

SOLUBLE MODELS AND THE MEANING OF NONRENORMALIZABILITY

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1. INTRODUCTION

At the present time there exists no satisfactory formulation of the so-called nonrenormalizable quantum field theories, formally or otherwise. Although the need for an infinite number of counterterms is suggested by perturbation theory, this conclusion is often justified by arguments based on the dimensionality of the coupling constant. Guided by the solution of certain idealized models, we believe that the present unsatisfactory situation can be understood and that an alternative picture suggests itself in which nonrenormalizable fields may effectively become "renormalizable".¹ This alternative picture differs fundamentally from the conventional one. Quite simply, in the language of functional integration over fields, nonrenormalizable interactions act, in part, formally like hard cores, so that once introduced their effects can never be completely turned off, but instead as the coupling constant tends to zero, the effects of the hard core invariably remain and give rise to a (zero-coupling) pseudofree theory, which includes the hard core, and which is generally very different than a conventional free theory. If an interacting theory is expanded about the pseudofree theory, rather than the free theory, then conceivably a better-behaved perturbation expansion may arise. As will be indicated, certain soluble models confirm this general view.

We begin with some heuristic considerations of covariant field theory. These are followed by a variety of idealized models the solutions of which exhibit the effect in question. Finally, we return to the covariant case to argue the plausibility of the effect in such cases.

Heuristic Remarks

Consider the functional integral formulation of a scalar field as described in an n-dimensional Euclidean space time. Formally, the generating functional of interest for a typical nonlinear interaction reads

$$\mathcal{N} \int e^{i \int h \phi dx - \int \left\{ \frac{1}{2} [(\nabla \phi)^2 + m^2 \phi^2] + \lambda \phi^p \right\} dx} \mathcal{D}\phi, \quad (1)$$

where $h(x)$ is a test function, $\phi(x)$ a random field, \mathcal{N} a normalization factor, and where $dx \equiv d^n x$, $\lambda > 0$, and p is even. Attempts to give meaning to this expression generally involve cutoffs, normal ordering,

renormalization counterterms, etc. Some improvement, at least in form, comes from grouping the quadratic terms into a normal probability measure μ_F with covariance

$$\langle \Phi(x)\Phi(y) \rangle_F = (2\pi)^{-n} \int \frac{e^{ik(x-y)}}{k^2 + m^2} dk \quad (2)$$

characteristic of the free (F) theory. Equation (1) then becomes

$$n, \int e^{i\int h\Phi dx - \lambda \int \Phi^p dx} d\mu_F(\Phi) \quad (3)$$

which still generally requires cutoffs, normal ordering, renormalization counterterms, limits, etc., for any kind of definition. The goal of these endeavors is to determine a proper probability measure μ incorporating the nonlinear interaction, and so that (1) or (3) becomes

$$\int e^{i\int h\Phi dx} d\mu(\Phi) \quad (4)$$

Consider, now, the dependence of the interacting measure μ on λ . We speak of a continuous perturbation whenever

$$w\text{-}\lim_{\lambda \downarrow 0} \mu = \mu_F \quad , \quad (5a)$$

and of a discontinuous perturbation whenever

$$w\text{-}\lim_{\lambda \downarrow 0} \mu = \mu_{PF} \neq \mu_F \quad , \quad (5b)$$

where the "pseudofree" (PF) measure μ_{PF} is generally orthogonal to μ_F . We believe that (5a) applies to renormalizable interactions while (5b) applies to nonrenormalizable interactions,¹ a point of view we shall attempt to make plausible. That the behavior in (5b) is at all possible may be heuristically understood by attributing the formal behavior of a hard core to the interaction which thus has the effect, after once being introduced, of creating a permanent change in the system.

