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Reentrant Hawking–Page phase transition of charged Gauss–Bonnet-AdS black holes in the grand canonical ensemble

Xiao-yan Hu[a], **Yuan-zhang Cui**[b], **Wei Xu**[c]

School of Mathematics and Physics, China University of Geosciences, Wuhan 430074, China

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Abstract In this paper, we study the reentrant Hawking–Page transition in the grand canonical ensemble of Gauss–Bonnet AdS spacetime. We find that the four-dimensional Gauss–Bonnet hyperbolic AdS black hole always has a reentrant Hawking–Page transition in the range of electric potential $0 < \Phi < \Phi_{tr}$, accompanied by the appearance of the triple point. However, once the potential exceeds a certain upper limit Φ_{tr}, i.e. $\Phi > \Phi_{tr}$, the Hawking–Page transition disappears. In the spacetime of five and higher dimensional Gauss–Bonnet hyperbolic AdS black hole, the reentrant Hawking–Page transition is solely observed to occur when the electric potential Φ lies between two specific thresholds ($\Phi_c < \Phi < \Phi_{tr}$). In scenarios where the electric potential is below Φ_c ($\Phi < \Phi_c$), only the standard Hawking–Page transition as in the the Einstein gravity is observed. Similar to the four-dimensional case, the Hawking–Page transition is negated when the electric potential exceeds Φ_{tr} ($\Phi > \Phi_{tr}$). We give the coexistence line, the triple point and critical point of the Hawking–Page transition in the phase diagram of the Gauss–Bonnet hyperbolic AdS black hole. The observed reentrant Hawking–Page transitions and triple points in the context of Gauss–Bonnet hyperbolic AdS black holes may correspond to the phase transitions and triple points in QCD phase diagrams, following the spirit of the AdS/CFT correspondence. To be a complete research, the Hawking–Page transition of d-dimensional charged spherical Gauss–Bonnet-AdS black hole in the grand canonical ensemble is also study in the Appendix, for which there exists a standard Hawking–Page transition with the transition temperature depending on the Gauss–Bonnet constant α.

[a] e-mail: 1202210958@cug.edu.cn
[b] e-mail: kylins@cug.edu.cn
[c] e-mail: xuwei@cug.edu.cn (corresponding author)

1 Introduction

Black hole thermodynamics is one of the central topics in modern theoretical physics. It is now widely accepted that a black hole is a complex thermodynamic system with thermodynamic variables such as entropy and temperature [1]. The establishment of the four laws of black hole thermodynamics makes the connection between thermodynamics, classical gravity and quantum mechanics more close [2]. It is believed that the study of black hole thermodynamics will help to further deepen the understanding of the quantum properties of gravity.

In recent years, black hole thermodynamics in the extended phase space of anti-de Sitter (AdS) spacetime has been established by treating the cosmological constant as the pressure, its conjugate as the thermodynamic volume, and the black hole mass as the enthalpy [3–9]. In the framework of the extended phase space, black hole thermodynamics exhibits very similar properties to non-ideal fluids. For example, the small black hole (SBH)/large black hole (LBH) phase transition of a four-dimensional Reissner–Nordström-AdS black hole is physically similar to the liquid/gas phase transition of a van der Waals gas [10–12]. Later, a series of interesting phase transitions have been studied in black hole thermodynamics, such as the reentrant phase transitions [13,14] and the superfluid phase transitions [15].

Due to the discovery of the AdS/CFT correspondence, black hole thermodynamics and phase transitions in AdS spacetime have attracted much attention [16–18]. In particular, the Hawking–Page phase transition [19], a first-order phase transition between a large black hole and a hot AdS vacuum, can be interpreted as a confinement/deconfinement phase transition in canonical field theory [20]. Generally, at low temperatures, the thermal AdS vacuum phase dominates the partition function. At high temperatures, the black hole phase dominates. Therefore, when the temperature is

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greater than the Hawking Page phase transition temperature, the hot AdS gas will collapse into a large stable black hole. Recently, there has been a lot of research on the Hawking–Page transition in different contexts, especially with respect to Hawking–Page phase transitions under the modified theory of gravity (e.g. [21–31]).

Gauss–Bonnet gravitation, as a special case of Lovelock gravitation, is the second order of higher order curvature gravitation, and it has the following action [32]:

$$S_d = \frac{1}{16\pi G_d}\int d^d x\sqrt{-g}(R - 2\Lambda + \alpha\mathcal{G}),\\ \mathcal{G} := R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho} - 4R_{\mu\nu}R^{\mu\nu} + R^2, \tag{1}$$

where $\mathcal{G}$ is the Gauss–Bonnet term, α is the Gauss–Bonnet coupling constant and Λ is the cosmological constant. The Gauss–Bonnet term appears naturally in low-energy efficient theories derived from string theory [33]. The solution of a black hole under Gauss–Bonnet gravity has been discovered [34–39], and the correction caused by the Gauss–Bonnet term plays an important role in spacetime. Although the Gauss–Bonnet term is a quadratic curvature tensor, the gravitational field equation still has a second-order form (a sufficient condition to prevent Ostrogradsky instability), so it is possible to have a definite solution. However, the Gauss–Bonnet term is often limited to higher dimensions (above five dimensions), because the integral of the Gauss–Bonnet term on a four-dimensional space-time manifold is equal to a constant whose value depends on the manifold's Euler property. Therefore, although $\mathcal{G} \neq 0$ in the four-dimensional case, the Gauss–Bonnet term is often ignored as a topological invariant in the four-dimensional space-time. However, in black hole thermodynamics, entropy is still modified by the Gauss–Bonnet constant α in the case of four dimensions, so it still has certain research significance.

Recently, a new class of Hawking–Page phase transitions has been proposed under the Gauss–Bonnet gravity, namely, the Hawking–Page phase transition with reentrant characteristics and triple points [31]. The Reentrant Hawking–Page phase transition consists of two HP phase transitions with different HP temperatures. When the temperature gradually increases, the small mass black hole/massless black hole phase and the massless black hole/large black hole phase will occur successively. The triple point corresponds to the small black hole phase, and the massless black hole phase and the large black hole phase coexist. According to the AdS/CFT holographic duality, this new class of Hawking–Page phase transition with reentrance and triple points may be associated with the reentrant phase transitions composed of deconfinement phase transitions and quark phase transitions in the QCD phase diagram, as well as the three-phase coexistence states of hadronic matter, quark matter and quark-gluon plasma [40–57].

In this paper, we will further extend this kind of Hawking–Page phase transition to the grand canonical ensemble in the Gauss–Bonnet gravity. The final results show that the reentrant Hawking–Page phase transition always exists when the electric potential is in the range $0 \leq \Phi < \Phi_{tr}$ in four-dimensional spacetime, and the Hawking–Page phase transition temperature and the position of the triple point in the phase diagram are also corrected by the electric potential. When the potential is $\Phi > \Phi_{tr}$, the Hawking–Page phase transition will disappear. Reentrant Hawking–Page phase transition always exists when the electric potential is in the range $\Phi_c \leq \Phi < \Phi_{tr}$ in high dimensional spacetime, and the electric potential has a notable effect on the temperature of the Hawking–Page phase transition and the position of the triple point in the phase diagram. When the electric potential is $\Phi > \Phi_{tr}$, the Hawking–Page phase transition in higher dimensions disappears as well. In order to give a complete research of the Hawking–Page phase transition of d-dimensional charged Gauss–Bonnet-AdS black hole in the grand canonical ensemble, the spherical case is also study in the Appendix A. For this case, there exists a standard Hawking–Page phase transition with the Hawking–Page transition temperature depending on the Gauss–Bonnet constant α. These findings may contribute to a deeper understanding of black hole thermodynamics in the framework of quantum gravity, and may provide new ideas for exploring AdS/CFT correspondences beyond the bounds of classical gravity.

The organization of this paper is as follows: In Sect. 2, we will revisit the extended thermodynamics of hyperbolic AdS black holes in the Gauss–Bonnet gravitity in the grand canonical ensemble. In Sects. 3 and 4, we will study the reentrant HP phase transition and the triple point of dimensions $d = 4$ and $d \geq 5$ respectively. Finally, some concluding remarks are given.

2 Extended thermodynamics of the charged hyperbolic AdS Black hole in Gauss–Bonnet gravity in the grand canonical ensemble

In this section, we will revisit the extended thermodynamics of charged hyperbolic AdS black holes in the d-dimensional Gauss–Bonnet gravitational theory in the grand canonical ensemble. This black hole solution takes the form [34–39],

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2\, d\Omega^2_{d-2,k}, \tag{2}$$

$$f(r) = k + \frac{r^2}{2\widetilde{\alpha}}\left(1 - \sqrt{1 + \frac{64\pi\widetilde{\alpha}M}{(d-2)r^{d-1}} - \frac{2\widetilde{\alpha}Q^2}{(d-2)(d-3)r^{2d-4}} - \frac{64\pi\widetilde{\alpha}P}{(d-1)(d-2)}}\right), \tag{3}$$

where $d\Omega^2_{d-2,k}$ is the line element of maximally symmetric Einstein manifold of the $d-2$ dimensional black hole topology of the horizon corresponding to the curvature k. Σ_k is the unit area of the $d-2$ dimensional topology, which is taken as 1 in this paper for simplicity. Note that M is the mass of the black hole. And $\widetilde{\alpha} = (d-3)(d-4)\alpha$ is a renormalized Gauss–Bonnet coupling constant. In this article, we only consider the case of $\alpha > 0$, because in string theory, α is proportional to the inverse string tension of the positive coefficient [26]. We choose the space-time dimension of $d \geq 4$, because in the four-dimensional space-time, the black hole entropy has a non-trivial effect, and the Gauss–Bonnet term will have a significant impact on the black hole thermodynamics. Under the extended thermodynamics, the thermodynamic pressure of a AdS black hole is associated with the negative cosmological constant Λ, i.e. $P = -\frac{\Lambda}{8\pi} = \frac{(d-1)(d-2)}{16\pi\ell^2}$, where l is the d dimensional AdS radius. In higher derivative gravity, black holes solutions always have two branches. In this paper, we only consider the case in Eq. (3), and another branch solution is not able to approach the Schwarzschild limit and is unstable. Finally, in order for the well-defined vacuum solution, i.e. $M = Q = 0$, it needs to be satisfied

$$0 < \frac{64\pi\widetilde{\alpha}P}{(d-2)(d-1)} \leq 1, \quad d \geq 5. \tag{4}$$

Subsequently, we present the mass, temperature, and entropy associated with the hyperbolic($k=-1$) AdS black hole [31,38]. (See Appendix A for a discussion of the spherical($k=1$) AdS black holes.)

$$M = \frac{r_+^{d-5}\left((d-1)(d-2)\cdot\left((d-3)(d-4)\alpha - r_+^2\right) + 16\pi P r_+^4\right)}{16\pi\cdot(d-1)} + \frac{Q^2 r_+^{-d+3}}{32\pi(d-3)}, \tag{5}$$

$$T = \frac{32\pi P r_+^4 + 2(d-2)(d-3)\left((d-4)(d-5)\alpha - r_+^2\right) - r_+^{-2d+8}Q^2}{8\pi r_+\left(r_+^2 - 2(d-3)(d-4)\alpha\right)(d-2)}, \tag{6}$$

$$S = \frac{r_+^{d-4}\left(r_+^2 - 2(d-2)(d-3)\alpha\right)}{4}. \tag{7}$$

where r_+ is the event radius of the black hole. The black hole entropy is corrected by the Gauss–Bonnet term, while the black hole mass and temperature are corrected by the Gauss–Bonnet term and the charge term simultaneously. In the extended phase space, the black hole mass is considered to be the enthalpy of the black hole rather than the internal energy.

Since the hot AdS vacuum is uncharged, a black hole with a fixed charge cannot undergo a Hawking–Page phase transition according to the conservation of charge, thus our studies should be all discussed in the grand canonical ensemble set with a fixed potential (where the charge Q can change), in the following papers. The relation between charge and potential in d dimension is [37],

$$Q = 16\pi\Phi(d-3)r_+^{d-3}. \tag{8}$$

Thus, the expression for the mass M and Hawking temperature T of a charged Gauss–Bonnet-AdS black hole with a fixed potential can be rewritten as,

$$M = \frac{r_+^{d-5}\left((d-1)(d-2)\left((d-3)(d-4)\alpha - r_+^2\right) + 16\pi P r_+^4\right)}{16\pi(d-1)} + 8\pi\Phi^2(d-3)r_+^{d-3}, \tag{9}$$

$$T = \frac{16\pi P r_+^4 + (d-2)(d-3)\left((d-4)(d-5)\alpha - r_+^2\right) - 128 r_+^2\pi^2\Phi^2(d-3)^2}{4\pi r_+\left(r_+^2 - 2(d-3)(d-4)\alpha\right)(d-2)}. \tag{10}$$

According to the thermodynamic equation $G = H - TS - Q\Phi$, the Gibbs free energy of a hyperbolic AdS black hole in the Gauss–Bonnet gravitational grand canonical ensemble is

$$G = \frac{(d-1)(d-2)\left((d-3)(d-4)\alpha - r_+^2\right) + 16\pi P r_+^4}{16\pi(d-1)r_+^{5-d}} - 8\pi\Phi^2(d-3)r_+^{d-3} - \left(r_+^2 - 2(d-2)(d-3)\alpha\right) \times \frac{\left(16\pi P r_+^4 + (d-2)(d-3)\left((d-4)(d-5)\alpha - r_+^2\right) - 128 r_+^2\pi^2\Phi^2(d-3)^2\right)}{16\pi r_+^{5-d}\left(r_+^2 - 2(d-3)(d-4)\alpha\right)(d-2)} \tag{11}$$

where the potential term Φ is a feature of the grand canonical ensemble.

