arise at the late-time boundary of de Sitter. We will demonstrate this by considering the correlation functions of a free field in the principal series.

3.5.2 Contact terms in the two-point function

Let us now return to the setting of a free-principal series field, with $\Delta = \frac{1}{2}(1+i\nu)$ as in section 3.2.2. There, we introduced the bulk scalar field $\phi(\tau,\vartheta)$ and its canonical conjugate

$$\pi(\tau,\vartheta) = \cosh\tau\,\partial_\tau\phi \ , \tag{3.5.16}$$

subject to the quantisation condition (3.2.18)

$$[\phi(\tau,\vartheta),\pi(\tau,\vartheta')] = i\delta(\vartheta-\vartheta') \ . \tag{3.5.17}$$

The late-time behaviour of this free field operator is

$$\lim_{\tau\to\infty}\phi(\tau,\vartheta) \approx e^{-\Delta\tau}\mathcal{O}_\Delta(\vartheta) + e^{-(1-\Delta)\tau}\mathcal{O}^\dagger_\Delta(\vartheta) \ , \tag{3.5.18}$$

$$\lim_{\tau\to\infty}\pi(\tau,\vartheta) \approx -\frac{1}{2}\left[\Delta\, e^{(1-\Delta)\tau}\mathcal{O}_\Delta(\vartheta) + (1-\Delta)\, e^{\Delta\tau}\mathcal{O}^\dagger_\Delta(\vartheta)\right] \ , \tag{3.5.19}$$

where the operator $\mathcal{O}_\Delta(\vartheta)$ and its complex conjugate $\mathcal{O}^\dagger_\Delta(\vartheta)$ transform as conformal quasi-primary operators in the principal series with weight $\frac{1}{2}(1+i\nu)$ and $\frac{1}{2}(1-i\nu)$ respectively [87,88]. Recalling the definitions (3.2.9)-(3.2.12) we write these operators as follows:

$$\mathcal{O}_\Delta(\vartheta) = \frac{\Gamma\left(\frac{1}{2}-\Delta\right)}{2\pi}\sum_{n=-\infty}^{\infty}\left[a_n^\Delta\, e^{i(\alpha_n+\beta_n)-i\frac{\pi}{2}(\Delta-|n|)-in\vartheta} + a_n^{\Delta\dagger}\, e^{-i(\alpha_n-\beta_n)+i\frac{\pi}{2}(\Delta-|n|)+in\vartheta}\right] , \tag{3.5.20}$$

$$\mathcal{O}^\dagger_\Delta(\vartheta) = \frac{\Gamma\left(\Delta-\frac{1}{2}\right)}{2\pi}\sum_{n=-\infty}^{\infty}\left[a_n^\Delta\, e^{i(\alpha_n-\beta_n)-i\frac{\pi}{2}(1-\Delta-|n|)-in\vartheta} + a_n^{\Delta\dagger}\, e^{-i(\alpha_n+\beta_n)+i\frac{\pi}{2}(1-\Delta-|n|)+in\vartheta}\right] , \tag{3.5.21}$$

where we have defined the following phase:

$$e^{2i\beta_n} = \frac{\Gamma(\Delta-|n|)}{\Gamma(1-\Delta-|n|)} . \tag{3.5.22}$$

Given (3.5.17), we must have that

$$[\mathcal{O}_\Delta(\vartheta), \mathcal{O}^\dagger_\Delta(\vartheta')] = \frac{i}{\Delta-\frac{1}{2}}\delta(\vartheta-\vartheta') = \frac{2}{\nu}\delta(\vartheta-\vartheta') , \tag{3.5.23}$$

as one can check using the algebra of creation and annihilation operators. As a result of this rigid structure, one readily finds:

$$\langle\Omega|\,\mathcal{O}_\Delta(\vartheta)\mathcal{O}^\dagger_\Delta(\vartheta')\,|\Omega\rangle = \frac{1+\coth\left(\frac{\pi\nu}{2}\right)}{\nu}\delta(\vartheta-\vartheta') , \tag{3.5.24}$$

$$\langle\Omega|\,\mathcal{O}_\Delta(\vartheta)\mathcal{O}_\Delta(\vartheta')\,|\Omega\rangle = \frac{\Gamma(\Delta)\Gamma\left(\frac{1}{2}-\Delta\right)}{4\pi^{3/2}}\left[\sin^2\left(\frac{\vartheta-\vartheta'}{2}\right)\right]^{-\Delta} . \tag{3.5.25}$$

Notably, the contact term is rigid, stemming from the canonical quantisation condition imposed on the bulk scalar field ϕ, reproducing the structure in (3.5.12).

There is a related discontinuity that one can extract when bulk operators are null separated. This can be seen in the Wightman two-point function, which for a free theory reads

$$\langle\Omega|\phi(\tau,\vartheta)\phi(\tau',\vartheta')|\Omega\rangle = \frac{\Gamma(\Delta)\Gamma(1-\Delta)}{4\pi}\,{}_2F_1\left(\Delta, 1-\Delta, 1, 1-\frac{u}{2}\right) , \tag{3.5.26}$$

where

$$u = 1 + \sinh\tau \sinh\tau' - \cos(\vartheta - \vartheta')\cosh\tau\cosh\tau' \; . \tag{3.5.27}$$

The two-point function (3.5.26) exhibits a branch cut along $u = 0$, that is, for null-separated points, which results in a discontinity across the cut as we approach from a spacelike direction ($u \to 0^+$), or a timelike direction ($u \to 0^-$):

$$\left(\lim_{u\to 0^-} - \lim_{u\to 0^+}\right)\langle\Omega|\phi(\tau,\vartheta)\phi(\tau',\vartheta')|\Omega\rangle = -\frac{i}{4} \; . \tag{3.5.28}$$

One can further compute

$$\langle\Omega|\phi(\tau,\vartheta)\pi(\tau',\vartheta')|\Omega\rangle = \frac{\Gamma(\Delta)\Gamma(1-\Delta)}{4\pi}\cosh\tau'\partial_{\tau'}\,{}_2F_1\left(\Delta, 1-\Delta, 1, 1-\frac{u}{2}\right) \; . \tag{3.5.29}$$

For the above correlator, the singular behaviour extends along the light-cone from the δ-function singularity on the equal time slice (3.5.17). This singular structure will be present in the operator algebra of any interacting quantum field theory on a rigid de Sitter background.

3.5.3 Contact terms in higher-point functions

The operator algebra (3.5.23), or somewhat more concretely (3.5.17), persists in an interacting theory. Consequently, higher point functions must obey these operator algebras. For perturbatively small interactions, the operators $\mathcal{O}_\Delta(x)$ and $\mathcal{O}^\dagger_\Delta(x)$ remain good conformal operators up to small corrections. Take, for example, an equal-time $2n$-point function

$$G^{(2n)}(\vartheta_1,\ldots,\vartheta_n;\vartheta_{n+1},\ldots,\vartheta_{2n}) \equiv \langle\Omega|\prod_{i=1}^{n}\mathcal{O}_\Delta(\vartheta_i)\prod_{i=1}^{n}\mathcal{O}^\dagger_\Delta(\vartheta_{n+i})|\Omega\rangle \; . \tag{3.5.30}$$

The above correlation function will be invariant under permutations of $\mathcal{X}_1 = \{\vartheta_1,\ldots,\vartheta_n\}$ and $\mathcal{X}_2 = \{\vartheta_{n+1},\ldots,\vartheta_{2n}\}$. However, exchanging elements between $\mathcal{X}_1$ and $\mathcal{X}_2$ leads to non-trivial structure. For example, at tree level order we have

$$\langle\Omega|\left(\prod_{i=1}^{n-1}\mathcal{O}_\Delta(\vartheta_i)\right)[\mathcal{O}_\Delta(\vartheta_n),\mathcal{O}^\dagger_\Delta(\vartheta_{n+1})]\left(\prod_{i=1}^{n}\mathcal{O}^\dagger_\Delta(\vartheta_{n+i})\right)|\Omega\rangle = \frac{2}{\nu}\delta(\vartheta_n - \vartheta_{n+1}) \times G^{(2n-2)}(\vartheta_1,\ldots,\vartheta_{n-1};\vartheta_{n+2},\ldots,\vartheta_{2n}) \; , \tag{3.5.31}$$

Relations such as the above, which follow from the *Lorentzian* canonical nature of the bulk de Sitter theory, are an important structural feature of the space of correlation functions at the late-time surface.

3.5.4 Integrated operators and gravity

We now consider what happens when we couple the quantum field theory in question to gravity, bearing in mind the aforementioned contact terms. Let us assume the existence of a semiclassical gravitational theory, which permits a de Sitter solution with small fluctuations. This might be, for example, a de Sitter version of JT gravity coupled to matter fields [79, 244, 250] or two-dimensional gravity with $\Lambda > 0$ coupled to a CFT with a large positive central charge as in [124, 126].

For the sake of simplicity, we take the matter theory to have a pair of free massive scalars ϕ and $\tilde{\phi}$, each with $\Delta = \frac{1}{2}(1 + i\nu)$. Observables must be diffeomorphism invariant, and consequently also de Sitter invariant since the $SL(2,\mathbb{R})$ de Sitter isometries are a subgroup of the diffeomorphism group. On the late-time surface at $\mathcal{I}^+$, we require that the invariant operators are $SL(2,\mathbb{R})$ invariant with respect to the conformal transformations of the boundary S^1. Recalling that the one-form $d\vartheta$ transforms with weight minus one, this can be achieved by integrating an operator of weight $\Delta = 1$ over the boundary direction. One such example of a **non-Hermitian** boundary operator invariant under the residual $SL(2,\mathbb{R})$, we take

$$\mathcal{O}_\Delta^{\text{grav}} = \int_0^{2\pi} d\vartheta : \mathcal{O}_\Delta(\vartheta)\tilde{\mathcal{O}}_\Delta^\dagger(\vartheta) : \ , \tag{3.5.32}$$

where $\mathcal{O}_\Delta(\vartheta)$ and $\tilde{\mathcal{O}}_\Delta(\vartheta)$ are two distinct principal series conformal operators as in (3.5.20), transforming with $\Delta = \frac{1}{2}(1 + i\nu)$. One can also build de Sitter invariant states. One of them is the Bunch-Davies vacuum $|\Omega\rangle$. Acting with $\mathcal{O}_\Delta^{\text{grav}}$ on $|\Omega\rangle$ yields a de Sitter invariant state, but one that is not normalizable. A more systematic way of constructing de Sitter invariant states is discussed in [145, 146, 148, 149] and specifically for dS_2 in [147]. We can consider expectation values of de Sitter invariant operators. For instance,

$$\langle\Omega|\mathcal{O}_\Delta^{\text{grav}}\mathcal{O}_\Delta^{\text{grav}}|\Omega\rangle = \frac{\text{csch}(\pi\nu)}{4\pi\nu}\int_{S^1\times S^1} \frac{d\vartheta d\vartheta'}{\sin^2\left(\frac{\vartheta-\vartheta'}{2}\right)} \ . \tag{3.5.33}$$

The above expression is divergent and sensitive to the coincident point limit but avoids the appearance of any contact terms. Additionally, this expression, and corresponding higher-point correlation functions, are of the type that appears when considering open string amplitudes on the disk. To regularize the integral, we can consider a point-splitting cutoff as in [251], namely whenever the points collide we split them by a small amount ε. Specifically, we define

$$x \equiv \frac{\vartheta + \vartheta'}{2} \qquad\qquad y \equiv \vartheta - \vartheta' \ , \tag{3.5.34}$$

and write

$$\begin{aligned} \langle \Omega | \mathcal{O}^{\text{grav}}_{\Delta} \mathcal{O}^{\text{grav}}_{\Delta} | \Omega \rangle_{\text{reg.}} &= \frac{\operatorname{csch}(\pi\nu)}{4\pi\nu} \int_{\frac{\varepsilon}{2}}^{2\pi - \frac{\varepsilon}{2}} dx \left(\int_{-x}^{-\varepsilon} \frac{dy}{\sin^2 \frac{y}{2}} + \int_{\varepsilon}^{x} \frac{dy}{\sin^2 \frac{y}{2}} \right) \ , \\ &\underset{\varepsilon \to 0}{\approx} \frac{4 \operatorname{csch}(\pi\nu)}{\nu} \left(\frac{1}{\varepsilon} - \frac{1}{2\pi} + O(\varepsilon) \right) \ . \end{aligned} \tag{3.5.35}$$

which diverges linearly in ε. Higher-point functions will exhibit the same type of divergence.

We can compare the structure of the $SL(2,\mathbb{R})$ invariant gravitational correlators in dS_2 to the invariant volume of $SL(2,\mathbb{R})$, as computed in [251]:

$$\text{vol}\, SL(2,\mathbb{R}) = \frac{1}{32} \int_{S^1 \times S^1 \times S^1} \frac{d\varphi_1 d\varphi_2 d\varphi_3}{|\sin \frac{\varphi_1 - \varphi_2}{2} \sin \frac{\varphi_1 - \varphi_3}{2} \sin \frac{\varphi_2 - \varphi_3}{2}|} \ . \tag{3.5.36}$$

Upon regularisation we note the same linear divergence as for (3.5.35). The work of [251] goes a step further (at least in the context of open string theory) and argues, based on Weyl invariance, that one can meaningfully extract a constant term from $\text{vol}\, SL(2,\mathbb{R})$, which turns out to be $-\pi^2/2$.

If one were concerned with the non-Hermiticity of the operators $\mathcal{O}^{\text{grav}}_{\Delta}$ defined above, we may also build the analogous diffeomorphism invariant operators out of the late-time operators of a single field in order to analyze the contribution of the contact term (3.5.25). In this case we take

$$\mathcal{Q}^{\text{grav}}_{\Delta} = \int_0^{2\pi} d\vartheta \, : \mathcal{O}_{\Delta}(\vartheta) \mathcal{O}^{\dagger}_{\Delta}(\vartheta) : \ , \tag{3.5.37}$$

leading to:

$$\langle \Omega | \mathcal{Q}^{\text{grav}} \mathcal{Q}^{\text{grav}} | \Omega \rangle = \frac{\operatorname{csch}(\pi\nu)}{4\pi\nu} \int_{S^1 \times S^1} \frac{d\vartheta d\vartheta'}{\sin^2\left(\frac{\vartheta - \vartheta'}{2}\right)} + \frac{\operatorname{csch}^2\left(\frac{\pi\nu}{2}\right)}{\nu^2} \int_{S^1 \times S^1} d\vartheta d\vartheta' \, \delta^2(\vartheta - \vartheta') \ . \tag{3.5.38}$$

A very similar calculation to (3.5.35) yields[10]

$$\langle\Omega|\mathcal{Q}^{\text{grav}}\mathcal{Q}^{\text{grav}}|\Omega\rangle \underset{\varepsilon\to 0}{\approx} \frac{4\,\text{csch}(\pi\nu)}{\nu}\left(\frac{1}{\varepsilon}+\frac{\pi}{\nu}\coth\left(\frac{\pi\nu}{2}\right)\delta(0)-\frac{1}{2\pi}\right)+O(\varepsilon)\ . \tag{3.5.39}$$

The contact term contribution, unlike the contribution of separated points, has not been previously discussed in the context of the volume of $SL(2,\mathbb{R})$. Nonetheless, upon replacing the δ function with a limiting Gaussian, it has a similar linearly divergent behaviour. The suggestion that this too could be an appearance of the volume of the group, and therefore regularizable in the same way is intriguing. It seems any such gravitational correlators built out of Hermitian combinations of these late-time operators ((3.5.37) is one example) must include contractions between $\mathcal{O}$ and $\mathcal{O}^\dagger$ and so cannot avoid contributions from these types of contact terms. The situation is reversed in the case of correlators built from late-time operators for complementary series fields, as detailed in [88], the operators for $\Delta\in\left(0,\frac{1}{2}\right)$ are

$$\lim_{\tau\to\infty}\phi_\Delta(\tau,\vartheta)=e^{-\tau\Delta}\alpha_\Delta(\vartheta)+e^{-(1-\Delta)\tau}\beta_{1-\Delta}(\vartheta)\ . \tag{3.5.40}$$

The operators α and β are Hermitian, but contact terms, such as the ones described above, can only arise in correlators that mix α and β. This implies that integrated gravitational correlators can be constructed of Hermitian observables that do not encounter contact terms, for example, for $\Delta=1/2$ the two-point function of

$$\mathcal{A}^{\text{grav}}=\int_{S^1}d\vartheta\,:\alpha_{\frac{1}{2}}(\vartheta)\alpha_{\frac{1}{2}}(\vartheta):\ , \tag{3.5.41}$$

will be free of contact term singularities.

To recap: The implementation of gravitational constraints in this way renders observables diffeomorphism invariant at the expense of locality. This is not to say that quasi-local physics is doomed. Since recent observations suggest that an exponentially expanding universe is a good model for the cosmological era we are currently entering, we must reflect on how one is to describe the quasi-local physics of everyday experience.

[10] We make the substitution $\int_{S^1\times S^1}d\vartheta d\vartheta'\,\delta^2(\vartheta-\vartheta')=2\pi\delta(0)$, for reasons we hope are not too obscure.

4. A quantum-mechanical model

In this chapter we explore the proposal that the highly constrained gauge theory in (A)dS_2 described in section 3.4.3 may have a microscopic description in terms of the $q = 2$ Sachdev-Ye-Kitaev (SYK) model [252, 253]. We initially make the proposal based on the real $q = 2$ SYK model for brevity, analysing the symmetry structure in the low energy regime. We then examine the complexified version of the SYK model. We summarise and extend the work of [254–257] in the $q = 2$ case and make new comments on the implications for conformal symmetry, the operator algebra and closed form expressions for OPE coefficients.

We examine the appearance of the discrete series in a microphysical SYK model endowed with a two-body random interaction and discuss how it may connect to the aforementioned (section 3.4) BF theories at large-N [240, 246]. More generally, the parallels between dS_2 and dS_4 motivate us to explore the space of integrable/solvable quantum field theories in dS_2 and, possibly, cosmological analogues of the Yang-Baxter equation. This is in the spirit of the Schwinger model as a toy model for various phenomena in four-dimensional quantum field theory as explored in dS in [68], and rhymes with recent efforts on quantum gravity in dS_2 [79, 121–126, 244, 250, 258].

Our interest in this particular microscopic theory is the result of evolving developments in three connected areas of research. Firstly, the IR, near-conformal limit of the generic $q > 2$ case of the SYK model has attracted attention in recent years as a dual description of the black hole in near-AdS_2 JT gravity [127, 128]. In the $q > 2$ case there is an emergent reparameterisation symmetry which exhibits a pattern of symmetry breaking described in terms of the Schwarzian derivative effective action. This, along with the appearance of chaotic behaviour in the out-of-time-order correlators of the theory is indicative of the relationship to the two-dimensional black hole [129, 253, 259–263], as reviewed in [264, 265]. This note performs analogous analysis to these references, and makes use of many of the results. However, we focus on the solvable $q = 2$ case which is non-chaotic and therefore not regularly studied in the n-AdS_2 JT context.

As described in the previous section, the bulk theory is an $SL(N,\mathbb{R})$ BF theory in two dimensions, first explored in the context of low dimensional analogues to higher spin gravity in (A)dS by [222, 240]. This extended earlier work [266, 267] to write higher spin equations of motion for two-dimensions. The theory includes JT gravity as a subsector, coupled to matter in the form of a tower of integer-weight fields transforming in the discrete series representations of the (A)dS$_2$ isometry group $SO(1,2)$. Formal aspects of the higher spin algebra and operator content of this theory were developed in [246]. In parallel to the story of SYK in n-AdS JT, potential holographic duals have been discussed in AdS for the higher spin generalisation of JT by [223, 268]. This section aspires to be a part of this discussion, with a particular focus on the higher spin theory in dS$_2$ and reference to a wider exploration of the physical consequences of de Sitter representation theory for quantum field theory [55, 92, 103, 104, 190, 269].

