

Remarks on Common Representation Theory of Gauge Theories

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Abstract. Whereas the standard representation (rep) theory of gauge theories relies on quantum field theory (QFT), based on point coordinates x^μ and certain Lie symmetries, we discuss some aspects of a geometrical background pattern able to *explain* such rep theory and the known unitary symmetries from scratch. Switching to lines as geometrical base elements of space-time instead of point coordinates, line geometry induces a concept to identify quantum theories directly with the description of forces and moments. Moreover, sets of lines and their properties correspond to known physical notions, and line Complexes allow a natural approach to gauge descriptions, chiral rep theory, 3-dimensional complex spaces, grouping into families, triality, etc. Beyond Riemannian geometry, line geometry naturally adds symplectic geometry in P^3 , and its rep theory yields cubic curves as well as quartic surfaces. Last not least, it not only respects quadratic invariants advocated by bi-linear symmetric forms and orthogonal or unitary transformations, but projective geometry also allows to incorporate the complete invariant theory in a well-defined manner.

1 Introduction

In previous papers [1], [2], [3], we have investigated different aspects when working with the Dirac equation in quantum mechanics and QFT. Logically, we began with Dirac's decomposition of a quadric into two linear factors, and also discussed 'reverse paths', i.e. how to generate a quadric of P^3 by linear factors. To overcome the limitations of QFT and (affine) gauge formulations, this has to take place, of course, by means of projective geometry and suitable reps. So to gain some more insight, here we collect some central aspects of popular representation theories, and argue to relate them by a common geometrical background in projective geometry. For illustration purposes, below we sometimes use real reps which is no drawback because we can use imaginary collineations or apply known transfer principles like e.g. Lie transfer [4] if we respect possible ambiguities like in line-sphere transfer correctly.

The line as basic building block in P^3 can be simply identified as a set of points, for which we have derived a corresponding $SU(2) \times U(1)$ symmetry [3] in conjunction with the Dirac spinor rep ψ . The dual picture in P^3 then interprets the line as the axis of a plane pencil, and the adjoint rep ψ^+ then enforces the additional consideration of conjugated lines. However, two such pencils – dependent on intersection properties of the lines – allow to apply the projective generation of the quadric in P^3 by two skew plane pencils as the geometric basis of Dirac theory. When intersecting, the lines automatically yield cones and further aspects of defining physical work (which we'll discuss elsewhere).

Transferring to P^5 , an analogous discussion can be held in terms of linear and quadratic Complexes where line geometry of P^3 is embedded by basic point reps on the Plücker-Klein quadric M_2^4 . Thus having the richer structures of P^5 at hand, the main idea is to subsume known rep theory into this



transfer picture. Accordingly we expect symmetries usually 'plugged in' by hand into Lagrangean or Hamiltonian descriptions to be related to the common underlying basis of projective geometry.

So in sec. 2 we summarize previous results on the Lorentz transformation and supplement them with general transformations of the line coordinates in P^3 , which lead to automorphisms of points of the Plücker-Klein quadric M_2^4 in P^5 . Essentially lines in P^3 are mapped to lines in P^3 which stabilizes linearity and supports linear rep theory in P^3 . Beginning with sec. 3, we discuss known compact symmetries usually required *as ingredient* of gauge theory constructions. Here, such symmetries occur naturally by the chosen geometric approach of lines in P^3 or Complexes in P^5 . $SU(3)$ can be identified either as a symmetry of Plücker and Klein coordinates or, in a local rep, by the Hermitean Riemannian space CI . $SU(2) \times U(1)$, too, can be interpreted in various ways. In sec. 4, an obvious and easy way stems from the double coset decomposition of $SU_*(4) \cong SL(2, \mathbb{H})$ with $SU(2) \times U(1)$ as final gauge group. However, we can obtain the Yang-Mills Ansatz also by simple geometrical considerations in P^3 or by restrictions in P^5 . Sec. 5 then, while discussing moments, gives a brief but exact geometrical example why $SU(4)$ bridges the worlds of classical physics in terms of points and (polar) planes and quantum physics in terms of lines and Complexes. We close with a brief outlook and discuss further paths.

2 Special Relativity and Lorentz Transformations

In [3], we have transformed spatial points, represented by quaternary homogeneous coordinates¹ x_α and y_α , by Lorentz transformations according to

$$\begin{aligned} x_0 &\longrightarrow x'_0 = \gamma x_0 - \gamma \beta x_1, & x_2 &\longrightarrow x'_2, \\ x_1 &\longrightarrow x'_1 = \gamma x_1 - \gamma \beta x_0, & x_3 &\longrightarrow x'_3. \end{aligned} \quad (1)$$

This transformation mixes the 0- and the 1-component, so in this rep the system propagates into x -direction whereas the 2-/3-plane remains invariant. We have then discussed the transformation properties of line rep coordinates, i.e. the antisymmetric bi-linear form $p_{\alpha\beta} = x_\alpha y_\beta - x_\beta y_\alpha \rightarrow p'_{\alpha\beta}$ of quaternary point coordinates. They may be interpreted geometrically by comparing the tetrahedral edges $p_{\alpha\beta}$ and $p'_{\alpha\beta}$. The main result, however, is the ('Lorentz') invariance of $P = p_{01}p_{23} + p_{02}p_{31} + p_{03}p_{12} = 0$, which defines the senary quadric M_4^2 in P^5 , the Plücker-Klein quadric.