According to the Hawking–Page phase transition, a stable large black hole (with a positive heat capacity and a large event radius) can exchange energy with the hot AdS background and establish an equilibrium. The thermodynamic stable state of the system is given by the global minimum between the Gibbs free energy of the black hole and the background spacetime. Therefore, fixing the remaining parameters to plot G–T diagram is an efficient method. The Gibbs free energy of the hot AdS background is zero, so the case of $G < 0$ corresponds to a more thermodynamically stable Gauss–Bonnet black hole phase, and on the contrary $G > 0$ corresponds to a more stable background spacetime (i.e. the massless Gauss–Bonnet black hole phase). Thus $G = 0$ represents the Hawking–Page phase transition between the background spacetime phase and the Gauss–Bonnet black hole phase. Noting that the cosmological constant of the background spacetime is then modified by the Gauss–Bonnet constant, and when the Gauss–Bonnet constant α disappears, the spacetime is reduced to a hyperbolic Schwarzschild AdS black hole without the HP phase transition.

3 Reentrant Hawking–Page phase transition and triple point of charged Gauss–Bonnet hyperbolic AdS-black hole in four dimensions

Initially, in the four-dimensional spacetime, the Gauss–Bonnet term exerts no influence on the geometry, thus simplifying the spacetime to that of a hyperbolic Schwarzschild RN-AdS black hole. The metric and the black hole solution are as follows,

$$\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + \frac{1}{f(r)}\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega_{2,-1}^2, \tag{12}$$

$$f(r) = \frac{32P\pi r^4 - 96M\pi r + 3Q^2 - 12r^2}{12r^2}. \tag{13}$$

According to Eq. (13), the black hole event horizon radius r_+ of an RN-AdS black hole should be determined by the largest root of $f(r) = 0$, and then the black hole mass can be expressed as,

$$M = \frac{32P\pi r_+^4 + 3Q^2 - 12r_+^2}{96\pi r_+}. \tag{14}$$

The Hawking temperature of this black hole can then be calculated,

$$T = \frac{f'(r_+)}{4\pi} = \frac{32P\pi r_+^4 + 48M\pi r_+ - 3Q^2}{24r_+^3\pi}. \tag{15}$$

The relation between charge and potential in four dimensions can be given by $Q = 16\pi\Phi r_+$. It can be seen that the mass and temperature of the black hole are not corrected by the Gauss–Bonnet term, but the entropy of the black hole breaks the law of the area of the black hole, showing the following formula [7,58],

$$S = \frac{r_+^2}{4} - \alpha. \tag{16}$$

Rewriting the mass and temperature of the black hole in the grand canonical ensemble,

$$M = \frac{32P\pi r_+^3 + 768\Phi^2\pi^2 r_+ - 12r_+}{96\pi}, \tag{17}$$

$$T = \frac{8P\pi r_+^2 - 64\Phi^2\pi^2 - 1}{4\pi r_+}. \tag{18}$$

Obviously, non-trivial black hole entropy will have an impact on the thermodynamics of black holes. To visualize this effect, we focused on Gibbs free energy,

$$G = \frac{-192\Phi^2\left(r_+^2 + 4\alpha\right)\pi^2 - 8Pr_+^2\left(r_+^2 - 12\alpha\right)\pi - 3r_+^2 - 12\alpha}{48\pi r_+}. \tag{19}$$

In order to observe the global stability of Gibbs free energy, we plot G–T diagram in Fig. 1. The existence of the Hawking Page phase transition depends on whether the extreme Gibbs free energy is greater than zero. When the pressure is low ($P < P_{tr}$), the Gibbs free energy is always negative, the global stable state of the system is the black hole phase, and there is no Hawking Page phase transition. Then we increase the pressure, and we find that when the pressure increases to a certain value ($P = P_{tr}$), there is still no Hawking Page phase transition, but there is a critical Hawking–Page transition temperature. We will denote the pressure at this point as P_{tr} because this pressure corresponds exactly to the triple point in the P–T phase diagram shown below.

Then the pressure continues to increase ($P > P_{tr}$), at which point the maximum value of Gibbs free energy

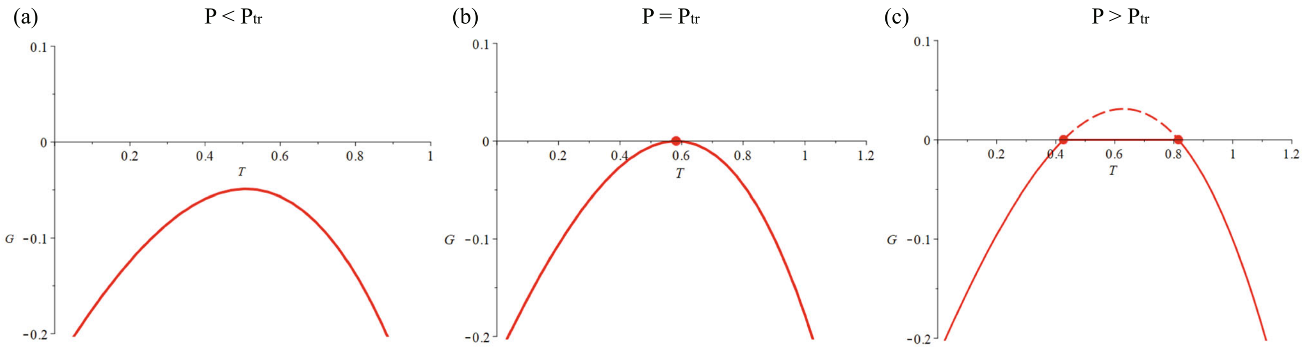

Fig. 1 The relationship between Gibbs free energy and temperature at different pressures (with α=1,$\Phi = 0.1$) in four-dimensional spacetime. The global stable states of the system are represented by the solid red lines, and the system undergoes a reentrant Hawking–Page phase transition when $P > P_{tr}$

is greater than zero, and since temperature is a monotone increasing function of r_+, the Gibbs free energy always has two zeros. According to Eq. (17), Eq. (18) can be found that the black hole mass is a single increasing function of temperature, so we can represent the left and right branches of the black hole as the small black hole phase with less mass and the large black hole phase with more mass, respectively. Considering the current global stability of the system, the stable state has been marked by the solid red line in the right one of Fig. 1. It can be seen that with the gradual increase in temperature, the global stable state evolution of the system meets the following conditions: small black hole phase (SBH) → massless black hole phase (MBH) → large black hole phase (LBH). For this case, the system underwent two Hawking–Page phase transitions, one with a low Hawking–Page temperature and another with a high Hawking–Page temperature. This behavior is called the reentrant Hawking–Page phase transition which consists of two first-order Hawking–Page phase transitions.

In fact, we can directly solve the Hawking–Page phase transition pressure P_{HP} by using $G = 0$. For simplicity we use r_{HP} to represent the radius of the black hole's event horizon, which obviously satisfies $r_{HP} > 0$. (There is an analytical solution for the black hole radius in the four-dimensional case, which will be given below.) Namely, we can get

$$P_{HP} = -\frac{3\left(64\Phi^2\pi^2+1\right)\left(r_{HP}^2+4\alpha\right)}{8\pi r_{HP}^2\left(r_{HP}^2-12\alpha\right)}, \ \frac{\partial P_{HP}}{\partial r_{HP}}\,|_{\alpha,\Phi} = \frac{48\left(\Phi^2\pi^2+\frac{1}{64}\right)\left(r_{HP}^2-4\alpha\right)\left(r_{HP}^2+12\alpha\right)}{r_{HP}^3\pi\left(r_{HP}^2-12\alpha\right)^2}. \quad (20)$$

By substituting P_{HP} into Eq. (18), we can give an expression for the HP phase transition temperature T_{HP},

$$T_{HP} = \frac{-64\Phi^2\pi^2 r_{HP} - r_{HP}}{\left(r_{HP}^2-12\alpha\right)\pi},$$

$$\frac{\partial T_{HP}}{\partial r_{HP}}\,|_{\alpha,\Phi} = \frac{\left(64\pi^2\Phi^2+1\right)\left(r_{HP}^2+12\alpha\right)}{\left(r_{HP}^2-12\alpha\right)^2\pi}. \quad (21)$$

Analysis of the first derivation of P_{HP} and T_{HP} with respect to r_{HP} shows that as r_{HP} increases ($r_{HP} > 0$) P_{HP} will have a minimum value of P_{tr}; T_{HP} is a monotonically increasing function of r_{HP}. This means that when the pressure is less than P_{tr}, there will be no HP phase transition, which is consistent with our previous analysis of the G–T diagram. We can directly use Eqs. (20), (21) to draw the coexistence lines of the reentrant HP phase transition in the P–T phase diagram shown in Fig. 2a, and comprehensively observe the reentrant Hawking–Page phase transition.

At a constant temperature, the system always has a single Hawking–Page phase transition: a Hawking–Page phase transition between a small black hole and a massless black hole at a lower temperature or a Hawking–Page phase transition between a large black hole and a massless black hole at a higher temperature. When the pressure is a constant, the system does not undergo Hawking–Page phase transition at low pressure, but will undergo Hawking–Page phase transition twice at high pressure, that is, reentrant Hawking–Page phase transition. There is a critical phase transition pressure P_{tr}, for which the system at this point is in a small black hole phase, a massless black hole phase and a large black hole phase coexist, that is, the triple point. By solving $\frac{\partial P_{HP}}{\partial r_{HP}}\,|_{\alpha,\Phi}= 0$, we can directly give the radius of the black hole corresponding to the triple point (only select the solution corresponding to $r_{tr} > 0$, another solution is not physical), Then substituted r_{tr} into Eqs. (20) and (21) to obtain the Hawking–Page temperature(T_{tr}) and Hawking–Page pressure (P_{tr}) of the triple point, i.e.

$$r_{tr} = 2\sqrt{\alpha}, \quad P_{tr} = \frac{3\left(64\Phi^2\pi^2+1\right)}{32\alpha\pi}, \quad T_{tr} = \frac{64\Phi^2\pi^2+1}{4\sqrt{\alpha}\pi}. \quad (22)$$

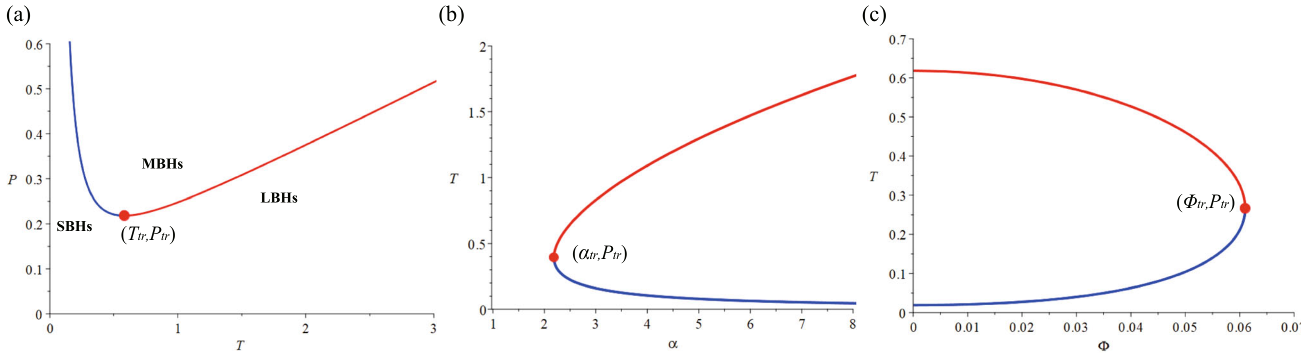

Fig. 2 **a** Coexistence line: P–T phase diagram (with $\alpha = 1$, $\Phi = 0.1$) in four dimensions. The triple point is highlighted. SBHs, LBHs, and MBHs correspond to the small black holes, large black holes, and massless black holes, respectively. **b** $T-\alpha$ diagram (with $P = 0.1$, $\Phi = 0.1$), α has a minimum of α_{tr}, and the Hawking–Page phase transition disappears when $\alpha < \alpha_{tr}$. **c** $T-\Phi$ diagram (with $P = 0.1$, $\alpha = 1$), Φ has a maximum of Φ_{tr}, and the Hawking–Page phase transition disappears when $\Phi > \Phi_{tr}$

In addition, it is quite interesting to introduce a relationship of the triple point,

$$\frac{P_{\rm Tr} r_{\rm Tr}}{T_{\rm Tr}} = \frac{3}{4}. \tag{23}$$

This relation is consistent with the uncharged Gauss–Bonnet black hole case, while Eq. (22) shows that the pressure and temperature at the triple point will be modified by the electric potential to become greater than the uncharged case.

