# HOLOGRAPHY IN DE-SITTER AND MINKOWSKI SPACE

by

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# Experience

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- **Holography in de-Sittes space using integral transform**, May 2014 – March 2015

  Atreya Chatterjee and David A. Lowe. Holographic operator mapping in dS/CFT and cluster decomposition. Phys. Rev., D92(8):084038, 2015.

  - We constructed the bulk to boundary mapping for massive scalar fields, providing a de Sitter analog of the LSZ reduction formula. The set of boundary correlators thus obtained defines a potentially new class of conformal field theories based on principal series representations of the global conformal group. Conversely, we show bulk field operators in de Sitter may be reconstructed from boundary operators. The resulting conformal field theory does not satisfy the Cluster decomposition, so is likely not well-defined once interactions are included.

- **de-Sitter gravity/Euclidean gravity correspondence**, March 2015 – October 2015

  Atreya Chatterjee and David A. Lowe. de Sitter gravity/Euclidean conformal gravity correspondence. Phys. Rev., D93(2):024038, 2016.

  - The holographic dual of a gravitational theory around the de Sitter background is argued to be a Euclidean conformal gravity theory in one fewer dimensions. The measure for the holographic theory naturally includes a sum over topologies as well as conformal structures.

- **dS/CFT and operator product expansion**, October 2015 – December 2016
  Atreya Chatterjee and David A. Lowe. dS/CFT and the operator product expansion. 2016. Phys.Rev. D96 (2017) no.6, 066031

  - Global conformal invariance determines the form of two and three-point functions of quasi- primary operators in a conformal field theory, and generates nontrivial relations between terms in the operator product expansion. These ideas are generalized to the principal and complementary series representations, which play an important role in the conjectured dS/CFT correspondence. OPEs of such conformal field theories have essential singularities, so cannot be realized as conventional field theories.

- **BMS symmetry, soft particles, Gravitational memory**, January 2016 – December 2017

  Atreya Chatterjee and David A. Lowe. BMS symmetry, soft particles, Gravitational memory 2017 arXiv:1712.03211

  - In this work, we revisit unitary irreducible representations of the Bondi-Metzner-Sachs (BMS) group discovered by McCarthy. Representations are labelled by an infinite number of super- momenta in addition to four-momentum. Tensor products of these irreducible representations lead to particle-like states dressed by soft gravitational modes. Conservation of 4-momentum and supermomentum in the scattering of such states leads to a memory effect encoded in the outgoing soft modes. We note there exist irreducible representations corresponding to soft states with strictly vanishing four-momentum, which may nevertheless be produced by scattering of particle-like states. This fact has interesting implications for the S-matrix in gravitational theories.

## Teaching

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- Physics Merit Fellowship 2017-2018: Awarded fellowship in 5th year of graduate school at Brown University based on research work done in previous years.
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- Institute Silver Medal 2013: For department rank 1 in B.Tech program in Engineering Physics.

# Preface and Acknowledgments

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# Chapter 1

# Introduction

Discovering the underlying oneness in the laws of nature is the goal of physics. Over the last century we have made progress which was unimaginable even couple of centuries ago. General relativity and quantum mechanics are the two pillars of physics today. Some form of Kaluza-Klein mechanism sheds light on classical unification of all forces in terms of geometry. However the gap between unification of various forces blows up when one tries to embed them in quantum framework. Quantum electrodynamics is one of the most precise theories, where as quantum chromodynamics and quantum gravity are filled with many conceptual difficulties.

Holography was discovered as a part of endeavour to solve the mysteries of quantum gravity. Principal of holography asserts that the theory of gravity in the bulk can be formulated as a dual theory on a manifold with one lesser dimension. Often it is convenient to think of this manifold as the boundary of the spacetime. In chapter 2 we introduce the basic ideas of holography and explain why and how it is possible to encode all the information of bulk in one lesser dimension. In this chapter we focus on the hologarphy in Anti-de-Sitter(AdS) space as it was the first model to be studied. AdS is a space with constant negative curvature. We mention some of the prescriptions which map bulk information to the dual theory.

Often one has to introduce quantities in calculation which are not observable in nature. Quantum field theory is generally described in terms of field operators. These fields are not observable . What we measure is there correlation function. It is often found that theory based on the observables is often simpler and conceptually more clear although it may not be intuitive. Arthur Wightman proposed a form of QFT starting with axioms on correlation functions. These reproduced quantum field theoretic results. This field of study is known as Axiomatic QFT. It turns out that in some prescriptions of holography correlation functions are easier to describe than the fields. So in chapter 3 we describe Axiomatic QFT and some of its important results and the holographic prescription.

Our universe has positive curvature. Such a space is known as de-Sitter(ds) space. In chapter 4 our first work on dS holography is presented. After brief introduction of various coordinate systems of dS, we mention the difficulties of analytically continuing AdS holography to dS. Then we explain our main mathematical tool of integral transform in section 4.3. Integral transform is nothing but generalization of Radon transform to curved spaces. Finally we use this machinery to understand holography in dS space in the large N limit, that is only for the free fields.

In chapter 5 we explore dS holography in the presence of gravity. We find that in the presence of gravity, boundary does not decouple from the bulk, as a result there is no well defined holographic map at finite N. We show this using various examples.

We continue exploring dS holography in presence of interaction in chapter 6. Assuming there exists a dual CFT to interacting bulk theory we calculate Operator Product Expansions (OPEs) of various operators. We show that such OPEs have increasingly singular behavior unlike OPEs of usual CFTs. This signals break down of CFT. This is another strong hint of breakdown of holographic description of dS space.

In chapter 7, we shift gears from dS space to Minkowski space. Asymptotic sym-

metry group of Minkowski space is known as Bondi-Metzner-Sachs (BMS) group. We review the representations of BMS group discovered by McCarthy. We consider representations of various little groups. It is shown that relation between BMS group, gravitational memory as proposed by Strominger et. al. is evident from the representations of BMS group.

# Chapter 2

# Anti-de-Sitter/Conformal Field Theory correspondence

Discovery of holography marks one of the biggest steps in theoretical physics. Almost all the fields in high energy theory and many fields outside have been effected by holography. There are excellent resources which give technical details [1, 2, 3, 4] of holography in details. However for a beginner it is often overwhelming to get an intuition of it. So before diving into mathematical formula, I will devote the first subsection in motivating holography. The second and third subsection introduces quantum field theory in Anti-de-Sitter space and conformal field theory respectively. Finally in fourth sub-section we present the precise statement of holography.

## 2.1 Motivation

A quantum theory is fully characterized by symmetry group. Eigenstates of the maximally commuting operators of the symmetry group are the observables. For example, symmetry group of Minkowski space is Poincare group. Quantum field theory of free particles in bulk is labelled by eigenstates of momentum. However, in reality there is always interaction, for example gravity. Individual momentum of each particle is not conserved due to interaction. Instead of individual particle one observes sum total of individual particle and the interaction between them. This suggests that symmetry group in presence of interaction is no more Poincare group. For example, position of the particle acts as a reference point and breaks translation invariance. It is still possible to describe the bulk by assuming free particles and interaction between them. If the interaction is weak one can do calculation which only gives perturbative power series expansion in coupling constant. The terms of the series, if known, can sometimes be resummed using various techniques. But even a bigger problem is that for most of the realistic theories terms beyond certain order are extremely difficult to calculate. This clearly indicates that there is something missing.

This motivates one to search for another approach which can give non-perturbative observable from first principal. Here it is important to elaborate more on "non-perturbative observable" and "first principal"? Although we do not have completely clear idea about them, we know something. Let us start from what we know. For a free theory, "first principal" is the symmetry group and "non-perturbative observable" are the eigenstates of the symmetry group. In presence of gravity (i.e interaction), symmetry group is not known. Even if it is known for certain cases, like Bondi-Metzner-Sachs group, it is not clear whether symmetry group is the correct first principal to start with. Observable is closely related to first principal. It is clear that individual particles cannot be observables. This suggests that spacetime including the particles and interactions has to be seen as one whole unit. But how can one "observe" the whole spacetime. This leads one to consider asymptotic spacetime. One can think of this as observing the whole spacetime from outside, seeing only its surface. So the natural choice is to understand theory of the boundary. In other words the "first principal" is the boundary theory and "non-perturbative observable" are states of the boundary.

To understand the theory of boundary one again looks at the free bulk theory for which symmetry group is known. It turns out to be easier to start with bulk with negative curvature called Anti-de-Sitter (AdS) space discussed in details in section 2.2. Symmetry group of $d+1$ dimensional AdS space is Lorentz group $SO(2,d)$. The boundary theory for the free bulk theory must have this symmetry. For a free two-point functions captures all the information. It turns out that $SO(2,d)$ is also symmetry group of conformal field theory(CFT) in $d$-dimension. In CFT, operators form a complete basis. In other words, product of any two operators can be expanded as a linear sum of operators in the CFT. This is is called Operator Product Expansion(OPE). Specifying all the operators and OPE uniquely specifies a CFT. We will explain more about conformal field theory in section 2.3. Hard thing is to describe interacting bulk theory. When interactions are turned on diagonalizing symmetry group is not the best strategy because symmetry group is itself not known.

Now comes the interesting part. Although interactions change the bulk symmetry, boundary theory is still conformally invariant. So boundary symmetry is still known. What changes in the boundary theory is the OPE structure. This non-trivial map between between bulk symmetry group and OPE on the boundary, forms the bedrock of holography. there are srong indications that even when interaction in the bulk are strong this duality will hold.

## 2.2 Anti-de-Sitter space (AdS)

Anti-de-Sitter space is maximally symmetric spacetime with constant negative curvature. Commonly used models have $l_{AdS}$ as the radius of curvature of AdS space, 2 time-like dimensions and $d$ space-like dimensions. Then the AdS space is realized on hyperbolic space

$$X_0^2 + X_{d+1}^2 - X_1^2 - ... - X_d^2 \quad = \quad l_{AdS}^2$$

Line element is given by

$$ds^2 = dX_0^2 + dX_{d+1}^2 - dX_1^2 - ... - dX_d^2$$

Throughout the thesis we will use positive signature for time-like coordinates and negative signature for space-like coordinates.

### 2.2.1 Global Coordinates

Using the variables

$$\begin{aligned} X_0 &= l \cos\tau \cosh\rho \\ X_{d+1} &= l \sin\tau \cosh\rho \\ X_i &= l \sinh\rho\omega_i \end{aligned}$$

where $1 \leq i \leq d$, $\omega_i$ parametrize the sphere.

$$\omega^1 = \cos\theta_1, \tag{2.2.1}$$

$$\omega^2 = \sin\theta_1 \cos\theta_2, \tag{2.2.2}$$

$$\vdots \tag{2.2.3}$$

$$\omega^{d-1} = \sin\theta_1 \cdots \sin\theta_{d-2} \cos\theta_{d-1}, \tag{2.2.4}$$

$$\omega^d = \sin\theta_1 \cdots \sin\theta_{d-2} \sin\theta_{d-1}, \tag{2.2.5}$$

where $0 \leq \theta_i < \pi$ for $1 \leq i < d-1$, but $0 \leq \theta_{d-1} < 2\pi$. Then it is clear that $\sum_{i=1}^d (\omega^i)^2 = 1$, and the metric on $S^{d-1}$ is

$$d\Omega_{d-1}^2 = \sum_{i=1}^{d} (d\omega^i)^2 = d\theta_1^2 + \sin^2\theta_1 d\theta_2^2 + \cdots + \sin^2\theta_1 \cdots \sin^2\theta_{d-2} d\theta_{d-1}^2. \tag{2.2.6}$$

In this coordinate system line element is given by

$$ds^2 = l^2\left(\cosh^2\rho d\tau^2 - d\rho^2 - \sinh^2\rho d\Omega_{d-1}^2\right)$$

Here $\tau$ is periodic with range $[0, 2\pi)$. To avoid time-like closed loops one relaxes the range to $[0, \infty)$. Using the variable transformation $r = l\sinh\rho, t = l\tau$ above line element can be written in another form

$$ds^2 = \left(1 + \frac{r^2}{l^2}\right)dt^2 - \left(1 + \frac{r^2}{l^2}\right)^{-1}dr^2 - r^2 d\Omega_{d-1}^2$$

### 2.2.2 Poincare coordinates

This is also known as horospherical coordinates. Using the following variables

$$X_0 = \frac{1}{2}(z + \frac{1}{z}) + \frac{\sum x_i^2 - t^2}{2z} \tag{2.2.7}$$

$$X_{d+1} = \frac{t}{z} \tag{2.2.8}$$

$$X_i = \frac{x_i}{z} \tag{2.2.9}$$

$$X_d = \frac{1}{2}(z - \frac{1}{z}) - \frac{\sum x_i^2 - t^2}{2z} \tag{2.2.10}$$

we get

$$ds^2 = \frac{l^2}{z^2}\left(dt^2 - d\overrightarrow{x}_{d-1}^2 - dz^2\right)$$

This is one of the most commonly used coordinate system. In this coordinate system boundary is at $z \to 0$.

## 2.3 Conformal Field Theory (CFT)

In this section we will give a short introduction of the conformal symmetry group. Under coordinate transformation $x \to x'$, if the metric transforms like

$$g'_{\alpha\beta}(x')\frac{\partial x'^{\alpha}}{\partial x^{\mu}}\frac{\partial x'^{\beta}}{\partial x^{\nu}} = \Lambda(x)g_{\mu\nu}(x)$$

then it is called a conformal transformation. Such mappings leave the angle between the tangents invariant at any point in manifold while changing their length. Generators of conformal group consists of translations, rotations, dilations and special conformal transformations

## 2.4 AdS/CFT correspondence

To keep the 2discussion simple and put across the main points we will only consider scalar field. We will follow the review [5] closely. Klein-Gordon equation in Poincare coordinates of AdS space is given by

$$\frac{z^2}{L^2}\left(\partial_z^2 - (d-1)\frac{\partial_z}{z} + \partial_x^2\right)\phi = m^2\phi \qquad (2.4.1)$$

We can separate the dependence along $z$ coordinate and transverse coordinates. $\phi(z,x) = f(z)e^{ikx}$. Substituting it in the equation (2.4.1) we get

$$z^2 f'' + (d-1)zgf' - \left(m^2L^2 + k^2z^2\right)f = 0$$

Solution of the above equation which is regular in the interior of the bulk is given by

$$f_k(z) = a_k(kz)^{d/2}K_\nu(kz) \qquad (2.4.2)$$

where $\nu = \sqrt{\frac{d^2}{4} + m^2L^2}$ and $K_\nu(kz)$ is modified Bessel function. $a_k$ is constant of integration which turns into creation and annihilation operator on quantization. Near the boundary $z \to 0$ solution behaves like

$$\begin{aligned} f(z) &= a_k (kz)^{d/2} \left[ \frac{\Gamma(\nu)}{2} \left( \frac{2}{kz} \right)^\nu + \frac{\Gamma(-\nu)}{2} \left( \frac{2}{kz} \right)^\nu \right] \\ &= \phi_0(k) z^{\Delta_-} + \phi_1(k) z^{\Delta_+} \end{aligned}$$

where

$$\phi_0(k) = a_k 2^{\nu-1} \Gamma(\nu) k^{\Delta_-} \qquad \phi_1(k) = a_k 2^{-(\nu+1)} \Gamma(-\nu) k^{\Delta_+}$$

The scaling exponents $\Delta_\pm$ are defined as

$$\Delta_\pm = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2L^2}$$

Finally in position space the solution can be written as

$$\phi(z, x) = \phi_0(x) z^{\Delta_-} + \phi_1(x) z^{\Delta_+} \tag{2.4.3}$$

where $\phi_0(x), \phi_1(x)$ are Fourier transform of $\phi_0(k), \phi_1(k)$. Now note that $\Delta_+ > 0, \Delta_- < 0$. Solution $z^{\Delta_+}$ is a normalizable mode and decays close to the boundary. Other solution $z^{\Delta_-}$ is non-normalizable mode and blows up near boundary. This allows us to define field on the boundary using the following ansatz

$$O(k) \equiv \phi_0(k) = \lim_{z\to 0} z^{-\Delta_-} f(z)$$

This can also be expressed in the position space

$$O(x) \equiv \phi_0(x) = \lim_{z\to 0} z^{-\Delta_-} \phi(z, x) \tag{2.4.4}$$

This is one of the ways of stating holography. We have found our first entry of the AdS/CFT dictionary. $O(x)$ defined in equation (2.4.4) is dual to bulk field (2.4.3). An important step in the above procedure is to use the non-normalizable mode as the boundary operator. This mode is otherwise thrown away if one looks only from the bulk.

Now we will give another prescription for the holography which was historically developed first. The difference between the two prescription is nicely elaborated in the paper by Harlow and Stanford [6]. The main idea is that boundary value of the bulk field acts as a source of operator on the boundary. Idea is to take the partition function of the bulk and substitute in it boundary value of the bulk field. Then that becomes generating function of boundary operator.

$$e^{\int d^4x \phi_0(\vec{x}) O(\vec{x})}{}_{CFT} = Z_{bulk}\left[\phi(\vec{x},z)|_{z=0} = \phi_0(\vec{x})\right]$$

One gets correlation function by successively differentiating with respect to the bulk field. We will show this by deriving CFT two point function from the bulk partition function. We will follow as done in [1, 2, 3, 4]. We have calculated in equation (2.4.2)

$$\phi(z,k) = a_k (kz)^{d/2} K_\nu(kz)$$

This blows up at the boundary $z \to 0$. Let us impose the boundary condition $\phi(\vec{x},z)|_{z=0} = \phi_0(\vec{x}) = e^{i\vec{p}.\vec{x}}$.

$$\phi(\vec{x},z) = \int d^d p a_k \frac{(kz)^{d/2} K_\nu(kz)}{(k\epsilon)^{d/2} K_\nu(k\epsilon)} e^{ipx}$$

Substitute this in the bulk action of massive scalar field in

$$S = \int d^{d+1}x \sqrt{g} \left[\frac{1}{2}(\partial\phi)^2 + \frac{1}{2} m^2 \phi^2\right]$$

After doing the integral, we expand in $\epsilon$ and retain only the leading non-analytic term that gives

$$O(p)O(q) \;=\; \epsilon^{2(\Delta-d)}(2\Delta-d)\frac{\Gamma(d-1-\Delta)}{\Gamma(\Delta-1)}\delta^d(\vec{p}+\vec{q})\left(\frac{\vec{p}}{2}\right)^{2\Delta-d}$$

where $\Delta = \frac{d}{2}+\sqrt{\frac{d^2}{4}+m^2l^2}$. Fourier transforming to position space we get

$$O(\vec{x})O(\vec{y}) \;\propto\; \frac{1}{|\vec{x}-\vec{y}|^{2\Delta}}$$

This is the expected two-point function of a CFT.

The main point to note is that in both the prescriptions we have to give a boundary condition. This by default removes half the degree of freedom. This does not create problem in AdS because anyway these non-normalizable modes are not part of inner product. However we will see that this is not possible in de-Sitter space. That will invalidate both these prescription. There we will need another prescription which is independent of boundary condition.

# Chapter 3

# Axiomatic Quantum Field Theory

Quantum field theory is most often introduced in textbooks as operator valued fields which extremizes the action and has appropriate commutation relations. Many difficulties arise from such a description. For example, exact QFT results are not analytic in a finite neighborhood of zero coupling.

Axiomatic QFT is an attempt to construct QFT based on some rigorous axioms. Arthur Wightman in 1950s was the first physicist to propose axioms for QFT. These are known as Wightman Axioms. These axioms define fields as operator valued distributions on Hilbert space. In this section we will state the axioms. Following that in section 3.1 we will give some of the relevant theorems following from these axioms. Wherever possible we will outline the proof of the theorems but will not prove any of them in details. Proofs are given in the book [7].

Definition: The space $S\left(\mathbb{R}^4\right)$ consists of infinitely differentiable real functions of four variables, which go to zero at infinite infinitely faster than any power of Euclidean distance. A tempered distribution is then a continuous linear map, $f : S \to \mathbb{C}$.

There are five axioms:

I. Assumptions of relativistic quantum theory: The states of the theory are described by unit rays in a separable Hilbert space $H$. Relativistic transformation law of the states is given by a continuous unitary representation of the inhomogenous $SL(2,C)$:

$$\{a,T\} \rightarrow U(a,T)$$

where $a$ is translation and $T$ is Lorentz transformation. $U(a,1)$ being unitary can be written as $U(a,1) = e^{iPa}$ where $P^\mu$ is an unbounded hermitian operator interpreted as the energy momentum operator of the theory. It satisfies spectral condition $P^0 \geq 0, P^\mu P_\mu = m^2 \geq 0$. There is an invariant state, $|0\rangle$ known as vacuum,

$$U(a,T)|0\rangle = |0\rangle$$

unique up to a constant phase factor. This is uniqueness of the vacuum.

II. Domain and continuity of the field: For each test function $f \in S$ there exists a set of operators $\phi_1(f), ..., \phi_n(f)$ and their adjoints $\phi_1^*(f), ..., \phi_n^*(f)$ defined by smeared field $\phi(f) = \int f(x)\phi(x)d^4x$. These operators are tempered distribution defined on a domain D of vectors in $H$. Furthermore D is a linear set containing $|0\rangle$. $U(a,T)$ and $\phi(f), \phi^*(f)$ carry vectors in $D$ into vectors in $D$

$$U(a,T)D \subset D \qquad \phi(f)D \subset D \qquad \phi^*(f)D \subset D$$

III. Transformation law of the field:

$$U(a,T)\phi(f)U(a,T)^{-1} = \phi(f_{a,T})$$

where the test function transforms as the inverse element of the group

$$f_{a,T}(x) = f\left(T^{-1}(x-a)\right)$$

IV. Local commutativity: If the support of $f$ and the support of $g$ are spacelike separated, then one or the other

$$[\phi_j(f), \phi_k(g)]_{\pm} \quad = \phi_j(f)\phi_k(g) \pm \phi_k(g)\phi_j(f) \quad = 0$$

for al $j, k$. In terms of unsmeared field this is

$$[\phi_j(x), \phi_k(y)]_{\pm} \quad = \quad 0$$

if $x$ and $y$ are spacelike separated.

There is another necessary assumption to make connection with the scattering problem.

V. Asymptotic completeness: $H = H^{in} = H^{out}$. This states that Hilbert space of incoming particles is same as Hilbert space of outgoing particles. This is necessary to describe any kind of collision. But this depends on some prescription for computing scattering states of elementary systems.

The compatibility of above axioms is seen from the existence of free field theory which satisfies all the axioms. Although this is a trivial example, it serves as good starting point. In free field theory number of particles is conserved. So the Hilbert space separates into different sectors with fixed number of particles

$$H \quad = \quad \oplus_{n=0}^{\infty} H^n$$

where $H^n$ is the subspace with exactly $n$ particles. A general state can be given by linear superposition of states from each of the subspace

$$\Phi \quad = \quad \sum_j \alpha^j \Phi^j$$
$$\Psi \quad = \quad \sum_k \beta^k \Psi^k$$

where $\Phi^j, \Psi^j \in H^j$. Their inner product is given by

$$\langle \Phi, \Psi \rangle = \sum_j \alpha^j \beta^j \langle \Phi^j, \Psi^j \rangle$$

. States which satisfy

$$\langle \Phi, \Phi \rangle = \sum_j \alpha^j \alpha^j \langle \Phi^j, \Phi^j \rangle < \infty$$

are the only acceptable states.

## 3.1 Vacuum expectation value

In this section we will discuss vacuum expectation values also known as Wightman function.

$$G^n(x_1, ..., x_n) = \langle 0|\phi_1(x_1)...\phi_n(x_n)|0\rangle$$

As before we will consider smeared vacuum expectation values.

$$\langle 0|\phi_1(f_1)...\phi_n(f_n)|0\rangle = \int dx_1...dx_n \langle 0|\phi_1(x_1)...\phi_n(x_n)|0\rangle f(x_{1.}...x_n)$$

where $f(x_1...x_n) = f_1(x_1)...f_n(x_n)$. Strictly speaking functions $f_j$ is a limit of sequence of functions $f_j^k \in S$ such that $f_j^k \to f_j$ as $k \to \infty$. Wightman function satisfies following properties which can be proved starting from the axioms given above. Proofs are given on page 107-110 in the book [7]. We will not give them here.

1. Relativistic transformation Law

$$G^n(x_1, ..., x_n) = G^n(Tx_1 + a, ..., Tx_n + a) \tag{3.1.1}$$

2. Spectral conditions

$$\begin{aligned} G^n\left(p_1, ...p_n\right) &= \delta\left(\sum p_i\right) G^n\left(p_1, p_1+p_2, ..., p_1+...p_n\right) \\ &= \int e^{i\sum p_i x_i} G^n\left(x_1, ..., x_n\right) \end{aligned}$$

Wightman function can be expressed in terms of relative coordinates $\xi_i = x_i - x_{i+1}$

$$G^n\left(\xi_i, ..., \xi_n\right)$$

Then the its Fourier transform is given by

$$G^n(q_1, ...q_n) = \int e^{\sum q_i \xi_i} G^n\left(\xi_i, ..., \xi_n\right)$$

It has the property that

$$G^n(q_1, ...q_n) = 0$$

if any of the $q_i$ lies outside the positive light cone.

3. Hermiticity condition

$$\langle 0|\phi_1(x_1)...\phi_n(x_n)|0\rangle = \overline{\langle 0|\phi_1^*(x_1)...\phi_n^*(x_n)|0\rangle} \tag{3.1.2}$$

4. Local commutativity relations

$$G^n(x_{i_1}, ..., x_{i_n}) = (-1)^m G^n(x_1, ..., x_n) \tag{3.1.3}$$

if the differences $x_i - x_j$ are space-like for all $j$ and $k$ and $m$ is the number of exchanges of anti-commuting fields necessary to permute $i_1...i_n \to 1...n$.

5. Positive definite condition:

$$\sum_{j,k=0} \int ... \int dx_1 ... dx_n dy_1 ... dy_n \overline{f_j \left(x_1, ... x_n\right)} G^{j+k} \left(x_1, .., x_j, y_1, ... y_k\right) f_k \left(y_1, ... y_k\right) \geq 0 \qquad (3.1.4)$$

6. Cluster decomposition Property:

All the properties from 1-4 guarantee that a set of tempered distributions gives vacuum expectation values except that the vacuum is unique. This condition of cluster decomposition is necessary for the uniqueness of the vacuum. If $a$ is a space-like vector , then

$$G\left(x_1, ..., x_j, x_{j+1} + \lambda a, ..., x_n + \lambda a\right) \quad \rightarrow \quad G\left(x_1, ... x_j\right) G\left(x_{j+1}, ..., x_n\right) \qquad (3.1.5)$$

as $\lambda \rightarrow \infty$, in the sense of convergence in $S$.

There is simple argument given by D. Rulle. For large $\lambda$ it is possible to reorder the positions inside the Wightman function to $G\left(x_{j+1} + \lambda a, ..., x_n + \lambda a, x_1, ..., x_j\right)$ with a possible change in sign. This action reverses the sign of momentum conjugate to $x_j - x_{j+1}$. The change in sign allows one to use spectral condition to prove the theorem.

Physically cluster decomposition is saying that when two systems are separated by large space-like separations the interaction between them vanishes. In other words, sufficiently separated regions behave independently. There are two important points to note. The above property holds only if the vacuum is pure state. The cluster decomposition property breaks down if the vacuum is degenerate and we have a mixed state. This will be important when we discuss CFTs dual to de-Sitter space. There we find that vacuum expectation value violates cluster decomposition signaling possible degeneracy of vacuum. Secondly, if the theory has a mass gap $M > 0$, then there is a value $a_0$ beyond which the connected correlation function is absolutely bounded by $Ce^{-M\lambda}$ where $C$ is some coefficient. Also if there are zero-mass particles in the theory the limit goes as slowly as $1/\lambda^2$. This is just the Coulomb force!

## 3.2 Wightman Reconstruction theorem

One of the holographic prescriptions, known as HKLL prescription, as we will explain in the next section constructs vacuum expectation value of the bulk using boundary correlation functions. Wightman reconstruction theorem ensures that the vacuum expectation values completely characterizes the bulk quantum field theory.

Let $G^n, n = 1, 2, ...$ be a sequence of tempered distributions, where $G^n$ depends on $n$ four-vector variables $x_1, ...x_n$. Suppose $G^n$ satisfies all the 6 properties mentioned in the previous section. Then there exists a separable Hlibert space $H$, a continuous unitary representation $U(a, T)$ of Poincare group in $H$, a unique vacuum state $|0\rangle$, invariant under $U(a, T)$, and a hermitian scalar field with domain $D_1$ and representation of Poincare group $U(a, T)$ such that

$$\langle 0|\phi(x_1)...\phi(x_n)|0\rangle = G^n\left(x_1, ..., x_n\right)$$

Additionally, any other field theory with these vacuum expectation values is unitary equivalent to this one. In other words, if $H_1$ is a Hilbert space, and $(a, T) \to U_1(a, T)$ is a continuous unitary representation of Poincare group in it, and $|0\rangle_1$ is unique vacuum vector in $H_1$ invariant under $U_1(a, T)$ and $\phi_1(x)$ is a scalar field with domain $D_{11}$with the property

$${}_1\langle 0|\phi_1(x_1)...\phi_1(x_n)|0\rangle_1 = G^n\left(x_1, ..., x_n\right)$$

then there exists a unitary transformation $V$ of $H$ onto $H_1$ such that

$$|0\rangle_1 = V|0\rangle \qquad U_1(a,T) = VU(a,T)V^{-1}$$
$$\phi_1(h) = V\phi(h)V^{-1} \qquad D_{11} = VD_1$$

The proof of the above theorem is lengthy but elegant. We will reproduce the proof given in the page 118 of [7].

One begins by explicit construction of Hilbert space $H$. Consider a vector space $H$ of all sequences $f = (f_0, f_1, ...)$ where $f_0$ is a complex constant and $f_k \in S$, $k = 1, 2, ...$ are non-zero except for finite number of them. Then we define various operations on this vector. Addition and multiplication is defined as

$$\begin{aligned} f + g = (f_0, f_1, ...) + (g_0, g_1, ...) &= (f_0 + g_0, f_1 + g_1, ...) \\ \alpha f = \alpha\,(f_0, f_1, ...) &= (\alpha f_0, \alpha f_1, ...) \end{aligned}$$

Scalar product between two vectors is defined by

$$\langle f, g\rangle = \sum_{j,k=0} \int ... \int dx_1...dx_n dy_1...dy_n \overline{f_j\,(x_1, ...x_n)} G^{j+k}\,(x_1, .., x_j, y_1, ...y_k)\, f_k\,(y_1, ...y_k) \tag{3.2.1}$$

where $G^0 \equiv 1$. This scalar product has the property that

$$\langle f, g\rangle \;=\; \overline{\langle g, f\rangle}$$

due to hermiticity condition (property 3) of $G$. Positive definite condition (property 5) ensures that $||f||^2 = \langle f, f\rangle \geq 0$. Action of $U(a,T)$ on the vector space is given by

$$U(a,T)\,(f_0, f_1, ...) \;=\; (f_0, U(a,T)f_1, U(a,T)f_2, ...) \tag{3.2.2}$$

where $U(a,T)f_k(x_1, ..., x_k) = f_k\,(T^{-1}(x_1 - a), ..., T^{-1}(x_k - a))$.

Now we begin construction of the states. We see that

$$U(a,T)(1,0,0,...) \;=\; (1,0,0,...)$$

This motivates us to define $|0\rangle = (1,0,0,...)$.

Equation 3.1.1 of relativistic transformation property of Wightman function guarantees that scalar product equation (3.2.1) is invariant under action of $U(a,T)$. Using equation (3.2.2) it is easy to prove

$$U(a_1,T_1)U(a_2,T_2) \;=\; U(a_1 + T_1 a_2, T_1 T_2)$$

Now we give a map which takes each test function $h$ to an operator field $\phi(h)$

$$\phi(h)\,(f_0, f_1, ...) \;=\; (0, h \otimes f_0, h \otimes f_1, ...) \tag{3.2.3}$$

where

$$h \otimes f_k\,(x_1, x_2, ...) \;=\; h(x_1) f_k\,(x_2, x_3, ..., x_{k+1})$$

is also a test function. Let us show that $\phi(h)$ satisfies the relativistic transformation

$$\begin{aligned} U(a,T)\phi(h)\,(f_0,f_1,...) &= U(a,T)\,(0,h\otimes f_0,h\otimes f_1,...) \\ &= \left(0,h_{(a,T)}\otimes f_{0(a,T)},h_{(a,T)}\otimes f_{1(a,T)},...\right) \\ &= \phi\left(h_{(a,T)}\right)\left(f_{0(a,T)},f_{1(a,T)},...\right) \\ &= \phi\left(h_{(a,T)}\right)U(a,T)\,(f_0,f_1,...) \\ \implies U(a,T)\phi(h)U^{-1}(a,T) &= \phi\left(h_{(a,T)}\right) \end{aligned}$$

These constructions have given us vacuum $|0\rangle$, field operators $\phi(h)$ and action of Poincare transformations $U(a,T)$ on them. $|0\rangle,\phi(h)$ forms a vector space $H$. To make it a Hilbert space, we have to do two things. First, we have to remove zero norm states from $H$. Second we have to complete the vector space $H$.