2. EXAMPLES OF DISCONTINUOUS PERTURBATIONS

Perturbed Free Particle

Differential Equation Formulation^{1,2}

The phenomena described above arises in simple quantum

mechanical problems as well. It is not difficult to show that the differential operator

$$H = -\frac{1}{2} \frac{d^2}{dx^2} + \frac{\lambda}{|x-c|^\alpha} \quad (6)$$

is essentially self adjoint for $\alpha > 2$ while for $\alpha < 2$ or $\alpha = 2$ (and $\lambda < 3/8$) this is not the case. In the range $\alpha \leq 2$, self-adjoint extensions may be chosen so that $H \rightarrow H_{PF} = -\frac{1}{2} d^2/dx^2$, but for $\alpha > 2$,

$$H \rightarrow H_{PF} = -\frac{1}{2} \frac{d^2}{dx^2} \Big|_{B.C.} \quad (7)$$

where the boundary conditions (B.C.) require that all $\psi \in \mathcal{D}(H_{PF})$ satisfy $\psi(c) = 0$.

Path Integral Formulation³

Feynman-Kac techniques help transcribe these results to a path space formulation. The interacting measure, related to the usual Wiener measure μ_W by

$$d\nu(x) \equiv e^{-\lambda \int_0^T |x(t)-c|^{-\alpha} dt} d\mu_W(x) \quad , \quad (8)$$

has the property for $\alpha > 2$ that

$$w\text{-}\lim_{\lambda \downarrow 0} \nu = \mu'_{W,c} \quad (9)$$

the measure appropriate to an absorbing Wiener process defined as a Wiener process in which all paths that reach $x = c$ are thenceforth neglected. The hard core picture is clearly applicable here.

For $1 \leq \alpha \leq 2$ it is possible to carefully choose "counterterms" at $x = c$ leading to a measure that weakly converges to μ_W . In particular, it suffices to define ν as the $\lambda \downarrow 0$ limit of measures in which the potential has the regularized form

$$(|x-c| + \varepsilon)^{-\alpha} - R_\varepsilon(\lambda, \alpha) \delta(x-c) \quad (10)$$

where [for simplicity we choose $(2-\alpha)^{-1}$ nonintegral]

$$R_\varepsilon(\lambda, \alpha) = \sum_{j=1}^{[(2-\alpha)^{-1}]^{-1}} k_j \lambda^{j-1} \varepsilon^{(2-\alpha)j-1} \quad (11)$$

and the coefficients k_j are determined recursively from $k_1 \equiv (\alpha-1)^{-1}$ and

$$k_j \equiv - \frac{2}{[1-(2-\alpha)j]} \sum_{n=1}^{j-1} k_n k_{j-n} \quad , \quad j = 2, 3, \dots \quad (12)$$

An interesting feature to observe is the essential λ dependence of the regularization term which can be likened to a need for counterterms expressed as a polynomial in the coupling constant familiar from renormalized perturbation theory.

Anharmonic Oscillator¹

A variation on the foregoing example is based on the stationary Ornstein-Uhlenbeck process rather than the Wiener process. Formally, we consider the path integration given by

$$\int e^{i \int s x dt - \int \left[\frac{1}{2} (\dot{x}^2 + x^2) + \lambda |x|^{-\alpha} \right] dt} d\mu_x \quad , \quad (13)$$

where $x(t)$ is a continuous sample path and $s(t)$ a test function, and we have chosen $c = 0$. The quadratic terms define the normal probability measure μ_F with covariance

$$\langle x(t)x(t') \rangle_F = (2\pi)^{-1} \int \frac{e^{i\omega(t-t')}}{\omega^2 + 1} d\omega = \frac{1}{2} e^{-|t-t'|} \quad (14)$$

so that (13) becomes

$$\int e^{i \int s x dt - \lambda \int |x|^{-\alpha} dt} d\mu_F(x) \quad . \quad (15)$$

Suppose the integrand here is defined through the regularization procedure described above (as $\lambda \downarrow 0$) to give the form

$$\int e^{i \int s x dt} d\mu(x) \quad . \quad (16)$$

If $\alpha \leq 2$, $\mu \rightarrow \mu_F$ as $\lambda \downarrow 0$, but if $\alpha > 2$ that is not the case. When $\alpha > 2$,

$$\lim_{\lambda \downarrow 0} \mu = \mu_{PF} \quad , \quad (17)$$

where μ_{PF} is concentrated on paths $x(t)$ for which either $x(t) > 0$ or $x(t) < 0$. Such paths have zero μ_F measure, and so μ_{PF} and μ_F are mutually singular (orthogonal). This is undoubtedly the typical case, and clearly the hard core analogy is largely heuristic in such cases.

