

Canonical quantization of some midi-superspace models in 2+1 and 3+1 dimensions

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Abstract. A proposal is put forward which enables the canonical quantization of a family of axially symmetric geometries in 2+1 dimensions and a corresponding spherically symmetric family in 3+1 dimensions. The proposal consists of a particular renormalization *assumption* and an accompanying *requirement* and results in a Wheeler–DeWitt equation which is based on a renormalized manifold parametrized by three smooth scalar functionals. The aforementioned equation is analytically solved for both the 2+1 and 3+1 case.

1. Introduction

After Dirac's pioneering work on the treatment of systems with constraints [1], [2], [3], [4] the way was open for a systematic treatment of constrained dynamics. Some of the landmarks in the study of constrained systems have been the connection between constraints and invariances [5], the extension of the formalism to describe fields with half-integer spin through the algebra of Grassmann variables [6] and the introduction of the BRST formalism [7]. All these results have helped the quantization of gauge theories. There exist several excellent reviews studying constraint systems with a finite number of degrees of freedom [8] or constraint field theories [9], as well as more general presentations [10], [11], [12], [13], [14], [15]. In particular, the conventional canonical analysis approach of quantum gravity has been initiated by B.S. DeWitt [16] based on earlier work of P.G. Bergmann [17].

Since a full theory of quantum gravity does not today exist, it is reasonably important to address the quantization of (classes of) simplified geometries. An elegant way to achieve a degree of simplification is to impose some symmetry. For example, the assumption of a G_3

symmetry group acting simply transitively on the surfaces of simultaneity, i.e. the existence of three independent space-like Killing vector fields, leads to classical and subsequently quantum homogeneous cosmology (see, e.g., [18], [19]). The imposition of lesser symmetry, e.g. fewer Killing vector fields, results in the various inhomogeneous cosmological models [20]. The canonical analysis under the assumption of spherical symmetry, which is a G_3 group acting multiply transitively on two-dimensional space-like subsurfaces of the three-slices, has been first considered in [21], [22].

This work consists in two parts. In the first part, we deal with the simplified case of 2+1 dimensions (see, e.g., [26], [27], [28] and references there in), under the assumption of one spacelike Killing vector field, while in the second part, the canonical quantization of all geometries admitting two-dimensional surfaces of maximal symmetry, i.e. spheres (constant positive curvature), planes (zero curvature) and Gauss-Bolyai-Lobachevsky (henceforth GBL) spaces (constant negative curvature) is considered. The structure of the paper is as follows: In Section 2 we give the reduced metrics, the space of classical solutions and the Hamiltonian formulation of the reduced Einstein-Hilbert action principle, resulting in one (quadratic) Hamiltonian and two (linear) momentum constraints, all being first class. In Section 3, we consider the quantization of this constraint system following Dirac's proposal of implementing the quantum operator constraints as conditions annihilating the wave-function [4]. Our guideline is a conceptual generalization of the quantization scheme developed in [30], [31] for the case of constraint systems with finite degrees of freedom, to the present case. Even though after the symmetry reduction the system still represents a field theory (all remaining metric components depend on time and the radial coordinate), we manage to extract and subsequently completely solve a Wheeler-DeWitt equation in terms of three unique smooth scalar functionals of the appropriate components of the reduced spatial metric. This is achieved through an appropriate re-normalization assumption we adopt. Finally, some concluding remarks are included in the discussion.

2. Possible Metrics, Classical Solutions and Hamiltonian Formulation

2.1. The 2+1 Case

Our starting point is the action principle:

$$I = \int d^3x \sqrt{-g}(R - 2\Lambda). \quad (2.1)$$

The equations of motion arising upon variation of this action are

$$R_{IJ} - \frac{1}{2}g_{IJ}R + \Lambda g_{IJ} = 0, \quad (2.2)$$

where $I, J = 0, 1, 2$. Of course, since in three dimensions the Riemann curvature tensor is expressible in terms of both the Ricci tensor and scalar, the space of solutions to (2.2) consists simply of all maximally symmetric 3D metrics (AdS3). If topological considerations are taken into account, the above space might be enriched containing, for example, the stationary BTZ "black" hole [32], [33]

$$ds^2 = -(M - \Lambda r^2)dt^2 - Jdt d\phi + \left(M - \Lambda r^2 + \frac{J^2}{4r^2}\right)^{-1} dr^2 + r^2 d\phi^2, \quad (2.3)$$

or the “cosmological” solutions [34], [35]

$$ds^2 = -\frac{1}{4t^2\Lambda}dt^2 + \frac{1}{2t\sqrt{\Lambda}}(dr^2 + d\phi^2), \quad (2.4)$$

$$ds^2 = -\left(\frac{4}{16t^2 - \Lambda}\right)^2 dt^2 + \frac{4}{16t^2 - \Lambda}dr^2 + \frac{4e^{-4r}}{16t^2 - \Lambda}d\phi^2. \quad (2.5)$$

The above metrics admit a manifest G_2 symmetry group. Here, we consider a generalization consisting in the imposition of a G_1 (axial) symmetry only, i.e we impose one Killing vector field, say $\xi = \frac{\partial}{\partial\phi}$. Subsequently, all components of the metric become functions of only the time and the radial coordinate. The canonical decomposition of such a metric is given in terms of the spatial metric $g_{ij}(t, r)$, the lapse function $N^o(t, r)$ and the shift “vector” $N_i(t, r)$ [10]:

$$ds^2 = -(N^o)^2 + g^{ij}N_iN_j dt^2 + 2N_idtdx^i + g_{ij}dx^i dx^j, \quad (2.6)$$

where

$$g_{ij} = \begin{pmatrix} \frac{\rho^2 + \sigma^2\chi^2}{\sigma} & \sigma\chi \\ \sigma\chi & \sigma \end{pmatrix}, \quad g^{ij} = \begin{pmatrix} \frac{\sigma}{\rho^2} & -\frac{\sigma\chi}{\rho^2} \\ -\frac{\sigma\chi}{\rho^2} & \frac{\rho^2 + \sigma^2\chi^2}{\sigma\rho^2} \end{pmatrix}, \quad (2.7)$$

with $i, j = 1, 2$, and $x^i = (r, \phi)$.

Utilizing the Gauss-Codazzi equation (see, e.g., [36]), we eliminate all second time-derivatives from the Einstein-Hilbert action, by subtracting an appropriate divergence, and arrive at an action quadratic in the velocities. The application of the Dirac algorithm results firstly in the three primary constraints $P_o \equiv \frac{\delta L}{\delta N^o} \approx 0$, $P^i \equiv \frac{\delta L}{\delta N_i} \approx 0$ and the Hamiltonian

$$H = \int (N^o\mathcal{H}_o + N^i\mathcal{H}_i) dr, \quad (2.8)$$

where $\mathcal{H}_o, \mathcal{H}_i$ are given by

$$\mathcal{H}_o = \frac{1}{2}G^{\alpha\beta}\pi_\alpha\pi_\beta + V, \quad (2.9a)$$

$$\mathcal{H}_1 = \sigma'\pi_\sigma - \rho\pi'_\rho - \chi\pi'_\chi, \quad (2.9b)$$

$$\mathcal{H}_2 = -\pi'_\chi, \quad (2.9c)$$

the indices (α, β) take the values (ρ, σ, χ) and $' \equiv \frac{\partial}{\partial r}$. The Wheeler-DeWitt midi-superspace metric $G^{\alpha\beta}$ reads

$$G^{\alpha\beta} = \begin{pmatrix} -\rho & -\sigma & 0 \\ -\sigma & 0 & 0 \\ 0 & 0 & \frac{\rho}{\sigma^2} \end{pmatrix}, \quad (2.10)$$

while the potential V is

$$V = 2\Lambda\rho + \left(\frac{\sigma'}{\rho}\right)'. \quad (2.11)$$

The requirement for preservation, in time, of the primary constraints leads to the secondary constraints

$$\mathcal{H}_o \approx 0, \quad \mathcal{H}_1 \approx 0, \quad \mathcal{H}_2 \approx 0. \quad (2.12)$$

At this stage, a tedious but straightforward calculation produces the following “open” Poisson bracket algebra of these constraints:

$$\begin{aligned} \{\mathcal{H}_o(r), \mathcal{H}_o(\tilde{r})\} &= [g^{1j}(r)\mathcal{H}_j(r) + g^{1j}(\tilde{r})\mathcal{H}_j(\tilde{r})]\delta'(r, \tilde{r}), \\ \{\mathcal{H}_1(r), \mathcal{H}_o(\tilde{r})\} &= \mathcal{H}_o(r)\delta'(r, \tilde{r}), \\ \{\mathcal{H}_2(r), \mathcal{H}_o(\tilde{r})\} &= 0, \\ \{\mathcal{H}_1(r), \mathcal{H}_1(\tilde{r})\} &= \mathcal{H}_1(r)\delta'(r, \tilde{r}) - \mathcal{H}_1(\tilde{r})\delta(r, \tilde{r})', \\ \{\mathcal{H}_1(r), \mathcal{H}_2(\tilde{r})\} &= \mathcal{H}_2(r)\delta'(r, \tilde{r}), \\ \{\mathcal{H}_2(r), \mathcal{H}_2(\tilde{r})\} &= 0, \end{aligned} \quad (2.13)$$

indicating that they are first class and also signaling the termination of the algorithm. Thus, our system is described by (2.12); the “dynamical” Hamilton-Jacobi equations $\frac{d\pi_\rho}{dt} = \{\pi_\rho, H\}$, $\frac{d\pi_\sigma}{dt} = \{\pi_\sigma, H\}$, $\frac{d\pi_\chi}{dt} = \{\pi_\chi, H\}$ are satisfied by virtue of the time derivatives of (2.12). These three equations, when expressed in the velocity phase-space with the help of the definitions $\frac{d\rho}{dt} = \{\rho, H\}$, $\frac{d\sigma}{dt} = \{\sigma, H\}$, $\frac{d\chi}{dt} = \{\chi, H\}$, are completely equivalent to the three independent Einstein’s field equations satisfied by (2.6).

Under changes of the radial variable r of the form $r \rightarrow \tilde{r} = h(r)$, it can easily be inferred from (2.6) and (2.7) that:

$$\begin{aligned} \tilde{\rho}(\tilde{r}) &= \rho(r) \frac{dr}{d\tilde{r}}, & \tilde{\sigma}(\tilde{r}) &= \sigma(r), & \tilde{\chi}(\tilde{r}) &= \chi(r) \frac{dr}{d\tilde{r}}, \\ \frac{d\tilde{\sigma}(\tilde{r})}{d\tilde{r}} &= \frac{d\sigma(r)}{dr} \frac{dr}{d\tilde{r}}, & \frac{d}{d\tilde{r}} \left(\frac{\tilde{\chi}(\tilde{r})}{\tilde{\rho}(\tilde{r})} \right) &= \frac{d}{dr} \left(\frac{\chi(r)}{\rho(r)} \right) \frac{dr}{d\tilde{r}}, \end{aligned} \quad (2.14)$$

where the t -dependence has been omitted for the sake of brevity. Thus, under the above coordinate transformations, σ , $\frac{\chi}{\rho}$ are scalars, while ρ , χ and the derivatives of σ , $\frac{\chi}{\rho}$ are covariant rank 1 tensors (one-forms), or, equivalently in one dimension, scalar densities of weight -1 . Therefore, the scalar derivative is not $\frac{d}{dr}$ but rather $\frac{d}{\rho dr}$ or $\frac{d}{\chi dr} \equiv \frac{\rho}{\chi} \frac{d}{dr}$. Finally, if we consider an infinitesimal transformation $r \rightarrow \tilde{r} = r - \eta(r)$, it is easily seen that the corresponding changes induced on the basic fields are:

$$\delta \rho(r) = (\rho(r) \eta(r))', \quad \delta \sigma(r) = \sigma'(r) \eta(r), \quad \delta \chi(r) = (\chi(r) \eta(r))', \quad (2.15)$$

i.e., nothing but the one-dimensional analogue of the appropriate Lie derivatives.