Note that all the zero norm states form a vector space. That is zero norm states are orthogonal to each other. For example consider two zero norm states $f,g,||f||=||G||=0$. Then

$$0\leq|\langle f,g\rangle|\leq||f||||g|| \;=\; 0 \tag{3.2.4}$$

by Schwartz inequality. Thus, if $f=(f_0,f_1,...)$ and $g=(g_0,g_1,...)$ are zero norm then $f$ is orthogonal to $g$and to $\alpha f+\beta g$. Now we define equivalence classes of sequences, $f=(f_0,f_1,...)$. Two sequences are equivalent if the differ by a sequence of zero norm. these equivalence classes form a vector space denoted by $H/H_0$. If $f\in F$ and $g\in G$ are two equivalence classes, then $\alpha f+\beta g$ belongs to the equivalence class$\alpha F+\beta G$ . The result does not depend on which representative is chosen because the set of vectors of zero length is a vector space. The set of sequence of zero norm is the zero in $H/H_0$ of equivalence classes. Also, if norm of a equivalence class is zero then it has no elements. That is $||F||=0 \implies F=0$. Finally we define scalar product between $F, G$ in $H/H_0$ by $\langle F, G\rangle = \langle f, g\rangle$. Once again it does not depend on the choice of representative sequence because a general sequence of $F$ can be written as $f + \alpha h$ where $f \in F$ and $||h|| = 0$. Then $\langle f + \alpha h, g\rangle = \langle f, g\rangle + \alpha\langle h, g\rangle = \langle f, g\rangle$ because of equation (3.2.4).

We also have to that $U(a, T)$ and $\phi(h)$ defined in equations (3.2.2) and (3.2.3) are mappings of equivalence classes. If $f_1, f_2 \in F$ then

$$\begin{aligned} ||f_1 - f_2|| &= 0 \\ \implies ||U(a,T)f_1 - U(a,T)f_2|| &= ||U(a,T)\left(f_1 - f_2\right)|| = ||\left(f_1 - f_2\right)|| = 0 \\ \implies U(a,T)f_1 \in F & \qquad U(a,T)f_2 \in F \end{aligned}$$

Next if $||f|| = 0$

$$0 \leq ||\phi(h)f||^2 = \langle \phi(h)f, \phi(h)f\rangle = \langle f, \phi(\overline{h})\phi(h)f\rangle \leq ||f||||\phi(\overline{h})\phi(h)f|| = 0$$

This implies $||\phi(h)f|| = 0$. Similarly $|0\rangle \in H/H_0$ will denote the equivalence class of $(1, 0, 0, ...)$.

Second requirement to make $H/H_0$ a Hilbert space is completeness. The proof is parallel to completing rational numbers to real numbers. Hence we will not show it here. It is given in the book [7].

Some other general theorems can be derived from the above axioms:

1. It is possible to show that there is general symmetry under parity, charge conjugation (matter-antimatter) and time reversal symmetry.

2. There is relation between spins and statistics. Particles with integer spin have Bose-Einstein statistics and particles with half-integer spin have Fermi-Dirac statistics.

3. Above axioms also prove that superluminal communication is not possible.

## 3.3 Haag's theorem

Any discussion of axiomatic quantum field theory is incomplete without Haag's theorem. In all interacting QFTs, canonical variables are assumed to be unitarily related to free field theory. That is in any interacting QFT like quantum electrodynaics or QCD

$$V(t)\phi_{free}(x,t)V(t)^{-1} = \phi_{int}(x,t)$$

Time dependence of the trasnformation operator gives rise to interaction. It was found that not only there are many inequivalent representations but also $\phi_{int}(x,t)$ is free field in disguise. In other words there can be no interaction ($\phi_{int}$) in an interaction picture ($V(t)$). Precise statement of the Haag's theorem is following:

Suppose that $\phi_1(x) \in H_1$ is a free hermitian scalar field of mass $m > 0$ and $\phi_2(x) \in H_2$ is a local field covariant under the inhomogenous $SL(2,C)$ transformations $U_1(a,T), U_2(a,T)$ and vacuum $|0\rangle_1, |0\rangle_2$ respectively. Suppose further that the fields $\phi_1, \dot{\phi}_1$ and $\phi_2, \dot{\phi}_2$ satisfy the hypotheses

$$\begin{aligned}
\phi_2 &= V(t)\phi_1 V(t)^{-1} \\
\dot{\phi}_2 &= V(t)\dot{\phi}_1 V(t)^{-1} \\
U_2(a,T) &= V(t)U_1(a,T)V(t)^{-1} \\
|0\rangle_2 &= |0\rangle_1
\end{aligned}$$

Then $\phi_2$ is a free field of mass $m$.

Above assumptions are very general. In fact, above result will hold in any theory where one can define correlation function that is vacuum expectation value. As has been pointed out by Haag's in his original paper [8] that the main reason is vacuum polarization of interacting theory. Any interacting theory polarizes the vacuum. In other words, interactions change the vacuum. Hence the vacuum is not unique and lies inside the renormalized Hilbert space. In that case, it is not possible to define vacuum expectation values $\langle 0|\phi_1(x_1)...\phi_n(x_n)|0\rangle$ because vacuum is not a fixed state any more. In fact, it is not clear what should be the observable in such case. In free field theory vacuum is an additional assumption lying outside the Hilbert space. One can always find an isomorphic map between the two Hilbert space but that will not be unitary and physical results will be ambiguous. This theorem goes to the heart of the problem of quantum gravity on how to include vacuum or the background metric in the Hilbert space.

## 3.4 HKLL prescription

We have seen two prescriptions to get boundary operator from the bulk field in AdS/CFT. One takes the boundary limit of the bulk field and the other takes the bulk partition function with the boundary value of the field as boundary partition function with source.

But we are more interested in constructing bulk from the boundary. How do the bulk degrees of freedom emerge from the boundary theory? That is the main reason of interest in holography. Specially, one is interested in understanding microscopic degrees of freedom of horizons. In this section we will discuss how to construct bulk fields from boundary operators which was developed by Alex Hamilton, Daniel Kabat, Gilad Lifschytz and David Lowe [9, 10]. This is known as HKLL prescription.

Basic idea is to express a bulk field as CFT operator smeared over the boundary.

$$\phi_{bulk}(x) = \int_{bound} K(x,y)O(y)$$

Kernel $K(x,y)$ is called smearing function. Here we will give the prescription only in large N limit and large t'Hooft coupling limit. So there is no interaction. Generalization to interacting scenario will be mentioned at the end. We know that scalar field of mass $m$, is dual to an operator with weight $\Delta = \frac{d}{2} + \sqrt{\frac{d^2}{4} + m^2 l^2}$. This hints that $O_\Delta$ must reproduce bulk field. Prescription for 3 dimensional AdS is the following

$$\phi(t,x,z) = \frac{\Delta - 1}{\pi} \int_{t'^2 + y'^2 < z^2} dt' dy' \left( \frac{z^2 - t'^2 - y'^2}{z} \right)^{\Delta - 2} O(t + t', x + iy') \,. \quad (3.4.1)$$

Note that integral is only over a disk on the boundary formed by the space-like lightcone of the bulk point. One may also note the similarity between the smearing function and bulk-boundary propagator of Witten. However these are completely different prescription. Witten's prescription give the non-normalizable mode of the bulk field which acts as a source that deforms the CFT. While in this case boundary operators of non-deformed CFT directly reproduce normalizable modes. Another important thing to note is that smearing function is not unique. One can smear it over other spacelike lightcones. One will also see that boundary coordinate is complexified. This plays an important role in restricting the integral to a finite support.

One way to verify the above formula is to check that it gives correct bulk-boundary propagator when applied on the boundary two-point function. Boundary two-point function is given by

$$\langle O(y,t) O(0,0) \rangle = \frac{1}{(y^2 - t^2)^\Delta}$$

Now applying equation (3.4.1) on the first operator we get

$$\langle \phi(z,0,t) \mathcal{O}(0,0) \rangle \quad (3.4.2)$$
$$= \frac{\Delta - 1}{\pi} \int_{y'^2 + t'^2 < z^2} dy' dt' \left( \frac{z^2 - y'^2 - t'^2}{z} \right)^{\Delta - 2} \langle \mathcal{O}(iy', t + t') \mathcal{O}(0,0) \rangle$$
$$= \frac{\Delta - 1}{\pi} (-1)^\Delta \int_0^z dr \int_0^{2\pi} d\theta \left( \frac{z^2 - r^2}{z} \right)^{\Delta - 2} \frac{r}{(r^2 + t^2 + 2rt\cos\theta)^\Delta} \quad (3.4.3)$$

we have set the bulk $x = 0$ and boundary $t' = r\cos\theta, y' = r\sin\theta$. Now one can use the result

$$\int_0^{2\pi} d\theta \frac{1}{(r^2+t^2+2rt\cos\theta)^\Delta} = 2\pi t^{-2\Delta}\,{}_2F_1\left(\Delta,\Delta;1;\frac{r^2}{t^2}\right) \tag{3.4.4}$$

to do the $\theta$ integral. After defining $q = r^2/Z^2$ and $y = Z^2/T^2$ we are left with

$$\begin{aligned} \langle \phi(z,0,t)\mathcal{O}(0,0)\rangle &= \frac{\Delta-1}{2\pi R}(-1)^\Delta t^{-2\Delta} z^\Delta \int_0^1 dq(1-q)^{\Delta-2}\,{}_2F_1(\Delta,\Delta;1;qy) \\ &= \frac{z^\Delta}{2\pi R}\frac{1}{(z^2-t^2)^\Delta}\,. \end{aligned}$$

For general $x, t$ one may use Lorentz invariance and analytic continuation. With a Wightman $i\epsilon$ prescription

$$\langle \phi(z,x,t)\mathcal{O}(0,0)\rangle = \frac{z^\Delta}{2\pi R}\frac{1}{(z^2+x^2-(t-i\epsilon)^2)^\Delta} \tag{3.4.5}$$

This is the correct bulk-boundary propagator. So the HKLL prescription has passed the first test. Next one would like to reproduce complete bulk propagator. This is interesting because it will give a way to describe bulk observable purely from boundary point of view. Although at present we are only discussing large N, but still this is a step forward in describing deep bulk. This is done on [11].

Now let us discuss some of the physical consequences of the prescription. First thing to see is whether the conventional wisdom of scale-radius duality holds? It is easy to see that taking bulk field close to the boundary shrinks the smearing integral to zero. As a result scale-radius duality is manifest. A bulk point at depth of $z$ gets smeared over a range of time $2z$ . This is just the elapsed time between the point on the boundary which is lightlike to the future of the bulk point at the same value of $\phi$, and the point on the boundary which is lightlike to the past at the same $\phi$.

Although finite calculations are very difficult, some of the finite N effects can be understood qualitatively. An important principal in bulk theory is that fields at spacelike separation should commute. At large N, this is also reproduced from the smearing integral because the commutator of boundary operators is a complex number and not an operator. But this does not hold in finite N. Commutators of boundary operators are now operators. As a result even space-like separated operators do not commute in the bulk. Causality is a very important principal of bulk. Only way to retain causality in holographic theory is by restricting the number of degrees of freedom in the bulk so that smearing integral is over disjoint regions of the boundary. So they will trivially commute. This gives a physical principal to count the number of degrees of freedom of the bulk. Consider two local bulk operators at the same values of $r$ and $t$ but different $\phi$ in equation (3.4.1). Up to $\frac{1}{N}$ corrections to the actual size of the region, these will correspond to boundary operators smeared in $t$ and imaginary $y$ directions according to (3.4.1). It is thus reasonable to assume that they will commute even at finite N if the $\phi$ separation is big enough. Working at large $r$, bulk operators are expected to commute if the separation $\delta\phi \geq \frac{2l}{r}$. Now consider the set of such operators at fixed $r$ and $t$. Operators at smaller values of $r$ and the same $t$ will be smeared over a larger disk on the boundary, so will not trivially commute with this set. Then the number of trivially commuting operators that can be localized to a radius $\leq r$, per radian along the boundary, per independent CFT degree of freedom is $r/2R$. Heuristically the number of CFT degrees of freedom is given by the central charge, so the maximum number of commuting operators per radian is of order $cr/2R$ . We get a very qualitative idea of why the number of freedom is reduced in the bulk. This indicates that HKLL construction is revealing something deep. The degrees of freedom on a Cauchy hyper-surface do not commute which breaks the canonical quantization of gravity.

# Chapter 4

# Holographic operator mapping in dS/CFT and cluster decomposition

The Bekenstein-Hawking entropy [12, 13]

$$S = \frac{A}{4G},$$

states that the entropy of any black hole is proportional to its surface area. This law is widely applicable for various kinds of black hole. This universality suggests that black holes can be described by microstates. In fact such a description is possible for any horizon. For example cosmological horizon can also be assigned thermodynamic description. But de-Sitter horizon and black hole horizon are not exactly on the same footing. De-Sitter horizon is observer dependent where as black hole horizon is not. So it is puzzling even to think about the microstates of dS horizon.

The only black hole example [14] where microstates are counted is with the help of supersymmetry. On the other hand supersymmetry is inconsistent with de-Sitter isometries. This can be understood using simple arguments. As we will discuss in detail later, de-Sitter does not admit positive conserved charges. Now if there is nonzero Hermitian supercharge $Q$, then it must be positive. Clearly there is no such object. This shows that supersymmetry is incompatible with de-Sitter space. In order to give microscopic description of de-Sitter horizon we must know how to get away from supersymmetry. Another related problem is absence of concrete examples from string theory. There are no go theorems which show that string theory solutions cannot contain dS vacua.

Another issue with dS holography is that dS has two boundaries. Dual theory on the full boundary will describe the whole of dS. On the other hand an observer can only observe half of the spacetime. Thus if dual theory on full boundary describes full spacetime that will go against the spirit of complementarity. There are many other difference and problems with de-Sitter which we will point out as we go along. But before we jump to calculations we will introduce de-Sitter space and various coordinate system in details in the next section. Then in section 4.2 we will discuss some of the earlier attempts to understand dS holography.

## 4.1 de-Sitter space

de-Sitter space is a maximally symmetric manifold of constant positive curvature. It can be realized on a hyperboloid

$$-X_0^2 + X_1^2 + \cdots + X_d^2 = \ell^2 \tag{4.1.1}$$

Flat metric is given by

$$ds^2 = -dX_0^2 + dX_1^2 + \cdots + dX_d^2, \tag{4.1.2}$$

Different coordinate systems give different insight into the manifold. As we saw in the case of AdS, coordinates on sphere are frequently used. For completeness we state here once again.

$$\omega^1 = \cos\theta_1, \tag{4.1.3}$$

$$\omega^2 = \sin\theta_1 \cos\theta_2, \tag{4.1.4}$$

$$\vdots \tag{4.1.5}$$

$$\omega^{d-1} = \sin\theta_1 \cdots \sin\theta_{d-2} \cos\theta_{d-1}, \tag{4.1.6}$$

$$\omega^d = \sin\theta_1 \cdots \sin\theta_{d-2} \sin\theta_{d-1}, \tag{4.1.7}$$

where $0 \leq \theta_i < \pi$ for $1 \leq i < d-1$, but $0 \leq \theta_{d-1} < 2\pi$. Then it is clear that $\sum_{i=1}^{d}(\omega^i)^2 = 1$, and the metric on $S^{d-1}$ is

$$d\Omega_{d-1}^2 = \sum_{i=1}^{d}(d\omega^i)^2 = d\theta_1^2 + \sin^2\theta_1 d\theta_2^2 + \cdots + \sin^2\theta_1 \cdots \sin^2\theta_{d-2} d\theta_{d-1}^2. \tag{4.1.8}$$

### 4.1.1 Global coordinates

Using the following variables

$$X^0 = \sinh\tau, \tag{4.1.9}$$

$$X^i = \omega^i \cosh\tau, \qquad i = 1, \ldots, d, \tag{4.1.10}$$

where $-\infty < \tau < \infty$ and the $\omega^i$ are as in (4.1.7). These satisfy equation (4.1.1). Substituting in (4.1.2) one obtains the line element

$$ds^2 = -d\tau^2 + (\cosh^2\tau)d\Omega_{d-1}^2. \tag{4.1.11}$$

At $\tau = -\infty, \tau = \infty$, dS is $d-1$ infinitely large sphere. It shrinks to miminum size at $\tau = 0$.

### 4.1.2 Conformal coordinates

These coordinates are related to the global coordinates by

$$\cosh\tau = \frac{1}{\cos T}, \tag{4.1.12}$$

so that we have $-\pi/2 < T < \pi/2$. The metric in these coordinates takes the form

$$ds^2 = \frac{1}{\cos^2 T}(-dT^2 + d\Omega_{d-1}^2). \tag{4.1.13}$$

Conformal coordinates are useful because they preserve the causal structure of the spacetime. One can remove any overall factor from the metric while preserving the light cone structures.

### 4.1.3 Poincare patch or Horospherical coordinates

$(\eta, y_i, i = 1, ..., d-1)$. It is possible to foliate the spacetime in the following way

$$\begin{aligned} x_0 &= \frac{1}{2}(\eta - \frac{1}{\eta}) - \frac{\sum y_i^2}{2\eta} \\ x_i &= \frac{y_i}{\eta} \\ x_d &= -\frac{1}{2}(\eta + \frac{1}{\eta}) + \frac{\sum y_i^2}{2\eta}, \end{aligned} \tag{4.1.14}$$

$\eta$ is known as conformal time. In these coordinates, line element looks like

$$ds^2 = \frac{1}{\eta^2}\left(-d\eta^2 + d\vec{y}^2\right)$$

This is the most commonly used coordinate system to describe holography. This covers only half of the global coordinate. Past boundary is at $\eta \to 0$. As one can see, metric blows up near the boundary.

### 4.1.4 Planar coordinates

$(t, y_i, i = 1, ..., d-1)$. This is commonly used coordinate system in cosmology literature. Using the variable transformation

$$\eta = e^t$$

the metric takes the form

$$ds^2 = -dt^2 + e^{-2t} dy_i dy^i. \tag{4.1.15}$$

This depicts exponentially expanding universe. As time goes on spacelike surfaces expand exponentially. This also covers only half the de-Sitter space.

### 4.1.5 Static coordinates

$(t, r, \theta_a)$, $a = 1, \ldots, d-2$. This coordinate system is constructed to have an explicit timelike Killing symmetry. If we write

$$X^0 = \sqrt{1-r^2} \sinh t, \tag{4.1.16}$$

$$X^a = r\omega^a, \qquad a = 1, \ldots, d-1, \tag{4.1.17}$$

$$X^d = \sqrt{1-r^2} \cosh t, \tag{4.1.18}$$

then the metric takes the form

$$ds^2 = -(1-r^2)dt^2 + \frac{dr^2}{1-r^2} + r^2 d\Omega^2_{d-2}. \tag{4.1.19}$$

In this coordinate system $\partial/\partial t$ is a Killing vector and generates the symmetry $t \to t + \text{constant}$. The horizons are at $r^2 = 1$, and the southern causal diamond has $0 \le r \le 1$, with the south pole at $r = 0$.

## 4.2 Analytic continuation to de sitter

In the last section we introduced all the coordinate systems and various insights that one gets from them. Now we are ready to explore holography in de-Sitter. In this section we will discuss works done by other people. As with AdS, many of the general features of holography are visible even in quantum field theory. Witten [15] and Strominger [16] were the first ones in early 2000s to study dS holography. They found many features of AdS/CFT to carry over to dS. But they also discovered many thorny issues which are new to de-Sitter. We will discuss them in this section. That will set the background to understand the relevance of our work. From now one all our discussions will be for 3 dimensional de-Sitter. This simplifies the boundary CFT considerably. Many of the results can be generalized to higher dimensions.

The first thing to check in a holography is the symmetry group. $d+1$ dimensional dS has $SO(1, d+1)$ symmetry group. On the other hand $d$-dimensional Minkowski CFT has $SO(2, d)$ symmetry group as we saw in AdS/CFT. Using Wick rotation of the time-like coordinate to spacelike coordinate it is possible to have d-dimensional Euclidean CFT which as $SO(1, d+1)$ symmetry. This change in symmetry group makes a lot of difference.

Next let us look at massive scalar field. We begin by noting the mode expansion for a bulk scalar field of mass $m$ [17]

$$\phi(\eta, y) = c_1 \int \frac{d^2k}{(2\pi)^2} \left( a_k \eta H^{(2)}_{i\mu}\left(|k|\eta\right) e^{ik.y} + a^\dagger_k \eta H^{*(2)}_{i\mu}\left(|k|\eta\right) e^{-ik.y} \right) , \qquad (4.2.1)$$

where $c_1 = \frac{\sqrt{\pi}}{2} e^{\frac{\pi\mu}{2}}$, $\mu = \sqrt{m^2 l^2 - 1}$, and $H^{(2)}_{i\mu}(|k|\eta)$ are Hankel functions of second kind. The operators $a_k$ and $a^\dagger_k$ are annihilation and creation operators, with the $a_k$ annihilating the Bunch-Davies vacuum, and

$$[a_k, a^\dagger_{k'}] = (2\pi)^2 \delta^{(2)}(k - k') \, .$$

Prescription of AdS/CFT suggests that we take the field to the boundary $\eta \to 0$ and absorb the blowing conformal factor. This prescription worked in AdS/CFT because the remaining part died away close to the boundary. Let us follow similar idea in dS. Taking the boundary limit we get

$$\phi(\eta, y) \sim A_k \eta^{h_-} + B_k \eta^{h_+}$$

where

$$\begin{aligned} h_\pm &= 1 \pm \sqrt{1 - m^2 l^2} \\ A_K &= \frac{i\Gamma(i\mu)}{2\sqrt{\pi}} e^{\pi\mu/2} \left(\frac{|k|}{2}\right)^{-i\mu} \\ B_K &= \frac{i\Gamma(-i\mu)}{2\sqrt{\pi}} e^{-\pi\mu/2} \left(\frac{|k|}{2}\right)^{i\mu} \end{aligned}$$

There are several striking observations that we find here. First note that for $ml < 1$, both the exponents are positive. $h_- < 1 < h_+$. As a result both the modes decay close to the boundary. This is in contrast to AdS where one of the modes grew and the other decayed. As a result there is no clear way to choose one over the other. In fact both the modes are present in the bulk in contrast to AdS where non-normalizable modes are thrown away. As a result we are not able to impose any boundary condition on the modes. In such a case, the second prescription of using the boundary value of the field as source of boundary operator through the partition function immediately breaks down. However, as done in [16] one may push the first prescription. Since the two modes decay at different rate, a natural suggestion is to consider slowly decaying part as the boundary operator. That is

$$\begin{aligned} O(y)_{bound} &= \lim_{\eta \to 0} \eta^{-h_-} \left(A_k \eta^{h_-} + B_k \eta^{h_+}\right) \\ &= A_k + \lim_{\eta \to 0} B_k \eta^{2\mu} \end{aligned}$$

This does not look satisfactory. It is not clear what to do with the second part even though on taking the limit it goes away. The same problem persists when we look at two-point function. Bulk two point function is given by

$$\langle\phi(\eta,y)\phi(\eta',y')\rangle \;=\; \frac{\Gamma(h_+)\Gamma(h_-)}{(4\pi)^{3/2}\Gamma(3/2)}{}_2F_1\left(h_+,h_-,\frac{3}{2},\frac{1+P}{2}\right)$$

where $P=\frac{(\eta-\eta')^2-(y-y')^2}{2\eta\eta'}$ is the geodesic distance between the two points. As we take $\eta\to 0,\eta'\to 0$ we get

$$\langle\phi(\eta,y)\phi(\eta',y')\rangle \;=\; \frac{c_+\left(\eta\eta'\right)^{h_+}}{|y-y'|^{2h_+}}+\frac{c_-\left(\eta\eta'\right)^{h_-}}{|y-y'|^{2h_-}}$$

One can see CFT two point function emerging. But there are two different weights corresponding to two different boundary behaviors of the scalar field. This clearly demonstrates that prescription of AdS/CFT cannot be used.

What can be done now? Remember the fundamental idea of holography is to map bulk fields to another manifold with conformal symmetry. The physical picture of taking the boundary limit of bulk field may not always be the correct mapping. This is the place where integral transform comes into play. As we will show in the next section integral transform solves the above problems.

## 4.3 Integral transform

In the section 3.4 a new prescription was given for holography. It gives a way to reconstruct bulk from the boundary. From the construction it is clear that all these prescriptions required dual manifold to be boundary of the bulk. The problem with this map is that it requires taking limit of one of the coordinates of the bulk. Often there are problems associated with taking limit. For example we saw that boundary limit of bulk field in dS has two fall offs. In such cases bulk-boundary map is not well defined. At a more fundamental level holography must be an invertible map from bulk to dual manifold. There is no reason or advantage, except as initial motivation, in thinking of the dual space as boundary.

As far as we know this is the first such approach to holography. In this section we give a detailed introduction to the maths behind it.

### 4.3.1 Radon transformation

Consider a sufficiently smooth function $f(x)$ on real affine $n-$dimensional manifold. Two most common ways of specifying a function is by giving its values at all the points in manifold or by giving all the derivatives at a point in the manifold. There is another way to capture all the information of a function by specifying their integrals over all possible hyper-surfaces of the manifold. Radon transform in layman's language is a map which encodes all the information of the function in terms of its integrals. Most of this section will closely follow the 1st chapter of 5th volume of the book by Gelfand [18].

Integral of a function is defined given the volume element of the n-dimensional real affine oriented space

$$dx = dx_1 ... dx_n$$

We now want to define the integral of $f(x)$ over the hyperplane whose equation is

$$\langle \xi, x \rangle = \xi_1 x_1 + ... + \xi_n x_n = p$$

To define the integral we give the volume element on the hyperplane, namely a differential form of degree $n-1$

$$d\langle \xi, x \rangle . \omega = dx_1 ... dx_n$$

By using $\delta$ function we may write the integral as

$$\hat{f}(\xi, p) \quad = \quad \int f(x)\delta(p - \langle \xi, x \rangle)dx \tag{4.3.1}$$

As we said in the beginning, with any function $f(x)$ we can associate another function $\hat{f}(\xi, p)$. This is known as Radon transform of $f(x)$.

Convergence of integral sets restriction on the integrand. We assume that $f(x)$ is infinitely differentiable and it is rapidly decreasing along with all the derivatives. Then the Radon transform is also infinitely differentiable function of $\xi, p$.

It might appear that $\hat{f}(\xi, p)$ depends on $n + 1$ variables where as $f(x)$ only on $n$ variables. Here we show that it is not correct. One can see from equation (4.3.1) that $\hat{f}(\xi, p)$ is homogenous function of degree $-1$ which means that for any real $\alpha \neq 0$

$$\hat{f}(\alpha\xi, \alpha p) \quad = \quad |\alpha|^{-1}\hat{f}(\xi, p)$$

This implies that given $\hat{f}(\xi, p)$ for any fixed $p$ and all $\xi$, say $p = 1$, $\hat{f}(\xi, p)$ is known for all values of $p$. Hence it also depends on $n$ variables.

To see the physical meaning of $\hat{f}(\xi, p)$ imagine $f(x)$ represent the mass density distribution through out the space. And let $M(\xi, p)$ be the total mass in the region $\langle \xi, x \rangle < p$. Then

$$M(\xi, p) \quad = \int_{\langle \xi, xp} f(x)dx \quad = \int f(x)\theta(p) - \langle \xi, x \rangle)dx \tag{4.3.2}$$

where $\theta(p)$ is the Heaviside step function. We know that $\theta'(p) = \delta(p)$. Thus the derivative of (4.3.2) with respect to $p$ is given by

$$\frac{\partial M(\xi, p)}{\partial p} \quad = \quad \int f(x)\delta(p - \langle \xi, x \rangle)dx = \hat{f}(\xi, p)$$

Thus we see that if $f(x)$ is the mass density distribution then its Radon transform is given by

$$\hat{f}(\xi, p) = \frac{\partial M(\xi, p)}{\partial p}$$

where $M(\xi, p)$ is the mass in the half-space $\langle \xi, p \rangle < p$. For example consider the constant function $f(x) = 1$ over a bounded region $V$. Then the Radon transform of $f(x)$ is given by

$$\hat{f}(\xi, p) = \frac{\partial V(\xi, p)}{\partial p}$$

where $V(\xi, p)$ is the volume enclosed by $\langle \xi, x \rangle < p$. Geometrically this gives the area of intersection of $V$ with $\langle \xi, x \rangle = p$.

Actually Radon transform is related to our old friend Fourier transform

$$\tilde{f}(\xi) = \int f(x) e^{i\langle \xi, x \rangle} dx \tag{4.3.3}$$

Note that exponential on the right side can be written as

$$e^{i\langle \xi, x \rangle} = \int e^{ip} \delta(p - \langle \xi, x \rangle) dp$$

Substituting in (4.3.3) we get

$$\begin{aligned} \tilde{f}(\xi) &= \int f(x) \int e^{ip} \delta(p - \langle \xi, x \rangle) dp dx \\ &= \int \hat{f}(\xi, p) e^{ip} dp \end{aligned}$$

Thus Fourier transform is obtained by integrating over the $p$ in the Radon transform. Also note that

$$
\begin{aligned}
\tilde{f}(\alpha\xi) &= \int \hat{f}(\alpha\xi, p) e^{ip} dp \\
&= \int_{-\infty}^{\infty} \hat{f}(\alpha\xi, \alpha p) e^{i\alpha p} d(\alpha p) \\
&= \int_{-\infty}^{\infty} \hat{f}(\xi, p) e^{i\alpha p} dp
\end{aligned}
$$

In the last step we have used the homogeneity condition $\hat{f}(\alpha\xi, \alpha p) = |\alpha|^{-1}\hat{f}(\xi, p)$. In other words, Fourier transform in $n$ dimension is given by Radon transform followed by one dimensional Fourier transform. Integral transform is nothing but generalization of Radon transform to curved spaces. In fact it is an advantage of Radon transform that it can be easily generalized. However the second step necessary to convert it to Fourier transform is not so easy to generalize. That is why study of Radon transform is important. Actually, analog of second step is related to representations of groups.

### 4.3.2 Inverse Radon transform

As with any map, usefulness is in being able to invert the map to get the final results in the original space. Here we discuss the inverse map of Radon transform. We want to invert equation (4.3.1) to get a formula of $f(x)$ in terms of $\hat{f}(\xi, p)$. We will just state the formula without giving any proof. One can look in [18] volume 5, chapter 1 for the proof.

Let us differentiate $\hat{f}(\xi, p)$ $(n-1)$ times with respect to $p$ where $n$ is the dimension of the space.

$$
\psi(\xi, p) = \hat{f}_p^{n-1}(\xi, p) \equiv \frac{\partial^{n-1}\hat{f}(\xi, p)}{\partial p^n}
$$

It is intuitively clear that value of the function at $x$ can be decoded only by analyzing all the $\hat{f}(\xi,p)$ passing through $x$. Indeed we do something very similar. We average $\psi(\xi,p)$ over all the hyperplanes passing through the point $x$ to get $f(x)$. Average is taken over a surface $\Gamma$ enclosing the point $\xi=0$ and with respect to the measure

$$\omega(\xi) = \sum_{k=1}^{n}(-1)^{k-1}\xi_k d\xi_1...d\xi_{k-1}d\xi_{k+1}...d\xi_n$$

It turns out that the inversion depends on whether the space is odd or even dimension. For odd dimension the inversion formula is given by

$$f(x) = \frac{\pi}{(2\pi)^n}(-1)^{\frac{n-1}{2}}\int_\Gamma \hat{f}_p^{n-1}(\xi,\langle\xi,x\rangle)\omega(\xi)$$

Inversion formula for even dimension is given by

$$f(x) = \frac{(n-1)!}{(2\pi)^n}(-1)^{\frac{n}{2}}\int_\Gamma \left[\hat{f}_p(\xi,p)(p-\langle\xi,x\rangle)^(-n)dp\right]\omega(\xi)$$

At the end we will give an example verifying the above formula. We will take a function, perform Radon transform on it and then Inverse Radon transform to get back the original function. Consider the following function $f(x)$ in odd dimension $n$

$$f(x) = e^{-x_1^2-...-x_n^2} = e^{-|x|^2}$$

Radon transform is given by

$$\hat{f}(\xi,p) = \int e^{-|x|^2}\delta(p-\langle\xi,x\rangle)dx$$

Rotating the coordinate system such that $\langle \xi, x\rangle = |\xi|x_1$ we get

$$\begin{aligned} \hat{f}(\xi,p) &= \int e^{-|x|^2}\delta(p-|\xi|x_1)dx \\ &= \frac{\pi^{\frac{n-1}{2}}}{|\xi|}e^{-p^2/|\xi|^2} \end{aligned}$$

This is the Radon transform. Now we wish to do inverse Radon transform. Let us choose $n = 3$.

$$\hat{f}_p^2(\xi,p) = = \frac{2\pi}{|\xi|^3}e^{-p^2/|\xi|^2}\left(\frac{2p^2}{|\xi|^2}-1\right)$$

Substitute $p = \langle \xi, x\rangle = |\xi||x|\cos\theta$ where $\theta$ is the angle between $\xi$ and $x$. Let $\Gamma$ be a unit sphere around the origin. Then $\omega(\xi) = \sin\theta d\theta d\phi$

$$\begin{aligned} f(x) &= \frac{\pi}{(2\pi)^3}(-1)\int_\Gamma \hat{f}_p^2(\xi,(\xi,x))\omega(\xi) \\ &= \frac{\pi}{(2\pi)^3}(-1)\int_{|\xi|=1}\frac{2\pi}{|\xi|^3}e^{-|x|^2\cos^2\theta}\left(2|x|^2\cos^2\theta-1\right)\sin\theta d\theta d\phi \\ &= e^{-|x|^2} \end{aligned}$$

Thus we get back the original function. This verifies that the inversion formula is working correctly. Note that $\hat{f}(\xi,p)$ and $f(x)$ both depend on $n$ variables. This is not surprising because they capture same amount of information.

