

Entanglement in Quantum Field Theory: particle mixing and oscillations

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Abstract. The phenomena of particle mixing and flavor oscillations in elementary particle physics are associated with multi-mode entanglement of single-particle states. We show that, in the framework of quantum field theory, these phenomena exhibit a fine structure of quantum correlations, as multi-mode multi-particle entanglement appears. Indeed, the presence of anti-particles adds further degrees of freedom, thus providing nontrivial contributions both to flavor entanglement and, more generally, to multi-partite entanglement. By using the global entanglement measure, based on the linear entropies associated with all the possible bipartitions, we analyze the entanglement in the multiparticle states of two-flavor neutrinos and anti-neutrinos. A direct comparison with the instance of the quantum mechanical Pontecorvo single-particle states is also performed.

1. Introduction

Quantum entanglement is a fundamental resource in quantum information and computation science. Topics concerning the study of quantum correlations in paradigmatic quantum systems have been addressed in several branches of condensed matter, atomic physics, and quantum optics [1]. Recently, some attention has been devoted to the investigation of entanglement in the context of elementary particles physics [2, 3, 4, 5, 6, 7, 8, 9]. In particular, in Refs. [5, 6], it has been studied the behavior of single-particle, multi-mode entanglement associated to particle mixing and oscillations. More specifically, in Ref. [6], by considering the physically relevant cases of two- and three-flavor neutrino oscillations, it has been shown that multi-mode single-particle entanglement can be expressed in terms of flavor transition probabilities. In the same paper, it is proposed a scheme for the transfer of the quantum information encoded in neutrino states to spatially delocalized two-flavor charged lepton states. It is worth to remark that, very recently, beams of neutrinos have been used to experimentally investigate the performance of a low-rate communications link, thus providing the feasibility of digital communication using neutrinos [10]. Therefore, at least in principle, the entangled states of oscillating neutrinos are legitimate physical resources for quantum information tasks. In the present work we extend the results of Ref. [6], obtained in the context of quantum mechanics (QM), to the framework of



quantum field theory (QFT). By exploiting tools of quantum information theory, we quantify the content of multi-particle flavor entanglement in the QFT system of oscillating neutrinos. Flavor states of mixed particles have been extensively discussed and a proper QFT framework for their description has been worked out in Refs. [11, 12, 13]. In this framework we show that the phenomena of flavor oscillations exhibit a richer structure of quantum correlations with respect to the corresponding QM (Pontecorvo) analog. Indeed, while the QM system of neutrino states are single-particle states possessing flavor entanglement, the QFT system of neutrino states are flavor-entangled multi-particle states, as the presence of anti-neutrino particle species provides further degrees of freedom to the oscillating neutrino system. It is important to notice that entanglement is a relative physical quantity, possessing a specific operational meaning according to the reference quantum observables, and to the quantum subsystems selected as suitable parties of the physical system. By assuming the particle-antiparticle species as further quantum modes, in section 2 we investigate the multiparticle flavor-species entanglement associated with flavor oscillations of the QFT neutrino system. Moreover, in the appendix, we also study the bipartite flavor entanglement associated with flavor oscillations, assuming, for instance, species-non-resolving detectors.

In the following we briefly review the background of the present analysis, i.e. the formalism and the achieved results associated with quantum mechanical instance. Flavor mixing of neutrinos for two generations is described by the 2×2 rotation matrix $\mathbf{U}(\theta)$ [14, 15]

$$\mathbf{U}(\theta) = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}, \quad (1)$$

where θ is the mixing angle. The two-flavor neutrino states are defined as

$$|\underline{\nu}^{(f)}\rangle = \mathbf{U}(\theta) |\underline{\nu}^{(m)}\rangle \quad (2)$$

where $|\underline{\nu}^{(f)}\rangle = (|\nu_e\rangle, |\nu_\mu\rangle)^T$ are the states with definite flavors e, μ and $|\underline{\nu}^{(m)}\rangle = (|\nu_1\rangle, |\nu_2\rangle)^T$ those with definite masses m_1, m_2 . Both $|\nu_\alpha\rangle$ ($\alpha = e, \mu$) and $|\nu_j\rangle$ ($j = 1, 2$) are orthonormal. By describing the free propagation of the neutrino mass-eigenstates with plane-waves of the form $|\nu_j(t)\rangle = e^{-i\omega_j t} |\nu_j\rangle$, ω_j denoting the frequency associated with the mass-eigenstate $|\nu_j\rangle$, the time evolution of the flavor states is given by

