$D = 11\, dS_5$ backgrounds with enhanced supersymmetry

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Abstract

We classify all warped dS_5 backgrounds in $D = 11$ supergravity with enhanced supersymmetry. We show that backgrounds preserving $N = 16$ supersymmetries consist of either a stack of M5 branes with transverse space $\mathbb{R}^5$, or a generalized M5-brane configuration with transverse space $\mathbb{R} \times N_4$, where N_4 is a hyper-Kähler manifold and the M5-brane harmonic function is determined by a hyper-Kähler potential on N_4. Moreover, we find that there are no backgrounds preserving exactly $N = 24$ supersymmetries. Backgrounds preserving $N = 32$ supersymmetries correspond to either $\mathbb{R}^{1,10}$ or $AdS_7 \times S^4$.

Keywords: supergravity, supersymmetry, M-branes

1. Introduction

Supersymmetry enhancement is known to play a particularly important role in the context of the geometric properties of supersymmetric black holes. It has been shown that for many supergravity theories, the near-horizon limits of supersymmetric extremal black holes[1] (and also branes) exhibit supersymmetry enhancement, which in turn imposes additional conditions on the geometry near to the event horizon. In particular, as a consequence of the enhanced supersymmetry, the isometry algebra enlarges in the near-horizon limit, containing a subalgebra isomorphic to $\mathfrak{sl}(2, \mathbb{R})$. In the context of $D = 11$ and type IIA (including massive IIA) supergravity, it has been shown that near-horizon geometries with smooth fields preserve an

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[1] In $N = 2, D = 4$ and $N = 2, D = 5$ supergravity, supersymmetric black holes are automatically extreme, however in $D = 11$ supergravity this need not be the case.

even number of supersymmetries [1–3]. The proof for this utilizes Lichnerowicz type theorems for certain generalized Dirac operators defined on the horizon spatial cross-section, which is assumed to be compact and without boundary. The index of these Dirac operators vanishes, which then establishes the supersymmetry enhancement.

Alternatively, one may consider the construction of a classification of highly supersymmetric solutions. In theories such as $D = 11$ supergravity, solutions preserving the minimal $N = 1$ supersymmetry have rather weak conditions on the geometry [4, 5]. In contrast, it is reasonable to expect that the classification of solutions with many supersymmetries will produce a much more restricted set of geometries. An important result is the homogeneity theorem, which states that backgrounds preserving $N > 16$ supersymmetry are locally homogeneous [6], i.e. the tangent space at each point is spanned by the Killing vectors which are constructed as bilinears of the Killing spinors. The theorem has been proven for $D = 11$ supergravity and type II $D = 10$ supergravities, and holds for many other theories as well. Using an adaptation of the homogeneity theorem, combined with an analysis of the associated superalgebras, it has been shown that there are no $N > 16$ smooth near-horizon geometries with non-trivial fluxes and also no warped AdS_2 backgrounds in ten or eleven dimensions [7]. This non-existence theorem applies provided that the horizon section, or the internal manifold, respectively, are compact and without boundary. Moreover, in [8] homogeneous $D = 11$ backgrounds which are symmetric have been classified up to local isometry. This provides a classification of $N > 16$ symmetric backgrounds in $D = 11$ supergravity. Furthermore, using spinorial geometry techniques it has been shown that all $D = 11$ backgrounds preserving $N = 30, 31$ supersymmetries and all type IIB backgrounds preserving $N > 28$ supersymmetries are maximally supersymmetric [9–11]; also there is a unique plane wave solution in IIB supergravity preserving $N = 28$ supersymmetry [12]. The spinorial geometry technique is a powerful tool to solve the Killing spinor equations (KSE) of supergravity theories and can be adapted to backgrounds with near maximal number of supersymmetries. It is based on the use of the gauge covariance of the KSE, together with a representation of the Clifford algebra, in an appropriate oscillator basis, acting on spinors which correspond to differential forms [13, 14].