In addition, by solving $G = 0$ directly, an analytical solution for the radius of the horizon can be given as

$$r_{HP} = \frac{\sqrt{P\left(-192\Phi^2\pi^2 + 96P\pi\alpha - 3 + \sqrt{3}\sqrt{\left(-64\Phi^2\pi^2 + 96P\pi\alpha - 1\right)\left(-192\Phi^2\pi^2 + 32P\pi\alpha - 3\right)}\right)}}{4\sqrt{\pi}P}. \tag{24}$$

Since the event radius r_+ must be positive. The Hawking–Page phase transition requires an additional condition,

$$\alpha P \geq \frac{3\left(64\Phi^2\pi^2 + 1\right)}{32\alpha\pi}. \tag{25}$$

In order to reflect the influence of the Gauss–Bonnet term, we draw the Hawking–Page transition temperature curve with α, as shown in Fig. 2b. Obviously, the reentrant Hawking–Page phase transition occurs only when Eq. (25). This is consistent with the discussion of the uncharged Gauss–Bonnet black hole case, but the range of α has been modified by electric potential Φ. On the other hand, the Gauss–Bonnet constant α increases the high Hawking–Page temperature and decreases the low Hawking–Page temperature.

The constraint on Φ to find the Hawking–Page phase transition can also be obtained from Eq. (24), i.e.

$$0 \leq \Phi \leq \Phi_{tr}, \quad \Phi_{tr} = \frac{\sqrt{3}\sqrt{32P\pi\alpha - 3}}{24\pi}. \tag{26}$$

We draw a graph showing the change of phase transition temperature with the electric potential Φ. The chart is labeled as graph Fig. 2c. We can observe that the reentrant Hawking–Page phase transition occurs only when the electric potential meets certain conditions, i.e, when the value of the electric potential lies in the interval ($0 \leq \Phi < \Phi_{tr}$). It can also be seen from the expression of Gibbs free energy Eq. (19) that when the electric potential is large enough, the Gibbs free energy of the system mainly depends on the electric potential term. In this case, the Gibbs free energy tends to be more negative, and the global stability of the system tends to be more like the black hole phase, and there is no Hawking–Page phase transition. In particular, as $\Phi \to 0$, the system can return to the uncharged Gauss–Bonnet black hole case. Besides, the electric potential Φ decreases the high Hawking–Page transition temperature and increases the low Hawking–Page transition temperature.

On the other hand, since the Gauss–Bonnet term is actually a total derivative in four-dimensional spacetime, it has no effect on the field equations and the structure of the spacetime. Interestingly, by scaling the Gauss–Bonnet coupling term in four dimensions $\alpha \to \alpha/(D-4)$, Glavan and Lin propose 4D Einstein-Gauss–Bonnet gravity (4D EGB) and apply it to maximum symmetric spacetime, spherically symmetric black holes, and cosmology [59]. Even in four dimensions, the Gauss–Bonnet term can have a non-trivial effect on gravitational dynamics. Later, many black hole solutions [60–63] were constructed under 4D EGB gravity. The thermodynamics and phase transition effects of black holes in the

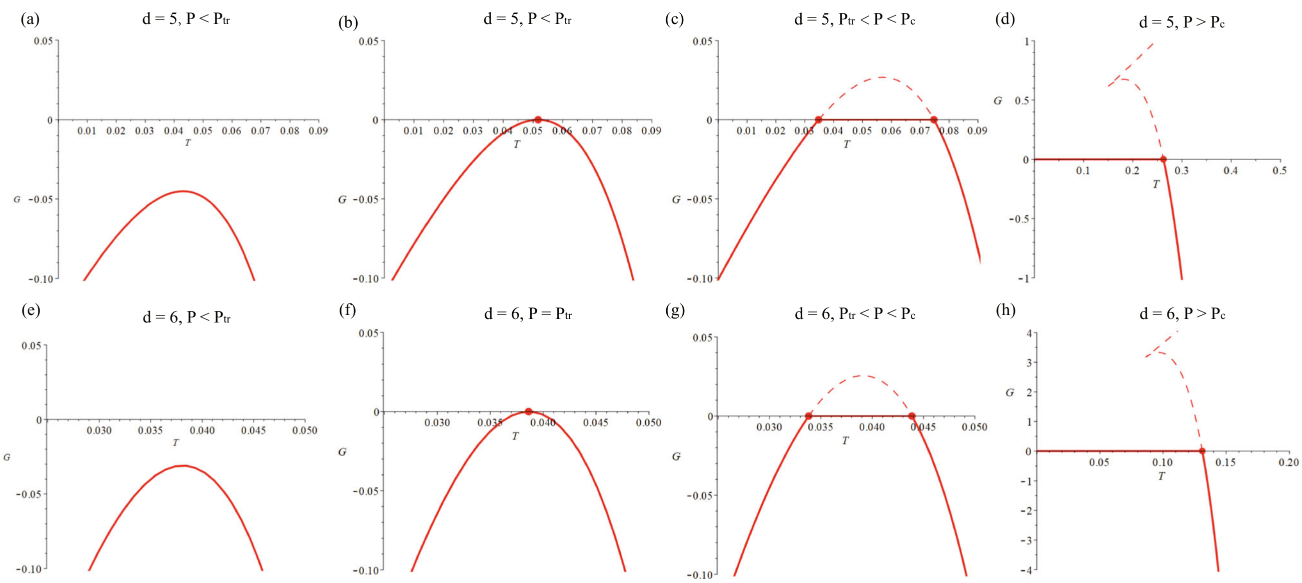

Fig. 3 The relationship between Gibbs free energy and temperature at different pressures (with $\alpha = 1, \Phi = 0.1$) in five-dimensional (**a**–**d**) and six-dimensional (**e**–**h**) spacetime. The global stable states of the system are all represented by the solid red lines. It is shown that the system undergoes a reentrant Hawking–Page phase transition when $P_{tr} < P < P_c$, and only a single Hawking–Page phase transition occurs when $P > P_c$

extended phase space are also studied [64–69] widely. One can see a recent review for detail [70]. This makes it important to study the Hawking–Page phase transition of AdS black holes in 4D EGB gravity. We leave it as a future work.

4 Reentrant Hawking–Page phase transition and triple point of charged Gauss–Bonnet hyperbolic AdS-black hole in $d \geq 5$ dimensions

There are some differences between the Hawking–Page phase transition in higher-dimensional spacetime and that in the four-dimensional case. One will find that the system always has a reentrant Hawking Page phase transition when the pressure is $P_{tr} \leq P < P_c$ in the grand canonical ensemble, which is consistent with the conclusion for the uncharged Gauss–Bonnet black hole case. Besides, in higher-dimensional spacetime, the reentrant Hawking–Page phase transition occurs only when the potential is in the range $\Phi_c \leq \Phi < \Phi_{tr}$. In this section, we first present the Hawking–Page phase transitions in five and six dimensions, with similar properties in other $d \geq 5$ dimensional spacetimes. Then, we derive the triple point of the Hawking–Page transition in d dimensions in the grand canonical ensemble.

4.1 Reentrant Hawking–Page transition of charged Gauss–Bonnet hyperbolic AdS black hole in five and six dimensions

The relationship between Gibbs free energy and temperature of Gauss–Bonnet hyperbolic AdS black hole in the grand canonical ensemble in five-dimensional space-time is reduced to,

$$G = \frac{-4P\pi r_+^6 + \left(-256\Phi^2\pi^2 + 144P\pi\alpha - 3\right) r_+^4 + \left(-3072\Phi^2\pi^2 - 18\right)\alpha r_+^2 - 72\alpha^2}{48\pi\left(r_+^2 - 4\alpha\right)}, \tag{27}$$

$$T = \frac{r_+\left(-8\pi P r_+^2 + 256\Phi^2\pi^2 + 3\right)}{6\pi\left(-r_+^2 + 4\alpha\right)}. \tag{28}$$

The corresponding thermodynamical quantities in six-dimensional space-time is

$$G = -\frac{r_+\left(P\pi r_+^6 + \left(120\Phi^2\pi^2 - 72P\pi\alpha + \frac{5}{4}\right) r_+^4 + \alpha\left(2880\Phi^2\pi^2 + \frac{15}{2}\right) r_+^2 + 180\alpha^2\right)}{20\pi\left(r_+^2 - 12\alpha\right)}, \tag{29}$$

$$T = \frac{4\pi P r_+^4 + \left(-288\Phi^2\pi^2 - 3\right) r_+^2 + 6\alpha}{4r_+\pi\left(r_+^2 - 12\alpha\right)}. \tag{30}$$

The behavior of Gibbs free energy under different pressures in five and six dimensions are plotted, which are similar and shown in the Fig. 3. At lower pressures, i.e., when $P < P_{tr}$, we observe that the Gibbs free energy of the system is always negative. This indicates that for this case, the Gibbs free energy of the hyperbolic Gauss–Bonnet AdS black hole phase in the grand canonical ensemble is always smaller than the background space-time phase, so the Hawking Page phase transition does not occur. Besides, as the pressure gradually increases, the situation changes when the range $P_{tr} < P < P_c$ is reached. Within this pressure interval, the system undergoes reentrant Hawking–Page phase transition consisting of two Hawking–Page transitions, as shown in Fig. 3c, g. Increasing the pressure further, when $P > P_c$, we noticed that one of the two Hawking–Page transition temperatures starts to turn negative, at which point there is only a single Hawking–Page phase transition in the system.

According to the method in the lase section, we can directly give the relationship between Hawking–Page phase transition pressure P_{HP}, temperature T_{HP} with the phase transition radius r_{HP} through $G = 0$. In the case of five dimensions, we can find,

$$P_{HP} = \frac{\left(-256\Phi^2\pi^2 - 3\right) r_{HP}^4 + \left(-3072\Phi^2\pi^2 - 18\right) \alpha r_{HP}^2 - 72\alpha^2}{4\pi r_{HP}^4 \left(r_{HP}^2 - 36\alpha\right)},$$

$$T_{HP} = \frac{\left(-256\Phi^2\pi^2 - 3\right) r^2 + 12\alpha}{2\pi r_{HP} \left(r_{HP}^2 - 36\alpha\right)}, \tag{31}$$

$$\frac{\partial P_{HP}}{\partial r_{HP}} = \frac{\left(r_{HP}^2 - 12\alpha\right)\left(256\Phi^2\pi^2 r_{HP}^4 + 9216\Phi^2\pi^2\alpha r_{HP}^2 + 3r_{HP}^4 + 72\alpha r_{HP}^2 + 432\alpha^2\right)}{2\pi r_{HP}^5 \left(r_{HP}^2 - 36\alpha\right)^2}, \tag{32}$$

$$\frac{\partial T_{HP}}{\partial r_{HP}} = \frac{\left(256\Phi^2\pi^2 + 3\right) r_{HP}^4 + \left(9216\Phi^2\pi^2 + 72\right)\alpha r_{HP}^2 + 432\alpha^2}{2\pi \left(r_{HP}^2 - 36\alpha\right)^2 r_{HP}^2}. \tag{33}$$

The corresponding quantities in six-dimensional spacetime are:

$$P_{HP} = \frac{\left(-480\Phi^2\pi^2 - 5\right) r_{HP}^4 + \left(-11520\Phi^2\pi^2 - 30\right) \alpha r_{HP}^2 - 720\alpha^2}{4\pi r_{HP}^4 \left(r_{HP}^2 - 72\alpha\right)},$$

$$T_{HP} = \frac{\left(-192\Phi^2\pi^2 - 2\right) r^2 + 24\alpha}{\pi r_{HP} \left(r_{HP}^2 - 72\alpha\right)}, \tag{34}$$

$$\frac{\partial P_{HP}}{\partial r_{HP}} = \frac{240\left(r_{HP}^2 - 24\alpha\right)\left(\left(\Phi^2\pi^2 + \frac{1}{96}\right) r_{HP}^4 + \alpha\left(72\Phi^2\pi^2 + \frac{3}{8}\right) r_{HP}^2 + 9\alpha^2\right)}{\pi r_{HP}^5 \left(r_{HP}^2 - 72\alpha\right)^2}, \tag{35}$$

$$\frac{\partial T_{HP}}{\partial r_{HP}} = \frac{\left(192\Phi^2\pi^2 + 2\right) r_{HP}^4 + \left(13824\Phi^2\pi^2 + 72\right)\alpha r_{HP}^2 + 1728\alpha^2}{\pi \left(r_{HP}^2 - 72\alpha\right)^2 r_{HP}^2}. \tag{36}$$

Similar to the four-dimensional case, the phase transition pressure has a minimum, while the phase transition temperature is a monotone increasing function of the black hole radius, and it is clear that in the grant canonical ensemble there is also a triple point in the higher dimensional case. To investigate the Hawking–Page phase structure more precisely, we plot the coexistence lines in the P–T phase diagram of Hawking–Page phase transitions in five- and six-dimensional spacetime, as shown in the Fig. 4. We find that the occurrence of the reentrant Hawking–Page phase transition is limited to a certain pressure range ($P_{tr} < P < P_c$) in high dimensional spacetime. In this range, when the black hole crosses the left branch of the solid line from left to right or bottom to top, it undergoes the Hawking–Page phase transition from small black hole to massless black hole; When a black hole crosses the real line in the right branch from right to left or from bottom to top, it undergoes a Hawking–Page phase transition from a large black hole to a massless black hole. When the pressure is greater than the critical pressure ($P > P_c$), there is only a single Hawking–Page phase transition. Since the corresponding phase transition temperature of the small branch is negative at this time, it is not physical. So the small mass black hole phase will disappear during the Hawking–Page phase transition in this region of pressure.