$$|\underline{\nu}^{(f)}(t)\rangle = \mathbf{U}(t) |\underline{\nu}^{(f)}\rangle \equiv \mathbf{U}(\theta) \mathbf{U}_0(t) \mathbf{U}^{-1}(\theta) |\underline{\nu}^{(f)}\rangle \quad (3)$$

where $|\underline{\nu}^{(f)}\rangle$ are the flavor states at $t = 0$, and $\mathbf{U}_0(t) = \text{diag}(e^{-i\omega_1 t}, e^{-i\omega_2 t})$. At time t the average neutrino number of the state $|\nu_\alpha(t)\rangle$ in the mode β is

$$\langle N_\beta \rangle_\alpha \equiv \langle \nu_\alpha(t) | N_\beta | \nu_\alpha(t) \rangle = |\mathbf{U}_{\alpha\beta}(t)|^2, \quad (\alpha, \beta = e, \mu). \quad (4)$$

By assuming the neutrino occupation number associated with a given flavor (mode) as reference quantum number, one can establish the following correspondence with two-qubit states: $|\nu_e\rangle \equiv |1\rangle_{\nu_e} |0\rangle_{\nu_\mu}$, $|\nu_\mu\rangle \equiv |0\rangle_{\nu_e} |1\rangle_{\nu_\mu}$, where $|j\rangle_{\nu_\alpha}$ stands for a j -occupation number state of a neutrino in mode α . Entanglement is thus established among flavor modes, in a single-particle setting. For instance, the free evolution of the electron-neutrino state $|\nu_e(t)\rangle$ can then be written in the form

$$|\nu_e(t)\rangle = \mathbf{U}_{ee}(t) |10\rangle + \mathbf{U}_{e\mu}(t) |01\rangle, \quad (5)$$

where $|ij\rangle$ denotes the two-qubit vector $|i\rangle_{\nu_e} |j\rangle_{\nu_\mu}$, and $|\mathbf{U}_{ee}(t)|^2 + |\mathbf{U}_{e\mu}(t)|^2 = 1$ due to normalization. Thus the time-evolved states $|\underline{\nu}^{(f)}(t)\rangle$ are entangled Bell-like superpositions of the two flavor eigenstates with time-dependent coefficients. Such an entanglement is in principle

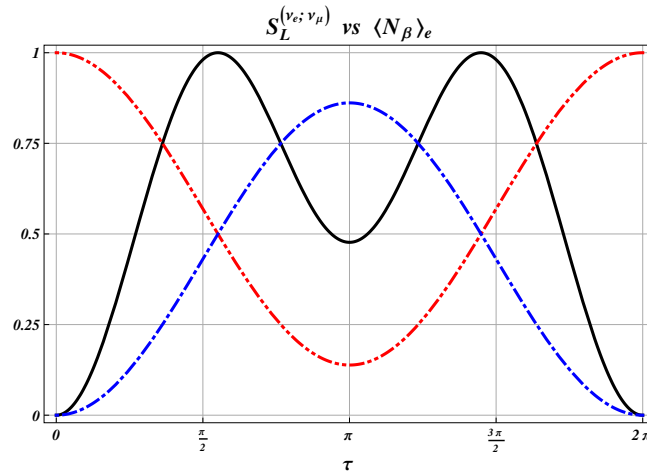


Figure 1. (Color online) QM instance: The linear entropy $S_L^{(\nu_e; \nu_\mu)}$ (full line) as a function of the scaled time $\tau = (\omega_2 - \omega_1)t$. The mixing angle θ is fixed at the experimental value $\sin^2 \theta = 0.314$. The average neutrino numbers $\langle N_e \rangle_e$ (double-dot-dashed line) and $\langle N_\mu \rangle_e$ (dot-dashed line) are also reported.

experimentally accessible, throughout a scheme for its transfer from single-neutrino states to two-flavor charged lepton states [6].