It is then obvious to ask for the transformation behaviour of P under general quaternary transformations $A_{4 \times 4}$, $x_\alpha \rightarrow x'_\alpha = A_{\alpha\beta} x_\beta$. Straightforward algebra shows that for each index pair $(\mu\nu)$ the quaternary transformation yields

$$p_{\mu\nu} \rightarrow p'_{\mu\nu} = \sum_{\alpha\beta} (A_{\mu\alpha} A_{\nu\beta} - A_{\mu\beta} A_{\nu\alpha}) p_{\alpha\beta} = \sum_{\alpha\beta} \begin{vmatrix} A_{\mu\alpha} & A_{\mu\beta} \\ A_{\nu\alpha} & A_{\nu\beta} \end{vmatrix} p_{\alpha\beta} =: \sum_{\alpha\beta} M_{\alpha\beta}^{\mu\nu} p_{\alpha\beta}. \quad (2)$$

The shorthand notation $M_{\alpha\beta}^{\mu\nu} := A_{\mu\alpha} A_{\nu\beta} - A_{\nu\alpha} A_{\mu\beta}$ for the 2×2 -sub-determinants (with fixed μ, ν) simplifies the insight into the algebraic structure and yields the important aspect that (for fixed values of μ, ν), the transformation parameters given by $A_{4 \times 4}$ also group into skew representations. Thus we can group the 16 components of $A_{4 \times 4}$ with respect to individual line coordinate transformations by $M_{\alpha\beta}^{\mu\nu}$ with fixed μ, ν and antisymmetric α, β . In general, $M_{\alpha\beta}^{\mu\nu}$ is antisymmetric in both index pairs, and we can establish various rules like

$$M_{\alpha\beta}^{01} M_{\alpha\beta}^{23} + M_{\alpha\beta}^{02} M_{\alpha\beta}^{31} + M_{\alpha\beta}^{03} M_{\alpha\beta}^{12} = 0, \quad \alpha, \beta \text{ fixed}, \quad (3)$$

$$M_{\alpha\beta}^{\mu\nu} = -M_{\beta\alpha}^{\nu\mu} = -M_{\beta\alpha}^{\mu\nu} = M_{\beta\alpha}^{\nu\mu}, \quad \text{etc.} \quad (4)$$

So in P^3 linear transformations of quaternary points or planes induce linear transformations of the skew line coordinates, and the individual transformations $p'_{\mu\nu}$ can be used to calculate the transformed Plücker-Klein quadric P' which – after lengthy but straightforward calculations – in terms of $A_{4 \times 4}$ reads as $P' = \tilde{a}(p_{01}p_{23} + p_{02}p_{31} + p_{03}p_{12}) = \tilde{a}P$, \tilde{a} being a polynomial of fourth order in $a_{\alpha\beta}$. Lorentz transformations according to eq. (1) yield $\tilde{a} = \gamma^2(\beta^2 - 1) = 1$, so $P' = P$ which proves automorphic transformations of P in P^5 . The same holds true when replacing the matrices $A_{4 \times 4}$ with the 16 operators of the matrix reps of the Dirac algebra $\{\gamma^\mu\}$. Rewriting P' in terms of the 2×2 -determinants $M_{\alpha\beta}^{\mu\nu}$ simplifies the discussion. P' then – arranged appropriately with respect to the index order – reads as

$$\begin{aligned} P' = & p_{01} p_{23} \left(M_{01}^{01} M_{23}^{23} + M_{01}^{02} M_{23}^{31} + M_{01}^{03} M_{23}^{12} + M_{01}^{23} M_{23}^{01} + M_{01}^{31} M_{23}^{02} + M_{01}^{12} M_{23}^{03} \right) \\ & + p_{02} p_{31} \left(M_{02}^{01} M_{31}^{23} + M_{02}^{02} M_{31}^{31} + M_{02}^{03} M_{31}^{12} + M_{02}^{23} M_{31}^{01} + M_{02}^{31} M_{31}^{02} + M_{02}^{12} M_{31}^{03} \right) \\ & + p_{03} p_{12} \left(M_{03}^{01} M_{12}^{23} + M_{03}^{02} M_{12}^{31} + M_{03}^{03} M_{12}^{12} + M_{03}^{23} M_{12}^{01} + M_{03}^{31} M_{12}^{02} + M_{03}^{12} M_{12}^{03} \right), \end{aligned} \quad (5)$$

¹In general, we use Greek indices to denote 0, 1, 2, 3 as well as 0 instead of the old notation 4.