With the use of (2.15), we can reveal the nature of the action of \mathcal{H}_1 on the basic configuration space variables as that of the generator of spatial diffeomorphisms:

$$\begin{aligned} \left\{ \rho(r), \int d\tilde{r} \eta(\tilde{r}) \mathcal{H}_1(\tilde{r}) \right\} &= (\rho(r) \eta(r))', \\ \left\{ \sigma(r), \int d\tilde{r} \eta(\tilde{r}) \mathcal{H}_1(\tilde{r}) \right\} &= \sigma'(r) \eta(r), \\ \left\{ \chi(r), \int d\tilde{r} \eta(\tilde{r}) \mathcal{H}_1(\tilde{r}) \right\} &= (\chi(r) \eta(r))'. \end{aligned} \quad (2.16)$$

Thus, we are justified to consider \mathcal{H}_1 as the representative, in phase-space, of an arbitrary infinitesimal reparametrization of the radial coordinate. As far as \mathcal{H}_2 is concerned, the situation is a little more complicated: the imposition of the symmetry generated by the Killing vector field $\xi = \partial/\partial\phi$ has left all configuration variables without any ϕ dependence; subsequently we can not expect \mathcal{H}_2 to generate arbitrary infinitesimal reparametrization of ϕ . Nevertheless, we can identify a property of \mathcal{H}_2 which links its existence to the existence of ξ . This property is described by the relation: $\{\mathcal{H}_2(r), \{\mathcal{H}_2(r), \{\mathcal{H}_2(r), g_{ij}(\tilde{r})\}\}\} = 0 \stackrel{\text{correspo}}{\underset{\text{ndence}}{\rightleftharpoons}} \mathcal{L}_\xi g_{ij} = 0$.

2.2. The 3+1 Case

Our starting point is the two-dimensional spaces of positive, zero or negative constant curvature. Their line elements are respectively:

$$ds^2 = d\theta^2 + \sin^2 \theta d\phi^2, \quad ds^2 = d\theta^2 + \theta^2 d\phi^2, \quad ds^2 = d\theta^2 + \sinh^2 \theta d\phi^2, \quad (2.17)$$

with an obvious range of the coordinates for each case. The corresponding (maximal) symmetry groups are generated by the following KVF's:

$$\xi_1 = \frac{\partial}{\partial\phi}, \quad \xi_2 = -\cos\phi \frac{\partial}{\partial\theta} + \cot\theta \sin\phi \frac{\partial}{\partial\phi}, \quad \xi_3 = \sin\phi \frac{\partial}{\partial\theta} + \cot\theta \cos\phi \frac{\partial}{\partial\phi}, \quad (2.18)$$

$$\xi_1 = \frac{\partial}{\partial\phi}, \quad \xi_2 = -\cos\phi \frac{\partial}{\partial\theta} + \frac{\sin\phi}{\theta} \frac{\partial}{\partial\phi}, \quad \xi_3 = \sin\phi \frac{\partial}{\partial\theta} + \frac{\cos\phi}{\theta} \frac{\partial}{\partial\phi}, \quad (2.19)$$

$$\xi_1 = \frac{\partial}{\partial\phi}, \quad \xi_2 = -\cos\phi \frac{\partial}{\partial\theta} + \coth\theta \sin\phi \frac{\partial}{\partial\phi}, \quad \xi_3 = \sin\phi \frac{\partial}{\partial\theta} + \coth\theta \cos\phi \frac{\partial}{\partial\phi}. \quad (2.20)$$

Trivially prolonging these KVF's to four-dimensional space-time, we arrive at the three classes of metrics described by the following line element:

$$ds^2 = \left(-\alpha(t, r)^2 + \frac{\beta(t, r)^2}{\gamma(t, r)^2} \right) dt^2 + 2\beta(t, r) dt dr + \gamma(t, r)^2 dr^2 + \psi(t, r)^2 d\theta^2 + \psi(t, r)^2 f(\theta)^2 d\phi^2, \quad (2.21)$$

where $f(\theta) = \sin\theta$ in the case of spherical symmetry, $f(\theta) = \theta$ in the case of plane symmetry, and $f(\theta) = \sinh\theta$ in the case of the GBL symmetry.

In order to attain the classical space of solution for these geometries one can, exploiting the freedom to change coordinates in the (t, r) subspace, bring the upper left block of the metric in conformally flat form and readily solve the vacuum Einstein's field equations. The result is given, in the light-cone coordinates $u = \frac{t+r}{2}$, $v = \frac{-t+r}{2}$, by the following two line elements:

$$ds^2 = 2\epsilon \frac{A + 2\psi(u, v)}{4\psi(u, v)} du dv + \psi(u, v)^2 d\theta^2 + \psi(u, v)^2 f(\theta)^2 d\phi^2, \quad (2.22)$$

where

$$\psi(u, v) = -\frac{A}{2} \left(1 + \text{ProductLog} \left(-\frac{\exp \left(-\frac{\lambda(A+u+\epsilon v)}{A} \right)}{A} \right) \right),$$

$\epsilon = \pm 1$, $\lambda = \pm 1$ for $f(\theta) = \sin \theta$, $\lambda = \pm i$, for $f(\theta) = \sinh \theta$ and $\text{ProductLog}(z)$ is the principal solution for w to the equation $z = w \exp^w$, and

$$ds^2 = 2\epsilon \frac{1}{\psi(u, v)} du dv + \psi(u, v)^2 d\theta^2 + \psi(u, v)^2 \theta^2 d\phi^2, \quad (2.23)$$

where

$$\psi(u, v) = \pm \sqrt{2u + 2\epsilon v}.$$

The canonical analysis proceeds similarly as the previous case and the reduced Hamiltonian reads

$$H = \int (N^o \mathcal{H}_o + N^i \mathcal{H}_i) dr, \quad (2.24)$$

where

$$N^o = \alpha(t, r), \quad N^1 = \frac{\beta(t, r)}{\gamma(t, r)^2}, \quad N^2 = 0, \quad N^3 = 0,$$

and $\mathcal{H}_o, \mathcal{H}_i$ are given by

$$\mathcal{H}_o = \frac{1}{2} G^{\alpha\beta} \pi_\alpha \pi_\beta + V, \quad (2.25a)$$

$$\mathcal{H}_1 = -\gamma \pi'_\gamma + \psi' \pi_\psi, \quad \mathcal{H}_2 = 0, \quad \mathcal{H}_3 = 0, \quad (2.25b)$$

while the indices $\{\alpha, \beta\}$ take the values $\{\gamma, \psi\}$ and $' = \frac{d}{dr}$. The reduced Wheeler-DeWitt super-metric $G^{\alpha\beta}$ reads

$$G^{\alpha\beta} = \begin{pmatrix} \frac{\gamma}{4\psi^2} & -\frac{1}{4\psi} \\ -\frac{1}{4\psi} & 0 \end{pmatrix}, \quad (2.26)$$

while the potential V is

$$V = -2\epsilon\gamma + 2\Lambda\gamma\psi^2 - 2\frac{\psi'^2}{\gamma} + 4\left(\frac{\psi\psi'}{\gamma}\right)', \quad (2.27)$$

with $\epsilon = \{1, 0, -1\}$ for the families of two-dimensional subspaces with positive, zero or negative constant curvature, respectively. The requirement for preservation, in time, of the primary constraints leads to the secondary constraints

$$\mathcal{H}_o \approx 0, \quad \mathcal{H}_1 \approx 0, \quad (2.28)$$

The ‘‘open’’ Poisson bracket algebra satisfied by these constraints is:

$$\begin{aligned} \{\mathcal{H}_o(r), \mathcal{H}_o(\tilde{r})\} &= \left[\frac{1}{\gamma^2(r)} \mathcal{H}_1(r) + \frac{1}{\gamma^2(\tilde{r})} \mathcal{H}_1(\tilde{r}) \right] \delta'(r, \tilde{r}), \\ \{\mathcal{H}_1(r), \mathcal{H}_o(\tilde{r})\} &= \mathcal{H}_o(r) \delta'(r, \tilde{r}), \\ \{\mathcal{H}_1(r), \mathcal{H}_1(\tilde{r})\} &= \mathcal{H}_1(r) \delta'(r, \tilde{r}) - \mathcal{H}_1(\tilde{r}) \delta(r, \tilde{r})', \end{aligned} \quad (2.29)$$

indicating that they are first class and also signaling the termination of the algorithm. Thus, our system is described by (2.28) and the dynamical Hamilton-Jacobi equations $\frac{d\pi_\gamma}{dt} = \{\pi_\gamma, H\}$, $\frac{d\pi_\psi}{dt} = \{\pi_\psi, H\}$. These equations, when expressed in the velocity phase-space with the help of the definitions $\frac{d\gamma}{dt} = \{\gamma, H\}$, $\frac{d\psi}{dt} = \{\psi, H\}$, are completely equivalent to the independent Einstein's field equations satisfied by (2.21).

Under changes of the radial variable r of the form $r \rightarrow \tilde{r} = h(r)$, it can easily be inferred from (2.21) that

$$\tilde{\gamma}(\tilde{r}) = \gamma(r) \frac{dr}{d\tilde{r}}, \quad \tilde{\psi}(\tilde{r}) = \psi(r), \quad \frac{d\tilde{\psi}(\tilde{r})}{d\tilde{r}} = \frac{d\psi(r)}{dr} \frac{dr}{d\tilde{r}}, \quad (2.30)$$

where the t -dependence has again been omitted. Thus, under the above coordinate transformations, ψ is a scalar, while γ , ψ' are covariant rank 1 tensors (one-forms), or, equivalently in one dimension, scalar densities of weight -1 . Therefore, the scalar derivative is not $\frac{d}{dr}$ but rather $\frac{d}{\gamma dr}$. Finally, if we consider an infinitesimal transformation $r \rightarrow \tilde{r} = r - \eta(r)$, it is easily seen that the corresponding changes induced on the basic fields are:

$$\delta \gamma(r) = (\gamma(r) \eta(r))', \quad \delta \psi(r) = \psi'(r) \eta(r), \quad (2.31)$$

i.e., nothing but the one-dimensional analogue of the appropriate Lie derivatives.

With the use of (2.31), we can reveal the nature of the action of \mathcal{H}_1 on the basic configuration space variables as that of the generator of spatial diffeomorphisms:

$$\left\{ \gamma(r), \int d\tilde{r} \eta(\tilde{r}) \mathcal{H}_1(\tilde{r}) \right\} = (\gamma(r) \eta(r))', \quad (2.32)$$

$$\left\{ \psi(r), \int d\tilde{r} \eta(\tilde{r}) \mathcal{H}_1(\tilde{r}) \right\} = \psi'(r) \eta(r).$$

Thus, we are justified to consider \mathcal{H}_1 as the representative, in phase-space, of an arbitrary infinitesimal reparametrization of the radial coordinate.