The von Neumann entropy, or any other monotonic function of it, quantifies the bipartite entanglement of pure states [16, 17]. In this paper we use a set of linear entropies to describe the quantum correlations of the neutrino oscillating system. Indeed, the linear entropy can be exploited to construct entanglement measure for multipartite systems [18, 19, 20]. For example, the global entanglement is associated to the set of linear entropies associated to all possible bi-partitions of the whole system. An alternative (geometric) characterization of multipartite entanglement is given in Refs. [21, 22]. For completeness, we recall the definition of such an entanglement measure. Let $\rho = |\psi\rangle\langle\psi|$ be the density operator corresponding to a pure state $|\psi\rangle$, describing the system S partitioned into N parties. Consider the bipartition of the N -partite system S in two subsystems S_{A_n} , constituted by n parties ($1 \leq n < N$), and $S_{B_{N-n}}$, constituted by the remaining $N - n$ parties. Let $\rho_{A_n} \equiv \text{Tr}_{B_{N-n}}[\rho]$ denote the reduced density matrix of subsystem S_{A_n} after tracing over subsystem $S_{B_{N-n}}$. The linear entropy associated to such a bipartition is defined as

$$S_L^{(A_n; B_{N-n})}(\rho) = \frac{d}{d-1}(1 - \text{Tr}_{A_n}[\rho_{A_n}^2]), \quad (6)$$

where the d is the Hilbert-space dimension given by $d = \min\{\dim S_{A_n}, \dim S_{B_{N-n}}\} = \min\{2^n, 2^{N-n}\}$. Finally, we introduce the average linear entropy

$$\langle S_L^{(n; N-n)}(\rho) \rangle = \binom{N}{n}^{-1} \sum_{A_n} S_L^{(A_n; B_{N-n})}(\rho), \quad (7)$$

where the sum is intended over all the possible bi-partitions of the system in two subsystems, respectively with n and $N - n$ elements ($1 \leq n < N$) [20]. The linear entropies (6), (7) can be easily computed for the two-qubit Bell state $|\nu_e(t)\rangle$, i.e. Eq. (5), with density matrix $\rho_e = |\nu_e(t)\rangle\langle\nu_e(t)|$.

The linear entropy $S_L^{(\nu_e;\nu_\mu)} \equiv S_L^{(\nu_e;\nu_\mu)}(\rho_e)$ associated to the reduced state after tracing over one mode (flavor) writes:

$$S_L^{(\nu_e;\nu_\mu)} = S_L^{(\nu_\mu;\nu_e)} = 4|\mathbf{U}_{ee}(t)|^2 |\mathbf{U}_{e\mu}(t)|^2 = 4|\mathbf{U}_{e\mu}(t)|^2 (1 - |\mathbf{U}_{e\mu}(t)|^2). \quad (8)$$

Therefore, the linear entropy (8) is strictly related to the average neutrino numbers, i.e. $S_L^{(\nu_e;\nu_\mu)} = \langle N_e \rangle_e \langle N_\mu \rangle_e$. Specifically the linear entropies $S_L^{(\nu_e;\nu_\mu)}$ coincide, apart from a constant factor, to the variance associated with the average neutrino number, i.e.

$$\langle (\Delta N_{\nu_e})^2 \rangle \equiv \langle N_e^2 \rangle_e - \langle N_e \rangle_e^2 = \langle N_e \rangle_e (1 - \langle N_e \rangle_e) = 4^{-1} S_L^{(\nu_e;\nu_\mu)}. \quad (9)$$

In Fig. 1 we show the behavior of $S_L^{(\nu_e;\nu_\mu)}$ and of $\langle N_\beta \rangle_e$ as functions of the scaled, dimensionless time $\tau = (\omega_2 - \omega_1)t$. At time $\tau = 0$, the two flavors are not mixed, the entanglement is zero, and the global state of the system is factorized. For $\tau > 0$, flavor oscillations occur; the entanglement is maximal at largest mixing: $\langle N_e \rangle_e = \langle N_\mu \rangle_e = 1/2$.

2. QFT flavor entanglement

In order to present a generalization of the above analysis to the QFT framework, we recall the essential features of a specific QFT model of particle mixing describing the phenomena of neutrino oscillations [11, 12]. The neutrino fields $\nu_e(x)$ and $\nu_\mu(x)$ are defined through the mixing relations:

$$\nu_e(x) = \cos \theta \nu_1(x) + \sin \theta \nu_2(x), \quad (10)$$

$$\nu_\mu(x) = -\sin \theta \nu_1(x) + \cos \theta \nu_2(x), \quad (11)$$

where, in standard notation, x stands for the four-vector $x \equiv (t, x_1, x_2, x_3) \equiv (t, \mathbf{x})$, and the free fields $\nu_1(x)$ and $\nu_2(x)$ denote the neutrino mass eigenstates. The generator of the mixing transformations is given by:

$$G_\theta(t) = \exp \left[\theta \int d^3 \mathbf{x} \left(\nu_1^\dagger(x) \nu_2(x) - \nu_2^\dagger(x) \nu_1(x) \right) \right] \quad (12)$$