The third development relevant to this work is that of low dimensional models of quantum gravity in dS. Since initial discussions of holographic dS quantum gravity [31, 139, 140], there has been a concerted effort to find explicit models of dS/CFT. The proof of principal case is the description of Vasiliev higher spin gravity in dS$_4$ by free bosonic degrees of freedom living at the late time boundary [154, 155], for which our proposed description may be thought of as a lower dimensional analogue. We are motivated to find holographic descriptions of discrete series models in particular, as these are the representations, in two- and four-dimensions, which are carried by the graviton in dS and AdS (as we described in section 3.2.5). The common existence of discrete series models in AdS$_2$ and dS$_2$ also allows for a more direct comparison, making these important tools in building models of dS/CFT. There have been a number of alternate research avenues opened in recent years [122–126, 258, 270], some of which attempt to find low-dimensional toy models which carry some of the properties of four-dimensional de Sitter quantum gravity. Examples which share some of the areas of interest in our work include interpretations of JT gravity in de Sitter [76, 81, 244, 250, 271], and considerations of embedding a piece of dS$_2$ in AdS [78, 79, 121, 272]. Within this area there has also been a recent interest in a potential connection between the double-scaled SYK model (DSSYK) and dS$_3$ [273–278], a separate but complementary conjecture to that presented here.

This chapter is structured as follows. We first make a conjecture, that the $q = 2$ SYK model is a potential microscopic description of the non-abelian BF gauge theory of section 3.4.2 in the limit of large $\mathbf{N}$. In order to justify this conjecture we make a short analysis of the infinite symmetries of the low energy regime of the real $q = 2$ SYK model. We also mention the existence of the tower of discrete series operators although we do not provide a proof of this in any model until section 4.2.2.

We define the complex $q = 2$ SYK model in section 4.2, describing the classical equations of motion, and rewriting the action in terms of the bilinear "master fields" (G, Σ). We also calculate the two-point function from the Schwinger-Dyson equations of motion in the large **N** limit. We consider correlation functions of the large **N** IR CFT in section 4.2.2. In this section we make the claim that there exist a tower of integer-weight operators in the CFT which contribute as conformal blocks to the four-point function of fermions, we contrast this with the case of general q in which the operators pick up non-integer weights, and comment on consequences for conformal symmetry near the IR fixed point. We also construct OPE coefficients for these operators in terms of a finite series. Finally in section 4.3.2 we comment on the thermodynamics of the model, using numerical results presented in appendix G and analytic arguments to comment on differences with the general q model.

4.1 A holographic proposal

We begin this chapter with a discussion of a microphysical model whose operator content furnishes an infinite tower of discrete series UIRs, namely $D_s^{\pm}$ with $s = 2, 4, \ldots$ and moreover has an infinitely large symmetry. This is the $q = 2$ SYK model, whose Hamiltonian governs $\mathbf{N}$ quantum mechanical Majorana fermions ψ_i, with $i = 1, \ldots, \mathbf{N}$ subject a random two-body interaction [254, 262, 279]. Here we initially justify the link between the two theories, taking a brief look at the real SYK model: Later in section 4.2 we analyse the complex version of this model, and prove the existence of the tower of operators that we conjecture in the case of the real model in this section.

Although free, the model exhibits an emergent conformal symmetry at low energies and the operator spectrum can be organised in terms of their properties under $SL(2, \mathbb{R})$. The discrete series operators take the schematic form $\mathcal{O}_\Delta(\vartheta) = \psi_i\, \partial_\vartheta^{\Delta-1} \psi_i$ with $\Delta = 2, 4, \ldots$. We discuss the potential role of $\mathcal{O}_\Delta$ in terms of a microphysical completion of a dS_2 theory endowed with an infinite tower of higher-spin fields; a de Sitter version of the theories discussed in [222, 223, 240, 246].

4.1.1 The real $q = 2$ SYK model

The real SYK model describes $\mathbf{N}$ Majorana fermions ψ_i interacting via a q-body interaction, here we consider the model on S^1. Following [254, 262], the Euclidean action is given by

$$S_{\mathrm{UV}} = \int_{S^1} d\vartheta \left[\frac{1}{2} \psi_i \partial_\vartheta \psi_i - i^{q/2} \sum_{1 \le i_1 < i_2 < \cdots < i_q \le \mathbf{N}} J_{i_1 \ldots i_q} \psi_{i_1} \ldots \psi_{i_q} \right] , \tag{4.1.1}$$

where the factor of $i^{q/2}$ in (4.1.1) is required by Hermiticity, and the couplings $J_{i_1 \ldots i_q}$ are sampled from a Gaussian with zero mean and

$$\left\langle J^2_{i_1 \ldots i_q} \right\rangle = \frac{J^2 (q-1)!}{\mathbf{N}^{q-1}} , \tag{4.1.2}$$

where J has units of energy and characterizes the variance of the distribution on $J_{i_1 \ldots i_q}$. The coordinate $\vartheta \sim \vartheta + 2\pi$ is a coordinate on a Euclidean S^1 which is meant to evoke the boundary coordinate of global dS_2. We can obtain an effective, disorder-averaged theory by integrating in a bilocal field

$$G(\vartheta,\vartheta') = \frac{1}{\mathbf{N}}\sum_{i=1}^{\mathbf{N}} \langle \psi_i(\vartheta)\psi_i(\vartheta')\rangle \, .$$

Integrating over both the couplings $J_{i_1 \dots i_q}$ and the fermions leads to the effective action

$$S^{\text{eff}}_{\text{UV}} = -\frac{\mathbf{N}}{2}\log\det\left(\partial_\vartheta - \Sigma\right) + \frac{\mathbf{N}}{2}\int_{S^1\times S^1} d\vartheta d\vartheta' \left[\Sigma(\vartheta,\vartheta')G(\vartheta,\vartheta') - \frac{J^2}{q}G(\vartheta,\vartheta')^q\right] \tag{4.1.3}$$

(We describe this in more detail for the complex version of the model in section 4.2).

The saddle point equations are given by

$$\partial_\vartheta G(\vartheta,\vartheta') - \int_{S^1} dv\, \Sigma(\vartheta,v)G(v,\vartheta') = \delta(\vartheta-\vartheta')\, , \qquad \Sigma(\vartheta,\vartheta') = J^2 G(\vartheta,\vartheta')^{q-1}\, . \tag{4.1.4}$$

At low energies , where $J\vartheta \gg 1$, we drop the local derivative term. Focusing on $q = 2$, we are left with the following equation of motion:

$$J^2\int_{S^1} dv\, G(\vartheta,v)G(v,\vartheta') = -\delta(\vartheta-\vartheta')\ . \tag{4.1.5}$$

As does the general q-body SYK model, the $q = 2$ theory exhibits a reparameterisation invariance

$$\vartheta \to f(\vartheta)\ , \qquad G(\vartheta_1,\vartheta_2) \to [f'(\vartheta_1)]^{1/2}G(f_1,f_2)[f'(\vartheta_2)]^{1/2}\ , \tag{4.1.6}$$

where $f(\vartheta)$ is a monotonic map. The fermion operators ψ_i transform as primaries of scaling dimension $\Delta = 1/2$. The saddle point solution is given by

$$G^{(cl)}(\vartheta_1,\vartheta_2) = \frac{1}{2\pi J \sin\frac{\vartheta_1-\vartheta_2}{2}}\ , \tag{4.1.7}$$

and at low energies all reparameterisations (4.1.6) of the above are also solutions. In section 4.2.2 we prove that the low energy sector of the complex version of the model contains a tower of conformal primary operators for $\Delta \in 2\mathbb{Z}^+$, where $\Delta = 1,2,3,4,\dots$ are the respective conformal dimensions. In the real case only those with $\Delta = 2,4,6,...$ are expected to appear in the algebra, they take the form

$$\mathcal{O}_\Delta(\vartheta) = \frac{1}{\mathbf{N}}\sum_i \psi_i \partial_\vartheta^{\Delta-1}\psi_i\ , \tag{4.1.8}$$

The presence of an infinite tower of conformal operators, each of integer weight, is suggestive of an integrable structure with an infinite enhancement of symmetries. We now show this is indeed the case.

4.1.2 Infinite symmetries of the $q = 2$ SYK model

The saddle point equation (4.1.5) is reminiscent of the matrix multiplication $G \cdot G^T = -J^{-2}\,\mathbb{1}$ which is invariant under

$$G \to O \cdot G \cdot O^T \tag{4.1.9}$$

with O an orthogonal matrix. The Lie algebra of the orthogonal matrices is spanned by the skew-symmetric matrices. In the continuum case, O is replaced by a function of two times, and each time coordinate serves the purpose of a continuous matrix index. We thus observe that (4.1.5) is invariant under the transformation

$$G(\vartheta, \vartheta') \to \int_{S^1 \times S^1} d\upsilon d\zeta \, O(\vartheta, \upsilon) G(\upsilon, \zeta) O(\vartheta', \zeta) \ , \tag{4.1.10}$$

where $O(\vartheta, \vartheta')$ satisfies

$$\int_{S^1} d\upsilon \, O(\vartheta, \upsilon) O(\vartheta', \upsilon) = \delta(\vartheta - \vartheta') \, . \tag{4.1.11}$$

At the infinitesimal level, we can expand $O(\vartheta, \upsilon) = \delta(\vartheta, \upsilon) + \xi(\vartheta, \upsilon)$, where it follows from (4.1.11) that $\xi(\vartheta, \upsilon)$ is an anti-symmetric function $\xi(\vartheta, \upsilon) = -\xi(\upsilon, \vartheta)$. To first order in $\xi(\vartheta, \upsilon)$, we find that the transformation of G is

$$\delta_\xi G(\vartheta, \vartheta') = \int_{S^1} d\upsilon \, \left(\xi(\vartheta, \upsilon) G(\upsilon, \vartheta') - G(\vartheta, \upsilon) \xi(\upsilon, \vartheta') \right) \ . \tag{4.1.12}$$

We can determine the commutator of two such transformations by computing $[\delta_{\xi'}, \delta_\xi] \equiv \delta_{\xi'} \delta_\xi - \delta_\xi \delta_{\xi'}$. A little algebra reveals

$$[\delta_{\xi'}, \delta_\xi] = \delta_{\xi' \circ \xi} \tag{4.1.13}$$

where

$$\left[\xi' \circ \xi \right] (\vartheta, \vartheta') \equiv \int d\upsilon \, \left[\xi'(\vartheta, \upsilon) \xi(\upsilon, \vartheta') - \xi(\vartheta, \upsilon) \xi'(\upsilon, \vartheta') \right] \ . \tag{4.1.14}$$

The commutator of an infinitesimal reparameterisation for ϑ, which is (4.1.6) with $f(\vartheta) = \vartheta + \varepsilon(\vartheta)$, with an orthogonal generator $[\delta_\varepsilon, \delta_\xi]$ yields

$$[\delta_\varepsilon, \delta_\xi] G(\vartheta, \vartheta') = \int_{S^1} dv \left[\left(\varepsilon(\vartheta)\partial_\vartheta + \varepsilon(v)\partial_v \right) \xi(\vartheta, v) + \frac{\varepsilon'(\vartheta) + \varepsilon'(v)}{2} \xi(\vartheta, v) \right] G(v, \vartheta')$$
$$- \int_{S^1} dv \, G(\vartheta, v) \left[\left(\varepsilon(v)\partial_v + \varepsilon(\vartheta')\partial_{\vartheta'} \right) \xi(v, \vartheta') + \frac{\varepsilon'(v) + \varepsilon'(\vartheta')}{2} \xi(v, \vartheta') \right] , \quad (4.1.15)$$

which is the action of a reparameterised ξ. As for the reparameterisation symmetries, although (4.1.10) is a symmetry of the strict low energy action, it is broken by the leading irrelevant contribution to the effective action

$$S^{\text{eff}}_{\text{UV}} = \frac{\mathbf{N}}{2} \int_{S^1 \times S^1} d\vartheta d\vartheta' \, \delta(\vartheta - \vartheta') \partial_\vartheta G^{(cl)}(\vartheta, \vartheta') \, , \quad (4.1.16)$$
$$= \frac{\mathbf{N}}{4\pi J} \int_{S^1 \times S^1} d\vartheta d\vartheta' \, \delta(\vartheta - \vartheta') \partial_\vartheta \int_{S^1 \times S^1} dv d\zeta \, \frac{O(\vartheta, v) O(\vartheta', \zeta)}{\sin \frac{v - \zeta}{2}} \, , \quad (4.1.17)$$

which diverges at coincident points. Extracting the soft mode action from this divergence can be done by a heuristic point-splitting analysis (see section 3 of [280]) where we split the coincident points by a small amount $\delta\varepsilon$. We find the following contribution

$$S^{\text{eff}}_{\text{breaking}} = \frac{\mathbf{N}}{2\pi J} \int_{S^1 \times S^1} d\vartheta d\nu \left[\text{p.v.} \int \frac{d\xi}{\sin \xi} O(\vartheta + \delta\varepsilon, \nu + \xi) \partial_\vartheta O(\vartheta, \nu - \xi) \right] , \quad (4.1.18)$$

where p.v. denotes the Cauchy principal value. The above is reminiscent of the contribution to the soft-mode sector in SYK theories with global symmetries [280,281].

Although many of the transformations (4.1.10) lead to a non-vanishing soft-mode action and are consequently softly broken, the SYK model with $q = 2$ retains an infinite number of physical symmetries due to the fact that the underlying theory is free. This is most easily seen from the perspective of the Euclidean fermionic action

$$S_{\text{UV}} = \sum_{i,j} \frac{1}{2} \int_{S^1 \times S^1} d\vartheta d\vartheta' \psi_i(\vartheta) \delta(\vartheta - \vartheta') \left(\delta_{ij} \partial_{\vartheta'} - i J_{ij} \right) \psi_j(\vartheta') \, , \quad (4.1.19)$$

where the couplings $J_{ij} = -J_{ji}$ are sampled from a Gaussian with zero mean and variance $\left\langle J_{ij}^2 \right\rangle = J^2/\mathbf{N}$. For any given realisation of the couplings, the following non-local transformation

$$\psi_i(\vartheta) \to \sum_j \int_{S^1} d\vartheta' Q_{ij}(\vartheta, \vartheta')\psi_j(\vartheta') \ , \tag{4.1.20}$$

transforms the action as follows

$$S_{\rm UV} = \sum_{i,j} \frac{1}{2} \int_{(S^1)^4} d\vartheta d\vartheta' d\vartheta'' d\vartheta''' Q_{il}(\vartheta, \vartheta'')\psi_l(\vartheta'')\delta(\vartheta - \vartheta') \left(\delta_{ij}\partial_{\vartheta'} - iJ_{ij}\right) Q_{jk}(\vartheta', \vartheta''')\psi_k(\vartheta''') \ . \tag{4.1.21}$$

Thus for $Q_{ij}(\vartheta, \vartheta')$'s satisfying

$$\sum_{i,j} \int_{S^1 \times S^1} d\vartheta d\vartheta' \delta(\vartheta - \vartheta') Q_{il}(\vartheta, \vartheta'') \left(\delta_{ij}\partial_{\vartheta'} - iJ_{ij}\right) Q_{jk}(\vartheta', \vartheta''') = \delta(\vartheta'' - \vartheta''') \left(\delta_{lk}\partial_{\vartheta'''} - iJ_{lk}\right) \ , \tag{4.1.22}$$

we have a symmetry of the action.

Ordinarily, non-local field transformations are not permitted but here the locality properties of fields may be relaxed by the application to the de Sitter case discussed in section 3.5. At low energies, and upon averaging over the couplings the above symmetry becomes (4.1.10). What we see here is that the infinite low energy symmetry is deformed into an infinite symmetry of the ultraviolet fermionic theory. Within this structure it is possible that the large **N** limit of the $q = 2$ SYK model describes a highly constrained theory like that of section 3.4.3.

4.2 Complex $q = 2$ SYK model

In order to show the existence of integer weight operators in these theories we focus on the complex SYK model in the special case of a quadratic interaction. This model is the same as that of [256, 257, 282] with the interaction parameter $q = 2$. We begin with a classical analysis of the equations of motion and a short discussion of quantisation. Following the usual approach for $q > 2$ [254], we write the model as an effective theory of a pair of bilinear operators (the (G, Σ) formalism).

We consider a theory of $\mathbf{N}$ complex fermions in a single Euclidean dimension parameterised by ϑ. In the following we choose to perform the analysis on the real line to simplify the correlation functions. To transform between the line and the circle we use the conformal transformation

$$\vartheta_{\text{line}} = \tan \frac{\pi}{\beta} \vartheta_{\text{circle}} \ , \tag{4.2.1}$$

where β is the periodicity of the thermal circle. The complex field ψ_i is built out of a pair of one-dimensional Grassman-odd Majorana fermions $\overline{\chi}_i^I = \chi_i^I$, such that

$$\psi_i = \chi_i^1 + i\chi_i^2 \ , \tag{4.2.2}$$

$$\overline{\psi}_i = \chi_i^1 - i\chi_i^2 \ . \tag{4.2.3}$$

We keep the convention that for any spinor, real or complex $\overline{(\xi\eta)} = \overline{\eta}\overline{\xi}$. The Dirac spinor field is also Grassman odd, as a result $(\psi_i)^2 = 0$ and $\overline{\psi}_i\psi_i = -\psi_i\overline{\psi}_i \neq 0$. The Hamiltonian is

$$H_{SYK} = \frac{i}{2} J_{ij} \overline{\psi}_i \psi_j \ , \tag{4.2.4}$$

where J_{ij} is an antisymmetric, real matrix to preserve hermiticity of the Hamiltonian and repeated indices are summed over. Throughtout this section any repeated appearance of an index i, j, k, l we be summed over. As above the J_{ij} are drawn from a Gaussian ensemble with variance $\frac{J^2}{\mathbf{N}}$. The theory is the result of taking an average over these quadratic "realisations" of the model.

The conjugate momentum of ψ_i is $\overline{\psi_i}$ and the Hamiltonian equations of motion are

$$\partial_\vartheta \psi_i = \frac{i}{2} J_{ij} \psi_j \ , \tag{4.2.5}$$

$$\partial_\vartheta \overline{\psi}_i = -\frac{i}{2} J_{ij} \overline{\psi}_j \ . \tag{4.2.6}$$

Taking the Legendre transform we have the Euclidean action

$$S_{SYK} = \frac{1}{2} \int d\vartheta \left[\bar{\psi}_i \partial_\vartheta \psi_i - i J_{ij} \bar{\psi}_i \psi_j \right] \ .$$

For a particular realisation of the model, with fixed J_{ij}, the equation of motion may be solved directly by diagonalising the matrix J_{ij}. As such, a single realisation is a model of $\mathbf{N}$ free fermions in one dimension with random masses, the only source of subtlety is the average over the J_{ij}, referred to here as the disorder average. The disorder average can either be performed as a “quenched” average, in which case the partition function will be averaged and thermodynamic quantities calculated from this partition function. Or, in the case of a few thermodynamic quantities in section 4.3.2, an “annealed” average where the calculation will be performed for the free random mass fermion and then averaged. As in the general q case we expect the two quantities to converge in the large $\mathbf{N}$ limit. In this note annealed average quantities will be denoted with angular brackets $\langle \mathcal{O} \rangle_J$, quenched averaged quantities will not.

The almost free nature of the theory is reflected in the enhanced symmetry at the classical and quantum level, as discussed in section 4.1. Our aim is to show the existence of a tower of operators in the IR spectrum of the theory. There is already a suggestion of this in the fact that the underlying theory is free, and therefore has an number of conserved charges. In addition to the $U(1)$ charge

$$\mathcal{O}_1 = \overline{\psi}_i \psi_i \ , \tag{4.2.7}$$

and the $h = 2$ mode of [254] (here generated classically by the global spacetime symmetry $\psi_i \to \psi_i + \epsilon \partial_\vartheta \psi_i$)

$$\mathcal{O}_2 = \overline{\psi}_i \partial_\vartheta \psi_i \ , \tag{4.2.8}$$

there are further conserved charges, for example

$$\mathcal{Q} = J_{ij} J_{jk} \overline{\psi}_i \psi_k \ . \tag{4.2.9}$$

These are only the local transformations, if we enlarge the set to non-local transformations of the fields we can find an infinite class of symmetries for the model, even away from the IR fixed point (see equation (4.1.22)).

On quantisation we apply the canonical anti-commutation relation

$$\{\overline{\psi}_i, \psi_j\} = \delta_{ij} \ . \tag{4.2.10}$$

Considering the vacuum $|0\rangle$ annihilated by ψ_i and, using $(\overline{\psi}_i)^2 = 0$, we build a finite basis of n-particle states

$$|i_\alpha; n\rangle = \prod_{\alpha=1}^{n} \overline{\psi}_i \, |0\rangle \ . \tag{4.2.11}$$

The states are characterised by the number of particles, $0 \geq n \geq \mathbf{N}$, the eigenvalue of the number operator $\overline{\psi}_i \psi_i$. The number of possible states at particle number n is $\binom{\mathbf{N}}{n}$, implying a $2^{\mathbf{N}}$-dimensional Hilbert space.