### 4.3.3 Integral transform

Having discussed Radon transform and its inverse, now we are ready to generalize the concept to curved spaces. Generalization of Radon transform to curved spaces is called Integral transform. Discussions in this chapter will mostly follow chapter V of volume 5 of [18]. First thing needed to define integral transform is the concept of hypersurfaces in curved spaces. What is the generalization of hyper-planes in

Euclidean space to curved spaces? Discussion of various hyper-surfaces will form the first part of this subsection. As we go on, we will also give formulas of integral transform and inverse integral transform for each hyper-surfaces. Our discussion will focus on spaces of positive curvature also known as de-Sitter spaces. We will not go into the discussion Lobachevskian space that is space of constant negative curvature also known as Anti-de Sitter space.

In this section we will introduce some notation, broadly following the integral geometry approach of [19] in imaginary Lobachevskian space, also known as elliptic de Sitter spacetime [20, 21]. Elliptic de Sitter is simply global de Sitter modulo the antipodal map. Our main focus will be global de Sitter. In some ways elliptic de Sitter is simpler because there is a single connected boundary at infinity, while in global de Sitter there are two disconnected boundaries, one in the distant past, and one in the distant future. We will find in global de Sitter a CFT may be defined on either boundary, and for the sake of definiteness we choose the past boundary. Our formulas will be explicitly written for the case of three-dimensional de Sitter spacetime, but the results generalize immediately to higher dimensions.

The de-Sitter space can be realized on a hyperboloid embedded in four-dimensional Minkowski spacetime

$$x_0^2 - x_1^2 - x_2^2 - x_3^2 = -R^2 \,, \tag{4.3.4}$$

where $R$ is some positive constant. The geodesic distance $r$ between any two points

$$\cosh^2 kr = \frac{\langle x, y\rangle^2}{\langle x, x\rangle \langle y, y\rangle} \,, \tag{4.3.5}$$

where

$$\langle x, y\rangle = x_0y_0 - x_1y_1 - x_2y_2 - x_3y_3 \,,$$

is the inner product of two vectors and $k = \frac{1}{R}$ is another positive constant.

The family of points satisfying $\langle x, x\rangle = -R^2$ with antipodal points $(x \sim -x)$ identified is called imaginary Lobachevskian space or elliptic de Sitter spacetime. Without the identification we have ordinary global de Sitter spacetime. The distance $r$ can be real and non-negative (if $1 \leq \cosh kr \leq \infty$) or imaginary in the interval $[0, \frac{\pi i}{2k}]$ (if $0 \leq \cosh kr \leq 1$). From now onwards any point on the light cone in the embedding space will be denoted by $\xi$, that is $[\xi, \xi] = 0$. In the next subsections we will discuss complete set of hyper-surfaces of de-Sitter space. Also we will set $R = 1$.

#### 4.3.3.1 Isotropic lines

Isotropic line is the set of points of the form $x = sa + tb$ where $a$ and $b$ are fixed vectors on the de-Sitter space and $s, t$ vary to generate the line. Since basis vectors lie on dS they satisfy $\langle a, a\rangle = \langle b, b\rangle = -1$. An isotropic line is defined to be set of points such that distance between any two of them vanishes that is $r = 0$. Since $a$ and $b$ lie on the line, distance between them must also vanish. That is $\langle a, b\rangle = \cosh kr = 1$. Thus $\xi = b - a$ is a point on light cone. Also

$$\begin{aligned} \langle x, x\rangle &= -1 \\ \implies \langle sa + t\xi, sa + t\xi\rangle &= -1 \\ \implies s &= 1 \end{aligned}$$

So isotropic line is

$$x = a + t\xi$$

$$\langle a, a\rangle = -1 \qquad \langle a, \xi\rangle = \langle \xi, \xi\rangle = 0$$

$\xi$ is called the direction vector of isotropic line and lies on the light cone. $a$ lies on the tangent plane to the cone because $\langle a, \xi\rangle = 0$. Thus isotropic lines are two dimensional planes tangent to the light cone.

#### 4.3.3.2 Horospheres

Now let us consider some surfaces in de Sitter with particularly simple transformation properties under the isometry group. The equation describing a sphere of radius $r$ with center at $a$ is given by

$$\langle x, a \rangle^{2} = c \langle a, a \rangle \langle x, x \rangle \ .$$

where $c = \cosh^2 kr$. If $c > 1 \implies r > 0$, $c = 1 \implies r = 0$ and $c < 1 \implies r$ is imaginary. Note that sphere with vanishing radius $r = 0$ are the set of points whose distance from $a$ vanishes. This is nothing but the surface generated by isotropic lies passing through $a$. Such a surface is also known as isotropic cone of dS space.

Consider taking the center to the infinity while ensuring that the sphere passes through a fixed point $b$. The surface obtained in this way is called a horosphere. In this limit, the product $c \langle a, a \rangle$ is fixed to some constant $c_1$ to obtain the surface

$$\langle x, \xi \rangle^{2} = c_1 \langle x, x \rangle \ . \tag{4.3.6}$$

When $c_1 < 0$ this is called a horosphere of the first kind. It is possible to normalize $c_1 = -1$ by normalizing $\xi$. If we set $R = 1$, so that $\langle x, x \rangle = -1$ then the horospheres of the first kind look like

$$| \langle x, \xi \rangle | = 1 \, . \tag{4.3.7}$$

Thus a horosphere of the first kind may be specified by choosing a point $\xi$ on the positive cone, $\langle \xi, \xi \rangle = 0 \, , \xi_0 > 0$.

When $c_1 = 0$ one gets a horosphere of the second kind

$$\langle x, \xi \rangle = 0 \,. \tag{4.3.8}$$

In this paper, our focus will be on the horospheres of the first kind, which will correspond to principal series representations of the de Sitter group [19]. We will consider the horospheres of the second kind in future work, which correspond to the discrete series representations.

Note that the horospheres of the first kind go over to themselves under the action of hyperbolic rotation. Same for horospheres of second kind.

#### 4.3.3.3 Integral transform and inverse integral transform

Now we will formally state the integral transform and its inverse in de-sitter space. We will not give any proof of the formula however will highlight some of the technical points of the proof.

Let $f(x)$ be an infinitely differentiable function of bounded support on a de-Sitter space $\langle x, x\rangle = -1$. We form the integrals of this function over the horospheres of the first kind and over the isotropic lines; these integrals are defined in the following way. The integral over the horosphere of the first kind whose equation is $|x, \xi| = 1$ is

$$h(\xi) \;=\; \int f(x)\delta(|\langle x, \xi\rangle| - 1)dx \tag{4.3.9}$$

, where $\langle \xi, \xi\rangle = 0, \xi_0 > 0$; the integral over the isotropic line $x = b + t\xi$ is

$$\varphi(\xi, b) \;=\; \int_{-\infty}^{\infty} f(b + t\xi)dt \tag{4.3.10}$$

, where $\langle b, b\rangle = -1, \langle b, \xi\rangle = \langle \xi, \xi\rangle = 0, b_0 = 0$.

Then the value of $f(x)$ at any point $a$of the de-Sitter space is given in terms of

$h(\xi)$ and $\varphi(\xi, b)$ by the inversion formula

$$f(a) = -(4\pi)^{-2} \int h(\xi)\delta''(|\langle a, \xi\rangle| - 1)d\xi \tag{4.3.11}$$
$$+(2\pi)^{-2} \int_0^\pi \cot^2\theta d\theta \int_\Gamma \varphi(\xi, \theta)d\omega \tag{4.3.12}$$

where $d\xi = |\xi_0|^{-1}d\xi_1 d\xi_2 d\xi_3$ is the invariant measure on the $\langle \xi, \xi\rangle = 0$ null cone. $\varphi(\xi, \theta)$ denotes the value of $\varphi(\xi, b)$ for an isotropic line $x = b + t\xi$ lying in the $\langle a, x\rangle = \cos\theta$ plane (that is $\langle a, b\rangle = \cos\theta$). $\Gamma$ is any surface on the $\langle \xi, \xi\rangle = 0$null cone that intersects all the generators of the cone, the measure $d\omega$ is defined by

$$d\omega = |\xi_0|^{-1}(\xi_1 d\xi_2 d\xi_3 - \xi_2 d\xi_1 d\xi_3 + \xi_3 d\xi_1 d\xi_2)$$

$h(\xi)$ satisfy following symmetry relation $H(a, t) = H(a, t^{-1})$ where

$$H(a, t) = \int h(\xi)\delta(|[a, \xi]| - t)d\xi$$

This will be used in section 4.6 to simplify transform. The other symmetry relation is

$$\int_\Gamma \varphi(\xi, \theta)\delta([a, \xi])d\omega = \int_\Gamma \varphi(\xi, \pi - \theta)\delta([a, \xi])d\omega$$

In the rest of the thesis we will be interested only in the integral transform of the first kind. Reason being that integral transforms are closely related to representations of symmetry group. Horospheres of first kind are related to principal series representations and horospheres of second kind are related to discrete series representations of de-Sitter group. These representations will be defined later, they represent different mass ranges of quantum fields in dS. Since we are only interested in understanding the principal series representation in our first paper [22] we will analyze only equations

(4.3.9) and (4.3.11) in the rest of the paper.

Although the integral transform as stated above requires the function in dS to have bounded support, we know that free fields in dS do not have bounded support. As we will see in this work, unbounded support results in poles in equation (4.5.9). Applying LSZ prescription relates the weights of the boundary operators to mass of the bulk field.

### 4.3.4 Integral transform and representations of group

At the end of the day we want to understand representations of the dS isometry group. Often it is difficult to understand representations of $f(x)$ in original space $X$. In such circumstances each $f(x)$ over $X$ is mapped to objects $\hat{f}(y)$ over $Y$ such that representations of $\hat{f}(y)$ is comparatively easier. The map is constructed to be invertible so that after understanding the representations, one can go back to original space $X$. Integral geometry is one such tool which greatly simplifies the representation theory of group. As we will now show horospheres have simple group theoretical understanding.

Consider a point $x_0$ in homogenous space $X$. Let $g$ be the transformations which take the point $x_0$ to $x$. Transformations $h$ which carry $x_0$ to $x_0$ are called stability subgroup of $x_0$. Then the transformations $hg$, also maps $x_0$ to $x$. These are all the transformations that map $x_0$ to $x$. Set of such transformation is a right coset of the stability subgroup of $x_0$. Thus there is one to one mapping between points of the homogenous space and and the right cosets of the stability subgroup with respect to $x_0$.

Let us now consider examples of spaces homogenous with respect to Lorentz group. These spaces are invariant under the motion of Lorentz group. Consider the two dimensional complex plane $(z_1, z_2)$ modulo origin $(0, 0)$. Elements of Lorentz group

$$g = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}, \alpha\beta - \gamma\delta = 1$$

act on the complex plane as

$$(z_1, z_2) \rightarrow (\alpha z_1 + \gamma z_2, \beta z_1 + \delta z_2)$$

This is called complex affine plane. To find stability subgroup of complex affine plane let us choose the point $(0, 1)$. Transformations that leave this point fixed are

$$g = \begin{bmatrix} 1 & \zeta \\ 0 & 1 \end{bmatrix}$$

Now let us consider a model of dS space (imaginary Lobachevskian space) given by set of all positive definite hermitian matrices in two dimensions

$$h = \begin{bmatrix} x_0 - x_3 & x_2 - ix_1 \\ x_2 + ix_1 & x_0 + x_3 \end{bmatrix}$$
$$x_0^2 - x_1^2 - x_2^2 - x_3^2 = -1$$

Under the action of Lorentz group the matrices transform as

$$h' = g * hg$$

Now lets choose the fixed matrix to be

$$\sigma = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

Stability subgroup of $\sigma$ are matrices $g$ which satisfy

$$g * \sigma g \quad = \quad \pm\sigma$$

Such matrices are of the form

$$\begin{bmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{bmatrix}, \qquad \begin{bmatrix} \beta & \alpha \\ -\bar{\alpha} & -\bar{\beta} \end{bmatrix}, \qquad |\alpha|^2 - |\beta|^2 = 1$$

We will now give a group theoretic definition of horosphere and show that it coincides with the geometrical definition of the horosphere. The advantage is that this definition is applicable for all homogenous spaces whereas geometrical picture though more intuitive has to be derived separately for each space. Definition is the following.

A horosphere in a space $X$, homogenous under the group of complex two-dimensional unimodular matrices (that is Lorentz group) is the orbit of any point $x \in X$ under the subgroup $Z$ of the matrices

$$\begin{bmatrix} 1 & \zeta \\ 0 & 1 \end{bmatrix}$$

or under any subgroup conjugate to it.

This implies that horospheres have a stationary subgroup $Z$ or some conjugate subgroup and its structure is isomorphic to complex affine line. Thus a horosphere $\omega$ consists of all points of the form $xg^{-1}\zeta g$ given by some point $x \in X$, some element $g$ of the Lorentz group and where $\zeta$ runs over $Z$.

To explicitly see the geometrical picture, let us consider the Hermitian matrix model of de-Sitter space. Horosphere $\omega$ consists of matrices of the form

$$x = \begin{bmatrix} 1 & 0 \\ \bar{\zeta} & 1 \end{bmatrix} \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} 1 & \zeta \\ 0 & 1 \end{bmatrix}$$

For all such transformation $x_{11} = h_{11} = C \neq 0$ is unchanged. This equation can also be written for the model $\langle x, x \rangle = -1$. Then $h_{11} = x_0 - x_3 = C$. That is $\langle x, \xi \rangle = C$ where $\xi = (1, 0, 0, 1)$ lies on the null cone. Thus $\langle x, \xi \rangle = C$ is the equation of horosphere. This gives the horospheres of the first kind.

If $h_{11} = C = 0$. Then

$$\begin{bmatrix} 1 & 0 \\ \bar{\zeta} & 1 \end{bmatrix} \begin{bmatrix} 0 & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} 1 & \zeta \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & x_{12} \\ x_{21} & x_{22} \end{bmatrix}$$

Thus we see $x_{11} = 0, x_{12} = C$ where $|C| = 1$. Thus $x_2 - ix_1 = C = C_1 - iC_2$. Thus the horosphere is given by $x_0 = x_3, x_1 = C_1, x_2 = C_2, C_1^2 + C_2^2 = 1$. This is nothing but isotropic line in deSitter space. This shows that horospheres of de-Sitter decompose into two classes. One given by horospheres of the first kind which will correspond to principal series representations. Second given by horospheres of the second kind, isotropic lines, which will correspond to discrete series representations.

## 4.4 Integral transform in de-Sitter space

There has been a lot of progress in understanding this correspondence between bulk quantum theory in anti-de Sitter spacetime and boundary conformal field theory [1]. One expects these ideas to carry over in some form to the cases of asymptotically flat spacetime and asymptotically de Sitter spacetime. In these cases the situation is much less clear, and our aim in the present work is to carefully set up the bulk/boundary correspondence in the de Sitter case. This will allow us to draw some interesting conclusions about the structure of the novel conformal field theories that must appear in this case, and their ultimate consistency.

Various different formulations of dS/CFT has been proposed, following Strominger's initial work [16]. Some formulations simply extend the AdS/CFT correspondence to the dS space via analytic continuation, which has been successful for massless fields (and possibly sub-Hubble mass fields) and the massless higher spin gravity theories [23]. Our goal in the present work is to investigate the situation for generic fields with masses larger than the Hubble scale, which are related by analytic continuation to tachyonic fields in anti-de Sitter spacetime. New methods must be developed to treat this case. It is worth noting that in the CFT these fields will correspond to quasi-primary fields with complex conformal weights. Nevertheless, these form unitary representations of the global conformal group [24, 25, 26, 27], opening the door to possibility that an entirely new class of conformal field theories might be defined based on these representations.

One of the key mysteries in the dS/CFT correspondence is the origin of bulk time, since the dual CFT is a purely Euclidean theory. In the AdS/CFT correspondence this is not an issue because the bulk time is parallel to the boundary time and the CFT lives in a spacetime with Lorentzian signature. As a result, it becomes more interesting to see how unitarity and time ordering in the bulk theory emerges from the Euclidean CFT, and we will obtain partial results in this direction.

The paper is organized as follows. We begin by presenting an analog of the LSZ construction [28] for quantum fields in de Sitter spacetime, which provides a clear definition of correlators in the boundary CFT. This step is necessary because the representations of the conformal group in question, the principal series, are not commonly studied in the context of conformal field theory. The construction is inspired by the integral geometry approach of Gelfand [19], and many of the results detailed there carry over to the present case. For the most part, our focus will be on three-dimensional de Sitter spacetime, though many of the ideas carry over to the higher dimensional case.

We then consider the inverse map, reconstructing bulk field operators in terms of the CFT data. At leading order (essentially the free level from the viewpoint of quantum field theory in the bulk) we construct bulk creation and annihilation field operators using operators in the CFT. Bulk operator ordering in correlators can be accomplished by adopting an $i\epsilon$-prescription, complexifying the radial direction in the CFT. This is sufficient to recover the bulk Wightman two-point correlation function, with the correct Hadamard singularity at light-like separations. This approach may also be used to build higher point correlators, for bulk theories with perturbative expansions, by using the creation and annihilation operators to reproduce the Wick expansion. However a completely general nonperturbative understanding of the bulk operator ordering, and hence the origin of bulk time, is elusive.

The construction we describe allows one to define a CFT from some set of bulk correlators in de Sitter spacetime. We may then proceed to analyze the basic consistency of the resulting CFT, to check whether it satisfies the basic axioms expected of a Euclidean quantum field theory. These are known as the Osterwalder-Schrader axioms [29, 30]. One of these axioms is the Euclidean version of cluster decomposition, which requires correlators to factorize in the limit of large separations. We find this fails in the case of the principal series, if, for example, operators of the form $L_1\bar{L}_1\mathcal{O}_\Delta$ are considered, where $L_1$ and $\bar{L}_1$ are conformal generators that raise the weight by 1, and $\mathcal{O}_\Delta$ is a quasi-primary operator with weight $\Delta$. Note in ordinary CFTs with $\mathcal{O}_\Delta$ a primary field with positive conformal weight, the combination $L_1\mathcal{O}_\Delta$ would vanish. The operator $L_1\bar{L}_1\mathcal{O}_\Delta$ will be dual to a graviton plus a massive matter field insertion. The failure of cluster decomposition signals that the vacuum of the CFT is not unique, i.e. there can be many excitations in the bulk that give rise to nontrivial operators on the boundary satisfying $L_0 = \bar{L}_0 = 0$. This follows from the lack of a positive energy theorem in the bulk theory [31],[1]. We note this does not immediately imply infrared divergences in the bulk theory. In fact, the classical stability of de Sitter spacetime for pure gravity or massless conformal matter coupled to gravity has been demonstrated [33, 34, 35]. Most likely, this result should be interpreted as an incompleteness in the CFT dual to an interacting theory in de Sitter spacetime, a point we hope to return to in future work.

A related construction of bulk operators from boundary operators in dS/CFT has been considered in [36, 37]. There are numerous differences in the details and conclusions with the present work.

## 4.5 Boundary CFT operators

It is useful to begin by reviewing the decomposition of some general bounded, normalizable function on de Sitter into components that transform as unitary irreducible representations of the conformal group [19]. For every $f(x)$ one constructs the integral transform

$$h(\omega) = \int_\omega f(x) d\sigma \,, \tag{4.5.1}$$

and $d\sigma$ is an invariant measure, and the integral is over a horosphere of first kind $\omega$. We require that these integrals are invariant under hyperbolic rotation.

$$\int_\omega f(xg) d\sigma \;\; = \;\; \int_{\omega g} f(x) d\sigma_g = \int_\omega f(x) d\sigma$$

Let the equation of a horosphere be $| \langle x, \xi \rangle | = 1$. Equation (4.5.1) can also be written as

[1] There is a positive energy theorem for the global timelike conformal Killing vector of de Sitter [32], but it is not clear if this is well-defined on the conformal compactification of de Sitter. So its relation to the dual CFT is not currently understood.

$$h(\xi) = \int f(x)\delta\left(|\langle x, \xi\rangle| - 1\right) dx\,, \tag{4.5.2}$$

where $dx$ is the invariant measure on the de Sitter spacetime. This above map is nothing but a generalization of the Fourier transform, which takes a function defined on the horosphere to a function defined on the light cone labelled by $\xi$. As we will see, $\xi$ can be used to parametrize the boundary at past infinity in de Sitter. Note that $f(x)$ and $(\xi)$ both are function of 3 variables. This guarantees that no information is lost and $h(\xi)$ can be inverted back to $f(x)$. Then we do Fourier transform in $\lambda$ coordinate

Now consider functions $h(\xi)$ over the positive sheet of the light cone where $\xi^0 > 0$. These functions may be decomposed into components with well-defined conformal weights by Fourier transforming

$$F(\xi;\rho) = \int_0^\infty h(t\xi)t^{-i\rho}dt\,, \tag{4.5.3}$$

where the complex conformal weight $\Delta$ is related to the real parameter $\rho$ via $i\rho = 1-\Delta$. Let us note that inserting (4.5.2) into (4.5.3) we have

$$F(\xi;\rho) = \int_0^\infty dt \int dx f(x)\delta\left(|\langle x, t\xi\rangle| - 1\right) t^{-i\rho}\,.$$

Performing the integral over $t$ we arrive at

$$F(\xi;\rho) = \int dx f(x)\left|\langle x, \xi\rangle\right|^{-\Delta}\,. \tag{4.5.4}$$

Again let us count the number of variables. $F(\xi;\rho)$ depends on 2- $\xi$ coordinates and $\rho$. So total 3 variables. This again ensures that $h(\xi;\rho)$ can be inverted to get back $f(x)$.

Generalizing $f$ to some bulk correlator of some scalar field of mass $m$, our goal

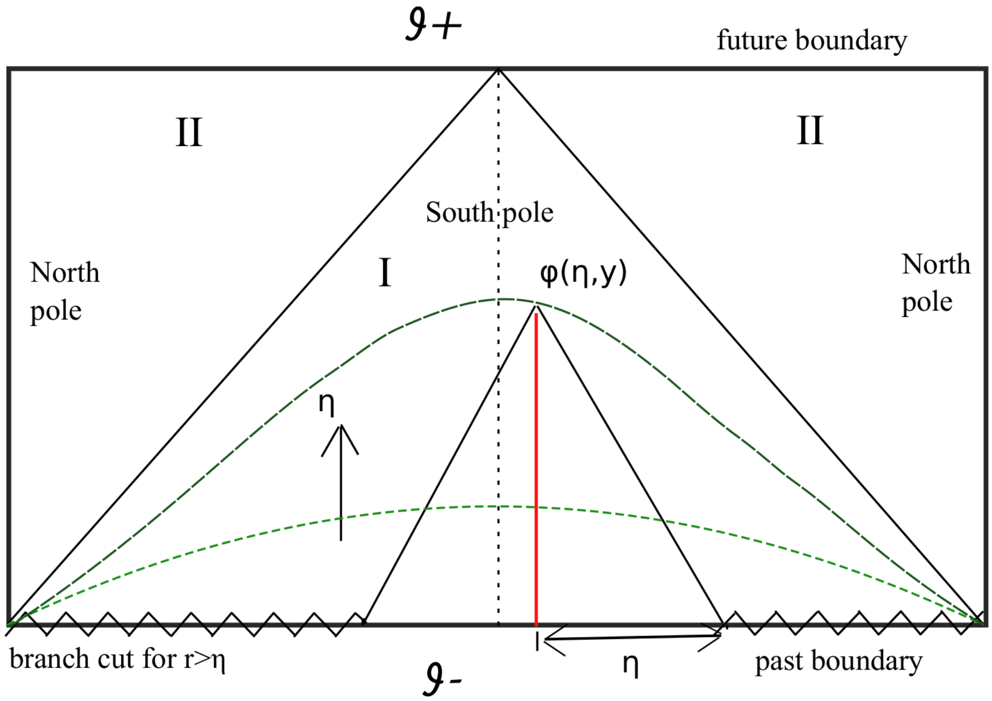


Figure 4.5.1: Penrose diagram for de-Sitter space. The vertical dashed line is the South pole. The left and right edges are North pole which are identified with each other. Horizontal dashed lines are constant $\eta$ slices. The figure shows the calculation of bulk field from boundary operator in the past boundary. The boundary operator is smeared over the whole boundary. There is branch cut in the smearing function for $r > \eta$. Continuing $r$ and $\eta$ to the complex plane via an $i\epsilon$ prescription selects the branch yielding a Bunch-Davies/Euclidean vacuum positive or negative frequency mode.

will then be to view the analog of $F$ as a boundary correlator. A key difference with the work of Gelfand is that we must give up the condition of normalizability (in the sense that $\int |f(x)|^2 dx$ is finite). As we will see, this the de Sitter isometry covariant component of (4.5.2) will correspond to the residue of a pole in $\rho^2 - m^2$ reminiscent of the LSZ reduction formula in flat spacetime [28].

### 4.5.1 Flat slicing

Horospheres of the first kind are diffeomorphic to flat spatial slices in de Sitter. It therefore will be convenient to express the general coordinate invariant expression (4.5.4) on flat slices. See [38] for some related work in the context of four-dimensional de Sitter. Setting $R = 1$, the 3-dimensional de Sitter hyperboloid can be parameterized by the coordinates $(\eta, y_1, y_2)$ via

$$\begin{aligned} x^0 &= \frac{1}{2}(\eta - \frac{1}{\eta}) - \frac{\sum y_i^2}{2\eta} \\ x^1 &= \frac{y_1}{\eta} \\ x^2 &= \frac{y_2}{\eta} \\ x^3 &= -\frac{1}{2}(\eta + \frac{1}{\eta}) + \frac{\sum y_i^2}{2\eta}\,, \end{aligned}$$

yielding the de Sitter metric with a flat spatial slicing and conformal time $\eta$

$$ds^2 = \frac{d\eta^2}{\eta^2} - \frac{1}{\eta^2}\left(dy_1^2 + dy_2^2\right) \ .$$

The volume measure is

$$dx = \frac{1}{\eta^3} d\eta dy_1 dy_2 \,. \tag{4.5.5}$$

A point on a light cone may be parameterized by

$$\xi = k\lambda(1 + z^2, 2z_1, 2z_2, 1 - z^2)\,, \tag{4.5.6}$$

where $z^2 = z_1^2 + z_2^2$. The coordinates $z_1, z_2$ label a point on the boundary at past infinity in de Sitter. In these coordinates we have

$$\langle x, \xi \rangle = \lambda\eta \left(1 - \frac{(y_1 + z_1)^2}{\eta^2} - \frac{(y_2 + z_2)^2}{\eta^2}\right) \ .$$

We will also need the measure on the cone

$$d\xi = \frac{d\xi_1 d\xi_2 d\xi_3}{\xi_0}\,,$$

and the measure on the boundary at infinity

$$d\omega = d^2 z\,.$$

### 4.5.2 Transform from bulk to boundary

Our aim is to use the transform of Gelfand [19] as a guide to constructing the transform for the class of functions that appear in correlation functions of quantum fields in de Sitter. In particular, these functions do not satisfy the compact support condition used in Gelfand's inversion theorem. This will lead us to build an analog of the flat-spacetime LSZ reduction formula for de Sitter spacetime, requiring some important differences with Gelfand's construction.

We begin by noting the mode expansion for a bulk scalar field of mass $m$ [17]

$$\phi(\eta, y) = c_1 \int \frac{d^2k}{(2\pi)^2} \left( a_k \eta H^{(2)}_{i\mu}\left(|k|\eta\right) e^{ik.y} + a^\dagger_k \eta H^{*(2)}_{i\mu}\left(|k|\eta\right) e^{-ik.y} \right)\,, \tag{4.5.7}$$

where $c_1 = \frac{\sqrt{\pi}}{2} e^{\frac{\pi\mu}{2}}$, $\mu = \sqrt{m^2 - 1}$, and $H^{(2)}_{i\mu}(|k|\eta)$ are Hankel functions of second kind. The operators $a_k$ and $a^\dagger_k$ are annihilation and creation operators, with the $a_k$ annihilating the Bunch-Davies vacuum, and

$$[a_k, a^\dagger_{k'}] = (2\pi)^2 \delta^{(2)}(k - k')\,.$$

To construct the boundary operator, we perform the following integral over region I of figure 4.5.1,

$$\begin{aligned} \Phi_\Delta(z) &= c_1 \int \frac{d^2k}{(2\pi)^2} (a_k \eta H^{(2)}_{i\mu}(|k|\eta) e^{ik.y} + a^\dagger_k \eta H^{*(2)}_{i\mu}(|k|\eta) e^{-ik.y}) \\ & \quad \left(1 - \frac{(y_1 + z_1)^2}{\eta^2} - \frac{(y_2 + z_2)^2}{\eta^2}\right)^{-\Delta} \eta^{-(3+\Delta)} d\eta d^2 y\,. \end{aligned} \tag{4.5.8}$$

We define the cut in the $x^{-\Delta}$ factor as

$$x^{-\Delta} = |x|^{-\Delta} e^{-i\Delta \arg x} ,$$

where $\arg x \in (-\pi, \pi]$. Note this choice of phase differs from the expression (4.5.4) and will be related to the choice of the Bunch-Davies/Euclidean vacuum for the free theory. Other phase conventions can lead to the more general $\alpha$-vacua [39] which are thought to be unphysical [40].