so that

$$\nu_\sigma^\alpha(x) = G_\theta^{-1}(t) \nu_i^\alpha(x) G_\theta(t), \quad (13)$$

where $(\sigma, i) = (e, 1), (\mu, 2)$, and the superscript $\alpha = 1, \dots, 4$ denotes the spinorial component. At finite volume, $G_\theta(t)$ is a unitary operator, i.e. $G_\theta^{-1}(t) = G_{-\theta}(t) = G_\theta^\dagger(t)$, preserving the canonical anticommutation relations. The generator $G_\theta^{-1}(t)$ maps the Hilbert space for free fields $\mathcal{H}_{1,2}$ to the Hilbert space for mixed fields $\mathcal{H}_{e,\mu}$: $G_\theta^{-1}(t) : \mathcal{H}_{1,2} \mapsto \mathcal{H}_{e,\mu}$. In particular, the flavor vacuum is given by $|0(t)\rangle_{e,\mu} = G_\theta^{-1}(t) |0\rangle_{1,2}$ at finite volume V . We denote by $|0\rangle_{e,\mu}$ the flavor vacuum at $t = 0$. It is worth noticing that, in the infinite volume limit, the flavor and the mass vacua are unitarily inequivalent. The free fields $\nu_i(x)$ ($i = 1, 2$) are given by the following expansions

$$\nu_i(x) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k}, r} \left[u_{\mathbf{k}, i}^r \alpha_{\mathbf{k}, i}^r(t) + v_{-\mathbf{k}, i}^r \beta_{-\mathbf{k}, i}^{r\dagger}(t) \right] e^{i\mathbf{k} \cdot \mathbf{x}}, \quad (14)$$

where $\mathbf{k} \equiv (k_1, k_2, k_3)$ is the momentum vector, $r = 1, 2$ denotes the helicity, $\alpha_{\mathbf{k}, i}^r(t) = \alpha_{\mathbf{k}, i}^r e^{-i\omega_{k,i}t}$, $\beta_{\mathbf{k}, i}^{r\dagger}(t) = \beta_{\mathbf{k}, i}^{r\dagger} e^{i\omega_{k,i}t}$, and $\omega_{k,i} = \sqrt{\mathbf{k}^2 + m_i^2}$. The operators $\alpha_{\mathbf{k}, i}^r$ and $\beta_{\mathbf{k}, i}^r$ are the

annihilation operators for the vacuum state $|0\rangle_m \equiv |0\rangle_1 \otimes |0\rangle_2$, i.e. $\alpha_{\mathbf{k},i}^r |0\rangle_m = \beta_{\mathbf{k},i}^r |0\rangle_m = 0$. The anticommutation relations are the usual ones; for further details, e.g. orthonormality and completeness relations, see Refs. [11, 12]. By use of $G_\theta(t)$, the flavor fields can be expanded as:

$$\nu_\sigma(\mathbf{x}) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k},r} \left[u_{\mathbf{k},i}^r \alpha_{\mathbf{k},\sigma}^r(t) + v_{-\mathbf{k},i}^r \beta_{-\mathbf{k},\sigma}^{r\dagger}(t) \right] e^{i\mathbf{k}\cdot\mathbf{x}}. \quad (15)$$

The flavor annihilation operators are defined as $\alpha_{\mathbf{k},\sigma}^r(t) \equiv G_\theta^{-1}(t) \alpha_{\mathbf{k},i}^r G_\theta(t)$ and $\beta_{\mathbf{k},\sigma}^{r\dagger}(t) \equiv G_\theta^{-1}(t) \beta_{\mathbf{k},i}^{r\dagger} G_\theta(t)$. Without any loss of generality, let us choose the reference frame such that $\mathbf{k} = (0, 0, |\mathbf{k}|)$, we have

$$\alpha_{\mathbf{k},e}^r(t) = \cos \theta \alpha_{\mathbf{k},1}^r(t) + \sin \theta \left(|U_{\mathbf{k}}| \alpha_{\mathbf{k},2}^r(t) + \epsilon^r |V_{\mathbf{k}}| \beta_{-\mathbf{k},2}^{r\dagger}(t) \right), \quad (16)$$

where $\epsilon^r = (-1)^r$ and

$$|U_{\mathbf{k}}| \equiv u_{\mathbf{k},i}^{r\dagger} u_{\mathbf{k},j}^r = v_{-\mathbf{k},i}^{r\dagger} v_{-\mathbf{k},j}^r = \frac{|\mathbf{k}|^2 + (\omega_{k,1} + m_1)(\omega_{k,2} + m_2)}{2\sqrt{\omega_{k,1}\omega_{k,2}(\omega_{k,1} + m_1)(\omega_{k,2} + m_2)}}, \quad (17)$$