4.2.1 (G, Σ) formalism and two-point function

The disorder average is performed by allowing the J_{ij} to vary in the path integral, in this case the quenched and annealed averages are equal by definition

$$Z = \langle \mathcal{Z} \rangle_J = \int [DJ_{ij}][D\psi_i][D\overline{\psi}_i] e^{-\frac{\sqrt{\mathbf{N}} J_{ij} J_{ij}}{2J}} e^{-S_{SYK}} \ . \tag{4.2.12}$$

This model is not quite quadratic, in the sense that it includes a cubic "interaction" term resulting in Feynman diagrams (figure 4.2.1) involving an effective vertex from the disorder average, which does not transfer momentum. A finite $\mathbf{N}$ analysis of

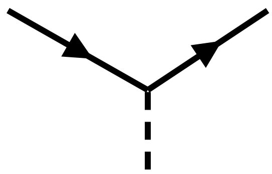

Figure 4.2.1: The effective interaction vertex of free fermions with a disorder line.

certain observables may be performed by rewriting the path integral over J_{ij} in terms of its eigenvalues, this approach is considered in appendix 2 of [283]. Here we work in the limit of large $\mathbf{N}$ to simplify the analysis of the two- and four-point function.

At infinite $\mathbf{N}$ the only contributing diagrams to the two-point function are non-crossing "rainbow" diagrams, the $q = 2$ version of the "melonic" diagrams for ordinary SYK

[264, 283]. One may perform the following analysis using these diagrams to gain the large $\mathbf{N}$ formula for the two-point function $G(\vartheta,\vartheta') = \langle\overline{\psi}_i(\vartheta)\psi_i(\vartheta')\rangle$, where the repeated index is summed over. Alternatively, one can start from the path integral (4.2.12) and perform the integral over J_{ij}, which appears quadratically. By integrating in the identity in terms of G

$$G(\vartheta,\vartheta') = \frac{\langle\overline{\psi}_i(\vartheta)\psi_i(\vartheta')\rangle}{\mathbf{N}} \implies 1 = \int [DG]\delta(\mathbf{N}G - \overline{\psi}_i(\vartheta)\psi_i(\vartheta')) \ , \tag{4.2.13}$$

we perform the path integral over ψ_i to give the final result

$$Z = \left(\frac{\mathbf{N}i\pi}{J^2}\right)^{\frac{\mathbf{N}(\mathbf{N}+1)}{2}} \int [DG][D\Sigma]e^{-\mathbf{N}I[G,\Sigma]} \ , \tag{4.2.14}$$

such that

$$I[G,\Sigma] = \frac{1}{2}\left(-\log[\det(\partial_\vartheta - \Sigma)] + \int d\vartheta d\vartheta' \left[\Sigma(\vartheta-\vartheta')G(\vartheta-\vartheta') - \frac{J^2}{2}G(\vartheta-\vartheta')G(\vartheta-\vartheta')\right]\right) \ . \tag{4.2.15}$$

The equations of motion for the above action are the Schwinger-Dyson equations of q = 2 SYK:

$$\partial_\vartheta G(\vartheta,\vartheta') - \int dv \Sigma(\vartheta,v)G(v,\vartheta) = \delta(\vartheta-\vartheta') \ , \tag{4.2.16}$$

$$\Sigma(\vartheta,\vartheta') = J^2 G(\vartheta,\vartheta') \ . \tag{4.2.17}$$

We are permitted to perform a semi-classical analysis due to the appearance of $\mathbf{N}$ multiplying the action $I[G,\Sigma]$. We solve the Schwinger-Dyson equations at finite cutoff, as well as considering the UV and IR limits.

Frequency space: Examining the Schwinger-Dyson equations in frequency space is simplest and provides a single quadratic equation of motion

$$i\omega G(\omega) - J^2 G(\omega)^2 = 1 \ , \tag{4.2.18}$$

the solution to which is

$$G(\omega) = \frac{i\ \mathrm{sgn}(\omega)}{2J^2}\left(|\omega| - \sqrt{\omega^2 + 4J^2}\right) \ . \tag{4.2.19}$$

Observing that J scales with energy, implying the UV exists at high frequency relative to J, $G(\omega)$ in this case cannot be obtained from (4.2.19). Instead, solving the Schwinger-Dyson equations once more gives

$$G_{UV}(\omega) = -\frac{i}{\omega} \ , \tag{4.2.20}$$

while in the IR ($J \to \infty$)

$$G_{IR}(\omega) = -\frac{i \ \mathrm{sgn}(\omega)}{J} \ . \tag{4.2.21}$$

Position Space: In the UV, the Hamiltonian is taken to be zero as the kinetic term dominates the action, this implies the field itself does not evolve in time $\psi_i(\vartheta) \approx \psi_i(0)$, we may them use the commutation relation, along with translation invariance and antisymmetry to find

$$G_{UV}(\vartheta) = \frac{\mathrm{sgn}(\vartheta)}{2} \ . \tag{4.2.22}$$

In the IR limit we would expect the effective description in terms of G to take the form of a CFT two-point function, using the ansatz

$$G_{IR}(\vartheta) = \frac{b}{|\vartheta|^{2\Delta}} \ , \tag{4.2.23}$$

and substituting this into the fourier transform for the IR two-point function in frequency space demands

$$G_{IR}(\vartheta) = \frac{1}{J\pi\vartheta} \ . \tag{4.2.24}$$

This implies the existence of an operator in the IR CFT which we continue to label ψ_i, which transforms as a $\Delta = \frac{1}{2}$ primary. In addition the full position space two-point function is accessible in this solvable case, taking the fourier transformation of (4.2.19)

$$G(\vartheta) = \frac{I_1(2J|\vartheta|) - \boldsymbol{L}_1(2J|\vartheta|)}{2J|\vartheta|} \ . \tag{4.2.25}$$

Where the I_1 and $\boldsymbol{L}_1$ are spherical Bessel and Struve special functions respectively. We note that we can take the large and small $J\vartheta$ limit to find the UV and IR formulae respectively.

4.2.2 Correlation functions in the large N limit

We now turn to the decomposition of the four-point function of fermions in terms of primaries in the IR CFT. Our objective is to show the existence of a tower of integer-weight operators in the spectrum of the CFT. We then consider the OPE coefficients of these operators.

Four-point function

Here we follow the analysis of [256], specialising the results of this work to the solvable case of $q = 2$. The object of interest is the four-point function

$$\frac{\langle \bar{\psi}_i(\vartheta_1)\psi_i(\vartheta_2)\bar{\psi}_j(\vartheta_3)\psi_j(\vartheta_4)\rangle}{\mathbf{N}^2} \ . \tag{4.2.26}$$

Schematically the leading and subleading terms of the four-point function are

$$G(\vartheta_1 - \vartheta_2)G(\vartheta_3 - \vartheta_4) + \frac{1}{\mathbf{N}}\sum_{n=0}^{\infty}\mathcal{F}_n(\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4) + \mathcal{O}\left(\frac{1}{\mathbf{N}^2}\right) \ . \tag{4.2.27}$$

Where the sum over n expands the four-point function in terms of ladder diagrams with n "rungs" constructed of disorder lines. $\mathcal{F}_n$ is organised using symmetry under the operation of swapping the fermions. Assuming the four-point function is even under time-reversal there may be only two contributions, using the notation of [256]:

$$\mathcal{F}_n \equiv \mathcal{F}_n^+ - \mathcal{F}_n^- \ , \tag{4.2.28}$$

where $\mathcal{F}_n^+$ and $\mathcal{F}_n^-$ are respectively even and odd under operations which act as either $\vartheta_1 \leftrightarrow \vartheta_2$ or $\vartheta_3 \leftrightarrow \vartheta_4$. Each is the contribution of a ladder diagram which may be constructed by repeated integration against a kernel K,

$$\mathcal{F}_n^{\pm}(\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4) = \int d\vartheta d\vartheta' K(\vartheta_1, \vartheta_2; \vartheta, \vartheta')\mathcal{F}_{n-1}^{\pm}(\vartheta, \vartheta', \vartheta_3, \vartheta_4) \ , \tag{4.2.29}$$

where

$$K(\vartheta_1, \vartheta_2; \vartheta, \vartheta') \equiv -J^2 G(\vartheta_1 - \vartheta)G(\vartheta_2 - \vartheta') \ . \tag{4.2.30}$$

The initial contribution, without any disorder connecting the two branches is

$$\mathcal{F}_0^{\pm}(\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4) = G(\vartheta_1 - \vartheta_3)G(\vartheta_2 - \vartheta_4) \pm G(\vartheta_1 - \vartheta_4)G(\vartheta_2 - \vartheta_3) \ . \tag{4.2.31}$$

Schematically, for both symmetric and antisymmetric contributions we resum the ladder diagrams using the following formula, as described in [254, 256, 264, 283] for the general q model,

$$\sum_{n=0}^{\infty}\mathcal{F}_n^{\pm} = \sum_h \frac{1}{1 - k(h)}\frac{\langle\langle \Psi_h^{\pm}, \mathcal{F}_0^{\pm}\rangle\rangle}{\langle\langle \Psi_h^{\pm}, \Psi_h^{\pm}\rangle\rangle}\Psi_h^{\pm} \ , \tag{4.2.32}$$

where $\langle\langle\ ,\ \rangle\rangle$ denotes theinner product on the space of eigenfunctions, and the sum includes a contribution for all non-zero eigenvalues, $k(h)$, of the operation of the kernel defined in (4.2.29). The eigenfunctions of the kernel, Ψ_h, for symmetric and antisymmetric contributions were calculated in [256]; their conformally invariant form is included in appendix I. Here we have been careful to take the large-$\mathbf{N}$ limit and resum the series of ladder diagrams (i.e. sum over n) before taking the CFT limit. Failure to use this order of limits results in ambiguities, indeed we must have conformal symmetry to make precise the meaning of h, which is the eigenvalue of the conformal Casimir.

An important simplification can be made in the case of the $q = 2$ model. None of the ladders have any transfer momentum between the branches: instead the propagator for the dashed line is a kronecker delta function and controls the expansion in $\mathbf{N}$. This is described for the real $q = 2$ SYK model in appendix C of [283]. In the above formalism, this fact is reflected in the independence of $k^{\pm}(h)$ from h,

$$k^{\pm}(h) = -1 \ . \tag{4.2.33}$$

This can be directly calculated from the three point function of operators of weight h with two fermions, as shown in appendix H. We immediately write the result

$$\sum_{n=0}^{\infty} \mathcal{F}_n^{\pm} = \frac{1}{2} \sum_h \frac{\langle\langle \Psi_h^{\pm}, \mathcal{F}_0^{\pm} \rangle\rangle}{\langle\langle \Psi_h^{\pm}, \Psi_h^{\pm} \rangle\rangle} \Psi_h^{\pm} = \frac{1}{2} \mathcal{F}_0^{\pm} \ . \tag{4.2.34}$$

Here we have effectively performed a regularisation, $\sum_n (-1)^n = \frac{1}{2}$, to give

$$\frac{\langle \bar{\psi}_i(\vartheta_1)\psi_i(\vartheta_2)\bar{\psi}_j(\vartheta_3)\psi_j(\vartheta_4)\rangle}{\mathbf{N}^2} = G(\vartheta_1-\vartheta_2)G(\vartheta_3-\vartheta_4)+\frac{1}{\mathbf{N}}G(\vartheta_1-\vartheta_4)G(\vartheta_2-\vartheta_3)+\mathcal{O}\left(\frac{1}{\mathbf{N}^2}\right) \ . \tag{4.2.35}$$

Harmonic analysis

The benefit of writing the four-point function as a sum over ladder contributions is the ease with which we may now decompose the four-point function into contributions of conformal primaries, as noted first in [254]. In this section we perform a harmonic analysis to prove the existence of a tower of integer-weight operators in the spectrum of the theory. The invariant cross ratio in one dimension is

$$x = \frac{\vartheta_{12}\vartheta_{34}}{\vartheta_{13}\vartheta_{24}} \qquad , \qquad 1 - x = \frac{\vartheta_{23}\vartheta_{14}}{\vartheta_{13}\vartheta_{24}} \ , \tag{4.2.36}$$

where $\vartheta_{ij} = \vartheta_i - \vartheta_j$. Considering the conformally invariant four-point function

$$\frac{\langle \bar{\psi}_i(\vartheta_1)\psi_i(\vartheta_2)\bar{\psi}_j(\vartheta_3)\psi_j(\vartheta_4)\rangle}{\mathbf{N}^2 \langle \overline{\psi}_i(\vartheta_1)\psi_i(\vartheta_2)\rangle \langle \overline{\psi}_i(\vartheta_3)\psi_i(\vartheta_4)\rangle} = 1 + \frac{1}{\mathbf{N}}\left(\frac{x}{1-x}\right) + \mathcal{O}\left(\frac{1}{\mathbf{N}^2}\right) , \qquad (4.2.37)$$

here we have used $1 - x = \frac{\vartheta_{23}\vartheta_{14}}{\vartheta_{13}\vartheta_{24}}$. In another form the four-point function may be written as a sum over conformal blocks,

$$\mathcal{F}(x) = \frac{x}{1-x} \equiv \left(\sum_h (c^h_{\psi\psi})^2 \tilde{\Psi}_h(x)\right) . \qquad (4.2.38)$$

Where h are the eigenvalues of the conformal Casimir,

$$\mathcal{C} = x^2(1-x)\partial_x^2 - x^2\partial_x . \qquad (4.2.39)$$

The functions $\tilde{\Psi}_h(x)$ are the eigenfunctions. As the conformal Casimir and the action of the kernel defined in (4.2.29) commute, these are precisely the functions used to decompose the four-point function in terms of ladder diagrams in (4.2.32), made conformally invariant.

$$\tilde{\Psi}_h(x) \equiv \frac{\Psi_h(\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4)}{\langle \overline{\psi}_i(\vartheta_1)\psi_i(\vartheta_2)\rangle \langle \overline{\psi}_i(\vartheta_3)\psi_i(\vartheta_4)\rangle} , \qquad (4.2.40)$$

We use $\sim$ to describe the conformally invariant versions of the functions above in the following. Comparing (4.2.38) to (4.2.34) we gain the following formula for the OPE coefficients,

$$(c^{\pm,h}_{\psi\psi})^2 = \pm\frac{\langle\langle \tilde{\Psi}^{\pm}_h, \tilde{\mathcal{F}}^{\pm}_0 \rangle\rangle}{2\,\langle\langle \tilde{\Psi}^{\pm}_h, \tilde{\Psi}^{\pm}_h \rangle\rangle} + \mathcal{O}\left(\frac{1}{\mathbf{N}}\right) . \qquad (4.2.41)$$

Noting there are separate sums for parity odd, and even terms in the complex case, which we specify using $\pm$ as described in appendix I. In appendix I there are formulae for $\langle \tilde{\Psi}^{\pm}_h, \tilde{\Psi}^{\pm}_{h'} \rangle$, the only calculation that remains is to find $\langle \tilde{\Psi}^{\pm}_h, \tilde{\mathcal{F}}^{\pm}_0 \rangle$. Combining (4.2.31) and (4.2.37), we write the odd and even contributions in terms of the cross ratio

$$\tilde{\mathcal{F}}^{\pm}_0 = \frac{1}{2}\left(x \pm \frac{x}{x-1}\right) . \qquad (4.2.42)$$

The inner product in the space of eigenfunctions is

$$\langle\langle f, g \rangle\rangle \equiv \int_0^\infty \frac{dx}{x^2} \bar{f} g . \qquad (4.2.43)$$

Making use of the symmetry of (4.2.42) under $x \to \frac{x}{x-1}$, these are calculated exactly in equation (A.11) of [256] using the split representation, and agree with [254] for the symmetric pieces, specialising to the $q = 2$ case

$$\langle\langle \tilde{\Psi}_h^{\pm}, \tilde{\mathcal{F}}_0^{\pm} \rangle\rangle = \int \frac{dx}{x} \tilde{\Psi}_h^{\pm}(x) = -\pi^2 \ . \tag{4.2.44}$$

Generally, the four-point function may be written with contributions split between the principal series $h = \frac{1}{2} + is$ and the discrete series $h \in \mathbb{Z}^+$, as written explicitly in equation (5.42) and (5.43) of [256]. We note that the wavefunctions $\tilde{\Psi}_h^{\pm}$ are symmetric under the "shadow" transformation $h \to 1 - h$. It is well known from [254], that for $q > 2$ the symmetric contribution may be rewritten as a single sum over a discrete set of poles with irrational weights. We now discuss the analogous construction for the $q = 2$ model. Combining these results

$$\mathcal{F}(x) = \frac{1}{4} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} dh \frac{2h-1}{\tan \pi h} \left(\tilde{\Psi}_h^{+}(x) - \tilde{\Psi}_{1-h}^{-}(x) \right) + \sum_{n=1}^{\infty} \frac{2n-1}{4} \tilde{\Psi}(x). \tag{4.2.45}$$

Where we have defined $\tilde{\Psi}_n(x) = \pm \tilde{\Psi}_n^{\pm}(x)$ for respectively even, and odd values of n.

For the principal series contribution, we have used the shadow symmetry $s \to -s$ to define the contour along the entire real axis in s. Using the appendix, in analogy to equation (3.84) in [254] we can write the contour integral over the principal series in terms of the hypergeometric functions

$$\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} dh \frac{2h-1}{\tan \pi h} \chi_h^P(x) \ , \tag{4.2.46}$$

where we can rewrite the wavefunctions using a hypergeometric identity such that

$$\chi_h^P(x) = \begin{cases} \frac{\Gamma(h)^2}{\Gamma(2h)} x^h \, {}_2F_1(h, h; 2h; x) & x < 1 \ , \\ \frac{\pi}{\sin \pi h} \, {}_2F_1(h, 1-h; 1; \frac{1}{x}) & x > 1 \ . \end{cases} \tag{4.2.47}$$

In both cases there are poles at integer points in h, which we pick up by deforming the contour onto the positive real axis, the result is

$$\mathcal{F}(x) = \sum_{n=1}^{\infty} \frac{(2n-1)}{2} \Psi_n(x) \ . \tag{4.2.48}$$

Ψ_n are the discrete series eigenfunctions of the conformal casimir, as defined in appendix I. It is possible to verify this relationship numerically by resumming a finite number of the hypergeometrics to recover (4.2.48).

This is the most important result of this section, and has not featured in previous literature. There is a significant difference in the $q = 2$ case displayed here from the generic $q > 2$ case of [254, 256, 283]. In the general model, the contour deformation replaces the principal **and** discrete series expansion by a sum over operators with irrational weights, here they all have the exact integer weights of the discrete series. The existence of a tower of operators transforming in a representation of $SL(2, \mathbb{R})$ with discrete series weights in the $q = 2$ model links this theory to the models discussed in section 3.4. In addition, the exact integer-weights are fundamentally important to dS_2. Usually in AdS the timelike direction is decompactified by taking the universal cover of the $SO(2, 1)$ isometry group. This group has a more general set of highest or lowest weight representations with weights in $\mathbb{R}^+$. In de Sitter no such universal cover is taken, and so highest or lowest weight representations have protected integer weights.

Additionally, a very important contribution to the four-point function in the real $q > 2$ case is the double pole at $h = 2$ [129, 262, 264]. In the complex model there is an infinite term in the sum at either $h = 1, 2$, as described in the generic q case for the complex model in [256]. These modes break the conformal reparameterisation invariance of the general model, leading to the domination of the leading order correction to the conformal action (the Schwarzian action) in the soft sector of the model and it's description as a "near-CFT" in the literature. The $q = 2$ model seems to sidestep this complication, instead we anticipate that moving away from the IR fixed point will involve a more equal contribution from the tower of operators derived in (4.2.48). The Schwarzian action will still make an appearance, however we anticipate additional structure in the form of "higher spin" versions of the Schwarzian similar to those discussed in [268].

OPE coefficients

Looking toward future discussion of the bulk theory, we now write a closed formula for the OPE coefficients c_{nmk} appearing in the three point function of the primary integer-weight operators to leading non-trivial order in $\frac{1}{\mathbf{N}}$.

$$\langle \mathcal{O}_n(\vartheta_1)\mathcal{O}_m(\vartheta_2)\mathcal{O}_k(\vartheta_3)\rangle = \frac{1}{\sqrt{\mathbf{N}}}\frac{c_{nmk}}{|\vartheta_{12}|^{n+m-k}|\vartheta_{13}|^{n+k-m}|\vartheta_{32}|^{m+k-n}}\ , \tag{4.2.49}$$

This calculation in the generic $q > 2$ case has been performed explicitly in [255], the formulae included simplify considerably for $q = 2$; some of their considerations are unnecessary due to the almost free nature of the theory.