This expression can be simplified further by introducing 6-vectors $\mathfrak{A}_{\alpha\beta} = (M_{\alpha\beta}^{01}, M_{\alpha\beta}^{02}, \dots, M_{\alpha\beta}^{12})^T$ constructed by means of the original 4×4 -transformation parameters $A_{4 \times 4}$. For a given index pair $\alpha\beta$, $\mathfrak{A}_{\alpha\beta}$ transforms as 6-vector because $M_{\alpha\beta}^{\mu\nu}$ is antisymmetric in both index pairs $\mu\nu$ and $\alpha\beta$ according to eq. (4), moreover eq. (3) shows that $\mathfrak{A}_{\alpha\beta}$ can be seen as a special linear Complex or an axis.

Using then the usual 6-vector product

$$[\mathfrak{A} \cdot \mathfrak{B}] = [\mathfrak{A}_{\alpha\beta} \cdot \mathfrak{B}_{\gamma\delta}] = M_{\alpha\beta}^{01} M_{\gamma\delta}^{23} + M_{\alpha\beta}^{02} M_{\gamma\delta}^{31} + \dots + M_{\alpha\beta}^{12} M_{\gamma\delta}^{03}, \quad (6)$$

eq. (5) can be formally² rewritten by using the six 6-vectors $\mathfrak{A}_{\alpha\beta}$ as

$$P' = p_{01} p_{23} [\mathfrak{A}_{01} \cdot \mathfrak{A}_{23}] + p_{02} p_{31} [\mathfrak{A}_{02} \cdot \mathfrak{A}_{31}] + p_{03} p_{12} [\mathfrak{A}_{03} \cdot \mathfrak{A}_{12}]. \quad (7)$$

This, however, traces back to Klein's notion of 6-vectors 'being in involution' or not, i.e. whether they 'commute', $[\mathfrak{A}_{\alpha\beta} \cdot \mathfrak{B}_{\gamma\delta}] = 0$, or not. Essentially the product (6) describes the simultaneous invariant of two 6-vectors, and besides its central rôle in linear and quadratic Complex geometry (see e.g. [5], p. 91ff) later on we are going to use it physically to investigate work related to variations in Hamiltonian frameworks. The key takeaway here, however, is the fact that transforming with general P^3 transformations $A_{4 \times 4}$, the quadric P' depends on the 3 simultaneous invariants of the 6-vectors pairs $(\mathfrak{A}_{01}, \mathfrak{A}_{23})$, $(\mathfrak{A}_{02}, \mathfrak{A}_{31})$, and $(\mathfrak{A}_{03}, \mathfrak{A}_{12})$ which themselves are assembled by the 2×2 -sub-determinants $M_{\alpha\beta}^{\mu\nu}$ of the underlying quaternary transformations parameters $A_{4 \times 4}$, and which 'scale' the products $p_{01} p_{23}$, $p_{02} p_{31}$, and $p_{03} p_{12}$ of the opposite tetrahedral edge pairs. Moreover, eq. (3) demonstrates the line character of the 'parameter'-6-vectors $\mathfrak{A}_{\alpha\beta}$, and we may arrange them into two groups of 3 lines (or 'generators') each.

The special case $[\mathfrak{A}_{01} \cdot \mathfrak{A}_{23}] = [\mathfrak{A}_{02} \cdot \mathfrak{A}_{31}] = [\mathfrak{A}_{03} \cdot \mathfrak{A}_{12}] = 1$ in eq. (7) yields the Plücker-Klein quadric, the (formal) general case $[\mathfrak{A}_{01} \cdot \mathfrak{A}_{23}] \neq [\mathfrak{A}_{02} \cdot \mathfrak{A}_{31}] \neq [\mathfrak{A}_{03} \cdot \mathfrak{A}_{12}]$ is comprised when treating quadratic (and especially tetrahedral) Complexes $a p_{01} p_{23} + b p_{02} p_{31} + c p_{03} p_{12}$ [6], and non-vanishing parameters a, b, c like in eq. (5) yield another approach to discuss non-commuting symmetry generators. So for general quaternary transformations $A_{4 \times 4}$, i.e. using 16 (15) independent real parameters $A_{\alpha\beta}$, the result is a 'scaled' Plücker-Klein quadric $P' = \tilde{a} P$ based on its basic summands $p_{01} p_{23}$, $p_{02} p_{31}$, and $p_{03} p_{12}$ only, other quadratic terms in $p_{\alpha\beta}$ vanish. Because the general transformation of six 6-vectors comprises 36 (35) parameters, even in the case of linear Complexes, general P^3 transformations will be at every stage a subset of these general Complex transformations, and a rescaling of the Plücker-Klein quadric preserves linearity in P^3 . This revives the original problem of determining the principal axes of quadrics, the eigenvectors and eigenvalues, and in P^3 as well as in P^5 we can work alternatively with corresponding second-order differential equations (like Laplace operators in P^3 or equations derived from them) against functional reps of the quadrics.