At the level of the bulk correlators, the operator ordering is determined by continuing the bulk time $\eta \to \eta \pm i\epsilon$. This then yields the distinctive signature of the Hadamard singularity of the two-point correlator in the light-like limit, which in turn matches the short-distance singularities of flat-spacetime [41]. This continuation determines the branch of the cut in (4.5.8), and as we will see projects onto the $a_k$ or the $a_k^\dagger$ terms dependent on the sign. Therefore we define $P_\Delta$ and $P_\Delta^\dagger$ as follows

$$\begin{aligned} P_\Delta(z) &= \Phi_\Delta(z) , \qquad \eta \to \eta + i\epsilon \\ P_\Delta^\dagger(z) &= \Phi_\Delta(z) , \qquad \eta \to \eta - i\epsilon \end{aligned}$$

with $\epsilon > 0$. Performing the integrals we then get

$$\begin{aligned} P_\Delta(z) &= d(\Delta) \frac{1}{(\Delta-1)^2+\mu^2} \int \frac{d^2k}{(2\pi)^2} a_k |k|^{-1+\Delta} e^{ik\cdot z} \\ P_\Delta^\dagger(z) &= \tilde{d}(\Delta) \frac{1}{(\Delta-1)^2+\mu^2} \int \frac{d^2k}{(2\pi)^2} a_k^\dagger |k|^{-1+\Delta} e^{-ik\cdot z} , \end{aligned} \tag{4.5.9}$$

where

$$\begin{aligned} d(\Delta) &= i 2^{2-\Delta} e^{-i\pi\Delta/2} \sqrt{\pi}\Gamma(1-\Delta) \sin(\pi\Delta) \\ \tilde{d}(\Delta) &= -i 2^{2-\Delta} e^{i\pi\Delta/2} \sqrt{\pi}\Gamma(1-\Delta) \sin(\pi\Delta) \end{aligned}$$

We note the prefactors of the boundary operators have poles when $\Delta = 1 \pm i\mu$, reminiscent of the poles arising in momentum space when one performs the LSZ reduction in flat spacetime, which yields the S-matrix. In the same way, we find by taking the residues of these poles, we are able to define conformally covariant operators on the boundary

$$\mathcal{O}_\Delta(z) = d(\Delta)\frac{i}{2(\Delta-1)}\int \frac{d^2k}{(2\pi)^2} a_k |k|^{-1+\Delta} e^{ik\cdot z}\,, \tag{4.5.10}$$

where now $\Delta = 1 - i\mu$. The other pole yields the operator $\mathcal{O}_{2-\Delta}(z)$. As we will see there is an equivalence between these two operators, since either may be used to reconstruct the bulk annihilation mode. A similar relation is found in the work of Gelfand. For the principal series, the representations corresponding to $\Delta$ and $2-\Delta$ are equivalent, so the minimal spectrum of representations corresponds to $\mu > 0$. The formulas carry over straightforwardly to the operators $\mathcal{O}^\dagger_\Delta$ and $\mathcal{O}^\dagger_{2-\Delta}$ . The pole at $\Delta = 1+i\mu$, determines the value of $\Delta$. In general integral transform, $\rho$ is a continuous coordinate taking values from $(-\infty, \infty)$. But for bulk fields, obeying Klein-Gordon equation, $\rho$ is restricted to just one value $\rho = \mu = \sqrt{m^2-1}$. It is at this point that one dimension is restricted to a particular value. With $\rho$ fixed, $\Phi_\Delta(z)$ which was a function of $d$ variables $(\lambda, z_i)$ is now just function of $d-1$ variables $(z_i)$. This happens only because of properties of $\phi(x)$ which obeys KG equation. This would not be true for any arbitrary function $f(x)$ in the bulk.

Using this construction, we may then build the boundary two-point correlators from the bulk Wightman function by plugging into (4.5.8). The bulk Wightman function is [41]

$$G_E(x,x') = \frac{\Gamma(\Delta)\Gamma(2-\Delta)}{(4\pi)^{3/2}\Gamma\left(\frac{3}{2}\right)}\,{}_2F_1\left(\Delta, 2-\Delta; \frac{3}{2}; \frac{1+\langle x, x'\rangle}{2}\right)\,, \tag{4.5.11}$$

where $x$ and $x'$ are complexified to give the correct $i\epsilon$ prescription near the light-like singularity. This may also be written in terms of the integral over mode functions as

$$G_E(x,x') = c_1^2 \int \frac{d^2k}{(2\pi)^2} \eta H_{i\mu}^{(2)}\left(|k|\eta\right) \eta' H_{i\mu}^{(2)*}\left(|k|\eta'\right) e^{ik.(y-y')} .$$

Performing the bulk to boundary transform on each mode function, and taking residues yields the non-vanishing two-point correlators

$$\begin{aligned} \left\langle \mathcal{O}_\Delta(z) \mathcal{O}_\Delta^\dagger(0) \right\rangle &= -\frac{\pi \sin\left(\pi\Delta\right)}{\left(\Delta-1\right)^2} \frac{1}{|z|^{2\Delta}} \\ \left\langle \mathcal{O}_{2-\Delta}(z) \mathcal{O}_{2-\Delta}^\dagger(0) \right\rangle &= \frac{\pi \sin\left(\pi\Delta\right)}{\left(\Delta-1\right)^2} \frac{1}{|z|^{2(2-\Delta)}} . \end{aligned}$$

It is helpful to recall that scalings and translations fix the form of the correlator, but only covariance under inversions gives the requirement that each operator in the two-point function have the same conformal weight. Potential off-diagonal contributions vanish as required when the integrals (4.5.8) are performed.

The operators $\mathcal{O}_\Delta$, etc. are quasi-primary operators, in the sense that they transform under $SL(2,C)$ transformations

$$\begin{aligned} z &\rightarrow \frac{\alpha z+\beta}{\gamma z+\delta}, \qquad \alpha\delta-\beta\gamma=1 \\ \mathcal{O}_\Delta(z) &\rightarrow |\gamma z+\delta|^{-2\Delta} \mathcal{O}_\Delta\left(\frac{\alpha z+\beta}{\gamma z+\delta}\right) . \end{aligned}$$

Note, however, that in the principal series, they are not annihilated by the positive weight generators of $SL(2,C)$. Thus $L_1\mathcal{O}_\Delta \neq 0$ and $\bar{L}_1\mathcal{O}_\Delta \neq 0$ so that the operators are not primary operators. The only representations of the conformal group that behave as the usual CFT primary operators are the discrete series.

The appearance of $\mathcal{O}_\Delta$ and $\mathcal{O}_\Delta^\dagger$ as separate operators in the CFT is somewhat unusual. The Hermitian conjugation is not the natural one typically used in conformal field theory, but rather refers to bulk Hermitian conjugation with respect to the Klein-

Gordon inner product. Likewise, it is with respect to this bulk inner product, the one typically used in quantum field theory in curved spacetime, that the representations are unitary.

Having performed this construction for a single set of de Sitter mode functions, and the two-point function, one can try to generalize to higher point functions. As is clear from the above discussion, the residue of the integral transform (4.5.8) essentially picks off a free ingoing or outgoing mode, depending on the branch of the integrand the $i\epsilon$ term picks. Therefore, if the bulk quantum field theory satisfies cluster decomposition, one may apply the transform to a multi-point correlation function to define a de Sitter version of the S-matrix, in analogy with the LSZ reduction formula. The resulting S-matrix should transform covariantly under global conformal transformations. As we will see shortly, the existence of this S-matrix will hinge on this assumption of cluster decomposition.

## 4.6 Reconstructing the Bulk

It is helpful to again recall the integral geometry construction of [19]. Having constructed the boundary function $h(\xi)$, the bulk function is reconstructed by the inverse transform

$$f(x) = -\frac{1}{16\pi^2}\int \delta''\left(|[x,\xi]|-1\right)h(\xi)d\xi\,, \tag{4.6.1}$$

where the measure $d\xi$ is described in more detail in [19]. This can also be written as

$$f(x) = -\frac{1}{16\pi^2}\int_0^\infty \delta''(t-1)H(x,t)dt = -\frac{1}{16\pi^2}H_t''(x,1)\,, \tag{4.6.2}$$

where

$$H(x,t) = \int h(\xi)\delta\left(|[x,\xi]|-t\right)d\xi\,.$$

Now consider functions $h(\xi)$ over the positive sheet of the light cone. These functions may be decomposed into components with well-defined conformal weights by Fourier transforming

$$F(\xi;\rho) = \int_0^\infty h(t\xi)t^{-i\rho}dt\,, \tag{4.6.3}$$

where the complex conformal weight $\Delta$ is related to the real parameter $\rho$ via $i\rho = 1-\Delta$. The inverse Fourier transform becomes

$$h(\xi) = \frac{1}{8\pi}\int_{-\infty}^\infty F(\xi;\rho)d\rho\,. \tag{4.6.4}$$

Using equation (4.6.3) and (4.6.4) we get

$$f(x) = -\frac{1}{2(4\pi)^3}\int_{-\infty}^\infty \int F(\xi;\rho)\delta''\left(|[x,\xi]|-1\right)d\xi d\rho\,. \tag{4.6.5}$$

This can be written in the form

$$f(x) = \frac{1}{4(8\pi)^3}\int_{-\infty}^\infty d\rho\,\rho(\rho+4i)\int_\Gamma d\omega\,F(\xi;\rho)\,|[x,\xi]|^{\,-i\rho-1}\,, \tag{4.6.6}$$

where $d\omega$ is a measure on the boundary at infinity, obtained by modding out the overall scale from $d\xi$. The surface $\Gamma$ is an arbitrary surface on the light-cone that intersects each of its generators, and $d\omega$ is defined by $d\xi = d\omega dP$ where $P(\xi) = 1$ is the equation of $\Gamma$. Thus we get a function in the bulk by applying the inverse integral transform to functions on the boundary transforming with well-defined conformal weights. Finally, a symmetry of this integral relates the integral over $\rho$ from $-\infty, 0$ to the range $0, \infty$, allowing the range to be collapsed to one copy of each irreducible principal series representation $\rho = 0\cdots\infty$.

### 4.6.1 Bulk operators

Again we will need to generalize these methods to the distributions encountered in quantum field theory. Our goal is to reconstruct the bulk field, at the free level (4.5.7) using only the covariant boundary operators (4.5.10). For simplicity we assume only a single mass field with mass $m$ is present. Generalization to the quasi-free case, where a superposition of masses is present is straightforward. The inverse transform of $\mathcal{O}_\Delta$ , in the flat-slicing, is

$$\phi_-(\eta, y) = -\frac{(\Delta-1)^2}{2\pi^2}(\cot\pi\Delta + i)\int d^2z\left(\frac{\eta^2 - z^2}{\eta}\right)^{\Delta-2}\mathcal{O}_\Delta(z+y)\,.$$

The continuation of $\eta \to \eta - i\epsilon$ defines the branch of the integrand. Likewise we define

$$\phi_+(\eta, y) = \frac{(\Delta-1)^2}{2\pi^2}(\cot\pi\Delta - i)\int d^2z\left(\frac{\eta^2 - z^2}{\eta}\right)^{\Delta-2}\mathcal{O}^\dagger_\Delta(z+y)\,,$$

where now $\eta \to \eta + i\epsilon$. Inserting the expression (4.5.10) and performing the integrals, one recovers (4.5.7) with $\phi = \phi_+ + \phi_-$.

The same method may be used to reconstruct the bulk Wightman function in the Bunch-Davies/Euclidean vacuum

$$\begin{aligned}\langle\phi(\eta_1, y_1)\,\phi(\eta_2, y_2)\rangle &= -\frac{(\Delta-1)^4}{4\pi^4}\csc^2(\pi\Delta)\int d^2z_1\left(\frac{\eta_1^2 - z_1^2}{\eta_1}\right)^{\Delta-2}\int d^2z_2\left(\frac{\eta_2^2 - z_2^2}{\eta_2}\right)^{\Delta-2} \\ &\times \left\langle\mathcal{O}_\Delta(z_1+y_1)\mathcal{O}^\dagger_\Delta(z_2+y_2)\right\rangle\,,\end{aligned}$$

where on the right-hand-side a CFT correlator appears, while on the left, a bulk Wightman function appears. In this formula, it is understood that $\eta_1 \to \eta_1 - i\epsilon$ and $\eta_2 \to \eta_2 + i\epsilon$. Likewise the boundary radial directions must be continued in the same way, which regulates the singularity in the integrand when points coincide. We emphasize this reproduces the full bulk Wightman function for general points in the bulk of de Sitter (4.5.11).

This construction allows us to build field operators at arbitrary bulk points in de Sitter yielding important insight into how the de Sitter time arises from the purely Euclidean CFT. Likewise, the Euclidean CFT does not have a natural operator ordering. In the bulk, this arises from the complexification of the radial direction in the CFT, combined with the branch choices in the smearing functions. This allows us to build ingoing or outgoing modes in the bulk. For a bulk theory with some perturbative expansion, this approach is sufficient to reconstruct the bulk correlators from the boundary correlators, by reconstructing the Wick expansion of the bulk correlators, using the building blocks we have presented.

## 4.7 Euclidean axioms

For a well-defined set of bulk correlators, we can use the prescription of section 4.5 to define a conformally covariant set of boundary correlators. These then may be viewed as a definition of some Euclidean conformal field theory that includes quasi-primary operators corresponding to the principal series.

The basic axioms of Euclidean quantum field theory were formulated long-ago by Ostwerwalder and Schraeder. One of the most elementary axioms needed for a consistent Euclidean theory is that of cluster decomposition, namely

$$\lim_{r\to\infty} \langle \phi(r)\phi'(0)\rangle = \langle \phi(r)|0\rangle \langle 0|\phi'(0)\rangle \ ,$$

so that correlators factorize when groups of insertions are separated by long distance. This is the Euclidean analog of uniqueness of the vacuum state in Lorentzian signature. It is straightforward to see this can never be the case for a CFT that contains operators based on the principal series. Consider the CFT correlator

$$\left\langle \left(L_1\bar{L}_1\right)^n \mathcal{O}_\Delta(z) \left(L_1\bar{L}_1\right)^n \mathcal{O}^\dagger_\Delta \right\rangle \propto \frac{1}{|z|^{2\Delta-4n}} .$$

This grows with distance for $n > 0$, violating cluster decomposition. If instead one had a typical CFT, and $\mathcal{O}$ was a primary operator, one would have the identity $L_n\mathcal{O} = 0$ for $n > 1$, avoiding this problem.

We interpret the results of this paper as a proof by contradiction that nontrivial CFTs based on the principal series cannot exist. Nevertheless, this result has important implications for theories in the bulk. In analogy with AdS/CFT, we can interpret the operator $\left(L_1\bar{L}_1\right)\mathcal{O}_\Delta(z)$ as dual to a composite of a bulk graviton and a scalar matter field. This violation of cluster decomposition on the boundary arises because the bulk theory has no positive energy theorem [31]. The Killing vector associated with $L_0 + \bar{L}_0$ is not globally timelike. There are therefore many bulk excitations satisfying $L_0 = \bar{L}_0 = 0$ at the boundary, which will appear as intermediate states when one tries to factorize a CFT correlator.

We conclude then that the Euclidean CFT associated with a free massive scalar in de Sitter violates the basic axioms of Euclidean quantum field theory. We take this as a sign that the holographic dual is incomplete as a CFT, and we hope to return to a more constructive approach to building the correct holographic dual in future work.

# Chapter 5

# de Sitter gravity/Euclidean conformal gravity correspondence

There has been much success in describing gravity in Anti-de-Sitter spacetime using a holographic description on the boundary at infinity. In the holographic description, a conformal field theory (CFT) lives on the boundary. This marks a major step in quantizing gravity in asymptotically anti-de-Sitter spacetime. However according to experimental observation, our universe has positive cosmological constant. Thus it is interesting to consider a holographic description of gravity in the de-Sitter spacetime.

One way of approaching this problem is to analytically continue the AdS/CFT correspondence to dS/CFT correspondence [16]. There are many successes in this approach but there are many conceptual difficulties as well. In this paper we will try to clarify some of these difficulties. In the paper [22], we constructed a mapping between bulk field operators and boundary operators. As we saw in that paper, the boundary CFT has operators which violate cluster decomposition. Cluster decomposition is one of the basic assumptions of any interacting quantum field theory [29, 30].

To set the stage for understanding the problem in de Sitter spacetime, be begin by considering the well-understood problem in anti de Sitter spacetime. In that case, there is a positive energy theorem [31, 42] and the unitary representations of the conformal group $SO(d-1,2)$ that appear are lowest weight. Moreover the boundary conditions on conformal infinity $I$ that preserve conformal flatness are compatible with the unitarity bound of [42]. In particular, with these boundary conditions, one obtains a complete set of modes for fluctuations around the anti-de Sitter background.

For de Sitter spacetime there is no global positive energy theorem [31] and the unitary representations of the conformal group $SO(d,1)$ corresponding to ordinary massive and massless fields are neither highest nor lowest weight, but are rather the principal series and the complementary series, which are unbounded. This leads to the problem of cluster decomposition violation in the boundary theory, noted in [22]. In the case of de Sitter, a complete set of modes (for the graviton) leads to configurations with a nontrivial conformal class at conformal infinity $\mathcal{I}$. Thus one cannot impose boundary conditions to maintain conformal flatness, without truncating the linearized spectrum of the theory [43, 44]. Therefore to describe a quantum theory with the full set of modes in a de Sitter background, the holographic description must accommodate a path integral over boundary metrics. The boundary theory will be invariant under the asymptotic symmetry group of the de Sitter spacetime that preserves this more general set of asymptotic boundary conditions. In this case, the asymptotic symmetry group is not just the conformal group, corresponding to isometries of de Sitter, but is rather the full group of diffeomorphisms of $\mathcal{I}$. This leads us to conjecture the holographic dual will be a theory of conformal gravity theory living at $\mathcal{I}$.

At first sight, this might seem a step backward, since theories of conformal gravity seem difficult to quantize [45]. Nevertheless, there are examples where progress has been made. For three dimensional pure conformal gravity, a Chern-Simons gauge formulation is available [46]. For conformal gravity arising in string theory, a twistor string formulation has been found [47]. So there is hope that the rather different conformal gravity theories considered here can be successfully quantized.

Having found a path integral over conformal classes of metric on $\mathcal{I}$ is needed to provide a holographic description of gravity in de Sitter, it is then natural to ask whether one must include a sum over topologies of $\mathcal{I}$ as well. In the case of anti-de Sitter, this question was addressed in [48]. There it was found that if $\mathcal{I}$ has positive curvature, it must be connected and cannot contain nontrivial topology, such as wormholes. This result is important for the basic consistency of AdS/CFT.

Some related questions have been considered in the context of dS/CFT in [49, 50]. However there it quickly becomes clear that ordinary matter will lead to nontrivial topology for $\mathcal{I}$ in four-dimensional de Sitter since a black hole already changes the topology from $S^3$ for empty de Sitter to $S^2 \times R$ for a black hole. Recall in AdS, the topology of $\mathcal{I}$ remains $S^2 \times R$ for empty AdS, or the AdS Schwarzschild black hole.

One can gain a more detailed understanding of this topology change in the case of three-dimensional de Sitter. As an example, we consider the solution for multi-black holes in three-dimensional de Sitter spacetime [51]. We show $\mathcal{I}$ can be mapped from a multi-sheeted sphere to a single cover with punctures. The resulting holographic dual is a theory of two-dimensional gravity, identical to a worldsheet string theory. At least in this example, there is a natural moduli space corresponding to a sum of worldsheet topologies. It remains an interesting open question whether such a sum over topologies can be defined in the higher dimensional case.

## 5.1 Asymptotic symmetry group

To specify the asymptotic structure of a spacetime we attempt to construct a set of boundary conditions that capture a wide-class of physically interesting solutions. The Penrose conformal compactification of the geometry provides an enormous simplification in treating these asymptotic boundary conditions, because solutions may more easily be studied on the compact unphysical spacetime (related by a Weyl transformation to the physical spacetime) where the group of diffeomorphisms is clearly defined [52].

If one considers linearized perturbations around de Sitter, the conformal group should have a well-defined action. In this limit, one can consider the perturbation on top of the fixed de Sitter background, which has as an isometry group $SO(d,1)$. These isometries induce a $SO(d,1)$ global conformal transformation on $I$.

### 5.1.1 Four dimensions

At first sight, the situation for nonlinear solutions appears much less clear. We will restrict our discussion to four-dimensional de Sitter, and discuss the very special features of three dimensions later. As mentioned in the introduction, already black holes will tend to change the very topology of $I$ and it is not clear if any precise asymptotic conditions can be formulated. Ashtekar et al. [43, 44] deal with this by focussing on isolated gravitating systems in de Sitter. Our approach will take a different viewpoint, and allow for arbitrary boundary metrics that respect the asymptotic de Sitter metric conditions locally

$$ds^2 = R_{dS}^2 \eta^{-2} \left(-d\eta^2 + (\delta_{ij} + h_{ij}) dx^i dx^j\right) \tag{5.1.1}$$

where we can perform a power series expansion of $h_{ij}$ as

$$h_{ij}(\eta, x) = h_{(0)ij}(x) + \eta^2 h_{(2)ij}(x) + \eta^3 h_{(3)ij}(x) + \mathcal{O}(\eta^4) \tag{5.1.2}$$

following [53]. For now we will take $I$ to have topology of the 3-sphere, thus we are considering globally asymptotically de Sitter spacetimes. We will consider more general topologies later in the paper. As we will see later, typical matter configurations only yield a single regular asymptotic region, so we take $I$ to refer to either $I^+$ or $I^-$

but not a disconnected union of the two.

One can largely separate the issue of topology change by first restricting considerations to theories of gravity with conformally coupled matter. There powerful nonlinear stability theorems have been proven by Friedrich [35]. In particular, for an open set of initial data, it has been shown that a past asymptotic de Sitter spacetime can smoothly evolve to a future asymptotic de Sitter spacetime. The solutions obtained involve a metric at $I$ in a nontrivial conformal class. These correspond to the usual long wavelength gravitons of the theory of inflation, which freeze out when stretched past the horizon scale. They induce a nontrivial Cotton tensor on $I$.

As pointed out in [43, 44], demanding conformal flatness of the boundary projects out these graviton modes from de Sitter. Therefore if the holographic theory of de Sitter gravity was simply a conformal field theory, living on a background with a fixed conformal structure, the CFT would not be able to reproduce the full set of graviton modes. One may of course perturbatively correct for this by introducing sources on the boundary, however then one must specify a path integral measure for such sources in order to reproduce bulk observables, such as in-in correlators.

Let us try to establish the gauge symmetries of the boundary theory. If we consider general asymptotic boundary conditions of the form (5.1.1) the asymptotic symmetry group is much larger than the global conformal group. Instead, it consists of the full group of diffeomorphisms of $I$. As we will see later, we can reconstruct part of the action of the holographic dual by considering the boundary action of the bulk theory, evaluated on solutions of the equation of motion. This boundary action then inherits the gauge symmetry of the bulk, associated with diffeomorphisms of $I$.

The construction of the boundary theory is predicated on the Penrose compactification of the bulk spacetime. This is achieved by performing a general Weyl transformation of the bulk metric $g_{(unphys)\mu,\nu} = \Omega^2(\eta, x) g_{(phys)\mu,\nu}$ for some choice of smooth function $\Omega$ that vanishes on $I$, but with non-vanishing normal derivative. Again, by reconstructing part of the action of the holographic dual involving the boundary metric, one sees the boundary theory must inherit this Weyl invariance as a gauge symmetry. We conclude then that the boundary theory must be a theory of Euclidean conformal gravity.

In many ways, this is not a new statement. It has been advocated that the dS/CFT correspondence be viewed as a computation of a wavefunction via a CFT partition function

$$\Psi(h) = Z_{CFT}[h] \tag{5.1.3}$$

where $h$ denotes the boundary metric. Our point is simply to compute bulk observables, one must make the further step of computing

$$\langle 0|O(x_1)O(x_2)|0\rangle = \int \mathcal{D}h\Psi^*(h)O_{CFT}(x_1)O_{CFT}(x_2)\Psi(h) = \int \mathcal{D}h\, Z^*_{CFT}[h]O_{CFT}(x_1)O_{CFT}(x_2)Z_{CFT}[$$

with some a priori unknown measure $\mathcal{D}h$, and some de Sitter spacetime operators $O$. Here the vacuum state $|0\rangle$ is to be understood as an interacting generalization of the Bunch-Davies vacuum. The operators $O_{CFT}$ are the dual CFT operators. For matter fields in a fixed de Sitter background, these can be constructed [22]. To formulate a complete holographic description, one instead must build the integration measure into the theory. This gives rise to our conjecture that dS gravity is dual to a theory of conformal gravity on $I$. In that case, the relevant correlator would be

$$\langle 0|O(x_1)O(x_2)|0\rangle = \langle O_{cgrav}(x_1)O_{cgrav}(x_2)\rangle \tag{5.1.4}$$

where the left-hand side is an in-in correlator in the bulk theory, and the right-hand side represents the map of these observables into the conformal gravity theory. The next goal is to try to specify as much as possible, this conformal gravity theory. If this can be established, it will then be necessary to revisit the boundary to bulk operator mapping after properly understanding the gauge invariant observables of the conformal gravity theory. In its current formulation [22], the mapping would only make sense for small perturbations around some classical background.

### 5.1.2 Quadratic action for holographic theory: 4d de Sitter

In general to build operators in the boundary theory from those in the bulk, one must use the integral transform method described in [22], and its generalizations. This can be viewed as an analog of the LSZ transform in constructing the S-matrix in asymptotically flat spacetime.

In anti-de Sitter spacetime, one has a much easier task, because the bulk to boundary mapping is much simpler, since the physical fields of interest have simple power law falloff, dependent on their masses. So while one must perform an integral transform to construct quasi-local bulk fields from boundary operators, the inverse operation reduces to taking a residue in the limit that the bulk operator approaches infinity.

Nevertheless, if we focus on the gravitational field, and massless minimally coupled scalars, for example, the results of AdS may be continued to de Sitter. This is the approach followed in [54, 55]. See also [6] for related discussion of these issues. Here let us generalize this to a massive scalar in de Sitter, with action

$$S_{mat} = \int d\eta d^3x \, \frac{1}{2} \sqrt{-\det g} \left( -g^{\mu\nu}\partial_\mu \phi \partial_\nu \phi - m^2 \phi^2 - \xi R \phi^2 \right) . \tag{5.1.5}$$

In empty de Sitter, with metric (5.1.1) the solution of the equation of motion may be decomposed into the Bunch-Davies [24, 56] mode functions

$$u_k = \frac{1}{2^{5/2}\pi} \eta^{3/2} H_\mu^{(2)}(-k\eta) e^{ik\cdot x}$$

where

$$\mu^2 = \frac{9}{4} - 12\left(\frac{m^2}{R} + \xi\right).$$

Let us for the moment take $\mu$ to be real, corresponding to the so-called complementary series representations of the conformal group. Note we will work with the future half of the slicing, so $-\infty < \eta < 0$. We wish to compute the on-shell action, which reduces to a boundary term as $\eta \to 0^-$. We take a solution with some fixed behavior on some late-time slice $\eta = \eta_c$

$$\phi(\eta, x) = \frac{\eta^{3/2} H_\mu^{(2)}(-k\eta)}{\eta_c^{3/2} H_\mu^{(2)}(-k\eta_c)} f_{\vec{k}} e^{i\vec{k}\cdot\vec{x}}$$

and substitute into (5.1.5) to obtain

$$\begin{aligned} iS_{mat} &= iR_{dS}^2 \int d^3x \frac{1}{2\eta^2}\phi\partial_\eta\phi|_{\eta=\eta_c} \\ &= iR_{dS}^2 \int \frac{d^3k}{2(2\pi)^3} f_{\vec{k}} f_{-\vec{k}} \frac{1}{4\eta_c^3}\left(3 - 2\mu - \frac{2\eta_c k H_{\mu-1}^{(2)}(-k\eta_c)}{H_\mu^{(2)}(-k\eta_c)}\right). \qquad (5.1.6) \end{aligned}$$

When $\mu$ is half-integer, this expression may be expanded near $\eta \to 0^-$ (i.e. $I^+$) and interpreted as a series of counter-terms that must be subtracted to yield a finite boundary action. For example, the massless minimally coupled scalar corresponds to $m = 0, \xi = 0$ giving $\mu = 3/2$ and

$$iS_{mat} = R_{dS}^2 \int \frac{d^3k}{2(2\pi)^3} f_{\vec{k}} f_{-\vec{k}} \left(\frac{ik^2}{2\eta_c} - \frac{k^3}{2}\right)$$

as $\eta_c \to 0$. The imaginary divergent term might then be subtracted with a local $\int d^3x(\partial\phi)^2$ counterterm. The finite piece yields the expected boundary propagator of a quasi-primary field with conformal weight $\Delta = 3 = \frac{3}{2} + \mu$. The boundary action for the scalar field then has the following form, which is non-analytic in momenta

$$S_{boundary} = R_{dS}^2 \int d^3x \frac{1}{2}\phi\,(\Box)^{3/2}\,\phi\,.$$

Another simple example is the massless conformally coupled scalar, with $m = 0, \xi = 1/6$ which gives $\mu = 1/2$ and

$$iS_{mat} = R_{dS}^2 \int \frac{d^3k}{2\,(2\pi)^3} f_{\vec{k}} f_{-\vec{k}} \left( \frac{i}{2\eta_c^3} - \frac{k}{2\eta_c^2} \right)$$

which has a vanishing finite boundary action after subtracting the divergent counterterms. We will comment on this and the case of more general mass in a moment.

In a transverse traceless gauge, the action for metric fluctuations matches that of the massless minimally coupled scalar with a different normalization, giving the boundary action

$$S_{grav,boundary} = \frac{R_{dS}^2}{64\pi G} \int d^3x \, h_{ij}^{TT} \, (\Box)^{3/2} \, h_{TT}^{ij} \, .$$

As noted in [54] this gives a negative contribution to the 2-point function of the boundary stress energy tensor proportional to the central charge.

So far we have seen the boundary counter-term approach seems to work well for the metric and massless minimally coupled scalar matter. As noted in [22] this approach of extracting boundary operators for more general matter in de Sitter, by simply taking asymptotic limits of the fields, fails in general. If one were to evaluate (5.1.6) one would get oscillating cutoff (i.e. $\eta_c$) dependent expressions[1]. The correct approach is to apply an analog of the LSZ reduction formula of asymptotically flat spacetime, by performing an integral transform on the bulk fields to obtain a boundary expression that transforms covariantly under the conformal group [22]. For scalar fields, this gives

$$S_{matter,boundary} = R_{dS}^2 \int d^3x \frac{1}{2} \phi \, (\Box)^{\mu} \, \phi$$

[1]These oscillations play an important role in the minisuperspace approach to solving the Wheeler-DeWitt equation [57]. Our philosophy in this section is to use the form of the square of this wavefunction to guess the form of the conformal gravity action living on $I$. The duality proposed in (5.1.4) then makes no reference to the phase of this wavefunction.

for $\mu > 0$ real, corresponding to the complementary series of the conformal group and

$$S_{matter,boundary} = R_{dS}^2 \int d^3x \frac{1}{2} \left( \phi \left( \Box \right)^{\mu} \phi + \phi^* \left( \Box \right)^{-\mu} \phi^* \right)$$

for $\mu$ imaginary, corresponding to the principal series of the conformal group.

In the above, we have derived the quadratic terms that appear in the conformal gravity theory coupled to matter. Already we see the conformal gravity theory seems to be of a new kind, due to the non-polynomial nature of its derivatives appearing in the quadratic term. The theory appears to be free of ghosts, at least in the case when the matter is restricted so $\mu > 0$. At least at bulk tree-level, one should be able to recover the higher order terms in the holographic Lagrangian. The classical stability of de Sitter would seem to indicate this procedure should be completely well-defined.

An easy generalization of the above is to break parity in the bulk by adding an $\int RR$ term, which corresponds to adding a Chern-Simons gravity term to the boundary. This yields the boundary Lagrangian for the well-studied case of topologically massive gravity [58], which is invariant under Weyl and diffeomorphism symmetries.

As has been emphasized in [22] the boundary theory violates cluster decomposition, which is one of the central axioms of Euclidean quantum field theory [29, 30]. It will be very interesting to construct interacting holographic duals. In the free limit, examples have been constructed in the context of higher spin gravity in de Sitter [23].

### 5.1.3 Three dimensions

The case of three-dimensional de Sitter is special, because then $I$ is two-dimensional, and always locally conformally flat. In this case, the expansion of the metric (5.1.1) takes the form

$$h_{ij} = h_{(0)ij} + \eta^2 h_{(2)ij} + \mathcal{O}(\eta^3)\,.$$

We can nevertheless follow the strategy described above to compute the boundary term arising from the on-shell bulk action. Now we will find the boundary counter-term action

$$iS_{grav} = \frac{i}{16\pi G_3} \int d^2x 2\sqrt{-\det g} - \frac{1}{2} \log\left(-\eta_c\right) \sqrt{\det h_{(0)}} R_{(0)} \,.$$

The anomalous contribution proportional to $\log \eta_c$ must be cancelled for the theory to be conformally invariant. One way to approach the problem is to couple the boundary theory to a Liouville field theory with central charge adjusted so that a Weyl transformation, shifting $\eta_c \to \alpha\eta_c$ is compensated by the anomaly term coming from the Liouville theory. This renders the boundary theory diffeomorphism invariant and Weyl invariant.

In this way, the boundary theory takes the form of the Polyakov string. The central charge induced by the gravitational contribution to the conformal anomaly is

$$c = -\frac{3R_{dS}}{2G_3} \,. \tag{5.1.7}$$

For the theory to be Weyl invariant at the quantum level, this central charge must be cancelled by that of the Liouville field, leading to a boundary theory with vanishing conformal anomaly.

In the usual conformal gauge of string theory, for fixed boundary topology, the theory reduces to an ordinary conformal field theory (coupled to the Liouville field) and the details of conformal gravity may be forgotten. Moreover in string theory there is a well-defined path integral involving sums over nontrivial worldsheet topologies. Each topology is equipped with a well-defined moduli space. We expect this sum over topologies is important to properly understand the holographic theory describing quantum gravity in de Sitter, a question we turn to in the next section.

## 5.2 Topology change

It is important for the consistency of AdS/CFT that there are strong restrictions on the topology of the bulk geometry $M$ given the boundary. For example, Witten and Yau [48] showed that the boundary must be connected, and that the bulk Euclidean geometry satisfies $H_n(M, \mathbb{Z}) = 0$ if the boundary has positive scalar curvature.