Fermion six-point function We first write a closed form expression for the connected part of the six point function of fermions, to leading order in $\frac{1}{\mathbf{N}}$ [255]. In this section "connected" is used in the sense that all propagators are connected by disorder lines. However, in the final result the connected piece will still be a reorganised product of propagators. This combinatorial exercise is accomplished by considering combinations of four-point functions.

In an individual realisation of the model, disconnected diagrams occur at every order in $\mathbf{N}$, they are represented by the type of diagram displayed in figure 4.2.2, where the grey shaded area is a sum over the entire expansion of the four-point function in all powers of $\frac{1}{\mathbf{N}}$. These disconnected diagrams do not contribute to the three point function of the integer-weight operators except as contact terms, as a consequence of the particular coincident point limits we take below, here we ignore them.

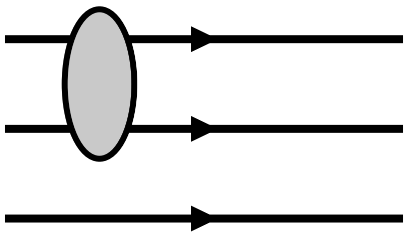

Figure 4.2.2: Diagrams of this form contribute to the six point function of fermions at first order in $\frac{1}{\mathbf{N}}$. However they only contribute as a contact term to the three point function of integer-weight operators.

The important contribution comes from connected diagrams. The first leading-order connected contribution is a sum over two sets of ladder diagrams which sequentially occur between two of the propagators, at order $\mathbf{N}^{-2}$. These terms give the leading order contribution to the connected six point function written in terms of the propagator,

$$\frac{\langle\bar{\psi}_i(\vartheta_1)\psi_i(\vartheta_2)\bar{\psi}_j(\vartheta_3)\psi_j(\vartheta_4)\bar{\psi}_k(\vartheta_5)\psi_k(\vartheta_6)\rangle_C}{\mathbf{N}^3} = \frac{3}{\mathbf{N}^2}\left(G(\vartheta_1-\vartheta_6)G(\vartheta_3-\vartheta_2)G(\vartheta_5-\vartheta_4)+G(\vartheta_1-\vartheta_4)G(\vartheta_3-\vartheta_6)G(\vartheta_5-\vartheta_2)\right)+\mathcal{O}(\mathbf{N}^{-3})\ . \tag{4.2.50}$$

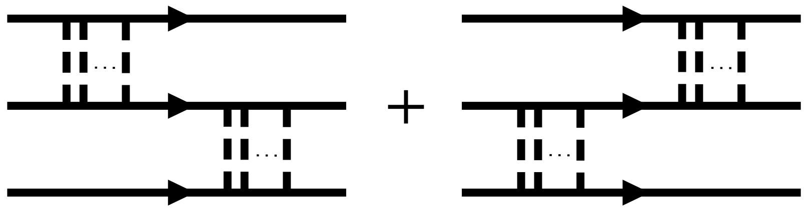

Figure 4.2.3: Resummation over two sets of non-crossing ladder diagrams gives the $\mathcal{O}(\mathbf{N}^{-2})$ contribution to the six-point function of fermions, these diagrams give the OPE coefficients of the integer-weight operators. Contributions from diagrams with crossing ladder diagrams contribute at higher orders in $\mathbf{N}^{-1}$

Three-point function of operators We can represent the operators of the tower in terms of the original field operators of the SYK, the fermions [254, 255, 284]. These are primary, gaining a dimension of $\frac{1}{2}$ in the case of $q = 2$. The operators are

$$\mathcal{O}_n(\vartheta) = \frac{i}{\sqrt{\mathbf{N}}} \sum_{r=0}^{n} d_{nr} \partial_\vartheta^r \bar{\psi}_i(\vartheta) \partial_\vartheta^{n-r} \psi_i(\vartheta) \ , \tag{4.2.51}$$

where the d_{nr} are chosen to ensure the operator is a primary,

$$d_{nr} = \frac{\pi J (-1)^r (-n)_r{}^2}{\sqrt{\Gamma(2n+1)}\Gamma(r+1)^2} \ . \tag{4.2.52}$$

Here $(x)_n$ is the Pochhammer symbol. These can be constructed from the normalisation of the two-point function such that

$$\langle \mathcal{O}_n(\vartheta_1) \mathcal{O}_m(\vartheta_2) \rangle = \frac{\delta_{n,m}}{|\vartheta_1 - \vartheta_2|^{2n}} \ . \tag{4.2.53}$$

The three point function can be retrieved from the six point function of the fermion operators in (4.2.50), by taking the appropriate derivatives and then the limits $\vartheta_1 \to \vartheta_2$, $\vartheta_3 \to \vartheta_4$ and $\vartheta_5 \to \vartheta_6$.

$$\langle \mathcal{O}_n(\vartheta_2) \mathcal{O}_m(\vartheta_4) \mathcal{O}_k(\vartheta_6) \rangle = \\ -i \lim_{\vartheta_1 \to \vartheta_2} \lim_{\vartheta_3 \to \vartheta_4} \lim_{\vartheta_5 \to \vartheta_6} \sum_{r_1=0}^{n} \sum_{r_2=0}^{m} \sum_{r_3=0}^{k} d_{nr_1} d_{mr_2} d_{kr_3} \partial_{\vartheta_1}^{r_1} \partial_{\vartheta_2}^{n-r_1} \partial_{\vartheta_3}^{r_2} \partial_{\vartheta_4}^{m-r_2} \partial_{\vartheta_5}^{r_3} \partial_{\vartheta_6}^{k-r_3} \mathcal{F}_C^6(\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4, \vartheta_5, \vartheta_6) \ . \tag{4.2.54}$$

c_{1mk}	$k=1$	2	3	4	5
$m=1$	0	$3i\sqrt{2}$	0	$\frac{3i}{\sqrt{5}}$	0
2	$3i\sqrt{2}$	0	$3i\sqrt{3}$	0	$3i\sqrt{\frac{5}{7}}$
3	0	$3i\sqrt{3}$	0	$i\sqrt{30}$	0
4	$\frac{3i}{\sqrt{5}}$	0	$i\sqrt{30}$	0	$3i\sqrt{\frac{7}{2}}$
5	0	$3i\sqrt{\frac{5}{7}}$	0	$3i\sqrt{\frac{7}{2}}$	0

Table 4.2.1: OPE coefficients c_{1mk} for integer-weight operators of the complex $q=2$ SYK model for some small values of weights with $1 < m, k < 5$.

Case by case the OPE coefficients can be constructed,

$$c_{nmk} = -i \sum_{r_1=0}^{n} \sum_{r_2=0}^{m} \sum_{r_3=0}^{k} d_{nr_1} d_{mr_2} d_{kr_3} \left(\frac{C_{r_1r_2r_3} \vartheta_{24}^{m+2r_1} \vartheta_{46}^{k+2r_2} \vartheta_{62}^{n+2r_3} + \tilde{C}_{r_1r_2r_3} \vartheta_{24}^{n+2r_2} \vartheta_{46}^{m+2r_3} \vartheta_{62}^{k+2r_1}}{\vartheta_{24}^{k+r_1+r_2} \vartheta_{46}^{n+r_3+r_2} \vartheta_{62}^{m+r_1+r_3}} \right) . \tag{4.2.55}$$

where

$$C_{r_1r_2r_3} = (-1)^{1+r_1+r_2+r_3+m+k+n} (k - r_3 + r_1)!(n - r_1 + r_2)!(m - r_2 + r_3)! \ , \tag{4.2.56}$$

$$\tilde{C}_{r_1r_2r_3} = (-1)^{r_1+r_2+r_3} (k - r_3 + r_2)!(n - r_1 + r_3)!(m - r_2 + r_1)! \ . \tag{4.2.57}$$

After resummation, c_{nmk} are independent of the coordinates ϑ_{ij}. OPE coefficients c_{1mk} with $1 < m, k < 5$ are displayed in table 4.2.1.

4.3 Discussion

4.3.1 Higher-spin dS_2 dual?

We have seen that the $q = 2$ model displays a highly symmetric low energy sector encoding an infinite tower of conformal primaries $\mathcal{O}_\Delta$ with $\Delta \in \mathbb{Z}^+$. It is tempting to suggest that the $\mathcal{O}_\Delta$ are captured by an underlying dS_2 theory with an infinite tower of operators in the discrete series UIR, echoing ideas expressed for the AdS_2 case in [223, 268]. A potentially relevant class of models exhibiting such properties are the $N \to \infty$ extensions of the $SL(N, \mathbb{R})$ BF-theories [240, 246] discussed in section 3.4.3. Guided by the linearized equations (3.4.34), the higher-spin bulk operators at $\mathcal{I}^+$ extend the $SL(2, \mathbb{R})$ $\Delta = 2$ operator (3.4.20) to an infinite tower of operators with $\Delta = 2, 3, \ldots, \infty$.

We propose that the bulk late-time conformal operators are microscopically constructed from two towers of $q = 2$ SYK conformal operators

$$\mathcal{O}_\Delta(\vartheta) = \frac{i}{\sqrt{\mathbf{N}}} \sum_{r=0}^{n} d_{nr} \partial_\vartheta^r \bar{\psi}_i(\vartheta) \partial_\vartheta^{n-r} \psi_i(\vartheta) \,, \quad \tilde{\mathcal{O}}_\Delta(\vartheta) = \frac{i}{\sqrt{\mathbf{N}}} \sum_{r=0}^{n} d_{nr} \partial_\vartheta^r \bar{\chi}_i(\vartheta) \partial_\vartheta^{n-r} \chi_i(\vartheta) \,, \text{ with } \Delta \in \mathbb{Z}^+ \,. \tag{4.3.1}$$

Here ψ_i and χ_i, with $i = 1, \ldots, \mathbf{N}$, are two collections of $q = 2$ SYK fermions. The reason we have two towers of operators in the bulk dS_2 is the higher spin extension of the observation that there are two collections of operator modes, α_n and β_n, associated with the mode expansion bulk operator $\omega(T, \vartheta)$ in (3.4.20).

In addition, there should be operators associated to the adjoint valued $B(T, \vartheta)$ field, which has non-trivial commutation relations with the $SL(N, \mathbb{R})$ gauge field. It is natural to construct these out of fermionic operators also, which have non-trivial commutation relations with the ψ_i and χ_i. To this end, we note that for a sufficiently large number $\mathbf{N}$ of fermionic operators satisfying the standard anti-commutation relations, one can approximate the bosonic creation/annihilation operator algebra with arbitrary precision [285].

Due to the presence of a timelike boundary, the AdS_2 version of the $SL(N, \mathbb{R})$ BF theory has a slightly broken infinite dimensional higher-spin algebra governed by a soft-sector [268], as well as an infinite set of conserved physical boundary charges extending the conservation of energy associated to ordinary JT gravity. In the dS_2 case, all physical symmetries must be further gauged. Given that the operators are built from quantum mechanical fermions subject to an infinite dimensional symmetry, such a gauging might result in a finite-dimensional Hilbert space [72, 111, 286, 287].

4.3.2 Note on the thermodynamics of $q = 2$ SYK

Our analysis has focused on aspects of the complex $q = 2$ SYK model relevant to identifying the holographic relationship it may have with the de Sitter model described in section 3.4.2. For completeness here we describe some aspects of the thermodynamics of the SYK model which differ from the generic q case. In future work, a thorough analysis of the thermodynamics of the disorder averaged model may provide a better description of the emergent structure of the model in the IR. In particular the (annealed) average free energy is calculated at temperature $\frac{1}{\beta}$ in appendix A of [279] at large $\mathbf{N}$. As described in [283, 288] in this limit the spectral function controlling the distribution of eigenvalues of the coupling iJ_{ij} approximates the Wigner semicircle. This allows an analytic calculation of the annealed thermodynamic quantities in various thermodynamic limits. The calculation in the low-temperature $\beta \to \infty$ limit gives a linear specific heat, and as the free energy tends to 0, this result agrees with the calculation of low temperature entropy for the real $q = 2$ model in equation (D.5) of [78], calculated from the master fields in (4.2.15)

$$\lim_{\beta\to\infty} \langle S[\beta] \rangle_J = \lim_{\beta\to\infty} \langle C[\beta] \rangle_J = \frac{\pi \mathbf{N}}{6J\beta} + \mathcal{O}(\mathbf{N}^2) \ . \tag{4.3.2}$$

As described in [78] the entropy of the $q = 2$ model is 0 in the large $\mathbf{N}$ limit at zero temperature in a significant departure from the general case. In appendix G the annealed disorder quantities are calculated and presented for a finite number of realisations of the real $q = 2$ model using excellent freely available code [289]. These results were then extrapolated to large $\mathbf{N}$ using the same procedure as in [260] to show the expected zero low temperature and linear behaviour of the entropy for the real model in figure G.2.1. We expect the complex model to retain many of the same features, particularly as the spectral function controlling the thermodynamics converges for large $\mathbf{N}$ [283].

The linear entropy behaviour (4.3.2) can be derived from the Schwarzian action in the $q = 2$ case, much like the general case. This would seem to disagree with the more democratic soft sector of the theory suggested by the absence of the double pole in the four-point function. However, it is consistent with our analysis if it can be shown that these other soft sectors contribute at higher order in $\frac{1}{\beta}$. Indeed the generality of the Schwarzian contribution for any q SYK is commented on in [290], in which a link is made to the $SL(2,\mathbb{R})$ BF theory, a subsector of the $SL(N,\mathbb{R})$ model of [222]. A more thorough analysis is required and it would seem that the numerical results presented in appendix G are a good starting point for this work.

5. Outlook

We have presented an approach to build understanding of quantum phenomena in de Sitter space. As described in section 1.3, this is a context in which the basic assumptions of quantum field theory in static spacetimes are replaced. Most importantly, unitary Hamiltonian evolution of a quantum system for the non-accelerating observer is replaced by covariance or invariance under the de Sitter group.

Our strategy was to begin with representation theory, applying Wigner's classification of single particle states to the de Sitter group, in order to gain a spectroscopic understanding of the types of fluctuations present in a fixed de Sitter universe. This lead us firstly to uncover embedding space methods for fermionic and tensorial fields in general de Sitter spacetimes in chapter 2.

In chapter 3 we began an in-depth exploration of two-dimensional de Sitter spacetime, focusing on the discrete series of representations of the de Sitter group in this case $SL(2,\mathbb{R})$. These representations were found to appear in a well defined way in the context of the gauged massless scalar field, as well as a generalisation of JT gravity in the form of the topological BF theory on dS_2.

The implication that the discrete series was associated to gravity was discussed and the consequences for late-time correlation functions were considered in section 3.5. Including gravity presents an interesting comparison to AdS, in this section we develop the necessity of non-local observables due to the lack of a stable boundary against gravitational fluctuations.

The final chapter of this thesis, chapter 4, was devoted to the study of the quantum mechanical SYK model, and its potential application as a microscopic description of the BF model of chapter 3. We considered the large-$\mathbf{N}$ behaviour of the complex $q = 2$ SYK model near the IR fixed point and found several differences from the general $q > 2$ case studied more regularly. This analysis included a description of the operator algebra of the SYK model, which was shown to match the BF theory in section 3.4.3, as well as a description of the emergent conformal symmetry of the IR in the large-$\mathbf{N}$ limit, which may prove to describe the large number of gauge constraints for the bulk theory.

We found the model contains an infinite set of integer-weight operators; unlike the general q model, there is no infinite $h = 1, 2$ contribution to the four-point function of fermions. The model has a symmetry breaking structure distinct from the general case, in which the Schwarzian soft sector generated by the $h = 2$ mode dominates the low energy behaviour of the model. Instead, we might imagine the deformation of the $q = 2$ theory away from the strict-IR limit contains contributions from the entire tower of operators.

As a result of its almost Gaussian structure, the model is solvable. As part of our analysis we calculated the OPE coefficients for the integer weight operators. We therefore collected the CFT data of the discrete-series sector of the complex $q = 2$ SYK model in preparation for further work on the bulk theory in section 4.2.2.

There is still work to do in furthering the possible description of the bulk $SL(N\mathbb{R}$ BF theory by the $q = 2$ SYK model, but the evidence provided here suggests that a microscopic description of dS_2 is attainable. These developments would construct a low-dimensional analogue to the four-dimensional higher-spin model of [154, 155]. The bulk theory is Lagrangian, unlike this four-dimensional example, with potentially accessible correlation functions. This implies a step forward in understanding holography in de Sitter with applications for both the inflationary paradigm, and for observables in the static patch.

5.1 Next steps

The complex $q = 2$ SYK model shares an operator algebra and enhanced symmetry structure with $SL(N, \mathbb{R})$ BF theory in the $N \to \infty$ limit of chapter 3, described in dS in section 3.4.3. Given the CFT data of the microscopic theory presented in section 4.2.2, the next obvious step is to match these correlation functions on to an observable calculated from the bulk theory. This is challenging for a theory without propagating dynamical degrees of freedom. In order to find a bulk observable which may be dual to the CFT data we display in table 4.2.1 there are several approaches that we would be interested in following in future work.

First, the bulk theory is a version of higher-spin gravity, appropriate for two dimensions [222]. In [223, 246] a further deformation of the bulk $SL(N, \mathbb{R})$ theory is suggested, in which an additional tower of matter fields is added, while preserving the structure of the higher-spin symmetries. This extended theory includes a sector with propagating degrees of freedom. It is therefore possible to imagine calculating the correlation functions of insertions of the matter integer weight operators to match onto the microscopic theory on the late-time slice.

The theory was initially defined on AdS [222] rather than dS. Taking inspiration from JT gravity on near-AdS [129], which forms a subset of $SL(N, \mathbb{R})$ BF theory, it may be possible to deform the bulk theory by introducing a boundary or defect. The subject of the higher spin analogue to the schwarzian contribution of JT has been discussed in AdS in [240]. A version of the boundary approach was first attempted for dS-JT in [244] where the Hartle Hawking wavefunction on a boundary near to the late-time slice was calculated. This is not the only location available for the boundary, and several works analysing JT gravity in the stretched horizon paradigm have been conducted [76, 80, 81, 121]. The boundary provides static patch observables for matching onto the boundary microscopic theory, the interpretation of which would differ significantly from the AdS case.

We began chapter 3 with a spectroscopic perspective, by considering the representations of the isometry group of dS_2, and allowing them to organise the quantum mechanical fluctuations of the spacetime. In particular, we considered the discrete series, a little studied unitary irreducible representation in this context. Extending this to all the representations of the algebra $\mathfrak{sl}(2, \mathbb{R})$ would also require a study of fermionic degrees of freedom in dS_2, for which we have defined representations in appendix B. This is a natural application of section 2.3, the developments of embedding methods for spinorial fields. Similarly to the bosonic case, the fermionic representations are divided into those of positive definite mass, the principle series, and those of half-integer, negative m^2. These degrees of freedom are expected to obey equations of motion

$$\left(\nabla\!\!\!\!/ - i\frac{r}{\ell}\right)\psi = 0 \ , \tag{5.1.1}$$

such that $r = 1, 2, 3, ...$. Given the tachyonic nature of the Klein-Gordon equation obtained by squaring this equation, it would be interesting to perform an analysis similar to section 3.4. This would clarify the existence of zero-modes for the Dirac action in this case. The partially-massless representations in four dimensions with which we draw a close comparison for the bosonic discrete series also appear in the fermionic case for higher dimensional dS [102–104].

It seems plausible that these spinorial equations can be embedded in theories with gauge redundancy, a particularly interesting possibility is the imposition of supersymmetry in this context. It is difficult to justify de Sitter in the context of supergravity in general dimension [291,292], but a special case exists for two dimensions [293]. Given the appearance of the (conformal) killing vectors as (un)physical degrees of freedom for the discrete series $\Delta = 2$, it is possible that the conformal Killing spinors will provide a similar structure in the supersymmetrisation. This would be clarified by an examination of possible generalisations of Super-JT in dS_2.

5.2 Future applications

We introduced this thesis by considering recent measurements of supernovae, and the building evidence that the expansion of the universe is accelerating (section 1.1). Despite the experimental nature of our motivation we have ended the thesis on a theoretical note, discussing relationships between quantum mechanical models in one dimension and higher-spin gravity in dS_2. A natural question at this point is how will our discussion make a impact on practical considerations?