As shown in the phase diagram Fig. 4, the reentrant Hawking–Page phase transition is represented by a thick line defined by two specific points, the triple point (T_{tr}, P_{tr}) and the critical point (T_c, P_c). At the triple point, the system exhibits a unique state in which the small black hole phase, the massless black hole phase, and the large black hole phase coexist. At the critical point, the system is in a state where the massless black hole phase coexists with the large black hole phase. We can use the property that the first derivative of the Hawking–Page phase transition pressure at the triple

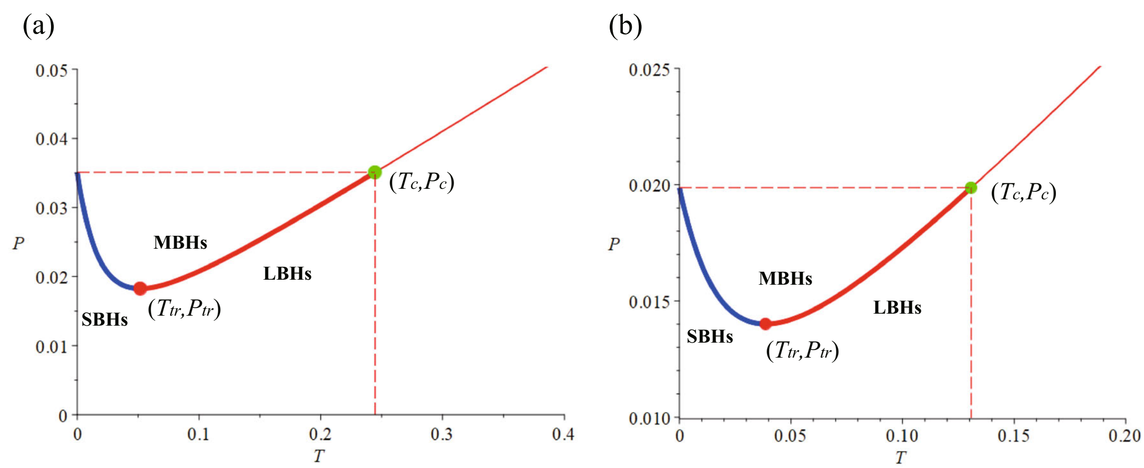

Fig. 4 Coexistence lines in the P–T phase diagram: The reentrant Hawking–Page phase transition is represented by a thick line, and the triple point and critical point are highlighted. SBHs, MBHs, and LBHs represent the small, massless, and large Gauss–Bonnet AdS black holes, respectively. **a** P–T phase diagram in five dimensions (with $\alpha = 1, \Phi = 0.01$). **b** P–T phase diagram in six dimensions (with $\alpha = 1, \Phi = 0.01$)

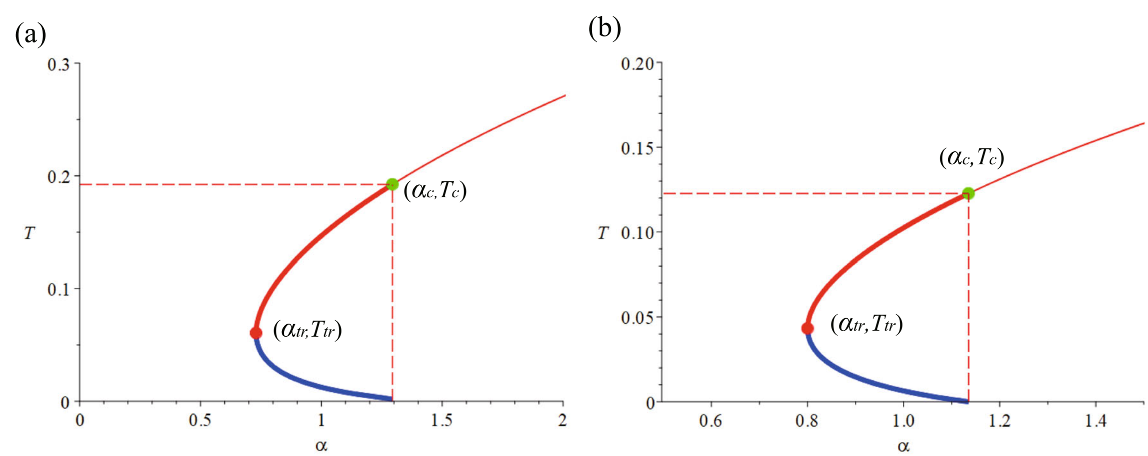

Fig. 5 The relationship between the Hawking–Page transition temperature and the Gauss–Bonnet constant at a given pressure and electric potential, and the triple point and critical point are highlighted in the figure. **a** $T-\alpha$ diagram in five dimensions (with $\Phi = 0.01$, $P = 0.025$). **b** $T-\alpha$ diagram in six dimensions (with $\Phi = 0.01$, $P = 0.0175$)

point is zero to calculate the triple point (Since we calculate the triple point in general dimensions in next subsection, we do not shown the results in five and six dimensions here.). In addition, for a charged Gauss–Bonnet-AdS black hole with a fixed electric potential Φ and a fixed Gauss–Bonnet constant α, the pressure makes the high Hawking–Page transition temperature larger and the low Hawking–Page transition temperature smaller.

Following the similar procedure for studying the effect of the pressure on the Hawking–Page phase transition, we can find the effect of the Gauss–Bonnet constant α on the Hawking–Page phase transition. The relationship between the two branches of the Hawking–Page transition temperature and the Gauss–Bonnet constant in five and six dimensions are also plotted in the Fig. 5. The reentrant Hawking–Page phase transition occurs only when the Gauss–Bonnet constant is in a specific range, i.e. $\alpha_{tr} < \alpha < \alpha_c$. However, when the Gauss–Bonnet constant is large ($\alpha > \alpha_c$), there is only a single Hawking–Page phase transition. If the Gauss–Bonnet constant is small ($\alpha < \alpha_{tr}$), there is no Hawking–Page phase transition, which is similar to the hyperbolic AdS case in Einstein gravity. Noting the values for α_{tr}, α_c are present in next subsection. Besides, for the systems with a fixed pressure and potential, the Gauss–Bonnet constant α makes the high Hawking–Page transition temperature higher and the low Hawking–Page transition temperature lower.

Finally, we show in Fig. 6 the relationship between the Hawking–Page phase transition temperature T_{HP} and the electric potential Φ in five and six dimensions. It is different from the four-dimensional case. When the electric poten-

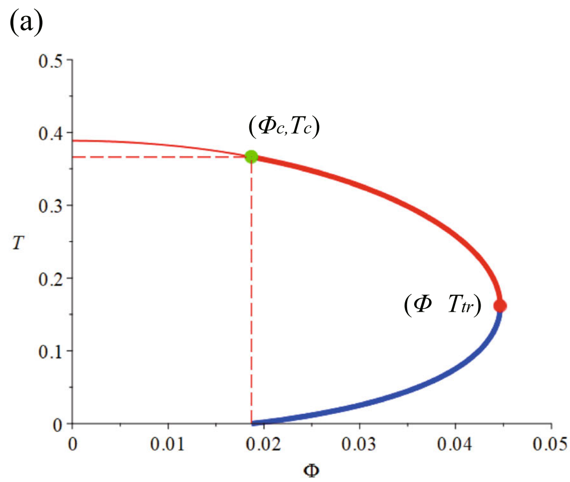

Fig. 6 The relationship between the Hawking–Page transition temperature and the electric potential at a given pressure and Gauss–Bonnet constant, and the triple point and critical point are highlighted in the figure. **a** $T-\Phi$ diagram in five dimensions (with $\alpha = 1$, $P = 0.05$). **b** $T-\Phi$ diagram in six dimensions (with $\alpha = 1$, $P = 0.025$)

tial is small ($\Phi < \Phi_c$), there is only a single Hawking–Page phase transition in the system. As the electric potential gradually increases, and when the electric potential is in the range of $\Phi_c \le \Phi < \Phi_{tr}$, the reentrant Hawking–Page phase transition will appear. When the electric potential is too large ($\Phi > \Phi_{tr}$), the global stability of the system is closer to the black hole phase because the Gibbs free energy mainly depends on the existence of the $Q\Phi$ term, so the Hawking–Page phase transition will not occur at this time. Noting the values for Φ_{tr}, Φ_c are present in next subsection. Contrary to the effects of the Gauss–Bonnet constant α and the pressure P, for a system with a constant pressure and Gauss–Bonnet constant, the electric potential tends to increase the low Hawking–Page phase transition temperature and decrease the high Hawking–Page phase transition temperature.

4.2 Triple points for the reentrant Hawking–Page transition of the charged Gauss–Bonnet hyperbolic AdS black hole in $d \ge 5$ dimensions

The triple point (P_{HP}, T_{HP}, r_{HP}) corresponds to a black hole phase with a maximum Gibbs free energy of zero, which can usually be calculated by the following formula,

$$G|_{P=P_{\mathrm{Tr}},T=T_{\mathrm{Tr}},r=r_{\mathrm{Tr}}} = 0, \quad \left.\frac{\partial G}{\partial r_+}\right|_{P=P_{\mathrm{Tr}},T=T_{\mathrm{Tr}},r_+=r_{\mathrm{Tr}}} = 0. \tag{37}$$

However, due to the influence of electric potential, we cannot directly calculate the thermodynamic quantity corresponding to the triple point from Eq. (37). Here we choose the method given in the above subsection to calculate. Namely, we can use the property that the first derivative of the Hawking–Page phase transition pressure at the triple point is zero, i.e.

$$G|_{P=P_{\mathrm{HP}},T=T_{\mathrm{HP}},r=r_{\mathrm{HP}}} = 0 \Rightarrow P_{HP}, \tag{38}$$

$$\left.\frac{\partial P_{HP}}{\partial r_{HP}}\right|_{r_{HP}=r_{\mathrm{tr}}} = 0, \quad \frac{\partial T_{HP}}{\partial r_{HP}} \ge 0, \quad \left.\frac{\partial^2 P_{HP}}{\partial r_{HP}^2}\right|_{r_{HP}=r_{\mathrm{tr}}} \ge 0. \tag{39}$$

The form of Gibbs free energy is given by Eq. (11) above. For simplicity, we only show P_{HP} and T_{HP} and their first derivative forms here,

$$P_{HP} = \frac{\left(2(d-4)(d-2)^2(d-3)^2\alpha^2 + \left(256\pi^2(d-3)\Phi^2 - d + 8\right)(d-2)(d-3)r_{HP}^2\alpha\right)(d-1)}{16\left(6(d-2)(d-3)\alpha - r^2\right)r_{HP}^4\pi} + \frac{\left(128\pi^2(d-3)\Phi^2 + d - 2\right)(d-1)}{16\left(6(d-2)(d-3)\alpha - r_{HP}^2\right)\pi}, \tag{40}$$

$$\frac{\partial P_{HP}}{\partial r_{HP}} = -\frac{\left(2(d-2)(d-3)\alpha - r_{HP}^2\right)(d-1)}{24r_{HP}^5\left((d-2)(d-3)\alpha - \frac{r_{HP}^2}{6}\right)^2\pi}\left(\frac{\left(128\pi^2(d-3)\Phi^2 + d - 2\right)r_{HP}^4}{12} + (d-4)(d-2)^2(d-3)^2\alpha^2 + \left(64\pi^2(d-3)\Phi^2 + 1\right)(d-3)r_{HP}^2(d-2)\alpha\right), \tag{41}$$

$$T_{HP} = \frac{-2\alpha(d-2)(d-3)(d-4) + \left(128\pi^2(d-3)\Phi^2 + d - 2\right)r_{HP}^2}{2r_{HP}\left(6(d-2)(d-3)\alpha - r_{HP}^2\right)\pi}, \tag{42}$$

$$\frac{\partial T_{HP}}{\partial r_{HP}} = \frac{1}{2\pi r_{HP}^2\left(6(d-2)(d-3)\alpha - r_{HP}^2\right)^2}\left(12(d-4)(d-2)^2(d-3)^2\alpha^2 + \left(12 + 768\pi^2(d-3)\Phi^2\right) \times (d-2)(d-3)\alpha r_{HP}^2 + \left(128\pi^2(d-3)\Phi^2 + d - 2\right)r_{HP}^4\right). \tag{43}$$