In this paper, we shall classify the warped product $dS_5 \times_w M_6$ solutions in $D = 11$ supergravity which exhibit enhanced supersymmetry. In [15] it was shown that supersymmetric warped product $dS_5 \times_w M_6$ solutions must preserve $N = 8k$ supersymmetries for $k = 1, 2, 3, 4$. Minimal supersymmetry therefore corresponds to $N = 8$ supersymmetry—these were fully classified in [15]. Furthermore, it was also noted that the only possible $N = 32$ warped-product $dS_5 \times_w M_6$ solutions are $\mathbb{R}^{1,10}$ with vanishing four-form, or the maximally supersymmetric $AdS_7 \times S^4$ solution. Hence, in this paper we shall primarily be concerned with the $N = 16$ and $N = 24$ cases. We shall prove that there are no exactly $N = 24$ solutions. This is somewhat analogous to the analysis for warped product $AdS_5 \times_w M_6$ solutions in [16], in which it was proven that there are no $N = 24$ warped product AdS_5 solutions. However, there are differences between the AdS_5 and dS_5 analysis. The non-existence of $N = 24$ AdS_5 solutions was established via an adapted version of the homogeneity theorem in [6], together with a maximum principle argument which utilizes certain (assumed) global properties of the internal space. In contrast, for the dS_5 solutions, the local geometric conditions are sufficiently strong to allow an explicit integration of the Killing spinor equation along two of the directions of M_6. The analysis of the Killing spinor equation then simplifies significantly to the counting of certain parallel spinors on a hyper-Kähler manifold. This enables the case of $N = 24$ supersymmetries to be excluded by direct inspection. Moreover, this notable simplification also allows for the full classification of the $N = 16$ dS_5 solutions. We find two classes of $N = 16$ $dS_5 \times_w M_6$ backgrounds. The first is a special class of solutions constructed in [15] for which the four-form is parallel; the geometry is a generalized M5-brane configuration with transverse space $\mathbb{R} \times N_4$, where N_4 is a

hyper-Kähler manifold and the $M5$-brane harmonic function is determined by a hyper-Kähler potential on N_4. The second class is a stack of M5 branes with transverse space $\mathbb{R}^5$ [17]. Further recent progress towards classifying the $N = 16$ AdS_5 warped product solutions has been made in [18], which develops a systematic examination of spacetimes, and other fields, which are invariant under the action of certain R-symmetry groups.

It is known that there are many no-go theorems which imply non-existence of de Sitter solutions in supergravity [19–21], in cases for which the warp factor and fluxes and are smooth, and the internal manifold is smooth and compact without boundary. Our motivation is therefore to construct a systematic classification of supersymmetric de Sitter solutions in supergravity theories from a purely local perspective. In addition, we shall not assume that the spinors factorize into products of spinors on dS_n and on the internal space, as it is known that such factorizations can produce a miscounting of supersymmetries [22]. This classification programme has already been fully completed in the case of heterotic supergravity, including the first order corrections in α' [23]. For heterotic warped product de Sitter geometries, the warped product dS_2 solutions are in 1–1 correspondence with the direct product AdS_3 solutions classified in [24]; moreover all warped product dS_n solutions for $3 \leqslant n \leqslant 9$ are direct product $\mathbb{R}^{1,n} \times M_{9-n}$ backgrounds. This is consistent with the restrictions on heterotic dS_n solutions for $n \geqslant 4$ found in [25]; it is also clear from the analysis of [23] that the warped product dS_2 and dS_3 solutions are also highly restricted in heterotic supergravity. The warped product dS_n solutions in $D = 11$ supergravity exhibit similar foliation properties for $5 \leqslant n \leqslant 10$. In the case of $n = 5$, dS_5 arises as a (conformal) foliation of $\mathbb{R}^{1,5}$, corresponding to the directions along the M5-brane worldvolume. In contrast, the warped product dS_4 solutions with minimal $N = 8$ supersymmetry have been classified in [26], and there is no analogous foliation of dS_4 into AdS_5 or $\mathbb{R}^{1,4}$ appearing.