Using eq. (2), we can associate linear Complexes and their transformations in P^5 as well, and we have an analytical tool-set at hand to relate both P^3 and P^5 transformations based on points/planes or lines in P^3 with linear Complexes. Eq. (5) yields an additional tool to investigate automorphisms of M_4^2 in P^5 . So the construction rule to build Langrangeans invariant with respect to Lorentz transformations is intrinsically respected when using lines in P^3 , or linear Complexes in P^5 as building blocks. As a byproduct, appropriately constructed Complex invariants also yield by the very definition a symplectic symmetry.

3 SU(3) Transformations

In [7], we have used Plücker's line coordinates $p_{\alpha\beta}$, located within the 4×2 -Dirac spinor ψ , to represent Klein coordinates x_{\aleph} , $1 \leq \aleph \leq 6$, and their SO(6) symmetry, and we have derived *real* senary coordinates π_{\aleph} by

$$\begin{aligned} \pi_1 &= 2x_1 = p_{01} + p_{23}, & \pi_2 &= 2I x_2 = p_{01} - p_{23}, \\ \pi_3 &= 2x_3 = p_{02} + p_{31}, & \pi_4 &= 2I x_4 = p_{02} - p_{31}, \\ \pi_5 &= 2x_5 = p_{03} + p_{12}, & \pi_6 &= 2I x_6 = p_{03} - p_{12}. \end{aligned} \quad (8)$$

These senary coordinates π_{\aleph} transform as:

π'_{\aleph}	$\mathbb{1} \psi$	$\gamma^1 \psi$	$\gamma^2 \psi$	$\gamma^3 \psi$	$\gamma^0 \psi$	$\gamma_5 \psi$	$\gamma_5 \gamma^1 \psi$	$\gamma_5 \gamma^2 \psi$	$\gamma_5 \gamma^3 \psi$	$\gamma_5 \gamma^0 \psi$
π'_1	$+\pi_1$	$-\pi_1$	$-\pi_1$	$-\pi_1$	$+\pi_1$	$+\pi_1$	$-\pi_1$	$-\pi_1$	$-\pi_1$	$+\pi_1$
π'_2	$+\pi_2$	$+\pi_2$	$+\pi_2$	$+\pi_2$	$+\pi_2$	$-\pi_2$	$-\pi_2$	$-\pi_2$	$-\pi_2$	$-\pi_2$
π'_3	$+\pi_3$	$+\pi_3$	$-\pi_3$	$+\pi_3$	$-\pi_3$	$-\pi_3$	$-\pi_3$	$+\pi_3$	$-\pi_3$	$+\pi_3$
π'_4	$+\pi_4$	$-\pi_4$	$+\pi_4$	$+\pi_4$	$-\pi_4$	$-\pi_4$	$+\pi_4$	$-\pi_4$	$-\pi_4$	$+\pi_4$
π'_5	$+\pi_5$	$+\pi_5$	$+\pi_5$	$-\pi_5$	$-\pi_5$	$-\pi_5$	$-\pi_5$	$-\pi_5$	$+\pi_5$	$+\pi_5$
π'_6	$+\pi_6$	$+\pi_6$	$+\pi_6$	$+\pi_6$	$-\pi_6$	$+\pi_6$	$+\pi_6$	$+\pi_6$	$+\pi_6$	$-\pi_6$

(9)

²Relations between products of the determinants $M_{\alpha\beta}^{\mu\nu}$ allow further reductions.

$$\text{and } \begin{array}{c|c|c|c|c|c|c|c} \pi'_N & \mathbb{1} \psi & \sigma^{01} \psi & \sigma^{02} \psi & \sigma^{03} \psi & \sigma^{12} \psi & \sigma^{13} \psi & \sigma^{23} \psi \\ \hline \pi'_1 & +\pi_1 & +\pi_1 & +\pi_1 & +\pi_1 & -\pi_1 & -\pi_1 & -\pi_1 \\ \pi'_2 & +\pi_2 & -\pi_2 & -\pi_2 & -\pi_2 & -\pi_2 & -\pi_2 & -\pi_2 \\ \pi'_3 & +\pi_3 & +\pi_3 & -\pi_3 & +\pi_3 & +\pi_3 & -\pi_3 & +\pi_3 \\ \pi'_4 & +\pi_4 & -\pi_4 & +\pi_4 & +\pi_4 & +\pi_4 & +\pi_4 & -\pi_4 \\ \pi'_5 & +\pi_5 & +\pi_5 & +\pi_5 & -\pi_5 & -\pi_5 & +\pi_5 & +\pi_5 \\ \pi'_6 & +\pi_6 & +\pi_6 & +\pi_6 & +\pi_6 & -\pi_6 & -\pi_6 & -\pi_6 \end{array} \quad (10)$$