$$|V_{\mathbf{k}}| \equiv \epsilon^r u_{\mathbf{k},1}^{r\dagger} v_{-\mathbf{k},2}^r = -\epsilon^r u_{\mathbf{k},2}^{r\dagger} v_{-\mathbf{k},1}^r = \frac{(\omega_{k,1} + m_1) - (\omega_{k,2} + m_2)}{2\sqrt{\omega_{k,1}\omega_{k,2}(\omega_{k,1} + m_1)(\omega_{k,2} + m_2)}} |\mathbf{k}|, \quad (18)$$

with $i, j = 1, 2$, $i \neq j$, $|U_{\mathbf{k}}|^2 + |V_{\mathbf{k}}|^2 = 1$. The coefficient $|V_{\mathbf{k}}|$ is associated with the condensation density of the flavor vacuum and is responsible for several phenomenological consequences [13, 23, 24, 25]. The explicit expression for the flavor states $|\nu_{\mathbf{k},e}^r\rangle$ at time $t = 0$ is

$$\begin{aligned} |\nu_{\mathbf{k},e}^r\rangle &\equiv \alpha_{\mathbf{k},e}^{r\dagger}(0) |0\rangle_{e,\mu} \\ &= [\cos \theta \alpha_{\mathbf{k},1}^{r\dagger} + |U_{\mathbf{k}}| \sin \theta \alpha_{\mathbf{k},2}^{r\dagger} - \epsilon^r |V_{\mathbf{k}}| \sin \theta \alpha_{\mathbf{k},1}^{r\dagger} \alpha_{\mathbf{k},2}^{r\dagger} \beta_{-\mathbf{k},1}^{r\dagger}] G_{\theta,\mathbf{k},r}^{-1} \prod_{\mathbf{p} \neq \mathbf{k}} G_{\theta,\mathbf{p},s}^{-1} |0\rangle_{1,2}, \end{aligned} \quad (19)$$

where $G_\theta = \prod_{\mathbf{p}} \prod_{s=1}^2 G_{\theta,\mathbf{p},s}$. In the state (19), a multiparticle component is present, disappearing in the relativistic limit $|\mathbf{k}| \gg \sqrt{m_1 m_2}$: indeed, for large $|\mathbf{k}|$, since one gets $|U_{\mathbf{k}}|^2 \rightarrow 1$ and $|V_{\mathbf{k}}|^2 \rightarrow 0$, the (quantum-mechanical) Pontecorvo states are recovered. In order to simplify the notation, we omit the superscript r (by fixing $r = 2$) and the subscript \mathbf{k} , thus restricting the analysis to the flavor neutrino state $|\nu_e\rangle$ of fixed momentum and helicity. Let us consider again the free evolution of the electron-neutrino state (19):

$$|\nu_e(t)\rangle = e^{-iH_{free}t} |\nu_e\rangle, \quad (20)$$

where H_{free} is the standard QFT free Hamiltonian. In the Hilbert space $\mathcal{H}_{e,\mu}$, Eq. (20) can be rewritten in the form:

$$|\nu_e(t)\rangle = [\mathbf{U}_{ee}(t) \alpha_e^\dagger + \mathbf{U}_{e\mu}(t) \alpha_\mu^\dagger + \mathbf{U}_{e\mu}^{e\bar{e}}(t) \alpha_e^\dagger \alpha_\mu^\dagger \beta_e^\dagger + \mathbf{U}_{ee}^{\mu\bar{\mu}}(t) \alpha_e^\dagger \alpha_\mu^\dagger \beta_\mu^\dagger] |0\rangle_{e,\mu} \quad (21)$$

where the time-dependent coefficients are given by:

$$\begin{aligned} \mathbf{U}_{ee}(t) &= e^{-i\omega_1 t} [\cos^2 \theta + \sin^2 \theta (e^{-i(\omega_2 - \omega_1)t} |U|^2 + e^{-i(\omega_2 + \omega_1)t} |V|^2)] \\ \mathbf{U}_{e\mu}(t) &= e^{-i\omega_1 t} U \cos \theta \sin \theta (e^{-i(\omega_2 - \omega_1)t} - 1) \\ \mathbf{U}_{e\mu}^{e\bar{e}}(t) &= e^{-i\omega_1 t} V \cos \theta \sin \theta (1 - e^{-i(\omega_2 + \omega_1)t}) \\ \mathbf{U}_{ee}^{\mu\bar{\mu}}(t) &= e^{-i\omega_1 t} UV \sin^2 \theta (e^{-i(\omega_2 + \omega_1)t} - e^{-i(\omega_2 - \omega_1)t}), \\ |\mathbf{U}_{ee}(t)|^2 + |\mathbf{U}_{e\mu}(t)|^2 + |\mathbf{U}_{e\mu}^{e\bar{e}}(t)|^2 + |\mathbf{U}_{ee}^{\mu\bar{\mu}}(t)|^2 &= 1. \end{aligned} \quad (22)$$