In order to construct the classification of all $D = 11$ dS_5 solutions with enhanced supersymmetry, we begin by taking the necessary conditions obtained for a supersymmetric solution to preserve (at least) the minimal $N = 8$ possible supersymmetry, which were obtained in [15]. Such conditions imply the existence of two preferred co-ordinates s and t in the internal manifold, and also impose certain conditions on the four-form flux, when written in terms of these co-ordinates. We then consider the conditions imposed on an arbitrary parallel (with respect to the supercovariant connection of $D = 11$ supergravity) spinor ε in such a background. Using the conditions on the flux and the geometry imposed by minimal $N = 8$ supersymmetry, we explicitly integrate up the Killing spinor equation along the dS_5 directions, as well as the s and t directions on M_6. After some manipulation, which depends on whether or not the Hessian of the warp factor satisfies a particular condition, this allows us, in all possible cases, to explicitly write the spinor ε in terms of certain covariantly constant spinors $\sigma^{\pm}$ on a four-dimensional hyper-Kähler manifold which is embedded in M_6. By counting such spinors, we show that the case of exactly $N = 24$ supersymmetries is excluded. Moreover, it also enables a complete classification of all solutions which preserve exactly $N = 16$ supersymmetries. If the Hessian of the warp factor satisfies the condition mentioned above, then we prove that for a $N = 16$ solution, there must be eight σ^+ spinors, and also eight σ^- spinors. The resulting conditions on the flux and geometry imply that such $N = 16$ solutions correspond to the class of solutions obtained in [15] for which the four-form is parallel. If, however, the Hessian of the warp factor does not satisfy the condition, then for a $N = 16$ solution, it follows that there must be 16 linearly independent σ^+ spinors on the hyper-Kähler manifold, which implies that this manifold must be $\mathbb{R}^4$. This case corresponds to the $N = 16$ solution which is a stack of M5-branes with transverse space $\mathbb{R}^5$ [17]. Full details of the integration of the Killing spinor equation are given in section 3.

The plan of this paper is as follows: in section 2 we summarize some key aspects of the classification of warped product dS_5 solutions, $dS_5 \times_w M_6$, preserving the minimal amount of $N = 8$ supersymmetry constructed in [15], in which it is shown that all such solutions are generalized $M5$-brane solutions for which the five-dimensional transverse manifold is $\mathbb{R} \times N_4$, where N_4 is a hyper-Kähler manifold. A particularly useful special case, which turns out to have enhanced supersymmetry, for which the four-form F is covariantly constant, is also presented. In section 3, we use the results presented in section 2 to explicitly integrate the Killing spinor equations, acting on a generic spinor, along M_6. This produces a gravitino type equation, and an algebraic condition. The analysis then splits into two subcases, depending on the properties of the algebraic condition. In each of these subcases, the Killing spinor equations can ultimately be reduced to counting parallel spinors on the hyper-Kähler manifold N_4, enabling all of the solutions with enhanced supersymmetry to be fully classified. The results are summarized in section 4.

2.　$D = 11$ warped product dS_5 backgrounds

In this section, we summarize the key results about warped $dS_5 \times_w M_6$ backgrounds in $D = 11$ supergravity [15], which were derived for solutions preserving the minimal $N = 8$ supersymmetry. First of all, the 11-dimensional metric is given by

$$ds^2(M_{11}) = A^2 ds^2(dS_5) + ds^2(M_6) \tag{1}$$

where A is a function of the co-ordinates of the internal Riemannian manifold M_6 and

$$ds^2(dS_5) = \frac{1}{\left(1 + \frac{k}{4}|x|^2\right)^2} \eta_{\mu\nu} dx^\mu dx^\nu, \quad \mu, \nu = 0, 1, \dots, 4 \tag{2}$$