Independent-Value Quantum Field Theory⁴

The next example to be treated is a field theory, albeit an extremely unphysical one. Suppose we take Eq. (1) and discard the space-time gradient terms to yield a model heuristically represented by

$$n \int e^{i \int h \phi dx - \int [\frac{1}{2} m^2 \phi^2 + \lambda \phi^p] dx} d\phi \quad (18)$$

When $\lambda \equiv 0$ in the integrand we generate the free measure μ_F with characteristic functional

$$e^{-\frac{1}{2} m^{-2} \int h^2(x) dx} = \int e^{i \int h \phi dx} d\mu_F(\phi) \quad (19)$$

and it is consistent with conventional ideas that (18) may be rewritten as

$$n' \int e^{i \int h \phi dx - \lambda \int \phi^p dx} d\mu_F(\phi) \quad (20)$$

This point of view is conceptually no different than in the covariant case except of course for the change in definition of μ_F . Ultimately, one expects to find a measure μ incorporating the interaction such that (18) reads

$$\int e^{i \int h \phi dx} d\mu(\phi) \quad (21)$$

If $p = 2$, and thus we deal with a mass perturbation, a consistent choice for μ is a normal distribution with variance parameter $(m^2 + 2\lambda)^{-1}$, and quite clearly $w\text{-lim } \mu = \mu_F$. However, if $p > 2$ the situation is completely changed.

Remarkably, when $p > 2$ a complete closed form solution to the formal expression (18) can be given,⁴ namely

$$\exp(-\int dx \int \{1 - \cos[h(x)u]\} e^{-\frac{1}{2} m^2 u^2 - \lambda u^p} du / |u|) \quad (22)$$

and this expression (being a positive-definite functional) implicitly defines the interacting measure μ . The limit $\lambda \downarrow 0$ may be taken directly and leads to the characteristic functional

$$\exp(-\int dx \int \{1 - \cos[h(x)u]\} e^{-\frac{1}{2}m^2 u^2} du / |u|) \quad , \quad (23)$$

which implicitly defines the measure μ_{PF} . Evidently

$$w\text{-lim } \mu = \mu_{PF} \neq \mu_F \quad , \quad (24)$$

and in fact $\mu_{PF} \perp \mu_F$.

Observe that every interacting model converges as $\lambda \rightarrow 0$ to one and the same pseudofree model. Also it is rather clear that a perturbation development of the interacting theories--i.e., expansion of (22) as a power series in λ --leads to an asymptotic series, and moreover one for which Borel resummation techniques are successful. In other words, when expanded about the pseudofree model, these "nonrenormalizable" models are as well behaved as could be expected.

Ultralocal Quantum Field Theory⁵

Slightly more realistic than the last model, an ultralocal scalar field is formally given by the functional integral

$$\int e^{i \int h \Phi dx - \int \{ \frac{1}{2} \dot{\Phi}^2 + \frac{1}{2} m^2 \Phi^2 + \lambda \Phi^p \} dx} \mathcal{D}\Phi \quad . \quad (25)$$

In the traditional view such an expression is formally equivalent to

$$\int e^{i \int h \Phi dx - \lambda \int \Phi^p dx} d\mu_F(\Phi) \quad (26)$$

where μ_F is a normal measure such that

$$e^{-\frac{1}{2} \int \frac{|\tilde{h}(k)|^2}{k_0^2 + m^2} dk} = \int e^{i \int h \Phi dx} d\mu_F(\Phi) \quad . \quad (27)$$

As before the aim is to find the interacting measure μ so that (25) is given by

$$\int e^{i \int h \Phi dx} d\mu(\Phi) \quad . \quad (28)$$

Clearly, if $p = 2$ it is consistent to choose μ as a normal measure (with shifted mass), and thus $w\text{-lim } \mu = \mu_F$. For $p > 2$ the situation changes completely.

The solution for $p > 2$ has been thoroughly discussed elsewhere in the framework of real-time quantum field theory.⁵ A direct discussion of the solution in the Euclidean framework is complicated because the time-ordered vacuum expectation values do not exist. However, this is not too serious since time ordering is sensible for certain renormalized field powers, and with these a connection to the present discussion can be made. On the other hand, it is not necessary to make this connection explicit since many pseudofree characteristics can be exhibited in the real-time formulation.