Similar topological restrictions have been explored in the context of four-dimensional asymptotically de Sitter spacetimes in [49, 50]. For example, if $I^+$ has infinite fundamental group, then one has the rather strong result if matter obeys the null energy condition, there is no regular $I^-$. Similarly if $I^+$ has positive first Betti number, then the bulk is past null geodesically incomplete. Nevertheless, there are many examples where at least $I^+$ is well-defined. The case we will be most interested in is the case where $I^+$ is a sphere with punctures. Isolated gravitating systems in de Sitter can reach $I^+$ where they appear as punctures. In the work of [43, 44] the focus is on a single isolated gravitating system. Since here we are interested in building a holographic dual applicable to cosmology, we will be most interested in is the case where $I^+$ is a sphere with multiple punctures.

If we wish to accommodate such isolated gravitating systems in the dual conformal gravity theory living on $I^+$, we must therefore include a sum over topologies of the boundary. In the case of three-dimensional Euclidean geometries, it is not clear whether a path integral of conformal gravity over such a space can be defined. Though it nevertheless appears to be a simpler problem than the original proposals for four-dimensional Euclidean quantum gravity as a path integral over geometries.

In the case of three-dimensional asymptotically de Sitter geometries things are much simpler. Again, $I^+$ is always conformally flat, but one nevertheless must deal with this sum over topologies. The sum over the moduli space of compact Riemann surfaces (including punctures), is well-understood in the context of string theory and leads to a complete proposal for the path integral of the conformal gravity theory.

That is, if we are given a Lagrangian for a CFT with central charge (5.1.7), we can couple it to conformal gravity by performing a Weyl rescaling, and add in the Liouville sector to cancel the overall conformal anomaly. One can then fix conformal gauge, and treat the theory as one would with any worldsheet string theory.

In the remainder of this section, we consider an example of a multi-black hole solution in three-dimensional asymptotically de Sitter spacetime [51]. If the above proposal is correct, it should be possible to view $I^+$ as a 2-sphere with punctures. However the original work [51] expressed the Cauchy slices as a multiple cover of a sphere with only two punctures at the north and south poles. In the following, we construct the covering space and show it is a single cover of a sphere with multiple non-degenerate punctures.

### 5.2.1 Example: multi-black hole solution in $dS_3$

Deser and Jackiw have found the metric of 2+1 dimensional gravity asymptotically de-Sitter spacetime [51] in the presence of $N$ stationary massive particles. It is given by

$$\begin{aligned}
ds^2 &= M^2(r)dt^2 + f(r)dzdz^* \\
f(z) &= \frac{\epsilon}{\lambda V(z)V^*(z^*)\cosh^2\left(\sqrt{\epsilon}\,(\zeta-\zeta_0)\right)} \\
M(z) &= \epsilon\tanh\left(\sqrt{\epsilon}\,(\zeta-\zeta_0)\right) \\
V(z) &= c^{-1}\prod_{n=1}^{N}(z-z_n) \\
\zeta(z) &= \frac{1}{2}\left(\int\frac{dz}{V(z)}+\int\frac{dz^*}{V^*(z^*)}\right) = \ln\left(\prod_n |z-z_n|^{c_n}\right) \\
c_n &= \prod_{n'\neq n}\frac{c}{z_n-z_{n'}} \\
\sum_{n=1}^{N} c_n &= 0\,.
\end{aligned} \tag{5.2.1}$$

Here $\lambda > 0$ is the cosmological constant. The first equation gives the metric in complex plane in terms of $f(z), M(z)$. $V(z)$ is the master function in terms of which the solution is given. $z_n$ are the punctures in the complex plane where particles are inserted and $c$ is a free parameter. We demand that $c_n$ be real for single valuedness of the solution. The coordinate transformation

$$\begin{aligned}\sin\omega &= \frac{1}{\cosh\left(\sqrt{\epsilon}(\zeta-\zeta_0)\right)} \\ \phi &= \frac{\epsilon}{2i}\left(\int\frac{dz}{V(z)}-\int\frac{dz^*}{V^*(z^*)}\right)\end{aligned} \tag{5.2.2}$$

takes us to the familiar static coordinates

$$ds^2 = -\cos^2\omega dt^2 + \lambda^{-1}\left(d\omega^2+\sin^2\omega d\phi^2\right) . \tag{5.2.3}$$

Note that all the particles are located at $\sin\omega = 0$ so that $\omega = 0, \pi$. The further coordinate change $\sqrt{\lambda}R = \sin\omega$ takes us to the static Schwarzschild-de-Sitter coordinates

$$ds^2 = -(1-\lambda R^2)dt^2 + (1-\lambda R^2)^{-1}dR^2 + R^2 d\phi^2$$

which covers the full space, but the range of $\phi$ goes from $[0, 2\pi\alpha_n)$ at the location of $n^{th}$ particle where $\alpha_n = \sqrt{\epsilon}c_n = 1-4Gm_n$. This is the familiar conical deficit of 3-dimensional gravity. Locally the metric is same as pure de-Sitter and has constant curvature.

Now we will investigate the geometry. First we will consider 3-particle case before generalizing to the $N$ particle case. Uniqueness of the solution requires that the 3 particles are all in a line and $c_n$ sum to zero. Let us take $c = 1, z_1 = -3, z_2 = 1, z_3 = 2$.

Then let us choose

$$\begin{aligned}
c_1 &= \frac{1}{(z_1 - z_2)(z_1 - z_3)} = \frac{1}{20} \\
c_2 &= -\frac{1}{4} \\
c_3 &= \frac{1}{5} \\
c_1 + c_2 + c_2 &= 0\,.
\end{aligned}$$

Thus this configuration satisfies all the constraints. Now let us look at the functions that determine the geometry

$$\begin{aligned}
V(z) &= (z+3)(z-1)(z-2) \\
\frac{1}{V(z)} &= \frac{1}{20(z+3)} - \frac{1}{4(z-1)} + \frac{1}{5(z-2)} \\
\zeta &= \frac{1}{2}\left(\int \frac{dz}{V(z)} + \int \frac{dz^*}{V^*(z^*)}\right) = \frac{1}{20}\ln\left(\frac{|z+3||z-2|^4}{|z-1|^5}\right)\,.
\end{aligned}$$

At $z = -3, 2$, $\zeta = -\infty$ and at $z = 1$, $\zeta = \infty$. In $z$ coordinates, we have punctures at 3 points. We now want to understand the picture in the $\omega, \phi$ coordinates using (5.2.2)..

The points $z = -3, 1, 2$ correspond to $\sin\omega = 0 \implies \omega = 0, \pi$. Thus two of the particles are at south pole and one at north pole. But then it is not immediately clear whether the particles at the south pole are overlapping or they are multiple disconnected sheets or they are sphere connected at some points etc. To understand the topology, we first note that we can have a path between any two particles without crossing the other particle. This implies that the sheets are connected. Secondly, the distance between any two particles is non-zero.

To see this we note that constant $\omega$ corresponds to constant $\frac{|z+3||z-2|^4}{|z-1|^5}$ curves in the complex plane.

1. $\frac{|z+3||z-2|^4}{|z-1|^5} = \pm\infty$ would correspond to north and south pole $\omega = 0, \pi$.

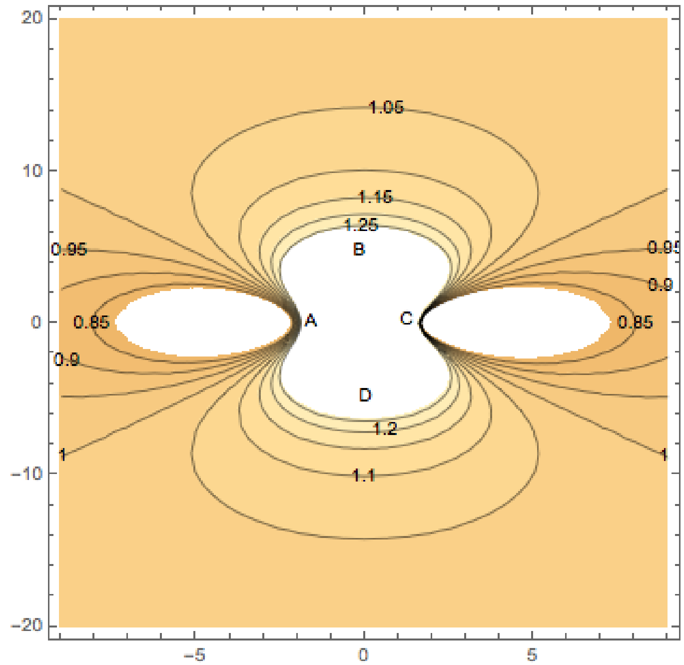


Figure 5.2.1: Contour plot of $e^{20\zeta} = \frac{|z+3||z-2|^4}{|z-1|^5}$ in the complex $z$-plane.

2. $\frac{|z+3||z-2|^4}{|z-1|^5} = 1$ corresponds to the equator $\omega = \pi/2$.

First we note that at $z = -3, 2$, $\frac{|z+3||z-2|^4}{|z-1|^5} = 0 < 1$ and at $z = 1$, $\frac{|z+3||z-2|^4}{|z-1|^5} = \infty > 1$. Thus we are sure that $\frac{|z+3||z-2|^4}{|z-1|^5} = 1$ contour will pass between ($-3$ and 1) and also between (1 and 2). We can verify it by plotting the contours as shown in Figure (5.2.1).

The plot clearly shows that from $z = -3$ to $z = 2$ we have to cross $\frac{|z+3||z-2|^4}{|z-1|^5} = 1$ contour at least twice. That is we have to cross equator at least twice. To go from $z = -3, 2$ to $z = 1$ we have to cross $\frac{|z+3||z-2|^4}{|z-1|^5} = 1$ contour or the equator at least once. Thus we are getting a picture where we have two spheres. The south pole of one sphere corresponds to $z = -3$ and the south pole of other sphere correspond to $z = 2$. The north pole of both the spheres correspond to $z = 1$. This means the two spheres have common northern hemispheres ($\zeta > 0$) and separate southern hemispheres ($\zeta < 0$).

How does this all look in the $\omega, \phi$ coordinate? First let us look at the contours

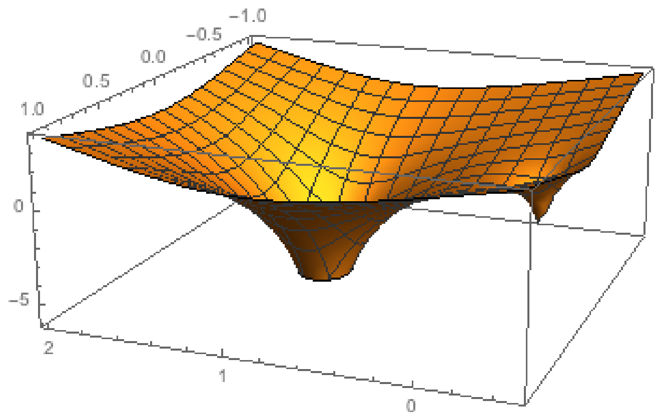


Figure 5.2.2: 3D plot of $e^{20\zeta}$ after the transformation $z \to \frac{1}{z'} + 1$ in z' plane. This shows that the topology of boundary is a pair of pants.

for $\zeta > 0 \implies \omega < \pi/2$. These curves are connected and close around $z = 1$. As we move along each contour $\phi$ ranges from $[0, \alpha_1 2\pi)$ where $1 - \alpha_1$ is the conical deficit of the particle at $z = 1$. For convenience let us mark the point $A$ as $\phi = 0$. Then as we move along the curve we reach $B(\phi = \alpha_1\pi/2), C(\phi = \alpha_1\pi), D(\phi = 3\alpha_1\pi/2)$ and when we come back to $A$, $\phi$ changes by $\alpha_1 2\pi$. These points are shown in figure (5.2.1). This is true for all the contours $\omega < \pi/2$. For $\omega = \pi/2$, contour splits at $C, D \to \infty$. Topologically one then has a sphere with 3 punctures, also know as the pants diagram. To see that explicitly, we do the transformation $z \to \frac{1}{z'} + 1$. This sends $z = 1, \infty$ to $z' = \infty, 0$ respectively. The new function that determines the geometry is shown in Figure (5.2.2).

Generalizing to $N$ particles this will be $N$ punctures on the Riemann sphere. Single valuedness of the solution requires that all the punctures (position of the particles) lie on a line. Without loss of generality we can take this line to be real axis. Mass defects are given by absolute value of the $c_n$, which should sum to 0. We can choose $c = 1$ in the solution (5.2.1). and take the positions $z_n$ such that they satisfy $\sum c_n = 0$. Then the solution is given by (5.2.1). Let us label the positions such that $z_1 > z_2 > ... >$

$z_{N-1} > z_N$. Then

$$\begin{aligned} c_1 &= \frac{1}{(z_1 - z_2)(z_1 - z_3)...(z_1 - z_N)} > 0 \\ c_2 &= \frac{1}{(z_2 - z_1)(z_2 - z_3)...(z_2 - z_N)} < 0 \\ c_3 &> 0 \\ &... \end{aligned}$$

$c_{2m+1} > 0, c_{2m} < 0$. Thus we see from equation (5.2.1) that $\zeta(z_{2m+1}) = -\infty, \zeta(z_{2m}) = \infty$. This solution in $z$ coordinate is transformed to the de-Sitter like metric (5.2.3) using (5.2.2). We see that at $z = z_n \implies \omega = 0, \pi$. That is particles are either at the south or north pole, corresponding to a multi-sheeted sphere with 2 punctures.

To see the geometry more clearly we look at the equator that is $\omega = \pi/2 \implies \sin\omega = 1 \implies \zeta(z) = \zeta_0$ contours. Let us check these contours generate the Riemann sphere with $N$ punctures.

Since this is a compact manifold, all the contours must be closed. The punctures live at $|\zeta(z_n)| = \infty$. So none of the contours with finite value of $\zeta_0$, end at the punctures. The second observation is that the $\zeta(z) = 0$ contour continuously extends to $z = \infty$. Thus all the contours with $\zeta(z) = 0$ are connected at $z = \infty$. A third observation is that $\zeta(z_{2m+1}) = -\infty, \zeta(z_{2m}) = \infty$. That is $\zeta(z)$ at successive punctures are of opposite sign. Thus, $\zeta(z) = 0$ contour separates any two successive punctures. Thus there are $N-1$ $\zeta(z) = 0$ contours joined at $z = \infty$. These contours divide the Riemann sphere into $N$ segments. Each segment contains exactly one puncture, and we have mapped the geometry to a single-cover of the $N$-punctured sphere.

## 5.3 Conclusion

We have conjectured the holographic dual of an asymptotically de Sitter spacetime in $d+1$-dimensions is a $d$-dimensional theory of Euclidean conformal gravity living on $I$. Various quadratic terms in the action of the conformal gravity have been constructed, which indicate the boundary metric becomes a dynamical variable. This then forces one to consider whether the path integral over the boundary metric includes a sum over topologies.

This is a sharp departure from the simplicity of the conformal field theory/anti-de Sitter correspondence, where we have many examples of suitable large $N$ conformal field theories and the boundary metric is not dynamical. In the case of de Sitter, we instead get holographic theories that violate the usual axioms of Euclidean field theory [22] and examples are hard to come by. The massless higher spin theories have provided some examples where these issues can be explored in detail [23, 59]. Optimistically one might hope that the new feature of coupling to conformal gravity solves some of these problems. More pessimistically it suggests that the natural UV completion of de Sitter gravity may not be some lower dimensional holographic theory, but is rather to be understood as an unstable background in some larger complete theory [60, 61].

# Chapter 6

# dS/CFT and the operator product expansion

The success of the anti-de Sitter/conformal field theory correspondence (AdS/CFT) has inspired applications to de Sitter spacetime (dS) [16]. This leads one to try to find conformal field theories of relevance to this correspondence which appear to exhibit novel properties, and many have questioned whether such theories can be defined at all.

In the context of AdS/CFT detailed dictionaries relating the bulk and boundary variables [1] were found at the free level. These ideas were generalized to a precision boundary/bulk correspondence, order by order in a $1/N$ expansion in HKLL [9, 10, 60, 62]. Our present goal is to attempt to extend such ideas to the dS/CFT correspondence.

However, it has been difficult to produce examples in Minkowski space and De-Sitter space because of various issues. Instability of string theory in de-Sitter background [63], and the compact spacelike boundary has made holography challenging in de-Sitter [15]. We have discussed many of these problems, in our previous papers [22, 64]. There we have shown how to extend the HKLL dictionary to dS space for the case of non-interacting bulk theory. Many people have contributed in understanding dS-holography including higher-spin holography for dS[16, 23, 59, 65, 66, 6, 37, 41, 36, 32, 67].

In section 6.1 we introduce principal series and discrete series representation and give a short description of the earlier work in maths literature. Then we show how the generators act on the bulk fields. We derive the bulk fields by solving the appropriate wave equation. Section 6.2, is devoted to the massless scalar field. We find that modes of the massless scalar include both the discrete series and a limit of the complementary series, which is an indecomposable representation of the conformal group. This work makes contact with recent work by Ashtekar et al. [43, 44] on the asymptotic boundary conditions in de Sitter spacetime. In particular the discrete series modes carry vanishing energy, while the indecomposable mode can carry energy, but changes the conformal structure of the boundary. Both sets of operators are needed in the CFT to reproduce a complete set of bulk modes.

It is the main goal of the present paper to construct the operator product expansion in the conformal field theory for operators dual to massive modes in the bulk. As is usual in conformal field theory, the two and three-point functions of quasi-primary operators are determined by conformal invariance. However when we explore the implications of this for the operator product expansion, some surprising results emerge, including the fact that the expansion involves terms with arbitrarily rapid short distance singularities determined by a seemingly infinite number of free parameters. This is in constrast to the more ordinary CFTs appearing in the AdS/CFT correspondence, where the most singular terms in the operator product expansion are determined by the weights of the operators, and conformal invariance implies a single parameter determines the full set of descendent couplings via conformal partial waves. This leads us to conclude that such conformal field theories do not exist in the space of ordinary renormalizable quantum field theories, but rather share many of the features of non-renormalizable field theories. For concreteness, many of our results are stated for three-dimensional de Sitter spacetime. However since we only use the global conformal group, the results are easily generalized to higher dimensions.

## 6.1 Principal and Discrete series representations of bulk states.

The isometries of 3-dimensional dS form the group $SO(1,3)$. This spacetime may be viewed as a hyperboloid embedded in 4-dimensional Minkowski spacetime. The generators are given by $J_i, K_i$ for $i = 1, 2, 3$. $J_i$ are the generators of rotation mixing three spacelike embedding dimensions. $K_i$ are the generators of boost mixing three spacelike dimension with the timelike dimension. There are various Cartan sub-algebras of $SO(1,3)$. Depending on which Cartan subgroup we choose, we get a different basis for the representations. One can choose $SO(3) = \{J_i\}$ as the Cartan subgroup. Most papers in 1950-70 by Naimark, Tagirov, Chernikov, Raczka et al[24, 25, 68, 69, 70] do that. So mode functions were labelled by quantum numbers $l, m$ (Eigenvalue of $\{J^2, J_3\}$ respectively). $SO(3)$ (compact group) has only finite dimensional representations $l = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, ..., m = -l, -l+1, ..., l$. So range of $m$ is bounded for a given $l$.

On the CFT side, states are usually chosen as eigenstates of the $SU(1,1)$ Cartan sub-group. So it is useful to write the bulk generators $SO(1,3)$ as $SU_L(1,1) \otimes SU_R(1,1)$. (Just like $SL(2,C) \cong SU(1,1) \otimes SU(1,1)$.) Combine the generators in the following way

$$
\begin{aligned}
K_{1L} &= \frac{1}{2}\left(-K_1 + iJ_1\right) & K_{2L} &= \tfrac{1}{2}\left(-K_2 + iJ_2\right) & J_L &= \frac{1}{2}\left(J_3 + iK_3\right) \\
K_{1R} &= \frac{1}{2}\left(K_1 + iJ_1\right) & K_{2R} &= \tfrac{1}{2}\left(K_2 + iJ_2\right) & J_R &= \frac{1}{2}\left(J_3 - iK_3\right) .
\end{aligned}
$$

Then

$$
\begin{aligned}
[J_{L(R)}, K_{1L(R)}] &= iK_{2L(R)} \\
[J_{L(R)}, K_{2L(R)}] &= -iK_{1L(R)} \\
[K_{1L(R)}, K_{2L(R)}] &= -iJ_{L(R)} \,.
\end{aligned}
$$

Left and right sectors commute. We can also form the raising and lowering operators $K_{\pm L(R)} = K_{1L(R)} \pm iK_{2L(R)}$.

$$
[J_R, K_{\pm R}] = \pm K_{\pm R} \qquad\qquad [J_L, K_{\pm L}] = \pm K_{\pm L} \,.
$$

Thus $\{J_{L(R)}, K_{\pm L(R)}\}$ form $SU_{L(R)}(1,1)$ group.

Now let us discuss unitary irreducible representations of $SU(1,1)$. States are labelled by eigenvalues of $\{C_L = J_L^2 - K_{1L}^2 - K_{2L}^2, J_L, J_R\}$

$$
\begin{aligned}
C_L|h,l\rangle &= h(h-1)|h,l\rangle \\
J_L|h,l\rangle &= l|h,l\rangle \,.
\end{aligned}
$$

Irreducible representations split into discrete series and continuous series (principal and complementary series) [68, 69, 71, 72, 73, 74, 75, 76, 77]. In the discrete series $h = n/2, n \in N$. $l = h, h+1, ...$ for positive discrete series $D^+$(lowest weight) and $l = -h, -h-1, ...$ for negative discrete series $D^-$(highest weight). For continuous series $h = \frac{1}{2} - i\rho$, $0 < \rho < \infty$ and $l = 0, \pm 1, \pm 2, ...$ or $l = \pm\frac{1}{2}, \pm\frac{3}{2}, ...$ corresponding to $C_\rho^0$ or $C_\rho^{1/2}$ respectively.

Similarly, $SU_R(1,1)$ sector can be constructed. For scalar fields $h_L = h_R = h$.

Casimir of $SO(1,3)$ is then given by

$$\begin{aligned} C &= C_L + C_R \\ &= 2h(h-1)\,. \end{aligned}$$

For the discrete series

$$C = -\frac{1}{2}n(2-n),\ n \in N\,.$$

For the continuous series

$$C = 2\rho^2 - \frac{1}{2},\ 0 < \rho < \infty\,.$$

As we will see, some modes of the massless scalar correspond to the $n = 2$ discrete series. There $l = \pm1, \pm2, ...$ for $D^{\pm}$ respectively.

### 6.1.1 Action of the generators on the states

Let us write below action of all the generators on the state $|h, l, r\rangle$

$$J_R|h,l,r\rangle = r|h,l,r\rangle \tag{6.1.1}$$

$$J_L|h,l,r\rangle = l|h,l,r\rangle \tag{6.1.2}$$

$$C_L|h,l,r\rangle = h\,(h-1)\,|h,l,r\rangle$$

$$C_R|h,l,r\rangle = h\,(h-1)\,|h,l,r\rangle$$

$$C|h,l,r\rangle = (C_L + C_R)\,|h,l,r\rangle = 2h\,(h-1)\,|h,l,r\rangle \tag{6.1.3}$$

$$K_{\pm L}|h,l,r\rangle = i\,(\pm\,(h-1) - l)\,|j_L, l\pm1, r\rangle \tag{6.1.4}$$

$$K_{\pm R}|h,l,r\rangle = i\,(\mp\,(h-1) - r)\,|j_L, l, r\pm1\rangle\,. \tag{6.1.5}$$

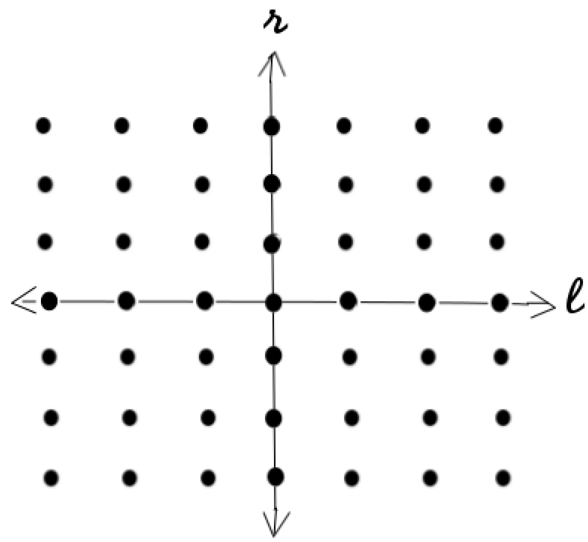


Figure 6.1.1: Weight space diagram for principal series. x-y axes are the $l, r$ values. Solid dots represent states for all $l, r \in \mathbb{Z}$. These states have both growing and decaying modes. $K_{\pm L}$ shift the states right and left respectively. Similarly $K_{\pm R}$ shift the states up and down respectively.

For a scalar field of mass $m$, $4h\,(h-1) = m^2 \implies h_{\pm} = \frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{m^2}{4}}$ and $l(r) = 0, \pm 1, \pm 2, ...$ or $l(r) = \pm\frac{1}{2}, \pm\frac{3}{2}, ....$ The principal series corresponds to $m > 1$ and the complementary series corresponds to $1 > m > 0$. A component of the massless scalar behaves like a discrete series with $h = 1$. Figure 6.1.1 and 6.1.2 show the weight space diagram for principal series and discrete series. Similar weight space diagrams for representation in Anti-de Sitter space was given by Dusedau and Freedman[78].

### 6.1.2 States in coordinate space

Now we know how the generators act on the states. To explore bulk-boundary correspondence, we want to see how the states behave close to the boundary. It is convenient to transform to a basis of eigenstates in coordinate space.

De Sitter space can be described by the flat slicing coordinates $\eta, z, \bar{z}$

$$ds^2 = \frac{1}{\eta^2}\left(-d\eta^2 + dz d\bar{z}\right) .$$

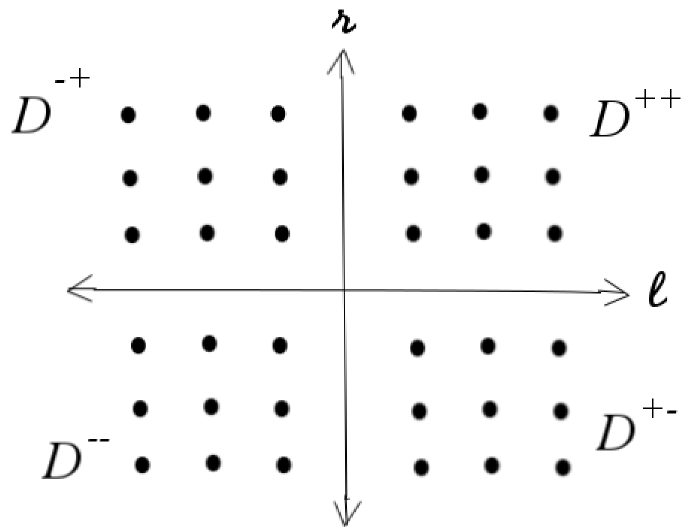


Figure 6.1.2: Weight space diagram for discrete series x-y axes are the $l, r$ values. Solid dots represent states for all non zero $l, r \in \mathbb{Z}$. These states contain only decaying modes. $K_{\pm L}$ shift the states right and left respectively. $K_{\pm L}$ annihilates $l = \mp 1$ states respectively. Similarly $K_{\pm R}$ shift the states up and down respectively. $K_{\pm R}$ annihilates $r = \mp 1$ states respectively.

There are many nice reviews of de-Sitter space [41]. $z$ is complexified spacelike coordinate. $\eta$ is timelike coordinate. De-Sitter has boundary at future and past infinity $\eta \to 0$. Bulk isometry generators are

$$
\begin{aligned}
J_L &= z\partial_z + \frac{\eta}{2}\partial_\eta \quad , \qquad K_{+L} = i\left(z^2\partial_z + \eta^2\partial_{\bar{z}} + z\eta\partial_\eta\right) \quad , \qquad K_{-L} = -i\partial_z \\
J_R &= -\bar{z}\partial_{\bar{z}} - \frac{\eta}{2}\partial_\eta \quad , \qquad K_{-R} = -i\left(\bar{z}^2\partial_{\bar{z}} + \eta^2\partial_z + \bar{z}\eta\partial_\eta\right) \quad , \qquad K_{+R} = i\partial_{\bar{z}}\,.
\end{aligned}
$$

Note that if we put $\eta \to 0$ and $\eta\partial_\eta \to 2h$ as we approach the boundary then

$$
\begin{aligned}
J_L &\to -L_0 \qquad K_{+L} \to -iL_1 \qquad K_{-L} \to iL_{-1} \\
J_R &\to \bar{L}_0 \qquad K_{+R} \to -i\bar{L}_{-1} \qquad K_{-R} \to i\bar{L}_1
\end{aligned}
$$

as shown in the appendix. Casimir operator is given by

$$
C \;=\; C_L + C_R\,.
$$

Simultaneous eigenstates of $J_L, J_R, C$ with eigenvalues $l, r, \frac{m^2}{4}$ ($m$ is mass) respectively, form the principal series representation. Solving the differential equations (6.1.1), (6.1.2) and (6.1.3) we get

$$
\begin{aligned}
&\phi_{l,r}(z,\bar{z},\eta) = \\
&\left(\frac{z}{\bar{z}}\right)^{\frac{l+r}{2}} \eta^{l-r}\Big[A_1 i^{-l-r}\left(\frac{z\bar{z}}{\eta^2}\right)^{-\frac{l+r}{2}} {}_2F_1\left(\frac{1-2l-\sqrt{1-m^2}}{2}, \frac{1-2l+\sqrt{1-m^2}}{2}, 1-l-r, \frac{z\bar{z}}{\eta^2}\right) \\
&+A_2 i^{l+r}\left(\frac{z\bar{z}}{\eta^2}\right)^{\frac{l+r}{2}} {}_2F_1\left(\frac{1+2r-\sqrt{1-m^2}}{2}, \frac{1+2r+\sqrt{1-m^2}}{2}, 1+l+r, \frac{z\bar{z}}{\eta^2}\right)\Big] \qquad (6.1.6)
\end{aligned}
$$

Near the boundary ($\eta \to 0$) it behaves like

$$
\begin{aligned}
\phi_{l,r}(z,\bar{z},\eta\to 0) &= \left(\frac{z}{\bar{z}}\right)^{\frac{l+r}{2}} \eta^{l-r}\Big[A_1 i^{-l-r}\left(\frac{z\bar{z}}{\eta^2}\right)^{-\frac{l+r}{2}}\left(a_1\left(\frac{\eta^2}{z\bar{z}}\right)^{-l+h_-} + a_2\left(\frac{\eta^2}{z\bar{z}}\right)^{-l+h_+}\right) \\
&\quad +A_2 i^{l+r}\left(\frac{z\bar{z}}{\eta^2}\right)^{\frac{l+r}{2}}\left(a_1\left(\frac{\eta^2}{z\bar{z}}\right)^{r+h_-} + a_2\left(\frac{\eta^2}{z\bar{z}}\right)^{r+h_+}\right)\Big] \\
&= b_-\eta^{2h_-}\left(\frac{1}{z^{h_- - l}\bar{z}^{h_- + r}}\right) + b_+\eta^{2h_+}\left(\frac{1}{z^{h_+ - l}\bar{z}^{h_+ + r}}\right) \\
&= b_-\eta^{2h_-}O_{l,r,h_-}(z,\bar{z}) + b_+\eta^{2h_+}O_{l,r,h_+}(z,\bar{z})
\end{aligned}
$$

where $b_\pm$ are some constants and $h_\pm = \frac{1\pm\sqrt{1-m^2}}{2}$. Here $-\infty \le l, r \le \infty$. Another important thing to note is that $\phi_{l,r} \sim z^l \bar{z}^r$ (power law).

$$
\phi_{l,r}(z,\bar{z},\eta\to 0) = \left(b_-\left(\frac{\eta^2}{z\bar{z}}\right)^{h_-} + b_+\left(\frac{\eta^2}{z\bar{z}}\right)^{h_+}\right) z^l \bar{z}^{-r}\,.
$$

For principal series $h_-, h_+$ are complex conjugate of each other. So the modes oscillate close to the boundary. For complementary series $h_- < 0 < h_+$ and real. So half of the modes grow $\left(\eta^{2h_-}\right)$ and other half of the modes decay $\left(\eta^{2h_+}\right)$ near the boundary. They are respectively called growing and decaying mode.

## 6.2 Massless scalar field

Now we are going to look into the massless case. There are two ways that representations contribute to the massless scalar.