Current research in quantum gravity is often inspired by tension in theoretical or experimental observations. For example unitary evolution in quantum mechanical systems seems to be in tension with semi-classical results on black hole information. This tension has sourced a huge body of continuing research, encompassing developments in string theory [294] and AdS/CFT [112], leading to the definition of low dimensional toy models [117] and building on considerations of non-local observables [295].

The data presented in section 1.1 from supernovae, baryon acoustic oscillations, weak gravitational lensing and the CMB significantly suggests the accelerating expansion of the universe. Nonetheless, there is room for tension in their interpretation:

Cosmological constant problem While the ΛCDM model (reviewed in [30]) provides an explanation for the observed flatness and large scale homogeneity of the universe, as well as the current redshift of supernovae, it does not provide a physical justification for the current value of the cosmological constant. This is widely discussed as part of the cosmological constant problem [296]. Considering the semi-classical Einstein field equation

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G \langle T_{\mu\nu} \rangle \tag{5.2.1}$$

the total observed (effective) cosmological constant when the effects of matter are included is naturally split into two pieces

$$\Lambda_{eff} = \Lambda + 8\pi G \langle \rho \rangle \tag{5.2.2}$$

such that the contribution of the zero-point energy to the right hand side is $T_{\mu\nu} = \langle \rho \rangle g_{\mu\nu} + \ldots$. Simple approximations in quantum field theory place the value of $\langle \rho \rangle \sim 10^8$ GeV^4 compared to the observed bounds from supernova surveys on $\Lambda_{eff} \sim 10^{-47}$ GeV^4 [297]. The cosmological constant is therefore many orders of magnitude smaller than the approximate quantum contributions to the zero-point energy of the standard model. Like the black hole information problem, the tension appears between a semi-classical prediction and quantum mechanical assumptions, and implies a gap in our understanding as gained by effective field theory intuition.

E-folds Details of the inflationary epoch have also been debated since the paradigm was suggested: consistency with observation implies constraints around the amount of inflation necessary to create a homogeneous universe. In order to solve the Horizon problem, inflation is generally expected to have expanded the primordial universe by a factor of $\sim e^{60}$, sometimes described as requiring 60 e-folds [298]. This is important when considering the initial conditions of a quantum mechanical model of the cosmos. In de Sitter the initial conditions are specified by the Bunch-Davies, or Euclidean vacuum. In inflation, this state is called the "No-Boundary proposal," and replaces the classical singularity at the Big Bang with a quantum mechanical vacuum state, regular over a smooth spacetime in the past, with no boundary on which to fix initial conditions [69]. This is a proposal for a fully quantum mechanical description of gravity in terms of a "wavefunctional of the universe" [299, 300]. Unfortunately, under some simple assumptions, this model can be seen to probabilistically favour solutions with shorter inflationary epochs than necessary to solve the horizon problem [301–304]. Only development of our understanding of quantum phenomena in positive curvature spacetimes can hope to elucidate these types of considerations.

Eternal inflation The simplest models of inflation, in which the vacuum is controlled by the slowly rolling inflaton potential, generically lead to situations in which inflation does not end across the entire spacetime [305, 306]. Instead, when combined with quantum mechanical fluctuations the nucleation of bubbles in which inflation has ended cause an eventual state characterised by patches of hot, dense matter separated by areas with the false vacuum field configuration still undergoing inflation. This leads to questions over the scientific meaning of measurements of our universe, and the predictive power of these models [307, 308]. It is not impossible to develop models in which this "eternal inflation" process does not occur [309, 310], but as physicists we await either new experimental data, or the development of new understanding of cosmological spacetimes to distinguish them.

Importantly this is an area of physics in which high energy processes have consequences for our observable, IR, scale. The tensions discussed here evidence gaps in our theoretical understanding at large energy scales and small, and present an opportunity to probe new physical regimes, just as in the black hole context. An example of this type of development is the possibility of using cosmological physics to probe energy scales far out of reach of current collider technology [32]. The models and techniques included in this thesis are a valuable toolkit for the continuation of these studies.

These areas of tension suggest our theory of cosmology is incomplete and, at times, contradictory. In this thesis we take the initial steps toward building a model of the microscopic degrees of freedom of an expanding spacetime and comprehending the processes by which these are reorganised into the emergent classical spacetime we observe. We see the existence of a relationship between a bulk theory in de Sitter and a quantum mechanical model as an opportunity to include expanding spacetime in the construction of a theory of emergent quantum gravity.

Appendices

A. Conventions

Throughout we obey the Einstein summation convention with repeated indices, unless otherwise specified. We use the mostly-plus convention, and indices ranges are given dependent on their alphabet:

- Greek indices are used as usual, $\mu, \nu, \rho, ... = 0, 1, \ldots d$.
- Latin indices from the start of the alphabet designate the d-spatial coordinates, $a, b, c, \ldots = 1, 2, \ldots d$.
- Latin indices from the middle of the alphabet, $i, j, k, ...$, range from 1 to $d + 1$.
- We use upper-case latin indices for the embedding coordinates in $\mathbb{R}^{1,d}$, i.e. $A, B, C \ldots = 0, 1, \ldots d + 1$.
- We also occasionally make use of light-cone variables and metric, where $A, B, C \ldots = +, -, 1, \ldots d$. This will be specified when appropriate. In our parametrisation, $\eta_{+,-} = -\frac{1}{2}$, i.e. $X^{\pm} = X^0 \pm X^{d+1}$.
- Antisymmetrisation and symmetrisation of indices are written using respectively square and round brackets, and have weight 1, i.e. $T_{(ab)} = \frac{1}{2}\left(T_{ab} + T_{ba}\right)$.

We note that in the above d=1 for most of chapter 3.

Throughout the de Sitter length is designated ℓ and the cosmological constant is Λ. The de Sitter manifold in dimension $(d + 1)$ is therefore defined as

$$\{X \in \mathbb{R}^{d,1} | \eta_{AB} X^A X^B = \ell^2\} \tag{A.0.1}$$

Throughout the thesis we make use of the embedding formalism for fields of fixed integer spin, or spin $\frac{1}{2}$. The action of abstract symmetry generators $\widehat{L}, \widehat{P}, \widehat{K}, \ldots$ on fields Φ are realised through differential operators $L, P, K, \ldots$ defined as

$$[\widehat{P}, \Phi] = P\Phi\,. \tag{A.0.2}$$

It follows from the Jacobi-identity that the $\hat{Q}$ have commutation relations given by minus those of the operators they represent, i.e. $[P, K]\phi = -[[\widehat{P}, \widehat{K}], \phi]$. These considerations are important should one wish to reproduce the detail of the derivations of the spin part of generators acting on fields on the slice. For the entire paper we work in conventions such that the Casimir on the de Sitter slice has the following differential action on scalar fields

$$\mathcal{C}\phi(x) = -\ell^2 \Box_{dS}\phi(x) \ , \tag{A.0.3}$$

where $\Box_{dS}$ is the $(d + 1)$ dimensional Laplacian.

A.1 Conventions for chapter 2

Throughout chapter 2 , the indices considered are flat, i.e. they are contracted using Minkowski or Euclidean metric, depending on the range. Indices are never contracted using the curved-space metric $g_{\mu\nu}$. In the slice they are contracted with the metric of the tangent space, $\eta_{\mu\nu}$, ambient space indices are contracted with the ordinary metric of flat space η_{AB}. We also set the de Sitter length $\ell = 1$ in this chapter as our focus is on efficient methods for computation, rather than the development of physical intuition.

We occasionally make use of the following notation for ambient space contraction of the indices $A, B, \ldots$

$$X \cdot Y = \eta_{AB} X^A Y^B = X^A Y_A \tag{A.1.1}$$

The generators we use in this chapter are anti-Hermitian, i.e. for the action on scalars we have

$$[\widehat{L}_{AB}, \phi] = L_{AB}\phi = \mathcal{L}_{AB}\phi = (X_A \partial_B - X_B \partial_A)\,\phi \tag{A.1.2}$$

This is convenient in this case to remove multiple extra appearances of i on finding the spinor constraints. The quadratic Casimir is

$$\mathcal{C} = -\frac{1}{2} L_{AB} L^{AB} = D^2 + \frac{1}{2}(P \cdot K + K \cdot P) - \frac{1}{2} M_{ab} M^{ab} \ .$$

Its eigenvalue can be found by considering primary fields of a CFT_d. The scaling part gives the usual $\Delta(\Delta - d)$, while the spin part for tensors of spin-j gives $j(j - d + 2)$, and for Dirac spinors $\frac{d(d-1)}{8}$.

We find the language of [199] convenient to refer to the different spinor representations. In even dimensions, the fundamental spinors are the left-handed (LH, $+$) and right-handed (RH,$-$) Weyl spinors. A Dirac spinor is the direct sum of a left and right Weyl spinor. In odd dimension, the irreducible representation (which is of Dirac type), is called a Pauli spinor. An $SU(2)$ doublet of Pauli spinors, form a Cartan spinor. This is of course a reducible representation, and the odd-dimensional analogue of the Dirac representation. We prove the following relations: Dirac spinors in dS_{2n} are uplifted to constrained Cartan spinors of $Spin(1, 2n)$, while Pauli spinors in dS_{2n+1} are uplifted to constrained Dirac spinors of $Spin(1, 2n + 1)$. In all cases, we consider a set of gamma matrices obeying $\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}$. The spin-matrix is then given by $\Sigma_{\mu\nu} = \frac{1}{4}[\gamma_\mu, \gamma_\nu]$. In the embedding space, we use Γ_A instead. Conjugation properties follow from $\gamma_\mu^\dagger = \gamma_0\gamma_\mu\gamma_0$. Conjugate spinors are defined through $\overline{\psi} = \psi^\dagger i\gamma_0$, and similarly in the embedding. This choice matches that of Weinberg [209], such that $(\overline{\alpha}\beta)^\star = \overline{\beta}\alpha$ and $(\overline{\alpha}\gamma_\mu\beta)^\star = -\overline{\beta}\gamma_\mu\alpha$. In even dimensions, we write the chiral matrix with $\gamma_\star^2 = 1$, and we use chiral-γ matrices, or σ-matrices. This means we consider

$$\gamma_\mu = \begin{pmatrix} 0 & \sigma_\mu \\ \overline{\sigma}_\mu & 0 \end{pmatrix} , \tag{A.1.3}$$

$$\sigma_\mu = (\mathbb{1}, \sigma_a) = \sigma_\mu^\dagger \,, \tag{A.1.4}$$

$$\overline{\sigma}_\mu = (-\mathbb{1}, \sigma_a) = \overline{\sigma}_\mu^\dagger \,, \tag{A.1.5}$$

$$\sigma_{(\mu}\overline{\sigma}_{\nu)} = \eta_{\mu\nu} \,, \tag{A.1.6}$$

$$\gamma_\star = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} , \tag{A.1.7}$$

$$\mathbb{P}_\pm = \frac{\mathbb{1} \pm \gamma_\star}{2} \,. \tag{A.1.8}$$

The σ matrices also give rise to chiral-rotation matrix, $\sigma_{\mu\nu} = \frac{1}{2}\sigma_{[\mu}\overline{\sigma}_{\nu]}$ and $\overline{\sigma}_{\mu\nu} = \frac{1}{2}\overline{\sigma}_{[\mu}\sigma_{\nu]}$, which appear when considering Weyl spinors. Embedding spinors are generically named Ψ and written in block form

$$\Psi = \begin{pmatrix} \chi \\ \rho \end{pmatrix} , \tag{A.1.9}$$

$$\overline{\Psi} = \begin{pmatrix} \overline{\rho}\gamma_0 & -\overline{\chi}\gamma_0 \end{pmatrix} \,. \tag{A.1.10}$$

When considering their transformation law, one has to take into account a sign and ordering difference

$$[\widehat{L}_{AB}, \Psi(P)] = -\mathcal{L}_{AB}\Psi(P) - \Sigma_{AB}\Psi(P) \,, \tag{A.1.11}$$

$$[\widehat{L}_{AB}, \overline{\Psi}(P)] = -\mathcal{L}_{AB}\overline{\Psi}(P) + \overline{\Psi}(P)\Sigma_{AB} \,, \tag{A.1.12}$$

and similarly for the commutation relations on the dS slice. dS spinors are usually named ψ, and are related non-trivially to χ and ρ, as shown in the text. Chiral spinors are preferably encoded using a whole Dirac spinor with eigenvalue equation $\gamma_\star\psi_\pm = \pm\psi_\pm$.

A.2 Conventions for chapter 3 and chapter 4

In this chapter we only consider the de Sitter group in two dimensions $G = SL(2,\mathbb{R})$. The embedding formalism is still used to discuss the geometry of dS_2, however the generators we use in this section are explicitly Hermitian

$$[\widehat{L}_{AB}, \phi] = L_{AB}\phi = \mathcal{L}_{AB}\phi = -i\left(X_A\partial_B - X_B\partial_A\right)\phi \tag{A.2.1}$$

The quadratic Casimir is

$$\mathcal{C} = \frac{1}{2}L_{AB}L^{AB}$$

and so, inspite of the two definitions of L_{AB}, $\mathcal{C}$ is consistent across the whole thesis (A.0.3).

B. Representation theory of $SL(2,\mathbb{R})$

We review the unitary irreducible representations (UIRs) of $SL(2,\mathbb{R})$. For more detailed reviews of this subject we refer the reader to [101, 229]. We provide a short summary here for convenience and to fix our conventions, making the link to the global coordinate parameterisation explicit.

Global decomposition. $SL(2,\mathbb{R})$ is a non-compact connected simple real Lie group. As such, it does not permit non-trivial finite dimensional UIRs. We can derive the UIRs of the group $SL(2,\mathbb{R})$ by induction on $K = SO(2)$, the maximal compact subgroup. This approach is reviewed in [87, 99–101]. We consider the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$. The complexified generators defined in equation (3.1.5) obey the commutation relations

$$[L_0, L_\pm] = \mp L_\pm \ , \qquad [L_+, L_-] = 2L_0 \ . \tag{B.0.1}$$

L_0 generates K and is associated to the spatial translations in the global coordinate parameterisation (3.1.10). We derive UIRs by insisting on the following reality conditions, derived from the hermiticity of the de Sitter generators in (3.1.3)

$$L_0^\dagger = L_0 \ , \qquad L_\pm^\dagger = L_\mp \ . \tag{B.0.2}$$

The quadratic Casimir is given, as in the text, by

$$\mathcal{C} \equiv L_0^2 - \frac{1}{2}\left(L_- L_+ + L_+ L_-\right) \ . \tag{B.0.3}$$

We consider states which simultaneously diagonalise L_0 and $\mathcal{C}$,

$$L_0 \left|n, \Delta\right\rangle = -n \left|n, \Delta\right\rangle \ , \quad \mathcal{C} \left|n, \Delta\right\rangle = \Delta(\Delta - 1) \left|n, \Delta\right\rangle \ . \tag{B.0.4}$$

Including half integer spin fields leads one to consider even and odd UIRs of $SL(2,\mathbb{R})$. These are defined respectively by the action of the exponentiation of L_0

$$e^{2\pi i L_0} |n,\Delta\rangle = \pm |n,\Delta\rangle \ . \tag{B.0.5}$$

It is clear that $n \in \mathbb{Z}$ or $n \in \mathbb{Z} + \frac{1}{2}$ for even and odd representations respectively. We can immediately see that the reality of the Casimir eigenvalue implies one of either $\Delta = \frac{1}{2} + i\nu$, $\nu \in \mathbb{R}$ or $\Delta \in \mathbb{R}$ holds. We further seek to normalise the states such that $\langle n,\Delta|n,\Delta\rangle \geq 0$ and the action of the ladder operators satisfies

$$L_{\pm} |n,\Delta\rangle = -(n \pm \Delta) |n \pm 1,\Delta\rangle \ . \tag{B.0.6}$$

Unitarity demands $\langle n,\Delta| L_- |n+1,\Delta\rangle = \langle n+1,\Delta| L_+ |n,\Delta\rangle^*$ and $\langle n,\Delta|n,\Delta\rangle > 0$, (see [101]) which leads to the following condition

$$\frac{\langle n+1,\Delta|n+1,\Delta\rangle}{\langle n,\Delta|n,\Delta\rangle} = \frac{n+1-\Delta}{n+\Delta^*} \equiv \lambda_n \ , \qquad \Longrightarrow \qquad \lambda_n > 0 \,, \tag{B.0.7}$$

for all n. For $\Delta \in \mathbb{R}$, it will be useful to rewrite

$$\lambda_n = \frac{\left(n+\frac{1}{2}\right)^2 - \left(\Delta - \frac{1}{2}\right)^2}{(n+\Delta)^2} > 0 \ , \qquad \Delta \in \mathbb{R} \ . \tag{B.0.8}$$

Thus, even UIRs are permitted for the principal series when $\Delta = \frac{1}{2}(1+i\nu)$, as well as for the complementary series when $\Delta \in (0,1)$, and for the discrete series $\Delta \in \mathbb{Z}^+$. The same follows for odd UIRs in the principal and discrete series. However the positivity condition (B.0.8) for $n = -\frac{1}{2}$ in the complementary series can not be satisfied and thus there are no odd complementary series UIRs. To summarise, the UIRs of $SL(2,\mathbb{R})$ are given by:

- **Even and odd principal series**, $\pi_\nu^\pm$: We have $\Delta = \frac{1}{2}(1+i\nu)$ and $n \in \mathbb{Z}$ or $n \in \mathbb{Z}+\frac{1}{2}$ for the even, resp., odd representations. The states can be consistently normalised as follows:

$$\langle n,\Delta|m,\Delta\rangle_{\pi_\nu} = \delta_{n,m} \,. \tag{B.0.9}$$

- **Complementary series**, γ_Δ: In this case we are confined to the range $\Delta \in (0,1)$. The complementary series must be even, and thus $n \in \mathbb{Z}$. For this representation, we can take:

$$\langle n, \Delta | m, \Delta \rangle_{\gamma_\Delta} = \frac{\Gamma(n+1-\Delta)}{\Gamma(n+\Delta)} \delta_{n,m} \,. \tag{B.0.10}$$

- **Even and odd discrete series**, $D^{\pm}_{\Delta,\pm}$: For the even and odd cases, where $\Delta \in \mathbb{Z}_+$ or $\Delta \in \mathbb{N} + \frac{1}{2}$, respectively, we have a pair of UIRs for which $L_\pm \left| n = \mp\Delta, \Delta \right\rangle = 0$, corresponding to $D^{\pm}_{\Delta}$: the highest- and lowest-weight representations. In this case we must normalize the states as follows:

$$\langle n, \Delta | m, \Delta \rangle_{D^{\pm}_{\Delta}} = \frac{\Gamma(\mp n+1-\Delta)}{\Gamma(\mp n+\Delta)} \delta_{n,m} \,. \tag{B.0.11}$$

- **Trivial Representation**: $\Delta = 0$ and $n = 0$.

C. Discrete symmetries in dS_2

In dS_2 there are three discrete spacetime symmetries, parity (P), time reversal (T) and the antipodal transformation (A), as well as charge conjugation (C) which has a less clear interpretation in dS. The PTA are clearly different symmetries when viewed from the perspective of the embedding formalism and map to well defined coordinate transformations in global coordinates. In planar and static coordinates some are unavailable, as the embedding transformations move the point to alternative patches on the slice. Throughout we consider the embedding formalism for spinors as defined in [82]. The discrete symmetries were considered in [228], we review this work for clarity here. The dS_2 Dirac fermion is written in terms of an $SU(2)$ doublet of ambient space Dirac fermions under the constraint as in chapter 2

$$\Gamma_A X^A \Psi = \Psi \,. \tag{C.0.1}$$

where $\Gamma'_A = \mathit{\Gamma}_A \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\mathit{\Gamma}$ are the ordinary gamma matrices in the ambient space. We can define a matrix which changes the sign of the constraint, therefore swapping the chiral parts of the full fermion in the chiral basis, $U\left(\mathbf{I} \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) U^\dagger \equiv -i\Gamma_\star$. The Dirac equation which is consistent with this constraint and maps to the Dirac equation in the slice is

$$\left(i\Gamma_\star \not{\nabla} - m\right) \Psi = 0 \,, \tag{C.0.2}$$

where

$$\begin{aligned} \not{\nabla} &= \Gamma^A \left(G_{AB}\partial^A - \Sigma_{AB} X^B \frac{1}{X \cdot X} \right) \,, &\text{(C.0.3)} \\ &= \not{\partial} - \not{X} X \cdot \partial - \frac{d+1}{2} \not{X} \,. &\text{(C.0.4)} \end{aligned}$$

We later assume the invariance of this equation under the discrete symmetries to derive the appropriate matrix transformation for the Dirac spinor. The dS slice in the embedding $X^A X_A = 1$, has three coordinate patches we are interested in including global [1]

$$X^A = (X^0, X^i) = (\sinh\tau, \cosh\tau\sin(\theta), \cosh\tau\cos(\theta))\,, \tag{C.0.5}$$

planar

$$X^A = (X^0, X^i) = (\frac{1+x^2-\eta^2}{\eta}, \frac{x}{\eta}, \frac{1-x^2+\eta^2}{\eta})\,, \tag{C.0.6}$$

and static

$$X^A = (X^0, X^i) = (\sqrt{1-\xi^2}\sinh t, \xi, \sqrt{1-\xi^2}\cosh t)\,. \tag{C.0.7}$$

[1] Where we have swapped the spacelike coordinate parameterisation to ensure the origin is in the same place for all three coordinate systems.