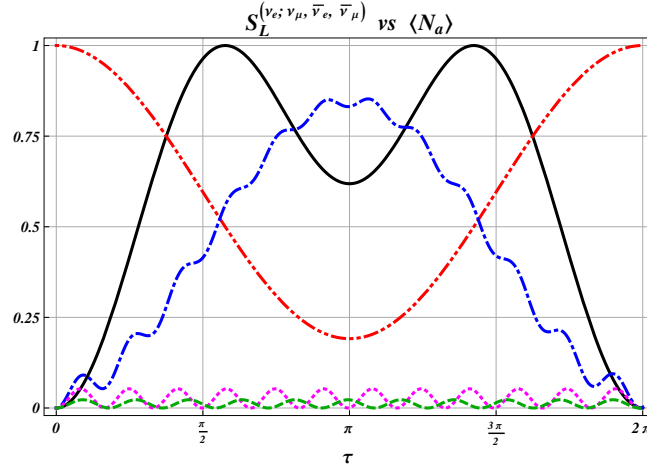


Figure 2. (Color online) QFT instance: The linear entropy $S_L^{(\nu_e; \nu_\mu, \bar{\nu}_e, \bar{\nu}_\mu)}$ (full line) as a function of the scaled time $\tau = (\omega_2 - \omega_1)t$. The mixing angle θ is fixed at the experimental value $\sin^2 \theta = 0.314$; the parameters x and p are fixed as $x = 10$ and $p = 5$. The average particle numbers $\langle N_{\nu_e} \rangle$ (double-dot-dashed line), $\langle N_{\nu_\mu} \rangle$ (dot-dashed line), $\langle N_{\bar{\nu}_e} \rangle$ (dotted line), and $\langle N_{\bar{\nu}_\mu} \rangle$ (dashed line) are also reported.

In the following, in order to conveniently parameterize the neutrino masses m_1 and m_2 , momentum $|\mathbf{k}|$, and the evolution time t , we use the real parameters $x = m_2/m_1$, $p = |\mathbf{k}|/\sqrt{m_1 m_2}$, and $\tau = (\omega_2 - \omega_1)t$. Therefore, x represents the ratio between the two masses eigenvalues; p expresses the ratio between the momentum and the masses geometrical mean and corresponds to the relativistic limit for $p \gg 1$. Evidently, the time-evolved flavor state (21) is a multi-particle entangled state. Analogously with the Pontecorvo states (5), we assume the neutrino occupation number as reference quantum number. However, with respect to Eq. (5), we have still two flavors, but we have a further degree of freedom that is the neutrino species, i.e. particles and anti-particles. In particular, the present analysis focuses on the multipartite entanglement both in flavors and in species (detectors resolving both the flavors and the species). Thus, we have truly multipartite entanglement in a four-qubit state that can be written in the form:

$$|\nu_e(t)\rangle = \mathbf{U}_{ee}(t)|1000\rangle + \mathbf{U}_{e\mu}(t)|0100\rangle + \mathbf{U}_{e\bar{e}}^{e\bar{e}}(t)|1110\rangle + \mathbf{U}_{e\bar{e}}^{\mu\bar{\mu}}(t)|1101\rangle, \quad (23)$$

where $|ijklh\rangle$ denotes the four-qubit vector $|i\rangle_{\nu_e}|j\rangle_{\nu_\mu}|k\rangle_{\bar{\nu}_e}|h\rangle_{\bar{\nu}_\mu}$ with $i, j, k, h = 0, 1$. For simplicity of notation, in Eq. (23) the time dependence has been omitted. In the appendix, we also consider the following instance: multipartite flavor entanglement (detectors resolving only the flavors independently of the species). Let us analyze the multipartite entanglement possessed by $|\nu_e\rangle$, i.e. Eq. (23), by using the global entanglement defined by Eqs. (6) and (7). Specifically, we compute the linear entropies $S_L^{(a;b,c,d)}$ associated with the bipartition of a single-particle subsystem and a three-particles subsystem, and the corresponding average linear entropy $\langle S_L^{(1;3)} \rangle$.