We content ourselves with a discussion of the spectrum of the free, pseudofree and interacting theories. It should be no surprise that there is only a discrete spectrum when the spacial gradients are absent. The free theory is simply a collection of independent, equivalent harmonic oscillators, and the energy spectrum is simply

$$0, m, 2m, 3m, 4m, 5m, \dots$$

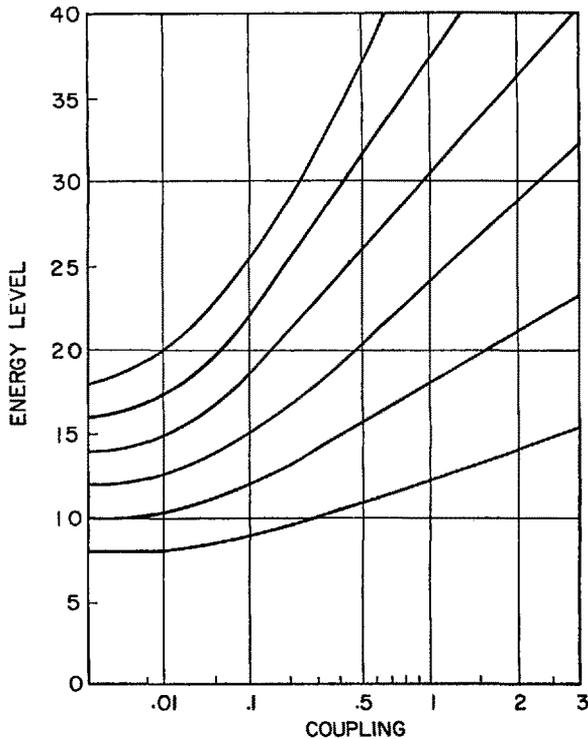
The pseudofree energy spectrum (the derivation of which is not presented here) reads

$$0, 8m, 10m, 12m, 14m, 16m, \dots$$

which are the values appropriate to a four-dimensional space time. The generalization to an n -dimensional space time reads

$$0, 2nm, (2n+2)m, (2n+4)m, \dots$$

In these spectra, the ground state has zero energy and is nondegenerate; all other energy levels are infinitely degenerate. When a nonlinear interaction is introduced, such simple numerical coefficients generally are lost, and analytic methods no longer suffice. We have calculated the first six energy levels numerically, based on equations given elsewhere,⁵ for the quartic interaction where $p = 4$ in a four-dimensional space time. Although this is a renormalizable interaction for covariant theories, it is a nonrenormalizable one for ultralocal theories. In the figure the first six energy levels are plotted (in units of m) as a function of the coupling constant (here 2λ). These levels have some of the qualitative behavior of a quartic anharmonic oscillator, e.g., asymptotically behaving as $\lambda^{1/3}$ for large λ ; but they differ significantly in that as $\lambda \downarrow 0$ the values 8, 10, 12, 14, ... emerge, as stated earlier. Although we have not verified the results explicitly we expect, from the type of equations involved, that a perturbation calculation of the energy levels about the pseudofree behavior leads to an asymptotic series and probably one summable by Padé approximants.



The interested reader may find further discussion of measures for ultralocal free and pseudofree theories in the references.

3. RELEVANCE FOR COVARIANT THEORIES

Discontinuous perturbations in quantum mechanics and in quantum field theory are a fact, and it is entirely possible that they may arise in covariant quantum field theory. No examples are known as yet, but certain plausible arguments suggest their existence and equivalence to conventional nonrenormalizable models. What are these arguments?

Consider the independent-value model, and focus on the two expressions $\int \phi^2 dx$ and $\int \phi^p dx$, which represent the free and interacting action, respectively. Formally, the L^2 functions control (in an unspecified way) the support of the free measure μ_F , while the $L^2 \cap L^p$ functions control the support of the interacting measure μ . If $p = 2$, there is no difference, but if $p > 2$, there is a profound difference. It is, heuristically at least, as if the presence of the interaction projects out ("hard core") all L^2 -dictated elements not also present among those that are $L^2 \cap L^p$ dictated. This projection persists for all $\lambda > 0$, and formally determines the pseudofree theory as $\lambda \downarrow 0$.