under action of the Dirac algebra, i.e. up to a sign we thus obtain irreps. Whereas deeper background relies on known and well-established projective geometry [7], a local representation in terms of Riemannian spaces AII and CI [8] is possible featuring the gauge group $SU(2) \times U(1)$. These Riemannian spaces have constant curvature which is essential for defining normal coordinates and geodesics, and relating them a priori to projective spaces [9] [3]. The two-step reduction by Riemannian spaces, besides their localized rep theory, allowed us to identify physics and to group the Klein coordinates with respect to an overall complex structure $I, I^2 = -1$, in CI which in turn groups the six coset coordinates into a 'complex 3-vector' X in \mathbb{C}^3 [7]. So in relativistic rep theories relying on the standard description by means of the Dirac algebra, $SU(3)$ thus enters *automatically* while leaving the quadratic expression $X^+ X$ invariant [7]. For later reasoning, it is important to remember that in the background we thus always use P^5 , and we may restrict it to P^3 in terms of points, planes or (Euclidean) lines.

This picture of fundamental Complexe ('6-vectors') and their linear combination allows for a natural decomposition into 3 Congruences ('pairs of skew lines') and to identify left-/right-handed spinorial reps, and to relate them to linear combinations of linear Complexe when representing forces and force pairs by 6-vectors. So we have a unique and common description of interactions and their resolution into 6-vectors. A possible interpretation (up to linear transformations) can then be given according to figure 1 which visualizes an arrangement of six fundamental Complexe and the 3 possible Congruences within the tetrahedron into 3 'families' derived from Dirac theory, for hadrons as well as leptons. The common binding mechanism is the representation of forces and force pairs in terms of 6-vectors, and their possible (linear) decompositions in terms of linear factors, which is formally suitable for classical as well as quantum rep theory.

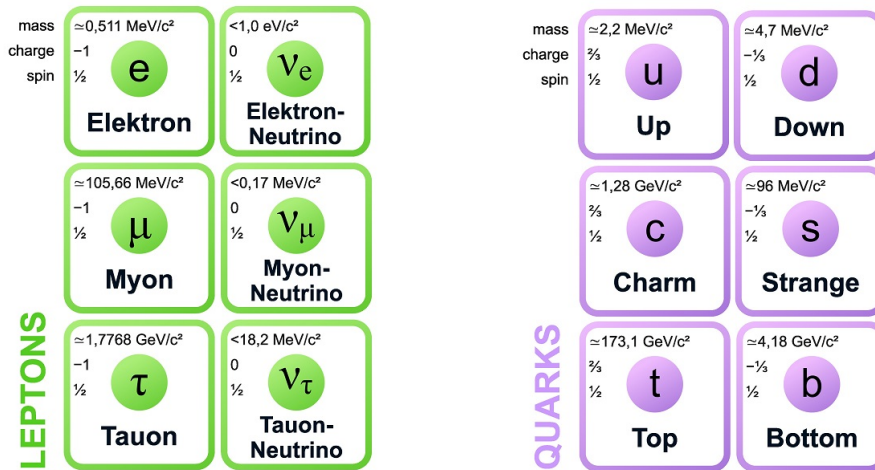


Figure 1: Grouping of 6-vector components (Klein coordinates) into Congruences with a possible 'physical' identification based on Dirac theory.

4 $SU(2) \times U(1)$ Transformations

Another necessary ingredient is an $SU(2) \times U(1)$ symmetry which, for complex representations, is one of the simplest symmetries and as such far from unique. There are several (of course related) ways – formally as well as geometrically – to realize and understand such low-dimensional symmetry behaviour, so the interpretation of the reps and their respective context matters. In the preceding section, we have

cited the local decomposition of $SU^*(4) \cong SL(2, \mathbb{H})$ according to $AII \times CI$ with a remaining $SU(2) \times U(1)$ gauge (isometry) group. Geometrically, we can discuss additional reps which result from restrictions of the full projective geometry of P^5 to P^3 , induced by choosing an appropriate metric and the remaining reps of transformations and states (usually in terms of relativistic quantum theories and gauge theory).

Whereas $U(1)$, understood as a phase transformation, represents a 1-dim rotation in the complex plane or around a fixed axis, the simplest approach to $SU(2)$ can be represented in terms of quaternions describing (dependent) S_2 rotations of two fundamental states $|0\rangle$ and $|1\rangle$ while preserving 'the metric' $\langle \cdot | \cdot \rangle$ (see also [10], ch. 5). For our purpose, it is sufficient to extract as formal requirements two (distinguishable) basic states, a conjugation and the construction of a quadratic invariant. Moreover, we generalize the sphere to quadrics. Last not least, it is important to recall that using projective geometry of real 3-space, complex numbers enter only after fixing the absolute plane $x_0 = 0$, i.e. given the quadric $\sum x_j^2 \pm x_0^2 = 0$, $1 \leq j \leq 3$, the absolute conic is given by $\sum x_j^2 = 0$, $x_0 = 0$ and is usually represented by spinors due to Hermite's trick. Moreover, the metric introduced as Cayley-Klein metric yields an additional (intrinsic) i [11]. Last not least symmetries like parity only occur *after* this reduction step to Euclidean reps and the remaining transformation groups. Here, we focus on (real) projective rep theory while keeping in mind that *in a final step* we reduce to the Euclidean geometry.