Obviously, for any dimension, the positive or negative value of $\frac{\partial P_{HP}}{\partial r_{HP}}$ depends only on the positive or negative value of $\left(r_{HP}^2 - 2\left((d-2)(d-3)\alpha\right)\right)$, so P_{HP} always has a minimum value located at $\left(r_{HP}^2 - 2\left((d-2)(d-3)\alpha\right)\right) = 0$ which corresponds to the triple point. However, $\frac{\partial T_{HP}}{\partial r_{HP}}$ is greater than zero in any dimension, so the phase transition temperature is a monotone increasing function of the phase transition radius. Now we can theoretically solve for the quantities that correspond to the triple point.

$$r_{tr} = \sqrt{2}\sqrt{(d-2)(d-3)\alpha},$$
$$P_{tr} = \frac{(256\pi^2(d-3)\Phi^2 + d)(d-1)}{64\alpha(d-2)(d-3)\pi}, \tag{44}$$
$$T_{tr} = \frac{16\sqrt{2}\left(\frac{1}{64} + \Phi^2(d-3)\pi^2\right)}{\sqrt{(d-2)(d-3)\alpha}\pi}.$$

Now we consider the critical points of the reentrant Hawking–Page phase transition. For the uncharged Gauss–Bonnet black hole case, we can calculate the temperature divergence of the Hawking–Page phase transition at the critical point (P_c, T_c, r_c). However, in the grand canonical ensemble, this condition becomes inapplicable due to the effect of the potential term on temperature. However, it is still feasible to solve the critical point theoretically, which can be given by the property that the Hawking–Page phase transition temperature of the left branch becomes zero. Since the expressions of the critical temperature T_c and critical radius r_c are too complicated, only the critical pressure P_c for general dimensions is given as follows.

$$P_c = \frac{(d-1)(128\pi^2(d-3)\Phi^2 + d - 2)}{64\pi\alpha(d-2)(d-3)(d-4)}. \tag{45}$$

In Appendix B, we give the values corresponding to the critical points of the reentrant Hawking–Page phase transition in diverse dimensions.

The cases with the dimensions $d > 6$ have the similar results. In conclusion, the hyperbolic Gauss–Bonnet AdS black holes in higher dimensions always exhibit a unique reentrant Hawking–Page phase transition in the grand canonical ensemble. Through the Figs. 7 and 8, we can see the triple point and the critical point in different dimensions. With the increase of the dimensions, the temperature of the triple point and the critical point both show a decreasing trend. We also draw the phase diagrams in different dimensions in the Fig. 9. Within a certain pressure range ($P_{tr} < P < P_c$), i.e. Eq. (46), the Gauss–Bonnet AdS black holes always have a reentrant Hawking–Page phase transition in $d \geq 4$ dimension,

$$\frac{(256\pi^2(d-3)\Phi^2 + d)(d-1)}{64(d-2)(d-3)\pi} < \alpha P < \frac{(d-1)(128\pi^2(d-3)\Phi^2 + d - 2)}{64\pi(d-2)(d-3)(d-4)}. \tag{46}$$

On the other hand, in view of the conditions for the occurrence of the reentrant Hawking–Page phase transition, namely Eq. (46), if the pressure and electric potential are kept as a constant, then only when the Gauss–Bonnet constant is in the range of $\alpha_{tr} < \alpha < \alpha_c$, where

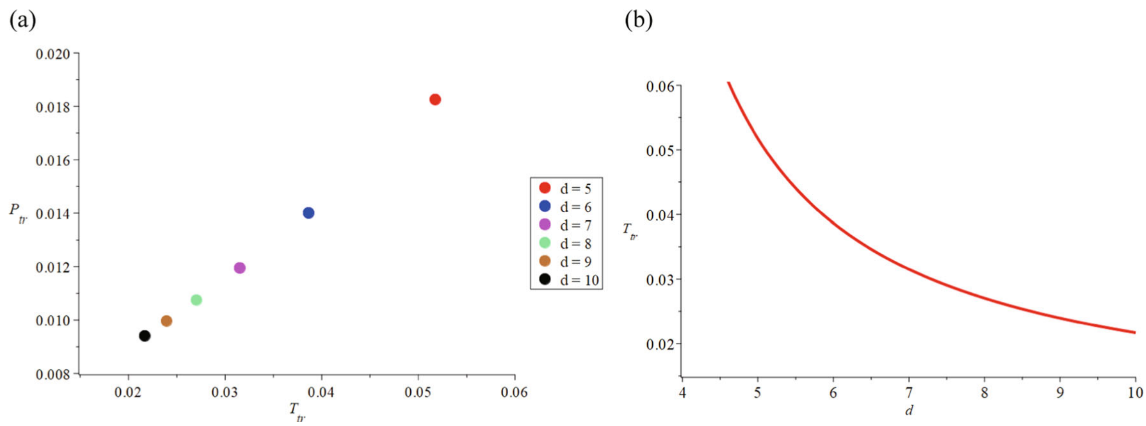

Fig. 7 Triple points in different dimensions (with $\alpha = 1$, $\Phi = 0.01$)

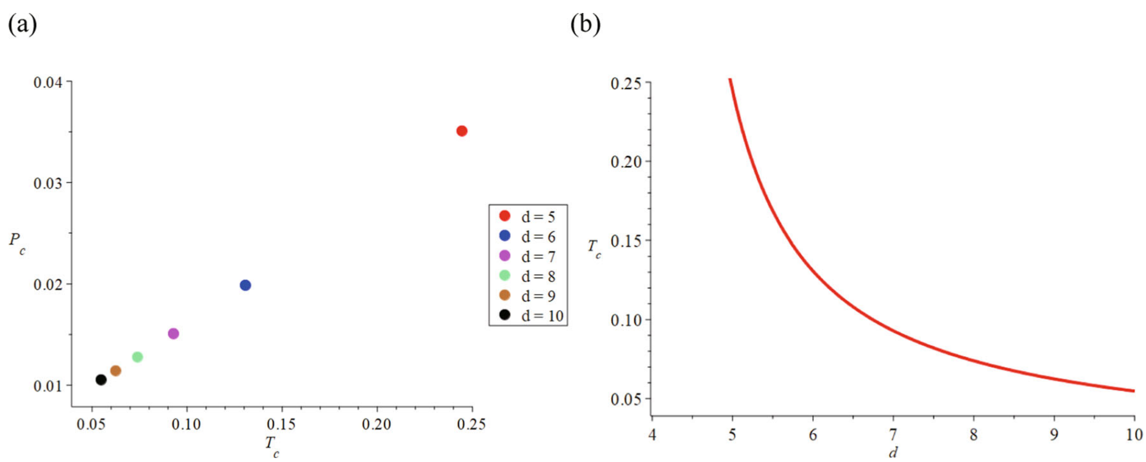

Fig. 8 Critical points in different dimensions (with $\alpha = 1$, $\Phi = 0.01$)

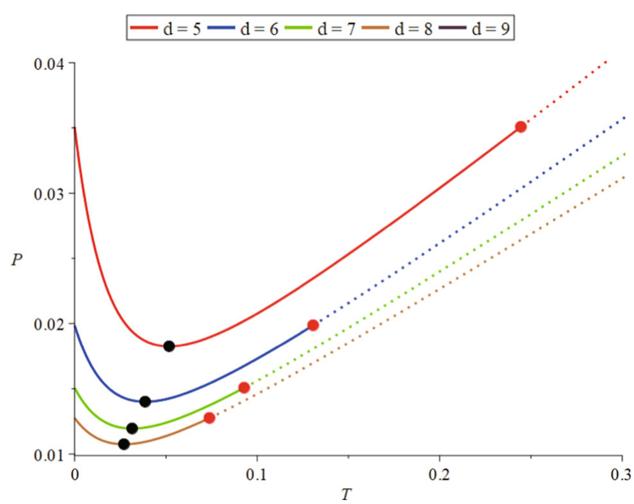

Fig. 9 P–T plots in different dimensions (with $\alpha = 1$, $\Phi = 0.01$). The reentrant Hawking–Page phase transition is represented by a straight line, and the single Hawking–Page phase transition is represented by a dashed line. The red and black dots represent the critical points and the triple point, respectively

$$\alpha_{tr} = \frac{(256\pi^2(d-3)\Phi^2 + d)(d-1)}{64\pi P(d-2)(d-3)},$$
$$\alpha_c = \frac{(d-1)(128\pi^2(d-3)\Phi^2 + d - 2)}{64\pi P(d-2)(d-3)(d-4)}, \tag{47}$$

the higher dimensional Gauss–Bonnet AdS black holes will have a reentrant Hawking–Page phase transition in the grand canonical ensemble. Similarly, if the pressure and Gauss–Bonnet constant are constant, it can be found that the reentrant Hawking–Page phase transition only occurs when the electric potential is in the range of $\Phi_c < \alpha < \Phi_{tr}$, where

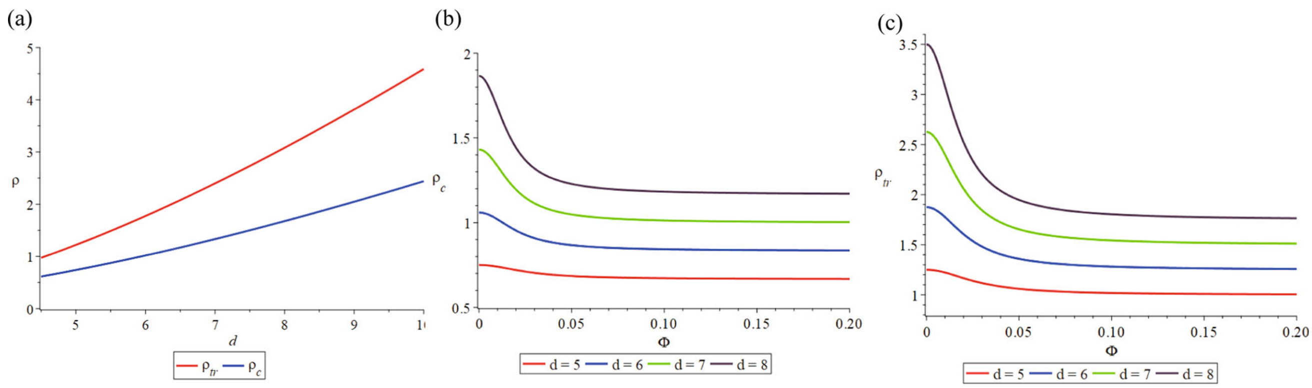

Fig. 10 **a** The relationship between the ratios and the dimensions. **b** The relationship between the ratio of the triple point and the electric potential. **c** The relationship between the ratio of the critical point and the electric potential

$$\Phi_c = \frac{\sqrt{2}\sqrt{-(d-1)(d-3)(d(d-3)-8\sqrt{\pi}\sqrt{\alpha P(d-1)(d-2)(d-3)(d-4)}+2)}}{16\pi(d-1)(d-3)},$$
$$\Phi_{tr} = \frac{\sqrt{(d-1)(d-3)(64\pi P\alpha(d-2)(d-3)-d(d-1))}}{16\pi(d-1)(d-3)}, \tag{48}$$

in general $d \geq 4$ dimensions.