The linear entropies $S_L^{(a;b,c,d)}$ associated with $|\nu_e(t)\rangle$ are given by simple generalizations of Eq. (8); we have:

$$S_L^{(\nu_e; \nu_\mu, \bar{\nu}_e, \bar{\nu}_\mu)} = 4|\mathbf{U}_{e\mu}(t)|^2(1 - |\mathbf{U}_{e\mu}(t)|^2), \quad (24)$$

$$S_L^{(\nu_\mu; \nu_e, \bar{\nu}_e, \bar{\nu}_\mu)} = 4|\mathbf{U}_{ee}(t)|^2(1 - |\mathbf{U}_{ee}(t)|^2), \quad (25)$$

$$S_L^{(\bar{\nu}_e; \nu_e, \nu_\mu, \bar{\nu}_\mu)} = 4|\mathbf{U}_{e\bar{e}}^{e\bar{e}}(t)|^2(1 - |\mathbf{U}_{e\bar{e}}^{e\bar{e}}(t)|^2), \quad (26)$$

$$S_L^{(\bar{\nu}_\mu; \nu_e, \nu_\mu, \bar{\nu}_e)} = 4|\mathbf{U}_{e\bar{e}}^{\mu\bar{\mu}}(t)|^2(1 - |\mathbf{U}_{e\bar{e}}^{\mu\bar{\mu}}(t)|^2). \quad (27)$$

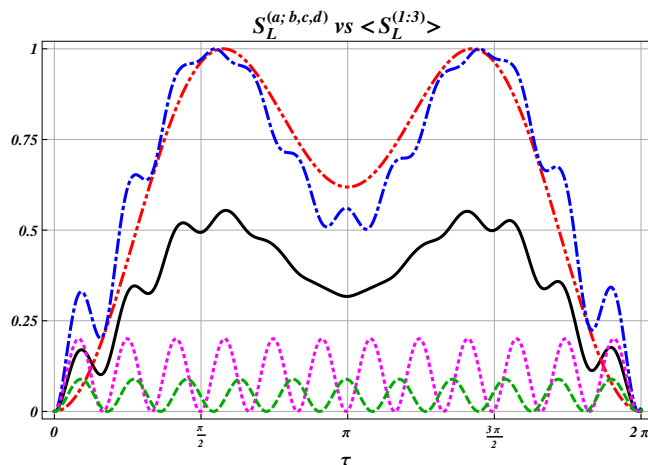


Figure 3. (Color online) QFT instance: The linear entropies $S_L^{(\nu_e; \nu_\mu, \bar{\nu}_e, \bar{\nu}_\mu)}$ (double-dot-dashed line), $S_L^{(\nu_\mu; \nu_e, \bar{\nu}_e, \bar{\nu}_\mu)}$ (dot-dashed line), $S_L^{(\bar{\nu}_e; \nu_e, \nu_\mu, \bar{\nu}_\mu)}$ (dotted line), $S_L^{(\bar{\nu}_\mu; \nu_e, \nu_\mu, \bar{\nu}_e)}$ (dashed line), and the average linear entropy $\langle S_L^{(1:3)} \rangle$ (full line) as functions of the scaled time $\tau = (\omega_2 - \omega_1)t$. The mixing angle θ is fixed at the experimental value $\sin^2 \theta = 0.314$; the parameters x and p are fixed as $x = 10$ and $p = 5$.

Of course, in the quantum mechanical limit, Eqs. (24) and (25) reduce to the Pontecorvo analogs, while Eqs. (26) and (27) go to zero. It is important to remark that, analogously to the quantum mechanics instance, the linear entropies $S_L^{(a;b,c,d)}$ are proportional to the variances associated with the particle number:

$$\langle (\Delta N_a)^2 \rangle \equiv \langle N_a^2 \rangle - \langle N_a \rangle^2 = 4^{-1} S_L^{(a;b,c,d)}. \quad (28)$$

Eq. (28) furnishes a clear operational meaning for the entanglement measures (24) to (27), throughout a direct connection with the average particle number $\langle N_a \rangle$, $a = \nu_e, \nu_\mu, \bar{\nu}_e, \bar{\nu}_\mu$.