C.1 Parity

The Parity transformation in global ($\theta \to 2\pi - \theta$), planar ($x \to -x$) and static ($\xi \to -\xi$) coordinate parameterisations are all achieved in the embedding by

$$(X^0, X^1, X^2) \to (X^0, -X^1, X^2)$$

If we transform the Dirac spinor in a general representations of the parity transformation

$$P^{-1}\Psi(X_P)P = S(P)\Psi(X)$$

We look for the precise representation by fixing the form of the Dirac equation and constraint, on the Dirac equation this implies

$$S(P)\left(S(P)^{-1}i\Gamma_\star \not{\nabla}_P S(P) - m\right)\Psi = 0 \tag{C.1.1}$$

For this equation to be invariant we require $S(P)^{-1}i\Gamma_\star\not{\nabla}_P S(P) = i\Gamma_\star\not{\nabla}$. Similarly on the constraint we have

$$S(P)\Psi(X) = \Gamma_A X_P^A S(P)\Psi(X)$$

Noting $-\Gamma_0\Gamma_2\Gamma_A X_P^A \Gamma_2\Gamma_0 = -\Gamma_A X^A$ with the minus included as part of the inverse, we can see that for the constraint not to swap sign under parity we add a the chiral matrix to give $S(P) = \Gamma_2\Gamma_0\Gamma_*$ this is consistent with the Dirac equation constraint, where we note a few properties of Γ_* which are not necessarily clear given the symplectic structure of the embedding setup.

$$\{\Gamma_*, \Gamma_A\} = 0 \tag{C.1.2}$$

$$\Gamma_*^{-1} = \Gamma_*. \tag{C.1.3}$$

We also note $S(P)^{-1} = -\Gamma_*\Gamma_0\Gamma_2$.

C.1.1 Parity on the states

The operator described as “Parity” in the literature acts on states $|\Delta, k, \pm\rangle$, where k is integer or half integer for parity even and odd cases respectively, is $e^{2\pi i L_0}$ in all representations, the SO(2) group element. It is trivial to show

$$e^{2\pi i L_0} |\Delta, k, \pm\rangle = e^{2\pi k i} |\Delta, k, \pm\rangle = \pm |\Delta, k, \pm\rangle \tag{C.1.4}$$

we can check the action is as expected for position eigenstates in the principal series representation

$$e^{2\pi i L_0} |\theta\rangle = e^{2\pi i L_0} \sum_k e^{-ik\theta} |\Delta, k, \pm\rangle = \sum_k e^{-ik\theta} e^{2\pi k i} |\Delta, k, \pm\rangle = \sum_k e^{-ik(\theta - 2\pi)} |\Delta, k, \pm\rangle = |\theta - 2\pi\rangle \tag{C.1.5}$$

This is inconsistent with the coordinate transformation in global coordinates for parity, $\theta \to 2\pi - \theta$, however it may be fixed by the assumption that the parity operator on these states is anti-unitary. It seems clear, and makes sense that we shouldn't identify the exponentiation of a generator with a discrete symmetry, so the above and below describe the action of a new operator which we define to have the action

$$P |\Delta, k, \pm\rangle = \pm |\Delta, -k, \pm\rangle \tag{C.1.6}$$

This operator exists for principal series states in the global patch, but its action is inconsistent in the discrete series.

C.2 Time Reversal

The embedding transformation is the inversion

$$(X^0, X^1, X^2) \to (-X^0, X^1, X^2).$$

This has a clear interpretation in global coordinates where the odd property of hyperbolic sine implies $\tau \to -\tau$ achieves this transformation. However, in planar coordinates there is no transformation in either of the coordinates which allows for this transformation, without also implying a change for the other two ambient space coordinates, instead the $\eta \to -\eta$ transformation implies the point moves out of the planar patch and performs the Antipodal transformation on the coordinates which we discuss below. In static coordinates, we can achieve the time reversal transformation in a similar way to the global coordinates via $t \to -t$. In the Minkowski case we also require that the time reversal operator is anti-unitary. As we do not necessarily require the boundedness of the Hamiltonian in this case we avoid this problem by including a choice of sign $\alpha = \pm 1$, for which the positive and negative cases imply unitarity and anti-unitarity. Following the analogous process to parity we require the transformation for time reversal on the spinor field

$$T^{-1}\Psi(X_T)T = S(T)\Psi(X).$$

Invariance of the equation of motion and constraint implies

$$S(T)^{-1}\Gamma_A X_T^A S(T) = \Gamma_A X_T^A$$

and

$$S(T)^{-1} i\Gamma_* \Gamma_A \nabla_T^A S(T) = \alpha i \Gamma_* \Gamma_A \nabla^A$$

this is achieved with $S(T) = \Gamma_0$, $\alpha = +1$.

C.3 Antipodal Transformation

The natural extension of Minkowski discrete symmetry to de Sitter includes the possibility of including another discrete transformation of the embedding space which appeared before in Planar coordinates, this implies the coordinate transform

$$(X^0, X^1, X^2) \to (-X^0, -X^1, -X^2)$$

This discrete transformation can clearly be partially reversed by subsequent P and T transformations when considered in the global coordinate patch, however there is a leftover change in the X^2 coordinate after application of PTA, which would imply it is independent of these, although potentially connected to T. In global coordinates the full coordinate change describing movement of a point to its antipode is achieved via $\tau \to -\tau$, $\theta \to \theta + \pi$ in global coordinates and the aforementioned $\eta \to -\eta$ in planar coordinates. Following the procedure above the transformation on the spinor

$$A^{-1}\Psi(X_A)A = S(A)\Psi(X).$$

is given by

$$S(A) = \Gamma_*.$$

Which potentially associates A with a change in chirality, as might be expected for the Parity transformation. The results in this section are summarised in the table in section 3.1.

D. STT Wightman function

The STT propagator in AdS has been calculated in [161]. Here we present a calculation of the Wightman function for symmetric traceless tensor fields in analogy. The case of symmetric traceless tensors in (A)dS, as well as the link to the euclidean sphere S^N and hyperbola H^N has been treated in [162, 165, 211, 217] among others. Here we perform the calculation explicitly in dS for reference and convenience, although the result follows in principle from the analytic continuation of the AdS Harmonic function [65]. The Wightman function between ambient space points X_1 and X_2 of a massive spin $s = J$ field in dS with polarisation vectors W_1 and W_2 respectively is dependent on the chordal distance $z = \frac{1}{2}(1 + X_1 \cdot X_2) = 1 - \frac{u}{4}$. We make use of two equivalent bases

$$
\begin{aligned}
\Pi_{J,\Delta}(X_1, W_1; X_2, W_2) &= \sum_{k=0}^{J} (W_1 \cdot W_2)^{J-k} ((W_1 \cdot X_2)(W_2 \cdot X_1))^k g_k(z) \\
&= \sum_{k=0}^{J} (W_1 \cdot W_2)^{J-k} ((W_1 \cdot \nabla_1)(W_2 \cdot \nabla_2))^k f_k(z).
\end{aligned}
$$

We can recover the first basis from the second via

$$
g_k(z) = \sum_{i=k}^{J} \left(\frac{1}{2}\right)^{i+k} \left(\frac{i!}{k!}\right)^2 \frac{1}{(i-k)!} \partial_z^{(i+k)} f_i(z)\,. \tag{D.0.1}
$$

The equation of motion provides the necessary differential equation which we seek to solve. For STT operators, the Laplacian is equal to the Casimir up to a constant shift. For this reason we solve for the Wightman function using the equation of motion according to [161],

$$
(\Box_{dS} - (\Delta(d-\Delta) + J))\Pi_{J,\Delta}(X_1, X_2) = 0\,. \tag{D.0.2}
$$

With the covariant derivative given as in (2.2.6). This equation is equivalent to the Casimir equation. To further simplify, we define $h_k(z) = \partial_z^k f_k(z)$. The equation of motion can be written recursively :

$$\Big((1-z)z\partial_z^2 + (d+1+2k)\left(\frac{1}{2}-z\right)\partial_z - \\ \Delta(d-\Delta) - 2k(k-J+1)\Big)h_k(z) = 4(J-k+1)h_{k-1}(z)\,, \\ h_{-1}(z) = 0\,. \tag{D.0.3}$$

The $k=0$ equation is simply the equation of motion for the scalar field and so is easily solved and normalised as described in section 2.2.3. The following equations can be solved recursively from the two previous ones as in EAdS,

$$h_k(z) = c_{J,k}\Big((d+2J-2k-1)\Big((2-d-J)h_{k-1}(z) + \left(\frac{1}{2}-z\right)h'_{k-1}(z)\Big) \\ +2(J-k+2)h_{k-2}(z)\Big)\,, \tag{D.0.4}$$

Where

$$c_{J,k} = \frac{-2(J-k+1)}{k(d+2J-k-2)(\Delta+J-k-1)(d-\Delta+J-k-1)}\,.$$

The recursion relations in AdS follow by changing variable $z \to -\sigma$ and the sign of the term multiplying h_{h-1} in (D.0.4).

E. Spinors in global coordinates

In the main text we showed that a generic Dirac spinor Ψ of $\mathbb{R}^{1,d+1}$ with eigenvalue equation $\Gamma^A X_A \Psi = \Psi$ encodes a Dirac spinor of the dS slice. Our proof uses flat-slicing, planar coordinates, but holds in general by the homogeneity of dS_{d+1}. It is convenient to have coordinate expression for some practical computations, and the one used previously does not cover the whole space, nor do they make the analytic continuation to S_{d+1} clear. This is why we devote this appendix to the analogous construction for global coordinates

$$X^A = (X^0, X^i) = (\sinh\tau, w^i \cosh\tau)\,, \tag{E.0.1}$$

with angular variables $w^i w_i = 1$. These have the benefit of both covering the whole space and making the analytic continuation $\tau = it$ straightforward. These coordinates treat the $(d+1)$-th component indistinctly from the others, and so the previous splitting of the $SO(1, d+1)$ Lorentz algebra into the conformal algebra is ill-suited to analyse the transformation law of fields induced on the slice. It is more natural to use a different decomposition of the group by identifying

$$\widehat{K}_i = \widehat{L}_{0i}\,, \tag{E.0.2}$$

$$\widehat{M}_{ij} = \widehat{L}_{ij}\,. \tag{E.0.3}$$

Note that this K_i is unrelated to the special conformal transformation generator of the conformal algebra. This splitting simply isolates the boosts and the rotations, and is precisely the one used to study the representations of $SO(1,3)$ in [209]. The commutation relations follow directly

$$[\widehat{M}_{ij}, \widehat{M}^{kl}] = -4\delta_{[i}{}^{[k}\widehat{M}_{j]}{}^{l]}\,, \tag{E.0.4}$$

$$[\widehat{M}_{ij}, \widehat{K}^{k}] = -2\delta_{[i}{}^{k}\widehat{K}_{j]}\,, \tag{E.0.5}$$

$$[\widehat{K}_i, \widehat{K}_j] = \widehat{M}_{ij}\,. \tag{E.0.6}$$

From the explicit form of the coordinate slices, we can find that acting on a scalar field $\phi(X(t,w))$ in the embedding, they act as the Killing vectors

$$M_{ij}\phi(X(\tau,w)) = w_j\frac{\partial}{\partial w^i}\phi - w_i\frac{\partial}{\partial w^j}\phi\,, \tag{E.0.7}$$

$$K_i\phi(X(\tau,w)) = w_i\frac{\partial}{\partial\tau}\phi + \tanh(\tau)\underbrace{(\delta_{ij} - w_iw_j)\frac{\partial}{\partial w_j}}_{=h_{ij}\partial^j=\nabla_i}\phi\,. \tag{E.0.8}$$

We see that the M_{ij} implement rotations while the boosts, K_i, contain both a time translation as well as a covariant derivative on the sphere. The covariant derivative is unsurprising as it is necessary for the generators to preserve the constraint $w^2 = 1$. Acting on a field in the slice $\psi(t,w)$ with definite spin representation specified by spin matrices Σ_{AB}, the orbital part of the generator for the scalar has to be supplemented by the spin part

$$M_{ij}\psi(\tau,w) = -\left(w_i\frac{\partial}{\partial w^j} - w_j\frac{\partial}{\partial w^i} + \Sigma_{ij}\right)\psi\,, \tag{E.0.9}$$

$$K_i,\psi(\tau,w) = w_i\frac{\partial}{\partial\tau}\psi + \tanh(\tau)\nabla_i\psi + S_i(\tau,w,\Sigma)\psi\,. \tag{E.0.10}$$

The vector $S_i(\tau,w,\Sigma)$ is non-trivial to compute, although it is entirely fixed by the Jacobi identity $[[A,B],\psi] = [A,[B,\psi]] - [B,[A,\psi]]$. We proceed in two steps. First, note that the Jacobi identity for $[M,K]$ is solved by the general ansatz $S_a = \Sigma_{ij}w^jg_1(\tau) + \Sigma_{i0}g_2(\tau)$. Secondly, one can use this ansatz in the Jacobi identity for $[K,K]$ and decompose the resulting equation in terms of independent structures multiplied by equations involving $g_1(\tau)$ and $g_2(\tau)$, which must all vanish. The output of this is that the action of the symmetry generators on fields with spin do generate a representation of the de Sitter algebra provided $g_1(\tau) = -\frac{1}{\tanh(\tau)}$ and $g_2(\tau) = -\frac{1}{\sinh(\tau)}$. From this analysis, we gather that on the slice, a generic spinning field transforms under the action of the generator according to

$$M_{ij}\psi(\tau,w) = -\left(w_i\frac{\partial}{\partial w^j} - w_j\frac{\partial}{\partial w^i} + \Sigma_{ij}\right)\psi\,, \tag{E.0.11}$$

$$K_i\psi(\tau,w) = \left(w_i\frac{\partial}{\partial\tau} + \tanh(\tau)\nabla_i - \frac{\cosh(\tau)\Sigma_{ij}\omega^j + \Sigma_{i0}}{\sinh(\tau)}\right)\psi\,. \tag{E.0.12}$$

This can, as previously, be compared with the transformation induced from the embedding on a generic spinor field Ψ as in (2.3.16). As argued in the main text, one can consider the case $d+1=2k+1$, and use the resulting formalism in any dimension, both odd and even. A parameterisation of the gamma matrices can be chosen similarly as before

$$\Gamma_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes \mathbb{1}\,, \tag{E.0.13}$$

$$\Gamma_i = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \gamma_0\gamma_i\,. \tag{E.0.14}$$

Using this convention, one can check that rotations behave as expected, however boosts require more care. Motivated by the planar case, we consider the commutation relation not for χ itself but for $f(\tau)\chi$, and leave $f(\tau)$ to be determined :

$$K_i\left(f(\tau)\chi\right) = \Big(w_i\partial_\tau + \tanh(\tau)\nabla_i\Big)(f\chi) - w_i\dot{f}(\tau)\chi + \frac{1}{2}\gamma_0\gamma_i f\rho \tag{E.0.15}$$

$$= \Big(w_i\partial_\tau + \tanh(\tau)\nabla_i\Big)(f\chi) - \frac{\gamma_i\not{w} - w_i}{2\tanh(\tau)}f(\tau)\chi - \frac{\gamma_i\gamma_0}{2\sinh(\tau)}f\chi\,. \tag{E.0.16}$$

Where the first line is derived from the explicit transformation law of the embedding space spinor, the second line is required for $f(\tau)\chi$ to transform as a dS_{d+1} spinor. This equality is solved provided

$$2\tanh(\tau)\dot{f}(\tau) + f(\tau) = 0 \Rightarrow f(\tau) = \frac{1}{\sqrt{\sinh(\tau)}}\,, \tag{E.0.17}$$

$$\rho = \frac{1-\cosh(\tau)\gamma_0\not{w}}{\sinh(\tau)}\chi\,. \tag{E.0.18}$$

This, as expected, corresponds to an embedding spinor satisfying $\Gamma_A X^A \Psi = \Psi$. Proceeding as before, one can define a polarisation vector such that

$$\overline{S}\Psi = \frac{1}{\sqrt{\sinh(\tau)}}\begin{pmatrix} \overline{s} & 0 \end{pmatrix}\left(\Gamma_A X^A + 1\right)\Psi = \overline{s}\psi\,. \tag{E.0.19}$$

The kinematics for global coordinates is then entirely fixed by using polarisations

$$X^A \Gamma_A S = -S \qquad \Longrightarrow \qquad S = \begin{pmatrix} -\gamma_0 \sinh(\tau) \\ \gamma_0 + \cosh(\tau) \not{\psi} \end{pmatrix} \frac{s}{\sqrt{\sinh(\tau)}} \,, \tag{E.0.20}$$

$$\overline{S} X^A \Gamma_A = \overline{S} \qquad \Longrightarrow \qquad \overline{S} = \frac{\overline{s}}{\sqrt{\sinh(\tau)}} \begin{pmatrix} \cosh(\tau)\gamma_0 \not{\psi} + 1 & \sinh(\tau) \end{pmatrix} \,. \tag{E.0.21}$$

One can use these expressions to find the different spinorial structures appearing in the correlation function. It is also straightforward to check that the boundary limit of these polarisations is as stated previously, and once rescaled we obtain a smooth limit to the polarisation spinor of a primary spinor on the lightcone.

F. Two conformal particles on the circle

In this appendix, we consider the two-particle Hilbert space content, for particles in the principal series UIR. For a related analysis see section 3.2 of [92]. We will be using the conventions of [91]. We start with the Hermitian generators acting on wavefunctions of a single degree of freedom $\theta \in [0, 2\pi)$

$$H_\Delta^\theta \equiv \quad 2i \cos\frac{\theta}{2}\left[\Delta \sin\frac{\theta}{2} - \cos\frac{\theta}{2}\partial_\theta\right] \ , \tag{F.0.1}$$

$$K_\Delta^\theta \equiv -2i \sin\frac{\theta}{2}\left[\Delta \cos\frac{\theta}{2} + \sin\frac{\theta}{2}\partial_\theta\right] \ , \tag{F.0.2}$$

$$D_\Delta^\theta \equiv -i\left[\Delta \cos\theta + \sin\theta\, \partial_\theta\right] \ . \tag{F.0.3}$$

These hermitian generators are related to the complexified ones of (3.1.5) by

$$H_\Delta^\theta = L_0^{\Delta,\theta} - \frac{1}{2}\left(L_+^{\Delta,\theta} + L_-^{\Delta,\theta}\right) \ , \quad K_\Delta^\theta = L_0^{\Delta,\theta} + \frac{1}{2}\left(L_+^{\Delta,\theta} + L_-^{\Delta,\theta}\right) \ , \quad D_\Delta^\theta = \frac{-i}{2}\left(L_+^{\Delta,\theta} - L_-^{\Delta,\theta}\right) \ . \tag{F.0.4}$$

The operators (F.0.1-F.0.3) obey the algebra

$$\left[D_\Delta^\theta, H_\Delta^\theta\right] = iH_\Delta^\theta \ , \qquad \left[D_\Delta^\theta, K_\Delta^\theta\right] = -iK_\Delta^\theta \ , \qquad \left[K_\Delta^\theta, H_\Delta^\theta\right] = 2iD_\Delta^\theta \ . \tag{F.0.5}$$

These square to the trivial quadratic Casimir

$$\mathcal{C}^\theta = \frac{1}{2}(H_\Delta^\theta K_\Delta^\theta + K_\Delta^\theta H_\Delta^\theta) - \left(D_\Delta^\theta\right)^2 = \Delta(\Delta - 1) \tag{F.0.6}$$

If we pick the standard inner product on this Hilbert space:

$$(f, g) = \int_0^{2\pi} d\theta\, f^*(\theta)\, g(\theta) \ , \tag{F.0.7}$$

then the operators (F.0.1-F.0.3) are self-adjoint with respect to this inner product if and only if $\Delta = \frac{1}{2}(1 + i\nu)$ with $\nu \in \mathbb{R}$, also known as the principal series.