3. Quantization

We are now interested in attempting to quantize these two Hamiltonian systems following Dirac's general spirit of realizing the classical first class constraints as quantum operator constraint conditions annihilating the wave function. The main motivation behind such an approach is the justified desire to construct a quantum theory manifestly invariant under the "gauge" generated by the constraints. To begin with, let us first note that, despite the simplification brought by the imposition of the symmetry $\xi = \partial/\partial\phi \Leftrightarrow \mathcal{L}_\xi g_{IJ} = 0$, the system is still a field theory in the sense that all configuration variables and canonical conjugate momenta depend not only on time (as is the case in homogeneous cosmology), but also on the radial coordinate r . Thus, to canonically quantize the system in the Schrödinger representation, we first realize the classical momenta as functional derivatives with respect to their corresponding conjugate fields

$$\hat{\pi}_\rho(r) = -i \frac{\delta}{\delta \rho(r)}, \quad \hat{\pi}_\sigma(r) = -i \frac{\delta}{\delta \sigma(r)}, \quad \hat{\pi}_\chi(r) = -i \frac{\delta}{\delta \chi(r)},$$

$$\hat{\pi}_\gamma(r) = -i \frac{\delta}{\delta \gamma(r)} \quad \hat{\pi}_\psi(r) = -i \frac{\delta}{\delta \psi(r)}.$$

We next have to decide on the initial space of state vectors. To elucidate our choice, let us consider the action of a momentum operator on some function of the configuration field variables, say

$$\hat{\pi}_\rho(r)\rho(\tilde{r})^2 = -2i\rho(\tilde{r})\delta(\tilde{r}, r).$$

The Dirac delta-function renders the outcome of this action a distribution rather than a function. Also, if the momentum operator were to act at the point at which the function is evaluated, i.e. if $\tilde{r} = r$, then its action would produce a $\delta(0)$ and would therefore be ill-defined. Both of these unwanted features are rectified, as far as expressions linear in momentum operators are concerned, if we choose as our initial collection of states all *smooth functionals* (i.e., integrals over r) of the configuration variables $\rho(r), \sigma(r), \chi(r), \gamma(r), \psi(r)$ and their derivatives of any order. Indeed, as we infer from the previous example,

$$\hat{\pi}_\rho(r) \int d\tilde{r} \rho(\tilde{r})^2 = -2i \int d\tilde{r} \rho(\tilde{r}) \delta(\tilde{r}, r) = -2i\rho(r);$$

thus the action of the momentum operators on all such states will be well-defined (no $\delta(0)$'s) and will also produce only local functions and not distributions. However, even so, $\delta(0)$'s will appear as soon as local expressions quadratic in momenta are considered, e.g.,

$$\hat{\pi}_\rho(r) \hat{\pi}_\rho(r) \int d\tilde{r} \rho(\tilde{r})^2 = \hat{\pi}_\rho(r) (-2i \int d\tilde{r} \rho(\tilde{r}) \delta(\tilde{r}, r)) = \hat{\pi}_\rho(r) (-2i\rho(r)) = -2i\delta(r, r).$$

Another problem of equal, if not greater, importance has to do with the number of derivatives (with respect to r) considered: A momentum operator acting on a smooth functional of degree n in derivatives of $\rho(r), \sigma(r), \chi(r), \gamma(r), \psi(r)$ will, in general, produce a function of degree $2n$, e.g.,

$$\hat{\pi}_\rho(r) \int d\tilde{r} \rho''(\tilde{r})^2 = -2i \int d\tilde{r} \rho''(\tilde{r}) \delta''(\tilde{r}, r) = -2i\rho^{(4)}(r).$$

Thus, clearly, more and more derivatives must be included if we desire the action of momentum operators to keep us inside the space of integrands corresponding to the initial collection of smooth functionals; eventually, we have to consider $n \rightarrow \infty$. This, in a sense, can be considered as the reflection to the canonical approach, of the non-renormalizability results existing in the so-called covariant approach. The way to deal with these problems is, loosely speaking, to regularize (i.e., render finite) the infinite distribution limits, and re-normalize the theory by, somehow, enforcing n to terminate at some finite value.

In the following, we are going to present a quantization scheme of our systems which: (a) avoids the occurrence of $\delta(0)$'s, (b) reveals the value $n = 1$, as the only possibility to obtain a closed space of state vectors, and (c) extracts a finite-dimensional Wheeler-DeWitt equation governing the quantum dynamics. The scheme closely parallels, conceptually, the quantization developed in [30],[31] for finite systems with one quadratic and a number of linear first class constraints. Therefore, we deem it appropriate and instructive to present a brief account of the essentials of this construction.

To this end, let us consider a system described by a Hamiltonian of the form

$$\begin{aligned} H &\equiv \mu X + \mu^i \chi_i \\ &= \mu \left(\frac{1}{2} G^{AB}(Q^\Gamma) P_A P_B + U^A(Q^\Gamma) P_A + V(Q^\Gamma) \right) + \mu^i \phi_i^A(Q^\Gamma) P_A, \end{aligned} \quad (3.1)$$

where $A, B, \Gamma \dots = 1, 2, \dots, M$ count the configuration space variables and $i = 1, 2, \dots, N < (M - 1)$ numbers the super-momenta constraints $\chi_i \approx 0$, which along with the super-Hamiltonian constraint $X \approx 0$ are assumed to be first class:

$$\{X, X\} = 0, \quad \{X, \chi_i\} = XC_i + C_i^j \chi_j, \quad \{\chi_i, \chi_j\} = C_{ij}^k \chi_k, \quad (3.2)$$

where the first (trivial) Poisson bracket has been included only to emphasize the difference from the first of (2.13).

The physical state of the system is unaffected by the ‘‘gauge’’ transformations generated by (X, χ_i) , **but also** under the following three changes:

- (I) Mixing of the super-momenta with a non-singular matrix

$$\bar{\chi}_i = \lambda_i^j(Q^\Gamma) \chi_j,$$

- (II) Gauging of the super-Hamiltonian with the super-momenta

$$\bar{X} = X + \kappa^{(Ai}(Q^\Gamma) \phi_i^{B)}(Q^\Gamma) P_A P_B + \sigma^i(Q^\Gamma) \phi_i^A(Q^\Gamma) P_A,$$

- (III) Scaling of the super-Hamiltonian

$$\bar{X} = \tau^2(Q^\Gamma) X.$$

Therefore, the geometrical structures on the configuration space that can be inferred from the super-Hamiltonian are really equivalence classes under actions (I), (II) and (III); for example (II), (III) imply that the super-metric G^{AB} is known only up to conformal scalings and additions of the super-momenta coefficients $\bar{G}^{AB} = \tau^2(G^{AB} + \kappa^{(Ai} \phi_i^{B)})$. It is thus mandatory that, when we Dirac-quantize the system, we realize the quantum operator constraint conditions on the wave-function in such a way as to secure that the whole scheme is independent of actions (I), (II), (III). This is achieved by the following steps:

- (1) Realize the linear operator constraint conditions with the momentum operators to the right

$$\hat{\chi}_i \Psi = 0 \leftrightarrow \phi_i^A(Q^\Gamma) \frac{\partial \Psi(Q^\Gamma)}{\partial Q^A} = 0,$$

which maintains the geometrical meaning of the linear constraints and produces the $M - N$ independent solutions to the above equations $q^\alpha(Q^\Gamma)$, $\alpha = 1, 2, \dots, M - N$ called physical variables, since they are invariant under the transformations generated by χ_i .

- (2) Define the induced structure $g^{\alpha\beta} \equiv G^{AB} \frac{\partial q^\alpha}{\partial Q^A} \frac{\partial q^\beta}{\partial Q^B}$ and realize the quadratic in momenta part of X as the conformal Laplace-Beltrami operator based on $g_{\alpha\beta}$.

We are now ready to proceed with the quantization of our systems (2.8-2.12), (2.24-2.28) respectively, in close analogy to the scheme above outlined.

3.1. The 2+1 Case

In order to realize the equivalent to step 1, we first define the quantum analogue of $\mathcal{H}_1(r) \approx 0$ as

$$\hat{\mathcal{H}}_1(r) \Phi = 0 \leftrightarrow -\rho(r) \left(\frac{\delta \Phi}{\delta \rho(r)} \right)' + \sigma'(r) \frac{\delta \Phi}{\delta \sigma(r)} - \chi(r) \left(\frac{\delta \Phi}{\delta \chi(r)} \right)' = 0. \quad (3.3)$$

As explained in the beginning of the section, the action of $\hat{\mathcal{H}}_1(r)$ on all smooth functionals is well defined, i.e., produces no $\delta(0)$'s. It can be proven that, in order for such a functional to be annihilated by this linear quantum operator, it must be scalar, i.e. have the form

$$\Phi = \int \rho(\tilde{r}) f\left(\Sigma^{(0)}, \Sigma^{(1)}, \dots, \Sigma^{(n)}, X^{(0)}, X^{(1)}, \dots, X^{(n)}\right) d\tilde{r}, \quad (3.4a)$$

$$\Sigma^{(0)} \equiv \sigma(\tilde{r}), \quad \Sigma^{(1)} \equiv \frac{\sigma'(\tilde{r})}{\rho(\tilde{r})}, \dots, \quad \Sigma^{(n)} \equiv \frac{1}{\rho(\tilde{r})} \frac{d}{d\tilde{r}} \left(\underbrace{\dots}_{n-1} \sigma(\tilde{r}) \right), \quad (3.4b)$$

$$X^{(0)} \equiv \frac{\chi(\tilde{r})}{\rho(\tilde{r})}, \quad X^{(1)} \equiv \frac{1}{\rho(\tilde{r})} \left(\frac{\chi(\tilde{r})}{\rho(\tilde{r})} \right)', \dots, \quad X^{(n)} \equiv \frac{1}{\rho(\tilde{r})} \frac{d}{d\tilde{r}} \left(\underbrace{\dots}_{n-1} \frac{\chi(\tilde{r})}{\rho(\tilde{r})} \right), \quad (3.4c)$$

where f is any function of its arguments. We note that, as it is discussed at the end of the previous section, $\frac{\sigma'}{\rho}$ is the only scalar first derivative of σ , and likewise for the higher derivatives. The proof of this statement is analogous to the corresponding result concerning full gravity [37]: consider an infinitesimal r -reparametrization $\tilde{r} = r - \eta(r)$. Under such a change, the left-hand side of (3.4a), being a number, must remain unaltered. If we calculate the change induced on the right-hand side we arrive at

$$0 = \int \left[f \delta\rho + \rho \frac{\delta f}{\delta\sigma} \delta\sigma + \rho \frac{\delta f}{\delta(\chi/\rho)} \delta \left(\frac{\chi}{\rho} \right) \right] dr = \int [\rho \hat{\mathcal{H}}_1(f)] \eta(r) dr, \quad (3.5)$$

where use of (2.15) and a partial integration has been made. Since this must hold for any $\eta(r)$, the result sought for is obtained.

We now turn to the second linear constraint and try to see what are the restrictions it brings into our space of state vectors. We define

$$\hat{\mathcal{H}}_2(r)\Phi = 0 \leftrightarrow \left(\frac{\delta\Phi}{\delta\chi(r)} \right)' = 0 \leftrightarrow \frac{\delta\Phi}{\delta\chi(r)} = k, \quad (3.6)$$

where k is any constant (with respect to r) independent of the basic fields and their derivatives, and Φ is given by (3.4a)–(3.4c). As we argued before, the functional derivative $\frac{\delta}{\delta\chi(r)}$ acting on $X^{(n)}$ will produce, upon partial integration of the n^{th} derivative of the Dirac delta function, a term proportional to $X^{(2n)}$. Since the arguments of f in (3.4a) reach only up to $X^{(n)}$, it is evident that f must be such that the coefficient of $X^{(2n)}$ vanishes; more precisely

$$\begin{aligned} \frac{\delta\Phi}{\delta\chi(r)} = k \leftrightarrow & \dots + \int \rho(\tilde{r}) \frac{\partial f}{\partial X^{(n)}(\tilde{r})} \frac{\delta X^{(n)}(\tilde{r})}{\delta\chi(r)} d\tilde{r} = k \leftrightarrow \\ & \dots + \int \rho(\tilde{r}) \frac{\partial f}{\partial X^{(n)}(\tilde{r})} \frac{1}{\rho(\tilde{r})} \frac{d}{d\tilde{r}} \left(\underbrace{\dots}_{n-1} \frac{\delta(r, \tilde{r})}{\rho(\tilde{r})} \right) d\tilde{r} = k \leftrightarrow \\ & \dots + (-1)^n \int \frac{\partial^2 f}{\partial (X^{(n)}(\tilde{r}))^2} X^{(2n)}(\tilde{r}) \delta(r, \tilde{r}) d\tilde{r} = k \leftrightarrow \\ & \dots + (-1)^n \frac{\partial^2 f}{\partial (X^{(n)})^2} X^{(2n)} = k. \end{aligned}$$