is the metric of five-dimensional de-Sitter spacetime, with $|x|^2 = \eta_{\mu\nu} x^\mu x^\nu$ and $k = \frac{1}{\ell^2}$. We require that the Lie derivative of the four-form flux F with respect to all of the isometries of dS_5 vanishes, and consequently

$$F = X \tag{3}$$

where X is a four-form on M_6 whose components depend only on the co-ordinates of M_6. Moreover, as we have mentioned previously, in what follows we do not make any assumption on smoothness of A or the four-form flux, neither do we require the internal manifold to be smooth or compact without boundary. Rather, the analysis is entirely local.

Let us introduce the space-time vielbein

$$\mathbf{e}^\mu = \frac{A}{1 + \frac{k}{4}|x|^2} dx^\mu, \quad \mathbf{e}^a = e^a{}_\Theta(y) dy^\Theta \tag{4}$$

where $a = 5, 6, \dots, \sharp$ is a frame index on M_6 and we have denoted by y^α the co-ordinates on M_6. The bosonic field equations of $D = 11$ supergravity

$$R_{AB} = \frac{1}{12} F_{AL_1 L_2 L_3} F_B{}^{L_1 L_2 L_3} - \frac{1}{144} g_{AB} F^2 \tag{5}$$

$$d \star_{11} F - \frac{1}{2} F \wedge F = 0 \tag{6}$$

and the Bianchi identities

$$dF = 0 \tag{7}$$

can be reduced on M_6 yielding

$$4kA^{-2} - A^{-1}\widetilde{\nabla}^a\widetilde{\nabla}_a A - 4A^{-2}(\widetilde{\nabla}A)^2 + \frac{1}{12}G^2 = 0 \tag{8}$$

$$\widetilde{R}_{ab} = 5A^{-1}\widetilde{\nabla}_a\widetilde{\nabla}_b A + \frac{1}{6}G^2\delta_{ab} - \frac{1}{2}G_{cb}G^c{}_a \tag{9}$$

$$\tilde{d}(A^5 G) = 0 \tag{10}$$

$$\tilde{d}\star_6 G = 0 \tag{11}$$

where $\widetilde{\nabla}$ denotes the Levi-Civita connection on M_6, $\tilde{d}$ is the exterior derivative on M_6, and

$$G = \star_6 X. \tag{12}$$

Furthermore, the KSE of $D = 11$ supergravity

$$\left(\nabla_A - \frac{1}{288}\Gamma_A{}^{B_1 B_2 B_3 B_4}F_{B_1 B_2 B_3 B_4} + \frac{1}{36}F_{AB_1 B_2 B_3}\Gamma^{B_1 B_2 B_3}\right)\epsilon = 0 \tag{13}$$

can be integrated along the de-Sitter directions yielding[2]

$$\epsilon = \left(1 + \frac{k}{4}|x|^2\right)^{-\frac{1}{2}}\left(1 + x^\mu\Gamma_\mu\left(-\frac{1}{2}\widetilde{\slashed{\nabla}}A + \frac{A}{288}\slashed{X}\right)\right)\psi \tag{14}$$

where ψ is a 32-component Majorana spinor on M_6 which satisfies

$$\widetilde{\nabla}_a\psi = \left(-\frac{1}{12}G_{ab}\Gamma^b + \frac{1}{12}\Gamma_a{}^{bc}G_{bc}\right)\Gamma^{(7)}\psi \tag{15}$$

and

$$\Gamma^{(7)} = \frac{1}{6!}\epsilon_{a_1 a_2 \ldots a_6}\Gamma^{a_1 a_2 \ldots a_6} \tag{16}$$

is the highest rank Gamma matrix on M_6. As shown in [15], we can pick co-ordinates $\{t, s, z^i\}$, with $i = 1, 2, 3, 4$ on M_6 such that