Single particle Hilbert space

To build the single particle Hilbert space, we construct the compact operator $L_0^{\Delta,\theta}$ and raising/lowering operators $L_\pm^{\Delta,\theta}$

$$L_0^{\Delta,\theta} = \frac{1}{2}\left(H_\Delta^\theta + K_\Delta^\theta\right) = -i\partial_\theta \ , \qquad L_\pm^{\Delta,\theta} = \frac{1}{2}\left(H_\Delta^\theta - K_\Delta^\theta\right) \mp iD_\Delta^\theta = e^{\mp i\theta}\left(\mp\Delta - i\partial_\theta\right) \ , \tag{F.0.8}$$

The Hilbert space is spanned by states of definite $\mathcal{C}^\theta$ and $L_0^{\Delta,\theta}$. These are states $\psi_n(\theta)$ satisfying:

$$L_0^{\Delta,\theta}\,\psi_n(\theta) = -n\,\psi_n(\theta) \ , \qquad \mathcal{C}^\theta\,\psi_n(\theta) = \Delta(\Delta-1)\,\psi_n(\theta) \tag{F.0.9}$$

with $n \in \mathbb{Z}$. These imply the action of the raising and lowering operators will be:

$$L_\pm^{\Delta,\theta}\psi_n(\theta) = -(n \pm \Delta)\psi_{n\pm 1}(\theta) \tag{F.0.10}$$

Since the Casimir is trivial, the wavefunctions are easy to compute

$$\psi_n(\theta) = \frac{1}{\sqrt{2\pi}}e^{-in\theta} \tag{F.0.11}$$

and are orthonormal with respect to the standard inner product

$$(\psi_k, \psi_n) = \int_0^{2\pi} d\theta\,\psi_k^*(\theta)\psi_n(\theta) = \delta_{kn} \ . \tag{F.0.12}$$

Two-particle Hilbert space

Let us now construct the two-particle Hilbert space, built out of the tensor product of two single-particle Hilbert spaces on the conformal circle. We would like to thus consider the diagonal $SL(2,\mathbb{R})$ algebra constructed as

$$H \equiv H^{\theta_1}_{\Delta_1} + H^{\theta_2}_{\Delta_2} \ , \qquad K \equiv H^{\theta_1}_{\Delta_1} + H^{\theta_2}_{\Delta_2} \ , \qquad D \equiv D^{\theta_1}_{\Delta_1} + D^{\theta_2}_{\Delta_2} \tag{F.0.13}$$

which obey the same algebra as before. Now, the quadratic Casimir operator

$$\mathcal{C} = \frac{1}{2}(HK + KH) - D^2 \tag{F.0.14}$$

is nontrivial and given explicitly by

$$\begin{aligned}\mathcal{C} = -4\sin^2\left(\frac{\theta_1-\theta_2}{2}\right)\partial_{\theta_1}\partial_{\theta_2} + 2\sin\left(\theta_1-\theta_2\right)\left[\Delta_2\partial_{\theta_1} - \Delta_1\partial_{\theta_2}\right] \\ + \left[\Delta_1(\Delta_1-1) + \Delta_2(\Delta_2-1) + 2\Delta_1\Delta_2\cos(\theta_1-\theta_2)\right] \ .\end{aligned} \tag{F.0.15}$$

To build the two-particle Hilbert space, we proceed exactly as before. First we construct the compact L_0 generator, as well as the raising/lowering operators $L_\pm$, defined as

$$L_0 = \frac{1}{2}\left(H + K\right) = -i\partial_{\theta_1} - i\partial_{\theta_2} \ , \qquad L_\pm = \frac{1}{2}\left(H - K\right) \mp iD \ . \tag{F.0.16}$$

The two-particle Hilbert space is spanned by wavefunctions that satisfy

$$L_0\,\psi^{\Delta}_n(\theta_1,\theta_2) = -n\,\psi^{\Delta}_n(\theta_1,\theta_2) \ , \qquad \mathcal{C}\,\psi^{\Delta}_n(\theta_1,\theta_2) = \Delta(\Delta-1)\,\psi^{\Delta}_n(\theta_1,\theta_2) \ . \tag{F.0.17}$$

What is interesting in the two-particle case, is that, while Δ_1 and Δ_2 are required to take values in the principal series by Hermiticity, the eigenvalue Δ is allowed to take values in any of the $SL(2,\mathbb{R})$ representations. It is possible to solve (F.0.17) explicitly. The solution is:

$$\psi_n^\Delta(\theta_1,\theta_2) = \frac{e^{-in\left(\frac{\theta_1+\theta_2}{2}\right)+i\Delta\left(\frac{\theta_1-\theta_2}{2}\right)}}{\left[\sin^2\left(\frac{\theta_1-\theta_2}{2}\right)\right]^{\frac{\Delta_1+\Delta_2-\Delta}{2}}}\times$$
$$\Bigg[\; c_1 e^{+i(n+\Delta_1-\Delta_2)\left(\frac{\theta_1-\theta_2}{2}\right)}\,{}_2F_1\left(\Delta+n,\, \Delta+\Delta_1-\Delta_2,\, 1+n+\Delta_1-\Delta_2,\, e^{i(\theta_1-\theta_2)}\right)$$
$$+\, c_2 e^{-i(n+\Delta_1-\Delta_2)\left(\frac{\theta_1-\theta_2}{2}\right)}\,{}_2F_1\left(\Delta-n,\, \Delta-\Delta_1+\Delta_2,\, 1-n-\Delta_1+\Delta_2,\, e^{i(\theta_1-\theta_2)}\right)\Bigg] \,. \tag{F.0.18}$$

The choices of c_1 and c_2 are predicated by the normaliseability of these wavefunctions using the inner product

$$(\psi_n^\Delta, \psi_m^{\Delta'}) = \int_0^{2\pi}\int_0^{2\pi} d\theta_1 d\theta_2\, (\psi_n^\Delta(\theta_1,\theta_2))^* \psi_m^{\Delta'}(\theta_1,\theta_2) = c_n\, \delta_{nm}\delta_{\Delta\Delta'} \,. \tag{F.0.19}$$

where $\delta_{\Delta\Delta'}$ is a stand-in for the appropriate Kronecker or Dirac delta function, depending upon which representation we are dealing with, and c_n is the appropriate coefficient that ensures that the generators act faithfully on the representation, see section 2.1. To actually check normalizability, it is easier to change coordinates to

$$x \equiv \frac{\theta_1+\theta_2}{2} \qquad\qquad y \equiv \theta_1 - \theta_2 \,, \tag{F.0.20}$$

and the integration measure becomes

$$\int_0^{2\pi}\int_0^{2\pi} d\theta_1 d\theta_2 \to \int_0^{2\pi} dx \int_{-x}^{x} dy \,. \tag{F.0.21}$$

F.1 Discrete highest weight $\Delta \in D^+_\Delta$

To check that the D^+_Δ UIR appears in the tensor product Hilbert space of two Principal Series quantum mechanical degrees of freedom, we simply need to check that the highest weight wavefunction $\psi^\Delta_{n=-\Delta}(\theta_1, \theta_2)$ is normalizable. This wavefunction satisfies

$$L_0 \psi^\Delta_{n=-\Delta}(\theta_1,\theta_2) = \Delta\, \psi^\Delta_{n=-\Delta}(\theta_1,\theta_2)\ , \qquad L_+ \psi^\Delta_{n=-\Delta}(\theta_1,\theta_2) = 0\ , \tag{F.1.1}$$

for $\Delta = 1+t$ and $t = 0, 1, 2, \ldots$. The rest of the highest weight module can be generated by acting successively with L_-. It is easy to find the wavefunction satisfying (F.1.1):

$$\psi^{1+t}_{-(1+t)}(\theta_1,\theta_2) = \frac{1}{N(t,\mu,\nu)}\, e^{i(1+t)\frac{\theta_1+\theta_2}{2} + \frac{\mu-\nu}{4}(\theta_1-\theta_2)} \left[\sin\left(\frac{\theta_1-\theta_2}{2}\right)\right]^{t-i\left(\frac{\mu-\nu}{2}\right)}\ , \tag{F.1.2}$$

where we have taken $\Delta_1 \equiv \frac{1}{2}(1+i\nu)$ and $\Delta_2 \equiv \frac{1}{2}(1+i\mu)$, and $N(t,\mu,\nu)$ is an overall factor that ensures that this wavefunction is properly normalized (see (B.0.11)). Normalizability of the wavefunction requires $t \geq 0$ and single-valuedness of the center-of-mass wavefunction fixes t to be an integer. It is straightforward to verify that these wavefunctions are square-integrable with respect to the inner product (F.0.19), and a simple calculation yields

$$N(t,\mu,\nu) = 2^{1-t}\sinh\left[\frac{\pi}{2}(\mu-\nu)\right]\sqrt{(2t+1)!\sum_{s=0}^{2t}\frac{(-1)^{1+s-t}\left[(s-t)^2-\left(\frac{\mu-\nu}{2}\right)^2\right]}{\left[(s-t)^2+\left(\frac{\mu-\nu}{2}\right)^2\right]^2}\binom{2t}{s}}\ . \tag{F.1.3}$$

With this normalization, we have verified that the action of L_- on this state follows (F.0.10). Thus we have shown that, in this quantum mechanics of two decoupled principal series degrees of freedom, the tensor product Hilbert space contains every possible discrete highest weight module. Let us now show that this is also true for the discrete lowest weight modules.

F.2 Discrete lowest weight $\Delta \in D^-_\Delta$

This exercise is exactly the same as before. Now we search for a wavefunction that satisfies

$$L_0 \psi^\Delta_{n=\Delta}(\theta_1, \theta_2) = -\Delta\, \psi^\Delta_{n=\Delta}(\theta_1, \theta_2)\ , \qquad L_- \psi^\Delta_{n=\Delta}(\theta_1, \theta_2) = 0\ , \tag{F.2.1}$$

for $\Delta = 1 + t$ and $t = 0, 1, 2, \ldots$. The remainder of the lowest weight module can be generated by acting successively with L_+. It is again easy to find the wavefunction satisfying (F.2.1):

$$\psi^{1+t}_{1+t}(\theta_1, \theta_2) = \frac{1}{N(t, \mu, \nu)}\, e^{-i(1+t)\frac{\theta_1+\theta_2}{2} - \frac{\mu-\nu}{4}(\theta_1-\theta_2)} \left[\sin\left(\frac{\theta_1 - \theta_2}{2}\right)\right]^{t-i\left(\frac{\mu-\nu}{2}\right)}\ , \tag{F.2.2}$$

where we have again taken $\Delta_1 \equiv \frac{1}{2}(1 + i\nu)$ and $\Delta_2 \equiv \frac{1}{2}(1 + i\mu)$, and $N(t, \mu, \nu)$ is given in (F.1.3) and ensures appropriate normalization according to (B.0.11).

F.3 Principal and complementary series

For normalizability of the Principal or complementary series, we will re-solve (F.0.17), this time for $n = 0$, specifically. The general solution can be written as

$$\psi_0^{\Delta} = \frac{1}{\left[\sin^2\left(\frac{\theta_1-\theta_2}{2}\right)\right]^{\frac{\Delta_1+\Delta_2-\frac{1}{2}}{2}}} \left\{ b_1\, P^{\Delta-\frac{1}{2}}_{\Delta_1-\Delta_2-\frac{1}{2}} \left[\cos\left(\frac{\theta_1-\theta_2}{2}\right)\right] + b_2\, Q^{\Delta-\frac{1}{2}}_{\Delta_1-\Delta_2-\frac{1}{2}} \left[\cos\left(\frac{\theta_1-\theta_2}{2}\right)\right] \right\} , \tag{F.3.1}$$

where $P^{\nu}_{\mu}(x)$ and $Q^{\nu}_{\mu}(x)$ are associated Legendre functions. It suffices to determine if ψ_0^{Δ} can be made normalizable for some choice of $b_{1,2}$. Specifically, we need, for the complementary series:

$$\int_0^{2\pi}\int_0^{2\pi} d\theta_1 d\theta_2\, (\psi_0^{\Delta}(\theta_1,\theta_2))^* \psi_0^{\Delta'}(\theta_1,\theta_2) = \delta(\Delta-\Delta') \ . \tag{F.3.2}$$

However, for $\Delta, \Delta' \in [0,1]$, it is easy to verify that the above wavefunctions are not oscillatory over the domain of $\theta_{1,2} \in [0, 2\pi]$, making it impossible for the wavefunctions to be plane-wave normalizable. This precludes the complementary series from appearing in the tensor product of two principal series Hilbert spaces.

For the principal series, we need:

$$\int_0^{2\pi}\int_0^{2\pi} d\theta_1 d\theta_2 \left(\psi_0^{\frac{1}{2}+i\gamma}(\theta_1,\theta_2)\right)^* \psi_0^{\frac{1}{2}+i\gamma'}(\theta_1,\theta_2) = \delta(\gamma-\gamma') \ . \tag{F.3.3}$$

For these parameters, the above wavefunctions are appropriately oscillatory, and if we were sufficiently patient, we could use the results of [311] to find which choice of $b_{1,2}$ gives the desired normalization.

F.4 Tensor products from Harish-Chandra characters

One way to demonstrate the presence of discrete series representations in the decomposition of the tensor product of principle series representations is a simple calculation using the Harish-Chandra characters. Starting from H^{θ}_{Δ} in (F.0.1), the principle series character for the representation $\Delta = \frac{1}{2}(1 + i\nu)$ can be derived as follows (see section in A.2 of [109] or section 3 of [91]):

$$\chi_{\pi_\nu}(t) \equiv \int d\theta \, \langle\theta| \, e^{-itH^{\theta}_{\Delta}} \, |\theta\rangle = \frac{e^{-\frac{1}{2}(1+i\nu)t} + e^{-\frac{1}{2}(1-i\nu)t}}{|1 - e^{-t}|} = \left|\operatorname{csch}\frac{t}{2}\right| \cos\frac{\nu t}{2} \, . \tag{F.4.1}$$

On the other hand, for $\Delta \in \mathbb{Z}_+$ the discrete series Harish-Chandra character is (see equation (5.18) of [101]):

$$\chi_{D^+_\Delta \oplus D^-_\Delta}(t) = \frac{2e^{-\Delta t}}{1 - e^{-t}} \, , \tag{F.4.2}$$

where the factor of 2 comes from the fact that D^+_Δ and D^-_Δ have the same character.

We will now show that the discrete series appears in the tensor product Hilbert space of two principal series representations. To do this, let us multiply the Harish-Chandra characters of two different principal series representations.

$$\chi_{\pi_\nu}(t) \times \chi_{\pi_\mu}(t) = \operatorname{csch}^2\frac{t}{2} \cos\frac{\nu t}{2} \cos\frac{\mu t}{2} \, , \tag{F.4.3}$$

$$= \int_{-\infty}^{\infty} d\lambda \, \rho(\lambda) \, \chi_{\pi_\lambda}(t) \, . \tag{F.4.4}$$

We can formally recover the density $\rho(\lambda)$ by dividing (F.4.3) by $\left|\operatorname{csch}\frac{t}{2}\right|$ and performing inverse cosine transform (see [92] for a similar discussion). The result is

$$\rho(\lambda) = -\frac{1}{8\pi}\left\{8\gamma + \sum_{\sigma_1,\sigma_2,\sigma_3=0}^{1} \psi\left(\frac{1}{2} + \frac{i}{2}\left[(-1)^{\sigma_1}\nu + (-1)^{\sigma_2}\mu + (-1)^{\sigma_3}\lambda\right)\right]\right)\right\} \, . \tag{F.4.5}$$

Here $\psi(x) \equiv \frac{\Gamma'(x)}{\Gamma(x)}$ is the digamma function, and γ The Euler-Mascheroni constant. The digamma function $\psi(x)$ has poles whenever x is a non-positive integer, with residue -1. Note, given the form or $\chi_{\pi_\lambda}(t)$ in (F.4.1), that $\rho(\lambda)$ and $\rho(\lambda) + c$ will give the same result when integrated against a character. Thus $\rho(\lambda)$ is not a well-defined concept in its own right, but makes sense when integrated against a certain class of distributions.

Let us consider the case $\mu = \nu$, where the appearance of the discrete series manifests itself most clearly since a piece of the density is independent of ν. In this case we can write

$$\rho_{\mu=\nu}(\lambda) \equiv \rho^{\text{indep.}}(\lambda) + \rho^{\text{dep.}}(\lambda) \tag{F.4.6}$$

with

$$\rho^{\text{indep.}}(\lambda) = -\frac{1}{4\pi}\left\{\psi\left(\frac{1}{2}(1+i\lambda)\right) + \psi\left(\frac{1}{2}(1-i\lambda)\right)\right\} \tag{F.4.7}$$

and

$$\rho^{\text{dep.}}(\lambda) = -\frac{1}{8\pi}\left\{8\gamma + \sum_{\sigma_1,\sigma_2=0}^{1}\psi\left(\frac{1}{2} + \frac{i}{2}\left[(-1)^{\sigma_1}2\nu + (-1)^{\sigma_2}\lambda\right]\right)\right\} \ . \tag{F.4.8}$$

Now we proceed to evaluate

$$\int_{-\infty}^{\infty} d\lambda\, \rho^{\text{indep.}}(\lambda)\, \chi_{\pi_\lambda}(t) \ . \tag{F.4.9}$$

The functions $\psi\left(\frac{1}{2}(1 \pm i\lambda)\right)$ have poles for $\lambda = \pm 2i\left(n + \frac{1}{2}\right)$, respectively, with $n = 0, 1, 2, \ldots$. Proceeding carefully, for terms multiplying $e^{-\frac{t}{2}(1+i\lambda)}$ we must close the contour in the lower-half λ-plane, and for terms multiplying $e^{-\frac{t}{2}(1-i\lambda)}$ we must close the contour in the upper-half λ-plane. Carefully noting the orientation of each of the contours, we obtain

$$\int_{-\infty}^{\infty} d\lambda\, \rho^{\text{indep.}}(\lambda)\, \chi_{\pi_\lambda}(t) = 2\sum_{n=0}^{\infty}\frac{e^{-(1+n)t}}{1-e^{-t}} = \sum_{\Delta=1}^{\infty}\chi_{D^+_\Delta \oplus D^-_\Delta}(t) \ . \tag{F.4.10}$$

Thus

$$\chi^2_{\pi_\nu}(t) = \sum_{\Delta=1}^{\infty}\chi_{D^+_\Delta \oplus D^-_\Delta}(t) + \int_{-\infty}^{\infty} d\lambda\, \rho^{\text{dep.}}(\lambda)\, \chi_{\pi_\lambda}(t) \ , \tag{F.4.11}$$

and we clearly see the discrete series Hilbert space emerging from an analysis of the character of the tensor product of two principal series representations. This nicely complements the analysis in appendices F.1 and F.2.

G. Numerical analysis of Majorana $q = 2$ SYK

The general q SYK model has specific thermodynamic properties arising because of the presence of the $h = 2$ mode and resultant domination of the low energy dynamics by the Schwarzian contribution as described in [254, 260]. In this appendix we use publicly available code [289] to repeat the analysis of this work in the real $q = 2$ case. We work with Majorana fermions in this appendix as the thermodynamic properties in the limits discussed in the text are the same as the complex model and as the Majorana case enjoys a simpler implementation. The Hamiltonian of the model is

$$H_{SYK} = i \sum_{i \leq j} J_{ij} \psi_i \psi_j \ . \tag{G.0.1}$$

J_{ij} is once again drawn from a real Gaussian distribution with variance $\langle J_{ij}^2 \rangle = J^2/N$. We directly calculated the spectrum of energy eigenvalues of this Hamiltonian for $J = 1$ and even N values between $N = 10$ and $N = 30$ for $2^{\frac{36-N}{2}}$ realisations of the model. As the convergence of the density of states is independent of the number of realisations, just as in the generic q case there is a "self averaging" property of the model which allows the scaling of the sample size with the size of the Hilbert space $2^{\frac{N}{2}}$.