Thus, since all the terms hidden in ... do not involve $X^{(2n)}$, then $\frac{\partial^2 f}{\partial (X^{(n)})^2} = 0$ in order for this equation to have a possibility to be satisfied. Subsequently:

$$f = f_1 \left(\Sigma^{(0)}, \dots, \Sigma^{(n)}, X^{(0)}, \dots, X^{(n-1)} \right) X^{(n)} + f_2 \left(\Sigma^{(0)}, \dots, \Sigma^{(n)}, X^{(0)}, \dots, X^{(n-1)} \right).$$

Now, the term in Φ corresponding to f_1 is, up to a surface term, equivalent to a general term depending on $X^{(0)}, \dots, X^{(n-1)}$ only:

$$\Phi_1 = \int \rho(\tilde{r}) f_1 \frac{1}{\rho(\tilde{r})} \frac{d}{d\tilde{r}} X^{(n-1)} d\tilde{r},$$

which upon subtraction of the surface term

$$A = \int d\tilde{r} \frac{d}{d\tilde{r}} \left(\int dX^{(n-1)} f_1 \right),$$

produces a smooth functional with arguments up to $X^{(n-1)}$ only. Thus, only f_2 is important for the local part of Φ . The entire argument can be repeated successively for $n-1, n-2, \dots, 1$; Therefore all $X^{(n)}$'s are suppressed from f except when $n=0$. Thus, finally, upon inserting into (3.6) the resulting functional:

$$\Phi = \int \rho(\tilde{r}) h \left(\Sigma^{(0)}, \dots, \Sigma^{(n)}, X^{(0)} \right) d\tilde{r},$$

we obtain

$$\begin{aligned} \frac{\delta \Phi}{\delta \chi(r)} &= k \leftrightarrow \int \rho(\tilde{r}) \frac{\partial h}{\partial X^{(0)}} \frac{\delta(r, \tilde{r})}{\rho(\tilde{r})} d\tilde{r} = k \leftrightarrow \frac{\partial h}{\partial X^{(0)}} = k \leftrightarrow \\ h &= k \frac{\chi(r)}{\rho(r)} + L \left(\Sigma^{(0)}, \dots, \Sigma^{(n)} \right). \end{aligned}$$

We have thus reached the conclusion that the imposition of both linear quantum operators $\hat{\mathcal{H}}_1$ and $\hat{\mathcal{H}}_2$ dictates the form of the smooth functional to be:

$$\Phi = k \int d\tilde{r} \chi(\tilde{r}) + \int d\tilde{r} \rho(\tilde{r}) L \left(\Sigma^{(0)}, \dots, \Sigma^{(n)} \right). \quad (3.7)$$

We now try to realize step 2 of the program previously outlined. We have to define the equivalent of Kuchař's induced metric on the so far "physical" space of states described by (3.7). Let us start our investigation by considering **one** initial candidate of the above form. Then the induced metric will be given by

$$g^{\Phi\Phi} = G^{\alpha\beta} \frac{\delta \Phi}{\delta x^\alpha} \frac{\delta \Phi}{\delta x^\beta}, \quad (3.8)$$

where $(x^\alpha, x^\beta) = (\rho, \sigma, \chi)$ and $G^{\alpha\beta}$ is given by (2.10). Note that this metric is well defined since it contains only first functional derivatives of the state vectors, as opposed to any second order functional derivative operator that might have been considered as a quantum analogue of the kinetic part of \mathcal{H}_o . Nevertheless, $g^{\Phi\Phi}$ will turn out to be a local function and not a smooth functional. It is thus clear that, if we want the induced metric $g^{\Phi\Phi}$ to be composed out

of the “physical” states annihilated by \hat{H}_1, \hat{H}_2 , we must establish a correspondence between local functions and smooth functionals. A way to achieve this is to adopt the following ansatz:

Assumption: *We assume that, as part of the re-normalization procedure, we are permitted to map local functions to their corresponding smeared expressions e.g., $\chi(r) \leftrightarrow \int d\tilde{r} \chi(\tilde{r})$.*

Let us be more specific, concerning the meaning of the above Assumption. Let \mathcal{F} be the space which contains all local functions, and define the equivalence relations

$$\sim: \{f_1(r) \sim f_2(\tilde{r}), \tilde{r} = g(r)\}, \quad \approx: \{h_1(r) \approx h_2(\tilde{r}) \frac{d\tilde{r}}{dr}, \tilde{r} = g(r)\}, \quad (3.9)$$

for scalars and densities respectively.

Now let $\mathcal{F}_o = \{f \in \mathcal{F}, \text{ mod } (\sim, \approx)\}$ and \mathcal{F}_I the space of the smeared functionals. We define the one to one maps $\mathfrak{G}, \mathfrak{G}^{-1}$

$$\mathfrak{G}: \mathcal{F}_o \mapsto \mathcal{F}_I: \quad \chi(r) \mapsto \int \chi(\tilde{r}) d\tilde{r}, \quad \mathfrak{G}^{-1}: \mathcal{F}_I \mapsto \mathcal{F}_o: \quad \int \chi(\tilde{r}) d\tilde{r} \mapsto \chi(r). \quad (3.10)$$

The necessity to define the maps $\mathfrak{G}, \mathfrak{G}^{-1}$ on the equivalence classes and not on the individual functions, stems out of the fact that we are trying to develop a quantum theory of the geometries (2.6) and not of their coordinate representations. If we had tried to define the map \mathfrak{G} from the original space \mathcal{F} to \mathcal{F}_I we would end up with states which would not be invariant under spatial coordinate transformations (\mathbf{r} -reparametrizations). Indeed, one can make a correspondence between local functions and smeared expressions, but smeared expressions *must* contain another arbitrary smearing function, say $s(r)$. Then the map between functions and smeared expressions is one to one (as is also the above map) and is given by multiplying by $s(r)$ and integrating over r ; while the inverse map is given by varying w.r.t. $s(r)$. However, this would be in the opposite direction from that which led us to the states (3.4a) by imposition of the linear operator constraint. As an example consider the action of this operator on one particular case of the states (3.4a), containing the structure $s(r)$:

$$\hat{\mathcal{H}}_1(r) \int s(\tilde{r}) \rho(\tilde{r}) \sigma(\tilde{r}) d\tilde{r} = -s'(r) \rho(r) \sigma(r) \neq 0 \quad \text{for arbitrary } s(r), \quad (3.11)$$

$$\hat{\mathcal{H}}_2(r) \int s(\tilde{r}) \chi(\tilde{r}) d\tilde{r} = s'(r) \neq 0 \quad \text{for arbitrary } s(r). \quad (3.12)$$

Thus, every foreign to the geometry structure $s(r)$ is not allowed to enter the physical states.

Now, after the correspondence has been established, we can come to the basic property the induced metric must have. In the case of finite degrees of freedom the induced metric depends, up to a conformal scaling, on the physical coordinates q^α by virtue of (3.2). In our case, due to the dependence of the configuration variables on the radial coordinate r , the above property is not automatically satisfied; e.g. the functional derivative $\frac{\delta}{\delta\sigma(r)}$ acting on $\Sigma^{(n)}$ will produce, upon partial integration of the n^{th} derivative of the Dirac delta function, a term proportional to $\Sigma^{(2n)}$. Therefore, since L in (3.7) contains derivatives of $\sigma(r)$ up to $\Sigma^{(n)}$, the above mentioned property must be *enforced*. The need for this can also be traced to the substantially different first Poisson bracket in (2.13), which signals a non trivial mixing between the dynamical evolution generator \mathcal{H}_o and the linear generators $\mathcal{H}_1, \mathcal{H}_2$.

Thus, according to the above reasoning, in order to proceed with the generalization of Kuchař's method, we have to demand that:

Requirement: $L(\Sigma^{(0)}, \dots, \Sigma^{(n)})$ must be such that $g^{\Phi\Phi}$ becomes a general function, say $F(k\chi(r) + \rho(r)L(\Sigma^{(0)}, \dots, \Sigma^{(n)}))$ of the integrand of Φ , so that it can be considered a function of this state: $g^{\Phi\Phi} \stackrel{\text{Assumption}}{\equiv} F(k \int \chi(\tilde{r})d\tilde{r} + \int \rho(\tilde{r})L(\Sigma^{(0)}, \dots, \Sigma^{(n)})d\tilde{r}) = F(\Phi)$.

At this point, we must emphasize that the application of the **Requirement** in the subsequent development of our quantum theory will result in very severe restrictions on the form of (3.4a). Essentially, all higher derivatives of $\sigma(r)$ (i.e. $\Sigma^{(2)} \dots \Sigma^{(n)}$) are eliminated from Φ . This might, at first sight, strike as odd; indeed, the common belief is that all the derivatives of the configuration variables should enter the physical states. However, before the imposition of both the linear *and* the quadratic constrains there are no truly physical states. Thus, no physical states are lost by the imposition of the **Requirement**; ultimately the only true physical states are the solutions to the Wheeler–DeWitt equation, which will be constructed at a later stage.

Having clarified the way in which we view the **Assumption** and **Requirement** above, we now proceed to the restrictions implied by their use.

In the case of finite degrees of freedom the induced metric depends, up to a conformal scaling, on the physical coordinates q^α by virtue of (3.2). The reason for this not being the case for our system might be traced to the substantially different first Poisson bracket in (2.13). A first consequence of the requirement that $g^{\Phi\Phi} = F(k\chi(r) + \rho(r)L(\Sigma^{(0)}, \dots, \Sigma^{(n)}))$ is the vanishing of k . This follows from (a) the property that $g^{\Phi\Phi}$ is homogenous in the functional derivative $\frac{\delta}{\delta\chi(r)}$, (b) that $G^{\alpha\beta}$ in (2.10) does not contain any $\chi(r)$; namely

$$g^{\Phi\Phi} = \dots + G^{33} \frac{\delta\Phi}{\delta\chi(r)} \frac{\delta\Phi}{\delta\chi(r)} \leftrightarrow g^{\Phi\Phi} = \underbrace{\dots}_{\text{no } \chi} + \frac{\rho}{\sigma^2} k^2 \equiv F(k\chi + \rho L).$$

Since \dots are terms not involving $\chi(r)$, the final identification is possible iff $k = 0$. Thus, Φ is reduced to:

$$\Phi = \int d\tilde{r} \rho(\tilde{r}) L(\Sigma^{(0)}, \dots, \Sigma^{(n)}). \tag{3.13}$$

We now turn to the degree of derivatives (n) of $\sigma(r)$. The situation is similar to the corresponding case with $X^{(n)}$ considered before; again the functional derivative $\frac{\delta}{\delta\sigma(r)}$ acting on Φ will bring a maximum term $\Sigma^{(2n)}$ while $\frac{\delta}{\delta\rho(r)}$ a corresponding term $\Sigma^{(2n-1)}$. More precisely

$$g^{\Phi\Phi} = \dots + 2G^{12} \frac{\delta\Phi}{\delta\rho(r)} \frac{\delta\Phi}{\delta\sigma(r)}.$$