### 6.2.1 Limit of complementary series

One is the $m \to 0$ limit of equation (6.1.6). This is the limit of complementary series representation.

$$\begin{aligned}\phi_{l,r}(z,\bar{z},\eta) &= \left(\frac{z}{\bar{z}}\right)^{\frac{l+r}{2}} \eta^{l-r}\Bigg[A_1 i^{-l-r}\left(\frac{z\bar{z}}{\eta^2}\right)^{-\frac{l+r}{2}} {}_2F_1\left(-l,1-l,1-l-r,\frac{z\bar{z}}{\eta^2}\right) \\ &\quad + A_2 i^{l+r}\left(\frac{z\bar{z}}{\eta^2}\right)^{\frac{l+r}{2}} {}_2F_1\left(r,1+r,1+l+r,\frac{z\bar{z}}{\eta^2}\right)\Bigg]. \qquad (6.2.1)\end{aligned}$$

Close to the boundary it goes like

$$\phi_{l,r}(z,\bar{z},\eta \to 0) = \left(b_- + b_+\frac{\eta^2}{z\bar{z}}\right) z^l \bar{z}^{-r}.$$

Note that, it has both the decaying mode and the constant mode.

### 6.2.2 Discrete series

Second is the Discrete series representation. There are two ways of deriving discrete series. Let us first see how it is derived in earlier math papers [68, 25, 24]. First find the eigenstates of $\Box|l,m\rangle = -m^2|l,m\rangle$ in the $|l,m\rangle$ basis (eigenstate of $\{J^2, J_3\}$). $\Box$ is second order differential equation and we get two independent solutions. Then choose only the decaying modes. This removes half of the solutions. This condition results in discrete eigenvalues $(-m^2)$ of $\Box$. Hence the representation is called Discrete series. Note that this is in agreement with the previous section where we said that for massless discrete series $h = 1$.

Now let us derive the discrete series in another way. Diagonalize the Hilbert space in the eigenstates of $J_L, J_R$. In addition to equations (6.1.1), (6.1.2) and (6.1.3)(with $h = 1$) states have to satisfy equations

$$\begin{aligned} C_L|1,l,r\rangle &= 0 \\ C_R|1,l,r\rangle &= 0 \\ K_{\pm L}|1,\mp 1,r\rangle &= 0 \qquad (6.2.2) \\ K_{\pm R}|1,l,\mp 1\rangle &= 0\,. \qquad (6.2.3) \end{aligned}$$

There are four sectors as shown in figure 6.1.2. $D^{m_L m_R}$ where $l(r) = -1$ is the lowest weight and $l(r) = 1$ is the highest weight state. Thus(6.2.2) and (6.2.3). In this basis, highest and lowest weight states are manifest. This is over-constrained set of equations. Equation (6.2.2) and (6.2.3) are first order differential equation which has only one solution. As a result, half of the general solution of equation (6.1.3) is removed. We find that eigenstates decay near boundary. To see this consider the following states

$$\begin{aligned} \phi^D_{-1,r}(z,\bar{z},\eta) &= A\left(\frac{z}{\eta}\right)^{r-1}\left(\frac{z\bar{z}}{\eta}-\eta\right)^{-1-r} \\ \phi^D_{1,r}(z,\bar{z},\eta) &= A\bar{z}^{-\frac{r+1}{2}}\eta^2 \\ \phi^D_{l,1}(z,\bar{z},\eta) &= A\left(\frac{\bar{z}}{\eta}\right)^{-l-1}\left(\frac{z\bar{z}}{\eta}-\eta\right)^{-1+l} \\ \phi^D_{l,-1}(z,\bar{z},\eta) &= Az^{\frac{l-1}{2}}\eta^2\,. \end{aligned}$$

Note that close to boundary all the above solutions go like $\eta^2$. All other states can be obtained by acting with $K_{\pm L}, K_{\pm R}$. Since $K_{\pm L}, K_{\pm R}$ do not decrease the power of $\eta$, all the states will have same $\eta$ dependence. Hence all the modes of the discrete series decay near the boundary. This also shows that $h = 1$. This suggests that these states are a linear combination of states found in the previous approach.

So either imposing regularity of the modes in the $|m, l\rangle$ basis is equivalent to requiring the existence of a highest weight state in $|1, l, r\rangle$ basis. It removes the half of the modes which stay constant near the boundary. On the other hand, the limit of the complementary series has both growing and decaying modes.

### 6.2.3 Discrete series cannot carry energy in dS.

Now that we have understood discrete series and limit of complementary series in more detail, what are the physical consequences? Does graviton belong to discrete series or complementary series? Ashtekar et al., in a series of papers [43, 44], has shown that gravity waves in de-Sitter cannot carry energy if the constant modes of the gravitons are removed. In light of this,

1. If the gravitons are described by discrete series then the constant modes are absent. Then gravity waves cannot carry energy.

2. If we want graviton modes to carry energy and a complete set of modes, the gravitons must contain modes from the limit of complementary series.

### 6.2.4 Indecomposiblity of limit of complementary series

In this section we will show that limit of complementary series is indecomposible. A representation is indecomposible[79] if it cannot be separated into two or more irreducible representations. We have already shown that decaying modes form the irreducible discrete series representation. Then the question is: Does the remaining constant mode also form irreducible representation?

To establish this we show that constant modes turn into decaying modes under the action of generators. Schematically,

$$\begin{aligned} K|decay\rangle &\rightarrow |decay\rangle \\ K|constant\rangle_{m_l\neq 0} &\rightarrow |constant\rangle \\ K|constant\rangle_{m_l=0} &\rightarrow |decay\rangle \end{aligned}$$

where $K$ is some ladder operator. Equation (6.2.1) is the general solution of the massless scalar field. Schematically the two independent solutions are

$$\begin{aligned} \phi_{l,r}(z,\bar{z},\eta\rightarrow 0) &= \left(b_- + b_+\frac{\eta^2}{z\bar{z}}\right) z^l \bar{z}^{-r} \\ &= \left(b_-|constant\rangle + b_+\frac{|decay\rangle}{z\bar{z}}\right) z^l \bar{z}^{-r} \\ |decay\rangle_{\eta\rightarrow 0} &= \eta^2 \\ |constant\rangle_{\eta\rightarrow 0} &= 1 \end{aligned}$$

where $b_-, b_+$ are some constants. $|decay\rangle$ modes form the irreducible discrete series. They are either highest or lowest weight representations. This we have discussed in previous section.

To understand the issue let us see the general $|-1, l=0, r\rangle$ mode

$$\begin{aligned} \phi_{0,r}(z,\bar{z},\eta) &= \left(\frac{z}{\bar{z}}\right)^{\frac{r}{2}} \eta^{-r}\Big[A_1 i^{-r}\left(\frac{z\bar{z}}{\eta^2}\right)^{-\frac{r}{2}} {}_2F_1\left(0,1,1-r,\frac{z\bar{z}}{\eta^2}\right) \\ &\quad + A_2 i^r\left(\frac{z\bar{z}}{\eta^2}\right)^{\frac{r}{2}} {}_2F_1\left(r,1+r,1+r,\frac{z\bar{z}}{\eta^2}\right)\Big] \\ &= A_1 i^{-r}\bar{z}^{-r} + A_2 i^r\left(\frac{z}{\eta^2}\right)^r\left(1-\frac{z\bar{z}}{\eta^2}\right)^{-r}. \end{aligned}$$

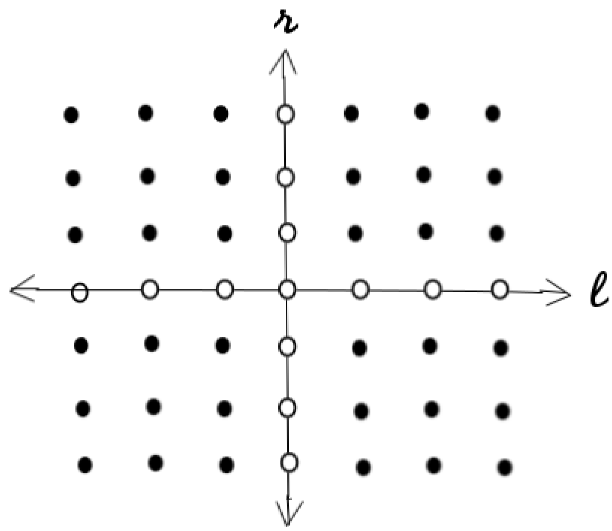


Figure 6.2.1: Weight space diagram for limit of complementary series. x-y axes are the $l, r$ values. Solid dots represent states for all non-zero $l, r \in \mathbb{Z}$. These states have both constant and decaying modes. Empty dots for $l = 0$ or $r = 0$ represent states which have only constant modes. $K_{\pm L}$ shift the states right and left respectively.$K_{\pm L}$ acting on constant modes of $l = 0$ states, convert them to decaying modes. Similarly $K_{\pm R}$ shift the states up and down respectively. $K_{\pm R}$ acting on constant modes of $r = 0$ states, convert them to decaying modes.

We see that there is only a constant part. Now let us apply $K_{-L}, K_{+L}$

$$
\begin{aligned}
K_{-L}|0, r\rangle &= A_2 r i^{-1+r} z^{-1+r} \eta^{2r} \left(1 - \frac{z\bar{z}}{\eta^2}\right)^{-r-1} = |-1, r\rangle_{decay} \\
K_{+L}|0, r\rangle &= A_1 r i^{-1-r} \bar{z}^{-1-r} \eta^2 = |1, r\rangle_{decay} .
\end{aligned}
$$

Thus we get only the decaying modes. This shows that the growing modes convert into decaying modes and proves that limit of complementary series is indecomposible representation. Figure 6.2.1 gives the weight space diagram for the limit of complementary series to illustrate this point.

## 6.3 Transformation from conformal basis to momentum basis

In this section we derive the transformation from a momentum basis (eigenstate of $L_{-1}$ operator) to the $l, r$ basis (eigenstates of $L_0, \bar{L}_0$ operator with eigenvalue $l, r$ respectively. ). One reason to do it is that the scalar field in the bulk is generally written in momentum basis but boundary operators are generally expressed in $l, r$ basis. Subsection (6.3.1) gives in detail the calculations for principal series. In subsection (6.3.2) we summarize the main results and compare the differences between the two representations.

### 6.3.1 Principal series

Momentum basis are eigenstates of $L_{-1} = -\partial_z, \bar{L}_{-1} = -\partial_{\bar{z}}$. $l, r$ basis are eigenstate of $L_0 = -\left(z\partial_z + h\right), \bar{L}_0 = -\left(\bar{z}\partial_{\bar{z}} + \bar{h}\right)$ respectively. We want to find the coefficients $c_{k,l,r}$ of the relation

$$|k, \bar{k}\rangle = \sum c_{k,l,r}|l, r\rangle \tag{6.3.1}$$

Our approach is similar to what Lindbad et al do in section (4A) of [71]. From commutation relation $[L_n, \phi_l] = \left((h-1)\,n - l\right)\phi_{n+l}$ we get

$$L_0|l, r\rangle = -l|l, r\rangle \tag{6.3.2}$$

$$L_1|l, r\rangle = (h - 1 - l)\,|l + 1, r\rangle \tag{6.3.3}$$

$$L_{-1}|l, r\rangle = (1 - h - l)\,|l - 1, r\rangle \tag{6.3.4}$$

$$L_{-1}|k, \bar{k}\rangle = \left(\frac{i\bar{k}}{2}\right)|k, \bar{k}\rangle\,. \tag{6.3.5}$$

To find $c_{k,l,r}$ we act with $L_{-1}$ on both the side of equation (6.3.1)

$$\begin{aligned} L_{-1}|k,\bar{k}\rangle &= \sum c_{k,l,r} L_{-1}|l,r\rangle \\ \left(\frac{i\bar{k}}{2}\right)\langle l',r'|k,\bar{k}\rangle &= \sum c_{k,l,r}(-l+1-h)\langle l',r'|l-1,r\rangle \\ \left(\frac{i\bar{k}}{2}\right) c_{k,l',r'} &= c_{k,l'+1,r'}(-l'-h)\,. \end{aligned}$$

Solving the recurrence relation we get

$$\begin{aligned} c_{k,l,r} &= c_{k,l-1,r}\left(\frac{-i\bar{k}}{2}\right)\frac{1}{(h+l-1)} \\ c_{k,l,r} &= \left(\frac{-i\bar{k}}{2}\right)^l \frac{h!}{(h+l-1)!} c_{k,0,r} \\ &= \left(\frac{-i\bar{k}}{2}\right)^l \frac{\Gamma(h+1)}{\Gamma(h+l)} c_{k,0,r} \\ &= \left(\frac{-i\bar{k}}{2}\right)^l \frac{\sin \pi h}{\pi}(-1)^l \Gamma(h+1)\Gamma(1-h-l) c_{k,0,r}\,. \end{aligned}$$

Similarly one can derive $c_{k,0,r}$ by the action of $\bar{L}_{-1}$. Finally one gets

$$c_{k,l,r} = \left(\frac{ik}{2}\right)^r \left(\frac{i\bar{k}}{2}\right)^l \left(\frac{\sin \pi h}{\pi}\right)^2 \Gamma(h+1)^2 \Gamma(1-h-l)\Gamma(1-h-r) c_{k,0,0}\,.$$

We choose normalization $c_{k,0,0} = \frac{\pi^2}{(\sin \pi h)^2(-2i)^{2h+1}\Gamma(h+1)^2|ik/2|}$. Plugging this back into equation (6.3.1) we get

$$|k,\bar{k}\rangle = \sum \left(\frac{ik}{2}\right)^{r-1/2} \left(\frac{i\bar{k}}{2}\right)^{l-1/2} (-2i)^{-2h-1}\Gamma(1-h-l)\Gamma(1-h-r)|l,r\rangle \tag{6.3.6}$$

We can now invert equation (6.3.6).

$$|l,r\rangle \;=\; \frac{1}{(2\pi i)^2}\oint dk d\bar{k}\left(\frac{ik}{2}\right)^{-r-1/2}\left(\frac{i\bar{k}}{2}\right)^{-l-1/2}\frac{(-2)^{2h+1}}{\Gamma(1-h-l)\Gamma(1-h-r)}|k,\bar{k}\rangle \tag{6.3.7}$$

One can check that this is consistent with equation (6.3.6). To see that start with the RHS of the above equation, substitute $|k,\bar{k}\rangle$ from equation (6.3.6) and we get the LHS of above equation

$$\begin{aligned}
& \frac{1}{(2\pi i)^2}\oint dk d\bar{k}\left(\frac{ik}{2}\right)^{-r'-1/2}\left(\frac{i\bar{k}}{2}\right)^{-l'-1/2}\frac{(-2i)^{2h-1}}{\Gamma(1-h-l')\Gamma(1-h-r')}|k,\bar{k}\rangle \\
=\; & \sum_{l.r}\frac{1}{(2\pi i)^2}\oint dk d\bar{k}\left(\frac{ik}{2}\right)^{-r'+r-1}\left(\frac{i\bar{k}}{2}\right)^{-l'+l-1}\frac{-\Gamma(1-h-l)\Gamma(1-h-r)}{4\Gamma(1-h-l')\Gamma(1-h-r')}|l,r\rangle \\
=\; & \sum_{l.r}\delta_{ll'}\delta_{rr'}\frac{\Gamma(1-h-l)\Gamma(1-h-r)}{\Gamma(1-h-l')\Gamma(1-h-r')}|l,r\rangle \\
=\; & |l',r'\rangle\,.
\end{aligned}$$

Now we know the basis transformations each way $|k,\bar{k}\rangle \leftrightarrow |l,r\rangle$, we can write this as a boundary operator/state correspondence as follows

$$\begin{aligned}
O(z,\bar{z})|0\rangle = |z,\bar{z}\rangle \;&=\; \sum_{l,r}\frac{1}{z^{h+l}\bar{z}^{h+r}}|l,r\rangle \\
O(z,\bar{z})|0\rangle = |z,\bar{z}\rangle \;&=\; \oint\frac{dkd\bar{k}}{(2\pi i)^2}e^{\frac{i}{2}\left(k\bar{z}+\bar{k}z\right)}|k|^{2h-1}|k,\bar{k}\rangle\,.
\end{aligned}$$

So RHS of the above two equations must be equal. That is

$$\sum_{l,r}\frac{1}{z^{h+r}\bar{z}^{h+l}}|l,r\rangle \;=\; \oint\frac{dkd\bar{k}}{(2\pi i)^2}e^{\frac{i}{2}\left(k\bar{z}+\bar{k}z\right)}|k|^{2h-1}|k,\bar{k}\rangle\,. \tag{6.3.8}$$

To verify that, substitute $|l,r\rangle$ from equation (6.3.7) in LHS to get the RHS.

$$
\begin{aligned}
|z,\bar{z}\rangle &= \sum_{l,r} \frac{1}{z^{h+l}\bar{z}^{h+r}} |l,r\rangle \\
&= \frac{1}{(2\pi i)^2} \oint dk d\bar{k} \sum_{l,r} \left(\frac{ik\bar{z}}{2}\right)^{-r-1/2} \left(\frac{i\bar{k}z}{2}\right)^{-l-1/2} \frac{(-2i)^{2h-1}}{\Gamma(1-h-l)\Gamma(1-h-r)} |k,\bar{k}\rangle \\
&= \frac{1}{(2\pi i)^2} \oint dk d\bar{k} |k|^{2h-1} \left( \sum_{l,r} \left(\frac{ik\bar{z}}{2}\right)^{-h-r} \left(\frac{i\bar{k}z}{2}\right)^{-h-l} \frac{1}{\Gamma(1-h-l)\Gamma(1-h-r)} \right) |k,\bar{k}\rangle \\
&= \frac{1}{(2\pi i)^2} \oint dk d\bar{k} |k|^{2h-1} e^{\frac{i}{2}\left(k\bar{z}+\bar{k}z\right)} |k,\bar{k}\rangle \, .
\end{aligned}
$$

Here we have used the identity

$$
e^z = \sum_{n\in Z} \frac{z^{h+n}}{\Gamma(h+n+1)} \, .
$$

When $h$ is integer, $\frac{1}{\Gamma(h+n+1)} = 0$ for $n < -h$. Thus

$$
\begin{aligned}
e^z &= \sum_{n\geq -h} \frac{z^{h+n}}{\Gamma(h+n+1)} \\
&= \sum_{m\geq 0} \frac{z^m}{\Gamma(m+1)}
\end{aligned}
$$

coincides with the usual definition of exponential function. When $h$ is non integer, negative powers of $z$ appear in the sum. Each such term diverges at the the origin but the sum is finite.

### 6.3.2 Summary

The boundary operator/state correspondence is

$$
O(z,\bar{z})|0\rangle = \begin{cases} \sum_{l,r\leq 0} \frac{1}{z^l\bar{z}^r} |l-h, r-h\rangle & \text{Highest Weight} \\ \sum_{l,r} \frac{1}{z^{h+l}\bar{z}^{h+r}} |l,r\rangle & \text{Principal Series} \end{cases}
$$

Transforming to the momentum basis we get

$$O(z,\bar{z})|0\rangle = \frac{1}{(2\pi i)^2}\oint dk d\bar{k}|k|^{2h-1}e^{\frac{i}{2}\left(k\bar{z}+\bar{k}z\right)}|k,\bar{k}\rangle\,.$$

We can also transform back. As we have already stated in subsection (6.3.1), the key identity is

$$e^z = \sum_{n\in Z}\frac{z^{h+n}}{\Gamma(h+n+1)} = \sum_{m\geq 0}\frac{z^m}{\Gamma(m+1)}$$

where $h$ is integer.

In the momentum basis, the expansion

$$O(z,\bar{z})|0\rangle = \oint\frac{dk d\bar{k}}{(2\pi i)^2}e^{\frac{i}{2}\left(k\bar{z}+\bar{k}z\right)}|k|^{2h-1}|k,\bar{k}\rangle$$

takes the same form for both the highest weight and principal series representation. So any two or three-point correlation function in the momentum or position basis is going to have the same scaling form for principal series and highest weight representation since the form is fixed by conformal symmetry. For example,

$$\begin{aligned}\langle O(z)O(w)\rangle_{\text{principal-series}} &= \frac{1}{(z-w)^{2h}}\\ \langle O(z)O(w)\rangle_{\text{highest-weight}} &= \frac{1}{(z-w)^{2h}}\end{aligned}$$

where $h$ is the weight of operator. It is real for highest weight rep but complex for principal series. Now consider the following 2-point function $\langle\oint w^{k+h}\phi(w)\oint z^k\phi(z)\rangle$ ($h$ is the weight of the operator). For highest weight representation

$$\langle\oint w^{k+h}\phi_H(w)\oint z^k\phi_H(z)\rangle = 0\,(\text{for } k\in\mathbb{Z}_+)$$

because $\oint z^k \phi_H(z)|0\rangle = 0$ for $k \in \mathbb{Z}_+$.

For principal series representation it gives

$$\begin{aligned}
& \langle \oint w^{k+h} \phi_{NH}(w) \oint z^k \phi_{NH}(z) \rangle \\
= & \langle \sum_{-\infty<m<\infty} \oint w^{k-m} dw \phi_m \sum_{-\infty<n<\infty} \oint z^{k-n-h} dz \phi_n \rangle \\
= & \langle \phi_{k+1} \sum_{-\infty<n<\infty} \frac{r^{k-n-h+1}\left(e^{2\pi i(k-n-h+1)}-1\right)}{i(k-n-h+1)} \phi_n \rangle \\
= & \frac{i\left(e^{-2\pi h i}-1\right)}{h r^h}
\end{aligned}$$

where $r$ is the radius of the circular loop around the origin. So we have constructed an observable which vanishes for highest weight CFT but does not vanish for non-highest weight CFT.

Another way to distinguish them is to compute the correlation function in $l, r$ basis

$$\begin{aligned}
\langle (L_0 O(l,r)) (L_0 O(l,r)) \rangle_{\text{principal-series}} &= \langle l, r | L_0^\dagger L_0 | l.r \rangle = l^2 \\
\langle (L_0 O(l,r)) (L_0 O(l,r)) \rangle_{\text{highest-weight}} &= \langle l-h, r-h | L_0^\dagger L_0 | l-h.r-h \rangle = (l-h)^2 \,.
\end{aligned}$$

For principal series, we get integer squared and is independent of the weight. Whereas for highest weight, it is non-integer and depends on the weight of the operator.

## 6.4 OPE of principal series operators

In this section we derive operator product expansion (OPE) for the principal series. First we will review the calculation for highest weight CFT from [80]. Then we will extend the derivation for principal series with suitable modification.

### 6.4.1 Highest weight OPE

Start with an ansatz

$$O_1(z)O_2(0) = \sum_{k\geq 0} \beta_k z^{\Delta_3-\Delta_1-\Delta_2+k} \left(\frac{\partial}{\partial\zeta}\right)^k O_3(\zeta)|_{\zeta\to 0}\,. \tag{6.4.1}$$

Then using symmetry we can determine the coefficients. Commute left side with $L_1$. Using the relation

$$[L_1, O_\Delta(z)] = \left[z^2\frac{\partial}{\partial z} + 2\Delta z\right] O_\Delta(z)$$

we get

$$[L_1, O_1(z)O_2(0)] = \left[z^2\frac{\partial}{\partial z} + 2\Delta_1 z\right] O_1(z)O_2(0)\,.$$

Substituting the ansatz from equation (6.4.1) in the right side we get

$$[L_1, O_1(z)O_2(0)] = \sum_{k\geq 0} \beta_k z^{\Delta_3-\Delta_1-\Delta_2+k+1} \left(\Delta_3+\Delta_1-\Delta_2+k\right) \left(\frac{\partial}{\partial\zeta}\right)^k O(\zeta)|_{\zeta\to 0} \tag{6.4.2}$$

Now commuting $L_1$ with the right side of equation (6.4.1) we get

$$\begin{aligned}&\sum_{k\geq 0} \beta_k z^{\Delta_3-\Delta_1-\Delta_2+k} \left(\frac{\partial}{\partial\zeta}\right)^k [L_1, O(\zeta)]\,|_{\zeta\to 0} = \\ &\sum_{k\geq 0} \beta_k z^{\Delta_3-\Delta_1-\Delta_2+k} \left(\frac{\partial}{\partial\zeta}\right)^k \left(\zeta^2\frac{\partial}{\partial\zeta} + 2\zeta\Delta_3\right) O(\zeta)|_{\zeta\to 0}\end{aligned} \tag{6.4.3}$$

We can now match the coefficient of power series of equations (6.4.2) and (6.4.3). Let us set $\Delta_1 = \Delta_2$ for simplicity. As an example, let us match the coefficient of

$z^{\Delta_3-\Delta_1-\Delta_2+1}$

$$\begin{aligned} \beta_0 \frac{\Gamma(\Delta_3+\Delta_1-\Delta_2+1)}{\Gamma(\Delta_3+\Delta_1-\Delta_2)} O(\zeta)|_{\zeta\to 0} &= \beta_1 \left(\frac{\partial}{\partial\zeta}\right)\left(\zeta^2\frac{\partial}{\partial\zeta}+2\zeta\Delta_3\right) O(\zeta)|_{\zeta\to 0} \\ \beta_0\Delta_3 O(\zeta)|_{\zeta\to 0} &= \beta_1\left(2\Delta_3+2\zeta\frac{\partial}{\partial\zeta}+\zeta^2\left(\frac{\partial}{\partial\zeta}\right)^2\right) O(\zeta)|_{\zeta\to 0} \end{aligned} \tag{6.4.4}$$

For highest weight $O(\zeta)|_{\zeta\to 0}$ is finite and $\zeta\frac{\partial}{\partial\zeta}O(\zeta)|_{\zeta\to 0}=0$. Thus we get

$$\beta_1 = \frac{\beta_0}{2}\,.$$

Similarly matching all the terms, we get

$$O_1(z)O_2(0) = \beta_{123}\sum z^{\Delta_3-2\Delta_1}\ {}_1F_1\left(\Delta_3, 2\Delta_3, z\frac{\partial}{\partial\zeta}\right) O_3(\zeta)|_{\zeta\to 0}\,. \tag{6.4.5}$$

The above equality can also be derived, starting from the 3 point function

$$\langle O(z_1)O(z_2\to 0)O(z_3)\rangle = \frac{\beta_{123}}{z_1^h(z_3-z_1)^h z_3^h} = \beta_{123} z_1^{-h}\left(1+h\frac{z_1}{z_3}+...\right)\frac{1}{z_3^{2h}}\,. \tag{6.4.6}$$

### 6.4.2 Principal series

For the principal series the OPE will take the form

$$O_1(z)O_2(0) = \sum_{k>0}\beta_{-k} z^{\Delta_3-\Delta_1-\Delta_2-k}(L_1)^k O_3(\zeta)|_{\zeta\to 0} + \sum_{k\geq 0}\beta_k z^{\Delta_3-\Delta_1-\Delta_2+k}(L_{-1})^k O_3(\zeta)|_{\zeta\to 0} \tag{6.4.7}$$

Here we have also added terms with $L_1 O$ because for principal series $L_1 O \neq 0$ in general. Again we commute with $L_1$ to determine $\beta_k$. Commuting the left side of equation (6.4.7) we get

$$[L_1, O_1(z)O_2(0)] =$$
$$\sum_{k>0} \beta_k z^{\Delta_3-\Delta_1-\Delta_2+k} \left( z\left(\Delta_3+\Delta_1-\Delta_2+k\right)\left(L_{-1}\right)^k + \left(\zeta^2 \frac{\partial}{\partial\zeta} + 2\Delta_2\zeta\right)\left(L_{-1}\right)^k \right) O(\zeta)\,.$$

Commuting the right side of equation (6.4.7) we get

$$[L_1, O_1(z)O_2(0)] \;=\; \sum_{k>0} \beta_k z^{\Delta_3-\Delta_1-\Delta_2+k} \left[L_1, \left(L_{-1}\right)^k O(\zeta)\right]\,.$$

Equating the above two equations gives

$$\begin{aligned} & \beta_{k+1}\left( L_{-1}^{k+1}\left(\zeta^2\frac{\partial}{\partial\zeta} + 2\Delta_3\zeta\right) - \left(\zeta^2\frac{\partial}{\partial\zeta} + 2\Delta_2\zeta\right)\left(L_{-1}\right)^{k+1}\right) O(\zeta) \\ = \; & \beta_k\left(\Delta_3+\Delta_1-\Delta_2+k\right)\left(L_{-1}\right)^k O(\zeta)\,. \end{aligned} \qquad (6.4.8)$$

Similarly, to determine $\beta_{-k}$ we can commute both sides with $L_{-1}$

$$[L_{-1}, O_1(z)O_2(0)] \;=\; \sum_{k>0} \beta_{-k} z^{\Delta_3-\Delta_1-\Delta_2-k-1} \left( \left(\Delta_3-\Delta_1-\Delta_2-k\right)\left(L_1\right)^k + z\left(L_1\right)^k L_{-1}\right) O(\zeta)\,.$$

Commuting the right side of equation (6.4.7) we get

$$[L_{-1}, O_1(z)O_2(0)] \;=\; \sum_{k<0} \beta_{-k} z^{\Delta_3-\Delta_1-\Delta_2-k} \left[L_{-1}, \left(L_1\right)^k O(\zeta)\right]\,.$$

Equating the two sides we get

$$\beta_{-k-1}\left[L_{-1}, L_1^{k+1}\right] O(\zeta)|_{\zeta\to 0} \;=\; \beta_{-k}\left(\Delta_3-\Delta_1-\Delta_2-k\right)\left(L_1\right)^k O(\zeta)\,. \quad (6.4.9)$$

Simplifying equation (6.4.8) gives

$$\begin{aligned} & \beta_{k+1}\left(2\left(k+1+\Delta_3-\Delta_2\right)\zeta\left(\frac{\partial}{\partial\zeta}\right)^{k+1}+\left(k+1\right)\left(k+2\Delta_3\right)\left(\frac{\partial}{\partial\zeta}\right)^{k}\right)O(\zeta) \\ = \;& \beta_k\left(\Delta_3+\Delta_1-\Delta_2+k\right)\left(\frac{\partial}{\partial\zeta}\right)^{k}O(\zeta)\,. \end{aligned} \tag{6.4.10}$$

An important thing to note is that recursion relations explicitly depend on $O(\zeta)$. Now we substitute the expansion

$$O_3(\zeta) \;=\; \sum_j \frac{O_{3j}}{\zeta^{h+j}}$$

and compare the coefficient of same power of $\zeta$, we get

$$\beta_{k+1} \;=\; \beta_k \frac{\left(\Delta_3+\Delta_1-\Delta_2+k\right)}{\left(2\left(k+1+\Delta_3-\Delta_2\right)\left(-h-j-k\right)+\left(k+1\right)\left(k+2\Delta_3\right)\right)}\,.$$

We find that $\beta_k$ depends on $j$. This suggests that we must start with an OPE of the form

$$O_1(z)O_2(\zeta) \;=\; \sum_j\left(\sum_{k>0}\beta_{-k,j}z^{\Delta_3-\Delta_1-\Delta_2-k}\left(L_1\right)^k\frac{O_{3j}}{\zeta^j}+\sum_{k>0}\beta_{k,j}z^{\Delta_3-\Delta_1-\Delta_2+k}\left(L_{-1}\right)^k\frac{O_{3j}}{\zeta^j}\right) \tag{6.4.11}$$

Then going through the above derivation we get

$$\begin{aligned} \beta_{k+1,j} &= -\beta_{k,j}\frac{\left(\Delta_3+\Delta_1-\Delta_2+k\right)}{\left(k-K_+\right)\left(k-K_-\right)} \\ \beta_{-k-1,j} &= \beta_{-k,j}\frac{\left(\Delta_3-\Delta_1-\Delta_2-k\right)}{(k+2\Delta_3+2j+2h)(k+1)} \end{aligned}$$

where

$$K_{\pm} = \frac{1}{2}\left(1+2\left(\Delta_2-j-h\right) \pm \sqrt{\left(1+\Delta_2\right)^2+4\left(2\Delta_3-j-h\right)\left(1-j-h\right)}\right) .$$

Calculations are shown in the appendix.

Then equation (6.4.11) can be written in terms of hypergeometric functions

$$\begin{aligned}
&O_1(z)O_2(\zeta) = \\
&\sum_j z^{\Delta_3-\Delta_1-\Delta_2}\beta_{0,j}\Bigg( {}_1F_1\left(\Delta_1+\Delta_2-\Delta_3; 2\Delta_3+2j+2h; -\frac{1}{z}\left(\zeta^2\frac{\partial}{\partial\zeta}+2\Delta_3\zeta\right)\right)\frac{O_{3j}}{\zeta^{h+j}} \\
&+ {}_2F_2\left(\Delta_3+\Delta_1-\Delta_2, 1; K_+, K_-; -z\frac{\partial}{\partial\zeta}\right)\frac{O_{3j}}{\zeta^{h+j}}\Bigg) .
\end{aligned}$$

This is the conformal partial wave expansion for the principal series.