Finally, we give the ratio relationship between the triple points of the reentrant Hawking-phase transition,

$$\rho_{\rm Tr} = \frac{P_{\rm Tr}r_{\rm Tr}}{T_{\rm Tr}} = \frac{\left(256\Phi^2(d-3)\pi^2+d\right)(d-1)}{16(1+64\Phi^2(d-3)\pi^2)}, \tag{49}$$

(The relationship $\rho_{\rm c} = \frac{P_{\rm c}r_{\rm c}}{T_{\rm c}}$ between the critical points is complicated and therefore not shown here. In Appendix B, we give the ratio values corresponding to the critical points in diverse dimensions.). The effect of the dimensions and the electric potential on the ratio at the triple point and the critical point are shown in the Fig. 10. When the electric potential and Gauss–Bonnet constant are fixed, the ratio of ρ to the triple point and the critical point both increase with the increase of the dimensions d. If the dimensions and Gauss–Bonnet constant are fixed, the ratio of ρ to the triple point and the critical point both decrease as the potential increases, and both tend to a constant when the electric potential is large enough. In addition, since the temperature and pressure between the triple point and the critical point have a similar dependence on the Gauss–Bonnet constant α, i.e. $T \sim \frac{1}{\sqrt{\alpha}}$, $P \sim \frac{1}{\alpha}$, one can find another relation in general dimensions,

$$\frac{P_{\rm Tr}}{P_c} = \frac{\left(256\pi^2(d-3)\Phi^2+d\right)(d-4)}{\left(128\pi^2(d-3)\Phi^2+d-2\right)^2}. \tag{50}$$

5 Conclusion

In this paper, we deeply investigate the Hawking–Page phase transition of hyperbolic Gauss–Bonnet-AdS black holes in the grand canonical ensemble of the extended phase space. In particular, for the case of $d \geq 4$ dimensions, we find that there can exist two Hawking–Page phase transitions with different temperatures, which merge to form a reentrant Hawking–Page phase transition. The remarkable feature of this phase transition is that there are three phases coexistence of small black hole, large black hole and massless black hole, i.e. the triple point. Through calculation, we find that the Hawking–Page transition temperatures of both branches are affected by the pressure, Gauss–Bonnet constant and electric potential. Concretely speaking, the pressure and Gauss–Bonnet constant raise the high Hawking–Page transition temperature and decrease the low Hawking–Page transition temperature, while the electric potential has the opposite effect.

Further studies have revealed the phase diagram properties of the hyperbolic Gauss–Bonnet black holes. In four-dimensional spacetime, as long as the electric potential remains in the range of $0 \leq \Phi < \Phi_{tr}$, the reentrant Hawking–Page phase transition becomes inevitable. However, when the electric potential exceeds Φ_{tr}, the Haw-age phase transition disappears due to the dominant role of the electric potential in Gibbs free energy, and the system is in the stable black hole phase. For the case of $d > 4$ dimensions, the reentrant Hawking–Page phase transition occurs only within a specific range of pressures or electric potentials, which corresponds to the range between the triple point and the critical point. The hyperbolic Gauss–Bonnet AdS black holes in $d > 4$ dimensions do not have the Hawking–Page transition at the pressures or electric potentials below the triple point, but undergo a Hawking–Page transition at pressures or electric potentials above the critical point. Finally, we present the triple point and the critical point of the reentrant Hawking–Page phase transition of the hyperbolic Gauss–Bonnet AdS black hole in general dimensions. For the future task, It is interesting to study the reentrant Hawking–Page phase transitions in different backgrounds and to generalize the study to the microcosmic and holographic frameworks. For example, one can consider the existence of the reentrant Hawking–Page transitions for the rotating AdS black holes and the AdS black hole in four dimensional 4D EGB gravity [59].

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Code Availability Statement This manuscript has no associated code/software. [Author's comment: Not applicable.]

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Appendix A: Hawking–Page phase transition of the charged Gauss–Bonnet spherical AdS black hole

In this appendix, we will focus on the Hawking–Page phase transition in the grand canonical ensemble of the charged Gauss–Bonnet spherical AdS black hole, i.e. the case with $k = 1$. We find that the spherical case only has a single Hawking–Page phase transition for any dimensions. And the Hawking–Page transition temperature depends on the Gauss–Bonnet constant α. We will discuss the Hawking–Page phase transition in four dimensions, $d = 5$ dimensions and $d > 5$ dimensions in turn.

In four dimensions

Firstly, the metric and black hole solution in four dimensional Gauss–Bonnet gravity should be the RN-AdS spherical black hole, i.e.

$$\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + \frac{1}{f(r)}\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega_{2,1}^2, \tag{51}$$

$$f(r_+) = \frac{8\pi P r_+^2}{3} - \frac{8M\pi}{r_+} + \frac{Q^2}{4r_+^2} + 1. \tag{52}$$

The thermodynamic quantities are shown below,

$$M = \frac{32P\pi r_+^4 + 3Q^2 + 12r_+^2}{96\pi r_+}, \tag{53}$$

$$T = \frac{32P\pi r_+^4 - Q^2 + 4r_+^2}{16r_+^3\pi}. \tag{54}$$

The entropy of a spherically symmetric AdS black hole in four-dimensional Gauss–Bonnet gravity is,

$$S = \frac{r_+^2}{4} + \alpha. \tag{55}$$

The relation between the electric charge Q and the electric potential Φ in four dimensions is $Q = 4\Phi\pi r_+$. Therefore, the black hole mass and Hawking temperature are rewritten in the case of fixed potential.

$$M = \frac{r_+\left(8P\pi r_+^2 + 192\Phi^2\pi^2 + 3\right)}{24\pi}, \tag{56}$$

$$T = \frac{8P\pi r_+^2 - 64\Phi^2\pi^2 + 1}{4\pi r_+}. \tag{57}$$

Thus, the Gibbs free energy $G(r_+, P, \Phi, \alpha)$ expression of the four dimensional spherically symmetric charged Gauss–Bonnet AdS black hole can be given,

$$G = \frac{-156\Phi^2\left(r_+^2 - \frac{4\alpha}{13}\right)\pi^2 - 8Pr_+^2\left(r_+^2 + 12\alpha\right)\pi + 3r_+^2 - 12\alpha}{48r_+\pi}. \tag{58}$$

The curve of Gibbs free energy with the temperature under the different pressures is plotted, as shown in Fig. 11. When the pressure is large, the Gibbs free energy of the system is negative, and the system is in the stable black hole phase. When the pressure is gradually decreased, the system will undergo a Hawking Page phase transition from the hot AdS vacuum phase to the large black hole phase. Note that there is another larger Hawking–Page phase transition temperature, for which the global stability of the system at this time depends on the minimum Gibbs free energy, thus this Hawking–Page phase transition temperature is a metastable state which is out of our discussion. Then, by substituting variables

$$R_{HP} = r_{HP}^2, \tag{59}$$

we can directly solve for $G = 0$ to get the radius of the Hawking–Page phase transition,

$$R_{HP} = \frac{-156\Phi^2\pi^2 - 96P\pi\alpha + 3}{16P\pi} + \frac{\sqrt{192\pi P\alpha\left(164\Phi^2\pi^2 + 48P\pi\alpha - 5\right) + 9\left(52\Phi^2\pi^2 - 1\right)^2}}{16\pi P}. \tag{60}$$

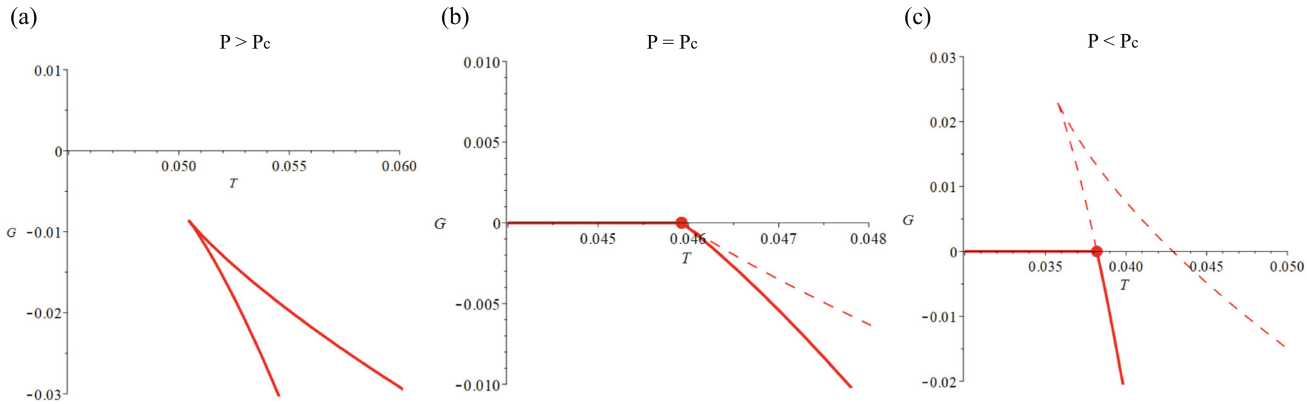

Fig. 11 *G–T* diagram of a four-dimensional spherically symmetric charged Gauss–Bonnet black hole. The global stable state of the system is represented by a solid red line. As the pressure increases, the system will undergo a Hawking–Page phase transition

The Hawking–Page temperature can be obtained by Eq. (57), namely,

$$T_{\mathrm{HP}} = T|_{r_+=r_{\mathrm{HP}}}. \tag{61}$$

To explore the phase structure more precisely, we plot the coexistence lines of the *P–T* phase diagram for the Hawking–Page phase transition of a four-dimensional spherically symmetric charged AdS black hole, as shown in Fig. 12a. When the temperature is constant, there is always a Hawking–Page phase transition from hot AdS vacuum to large black hole state in a small temperature range ($T < T_c$). Note that compared with the *G–T* diagram, the state represented by the blue dashed line is an metastable small black hole state. Since the global stability of the system at this time is biased towards the large black hole state, this phase disappears from the phase diagram. When the pressure is constant but smaller than the pressure at the critical point, although there are two Hawking–Page transition temperatures, there is only one Hawking–Page phase transition due to the less stability of the small black hole state. When the pressure is greater than the pressure at the critical point, there is no Hawking–Page phase transition. By making the two phase transition radii coincide, the critical point is easily derived, as shown below,

$$r_c = 2\sqrt{3}\sqrt{\alpha}, \quad P_c = \frac{1-64\Phi^2\pi^2}{96\pi\alpha},$$
$$T_c = \frac{\sqrt{3}\left(1-64\Phi^2\pi^2\right)}{12\sqrt{\alpha}\pi}. \tag{62}$$

Another critical point has a negative radius and is non-physical. We also introduce the universal relationship of critical points,

$$\frac{P_c r_c}{T_c} = \frac{1}{4}. \tag{63}$$

We plotted the Hawking–Page transition temperature curve with Gauss–Bonnet constant and the electric potential, as shown in Fig. 12b, c. Since the radius of the Hawking–Page transition is non-negative, the Gauss–Bonnet constant and the electric potential should satisfy the following requirements,

$$\alpha < \alpha_c = \frac{1-64\Phi^2\pi^2}{96\pi P}, \quad \Phi < \Phi_c = \frac{\sqrt{1-96P\pi\alpha}}{8\pi}. \tag{64}$$

When the Gauss–Bonnet constant or the electric potential is large, there is no Hawking–Page phase transition. This is consistence with the result in [24]. Noting that it is also interesting to study the Hawking–Page phase transition of spherical AdS black holes in 4D EGB gravity which is left as a future work.

The Hawking Page phase transition of charged spherically AdS black holes in high-dimensional spacetime is different from that in the four-dimensional case. We will analyze it from the perspectives of five and higher dimensions respectively. The results show that there is only a single Hawking Page phase transition and no reentrant HP phase transition for the spherically symmetric charged AdS black holes in any dimension.

In $d = 5$ dimensions

The Gibbs free energy and Hawking temperature of the 5-dimensional charged Gauss–Bonnet spherical AdS black hole in the grand canonical ensemble are shown as follows,

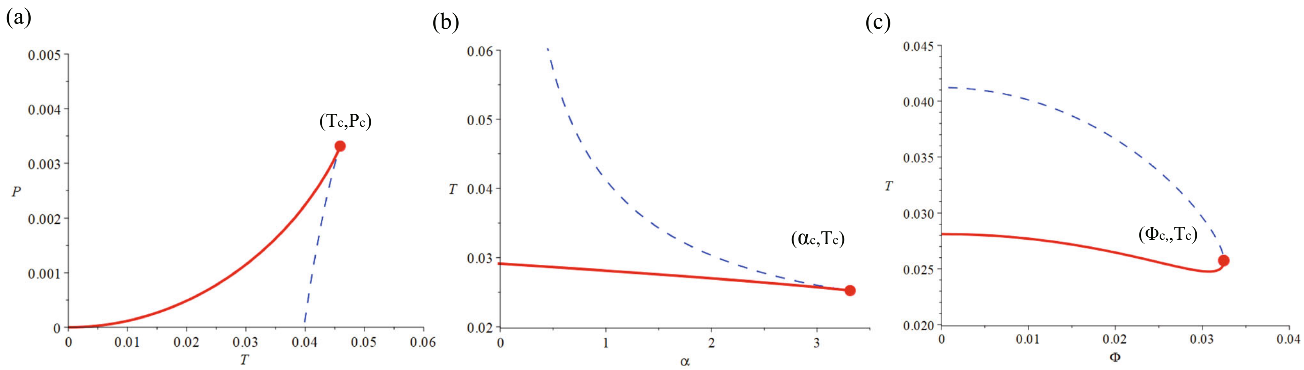

Fig. 12 **a** Coexistence line: P–T phase diagram (with $\alpha = 1, \Phi = 0.001$) in four dimensions. The critical points have been indicated in the diagram. The solid red line represents the stable large black hole state, and the dashed blue line represents the metastable small black hole state. **b** $T-\alpha$ diagram (with $P = 0.001$, $\Phi = 0.001$). When $\alpha > \alpha_c$, there is no HP phase transition. **c** $T-\Phi$ diagram (with $\alpha = 1$, $P = 0.001$). When $\Phi > \Phi_c$, there is no HP phase transition