Now the arguments given in [3] were related to the reverse of Dirac's top-down process, i.e. given a (special Lorentz) quadric, Dirac decomposed it by matrix reps into its (conjugated) linear factors. In [3], we have discussed a bottom-up process based on two plane pencils and used reps by the rank-2 spinor³ ψ . Besides understanding the two lines g, h in figure 2 both as axis of a plane pencil or a point sequence, intersection points of one line with the planes of the second pencil are projectively related. This simple

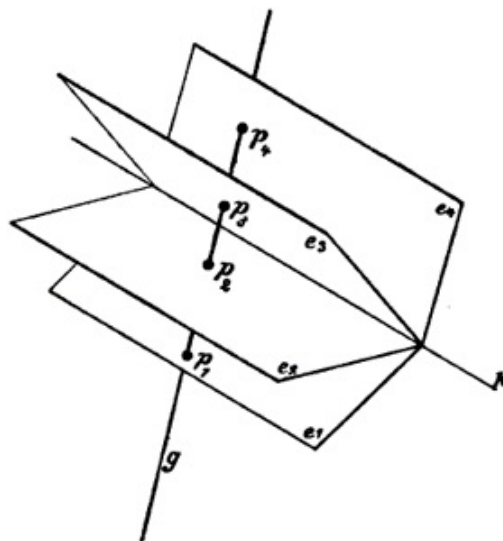


Figure 2: Visualization of related null-system transformations using two conjugate skew lines. Note that the rôle of the lines g and h serving either as point set or axis of a plane pencil can be exchanged!

picture illustrates some basic underlying principles of the Dirac picture because it yields two conjugated lines where polarity (and conjugation) is not attributed to usual polar systems known from differential or Riemannian geometry but conjugation here relates to null systems, incidence and dynamics.

The simplest generation of a (ruled) quadric, attributed to Plücker, is now to fix one of the lines and rotate the second one around this axis. The result is a (doubly) ruled quadric surface, the one-sheeted hyperboloid. Here, we find one generator of the rotation whereas in 3-space we need 3 further (dependent) generators to rotate the rotation axis freely. Right from beginning it is obvious that we find various additional reps to describe the quadric (i.e. the quadratic invariant) by points, by planes, by spheres and potencies, etc.

³The rank-1 case of singular solutions of $A|\psi\rangle$ in the matrix approach, the cone, can be associated to linear as well as quadratic Complexes and will be treated later.

The picture discussed in [3] is more sophisticated, because it used two lines conjugate with respect to a null system. So if we shift points x on one line, they can be seen as intersection points of a plane u belonging to a plane pencil where the second line represents the axis of the pencil [12]. Besides the incidence relation $x \cdot u = 0$, here the null-polarity relates translations of points with rotations of planes ('duality'). We are free, of course, to exchange the rôle of the lines. Moreover, when introducing absolute elements, we are free to arrange lines in the absolute plane and check the consequences, e.g. in choosing dynamical reps like screws with their intrinsic rotation and translation invariance. This gives also additional insight when we interpret lines of the two families as generators of a quadric (see figure 1), i.e. while 3 skew lines generate the quadric, the second family traverses these generators on the quadric.

Another generation of such a surface has been given in [3] and associated to the 4×2 -Dirac spinor ψ . Various construction details of orthogonal and simple hyperboloids (and a lot of further properties with respect to associated hyperboloid pencils, line quadrangles and usual polarity) are described at length in [13], p. 43ff, p. 69ff, and p. 78. Again, one of the basic building blocks is the Congruence with directrices g and h , and its inherent relation between point sets (line) and planes (pencils), visualized in figure 2.

Generally, besides the cones entering directly through *quadratic* Complexe, Schröter's approach [13] by two *linear* Complexe yields the connection to Congruences (or ray systems) and thus to classical treatments of optics and light. Thus it contains an additional, degenerate possibility to represent cones *linearly*. Recalling the possibility to also ask for lines contained in quadratic Complexe, both descriptions are not only connected, but we have a common geometrical method at hand to relate both the Dirac picture and classical ray systems.