In Fig. 2 we plot the quantity (24) as a function of the scaled time τ for $x = 10$ and $p = 5$; it is worth noticing that, such a choice of the parameters corresponds to the following assumptions: mass m_2 greater than mass m_1 of one order of magnitude, and momentum of the same order of magnitude as the masses geometrical mean. In the same plot, the average particle numbers $\langle N_a \rangle$ are also reported. Comparing the QM and QFT instances, i.e. Figs. 1 and 2 respectively, the linear entropies $S_L^{(\nu_e; \nu_\mu)}$ and $S_L^{(\nu_e; \nu_\mu, \bar{\nu}_e, \bar{\nu}_\mu)}$ show a very similar shape and behavior; however, the common minimum at $\tau = \pi$ is greater in the QFT instance. Moreover, the presence of antineutrinos gives rise to rapidly oscillating components characterizing the average particle numbers $\langle N_{\nu_\mu} \rangle$, $\langle N_{\bar{\nu}_e} \rangle$, and $\langle N_{\bar{\nu}_\mu} \rangle$. In Fig. 3 we plot the linear entropies of the form $S_L^{(a;b,c,d)}$, i.e. associated with the bipartition formed by a single party versus the other three parties; the average linear entropy (7) is also plotted. All the linear entropies, except $S_L^{(\nu_e; \nu_\mu, \bar{\nu}_e, \bar{\nu}_\mu)}$, exhibit rapidly oscillating components, due to the phenomenon of oscillations involving antineutrinos.

In conclusion we have investigated the multiparticle entanglement in the phenomenon of the neutrino flavor oscillations in the arena of QFT. The analysis represents an extension of the results reported in Refs. [6, 26]. An interesting outlook for the future developments concerns the study of quantum information protocols exploiting the entanglement associated with neutrino oscillations.

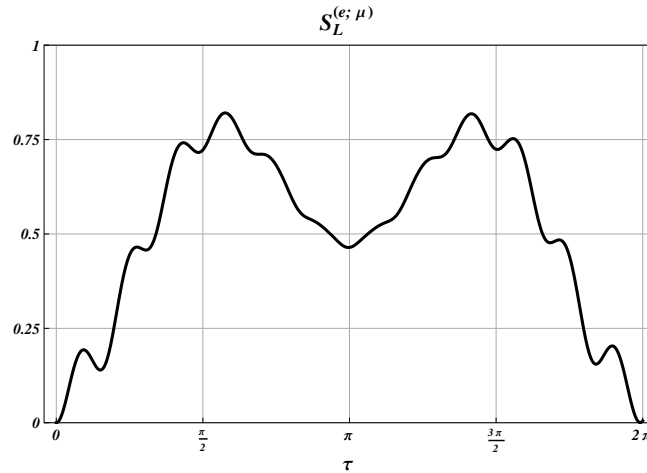


Figure A1. (Color online) QFT instance: The linear entropy $S_L^{(e;\mu)}$ (full line) as a function of the scaled time $\tau = (\omega_2 - \omega_1)t$. The mixing angle θ is fixed at the experimental value $\sin^2 \theta = 0.314$; the parameters x and p are fixed as $x = 10$ and $p = 5$.

Appendix

We present a further operational configuration: only flavor entanglement is considered and the species represents an internal degree of freedom; thus, one has bipartite entanglement in a two-qutrit state that can be written in the form:

$$|\nu_e\rangle = \mathbf{U}_{ee}(t)|10\rangle + \mathbf{U}_{e\mu}(t)|01\rangle + \mathbf{U}_{e\bar{e}}(t)|21\rangle + \mathbf{U}_{e\bar{\mu}}(t)|12\rangle, \quad (\text{A.1})$$

where $|ij\rangle$ denotes the two-qutrit vector $|i\rangle_{\nu_e}|j\rangle_{\nu_\mu}$ with $i, j = 0, 1, 2$, the level 2 representing the anti-neutrino species. In this case, it is assumed that the detection does not resolve the species (particle or antiparticle); then the entanglement is established between the flavors e and μ . It is easy to show that the linear entropy associated with the bipartition $(e; \mu)$ is given by:

$$S_L^{(e;\mu)} = (|\mathbf{U}_{ee}(t)|^2 + |\mathbf{U}_{e\mu}(t)|^2) \left[1 - (|\mathbf{U}_{ee}(t)|^2 + |\mathbf{U}_{e\mu}(t)|^2) \right]. \quad (\text{A.2})$$

$S_L^{(e;\mu)}$, i.e. Eq. (A.2), is plotted in Fig. A1. In this case, the neutrino oscillations associated with the antineutrino channels strongly affects the behavior of the linear entropy which exhibits oscillating components.

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