Similar arguments can be advanced for the ultralocal case, and, suitably modified, even for the anharmonic oscillator discussed earlier. Let us ask the same type of question in the covariant case.

If we let $W_0 = \frac{1}{2} \int [(\nabla\phi)^2 + m^2\phi^2] dx$, and $W_I = \int |\phi|^p dx$, where $dx \equiv d^n x$, then Sobolev-type inequalities^{6,1} show that $W_I \leq KW_0^{p/2}$ where K is a ϕ -independent, finite constant, provided $p \leq 2n/(n-2)$. On the other hand, if $p > 2n/(n-2)$ no such bound exists, and there are uncountably many ϕ with $W_0 < \infty$ and $W_I = \infty$. Formally, at least, in Eq. (1) the term W_I projects out fields inconsistent with the combination of both W_0 and W_I whenever $p > 2n/(n-2)$, and this "hard core" effect remains even after $\lambda \downarrow 0$. For $p \leq 2n/(n-2)$ no such phenomena should arise since $W_0 < \infty$ implies $W_I < \infty$. To conclude the argument we need only point out that conventional renormalized perturbation theory asserts that the models with $p \leq 2n/(n-2)$ are renormalizable, while those with $p > 2n/(n-2)$ are nonrenormalizable. These arguments tend to suggest, therefore, that discontinuous perturbations and the associated pseudofree theories may have relevance in the general study of covariant quantum field theory, especially for the so-called nonrenormalizable theories.

Generalization

Let us consider the generalized class of models for which $W_0 = \frac{1}{2} \int (k^{2\xi} + 1) |\tilde{\phi}(k)|^2 dk$ and $W_I = \int |\phi(x)|^p dx$. If $\xi = 1$ these are the covariant models (with $m = 1$) discussed above. Generalized Sobolev-type inequalities⁷ state that a ϕ -independent, finite constant K exists such that $W_I \leq KW_0^{p/2}$ provided $p \leq 2n/(n-2\xi)$. No such bound exists when $p > 2n/(n-2\xi)$. Remarkably, only the ratio ξ/n enters these conditions, and this suggests⁸ that in some sense we can simulate n -dimensional covariant quantum field theory in a one-dimensional space-time, i.e. time alone, if we simply set $\xi = 1/n$. What is simulated is the convergence or divergence of analogous graphs and the corresponding division into renormalizable and nonrenormalizable cases, which implies that these properties may also be studied in the context of one-dimensional stochastic processes, i.e., "noise". Finally we note that $\xi/n = 0$ for any n if $\xi = 0$. For $n = 1$, shot noise is a well-known δ -correlated process; for $n \geq 2$, the independent-value model discussed above is just the field-theory analogue of a special shot noise process.

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DISCUSSION

J. L. Lebowitz: In removing the regularization and then the coupling for the perturbed free particle, which links in the chain do not hold when the regularization is not the right one?

J. R. Klauder: Subject to minor restrictions, any regularization should at least have a convergent subsequence as $\epsilon \downarrow 0$. However, in order to obtain Wiener measure as $\lambda \downarrow 0$ only a limited class of regularizations will work, which are effectively as indicated [Eqs. (10)-(12)] plus what might be termed finite renormalizations.

K. Symanzik: Is your perturbation expansion about the nonfree theory with $\lambda = 0$ in terms of integer powers of λ , without nonanalytic terms?

J. R. Klauder: For the model field theories I discussed (independent-value; ultralocal) the perturbation expansion in λ about the pseudofree theory is asymptotic so long as $m > 0$. However, this property fails if $m = 0$ [as is clear say from Eq. (22)]. Also it is clear that the perturbation expansion of the anharmonic oscillator [Eqs. (13)-(17)] about the pseudo-free theory is asymptotic for $2 < \alpha < 3$, while for $\alpha \geq 3$ non-analytic terms arise.

A. Verbeure: Have you any idea how to find the measure for $\lambda = 0$ and for nonrenormalizable fields, without first solving the whole problem?

J. R. Klauder: Unfortunately, there are only a few general principles presently known that characterize pseudofree theories. For the special model field theories treated here, these principles plus the symmetry of the model determine the result, but for covariant models this is, at least so far, not the case.