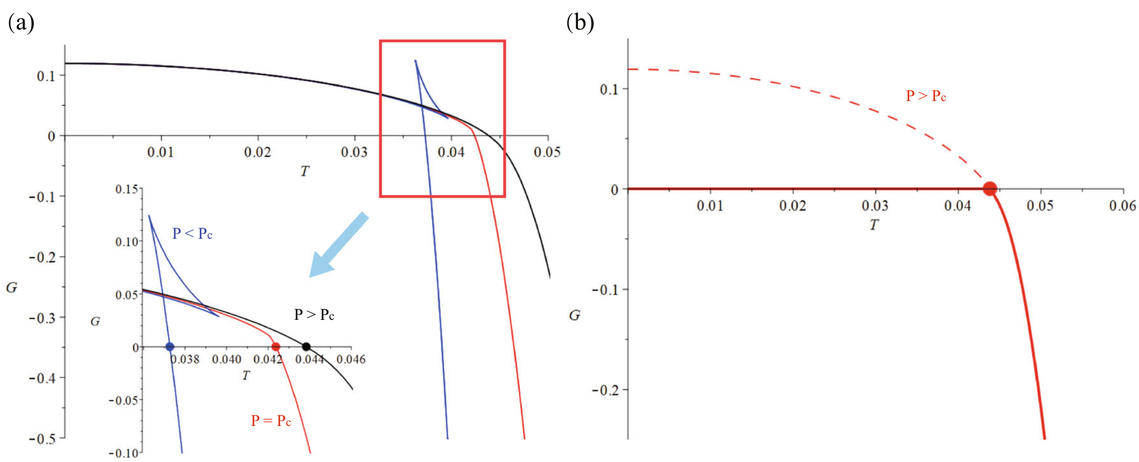

Fig. 13 G–T diagram of a spherically symmetric charged Gauss–Bonnet-AdS black hole in five-dimensional spacetime. The Hawking–Page phase transition points are marked in the diagram. **a** Critical behavior. When the pressure is small, the system undergoes a first order large/small black hole phase transition and a Hawking–Page phase transition. At high pressure, the system only undergoes a Hawking–Page phase transition. **b** The Hawking–Page phase transition diagram, the global stable state of the system is represented by the solid red line, while the dotted red line represents the unstable black hole state

$$G = \frac{-4P\pi r_+^6 + \left(-256\Phi^2\pi^2 - 144P\pi\alpha + 3\right) r_+^4 + \left(3072\Phi^2\pi^2 - 18\right)\alpha r_+^2 + 72\alpha^2}{48\left(r_+^2 + 4\alpha\right)\pi}, \tag{65}$$

$$T = -\frac{r_+\left(-8\pi P r_+^2 + 256\Phi^2\pi^2 - 3\right)}{6\left(r_+^2 + 4\alpha\right)\pi}. \tag{66}$$

We plot the Gibbs free energy curve with temperature in five dimensions. We find that when the pressure is low, the Gibbs free energy of the system has a classical swallow-tail behavior, that is, a first-order large/small black hole phase transition (see detail in [71].), as shown in the Fig. 13. The global stable state of the system depends on the global minimum of the Gibbs free energy. Thus the system actually undergoes only the Hawking–Page phase transition between the hot AdS vacuum phase and the large black hole phase.

In order to explore the phase structure of five-dimensional spacetime in more detail, we draw the P–T phase diagram of the Hawking–Page phase transition. It is obvious that 5-dimensional spherically symmetric charged Gauss–Bonnet AdS black hole always has a Hawking–Page phase transitions. On the other hand, the critical point for the first-order large/small black hole phase transition is highlight in Fig. 14. When the pressure is low (the red line in the Fig. 14), the

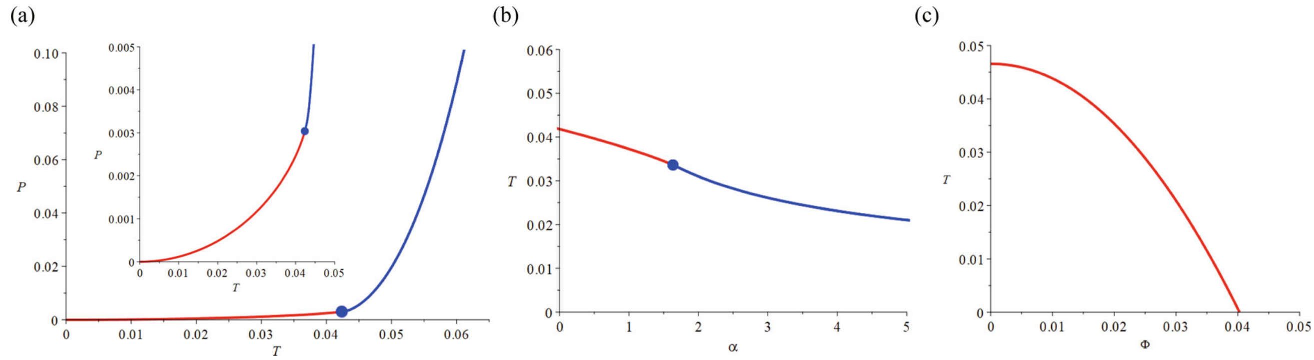

Fig. 14 There is always a Hawking–Page phase transition in the system. **a** Coexistence line: P–T phase diagram (with $\alpha = 1$, $\Phi = 0.01$) in five dimensions. The critical point for the first-order large/small black hole phase transition is highlight. **b** $T-\alpha$ diagram (with $P = 0.002$, $\Phi = 0.01$). **c** $T-\Phi$ diagram (with $\alpha = 1$, $P = 0.004$)

system also has a first-order large/small black hole phase transition in addition to a Hawking–Page phase transition. However, since the global stability of the vacuum phase is greater than that of the small black hole, only the Hawking–Page phase transition from the hot AdS vacuum to the large black hole exists in the system.

To show the influence of the Gauss–Bonnet constant, we plot the $T-\alpha$ diagram in Fig. 14b. Obviously, with the increase of Gauss–Bonnet constant, the Hawking–Page phase transition temperature gradually decreases. And for arbitrary Gauss–Bonnet constant, there is always a Hawking–Page phase transition. Finally, we plot $T-\Phi$ diagram in Fig. 14c, and as the electric potential Φ increases, the Hawking–Page transition temperature decreases. But when the electric potential is large enough, the Hawking–Page transition temperature will become negative, which makes that the electric potential needs to be constrained. Using a variable substitution, $\beta = \sqrt{\alpha P}$, this range can be given more easily by calculating the zero Hawking–Page transition temperature, i.e.

$$0 < \Phi < \frac{\sqrt{8\sqrt{6}\sqrt{\pi}\beta + 96\pi\beta^2 - 36}^{1/4}}{16\pi\sqrt{24\sqrt{\pi}\beta - \sqrt{6}}}. \tag{67}$$

In $d > 5$ dimensions

The Gibbs free energy and Hawking temperature in six dimensions are as following,

$$G = -\frac{r\left(P\pi r^6 + \left(120\pi^2\Phi^2 + 72P\pi\alpha - \frac{5}{4}\right)r^4 + \alpha\left(-2880\pi^2\Phi^2 + \frac{15}{2}\right)r^2 - 180\alpha^2\right)}{20\left(r^2 + 12\alpha\right)\pi}, \tag{68}$$

$$T = \frac{4\pi Pr^4 + \left(-288\Phi^2\pi^2 + 3\right)r^2 + 6\alpha}{4r\left(r^2 + 12\alpha\right)\pi}. \tag{69}$$

In seven dimensions, the corresponding quantities are,

$$G = -\frac{r^2\left(P\pi r^6 + \left(192\pi^2\Phi^2 + 120P\pi\alpha - \frac{15}{8}\right)r^4 + \alpha\left(-7680\pi^2\Phi^2 + \frac{15}{2}\right)r^2 - 900\alpha^2\right)}{30\pi\left(r^2 + 24\alpha\right)}, \tag{70}$$

$$T = \frac{4\pi Pr^4 + \left(-512\pi^2\Phi^2 + 5\right)r^2 + 30\alpha}{5\left(r^2 + 24\alpha\right)r\pi}. \tag{71}$$

We have plotted the G–T diagrams in six and seven dimensional spacetime in Fig. 15a, e. Obviously, in the case of higher dimensions (with $d > 5$), there is no first-order large/small black hole phase transition in the system (see detail in [71].), only a Hawking–Page phase transition. Corresponding to the P–T diagram in Fig. 15b, f, the critical point of a first-order large/small black hole phase transition existing in the five-dimensional spacetime disappears. In this case, the Hawking–Page phase transition always exists at arbitrary

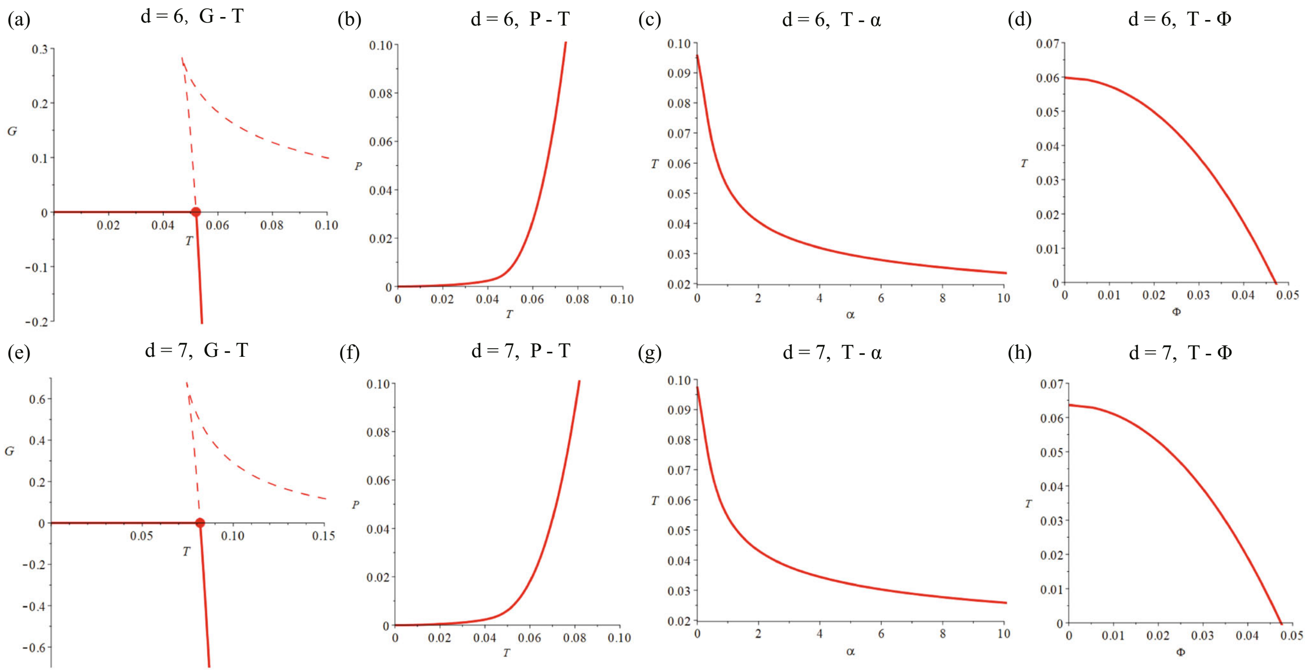

Fig. 15 **a**, **e** G–T diagram of a six- and seven-dimensional spherically symmetric charged Gauss–Bonnet AdS black hole (with $\alpha = 1$, $\Phi = 0.01$, $P = 0.01$). The system has a Hawking–Page phase transition from hot AdS vacuum phase to a large black hole phase. **b**, **f** P–T diagram of a six- and seven-dimensional spherically symmetric charged Gauss–Bonnet AdS black hole (with $\alpha = 1$, $\Phi = 0.01$). The Hawking–Page phase transition always exists for arbitrary temperature and pressure. **c**, **g** $T-\alpha$ diagram (with $P = 0.01$, $\Phi = 0.01$). With the increase of the Gauss–Bonnet constant, the Hawking–Page transition temperature decreases. **d**, **h** $T-\Phi$ diagram (with $\alpha = 1$, $P = 0.02$). With the increase of the electric potential, the Hawking–Page transition temperature decreases

temperature and pressure. The Hawking–Page phase transition temperature increases with the increase of the pressure. The effect of six - and seven-dimensional Hawking–Page phase transition temperatures with the Gauss–Bonnet constant and electric potential are also plotted, as shown in the Fig. 15c, d, g, h. Both the Gauss–Bonnet constant and the electric potential reduce the Hawking–Page transition temperature. For any Gauss–Bonnet constant, the Hawking–Page phase transition is always present, which means that the Gauss–Bonnet constant does not affect the existence of the Hawking–Page phase transition, only its temperature. However, when the electric potential is increased, the Hawking–Page temperature may become less than zero, which is physically unreasonable. Therefore, we need to put a certain constraint on the electric potential to ensure that the Hawking–Page transition temperature is always physical. Similar to the five-dimensional case, using variable substitution, $\beta = \sqrt{\alpha P}$, and calculate the zero Hawking–Page transition temperature, we give the range of the electric potentials required for a spherically symmetric charged Gauss–Bonnet AdS black hole to undergo a Hawking–Page phase transition in six and seven dimensions, (In fact, following the same procedure, the constraints of the electric potential on the occurrence of a Hawking–Page phase transitions in $d \geq 5$ dimensions can be analytically obtained, but the results are too complicated to be shown here.) i.e.

$$0 < \Phi < \frac{30^{3/4}\sqrt{4\beta\sqrt{\pi}\sqrt{30} + 192\beta^2\pi - 5}}{120\pi^{3/4}\sqrt{48\beta\sqrt{\pi} - \sqrt{30}}}, \quad \text{in six dimension,} \tag{72}$$

$$0 < \Phi < \frac{5^{1/4}2^{3/4}\sqrt{16\beta\sqrt{\pi}\sqrt{10} + 640\beta^2\pi - 15}}{32\pi^{3/4}\sqrt{-3\sqrt{10} + 80\sqrt{\pi}}}, \quad \text{in seven dimension.} \tag{73}$$

Finally, we show the P–T diagrams of the spherically symmetric charged Gauss–Bonnet AdS black holes in different dimensions in Fig. 16a. With the increase of the dimensions, the Hawking–Page phase transition temperature increases gradually under the same pressure. In addition, the relation between the dimensions and the maximum electric potential of the Hawking–Page phase transition is also plotted in Fig. 16b. As the dimensions increases, so does the maximum potential.