Where the functional derivatives are:

$$\begin{aligned}
 \frac{\delta\Phi}{\delta\sigma} &= \dots + \int \rho \frac{\partial L}{\partial \Sigma^{(n)}} \frac{\delta \Sigma^{(n)}}{\delta \sigma} d\tilde{r} = \dots + \int \rho \frac{\partial L}{\partial \Sigma^{(n)}} \frac{1}{\rho} \frac{d}{d\tilde{r}} \left(\underbrace{\dots}_{n-1} \delta(r, \tilde{r}) \right) d\tilde{r} = \\
 &= \dots - \int \frac{d}{d\tilde{r}} \left(\frac{\partial L}{\partial \Sigma^{(n)}} \right) \frac{1}{\rho} \frac{d}{d\tilde{r}} \left(\underbrace{\dots}_{n-2} \delta(r, \tilde{r}) \right) d\tilde{r} = \\
 &= \dots - \int \rho \frac{\partial^2 L}{\partial (\Sigma^{(n)})^2} \Sigma^{(n+1)} \frac{1}{\rho} \frac{d}{d\tilde{r}} \left(\underbrace{\dots}_{n-2} \delta(r, \tilde{r}) \right) d\tilde{r} = \\
 &= \dots + (-1)^n \int \rho(\tilde{r}) \frac{\partial^2 L}{\partial (\Sigma^{(n)})^2} \Sigma^{(2n)} \delta(r, \tilde{r}) d\tilde{r} = \\
 &= \dots + (-1)^n \rho \frac{\partial^2 L}{\partial (\Sigma^{(n)})^2} \Sigma^{(2n)},
 \end{aligned}$$

and

$$\begin{aligned}
 \frac{\delta\Phi}{\delta\rho} &= \dots + \int \rho \frac{\partial L}{\partial \Sigma^{(n)}} \frac{\delta \Sigma^{(n)}}{\delta \rho} d\tilde{r} = \dots + \int \rho \frac{\partial L}{\partial \Sigma^{(n)}} \frac{1}{\rho} \frac{d}{d\tilde{r}} \left(\underbrace{\dots}_{n-2} - \frac{\delta(r, \tilde{r})}{\rho(\tilde{r})^2} \sigma'(\tilde{r}) \right) d\tilde{r} = \\
 &= \dots + \int \rho \frac{\partial L}{\partial \Sigma^{(n)}} \frac{1}{\rho} \frac{d}{d\tilde{r}} \left(\underbrace{\dots}_{n-2} - \frac{\delta(r, \tilde{r})}{\rho(\tilde{r})} \Sigma^{(1)} \right) d\tilde{r} \\
 &= \dots - \int \frac{d}{d\tilde{r}} \left(\frac{\partial L}{\partial \Sigma^{(n)}} \right) \frac{1}{\rho} \frac{d}{d\tilde{r}} \left(\underbrace{\dots}_{n-3} - \frac{\delta(r, \tilde{r})}{\rho(\tilde{r})} \Sigma^{(1)} \right) d\tilde{r} = \\
 &= \dots + (-1)^{n-1} \int \frac{\partial^2 L}{\partial (\Sigma^{(n)})^2} \Sigma^{(2n-1)} \Sigma^{(1)} \delta(r, \tilde{r}) d\tilde{r} = \\
 &= \dots + (-1)^{n-1} \frac{\partial^2 L}{\partial (\Sigma^{(n)})^2} \Sigma^{(2n-1)} \Sigma^{(1)}.
 \end{aligned}$$

Therefore

$$g^{\Phi\Phi} = \dots - 2\sigma (-1)^{2n-1} \left(\frac{\partial^2 L}{\partial (\Sigma^{(n)})^2} \right)^2 \Sigma^{(1)} \Sigma^{(2n-1)} \Sigma^{(2n)},$$

where the ... stand for all other terms, not involving $\Sigma^{(2n)}$. Now, according to the aforementioned **Requirement** we need this to be a general function, say $F(\rho L)$, and for this to happen the coefficient of $\Sigma^{(2n)}$ must vanish, i.e.

$$\frac{\partial^2 L}{\partial (\Sigma^{(n)})^2} = 0 \Leftrightarrow L = L_1 \left(\Sigma^{(0)}, \dots, \Sigma^{(n-1)} \right) \Sigma^{(n)} + L_2 \left(\Sigma^{(0)}, \dots, \Sigma^{(n-1)} \right).$$

Again the term of Φ corresponding to L_1 is, up to a total derivative, equivalent to a local smooth functional containing $\Sigma^{(0)}, \dots, \Sigma^{(n-1)}$. The argument can be repeated for $(n-1), (n-2), \dots, 2$.

The case $n = 1$ needs separate consideration since, upon elimination of the linear in $\Sigma^{(2)}$ term we are left with a local function of $\Sigma^{(1)}$, and thus the possibility arises to meet the **Requirement** by solving a differential equation for L . In more detail, if

$$\Phi \equiv \int \rho(\tilde{r}) L(\sigma, \Sigma^{(1)}) d\tilde{r}, \quad (3.14)$$

$g^{\Phi\Phi}$ reads

$$\begin{aligned} g^{\Phi\Phi} = & -\rho \left(L - \Sigma^{(1)} \frac{\partial L}{\partial \Sigma^{(1)}} \right) \left[L - \Sigma^{(1)} \frac{\partial L}{\partial \Sigma^{(1)}} + 2\sigma \left(\frac{\partial L}{\partial \sigma} - \Sigma^{(1)} \frac{\partial^2 L}{\partial \sigma \partial \Sigma^{(1)}} \right) \right] + \\ & + 2\rho\sigma \left(L - \Sigma^{(1)} \frac{\partial L}{\partial \Sigma^{(1)}} \right) \frac{\partial^2 L}{\partial (\Sigma^{(1)})^2} \Sigma^{(2)}. \end{aligned} \quad (3.15)$$

Through the definition

$$H \equiv L - \Sigma^{(1)} \frac{\partial L}{\partial \Sigma^{(1)}}, \quad (3.16)$$

we obtain

$$\begin{aligned} \frac{\partial H}{\partial \sigma} &= \frac{\partial L}{\partial \sigma} - \Sigma^{(1)} \frac{\partial^2 L}{\partial \sigma \partial \Sigma^{(1)}}, \\ \frac{\partial H}{\partial \Sigma^{(1)}} &= -\Sigma^{(1)} \frac{\partial^2 L}{\partial (\Sigma^{(1)})^2}. \end{aligned}$$

Thus $g^{\Phi\Phi}$ assumes the form

$$g^{\Phi\Phi} = -\rho \left(H^2 + 2\sigma H \frac{\partial H}{\partial \sigma} + \frac{2\sigma}{\Sigma^{(1)}} H \frac{\partial H}{\partial \Sigma^{(1)}} \Sigma^{(2)} \right),$$

which upon addition, by virtue of the **Assumption**, of the surface term

$$A = \frac{d}{dr} \left(\int \frac{2\sigma}{\Sigma^{(1)}} H \frac{\partial H}{\partial \Sigma^{(1)}} d\Sigma^{(1)} \right),$$

gives

$$g^{\Phi\Phi} = -\rho \left(H^2 + 2\sigma H \frac{\partial H}{\partial \sigma} - \Sigma^{(1)} \frac{\partial}{\partial \sigma} \int \frac{2\sigma}{\Sigma^{(1)}} H \frac{\partial H}{\partial \Sigma^{(1)}} d\Sigma^{(1)} \right). \quad (3.17)$$

Since in the last expression we have only a multiplicative $\rho(r)$, it is obvious that the **Requirement**

$$g^{\Phi\Phi} = F(\rho L),$$

can be satisfied only by

$$g^{\Phi\Phi} = -\kappa \rho L, \quad (3.18)$$

with $g^{\Phi\Phi}$ given by (3.17). Upon differentiation of this equation with respect to $\Sigma^{(1)}$ we get

$$-\frac{\partial}{\partial \sigma} \int \frac{2\sigma}{\Sigma^{(1)}} H \frac{\partial H}{\partial \Sigma^{(1)}} = \kappa \frac{\partial L}{\partial \Sigma^{(1)}}.$$

Multiplying the last expression by $\Sigma^{(1)}$ and subtracting it from (3.18) we end up with the autonomous necessary condition for $H(\sigma, \Sigma^{(1)})$:

$$H \left(H + 2\sigma \frac{\partial H}{\partial \sigma} - \kappa \right) = 0,$$

where the definition of H was also used. The above equation can be readily integrated giving

$$\begin{aligned} H &= 0, \\ H &= \kappa + \frac{a(\Sigma^{(1)})}{\sqrt{\sigma}}, \end{aligned}$$

where $a(\Sigma^{(1)})$ is an arbitrary function of its argument. The first possibility gives according to the definition of H , $L = \lambda \Sigma^{(1)}$ which, however, contributes to Φ a surface term, and can thus be ignored. Inserting the second solution into the definition of H we construct a partial differential equation for L , namely

$$L - \Sigma^{(1)} \frac{\partial L}{\partial \Sigma^{(1)}} = \kappa + \frac{a(\Sigma^{(1)})}{\sqrt{\sigma}},$$

which upon integration gives

$$L = \kappa - \frac{\Sigma^{(1)}}{\sqrt{\sigma}} \int \frac{a(\Sigma^{(1)})}{\Sigma^{(1)2}} d\Sigma^{(1)} + c_1(\sigma) \Sigma^{(1)}.$$

Since this form of L emerged as a necessary condition, it must be inserted (along with H) in (3.18). The result is that $c_1(\sigma) = 0$. Thus L reads

$$L = \kappa - \frac{\Sigma^{(1)}}{\sqrt{\sigma}} \int \frac{a(\Sigma^{(1)})}{\Sigma^{(1)2}} d\Sigma^{(1)}. \quad (3.19)$$

By assuming that the $\Sigma^{(1)}$ -dependent part of L equals $b(\Sigma^{(1)})$, i.e.

$$-\Sigma^{(1)} \int \frac{a(\Sigma^{(1)})}{\Sigma^{(1)2}} d\Sigma^{(1)} = b(\Sigma^{(1)}),$$

we get, upon a double differentiation with respect to $\Sigma^{(1)}$, the ordinary differential equation

$$-\frac{a'(\Sigma^{(1)})}{\Sigma^{(1)}} = b''(\Sigma^{(1)}),$$

with solution

$$a(\Sigma^{(1)}) = b(\Sigma^{(1)}) + \kappa_1 - \Sigma^{(1)} b'(\Sigma^{(1)}),$$

where κ_1 is a constant. Substituting this equation into (3.19) and performing a partial integration we end up with

$$L = \kappa + \frac{\kappa_1}{\sqrt{\sigma}} + \frac{b(\Sigma^{(1)})}{\sqrt{\sigma}}. \quad (3.20)$$

κ , κ_1 and $b(\Sigma^{(1)})$ being completely arbitrary and to our disposal; the two simpler choices $\kappa_1 = 0$, $b(\Sigma^{(1)}) = 0$ and $\kappa = 0$, $b(\Sigma^{(1)}) = 0$ lead respectively to the following two basic local smooth functionals:

$$q^1 = \int d\tilde{r} \rho(\tilde{r}), \quad q^2 = \int d\tilde{r} \frac{\rho(\tilde{r})}{\sqrt{\sigma(\tilde{r})}}. \quad (3.21)$$

The next simplest choice $\kappa = 0$, $\kappa_1 = 0$ and $b(\Sigma^{(1)})$ arbitrary leads to a generic $q^3 = \int d\tilde{r} \rho(\tilde{r}) \frac{b(\Sigma^{(1)})}{\sqrt{\sigma(\tilde{r})}}$. However, it can be proven that, for any choice of $b(\Sigma^{(1)})$, the corresponding renormalized induced metric

$$g^{AB} = G^{\alpha\beta} \frac{\delta q^A}{\delta x^\alpha} \frac{\delta q^B}{\delta x^\beta} \quad \text{where } A, B = 1, 2, 3$$

is singular. The calculation of g^{AB} gives:

$$\begin{aligned} g^{11} &= G^{\alpha\beta} \frac{\delta q^1}{\delta x^\alpha} \frac{\delta q^1}{\delta x^\beta} = -\rho \stackrel{\text{Assumption}}{\iff} g_{ren}^{11} = -q^1, \\ g^{12} &= G^{\alpha\beta} \frac{\delta q^1}{\delta x^\alpha} \frac{\delta q^2}{\delta x^\beta} = -\frac{\rho}{2\sqrt{\sigma}} \stackrel{\text{Assumption}}{\iff} g_{ren}^{12} = -\frac{q^2}{2}, \\ g^{22} &= G^{\alpha\beta} \frac{\delta q^2}{\delta x^\alpha} \frac{\delta q^2}{\delta x^\beta} = 0 \stackrel{\text{Assumption}}{\iff} g_{ren}^{22} = 0, \\ g^{13} &= G^{\alpha\beta} \frac{\delta q^1}{\delta x^\alpha} \frac{\delta q^3}{\delta x^\beta} = \rho \left(-\frac{b}{2\sqrt{\sigma}} + \frac{\Sigma^{(1)}}{2\sqrt{\sigma}} b' + \sqrt{\sigma} \Sigma^{(2)} b'' \right) \stackrel{\text{Assumption}}{\iff} \\ g_{ren}^{13} &= \int dr \rho \left(-\frac{b}{2\sqrt{\sigma}} + \frac{\Sigma^{(1)}}{2\sqrt{\sigma}} b' + \sqrt{\sigma} \Sigma^{(2)} b'' \right) - \int dr \frac{d}{dr} \left(\int d\Sigma^{(1)} \sqrt{\sigma} b'' \right) = \\ &= -\int dr \rho \frac{b}{2\sqrt{\sigma}} = -\frac{q^3}{2}, \\ g^{23} &= G^{\alpha\beta} \frac{\delta q^2}{\delta x^\alpha} \frac{\delta q^3}{\delta x^\beta} = \rho \Sigma^{(2)} b'' = \frac{d}{dr} b' \stackrel{\text{Assumption}}{\iff} g_{ren}^{23} = 0, \\ g^{33} &= G^{\alpha\beta} \frac{\delta q^3}{\delta x^\alpha} \frac{\delta q^3}{\delta x^\beta} = 2\rho (b - \Sigma^{(1)} b') \Sigma^{(2)} b'' \stackrel{\text{Assumption}}{\iff} \\ g_{ren}^{33} &= 2 \int dr \rho (b - \Sigma^{(1)} b') \Sigma^{(2)} b'' - 2 \int dr \frac{d}{dr} \left[\int d\Sigma^{(1)} (b - \Sigma^{(1)} b') b'' \right] = 0, \end{aligned}$$

where by $'$ we denote differentiation with respect to $\Sigma^{(1)}$. Thus the renormalized induced metric reads

$$g_{ren}^{AB} = -\frac{1}{2} \begin{pmatrix} 2q^1 & q^2 & q^3 \\ q^2 & 0 & 0 \\ q^3 & 0 & 0 \end{pmatrix}.$$

Effecting the transformation $(\tilde{q}^1, \tilde{q}^2, \tilde{q}^3) = (q^1, q^2, f(\frac{q^3}{q^2}))$ we bring g_{ren}^{AB} into a manifestly degenerate form:

$$g_{ren}^{AB} = -\frac{1}{2} \begin{pmatrix} 2q^1 & q^2 & 0 \\ q^2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

So, it seems as though the relevant part of the renormalized metric is described by the upper 2×2 block. This fact is consistent with the form of the renormalized potential $V = 2\Lambda q^1$ which indeed does not contain any $\Sigma^{(1)}$ term.

However, this is not the end of our investigation for a suitable space of state vectors: the argument leading to q^1, q^2 depends upon the original choice of one initial candidate smooth scalar functional (3.14); to complete the search we must close the circle by starting with the two already secured smooth functionals (q^1, q^2), and a third of the general form

$$q^3 = \int dr \rho L(\Sigma^{(1)}),$$

since the σ dependence has already been fixed to either 1 or $\frac{1}{\sqrt{\sigma}}$. The calculation of the, related to q^3 , components of the induced metric g^{AB} gives:

$$\begin{aligned} g^{13} &= \rho \left(-L + \Sigma^{(1)} L' + \sigma \Sigma^{(2)} L'' \right) \stackrel{Assumption}{\iff} \\ g_{ren}^{13} &= \int dr \rho \left(-L + \Sigma^{(1)} L' + \sigma \Sigma^{(2)} L'' \right) - \int dr \frac{d}{dr} \left(\int d\Sigma^{(1)} \sigma L'' \right) = - \int dr \rho L = \\ &= -q^3, \\ g^{23} &= \rho \left(-\frac{L}{2\sqrt{\sigma}} + \frac{\Sigma^{(1)}}{2\sqrt{\sigma}} L' + \sqrt{\sigma} \Sigma^{(2)} L'' \right) \stackrel{Assumption}{\iff} \\ g_{ren}^{23} &= \int dr \rho \left(-\frac{L}{2\sqrt{\sigma}} + \frac{\Sigma^{(1)}}{2\sqrt{\sigma}} L' + \sqrt{\sigma} \Sigma^{(2)} L'' \right) - \int dr \frac{d}{dr} \left(\int d\Sigma^{(1)} \sqrt{\sigma} L'' \right) = \\ &= - \int dr \rho \frac{L}{2\sqrt{\sigma}} \stackrel{Assumption}{=} -\frac{q^2 q^3}{2q^1}, \\ g^{33} &= -\rho \left(L - \Sigma^{(1)} L' \right)^2 + 2\rho \sigma \left(L - \Sigma^{(1)} L' \right) \Sigma^{(2)} L''. \end{aligned}$$

By following the procedure presented between (3.15) and (3.17) we end up with the expression

$$g^{33} = -\rho \left[\left(L - \Sigma^{(1)} L' \right)^2 - \Sigma^{(1)} \int \frac{d\Sigma^{(1)}}{\Sigma^{(1)}} \frac{\partial}{\partial \Sigma^{(1)}} \left(L - \Sigma^{(1)} L' \right)^2 \right],$$

which is a generic function of $\Sigma^{(1)}$ and therefore, also of L : let this function be parameterized as $L(\Sigma^{(1)})^2 - \frac{1}{H'(L(\Sigma^{(1)}))}$; this ‘‘peculiar’’ parametrization of the arbitrariness in $L(\Sigma^{(1)})$ has been chosen in order to facilitate the subsequent proof that this freedom is a pure general

coordinate transformation (gct) of the induced renormalized metric. Indeed, let us first take the simple choice $L(\Sigma^{(1)}) \equiv \Sigma^{(1)2}$ which results in the renormalized metric

$$g_{ren}^{AB} = -\frac{1}{2} \begin{pmatrix} 2q^1 & q^2 & 2q^3 \\ q^2 & 0 & \frac{q^2 q^3}{q^1} \\ q^3 & \frac{q^2 q^3}{q^1} & -\frac{2(q^3)^2}{3q^1} \end{pmatrix}, \quad g_{ABren} = \frac{1}{2} \begin{pmatrix} \frac{3}{2q^1} & -\frac{4}{q^2} & -\frac{3}{2q^3} \\ -\frac{4}{q^2} & \frac{8q^1}{(q^2)^2} & 0 \\ -\frac{3}{2q^3} & 0 & \frac{3q^1}{2(q^3)^2} \end{pmatrix}. \quad (3.22)$$

Considering a generic $L(\Sigma^{(1)})$, i.e. $x^3 = \int dr \rho L(\Sigma^{(1)})$ (along with (3.21)) we are led to

$$g^{13} = -x^3, \quad g^{23} = -\frac{q^2 x^3}{2q^1},$$

and

$$g^{33} = -\rho \left[L^2 - \frac{1}{H'(L)} \right] = -\frac{(\rho L)^2}{\rho} + \frac{\rho}{H'(\frac{\rho L}{\rho})} \stackrel{\text{Assumption}}{\iff} g_{ren}^{33} = -\frac{(x^3)^2}{q^1} + \frac{q^1}{H'(\frac{x^3}{q^1})}.$$

Remarkably enough, the new induced renormalized metric can be put in gct equivalence with the metric (3.22) through the transformation

$$(q^1, q^2, x^3) = (q^1, q^2, q^1 H^{-1}(\frac{\sqrt{3}}{2} \ln \frac{q^3}{q^1})),$$

with H^{-1} denoting the function inverse to H .

We can therefore consider, without loss of generality, the reduced renormalized manifold to be parameterized by the following three smooth scalar functionals:

$$q^1 = \int d\tilde{r} \rho(\tilde{r}), \quad q^2 = \int d\tilde{r} \frac{\rho(\tilde{r})}{\sqrt{\sigma(\tilde{r})}}, \quad q^3 = \int d\tilde{r} \frac{\sigma'(\tilde{r})^2}{\rho(\tilde{r})}. \quad (3.23)$$

Any other functional, say $q^4 = \int d\tilde{r} \rho(\tilde{r}) L(\sigma(\tilde{r}), \Sigma^{(1)}(\tilde{r}))$, can be considered as a function of q^1, q^2, q^3 ; indeed, since the scalar functions appearing in the integrands of q^2, q^3 form a base in the space of $\sigma, \Sigma^{(1)}$, we can express the generic L in q^4 as $F(\frac{\rho}{\rho\sqrt{\sigma}}, \frac{\rho\Sigma^{(1)2}}{\rho})$, which (through the **Assumption**) gives $q^4 = q^1 F(\frac{q^2}{q^1}, \frac{q^3}{q^1})$.

The geometry of this space is described by the induced renormalized metric (3.22). Any function $\Psi(q^1, q^2, q^3)$ on this manifold is of course annihilated by the quantum linear constraints, i.e.

$$\hat{\mathcal{H}}_1 \Psi(q^1, q^2, q^3) = \frac{\partial \Psi(q^1, q^2, q^3)}{\partial q^1} \hat{\mathcal{H}}_1 q^1 + \frac{\partial \Psi(q^1, q^2, q^3)}{\partial q^2} \hat{\mathcal{H}}_1 q^2 + \frac{\partial \Psi(q^1, q^2, q^3)}{\partial q^3} \hat{\mathcal{H}}_1 q^3 = 0,$$

$$\hat{\mathcal{H}}_2 \Psi(q^1, q^2, q^3) = \frac{\partial \Psi(q^1, q^2, q^3)}{\partial q^1} \hat{\mathcal{H}}_2 q^1 + \frac{\partial \Psi(q^1, q^2, q^3)}{\partial q^2} \hat{\mathcal{H}}_2 q^2 + \frac{\partial \Psi(q^1, q^2, q^3)}{\partial q^3} \hat{\mathcal{H}}_2 q^3 = 0,$$

since the derivatives with respect to r are transparent to the partial derivatives of Ψ (which are, just like the q^A 's, r-numbers).

The final restriction on the form of Ψ will be obtained by the imposition of the quantum analog of the quadratic constraint \mathcal{H}_o . According to the above exposition we postulate that the quantum gravity of the geometries given by (2.6), (2.7) will be described by

$$\hat{\mathcal{H}}_o \Psi \equiv \left[-\frac{1}{2} \square_c + 2 \Lambda q^1\right] \Psi(q^1, q^2, q^3) = 0, \quad (3.24)$$

with

$$\square_c = \square + \frac{d-2}{4(d-1)} R, \quad (3.25)$$

being the conformal Laplacian based on g_{ABren} , R the Ricci scalar, and d the dimensions of g_{ABren} . Metric (3.22) is conformally flat with Ricci scalar $R = \frac{1}{2q^1}$, its dimension is $d = 3$, and therefore the Wheeler–DeWitt equation becomes

$$\begin{aligned} & 2q^1 \Lambda \Psi(q^1, q^2, q^3) - \frac{1}{32q^1} \Psi(q^1, q^2, q^3) + \frac{q^3}{12q^1} \frac{\partial \Psi(q^1, q^2, q^3)}{\partial q^3} + \frac{q^2}{8q^1} \frac{\partial \Psi(q^1, q^2, q^3)}{\partial q^2} + \\ & \frac{3}{4} \frac{\partial \Psi(q^1, q^2, q^3)}{\partial q^1} + \frac{q^2 q^3}{2q^1} \frac{\partial^2 \Psi(q^1, q^2, q^3)}{\partial q^2 \partial q^3} + q^3 \frac{\partial^2 \Psi(q^1, q^2, q^3)}{\partial q^1 \partial q^3} + \frac{q^2}{2} \frac{\partial^2 \Psi(q^1, q^2, q^3)}{\partial q^1 \partial q^2} + \\ & \frac{q^1}{2} \frac{\partial^2 \Psi(q^1, q^2, q^3)}{\partial (q^1)^2} - \frac{(q^3)^2}{6q^1} \frac{\partial^2 \Psi(q^1, q^2, q^3)}{\partial (q^3)^2} = 0. \end{aligned} \quad (3.26)$$