Using equation (6.3.8), we can show that above equation is equivalent to three point function

$$\langle O(z_1)O(z_2\to 0)O(z_3)\rangle = \frac{\beta_{123}}{z_1^h\left(z_3-z_1\right)^h z_3^h}$$

given $\beta_{0,j} = \beta_{123}$.

The main conclusion is that there are infinitely many singular terms coming from terms like $L_1^k O$ in the OPE. The OPE therefore has an essential singularity, unlike any known conformal field theory that may be viewed as arising from a renormalizable field theory. This puts the set of interacting conformal field theories based on representations containing the principal series well outside the class of conventional quantum field theories. The OPE also depends on an infinite number of parameters that are free at this level of analysis, compared to the single parameter one normally encounters in CFT. If these CFTs of relevance for de Sitter space exist, it seems they have more in common with non-renormalizable theories, than with conventional

CFTs.

Finally we note all the conclusions of the present section carry over to the complementary series, provided we take $h$ in the appropriate range $1 > h > 1/2$.

## 6.5 Conclusion

de-Sitter holography implies that bulk and boundary states should be in principal, complementary, discrete series and indecomposible representations. Some of the details of these representations were studied from the conformal field theory perspective. In particular, we analyzed the implications of global conformal invariance for the operator product expansion. Because the weights of the principal and complementary series are unbounded, there end up being infinitely many singular terms in the operator product expansion. Nevertheless, this is compatible with the usual simplifications of the two and three-point functions of quasi-primary operators. The essential singularity present in these operator product expansions is not reproducible from conventional quantum field theories.

# Chapter 7

# BMS symmetry, soft particles and memory

One of the first breakthroughs in laying the foundation for an understanding of holography in Minkowski space was the work of Bondi-Metzner-Sachs [81, 82]. It revealed that asymptotic symmetry group of Minkowski space is a group of large diffeomorphisms called the BMS group. Representations of the Poincare group [83] have played an important role in classifying elementary particles by their mass and spin. That motivates understanding the representations of the BMS group and its connection to elementary particles. In the 1970s McCarthy studied the positive energy unitary irreducible representations of BMS group [84, 85, 86, 87, 88, 89]. But after this initial work, the subject has received little attention. The physical interpretation of the representations was not entirely clear at the time. In this work, we study from a physical viewpoint most of the interesting representations with the aim of identifying the interesting representations needed to construct a holographic dual. These include massive and massless particles and also soft particles with vanishing four-momentum. We find that in addition to zero momentum limit of massless particles there are many new soft modes predicted by BMS group which are related to gravitational memory

[90, 91].

Recently, Strominger et. al. have discovered a relation between the BMS group, soft theorems and the memory effect [92, 93, 94]. They related supertranslations to memory effect [94] which led them to propose that black hole carries soft hair [95]. In this work we show that supertranslation charges indeed retain information about the initial states via a straightforward group theory construction. We consider a case where two particles collide and move away in different directions. Conservation of momenta (including supermomenta) reveals that final state has information about soft particles that stores information about the initial state. Another interesting discussion of the memory effect in electromagnetism appears in [96, 97].

The BMS charge algebra has been studied in [98, 99, 100, 101] and BMS representations in three dimensions have been explored in the following papers [102, 103, 104, 105]. Relation of BMS group to soft theorems has also been explored [106]. Other recent papers on the BMS group include a realization [107, 108] on a scalar field, and more generally relation between the BMS group and elementary particles[109]. The connection between BMS group and non-relativistic conformal group, also known as Galilean group [110, 111, 112, 113] has also been explored. The BMS group has also been realized as a conformal extension of the Carroll group [114, 115]. Interestingly, contrary to most of the literature Bousso and Porrati argue that soft modes do not constrain hard scattering problem [116, 117].

In the present work we begin by reviewing the BMS group and establishing notation. We then revisit some of the most relevant results from McCarthy's classification of unitary irreducible representations of the BMS group and connect the Bondi mass aspect to the function space on which BMS is realized. We try to highlight only the physically important representations and find all the massive and massless representations that appear in the usual Wigner classification of the Poincare group, as well as extra representations with differing supermomenta structures. The group invariant norms associated with these families of representations are constructed, which is an essential step in any attempt at capturing the bulk dynamics via a holographic description. We then consider tensor products/scattering of these states which allows us to explore the extent to which gravitational memory allows the initial state to be reconstructed from a final state. We conclude with some comments on the relevance of the results to general gravitational S-matrix theories in asymptotically flat space-time such as string theory, and the prospects for developing holographic models with BMS as a fundamental symmetry group.

## 7.1 Representations of the BMS group

Asymptotic flatness requires that the Weyl tensor of the metric must fall off like $O\left(r^{-3}\right)$ for large $r$ [82] (for a recent review see [118]) which allows the choice of the following asymptotically flat coordinates at leading order at large $r$

$$\begin{aligned} ds^2 &= -du^2 - 2dudr + 2r^2\gamma_{z\bar{z}}dzd\bar{z} + 2\frac{m_B(u,z,\bar{z})}{r}du^2 + rC_{zz}dz^2 \\ &\quad +D^zC_{zz}dud\bar{z} + c.c + ... \end{aligned} \tag{7.1.1}$$

The function $m_B(u,z,\bar{z})$ is called the Bondi mass aspect and the other coefficients are functions only of $u, z$ and $\bar{z}$. The covariant derivative $D^z$ is defined with respect to the metric on the unit sphere $\gamma_{z\bar{z}} = \frac{2}{(1+z\bar{z})^2}$. In the next subsection we give a brief introduction of the BMS group. Then we show that the invariant mass function introduced in [87] and the Bondi mass aspect are to be identified.

In this section, we will do warm up exercise to understand McCarthy's construction, focussing on his first paper. We summarize his paper mostly reproducing impotant calculations and conclusions. The reason we elaborate his earlier method is because it gives a nice playground to showcase some of the salient features of representations of BMS group.

The group of diffeomorphisms which preserve the form of the metric (7.1.1) is called the BMS group. It is given by

$$B = A \ltimes G$$

where $G = SL(2,\mathbb{C})$ and $A$ is the abelian group of pointwise addition of square integrable functions on a 2-sphere [89]. Scalar product into $A$ is defined by

$$\langle \alpha, \beta \rangle = \int_{S^2} \alpha(x)\beta(x)d\mu(x)$$

where $d\mu = \frac{1}{4\pi}\sin\theta d\theta d\phi = \frac{1}{2\pi i}\frac{dzd\bar{z}}{(1+|z|^2)^2}$ is the usual area measure on $S^2$. This gives Hilbert space structure to A which is an Abelian topological group. Then one defines a semi-direct product between $A$ and group $G$ of $2X2$ complex matrices of unit determinants.

$$B = A \ltimes G$$

Elements $\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$ of $G$ act on elements of $A$ in the following way

$$T(g)\alpha(z,\bar{z}) = K_g(z,\bar{z})\alpha(zg,\bar{z}g) \tag{7.1.2}$$

where $z$ is the complex coordinate on $S^2$ and

$$zg = \frac{az+c}{cz+d}$$
$$K_g(z,\bar{z}) = \frac{|az+c|^2 + |bz+d|^2}{1+|z|^2}$$

$\alpha(z,\bar{z})$ being functions on sphere can be expanded in terms of spherical harmonics

$$\alpha(\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\alpha_{lm}P_{lm}(\theta,\phi)$$

Now $A$ can be written as direct sum of subspaces $V$ and $\Sigma$ called translation and super-translation respectively

$$\begin{aligned} A &\approx V\oplus\Sigma \\ V: v(\theta,\phi) &= \sum_{l=0}^{1}\sum_{m=-l}^{l}\alpha_{lm}P_{lm}(\theta,\phi) = a^0 + a^1\sin\theta\cos\phi + a^2\sin\theta\sin\phi + a^3\cos\theta \\ \Sigma:\sigma(\theta,\phi) &= \sum_{l=2}^{\infty}\sum_{m=-l}^{l}\alpha_{lm}P_{lm}(\theta,\phi) \end{aligned}$$

Similarly one defines the dual group $A'$ of $A$ consisting of square integrable function $\phi(\theta,\phi)$ defined on sphere $S^2$. They can also be expanded in terms of spherical harmonics

$$\phi(\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}p_{lm}P_{lm}(\theta,\phi)$$

It can also be written as a direct sum of subspace $V^0$ and $\Sigma^0$ called momentum and super-momenta respectively.

$$\begin{aligned} \phi(\theta,\phi) &= p_0 + p_1\sin\theta\cos\phi + p_2\sin\theta\sin\phi + p_3\cos\theta + \sum_{l=2}^{\infty}\sum_{m=-l}^{l}p_{lm}P_{lm}(\theta,\phi) \\ A' &\approx V^0\oplus\Sigma^0 \end{aligned}$$

Using equation (7.1.2) we will determine the action of Lorentz generators on $\phi(\theta,\phi)$. Consider

$$\langle g\phi,\alpha\rangle = \int\phi(z,\bar{z})(g^{-1}\alpha)(z,\bar{z})d\mu(z,\bar{z}) = \int\phi(z,\bar{z})K_{g^{-1}}(z,\bar{z})\alpha(zg^{-1},\bar{zg}^{-1})d\mu(z,\bar{z})$$

Using the formula

$$d\mu(z,\bar{z}) \;=\; K_g^2(z,\bar{z})d\mu(zg,\bar{z}g)$$

we get

$$\langle g\phi,\alpha\rangle \;=\; \int K_g^{-3}(z,\bar{z})\phi(zg,\bar{z}g)\alpha(z,\bar{z})d\mu(z,\bar{z})$$

This gives

$$g\phi(z,\bar{z}) \;=\; K_g^{-3}(z,\bar{z})\phi(zg,\bar{z}g)$$

Now we are ready to discuss little groups of BMS group. If we recollect Wigner's work on representations of Poincare group basically has three steps. First, find out all the possible orbits inside Poincare group. In Poincare group, orbits are completely classified $p.p = m^2, p_0$ and spin, where $m$ is the mass and $p_0$ is the energy. Once the orbits are classified, next find the corresponding little groups. Third and final step is to find all the irreducible representations of the little group. Representations of BMS group also has three steps but they are slightly in different order. The reason for this difference is that finding orbits of $A'$, which is infinite dimensional, is very difficult. Orbits are homogenous spaces of $G$ and we have shown in subsection 4.3.4 that homogeneous spaces can be identified with the coset spaces of the group under certain conditions [119]. These conditions are satisfied in these cases. If $M$ is a homogenous space of $G$ and point $p \in M$ is fixed under the motion of largest subgroup $G_p$. Then there is one-one mapping between $M$ and $G/G_p$. Conversely if $L$ is a subgroup of $G$, then $G/L$ is a homogenous space of $G$ under the usual action of $G$ on cosets. So the homogenous spaces of $G$ can be classified by finding the non-conjugate subgroups of $G$. This gives the following prescription to find the orbits and little group:

1. Determine all non-conjugate subgroups $L \subset G$ .

2. For each subgroup $L \subset G$, find the orbit $\phi \in A'$ invariant under $L$. We will call $\phi$ invariant function.

3. Identify $L$ with the little group of $\phi$ associated with the orbit $G\phi \approx G/L$.

As one notices, we find sub-group first and then its orbit, unlike Poincare case where we find orbits first and then its little groups. Apart from that all the steps are similar. All the connected subgroups of $G$ are already found by Shaw 1970 [120]. So the first step is already done. Now we move to the second step. Not all the subgroups will have non-trivial invariant function. Our goal is to find those connected subgroups which have non-trivial invariant function and the corresponding invariant function.

We will show that any subgroup which has the boost generator $M_{03} = \frac{\sigma_3}{2} = \begin{bmatrix} 1/2 & 0 \\ 0 & -1/2 \end{bmatrix}$ (this is boost along $z$-axis) cannot have non-trivial invariant function. Corresponding finite generator is

$$g_t = \begin{bmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{bmatrix}$$

Its action on $\phi$ is given by equation (7.1.2). Invariance condition is

$$\phi(z, \bar{z}) = g_t \phi(z, \bar{z}) = \left( \frac{e^{-t} + e^t |z|^2}{1 + |z|^2} \right)^{-3} \phi(e^t z, e^t \bar{z})$$

Redefining $z = e^{s+i\psi}$ and substituting $t = -s$ we get

$$\begin{aligned} \phi(s,\psi) &= \left(\frac{e^s + e^s}{1+e^{2s}}\right)^{-3} \phi(e^{i\psi}, e^{-i\psi}) \\ \implies \phi(s,\psi) &= \cosh^3 s\phi(0,\psi) \\ &= \cosh^3 s\xi(\psi) \end{aligned}$$

where $\xi(\psi)$ is any function. Integration measure is given by

$$d\mu(s,\psi) = \frac{1}{(e^{-s}+e^s)^2} ds d\psi$$

Only square integrable functions are admissible in $A$. But we find that

$$\int \phi^2 d\mu = \int_0^\infty \int_0^{2\pi} \cosh^3 s\xi(\psi) \frac{1}{(e^{-s}+e^s)^2} ds d\psi$$

diverges for non-zero $\xi(\psi)$. Thus $\phi(s,\psi) = \cosh^3 s\xi(\psi)$ does not belong to $A$. Similar calculation shows that any subgroup with any of the boost generators cannot have non-trivial invariant function.

Now let us check for rotation generator. Consider generator of rotation along $z$ axis $M_{12}$.

$$g_t = \begin{bmatrix} e^{it/2} & 0 \\ 0 & e^{-it/2} \end{bmatrix}$$

For $\phi(z,\bar{z})$ to be fixed under $M_{12}$ it has to satisfy condition

$$\phi(z,\bar{z}) = g_t\phi(z,\bar{z}) = \phi(e^{it}z, e^{-it}\bar{z})$$

Using the polar coordinates $z = re^{it}$ the above condition gives

$$
\begin{aligned}
\phi(r,t) &= \phi(r,2t) \\
\Longrightarrow \phi(r,t) &= \phi(r,0) \equiv \xi(r)
\end{aligned}
$$

Integration measure is given by

$$
d\mu = \frac{rdrdt}{(1+r^2)^2}
$$

Clearly there are many nontrivial functions $\xi(r)$ which are square integrable

$$
\int \xi^2(r) \frac{rdrdt}{(1+r^2)^2} < \infty
$$

Thus we conclude that sub-groups consisting of generators of rotation but not of boost have non-trivial invariant function. There are three such connected sub-groups

1. $Z_2$. Invariant function $\phi(z, \bar{z})$. Coset space $G/Z_2$.

2. $\Gamma = \{M_{12}\}$. Invariant function $\phi(|z|)$. Coset space $G/\Gamma$.

3. $SU(2) = \{M_{12}, M_{23}, M_{31}\}$. Invariant function $\phi = K(constant)$. Coset space $G/SU(2)$.

One can see that the above invariant function crucially depends on what topology we impose on $A$. Here we chose square integrable functions. One of the important little groups that is missing is $\Delta$, the little group of massless particles. In the next section we will see that weakening the topology to Nuclear topology allows one to have more general non-trivial invariant functions. This will then include $\Delta$ and many other little groups.

## 7.2 BMS group in Nuclear topology

As mentioned at the end of precious section, square integrable functions on sphere is not the best choice. From this section on-wards we follow the definition of [87] and take these to be $C^\infty$ which implies that the representation of $G$ on $A$ is equivalent to the operator representation of $G$ on the space $D_{(2,2)}$ [18]. This is known as Nuclear topology.

The space $D_{(2,2)}$ consists of pair of functions $\xi(z)$ and $\hat{\xi}(z)$ on the complex plane, which may be thought of as functions on patches centered at the north and south poles of the sphere respectively. These functions are $C^\infty$ everywhere except at the origin and are related by the overlap condition

$$\hat{\xi}(z) = |z|^2 \xi(z^{-1})$$

The action of $SL(2,\mathbb{C})$ element $\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$ is given by

$$\begin{aligned} g\xi(z) &= |\alpha + \gamma z|^2 \xi \left( \frac{\beta + \delta z}{\alpha + \gamma z} \right) \\ g\hat{\xi}(z) &= |\beta + \delta z|^2 \hat{\xi} \left( \frac{\alpha + \gamma z}{\beta + \delta z} \right) \end{aligned} \tag{7.2.1}$$

We see that it is more appropriate to view the functions as quasi primary fields. We will be mostly interested in the dual space of $A$. As we will see this corresponds most directly to the class of functions $m_B(u, z, \bar{z})$ that appear for some fixed value of $u$. The dual space corresponds to the space $D_{(-2,-2)}$ in the notation of [18] and, as we will see, is a space of distributions with a class of allowed singularities. It is specified again by a pair of functions satisfying the matching condition

$$\hat{\phi}(z) = |z|^{-6} \phi(z^{-1}) \tag{7.2.2}$$

The action of $G$ is given by

$$g\phi(z) = |\alpha+\gamma z|^{-6}\phi\left(\frac{\beta+\delta z}{\alpha+\gamma z}\right) \qquad (7.2.3)$$
$$g\hat{\phi}(z) = |\beta+\delta z|^{-6}\hat{\phi}\left(\frac{\alpha+\gamma z}{\beta+\delta z}\right)$$

Representations of BMS group are labelled by infinite number of charges called supermomenta in addition to four-momenta. Together they are denoted by $(p_{00}, p_{11}, p_{1-1}, p_{10}, \{p_{lm}\})$. We will call this generalized momentum or just momenta. Four-momenta are related by $(E = p_{00}, p_x + ip_y = p_{11}, p_x - ip_y = p_{1-1}, p_z = p_{10})$. One can read off $p_{lm}$ from the function $\phi(\theta,\phi)$ by expanding in terms of spherical harmonics $P_{lm}$

$$\phi(\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} p_{lm}\cos^6\frac{\theta}{2}P_{lm}(\theta,\phi) \qquad (7.2.4)$$

The 4-momentum associated with the functions $\phi(z)$ may also be extracted via the projector $\Pi$ expressed as the integral

$$\Pi\phi(z') = \frac{i}{\pi}\int dz d\bar{z}(z-z')(\bar{z}-\bar{z}')\phi(z) \qquad (7.2.5)$$
$$= \frac{i}{\pi}\left((p^0+p^3)+(p^0-p^3)z'\bar{z}'-(p^1-ip^2)z'-(p^1+ip^2)\bar{z}'\right)$$

which is a polynomial of weight 2 in $z'$, with coefficients corresponding to the 4-momenta [87] $p^\mu$. For this to be well-defined, the regulator as $|z|\to\infty$ implicit in the definition of the $D_{(-2,-2)}$ distributions must be taken into account. This can therefore be rewritten in terms of convergent integrals as

$$\Pi\phi(z') = \frac{i}{\pi}\int_{|z|<1} dz d\bar{z}\left((z-z')(\bar{z}-\bar{z}')\phi(z)+(1-zz')(1-\bar{z}\bar{z}')\hat{\phi}(z)\right) \qquad (7.2.6)$$

The higher order terms in $\phi(z)$ are labelled by the supermomenta. The supermomenta form a $G$ invariant subspace, implying that an irreducible representation of the BMS group describes states with the same mass (i.e. 4-momentum squared). Equation (7.2.5) matches equation 72 in [82] which gives the Bondi 4-momentum in terms of an integral of the Bondi mass aspect $m_B(u, z, \bar{z})$. Thus we may identify $\phi(z)$ with $m_B$ up to a rescaling factor, and the derived 4-momenta behave as expected under $G$.

In turn, this provides a more physical justification for the choice of the space of functions $D_{(-2,-2)}$. This space of distributions yield 4-momenta corresponding to finite center of mass energies, as well as finite supermomenta, and prescribed fall-off conditions [18] that guarantee integrals such as (7.2.5) are well-defined.

### 7.2.1 Little groups

As with Wigner's classification of the irreducible representations of the Poincare group, the first step in understanding representations is to understand little groups. One may then construct the irreducible representations via the method of induced representations [83, 121], lifting representations of the subgroup to representations of the group.

In Wigner's classification, one identifies classes of four-momenta invariant under Poincare subgroups. For BMS the goal is to find functions $\phi(z)$ invariant under the little groups of BMS. McCarthy give a detailed list of most of the little groups [87]. Here we discuss some of them in detail. We want to find functions which are invariant under (7.2.3) of subgroups. These solutions will not contain the singular distributions like $\delta$ functions and its derivatives. So we add to these solutions, singular distributions satisfying appropriate differential equation. We show the calculations explicitly for some of the little groups.

#### 7.2.1.1 $SU(2)$

The first important little group is $SU(2)$. Generators of $SU(2)$ are well known Pauli matrices. We shall check for $\sigma_1, \sigma_2, \sigma_3$ separately. First let us look at $\Sigma_3 = \begin{bmatrix} \omega & 0 \\ 0 & \bar{\omega} \end{bmatrix}$. This is same as $\Gamma$. Which says that $\phi, \hat{\phi}$ must be function of $|z|$.

For $\Sigma_1 = e^{it\sigma_1} = \cos t + i \sin t\sigma_1$. For small $t$, $\Sigma_1 = \begin{bmatrix} 1 & it \\ it & 1 \end{bmatrix}$. we get

$$\begin{aligned} |1+itz|^{-6}\phi\left(\frac{it+z}{1+itz}\right) &= \phi(z) \\ |1+itz|^{-6}\hat{\phi}\left(\frac{it+z}{1+itz}\right) &= \hat{\phi}(z) \end{aligned}$$

Only function which satisfies the above equation is

$$\phi(z) = \hat{\phi}(z) = m\left(1+|z|^2\right)^{-3} \tag{7.2.7}$$

For consistency one can also check that above functions are invariant under $\Sigma_2 = e^{it\sigma_2} = \cos t + i \sin t\sigma_2$. For small $t$, $\Sigma_2 = \begin{bmatrix} 1 & t \\ -t & 1 \end{bmatrix}$. $\phi, \hat{\phi}$ does not diverge for any value of $z$. So there is no singular distribution. One can verify that this represents a particle of mass $m$ at rest

$$\begin{aligned}
\pi\phi &= \frac{2}{\pi}\int_{|z|<1}\left[(z-z')(\bar{z}-\bar{z}')\phi(z)+(1-zz')\left(1-\bar{z}\bar{z}'\right)\hat{\phi}(z)\right]dzd\bar{z}\\
&= \frac{2}{\pi}\int_{|z|<1}\left[(1+|z'|^2)|z|^2-(z'+\bar{z}')z+(z'+\bar{z}')\bar{z}+(1+|z'|^2)\right]m\left(1+|z|^2\right)^{-3}dzd\bar{z}\\
&= \frac{2}{\pi}\int\left[(1+|z'|^2)r^2-(z'+\bar{z}')re^{i\theta}+(z'+\bar{z}')re^{-i\theta}+(1+|z'|^2)\right]m\left(1+r^2\right)^{-3}rdrd\theta\\
&= \frac{2}{\pi}\int\left[(1+|z'|^2)r^2+(1+|z'|^2)\right]m\left(1+r^2\right)^{-3}rdrd\theta\\
&= \frac{2}{\pi}\int(1+|z'|^2)m\left(1+r^2\right)^{-2}rdrd\theta\\
&= m(1+|z'|^2)
\end{aligned}$$

Comparing this with equation (7.2.5) we get $p_0=m, p_1=p_2=p_3=0$. This shows that mass of the particle is $m$. One can check that the 4-momentum $(p_0,p_1,p_2,p_3)$ indeed transforms correctly under the action of Lorentz generators (7.2.3). As an example, let us look at the action of boost $g_t=\begin{bmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2}\end{bmatrix}$. Acting on (7.2.7)

$$g\phi(z) = me^{-3t}\left(1+|z|^2e^{-2t}\right)^{-3}=m\left(e^{-t}|z|^2+e^{t}\right)^{-3} \tag{7.2.8}$$

First half of the integral (7.2.6) gives

$$\begin{aligned}
&\frac{2}{\pi}\int_{|z|<1}(z-z')(\bar{z}-\bar{z}')g\phi(z)rdrd\theta\\
&= 2\int_{|z|<1}(r^2+|z'|^2)e^{-3t}m\left(1+e^{-2t}r^2\right)^{-3}dr^2\\
&= 2\int_{x<1}(x+|z'|^2)e^{3t}\frac{mx^{-3}}{(1+e^{2t}/x)^3}dx\\
&= 2\int_{\infty}^{1}(1+|z'|^2y)e^{3t}\frac{my^2}{(1+e^{2t}y)^3}\frac{-dy}{y^2}\\
&= 2\int_{1}^{\infty}(1+|z'|^2y)e^{3t}\frac{m}{(1+e^{2t}y)^3}dy
\end{aligned}$$

Second half of the integral (7.2.6) gives

$$\begin{aligned} & \frac{1}{\pi}\int_{r^2<1}\left(1+r^2|z'|^2\right)e^{3t}m\left(1+e^{2t}r^2\right)^{-3}dr^2d\theta \\ = & \; 2\int_0^1\left(1+y|z'|^2\right)e^{3t}\frac{m}{\left(1+e^{2t}y\right)^3}dy \end{aligned}$$

So

$$\begin{aligned} \pi\phi &= 2\int_0^\infty\left(1+y|z'|^2\right)\frac{e^{3t}m}{\left(1+e^{2t}y\right)^3}dy \\ &= 2\int_0^\infty\left(\frac{e^{3t}m}{\left(1+e^{2t}y\right)^3}+|z'|^2\frac{e^{-3t}m}{\left(1+e^{-2t}y\right)^3}\right)dy \\ &= m\left(e^t+|z'|^2e^{-t}\right) \end{aligned}$$

which leads via (7.2.5) to $p^0 = m\cosh t$ and $p^3 = m\sinh t$ as expected for a boost.

Also note that the super-momenta get populated by the action of the boost due to the higher order terms present in (7.2.8) beyond order 2. Thus the simplest representation of BMS is that of a massive particle, matching what one expects of the Poincare group, but the representation traces out an orbit in the infinite dimensional space of supermomenta as one acts with Lorentz generators.

The representations of the little group may also carry spin $\ell$ which is half-integer. As shown in [84] this yields a single spin $\ell$ representation of the Poincare subgroup of BMS.

#### 7.2.1.2 $\Delta$

The second important little group is

$$\Delta = \begin{bmatrix} \omega & \beta \\ 0 & \bar{\omega} \end{bmatrix}$$

, in the notation of [87], or more commonly the Euclidean group in two dimensions $E(2)$. It yields usual massless particles, and as above, Lorentz transformation fill out an orbit in supermomentum space. Using equation (7.2.3) we get

$$\begin{aligned} \phi\left(\frac{\bar{\omega}z+\beta}{\omega}\right) &= \phi(z) \\ \hat{\phi}\left(\frac{\omega z}{\beta z+\bar{\omega}}\right) &= |\beta z+\bar{\omega}|^{6}\hat{\phi}(z) \end{aligned} \tag{7.2.9}$$

Looking at the first equation for $\beta = 0$ implies that $\phi$ must be function of $|z| = r$. Then for $\omega = 1$ and $\beta = ib$ (same as $\Lambda$ done in the paper) we deduce that $\phi$ must be function of $z + \bar{z} = r\cos\theta$. Only function which satisfies these two conditions is constant function. Together with equation (7.2.2) we get

$$\begin{aligned} \phi(z) &= K \\ \hat{\phi}(z) &= K|z|^{-6} \end{aligned}$$

The invariance condition has no singular points. The general solution for $\hat{\phi}$ is thus above function plus any linear combination of $\delta$ function and its derivative satisfying the appropriate differential equation coming from (7.2.9). Second equation of (7.2.9) can be written as

$$\begin{aligned} \hat{\phi}(z) &= |1+\omega ibz|^{-6}\hat{\phi}\left(\omega^{2}z\left(1-\omega ibz\right)\right) \\ &\approx \left(1+ibz\right)^{-3}\left(1-ib\bar{z}\right)^{-3}\hat{\phi}\left(z+2i\delta z-ibz^{2}\right) \\ &\approx \hat{\phi}(z)+2i\delta\left(z\partial_{z}-\bar{z}\partial_{\bar{z}}\right)\hat{\phi}(z)+ib\left(z^{2}\partial_{z}+3z-\bar{z}^{2}\partial_{\bar{z}}-3\bar{z}\right)\hat{\phi}(z) \end{aligned} \tag{7.2.10}$$

Solution of this gives the invariant functions

$$\begin{aligned}\phi(z) &= K \\ \hat{\phi}(z) &= K|z|^{-6} + A\frac{\partial^2}{\partial z^2}\frac{\partial^2}{\partial \bar{z}^2}\delta(z) + C\delta(z)\end{aligned} \tag{7.2.11}$$

Note here $\delta(z) \equiv \delta(\mathrm{Re}z)\delta(\mathrm{Im}z)$, and likewise we suppress the $\bar{z}$ dependence of $\phi$,$\hat{\phi}$. Here $A$ and $C$ are real. This clearly illustrates the need for the $D_{(-2,-2)}$ space of generalized functions to correctly accommodate massless particles. These representations were not present in the earlier studies [86, 85, 84]. To evaluate four momentum on such a representation one must use the formula (7.2.6) to properly regulate the otherwise divergent expression (7.2.5). Finite 4-momenta are obtained provided $K = 0$. In this case, $C$ is proportional to the light-like 4-momentum.

The spin of these representations has been studied in [87] and as expected one gets either a chiral massless representation with a single Poincare spin $s = 0, 1/2, \cdots$. Alternatively one may get one of the massless continuous spin representations of Wigner's classification, whose physical significance remains unclear.

#### 7.2.1.3 $SL(2,C)$

In general one may take the entire group of Lorentz transformations to be a little group, in which case the invariant functions take the form

$$\phi(z) = \hat{\phi}(z) = 0$$

which implies vanishing of the 4-momentum and of all the supermomentum. Nevertheless, one may pick a unitary representation of the little group and lift it to a representation of BMS. It is natural to think of such representations as arising from a unitary irreducible representation corresponding to a massive (or massless) field on an internal three-dimensional de Sitter spacetime $dS_3$ [122]. Such representations are infinite-dimensional. In any case, the situation here is unchanged from the usual Poincare group. The standard procedure is to throw out all but the trivial representation, leaving the Poincare invariant vacuum as the unique state with vanishing 4-momentum. Lifting to BMS, we obtain a unique state with vanishing 4-momentum and supermomentum. Since the other infinite-dimensional families of states are not generated from tensor products of the other states we will consider with the vacuum, we can safely ignore these exotic infinite dimensional representations with vanishing momentum.

#### 7.2.1.4 $SL(2,R)$

The situation is more interesting for this maximal little group. In this case the invariant functions take the form

$$\begin{aligned} \phi(z) &= K\left(\frac{z-\bar{z}}{i}\right)^{-3} + A\delta^2\left(\frac{z-\bar{z}}{i}\right) \\ \hat{\phi}(z) &= K\left(\frac{\bar{z}-z}{i}\right)^{-3} + A\delta^2\left(\frac{\bar{z}-z}{i}\right) \end{aligned} \tag{7.2.12}$$

where $K$ and $A$ are real parameters. For the Poincare group, this little group would usually give rise to the tachyonic representations where $p_\mu p^\mu < 0$. Here the nuclear topology restricts the class of distributions to those with vanishing 4-momentum when inserted into (7.2.6). Nevertheless, the higher order terms present in the invariant functions generate a nontrivial orbit corresponding to nonvanishing supermomentum.

As with the case of $SL(2,C)$ one can assign such representations a nontrivial representation of the little group. In this case it would correspond to a massive or massless field on an internal two-dimensional de Sitter spacetime, which has the isometry group $SL(2,R)$. However again such representations are infinite dimensional, and will not arise from tensor products of the elementary representations we will consider. These representations arise already in Wigner's classification of the representations of the

Poincare group, and are likewise not thought to be physically relevant, because one can construct self-consistent theories where they do not appear.

An exception is the trivial representation of the little group $SL(2,R)$. Under the usual Poincare classification, these would be invariant under a larger little group $SL(2,C)$ and so would be equivalent to the $SL(2,C)$ invariant vacuum state. However under BMS such modes carry nontrivial supermomentum. This leads to a class of "soft modes" which in general will be produced in the scattering of particle-like states, and are in general necessary to enforce conservation of supermomentum.