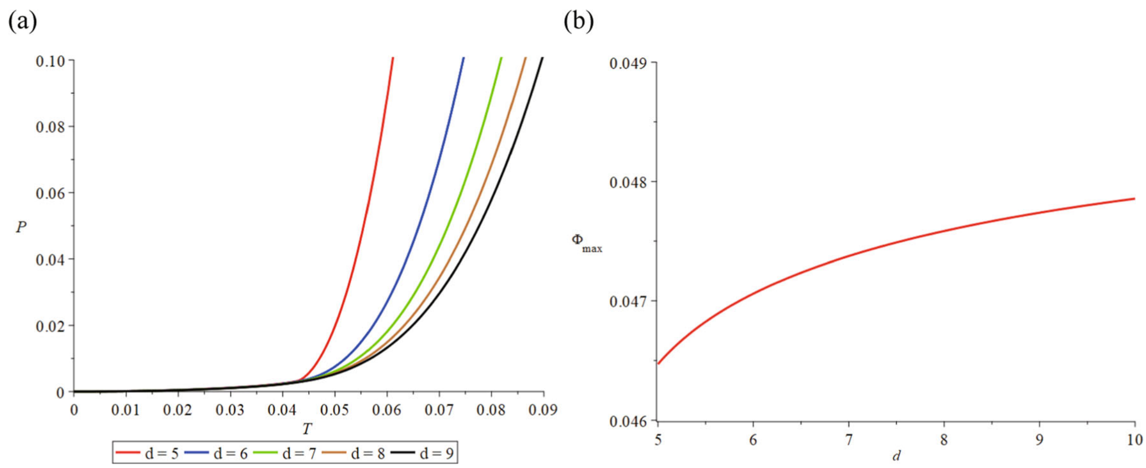

Fig. 16 **a** P–T diagram in different dimensions ($\alpha = 1$, $\Phi = 0.01$). **b** The relationship between dimensionality and maximum potential of Hawking–Page phase transition

Appendix B: The quantities at the critical point of the reentrant Hawking–Page phase transition for the Gauss–Bonnet charged hyperbolic AdS black holes in diverse dimensions

In this Appendix, we list the quantities at the critical point of the reentrant Hawking–Page phase transition for the Gauss–Bonnet charged hyperbolic AdS black holes in diverse dimensions. One can see some numerical results about the critical point of the reentrant Hawking–Page phase transition in diverse dimensions.

In D = 5 dimensions

$$r_c = \frac{2\sqrt{3}\sqrt{\alpha}\sqrt{384\Phi^2\pi^2 + 2\sqrt{36864\Phi^4\pi^4 + 640\Phi^2\pi^2 + 3} + 3}}{\sqrt{256\Phi^2\pi^2 + 3}}, \tag{74}$$

$$P_c = \frac{\left(256\Phi^2\pi^2 + 3\right)^2}{96\alpha\pi}, \tag{75}$$

$$\begin{aligned} T_c = \sqrt{3}&\left(192\Phi^2\pi^2 + \sqrt{36864\Phi^4\pi^4 + 640\Phi^2\pi^2 + 3} + 1\right)\left(256\Phi^2\pi^2 + 3\right)^{3/2} \\ &\times\Bigg(12\pi\sqrt{\alpha}\sqrt{384\Phi^2\pi^2 + 2\sqrt{36864\Phi^4\pi^4 + 640\Phi^2\pi^2 + 3} + 3} \\ &\times(192\Phi^2\pi^2 - \sqrt{36864\Phi^4\pi^4 + 640\Phi^2\pi^2 + 3} + 3)\Bigg)^{-1}, \end{aligned} \tag{76}$$

$$\frac{P_c r_c}{T_c} = \frac{448\Phi^2\pi^2 + 3 + 3\sqrt{36864\Phi^4\pi^4 + 640\Phi^2\pi^2 + 3}}{768\Phi^2\pi^2 + 4\sqrt{36864\Phi^4\pi^4 + 640\Phi^2\pi^2 + 3} + 4}. \tag{77}$$

In D = 6 dimensions

$$r_c = \frac{\sqrt{6\alpha}\sqrt{576\Phi^2\pi^2+\sqrt{331776\Phi^4\pi^4+4224\Phi^2\pi^2+17}+3}}{\sqrt{96\Phi^2\pi^2+1}}, \tag{78}$$

$$P_c = \frac{5\left(96\,\Phi^2\pi^2+1\right)^2}{96\,\alpha\,\pi}, \tag{79}$$

$$T_c = \sqrt{6}\left(576\Phi^2\pi^2 + 1 + \sqrt{331776\Phi^4\pi^4 + 4224}\Phi^2\pi^2 + 17\right)\left(96\Phi^2\pi^2 + 1\right)^{3/2} \times \left(3\pi\sqrt{\alpha}\sqrt{576\Phi^2\pi^2 + \sqrt{331776\Phi^4\pi^4 + 4224\Phi^2\pi^2 + 17} + 3} \times (-\sqrt{331776\Phi^4\pi^4 + 4224\Phi^2\pi^2 + 17} + 576\Phi^2\pi^2 + 9)\right)^{-1}, \tag{80}$$

$$\frac{P_c r_c}{T_c} = \frac{13440\Phi^2\pi^2 + 50 + 30\sqrt{331776\Phi^4\pi^4 + 4224\Phi^2\pi^2 + 17}}{18432\Phi^2\pi^2 + 32\sqrt{331776\Phi^4\pi^4 + 4224\Phi^2\pi^2 + 17} + 32}. \tag{81}$$

In D = 7 dimensions

$$r_c = \frac{2\sqrt{10\alpha}\sqrt{768\Phi^2\pi^2 + 2\sqrt{147456\Phi^4\pi^4 + 1536\Phi^2\pi^2 + 6} + 3}}{\sqrt{512\Phi^2\pi^2 + 5}}, \tag{82}$$

$$P_c = P_c = \frac{\left(512\Phi^2\pi^2 + 5\right)^2}{640\alpha\pi}, \tag{83}$$

$$T_c = -\frac{\sqrt{10}\left(384\Phi^2\pi^2 + \sqrt{147456\Phi^4\pi^4 + 1536\Phi^2\pi^2 + 6}\right)\left(512\Phi^2\pi^2 + 5\right)^{3/2}}{} \times \left(40\pi\sqrt{\alpha}\sqrt{768\Phi^2\pi^2 + 2\sqrt{147456\Phi^4\pi^4 + 1536\Phi^2\pi^2 + 6} + 3} \times (-384\Phi^2\pi^2 + \sqrt{147456\Phi^4\pi^4 + 1536\Phi^2\pi^2 + 6} - 6)\right)^{-1}, \tag{84}$$

$$\frac{P_c r_c}{T_c} = \frac{2688\Phi^2\pi^2 + 6 + 9\sqrt{147456\Phi^4\pi^4 + 1536\Phi^2\pi^2 + 6}}{3072\Phi^2\pi^2 + 8\sqrt{147456\Phi^4\pi^4 + 1536\Phi^2\pi^2 + 6}}. \tag{85}$$

In D = 8 dimensions

$$r_c = \frac{\sqrt{30}\sqrt{\alpha}\sqrt{960\Phi^2\pi^2 + \sqrt{921600\Phi^4\pi^4 + 8320\Phi^2\pi^2 + 33} + 3}}{\sqrt{320\Phi^2\pi^2 + 3}}, \tag{86}$$

$$P_c = \frac{7\left(320\Phi^2\pi^2 + 3\right)^2}{1920\alpha\pi}, \tag{87}$$

$$T_c = \frac{\sqrt{30}\left(960\Phi^2\pi^2 - 1 + \sqrt{921600\Phi^4\pi^4 + 8320\Phi^2\pi^2 + 33}\right)\left(320\Phi^2\pi^2 + 3\right)^{3/2}}{} \times \left(30\pi\sqrt{\alpha}\sqrt{960\Phi^2\pi^2 + \sqrt{921600\Phi^4\pi^4 + 8320\Phi^2\pi^2 + 33} + 3} \times (-\sqrt{921600\Phi^4\pi^4 + 8320\Phi^2\pi^2 + 33} + 960\Phi^2\pi^2 + 15)\right)^{-1}, \tag{88}$$

$$\frac{P_c r_c}{T_c} = \frac{-62720\Phi^2\pi^2 - 84 - 84\sqrt{921600\Phi^4\pi^4 + 8320\Phi^2\pi^2 + 33}}{61440\Phi^2\pi^2 + 64\sqrt{921600\Phi^4\pi^4 + 8320\Phi^2\pi^2 + 33} - 64}. \tag{89}$$

Table 1 When $\alpha = 1$, $\Phi = 0.01$, the quantities at the critical point of the reentrant Hawking–Page phase transition in diverse dimensions

Dimension (d)	Five	Six	Seven	Eight	Nine
Critical radius (r_c)	5.15719776	6.70568504	8.21714300	9.71728910	11.21384121
Critical pressure (P_c)	0.03507977	0.01986906	0.01507426	0.01275941	0.01140352
Critical temperature (T_c)	0.24453459	0.13073274	0.09290687	0.07391669	0.06245580
Universal ratio ($\frac{P_c r_c}{T_c}$)	0.73982707	1.01914544	1.33324172	1.67738748	2.04748352

In D = 9 dimensions

$$r_c = \frac{2\sqrt{21}\sqrt{\alpha}\sqrt{1152\Phi^2\pi^2 + 2\sqrt{331776\Phi^4\pi^4 + 2688\Phi^2\pi^2 + 11} + 3}}{\sqrt{768\Phi^2\pi^2 + 7}}, \tag{90}$$

$$P_c = \frac{\left(768\Phi^2\pi^2 + 7\right)^2}{1680\alpha\pi}, \tag{91}$$

$$\begin{aligned} T_c = & \frac{\sqrt{21}\left(576\Phi^2\pi^2 + \sqrt{331776\Phi^4\pi^4 + 2688\Phi^2\pi^2 + 11} - 1\right)\left(768\Phi^2\pi^2 + 7\right)^{3/2}}{} \\ & \times \left(84\pi\sqrt{\alpha}\sqrt{1152\Phi^2\pi^2 + 2\sqrt{331776\Phi^4\pi^4 + 2688\Phi^2\pi^2 + 11} + 3}\right. \\ & \left. \times (576\Phi^2\pi^2 - \sqrt{331776\Phi^4\pi^4 + 2688\Phi^2\pi^2 + 11} + 9)\right)^{-1}, \end{aligned} \tag{92}$$

$$\frac{P_c r_c}{T_c} = \frac{1344\Phi^2\pi^2 + 3\sqrt{331776\Phi^4\pi^4 + 2688\Phi^2\pi^2 + 11} + 1}{1152\Phi^2\pi^2 - 2 + 2\sqrt{331776\Phi^4\pi^4 + 2688\Phi^2\pi^2 + 11}}. \tag{93}$$

To have a quantitative picture for the quantities at the critical point, one can see the following table.

When $\alpha = 1$, $\Phi = 0.01$, the quantities at the critical point of the reentrant Hawking–Page phase transition in diverse dimensions (Table 1).

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