The change to new coordinates (x^1, x^2, x^3) described by

$$(q^1, q^2, q^3) = \left(e^{x^2}, e^{\frac{1}{2}(x^1+x^2)}, e^{x^2 + \frac{2}{\sqrt{3}}x^3} \right),$$

transforms the metric into the manifestly conformally flat form $diag\{e^{x^2}, -e^{x^2}, -e^{x^2}\}$ and brings (3.26) into the form

$$\begin{aligned} & 2e^{2x^2} \Lambda \Psi(x^1, x^2, x^3) - \frac{1}{32} \Psi(x^1, x^2, x^3) + \frac{1}{4} \frac{\partial \Psi(x^1, x^2, x^3)}{\partial x^2} - \frac{1}{2} \frac{\partial^2 \Psi(x^1, x^2, x^3)}{\partial (x^1)^2} + \\ & \frac{1}{2} \frac{\partial^2 \Psi(x^1, x^2, x^3)}{\partial (x^2)^2} + \frac{1}{2} \frac{\partial^2 \Psi(x^1, x^2, x^3)}{\partial (x^3)^2} = 0. \end{aligned} \quad (3.27)$$

This equation is readily solved by the method of separation of variables: assuming $\Psi(x^1, x^2, x^3) = X^1(x^1) X^2(x^2) X^3(x^3)$ and dividing (3.27) by Ψ we get the three ordinary differential equations:

$$\frac{1}{2X^1(x^1)} \frac{d^2 X^1(x^1)}{d(x^1)^2} + \frac{1}{32} = m + n, \quad (3.28a)$$

$$\frac{1}{2X^2(x^2)} \frac{d^2 X^2(x^2)}{d(x^2)^2} + \frac{1}{4X^2(x^2)} \frac{dX^2(x^2)}{dx^2} + 2e^{2x^2} \Lambda = n, \quad (3.28b)$$

$$\frac{1}{2X^3(x^3)} \frac{d^2 X^3(x^3)}{d(x^3)^2} = m, \quad (3.28c)$$

where m and n are separation constants. Their solutions are:

$$\begin{aligned} X^1(x^1) &= c_1 e^{\frac{1}{4}\sqrt{32m+32n-1} x^1} + c_2 e^{-\frac{1}{4}\sqrt{32m+32n-1} x^1}, \\ X^2(x^2) &= c_3 e^{-x^2/4} \Gamma\left(1 - \frac{1}{4}\sqrt{32n+1}\right) J_{-\frac{1}{4}\sqrt{32n+1}}\left(2e^{x^2}\sqrt{\Lambda}\right) + \\ &+ c_4 e^{-x^2/4} \Gamma\left(1 + \frac{1}{4}\sqrt{32n+1}\right) J_{\frac{1}{4}\sqrt{32n+1}}\left(2e^{x^2}\sqrt{\Lambda}\right) \\ X^3(x^3) &= c_5 e^{\sqrt{2m} x^3} + c_6 e^{-\sqrt{2m} x^3}, \end{aligned}$$

where $\Gamma\left(1 \pm \frac{1}{4}\sqrt{32n+1}\right)$ are Gamma functions and $J_{\pm\frac{1}{4}\sqrt{32n+1}}\left(2e^{x^2}\sqrt{\Lambda}\right)$ Bessel functions of the first kind and of non-integer order.

3.2. The 3+1 Case

The quantization of the Hamiltonian system (2.25a), (2.25b) proceeds in exact analogy to the 2+1 case described above. In order to realize the equivalent to step 1, we first define the quantum analogue of $\mathcal{H}_1(r) \approx 0$ as

$$\hat{\mathcal{H}}_1(r)\Phi = 0 \leftrightarrow -\gamma(r) \left(\frac{\delta\Phi}{\delta\gamma(r)}\right)' + \psi'(r) \frac{\delta\Phi}{\delta\psi(r)} = 0. \quad (3.29)$$

The general solution to this equation is given by the following family of smooth scalar functionals

$$\Phi = \int \gamma(\tilde{r}) L\left(\Psi^{(0)}, \Psi^{(1)}, \dots, \Psi^{(n)}\right) d\tilde{r}, \quad (3.30a)$$

$$\Psi^{(0)} \equiv \psi(\tilde{r}), \quad \Psi^{(1)} \equiv \frac{\psi'(\tilde{r})}{\gamma(\tilde{r})}, \dots, \Psi^{(n)} \equiv \frac{1}{\gamma(\tilde{r})} \frac{d}{d\tilde{r}} \left(\underbrace{\dots}_{n-1} \psi(\tilde{r}) \right). \quad (3.30b)$$

In trying to realize step 2 of the program, we have to define the equivalent of Kuchař's induced metric on the so far space of "physical" states Φ described by (3.30a) which are the analogues, in our case, of Kuchař's physical variables q^α . The imposition of the **Assumption** and the **Requirement** described in detail in the previous section results in the following three smooth scalar functionals which parametrize the reduced renormalized manifold:

$$y^1 = \int \gamma(\tilde{r}) d\tilde{r}, \quad y^2 = \int \gamma(\tilde{r}) \psi(\tilde{r})^2 d\tilde{r}, \quad y^3 = \int \gamma(\tilde{r}) L(\Psi^{(1)}) d\tilde{r}. \quad (3.31)$$

Any other functional, say $y^4 = \int d\tilde{r} \gamma(\tilde{r}) K[\psi(\tilde{r}), \Psi^{(1)}(\tilde{r})]$, can be considered as a function of y^1, y^2, y^3 ; indeed, since the scalar functions appearing in the integrands of y^2, y^3 form a base in the space spanned by $\psi, \Psi^{(1)}(r)$, we can express the generic K in y^4 as $K\left[\sqrt{\frac{\gamma\psi^2}{\gamma}}, \Psi^{(1)}\right]$, which (through the **Assumption**) gives $y^4 = y^1 K\left[\sqrt{\frac{y^2}{y^1}}, L^{-1}\left(\frac{y^3}{y^1}\right)\right]$.

The geometry of this space is described by the induced renormalized metric

$$g_{ren}^{AB}(y^1, y^2, y^3) = -\frac{1}{4} \begin{pmatrix} -\frac{(y^1)^2}{y^2} & y^1 & -\frac{y^1 y^3}{y^2} \\ y^1 & 3y^2 & y^3 \\ -\frac{y^1 y^3}{y^2} & y^3 & -\frac{(y^3)^2}{y^2} + \frac{4(y^1)^2 F\left(\frac{y^3}{y^1}\right)^2}{3y^2 F' \left[F\left(\frac{y^3}{y^1}\right)\right]^2} \end{pmatrix},$$

$$g_{ABren}(y^1, y^2, y^3) = \begin{pmatrix} \frac{3y^2}{(y^1)^4} \left((y^1)^2 - \frac{(y^3)^2 F' \left[F\left(\frac{y^3}{y^1}\right)\right]^2}{F\left(\frac{y^3}{y^1}\right)^2} \right) & -\frac{1}{y^1} & \frac{3y^2 y^3 F' \left[F\left(\frac{y^3}{y^1}\right)\right]^2}{(y^1)^3 F\left(\frac{y^3}{y^1}\right)^2} \\ -\frac{1}{y^1} & -\frac{1}{y^2} & 0 \\ \frac{3y^2 y^3 F' \left[F\left(\frac{y^3}{y^1}\right)\right]^2}{(y^1)^3 F\left(\frac{y^3}{y^1}\right)^2} & 0 & -\frac{3y^2 F' \left[F\left(\frac{y^3}{y^1}\right)\right]^2}{(y^1)^2 F\left(\frac{y^3}{y^1}\right)^2} \end{pmatrix}. \quad (3.32)$$

Any function $\Psi(y^1, y^2, y^3)$ on this manifold is of course annihilated by the quantum linear constraint, i.e.

$$\hat{\mathcal{H}}_1 \Psi(y^1, y^2, y^3) = \frac{\partial \Psi(y^1, y^2, y^3)}{\partial y^1} \hat{\mathcal{H}}_1 y^1 + \frac{\partial \Psi(y^1, y^2, y^3)}{\partial y^2} \hat{\mathcal{H}}_1 y^2 + \frac{\partial \Psi(y^1, y^2, y^3)}{\partial y^3} \hat{\mathcal{H}}_1 y^3 = 0,$$

since the derivatives with respect to r are transparent to the partial derivatives of Ψ (which are, just like the y^A 's, r -numbers).

The covariant metric (3.32) describes a three dimensional conformally flat geometry, since the Cotton-York tensor vanishes. The Ricci scalar is $R = \frac{3}{8y^2}$, indicating that the arbitrariness in F (and thus also in L) is a *pure gauge*. The change of coordinates

$$(y^1, y^2, y^3) = (e^{-\frac{1}{8}(5Y^1+3Y^3)}, e^{Y^1+Y^2+Y^3}, e^{-\frac{1}{8}(5Y^1+3Y^3)} F^{-1}(e^{\frac{1}{24}(-9Y^1+8Y^2-15Y^3)})), \quad (3.33)$$

(where F^{-1} denotes the function inverse to F , i.e. $F^{-1}(F(x)) = x$) brings the metric to the manifestly conformally flat form:

$$g_{ABren}(Y^1, Y^2, Y^3) = \begin{pmatrix} e^{Y^1+Y^2+Y^3} & 0 & 0 \\ 0 & -\frac{4}{3} e^{Y^1+Y^2+Y^3} & 0 \\ 0 & 0 & -e^{Y^1+Y^2+Y^3} \end{pmatrix}, \quad (3.34)$$

in which all the F dependence has indeed disappeared.

The final restriction on the form of Ψ will be obtained by the imposition of the quantum analog of the quadratic constraint \mathcal{H}_o . According to the above exposition we postulate that

the quantum gravity of the geometries given by (2.21) will be described by the following partial differential equation (in terms of the $Y^{A,s}$)

$$\hat{\mathcal{H}}_0 \Psi \equiv \left[-\frac{1}{2} \square_c + V_{ren}\right] \Psi(Y^1, Y^2, Y^3) = 0, \quad (3.35)$$

with

$$\square_c = \square + \frac{d-2}{4(d-1)} R, \quad (3.36)$$

being the conformal Laplacian based on $g_{ABren}(Y^1, Y^2, Y^3)$, R the Ricci scalar, and d the dimensions of g_{ABren} . The metric (3.34) is conformally flat with Ricci scalar $R = \frac{3}{8} e^{-Y^1-Y^2-Y^3}$, and its dimension is $d = 3$. The renormalized form of the potential (2.27) offers us the possibility to introduce, in a dynamical way, topological effects into our wave functional: Indeed, under our **Assumption**, the first two terms become $-2\epsilon y^1$ and $2\Lambda y^2$, respectively, while the last, being a total derivative, becomes $A_T \equiv 4 \frac{\psi \psi'}{\gamma} \Big|_{\alpha}^{\beta}$ (*if* $\alpha < r < \beta$). In the spirit previously explained we should drop this term, however one could also keep it. The renormalized form of the remaining, third, term of the potential can be obtained as follows

$$y^3 = \gamma L(\Psi^{(1)}) \Leftrightarrow L(\Psi^{(1)}) = \frac{y^3}{\gamma} \stackrel{\text{Assumption}}{\Leftrightarrow} L(\Psi^{(1)}) = \frac{y^3}{y^1} \Leftrightarrow \Psi^{(1)} = L^{-1} \left(\frac{y^3}{y^1} \right),$$

thus finally

$$\frac{\psi'}{\gamma} = L^{-1} \left(\frac{y^3}{y^1} \right),$$

and the third term becomes $-2y^1 \left[L^{-1} \left(\frac{y^3}{y^1} \right) \right]^2$. Finally, effecting the transformation (3.33) the form of the renormalized potential is

$$V_{ren} = -2\epsilon e^{-\frac{1}{8}(5Y^1+3Y^3)} - 2e^{-\frac{1}{8}(5Y^1+3Y^3)} \left[L^{-1} \left(F^{-1} \left(e^{\frac{1}{24}(-9Y^1+8Y^2-15Y^3)} \right) \right) \right]^2 + 2\Lambda e^{Y^1+Y^2+Y^3} + A_T, \quad (3.37)$$

and the Wheeler-deWitt equation is given as

$$\begin{aligned} & -2\epsilon e^{\frac{1}{8}(3Y^1+5Y^3)+Y^2} \Psi(Y^1, Y^2, Y^3) + 2\Lambda e^{2(Y^1+Y^2+Y^3)} \Psi(Y^1, Y^2, Y^3) - \\ & 2e^{\frac{1}{8}(3Y^1+5Y^3)+Y^2} \left[L^{-1} \left(F^{-1} \left(e^{\frac{1}{24}(-9Y^1+8Y^2-15Y^3)} \right) \right) \right]^2 \Psi(Y^1, Y^2, Y^3) + \\ & A_T e^{Y^1+Y^2+Y^3} \Psi(Y^1, Y^2, Y^3) - \frac{3}{128} \Psi(Y^1, Y^2, Y^3) - \frac{1}{4} \frac{\partial \Psi(Y^1, Y^2, Y^3)}{\partial Y^1} + \\ & \frac{3}{16} \frac{\partial \Psi(Y^1, Y^2, Y^3)}{\partial Y^2} + \frac{1}{4} \frac{\partial \Psi(Y^1, Y^2, Y^3)}{\partial Y^3} - \frac{1}{2} \frac{\partial^2 \Psi(Y^1, Y^2, Y^3)}{\partial (Y^1)^2} + \frac{3}{8} \frac{\partial^2 \Psi(Y^1, Y^2, Y^3)}{\partial (Y^2)^2} + \\ & \frac{1}{2} \frac{\partial^2 \Psi(Y^1, Y^2, Y^3)}{\partial (Y^3)^2} = 0. \end{aligned}$$