As a last important property of such a construction, we mention the use of planes as basic geometrical elements in P^3 to construct the lines of a hyperboloid. So a point on the quadric is constructed by two intersecting lines, each of which has a skew representation by means of two intersecting planes q and p of the two plane pencils. Formally, setting $p_\nu = \lambda_1 u_\nu^1 + \lambda_2 u_\nu^2$ and $q_\nu = \mu_1 v_\nu^1 + \mu_2 v_\nu^2$, one intersecting line on the quadric will have coordinates $p_{\alpha\beta} \sim p_\alpha q_\beta - p_\beta q_\alpha$, and may be resolved by P^1 coordinates $\frac{\lambda_2}{\lambda_1}$ and $\frac{\mu_2}{\mu_1}$ with respect to the base planes u^1, u^2, v^1 and v^2 of the pencils. The exponential then 'transports' such expressions via the Cayley-Klein mechanism to metric reps and groups [11]. An obvious, major issue emerges here when recalling the usual Ansatz by second quantization, i.e. by expressing operators which represent wave functions according to $\phi(\vec{x}, t) \sim \int \frac{d^3k}{\sqrt{(2\pi)^3 2\omega_k}} [a(k) \exp(ik \cdot x) + (h.c.)]$ at the space-time 'point' x_μ . Besides additional assumptions on the operator statistics (of $a(k)$ and $a^+(k)$) and a formal-only use of conjugation, in usual rep theories one has to introduce additional concepts like 'entanglement' or assumptions and rules on the treatment of expectation values or infinite series of such representations to compare to experiments. These issues are formally superseded when using a priori non-local line and plane reps as well as known geometrical transfer principles to restrict to unique reps, e.g. by incidence relations or by projections.

The takeaway here is the necessary use of plane reps, especially since planar projections are essential for the very definition of masses and the treatment of the various moments of general systems.

5 Moments

In [3], we have discussed already some aspects of comparing classical 3-vectors \vec{L} and 'quantum' or spin 6-vectors S_6 , and necessary factors 2 when comparing the respective representations. This – from the viewpoint of angular momentum theory – lead us to more general considerations of moments, so in order to compare or unify the treatment of moments, we have argued to consider combinations by choosing *equal* reps of L and S in terms of 6-vectors, i.e. L_6 and S_6 instead of \vec{L} and \vec{S} (or $\vec{\sigma}$).

Now referring in general to Poincaré's and Plücker's force reps by 6-vectors, citing Dirac's claim that spin is a 6-vector [14], sec. 2, and using the reps given originally by Complexe above, we can assemble already a larger picture combining some of the relevant properties. In conjunction with Möbius' mass definitions (including the distance to a common arbitrary plane or by projecting to a bundle of parallel lines) and his barycentric coordinates [15], we can use the more general and unified point of view originally based on planes, once again embedded deeply in classical geometry. This allows to recall Reye's Ansatz [16] in order to describe extended systems in 3-space *in general* by constructing a tetrahedron with 4 masses which is able to simulate *all the moments* and thus to replace each extended spatial system. The details can be found in [16] and throughout a sequence of additional papers, however, for us using the tetrahedron as base element of P^3 , Reye's arguments connect automatically to $SU(4)$ and its fundamental (tetrahedral) rep **4**.

Here, due to space restriction, we cite few of the central steps and refer to [16]:

- Reye's major objective was to show with respect to the motion of a solid body, that in addition

to describing linear motion of a body by its mass and the resulting force of all forces acting on the body, the general case can be treated by considering in addition the moment of rotation of the forces as well as the moment of inertia of the body in relation to a variable axis of rotation.

- Reyé then proposes a construction scheme which fixes after the choice of an arbitrary space point the mass of this arbitrarily chosen point as well as the plane with the remaining three points.
- It is sufficient to consider the moments of inertia for the two systems – the original one and its replacement – with respect to an arbitrary plane because the moments of inertia with respect to two orthogonal planes are equivalent to the moment of inertia with respect to their intersection line. If x and y are the distances to two orthogonal planes and m is the mass of a constituent, $m r^2 = m x^2 + m y^2$ holds per constituent, and $\sum m r^2 = \sum m x^2 + \sum m y^2$ holds for the system. r denotes the distance from the axis where the two planes intersect. If then the moments of inertia are the same with respect to an arbitrary plane, they are the same for an arbitrary axis in 3-space. So it is sufficient to consider moments with respect to planes.
- An additional step introduces a spatial ellipsoid by means of a 'radius ρ of inertia' $M \rho^2 = \sum m r^2$ to factor out the overall mass $M = \sum m$ like treating the center of mass when considering translations.
- Reyé's proof follows a comparison of the ellipsoids of inertia of both systems using polar theory while requiring equal total mass and identical center of mass.
- Reyé proves that an appropriate tetrahedron can fulfill these requirements while giving the relations to a 'central ellipsoid' and discussing polar properties of the ellipsoid versus properties of the 'mass tetrahedron'.

In this conference context, for now it is sufficient to acknowledge the possibility of such a replacement for an arbitrary extended system in P^3 . This is of twofold general interest for upcoming work: In P^3 (or in affine/Euclidean 3-space) it shifts the focus from usual point reps towards an accompanying plane geometry because planar projections govern mass and kinematic definitions whereas standard text books don't even respect plane coordinates and masses but work with '4-momenta'. However, a thorough geometrical treatment has to differentiate between the symbolism of 4-momenta on the one hand and masses and planar coordinates on the other hand. The second important aspect relies on the tetrahedral basis and thus endorses the principal use of SU(4) rep theory. Using SU(4) as a guiding tool, we have discussed some examples, see e.g. [11], sec. III, with respect to the threefold symmetric rep **20**. While the point of view there was dominated by physical associations, from a projective position, the tetrahedron **4** has a dominant rôle in P^3 just from the central meaning as coordinate tetrahedron, with respect to polar properties e.g. when discussing Möbius tetrahedra, or in advanced tetrahedral constructions like the 15 fundamental tetrahedra in line and Complex geometry. Moreover, the 3rd order norm curve is relevant for coordinates in P^3 and relates to binary forms, null systems and tetrahedral Complexe.