#### 7.2.1.5 $\Gamma$

For the Poincare group, the maximal little groups exhaust the set of little groups. However for BMS it is also necessary to consider the group $\Gamma$ which is a subgroup of all the above little groups corresponding to rotations in a plane $\begin{pmatrix} \omega & 0 \\ 0 & \bar{\omega} \end{pmatrix}$ with $\omega$ a complex number of unit modulus. While the 4-momenta invariant with respect to this little group are actually invariant under a larger little group, this is no longer the case when the supermomenta are included. Using equation (7.2.3) we get

$$\begin{aligned} \phi\left(\frac{\bar{\omega}z}{\omega}\right) = \phi\left(\bar{\omega}^2 z\right) &= |\omega|^6\phi(z) = \phi(z) \\ \hat{\phi}\left(\frac{\omega z}{\bar{\omega}}\right) = \hat{\phi}\left(\omega^2 z\right) &= |\bar{\omega}|^6\hat{\phi}(z) = \hat{\phi}(z) \end{aligned}$$

Above conditions imply that $\phi, \hat{\phi}$ must be function of $|z|$. Secondly, $\hat{\phi}(z) = |z|^{-6}\phi(z)$. Thus we get

$$\phi(z) = \beta(r) \tag{7.2.13}$$

$$\hat{\phi}(z) = r^{-6}\beta\left(r^{-1}\right) \tag{7.2.14}$$

Now we have to check if there is any singular distribution solution possible. General solution of $\hat{\phi}$ must satisfy

$$(z\partial_z - \bar{z}\partial_{\bar{z}})\, f \quad = \quad 0$$

Using the results $z\partial_z\delta^{\mu,\nu} = -(\mu+1)\delta^{\mu,\nu}$ and $\bar{z}\partial_{\bar{z}}\delta^{\mu,\nu} = -(\nu+1)\delta^{\mu,\nu}$ we get following solutions

$$\begin{aligned} \phi(z) &= \beta(r) \\ \hat{\phi}(z) &= r^{-6}\beta\left(r^{-1}\right) + A\delta^{2,2} + C\delta \end{aligned} \tag{7.2.15}$$

where $z = re^{i\phi}$ with $\phi = [0, 2\pi)$ and $r \geq 0$. Here $\beta$ is any distribution of radial coordinate such that $\phi, \hat{\phi}$ are well defined distribution in radial coordinate satisfying the conditions above. The 4-momenta corresponding to these representations may have $m^2 = 0$, $m^2 > 0$ or $m^2 < 0$. Enhancing the little group to $SU(2)$ restricts invariant function to (7.2.7) which is a special case of above invariant function.

For $m^2 > 0$ the representation corresponds [84] to an infinite tower of Poincare spins labelled by some integer/half-integer $j$ with the tower corresponding to all spins $s = j, j+1, \cdots$.

For $m^2 = 0$ and $m^2 < 0$ the Poincare representations are more exotic, with integrals over continuous spins needed to generate the BMS representation.

#### 7.2.1.6 $\Lambda$

Invariant functions under this group is given in [86]. We are reproducing it here for the sake of completeness. Orbits in $\Lambda$ little group are nothing but functions invariant under the motion generated by linear combination of following rotation generators

$$\frac{1}{2}\left(M_{23} + iM_{31}\right) \quad = \quad \frac{1}{2}\left(\sigma_1 + i\sigma_2\right) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Corresponding finite operator is

$$\Lambda = e^{ib\frac{1}{2}(\sigma_1+i\sigma_2)} = \begin{bmatrix} 1 & ib \\ 0 & 1 \end{bmatrix}$$

The invariance conditions are

$$\begin{aligned} \phi(z+ib) &= \phi(z) \\ \hat{\phi}\left(\frac{z}{ibz+1}\right) &= |ibz+1|^6\hat{\phi}(z) \end{aligned} \qquad (7.2.16)$$

This is a special case $\omega = 1$ of $\Delta$ little group. Solutions satisfying above conditions and (7.2.2) are

$$\begin{aligned} \phi(z) &= \beta(z+\bar{z}) \\ \hat{\phi}(z) &= |z|^{-6}\beta(z^{-1}+\bar{z}^{-1}) \end{aligned}$$

The invariance condition (7.2.16) has no singular points. Thus $\beta$ is any distribution such that $\phi, \hat{\phi}$ are well defined. The general solution for $\hat{\phi}$ is the above one plus any linear combination of $\delta$function and its derivatives satisfying the differential equation coming from (7.2.16). Second equation (7.2.16) can be written as which can also be found by substituting $\omega = 1$ in equation (7.2.10)

$$\begin{aligned} \hat{\phi}(z) &= |1+ibz|^{-6}\hat{\phi}\left(z\left(1-ibz\right)\right) \\ &\approx \left(1+ibz\right)^{-3}\left(1-ib\bar{z}\right)^{-3}\hat{\phi}\left(z-ibz^2\right) \\ &\approx \hat{\phi}(z) + ib\left(z^2\partial_z + 3z - \bar{z}^2\partial_{\bar{z}} - 3\bar{z}\right)\hat{\phi}(z) \end{aligned}$$

Solution of this gives the invariant functions

$$\phi(z) = \beta(z+\bar{z})$$
$$\hat{\phi}(z) = |z|^{-6}\beta(z^{-1}+\bar{z}^{-1}) + A\frac{\partial^2}{\partial z^2}\frac{\partial^2}{\partial \bar{z}^2}\delta(z,\bar{z}) + B\frac{\partial^2}{\partial z^2}\delta(z,\bar{z}) + \bar{B}\frac{\partial^2}{\partial \bar{z}^2}\delta(z,\bar{z}) + C\delta(z,\bar{z}) \tag{7.2.17}$$

$A, C$ are real and $B$ is complex.

#### 7.2.1.7 Indecomposable

In the present work we are restricting our consideration to unitary irreducible representations of the BMS group. It is possible this is too restrictive a class of representations to build a useful holographic description of asymptotically flat space. Because the BMS group is non-compact, representations that may be decomposed into irreducible representations are actually rather special, and more generally one should consider indecomposable representations. As far as we are aware, the classification of such representations for non-compact groups is still relatively undeveloped.

#### 7.2.1.8 Non-connected subgroups

There are a variety of non-connected little groups that can appear as subgroups of the BMS group [87]. For simplicity we do not consider these in the present work.

## 7.3 Invariant norms and Holography

One of the main motivations for considering the irreducible representations of the BMS group, is to get a better understanding of the basic ingredients needed to build a holographic description of the theory on null infinity $\mathcal{I}$. The same considerations also apply when considering the allowed set of asymptotic states in an S-matrix description of a gravitational theory. At large N holography amounts to change of basis between momentum basis and conformal basis. This is done by smearing the field using a conformally invariant kernel over the whole space, as shown in our earlier works on de-Sitter holography [22]. Only difference in the case of Minkowski space is the integration measure. As such, we now turn our attention to defining BMS invariant norms for the representations of interest, and see that these may be realized as integrals on $\mathcal{I}$.

In the general case the norm is defined using the group invariant measure on the coset space $G/H$ where $G = SL(2, C)$ and $H$ is the little group [86]

$$\int f(g) d\mu(g) \;\; = \;\; \int_{G/H} \left( \int_H f(gh) d\mu(h) \right) d\mu_{G/H} \, .$$

### 7.3.1 $SU(2)$ and $\Delta$

It is perhaps simplest to begin in momentum space. As we have seen for the $SU(2)$ little group, we have representations of BMS that essentially coincide with ordinary massive particle representations of the Poincare group. The same is true for massless particles and the little group $\Delta$. Consider wavefunction in momentum space $\psi(p)$. Transformation of $\psi(p)$ under the action of BMS group is given in section 3 of [85]. Wigner has given the invariant norm for these two subgroups as

$$(\psi_1, \psi_2) \;\; = \;\; \int_0^\infty \psi_1(p) \psi_2(p) \frac{dp_1 dp_2 dp_3}{p_4} \, . \qquad (7.3.1)$$

As we see, this integral may be viewed as an on-shell integral in the bulk $p_4^2 = m^2 + \sum_i p_i^2$, or as an off-shell integral over the holographic boundary $\mathcal{I}$.

### 7.3.2 $SL(2, R)$

Again the little group is three-dimensional, but now the invariant norm can be interpreted as an integral over three-dimensional de Sitter spacetime which corresponds to the coset $SL(2,C)/SL(2,R)$

$$(\psi_1, \psi_2) = \int_{\sum_i p_i^2=1}^{\infty} \psi_1(p)\psi_2(p)\frac{dp_1dp_2dp_3}{\sqrt{\sum_i p_i^2 - 1}} \tag{7.3.2}$$

### 7.3.3 Γ

Since Γ is only one-dimensional the coset space will be five-dimensional and may be written as an integral over on-shell 4-momenta ($p_4^2 = \sum_i p_i^2 + m^2$) supplemented by a pair of angles

$$(\psi_1, \psi_2) = \int_0^{\infty} \psi_1(p,\theta)\psi_2(p,\theta)\frac{dp_1dp_2dp_3d\theta_1d\theta_2}{p_4} \tag{7.3.3}$$

which may be interpreted as an integral over $\mathcal{I}$ and two internal degrees of freedom $\theta$.

### 7.3.4 Λ

Here instead of three rotation generators $M_{12}, M_{23}, M_{31}$, we will choose the following linear combinations $M_{12}, \frac{1}{2}(M_{23} + iM_{31}), \frac{1}{2}(M_{23} - iM_{31})$.

$$\begin{aligned} M_{12} &= \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\ \frac{1}{2}(M_{23} + iM_{31}) &= \frac{1}{2}(\sigma_1 + i\sigma_2) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\ \frac{1}{2}(M_{23} - iM_{31}) &= \frac{1}{2}(\sigma_1 - i\sigma_2) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \end{aligned}$$

Corresponding finite operators are

$$\Sigma_3 = e^{i\theta_3\sigma_3} = \begin{bmatrix} e^{i\theta_3} & 0 \\ 0 & e^{-i\theta_3} \end{bmatrix}$$
$$L_+ = e^{ib\frac{1}{2}(\sigma_1+i\sigma_2)} = \begin{bmatrix} 1 & ib \\ 0 & 1 \end{bmatrix}$$
$$L_- = e^{ic\frac{1}{2}(\sigma_1-i\sigma_2)} = \begin{bmatrix} 1 & 0 \\ ic & 1 \end{bmatrix}$$

Boost generators stay as it is. So the coordinates are $\theta_3, b, c$ and three for boost $p_1, p_2 p_3$. Then

$$d\mu(g) = dbdcd\theta_3 \frac{dp_1 dp_2 dp_3}{p_4}$$

Note that $\Lambda$ subgroup contains operators exactly of the form $L_+$. So

$$d\mu(\Lambda) = db$$
$$\Longrightarrow d\mu_{G/\Lambda} = dcd\theta_3 \frac{dp_1 dp_2 dp_3}{p_4}$$

So invariant norm is

$$(\psi_1, \psi_2) = \int \psi_1(p, c, \theta_3)\psi_2(p, c, \theta_3) dcd\theta_3 \frac{dp_1 dp_2 dp_3}{p_4}$$

which may be interpreted as an integral over $\mathcal{I}$ and $c, \theta_3$.

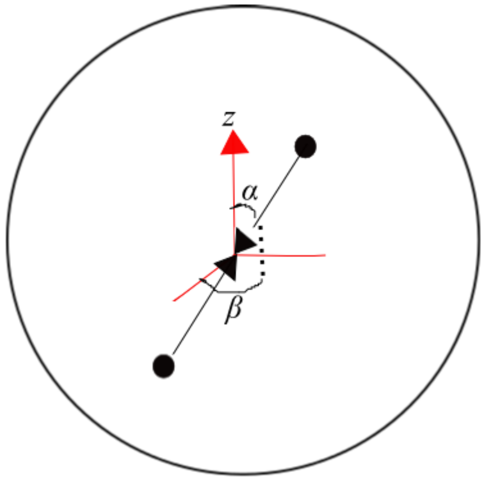


Figure 7.4.1: The figure depicts two particles of mass $M$ moving in opposite directions forming a bound state.

## 7.4 Scattering examples

By taking tensor products of the above representations of BMS we can gain insight into how the symmetry constrains the scattering of particle-like representations and study what BMS representations appear when ordinary particles undergo scattering.

### 7.4.1 Particles forming bound state

Consider a representation of the $SU(2)$ little group corresponding to two particles with mass $M$. One is moving in angular direction $(\theta, \phi) = (\alpha, \beta)$ and the other in the opposite direction $(\alpha - \pi, \beta)$. After sometime they collide and form a bound state as shown in figure(7.4.1) The mass aspect functions of a particle can be found by boosting (7.2.7) along $z$-axis and then rotating by $\alpha$ around $y$-axis followed by rotation around $z$-axis by $\beta$. That is

$$\phi_{+\alpha} = g_\alpha \phi_{z+}$$

where

$$\phi_{+\alpha}(\theta,\phi) = \begin{bmatrix} \cos\frac{\alpha}{2} & \sin\frac{\alpha}{2} \\ -\sin\frac{\alpha}{2} & \cos\frac{\alpha}{2} \end{bmatrix} \phi_{z+} = \phi_{z+}\left(\frac{z\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}}{z\sin\frac{\alpha}{2} + \cos\frac{\alpha}{2}}\right) = \left(\frac{e^t\left|\frac{z\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}}{z\sin\frac{\alpha}{2}+\cos\frac{\alpha}{2}}\right|^2 + e^{-t}}{1+\left|\frac{z\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}}{z\sin\frac{\alpha}{2}+\cos\frac{\alpha}{2}}\right|^2}\right)^{-3} M$$
$$= \frac{M}{(\cosh t - \sinh t\cos\theta\cos\alpha - \sinh t\sin\theta\cos\phi\sin\alpha)^3}$$

and rotating by $\beta$ around $z$-axis gives

$$\phi_{+\alpha,\beta}(\theta,\phi) = \frac{M}{(\cosh t - \sinh t\cos\theta\cos\alpha - \sinh t\sin\theta\cos(\phi+\beta)\sin\alpha)^3} \tag{7.4.1}$$

Now consider a particle moving in opposite direction. That is angular coordinates $(\alpha-\pi,\beta)$.

$$\phi_{-\alpha,\beta}(\theta,\phi) = \frac{M}{(\cosh t + \sinh t\cos\theta\cos\alpha + \sinh t\sin\theta\cos(\phi+\beta)\sin\alpha)^3} \tag{7.4.2}$$

At large N, mass aspect function of the whole system is given by the sum of mass aspect functions of individual particles

$$\begin{aligned}\phi_{\alpha,\beta}(\theta,\phi) &= \phi_{+\alpha,\beta}(\theta,\phi) + \phi_{-\alpha,\beta}(\theta,\phi) \\ &= \frac{M}{(\cosh t - \sinh t\cos\theta\cos\alpha - \sinh t\sin\theta\cos(\phi+\beta)\sin\alpha)^3} \\ &\quad + \frac{M}{(\cosh t + \sinh t\cos\theta\cos\alpha + \sinh t\sin\theta\cos(\phi+\beta)\sin\alpha)^3}\end{aligned} \tag{7.4.3}$$

Since two particles are moving in opposite directions, in no frame will both the particles be at rest together. Neither boosts nor rotations can transform the above function into a constant. The 4-momenta may be evaluated using (7.2.5) and are given by $p_0 = 2M \cosh t, p_1 = p_2 = p_3 = 0$. The higher momenta corresponding to supermomenta are nontrivial, and are functions of $\alpha, \beta$. Performing a rotation of $-\beta$ around $z$-axis followed by $-\alpha$ around $y$-axis transforms (7.4.3) to a function of $\cos\theta$ only. This implies the function is invariant under $\Gamma$ little group and no bigger subgroup of Lorentz group.

This construction also provides insight into the invariant norm for the $\Gamma$ representations (7.3.3). While boosts fill out three dimensions of the associated states as usual, one needs an extra integral over the angular directions corresponding to $(\alpha, \beta)$ to generate the complete set of associated states, yielding the five-dimensional integral in (7.3.3).

So we come to an interesting conclusion. The mass aspect functions of $\Gamma$ can be viewed as sum of mass aspect functions of $SU(2)$. In other words, the tensor product of two massive irreducible, unitary representations of BMS can be decomposed as a direct sum of irreducible representations, one of which is specified by a supermomentum orbit whose little group is G. One may perform essentially the same computation for the massless representations associated with the little group $\Delta$ replacing those of $SU(2)$. BMS representation of the final system retains memory about the direction of the incoming particles. In this case of two-body scattering, the supermomenta allow all the information about the initial state of the system to be retrieved from the final bound state. This is in line with the soft hair proposal of Strominger et al [95].

### 7.4.2 Soft modes in scattering

Extending the above considerations, we now consider $2 \to 2$ scattering. Consider an initial state $\phi_z$, and the final state $\phi_x$ and soft modes. Figure (7.4.2) shows the process.

$$\begin{aligned}\phi_{initial}(\theta,\phi)=\phi_z(\theta,\phi) &= (2M,0,0,0,\{p_{l,m}\}_z)\\ &= \frac{M}{(\cosh t-\sinh t\cos\theta)^3}+\frac{M}{(\cosh t+\sinh t\cos\theta)^3}\in\Gamma\end{aligned}$$

Final state contains two massive particles moving along the x-axis

$$\phi_x(\theta,\phi) = \frac{M}{(\cosh t-\sinh t\sin\theta\cos\phi)^3}+\frac{M}{(\cosh t+\sinh t\sin\theta\cos\phi)^3} \quad (7.4.4)$$

and soft modes. Soft mode can be found by conservation of supermomenta. The initial mass aspect should match the final mass aspect

$$\begin{aligned}\phi_{initial} &= \phi_{final}\\ \implies \phi_z(\theta,\phi) &= \phi_x(\theta,\phi)+\phi_{soft}\\ \implies (2M,0,0,0,\{p_{l,m}\}_z) &= (2M,0,0,0,\{p_{lm}\}_x)+(0,0,0,0,\{p_{lm}\}_z-\{p_{lm}\}_x)\\ \implies \phi_{soft} &= (0,0,0,0,\{p_{lm}\}_z-\{p_{lm}\}_x) \quad (7.4.5)\end{aligned}$$

In this case, while the outgoing massive particles transform under the standard $SU(2)$ little groups, there is an additional soft mode with vanishing 4-momentum but non–vanishing supermomentum. In this case the soft mode transforms under the $\Gamma$ little group and represents the gravitational memory effect.

## 7.5 Conclusion

Many of the results we have discussed appear in McCarthy's original works but have been passed over in much of the subsequent literature, and our goal was to cast the most relevant selection of these results in a modern context, where they may be of use to researchers attempting holographic formulations of asymptotically flat spacetime,

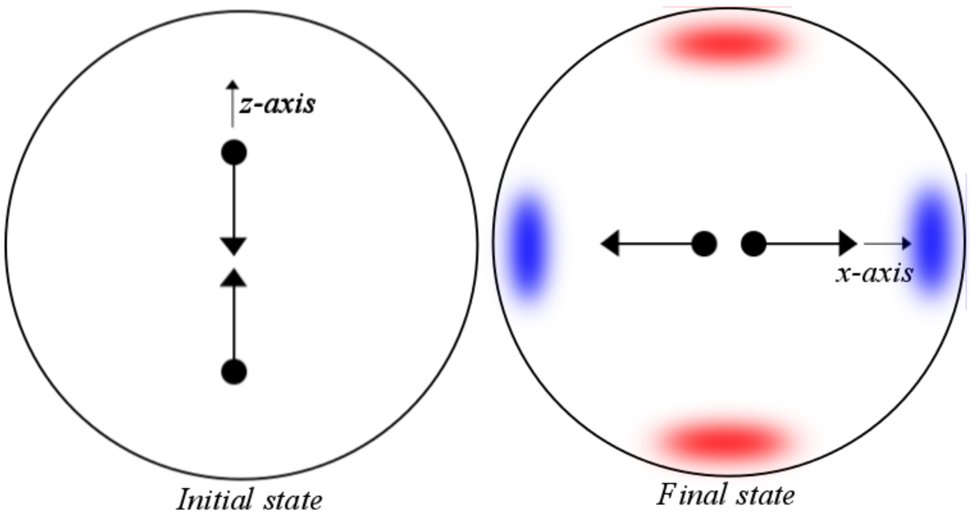


Figure 7.4.2: The left figure represent two particle of mass $m$ moving along $z$-axis. They collide and move out along $x$-axis. The figure on the right represents the final state. Subtracting blue patch from the red patch on the celestial sphere gives the soft mode in the final state.

or simply trying to understand gravitational memory from the perspective of the BMS group. We started with a brief introduction to the BMS group and identified 4-momenta and the supermomenta. Representations of $\Delta, SU(2)$ represent massless and massive particles respectively corresponding directly to Wigner's original classification of the Poincare group. Then we derive the invariant measure and invariant norm for some of the little groups. This revealed that invariant norm of little groups other than $SU(2), \Delta$ involves integrating over a larger phase space. Specifically for $\Gamma$ one encounters integrals over 5 dimensions. Starting with a representative state of $\Gamma$, both rotation and boosts are required to traverse complete orbit inside $\Gamma$. This implies that rotations produce states which cannot be obtained just by boosts. This is related to the fact that representations of $\Gamma$ can be expressed as bound state of rep of $SU(2), \Delta$. To explore this point we considered two particles moving in opposite directions forming a bound state. Momenta of final state depend on the direction of initial particles. In other words, BMS representations store not just the total 4-momenta of the system but also retain information about the individual 4-momenta of the initial state. This is in contrast to Poincare representations where the final state just depends on total energy.

These results should have important implications for any $S-$matrix theory of gravity in asymptotically flat spacetime. In string theory, for example, these $S-$matrix elements are built using vertex operators corresponding to representations of the Poincare group. For such a description to be consistent it is implicit that the scattering states of such particles form a complete set. According to our analysis of the BMS group, that is not the case. For example, there exist unitary irreducible representations of the BMS group with vanishing 4-momenta but non-vanishing supermomenta that are not limits of massless particles (with non-vanishing light-like 4-momentum) such as the soft mode representations of the $SL(2, R)$ little group that we discussed. One also has irreducible representations of the little group $\Gamma$ that can also generate soft modes with vanishing 4-momenta, but non-vanishing supermomenta. On the other hand, it is clear there is a unique vacuum state, the trivial representation of the BMS group, which is of course invariant under all the asymptotic symmetries. There has been some preliminary discussion of some of these issues in the bosonic string [123] but we believe the present results warrant further study of the spectrum of string theory to obtain a more complete understanding of the soft modes.

From the perspective of holography the present work shows what irreducible representations of the BMS group are needed to formulate the elementary ingredients of such a description. There is some commonality with the AdS/CFT approach, namely a holographic "operator" transforming as an irreducible representation of BMS in one-to-one correspondence with bulk fields with fixed mass and spin. Such operators naturally live in a three-dimensional space according to the norms described in section 7.3. However the existence of the more exotic representations discussed above suggest this picture in not complete in the case of BMS. For example if representations of the little group $\Gamma$ must be introduced as elementary operators in the holographic description, they naturally live in a five-dimensional space. Furthermore the operators corresponding to the $SL(2,R)$ representations will serve to generate states with nontrivial supermomenta, with no cost in 4-momentum. These representations appear to live in an auxiliary three-dimensional de Sitter spacetime. From the usual perspective, this would imply the vacuum is highly degenerate, making it difficult to construct a reasonable interacting theory based on such operators at the quantum level. In any case, we hope the present work goes some way to highlighting the obstacles that need to be addressed in formulating holography in asymptotically flat spacetime.

Over the past few years many extensions of BMS group have been discovered. For example, BMS algebra is extended to full Virasoro algebra. This introduces superrotation. It is also possible to consider central extension of Virasoro algebra [99]. It would be very interesting to understand representations of extended BMS group. However, at present physical interpretations of representation of even the BMS group is not fully understood. So in this work we have mainly focussed on representations of BMS group. We hope to come back to representations of BMS group in future.

# Chapter 8

# Conclusion

We started with a brief introduction of AdS/CFT holography in chapter 2. Then in chapter 3 we reviewed Axiomatic QFT. Wightman reconstruction theorem forms the bedrock of our work on de-Sitter holography.

In chapter 4, we described holography in dS space using integral transform. We gave necessary introduction to integral transform in section 4.3. Using integral transform it was shown to be possible to give well defined bulk-boundary map for dS space at least in the large N limit. However, we also found that boundary correlators violate cluster decomposition, signalling breakdown of CFT once interactions are included.

To explore dS holography in the presence of interaction we considered some simple examples in chapter 5. We looked at multi-black hole solutions in 3 dimensional dS space. It was clearly seen that boundary does not decouple from the bulk. Bulk and boundary have to be seen together. This led us to conjecture that conformal gravity is a better candidate to describe quantum gravity in dS.

We continued our exploration of interacting bulk theory in chapter 6. In dS space fields are represented by Principal series and Discrete series representations. We computed their boundary OPEs which were found to have essential singularities. This implied that CFT dual to dS bulk is not well defined.

In chapter 7 we changed track to Minkowski space holography. Our main focus was to understand representations of BMS group which were already discovered by McCarthy in 1972. We reviewed the little group construction of BMS group. Then we explicitly considered some of the interesting little groups. Soft particles were naturally found emerge from the representations. Some of the scattering examples showed that conservation of super-momentum has lot of similarity to gravitational memory effects.

Holography is essentially a map from bulk theory to another manifold. Group theory is expected to play an important role in understanding such a map. We hope that insights obtained from this work will shed some light on holography in dS and Minkowski space. There are many possible directions that can be explored in future. There is lot more to be understood in representations of BMS group and how to apply integral transform to Minkowski space. One can also explore quantum gravity in dS space by studying conformal gravity theory with dS solutions. Lately BMS groups has been extended to full Virasoro algebra. Understanding representations of superrotations can be another exciting future project.

# Chapter 9

# Appendix

## 9.1 Bulk isometries

Bulk $SO(3,1)$ isometries can be expressed in terms of embedding coordinates $X_A = (Y_1, Y_2, Z, T)$

$$z_{AB} = i\,(X_B\partial_A - X_A\partial_B) \tag{9.1.1}$$

where de Sitter spacetime is the hyperboloid

$$R^2 = Y_1^2 + Y_2^2 + Z^2 - T^2\,.$$

Poincare coordinates $(y_1, y_2, \eta)$ are given by

$$\begin{aligned} T &= \frac{R}{2}(\eta - \frac{1}{\eta}) - \frac{1}{2R\eta}(y_1^2 + y_2^2) \\ Y_1 &= \frac{y_1}{\eta} \\ Y_2 &= \frac{y_2}{\eta} \\ Z &= \frac{R}{2}(\eta + \frac{1}{\eta}) - \frac{1}{2R\eta}(y_1^2 + y_2^2)\,. \end{aligned}$$

With inverse relations

$$\begin{aligned}
\eta &= \frac{\sqrt{Y_1^2 + Y_2^2 + Z^2 - T^2}}{Z - T} \\
y_1 &= \frac{Y_1}{Z - T} \\
y_2 &= \frac{Y_2}{Z - T}\,.
\end{aligned}$$

Equation (9.1.1) in $R, \eta, y_1, y_2$ coordinates becomes

$$\begin{aligned}
J_3 \equiv J_{Y_1 Y_2} &= i\left(y_2 \partial_{y_1} - y_1 \partial_{y_2}\right) \\
J_2 \equiv J_{Z Y_1} &= -i\left(\frac{1 + y_1^2 - y_2^2 + \eta^2}{2}\partial_{y_1} + y_1 y_2 \partial_{y_2} + y_1 \eta \partial_\eta\right) \\
-J_1 \equiv J_{Z Y_2} &= -i\left(\frac{1 - y_1^2 + y_2^2 + \eta^2}{2}\partial_{y_2} + y_1 y_2 \partial_{y_1} + y_2 \eta \partial_\eta\right) \\
K_1 \equiv K_{Y_1 T} &= -i\left(\frac{-1 + y_1^2 - y_2^2 + \eta^2}{2}\partial_{y_1} + y_1 y_2 \partial_{y_2} + y_1 \eta \partial_\eta\right) \\
K_2 \equiv K_{Y_2 T} &= -i\left(\frac{-1 - y_1^2 + y_2^2 + \eta^2}{2}\partial_{y_2} + y_1 y_2 \partial_{y_1} + y_2 \eta \partial_\eta\right) \\
K_3 \equiv K_{ZT} &= -i\left(y_1 \partial_{y_1} + y_2 \partial_{y_2} + \eta \partial_\eta\right)\,.
\end{aligned}$$

We can go to the complex coordinate $z = y_1 + i y_2$ and define

$$\begin{aligned}
J_L = z\partial_z + \frac{\eta}{2}\partial_\eta \quad &, \qquad K_{+L} = i\left(z^2 \partial_z + \eta^2 \partial_{\bar z} + z\eta\partial_\eta\right) \quad , \qquad K_{-L} = -i\partial_z \\
J_R = -\bar z\partial_{\bar z} - \frac{\eta}{2}\partial_\eta \quad &, \qquad K_{-R} = -i\left(\bar z^2 \partial_{\bar z} + \eta^2 \partial_z + \bar z\eta\partial_\eta\right) \quad , \qquad K_{+R} = i\partial_{\bar z}\,.
\end{aligned}$$

We see that they take very simple form in Poincare coordinates compared to spherical coordinates.

## 9.2 OPE calculation for principal series

Calculation of $\beta_{k,j}$ is same as in equation (6.4.10).

$$\begin{aligned}
& \beta_{k+1,j}\left(2\left(k+1+\Delta_3-\Delta_2\right)\zeta\left(\frac{\partial}{\partial\zeta}\right)^{k+1}+\left(k+1\right)\left(k+2\Delta_3\right)\left(\frac{\partial}{\partial\zeta}\right)^{k}\right)\zeta^{-h-j} \\
= \quad & \beta_{k,j}\left(\Delta_3+\Delta_1-\Delta_2+k\right)\left(\frac{\partial}{\partial\zeta}\right)^{k}\zeta^{-h-j} \\
\Longrightarrow \quad & \beta_{k+1,j}2\left(k+1+\Delta_3-\Delta_2\right)\left(-h-j\right)...(-h-j-k)+ \\
& \beta_{k+1,j}\left(k+1\right)\left(k+2\Delta_3\right)\left(-h-j\right)...(-h-j-k+1)\zeta^{-h-j-k} \\
= \quad & \beta_{k,j}\left(\Delta_3+\Delta_1-\Delta_2+k\right)\left(-h-j\right)...(-h-j-k+1)\zeta^{-h-j-k} \\
\Longrightarrow \quad & \beta_{k+1,j}\left(2\left(k+1+\Delta_3-\Delta_2\right)\left(-j-h-k\right)+\left(k+1\right)\left(k+2\Delta_3\right)\right)\zeta^{-h-j-k} \\
= \quad & \beta_{k,j}\left(\Delta_3+\Delta_1-\Delta_2+k\right)\zeta^{-h-j-k} \\
\Longrightarrow \quad & \beta_{k+1,j} \\
= \quad & -\beta_{k,j}\frac{\left(\Delta_3+\Delta_1-\Delta_2+k\right)}{\left(k-K_+\right)\left(k-K_-\right)}
\end{aligned}$$

where

$$K_\pm = \frac{1}{2}\left(1+2\left(\Delta_2-j-h\right)\pm\sqrt{\left(1+\Delta_2\right)^2+4\left(2\Delta_3-j-h\right)\left(1-j-h\right)}\right).$$

Calculation of $\beta_{-k,j}$ is

$$
\begin{aligned}
& \beta_{-k-1,j}\left[L_{-1},L_1^{k+1}\right]\zeta^{-h-j} \\
= \quad & \beta_{-k,j}\left(\Delta_3-\Delta_1-\Delta_2-k\right)\left(L_1\right)^k\zeta^{-h-j} \\
\Longrightarrow \quad & \beta_{-k-1,j}\left(2\Delta_3+k-h-j-1\right)...\left(2\Delta_3-h-j-1\right)\left(-h-j\right) \\
& -\beta_{-k-1,j}\left(2\Delta_3+k-h-j\right)...\left(2\Delta_3-h-j\right)\left(k-h-j+1\right)\zeta^{-h-j+k} \\
= \quad & \beta_{-k,j}\left(\Delta_3-\Delta_1-\Delta_2-k\right)\left(2\Delta_3+k-1-h-j\right)...\left(2\Delta_3-h-j\right)\zeta^{-h-j+k} \\
\Longrightarrow \quad & \beta_{-k-1,j}\left(\left(2\Delta_3-h-j-1\right)\left(-h-j\right)-\left(2\Delta_3+k-h-j\right)\left(k-h-j+1\right)\right)\zeta^{-h-j+k} \\
= \quad & \beta_{-k,j}\left(\Delta_3-\Delta_1-\Delta_2-k\right)\zeta^{-h-j+k} \\
\Longrightarrow \quad & \beta_{-k-1,j} \\
= \quad & \beta_{-k,j}\frac{(\Delta_3-\Delta_1-\Delta_2-k)}{(k+2\Delta_3+2j+2h)(k+1)}\,.
\end{aligned}
$$

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