Since F^{-1} is an arbitrary function of its arguments, we may contemplate the choice:

$$F^{-1} \left(e^{\frac{1}{24}(-9Y^1+8Y^2-15Y^3)} \right) = L \left(\sqrt{e^{\frac{1}{24}(-9Y^1+8Y^2-15Y^3)} - \epsilon} \right).$$

Of course, there is a question of existence for such a choice: any demand that F has a specified form (much more in terms of L) constitutes a restriction on the form of L . Consequently, one has to prove that such an L indeed exists. The appropriate form of L is implicitly given by

$$L(\omega) = m + \int \frac{(\epsilon + \omega^2)^{1/4}}{\omega^{5/8}} e^{k - \frac{3\epsilon}{16\omega^2}} d\omega,$$

where $\omega \equiv \sqrt{\Psi^{(1)^2} - \epsilon}$ and m, k are related by

$$c_1 m + c_2 + c_3 e^k = 0.$$

For a detailed elaboration of the results, see [39].

Anyway, the above choice of F reduces the Wheeler–DeWitt equation to the final separable form:

$$\begin{aligned} & \left[2\Lambda e^{2(Y^1+Y^2+Y^3)} - 2e^{\frac{4}{3}Y^2} + A_T e^{Y^1+Y^2+Y^3} - \frac{3}{128} \right] \Psi(Y^1, Y^2, Y^3) - \\ & \frac{1}{4} \frac{\partial \Psi(Y^1, Y^2, Y^3)}{\partial Y^1} + \frac{3}{16} \frac{\partial \Psi(Y^1, Y^2, Y^3)}{\partial Y^2} + \frac{1}{4} \frac{\partial \Psi(Y^1, Y^2, Y^3)}{\partial Y^3} - \\ & \frac{1}{2} \frac{\partial^2 \Psi(Y^1, Y^2, Y^3)}{\partial (Y^1)^2} + \frac{3}{8} \frac{\partial^2 \Psi(Y^1, Y^2, Y^3)}{\partial (Y^2)^2} + \frac{1}{2} \frac{\partial^2 \Psi(Y^1, Y^2, Y^3)}{\partial (Y^3)^2} = 0. \end{aligned} \quad (3.38)$$

This equation is readily solved, for $\Lambda = 0$ and $A_T = 0$, by the method of separation of variables: assuming $\Psi(Y^1, Y^2, Y^3) = \Psi^1(Y^1) \Psi^2(Y^2) \Psi^3(Y^3)$ and dividing (3.38) by Ψ we get the three ordinary differential equations:

$$\begin{aligned} & \frac{1}{4\Psi^1(Y^1)} \frac{d\Psi^1(Y^1)}{dY^1} + \frac{1}{2\Psi^1(Y^1)} \frac{d^2\Psi^1(Y^1)}{d(Y^1)^2} = m, \\ & \frac{3}{16\Psi^2(Y^2)} \frac{d\Psi^2(Y^2)}{dY^2} + \frac{3}{8\Psi^2(Y^2)} \frac{d^2\Psi^2(Y^2)}{d(Y^2)^2} - 2e^{\frac{4}{3}Y^2} = n, \\ & \frac{1}{4\Psi^3(Y^3)} \frac{d\Psi^3(Y^3)}{dY^3} + \frac{1}{2\Psi^3(Y^3)} \frac{d^2\Psi^3(Y^3)}{d(Y^3)^2} - \frac{3}{128} = m - n, \end{aligned}$$

where m and n are separation constants. Their solutions are:

$$\begin{aligned} \Psi^1(Y^1) &= c_1 e^{\frac{1}{4}(-1-\sqrt{1+32m})Y^1} + c_2 e^{\frac{1}{4}(-1+\sqrt{1+32m})Y^1}, \\ \Psi^2(Y^2) &= c_3 e^{-Y^2/4} I_{-\frac{\sqrt{3}}{8}\sqrt{3+128n}} \left(2\sqrt{3} e^{2Y^2/3} \right) + c_4 e^{-Y^2/4} I_{\frac{\sqrt{3}}{8}\sqrt{3+128n}} \left(2\sqrt{3} e^{2Y^2/3} \right), \\ \Psi^3(Y^3) &= c_5 e^{\frac{1}{8}(-2-\sqrt{7+128m-128n})Y^3} + c_6 e^{\frac{1}{8}(-2+\sqrt{7+128m-128n})Y^3}, \end{aligned}$$

where $I_{\pm \frac{\sqrt{3}}{8}\sqrt{3+128n}} \left(2\sqrt{3} e^{2Y^2/3} \right)$ are modified Bessel functions of the first kind and of non-integer order.

4. Discussion

We have considered the canonical analysis and subsequent quantization of two midi-superspace models in 2+1 and 3+1 dimensions. The simplifying assumption was the existence of appropriate Killing fields, which reduced the components of the metrics to functions of time

and a radial coordinate. Thus, the emanating field theory is effectively one-dimensional. The quantum linear constraints reduced the initial Hilbert space consisting of all smooth functionals to the space of all smooth scalar functionals. The appropriate imposition of the quadratic constraint leads us to the **Assumption** and the **Requirement**, which strongly further restrict the space of functionals to the three functionals (3.23) and (3.31) for the 2+1 and 3+1 system, respectively. It is quite interesting that, although the systems considered are different, the resulting renormalized manifolds, their geometry, described by (3.22) and (3.32), and the corresponding Wheeler–DeWitt equations are strikingly similar. This points to the fact that the imposition of the **Assumption** and the **Requirement** may not be simply an elegant way to reduce the number of spatial derivatives of the configuration fields, contained in the scalar functional, but is rather a tool for unraveling the underlying geometrical structure of Quantum Gravity (in the approximation considered). We have reasons to believe that our method can be applied to more general models, even in full pure gravity, and we hope to return with relevant investigations in the very near future.

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References

- [1] Dirac P A M 1950 *Can. J. Math.* **2** 129
- [2] Dirac P A M 1951 *Can. J. Math.* **3** 1
- [3] Dirac P A M 1958 *Proc. R. Soc. (London)* A **246** 326
- [4] Dirac P A M 1964 *Lectures on Quantum Mechanics* (New York: Yeshiva University, Academic Press)
- [5] Bergmann P G 1961 *Rev. Mod. Phys.* **33** 510
- [6] Berezin F A and Marinov M S 1977 *Ann. Phys. (N.Y.)* **104** 336
- [7] Becchi C, Rouet C and Stora R 1976 *Ann. Phys. (N.Y.)* **98** 287
- [8] Sudarshan E C G and Mukunda N 1974 *Classical Dynamics: A Modern Perspective* (New York: Wiley)
- [9] Hanson A J, Regge T and Teitelboim C 1976 *Constrained Hamiltonian Systems* (Vatican: Accademia Nazionale dei Lincei)
- [10] Sundermeyer K 1982 *Lecture Notes in Physics* **169** *Constrained Dynamics* (Heidelberg: Springer-Verlag)
- [11] Gitman D M and Tyutin I V 1991 *Quantization of Fields with Constraints* Springer Series in Nuclear and Particle Physics
- [12] Govaers J 1991 *Hamiltonian Quantization and Constrained Dynamics* (Leuven: Leuven University Press)
- [13] Henneaux M and Teitelboim C 1992 *Quantization of Gauge Systems* (Princeton: Princeton University Press)
- [14] Wipf A *Hamilton's Formalism for Systems with Constraints* (Preprint hep-th/9312078)
- [15] Thiemann Th 2007 *Modern Canonical Quantum General Relativity* (Cambridge: Cambridge University Press)
- [16] DeWitt B S 1967 *Phys. Rev.* **160**(5) 1113
- [17] Bergmann P G 1966 *Phys. Rev.* **144** 1078
- [18] Ryan M P Jr and Shepley L C 1975 *Homogeneous Relativistic Cosmologies* (Princeton: Princeton University Press)
- [19] Christodoulakis T 2002 *Quantum Cosmology Lect. Notes Phys.* **592** Springer Verlag 318-350
- [20] Krasinski A 1997 *Inhomogeneous Cosmological Models* (Cambridge: Cambridge University Press)
- [21] Thomi P, Isaak B and Hajicek P 1984 *Phys. Rev. D* **30** 1168
- [22] Hajicek P 1984 *Phys. Rev. D* **30** 1178
- [23] Rodrigues L M C S, Soares I D and Zanelli J 1989 *Phys. Rev. Lett.* **62** 989
- [24] Kiefer C 2007 *Quantum Gravity* 2nd ed (Oxford: Oxford University Press) ch 7
- [25] Kiefer C, Müller-Hill J and Vaz C 2006 *Phys. Rev. D* **73** 044025
- [26] Carlip S 2003 *Quantum Gravity in 2+1 dimensions* (Cambridge: Cambridge University Press)
- [27] Carlip S 2005 *Class. Quantum Grav.* **22** R85
- [28] Carlip S 2005 *Living Rev. Rel.* **8** 1
- [29] Schroers B J, 1990 *Lessons from (2+1)-dimensional quantum gravity* (Preprint gr-qc/0710.5844)

- [30] Hajicek P and Kuchař K V 1990 *Phys. Rev. D* **41** 1091
- [31] Hajicek P and Kuchař K V 1990 *J. Math. Phys.* **31** 1723
- [32] Bañados M, Teitelboim C, and Zanelli J 1992 *Phys. Rev. Lett.* **69** 1849
- [33] Bañados M, Henneaux M, Teitelboim C, and Zanelli J 1993 *Phys. Rev. D* **48** 1506
- [34] Christodoulakis T and Papadopoulos G O *Preprint* gr-qc/0405144v2
- [35] Barrow J, Shaw D and Tsagas C 2006 *Class. Quantum Grav.* **23** 5291
- [36] Eisenhart L P 1964 *Riemannian Geometry*, (Princeton: Princeton University Press, New Jersey) 5th printing pp.146
- [37] Thomas T Y 1991 *The Differential Invariants of Generalized Spaces*, (New York: Chelsea Publishing Company) pp.140
- [38] Bojowald M and Swiderski R 2006 *Class. Quantum Grav.* **23** 2129
- [39] Christodoulakis T et al. *Towards a Canonical Quantum Gravity for Geometries Admitting Maximally Symmetric Two-Dimensional Surfaces* work in progress