6 Outlook

With respect to the objectives of this paper given above, it is obvious that and how we can derive central aspects of quantum rep theory by means of advanced projective geometry. Moreover, in order to describe massive systems as well as linear dynamics and moments of inertia, the usual approach by point geometry in P^3 must be completed with its dual picture by considering planes, too. However, the reduction scheme of SU*(4) via the Riemannian spaces AII and CI enforced the rep theory of P^5 , required by two ways: The coset spaces when reducing to an observable SU(2)×U(1) symmetry map to (symplectic) Complex geometry [7] and can be broken down to P^3 and its elements, denoted by 'points' and 'planes', or if we switch the space element in 3-space by 'lines' (or spheres after Lie transfer, or other objects using further known transfer principles...). It is here where we have used SU(4) as an approximate structure theory which essentially uses tetrahedra and their compositions, however, the main background is the rank-3 group and its fundamental rep versus the coordinatization of 3-space. The second, bottom-up approach when generalizing physical notion and observations relies on the switch of the space element from points and planes to lines and their projective relation to the Plücker-Klein quadric in P^5 . For us, based on the common reasoning for forces, force pairs and moments in both the classical and the quantum world, the embedding of classical geometry via the Plücker-Klein quadric allows to access more advanced physics by using more advanced geometrical methods whose extraordinary properties like non-commutativity, symplectic symmetry, transfer principles, etc. reflect in P^3 .

A typical problem of this kind occurs when discussing triality. Whereas naive discussions of triality sometimes end up discussing threefold structures in singlet reps of relativistic quantum theory, starting

from a trilinear map $t : V_1 \times V_2 \times V_3 \rightarrow \mathbb{R}$ of given vector spaces V_1 , V_2 , and V_3 can be cast into discussing a bi-linear map $b : V_1 \times V_2 \rightarrow V_3^*$ where V_3^* denotes the dual of V_3 to produce known 'quantum' rep theory $\langle \cdot | \cdot \rangle$ by $b' : V_3^* \times V_3 \rightarrow \mathbb{R}$ to recover t . Now recalling the action of null systems, we can 'act' with a Complex of P^5 on points in P^3 , the result is an incident planar coordinatization $u^\mu = F^{\mu\nu} x_\nu$ connecting the point x_μ with the tetrahedral coordinate edges. So choosing V_3^* as P^3 represented by planes u (i.e. the dual of the original point space), we may understand t as mapping by b and b' above and as the inner product of points. There exist, however, more direct and thorough discussions of triality in terms of Complexes which we'll discuss later in more detail. Here, we want to use the example above to propose a parallel discussion of *NOT* following necessarily spinor reps, Spin groups and Clifford or hypercomplex number reps, but to consider classical projective mechanisms, too. Because $P^n \times P^n \rightarrow P^{2n+1}$, we have a series of mappings at hand which for $n=0, 1, 2, 3$ yield P^1 , P^3 , P^5 , and P^7 . In P^1 we can use binary forms and point pairs on a line which provide generalized Legendre polynomials with respect to the line as axis [3] [17]. Mathematically suitable for Hesse transfer, physical identification of one of two point pairs on the line yields the polynomial rep of an electromagnetic field caused by charges [3].

P^3 , generated by $P^1 \times P^1$ reflects exactly the approach discussed above when constructing quadrics by plane pencils. Some of the applications are also discussed above in the context of Dirac theory and the decomposition of a quadric into linear factors, or when identifying moments or ellipsoids of inertia.

P^5 comprises Complexes and null systems, i.e. forces, force pairs and associated reps of moments, usually attributed to statics or the 'Geometrie der Lage'. Throughout this paper and some of our previous references, we have endorsed the use of linear and quadratic Complexes to approach common dynamical rep theory with intrinsic symplectic symmetry.

Following this sequence, we favour P^7 and $SO(8)$ from a geometrical view to represent generalizations of Plücker's and Study's Dynamen, i.e. in order to unify P^5 and general dynamical descriptions. We leave open whether or not it is necessary to follow division or Clifford algebras or hypercomplex number systems, or whether at a first glance it is sufficient to use Segre manifolds or Study's geometrical approach to Dynamen. For now, we have given enough evidence for the projective approach discussed above, and thus we can rely on following $P^n \times P^n \rightarrow P^{2n+1}$ as guideline towards a structure theory of physics and observations, especially when working out theoretical descriptions coined by physical/geometrical notion.

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