G.1 Density of states

At finite N an individual realisation of the $q = 2$ model is a model of free fermions with random masses. It is possible to numerically evaluate the partition function of a large number of realisations and perform the disorder average directly for low values of N. In the $q = 2$ case the density of states is fundamentally different at all N to that of larger finite values of $q > 2$. figure G.1.1 displays the normalised density of states for increasing even N over $2^{\frac{36-N}{2}}$. The most important aspect of this difference is the smooth decrease in the density of states at small energies. Each realisation has a single lowest energy state given by the creation of the lowest random mass particle. This results in lengthening "tails" of low energy, low density states, unlike the $q > 2$ case displayed in figure 15 of [260]. Much like the result for higher q, low N realisations of the model have oscillations in the density of states caused by level repulsion. These are smoothed out as the eigenvalue spectrum of the coupling (the spectral function denoted by $\rho(\lambda)$ in [283], note this is not the density of states discussed here) approaches the Wigner semicircle [279, 288].

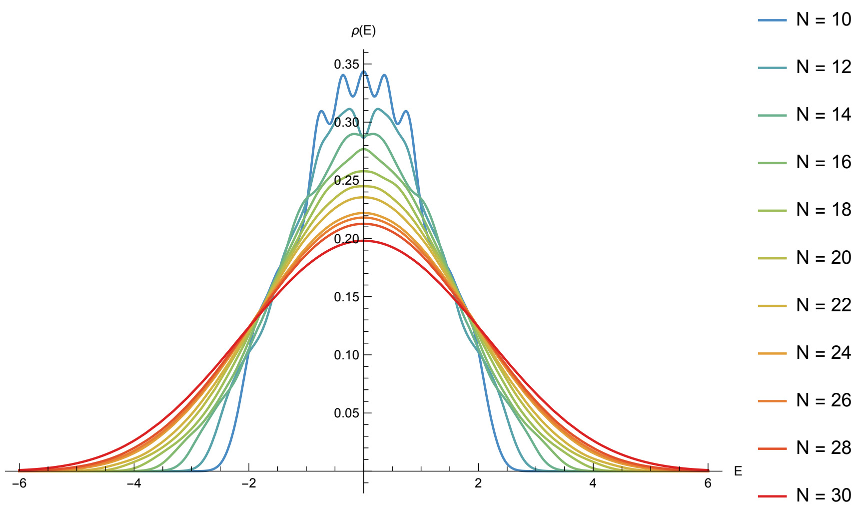

Figure G.1.1: Normalised disorder average density of states $\rho(E)$ plotted as a smooth histogram against energy for the $J = 1$, $q = 2$ SYK model with increasing values of N. In each instance $2^{\frac{36-N}{2}}$ realisations of the model were averaged. The density of states tends to zero smoothly with a "tail" indicative of a solvable theory, unlike the semi-circle $\rho(E)$ approached in the large-N limit for the $q > 2$ model.

G.2 Entropy

The tails of the distribution of states cause the low temperature entropy to be fixed at zero. Using the above data we calculated the disorder averaged entropy as a function of temperature for each N. We then used a large N fitting adapted from that in [260]. Extrapolating from a fit of this data at each T to the polynomial

$$a(T) + \frac{b(T)}{N} + \frac{c(T)}{N^2} \ , \tag{G.2.1}$$

we took the value of a(T) as the large N approximation to the curve. This is displayed in figure G.2.1 as the dashed plot. It is clear that in the $q = 2$ case the low temperature entropy converges on

$$\lim_{\beta\to\infty} \lim_{N\to\infty} S \approx 0 \ . \tag{G.2.2}$$

This is consistent with the analytic expectation of the entropy in this limit (to this order) derived in the appendix of [272] for the complex model. This graph should be compared to fig. 6 of [260], in which the same extrapolation was performed for the $q = 4$ Majorana model. A finite, non-zero, low temperature, large N entropy was observed in this case, consistent with

$$\lim_{\beta\to\infty} \lim_{N\to\infty} S_{q>2} \approx N \ . \tag{G.2.3}$$

At low temperature the model exhibits a linear entropy and a linear specific heat with respect to time this is shown in figure G.2.2 using the same extrapolation as for the entropy. It is noted here that the large N suppressed $\log(T)$ contribution to the entropy discussed in [254, 260] was not observed in the $q = 2$ case using this numerical procedure.

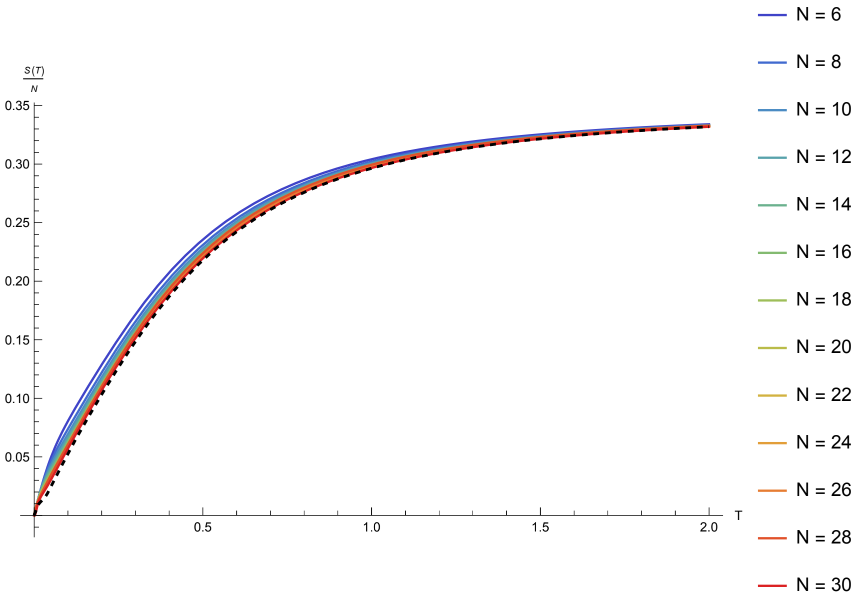

Figure G.2.1: Entropy with respect to temperature $\frac{S(T)}{N}$ plotted from the disorder average Majorana SYK model with $q = 2$ and $J = 1$ for various N. In each instance $2^{\frac{36-N}{2}}$ realisations of the model were averaged as described above. Fitted with a quadratic polynomial in N^{-1}, the extrapolated large N result is plotted as a dashed line.

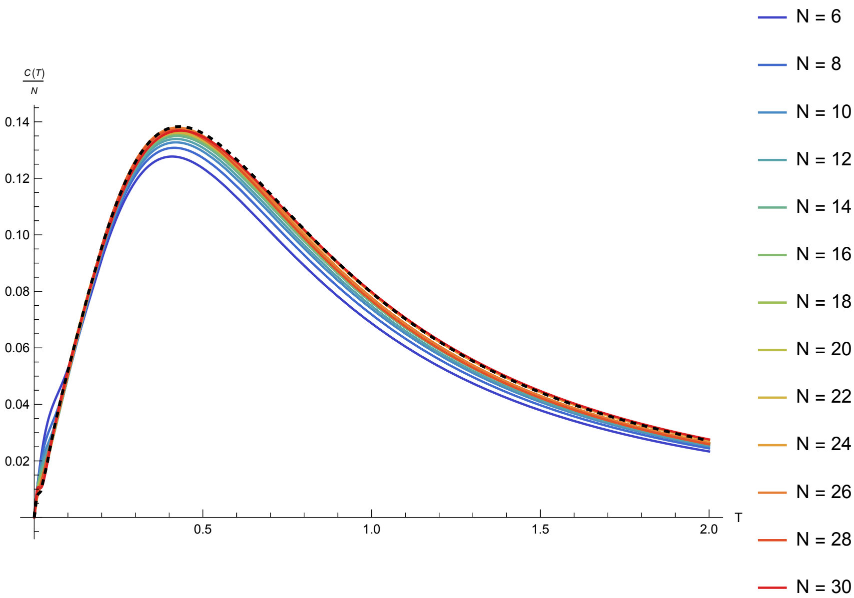

Figure G.2.2: Specific heat with respect to temperature $\frac{C(T)}{N}$ plotted from the disorder average Majorana SYK model with $q = 2$ and $J = 1$ for various N. In each instance $2^{\frac{36-N}{2}}$ realisations of the model were averaged as described above. Fitted with a quadratic polynomial in N^{-1}, the extrapolated large N result is plotted as a dashed line.

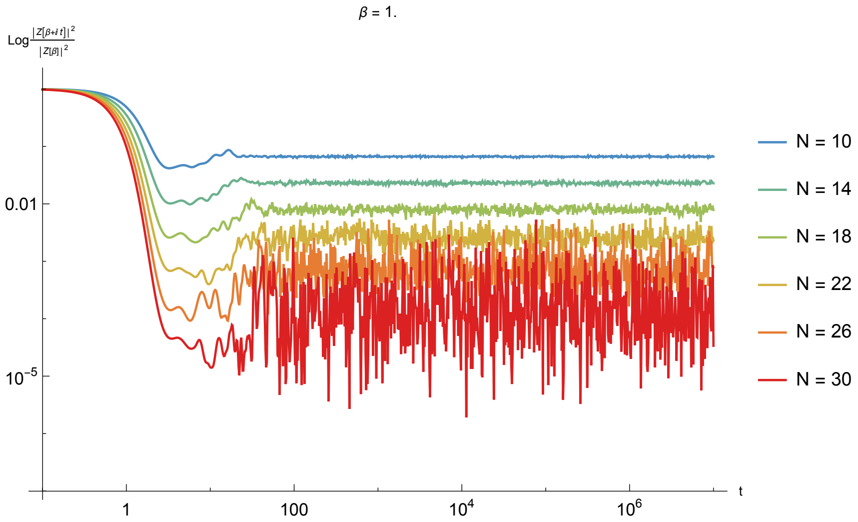

Figure G.3.1: The spectral form factor $\log\left(\frac{|Z(\beta+it)|^2}{|Z(\beta)|^2}\right)$ plotted w.r.t $t \geq 0.1$, with $\beta = 1$ for the disorder averaged Majorana $q = 2$ SYK model with $J = 1$ and various N with $2^{\frac{36-N}{2}}$ realisations.

G.3 Spectral Form Factor

The spectral form factor of the $q = 2$ model has been discussed in detail in appendix E of [260]. It is calculated analytically and a comment is made, that while the model is not chaotic there is evidence of a "mini ramp" and "mini plateau" in the squared disorder averaged correlation function. We calculated the disorder average square partition function and do indeed observe a slow increase after a minimum, with a "plateau" at $t \sim N$. However, the increasing phase is not an obviously linear "ramp". With increasing N it also seems that this squared quantity has an increasing amount of noise around the "plateau". The spectral form factor for the real Majorana $q = 2$ SYK model is calculated for $N = 10, 16, 20, 24, 30$ in figure G.3.1 in agreement with [260].

H. Eigenvalues of the ladder generating kernel

We make use of a result from [256], that the eigenvalues of the action of the kernel on one of the eigenfunctions given in appendix I are independent of h. In this appendix we show this. In the CFT limit the quadratic Casimir of the conformal algebra in the space of bilinear functions operates as

$$\mathcal{C}_{12} = 2\Delta(\Delta - 1) - \hat{K}_1\hat{P}_2 - \hat{P}_1\hat{K}_2 \ + 2\hat{D}_1\hat{D}_2 \ , \tag{H.0.1}$$

where

$$\hat{D}_i = -\vartheta_i \frac{\partial}{\partial \vartheta_i} - \Delta \ , \tag{H.0.2}$$

$$\hat{P}_i = \frac{\partial}{\partial \vartheta_i} \ , \tag{H.0.3}$$

$$\hat{K}_i = \vartheta_i^2 \frac{\partial}{\partial \vartheta_i} + 2\vartheta_i\Delta \ . \tag{H.0.4}$$

The operation of integrating against the kernel commutes with the action of the Casimir [254]. In fact as we are now considering a CFT all correlation functions are eigenfunctions of the Casimir, this includes the three point functions of two fermions (weight $\frac{1}{2}$) with another operator of weight h

$$\langle \overline{\psi}_i(\vartheta_1)\psi_i(\vartheta_2)\mathcal{O}_h(\vartheta_0)\rangle = \frac{\mathrm{sgn}(\vartheta_1 - \vartheta_2) - i\mathrm{sgn}(\vartheta_0 - \vartheta_1)\mathrm{sgn}(\vartheta_0 - \vartheta_2)}{|\vartheta_1 - \vartheta_0|^h |\vartheta_2 - \vartheta_0|^h |\vartheta_1 - \vartheta_2|^{1-h}} \tag{H.0.5}$$

We note that the symmetric and antisymmetric in $\vartheta_1 \leftrightarrow \vartheta_2$ parts of this three point function are clearly the real and imaginary pieces. These cannot, by themselves, form the basis functions for the four-point function Ψ_h. However, as we noted above, the kernel (4.2.29) only acts on two of the arguments of these basis four-point functions. However, we may write the four-point function as an integral over these three point functions,

$$\Psi_h(\vartheta_1,\vartheta_2,\vartheta_3,\vartheta_4) \approx \int d\vartheta_0 \left\langle \overline{\psi}_i(\vartheta_1)\psi_i(\vartheta_2)\mathcal{O}_h(\vartheta_0)\right\rangle \left\langle \mathcal{O}_h(\vartheta_0)\overline{\psi}_i(\vartheta_3)\psi_i(\vartheta_4)\right\rangle \ . \tag{H.0.6}$$

As we are only currently interested in the eigenvalue $k(h)$ we can ignore this decomposition for now, it is used to calculate the Ψ_h in [256], which we have summarised in appendix I. The Casimir is a differential operator acting on the variables outside of this integration, so $k(h)$ can be calculated by integrating the three-point functions against the kernel

$$\int d\vartheta d\vartheta' K(1,0,\vartheta,\vartheta') \left\langle \overline{\psi}_i(\vartheta)\psi_i(\vartheta')\mathcal{O}_h(\vartheta_0)\right\rangle^{\pm} = k^{\pm}(h) \left\langle \overline{\psi}_i(1)\psi_i(0)\mathcal{O}_h(\vartheta_0)\right\rangle \ , \tag{H.0.7}$$

defining

$$\left\langle \overline{\psi}_i(\vartheta)\psi_i(\vartheta')\mathcal{O}_h(\vartheta_0)\right\rangle^{+} = \mathrm{Re} \left\langle \overline{\psi}_i(\vartheta)\psi_i(\vartheta')\mathcal{O}_h(\vartheta_0)\right\rangle \ , \tag{H.0.8}$$

$$\left\langle \overline{\psi}_i(\vartheta)\psi_i(\vartheta')\mathcal{O}_h(\vartheta_0)\right\rangle^{-} = i \,\mathrm{Im} \left\langle \overline{\psi}_i(\vartheta)\psi_i(\vartheta')\mathcal{O}_h(\vartheta_0)\right\rangle \ . \tag{H.0.9}$$

Using the limit $\vartheta_0 \to \infty$ and (4.2.30)

$$k^{+}(h) = \pi^{-2} \int d\vartheta d\vartheta' \frac{\mathrm{sgn}(\vartheta-1)\mathrm{sgn}(\vartheta')\mathrm{sgn}(\vartheta-\vartheta')}{|\vartheta-1||\vartheta'||\vartheta-\vartheta'|^{1-h}} \tag{H.0.10}$$

$$k^{-}(h) = \pi^{-2} \int d\vartheta d\vartheta' \frac{\mathrm{sgn}(\vartheta-1)\mathrm{sgn}(\vartheta')}{|\vartheta-1||\vartheta'||\vartheta-\vartheta'|^{1-h}} \tag{H.0.11}$$

Using

$$\frac{1}{\vartheta} = i\pi \int \frac{d\omega}{2\pi} \mathrm{sgn}(\omega) e^{-i\omega\vartheta} \ , \tag{H.0.12}$$

we retrieve,

$$k^{\pm}(h) = -\int d\omega f^{\pm}(\omega) e^{-i\omega} \equiv -1 \ , \tag{H.0.13}$$

for all values of h. Here $f^{\pm}$ is the frequency space representation of the remaining function in each of the integrals above. We have made use of the fact here that both of the remaining functions evaluate to 1 at $\vartheta - \vartheta' = 1$.

I. Eigenfunctions of the $SL(2,\mathbb{R})$ Casimir

Our analysis requires a complete set of complex one-dimensional eigen-functions $\tilde{\Psi}_h$ of the $SL(2,\mathbb{R})$ Casimir.

$$\mathcal{C} = x^2(1-x)\partial_x^2 - x^2\partial_x \ . \tag{I.0.1}$$

These were calculated in [256], for convenience a short summary is included here. The appropriate functions are found by solving the eigenproblem

$$\mathcal{C}\tilde{\Psi}_h(x) = h(h-1)\tilde{\Psi}_h(x) \ , \tag{I.0.2}$$

and looking for normalisable functions under the inner product

$$\langle f, g\rangle = \int_{-\infty}^{\infty} \frac{dx}{x^2}\overline{f}g \ . \tag{I.0.3}$$

We are interested in two sets of eigen-functions, those even and odd under the transformation $x \to \frac{x}{x-1}$, we label these

$$\tilde{\Psi}_h^+\left(\frac{x}{x-1}\right) = \tilde{\Psi}_h^+(x) \ , \qquad \tilde{\Psi}_h^-\left(\frac{x}{x-1}\right) = -\tilde{\Psi}_h^-(x) \ . \tag{I.0.4}$$

The differential equation (I.0.2) has solutions

$$F_h = \frac{\Gamma(h)^2}{\Gamma(2h)}x^h \, {}_2F_1(h,h;2h;x) \tag{I.0.5}$$

the other solution can be reached using the symmetry of the differential equation under $h \to 1-h$. As there is a branch cut some care needs to be taken to ensure the solutions combine with the correct behaviour under the transformation (I.0.4), the results are

$$\tilde{\Psi}_h^+(x) = \begin{cases} \frac{2}{\cos\pi h}\left(\cos^2\frac{\pi h}{2}F_h(x) - \sin^2\frac{\pi h}{2}F_{1-h}(x)\right) & x<1 \ , \\ \frac{2}{\sqrt{\pi}}\Gamma\left(\frac{h}{2}\right)\Gamma\left(\frac{1-h}{2}\right) {}_2F_1\left(\frac{h}{2},\frac{1-h}{2};\frac{1}{2};\frac{(2-x)^2}{x^2}\right) & x>1 \ , \end{cases} \tag{I.0.6}$$

and

$$\tilde{\Psi}_h^-(x) = \begin{cases} \frac{2}{\cos \pi h}\left(\sin^2 \frac{\pi h}{2} F_h(x) - \cos^2 \frac{\pi h}{2} F_{1-h}(x)\right) & x < 1\ , \\ \frac{4}{\sqrt{\pi}}\left(\frac{2-x}{x}\right)\Gamma\left(1-\frac{h}{2}\right)\Gamma\left(\frac{1+h}{2}\right) {}_2F_1\left(1-\frac{h}{2}, \frac{1+h}{2}; \frac{3}{2}; \frac{(2-x)^2}{x^2}\right) & x > 1\ . \end{cases} \tag{I.0.7}$$

Which are normalisable up to boundary terms in the cases $h \in \mathbb{Z}$ and $h = \frac{1}{2} + is$, where $s \in \mathbb{R}$. The inner products are respectively

$$\langle \tilde{\Psi}_h^+(x), \tilde{\Psi}_{h'}^+(x) \rangle = \langle \tilde{\Psi}_h^-(x), \tilde{\Psi}_{h'}^-(x) \rangle = \frac{2\pi^2 \delta_{h,h'}}{|h - \frac{1}{2}|} \qquad h \in \mathbb{Z}\ , \tag{I.0.8}$$

$$\langle \tilde{\Psi}_s^+(x), \tilde{\Psi}_{s'}^+(x) \rangle = \langle \tilde{\Psi}_s^-(x), \tilde{\Psi}_{s'}^-(x) \rangle = \frac{4\pi^2 \coth \pi s}{s} \delta(s - s') \qquad h = \frac{1}{2} + is, \tag{I.0.9}$$

$$\langle \tilde{\Psi}_h^+(x), \tilde{\Psi}_{h'}^-(x) \rangle = 0\ . \tag{I.0.10}$$

Throughout the above we make use of the symmetry under $h \to 1 - h$ to limit the ranges of the weights to $s > 0$ and $h \in \mathbb{Z}^+$.

As a note on the group theory, we might expect a third case in which there are normalisable wavefunctions for the above Casimir, with real weights in the complementary series $0 < h < \frac{1}{2}$, as these weights also provide admissible unitary representations of $SL(2, \mathbb{R})$. However these are not required to form a complete basis of functions on $\mathbb{R}$ with this inner product [254].

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