$$ds^2(M_6) = f^{-4}(s,z)ds^2 + \frac{1}{k}f^2(s,z)dt^2 + f^{-4}(s,z)ds^2(N_4) \tag{17}$$

where N_4 is a four-dimensional hyperKähler manifold, with metric tensor

$$ds^2(N_4) = h_{ij}(z)dz^i dz^j. \tag{18}$$

Moreover, the warp factor A is given by

$$A = t \cdot f(s,z) \tag{19}$$

and the two-form flux G is

$$G = \frac{6}{\sqrt{k}}df \wedge dt. \tag{20}$$

Setting $f = H^{-\frac{1}{6}}$, and using (17) and (19), the 11-dimensional metric tensor (1) is given by

$$ds^2(M_{11}) = H^{-\frac{1}{3}}ds^2(\mathbb{R}^{1,5}) + H^{\frac{2}{3}}ds^2(\mathbb{R} \times N_4) \tag{21}$$

and the Einstein equation (8) simplifies to

$$\Box_5 H = 0 \tag{22}$$

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[2] If ω is a p-form on M_6 then $\slashed{\omega} = \omega_{a_1 \ldots a_p}\Gamma^{a_1 \ldots a_p}$.

where $\square_5$ denotes the Laplacian on $\mathbb{R} \times N_4$. Moreover, (20) yields

$$F = \star_5 dH \tag{23}$$

where $\star_5$ is the Hodge dual on $\mathbb{R} \times N_4$. The geometry given by (21)–(23) corresponds to that of a generalized M5-brane configuration, with transverse space $\mathbb{R} \times N_4$ [27].

2.1. Solutions with parallel four-form

A sub-class of these solutions corresponds to those backgrounds for which the four-form F is covariantly constant with respect to the 11-dimensional Levi-Civita connection; as we shall prove in the section 3.1, this class of solutions actually has enhanced supersymmetry. These backgrounds satisfy

$$\widetilde{\nabla} G = 0. \tag{24}$$

Note that (24) implies

$$G^2 = c^2 \tag{25}$$

with c constant. The condition (24) yields the following set of PDEs

$$\frac{\partial^2 f}{\partial s^2} + f^{-1}\left(\frac{\partial f}{\partial s}\right)^2 - 2f^{-1} h^{ij}\frac{\partial f}{\partial z^i}\frac{\partial f}{\partial z^j} = 0 \tag{26}$$

$$\frac{\partial f}{\partial s \partial z^i} + 3f^{-1}\frac{\partial f}{\partial s}\frac{\partial f}{\partial z^i} = 0 \tag{27}$$

$$\overset{\circ}{\nabla}_i \overset{\circ}{\nabla}_j f + 3f^{-1}\frac{\partial f}{\partial z^i}\frac{\partial f}{\partial z^j} - 2f^{-1}h_{ij}\left(\left(\frac{\partial f}{\partial s}\right)^2 + h^{kl}\frac{\partial f}{\partial z^k}\frac{\partial f}{\partial z^l}\right) = 0. \tag{28}$$

Equations (26)–(28), supplemented by (25), are equivalent to

$$f(s,z) = \sqrt{2}\left(c^2 s^2 + P(z)\right)^{\frac{1}{4}} \tag{29}$$

$$(\overset{\circ}{\nabla} P)^2 = 4c^2 P \tag{30}$$

$$\overset{\circ}{\nabla}_i \overset{\circ}{\nabla}_j P = 2c^2 h_{ij} \tag{31}$$

where $\overset{\circ}{\nabla}$ denotes the Levi-Civita connection on N_4. In particular, by taking $c = 0$ and $N_4 = \mathbb{R}^4$, we recover the maximally supersymmetric solution $\mathbb{R}^{1,10}$ with vanishing four-form[3]. In particular, (31) implies that $\frac{P}{2c^2}$ is a hyper-Kähler potential for N_4 [28].

3. Integration of the KSE on M_6

In this section, we integrate the KSE (15) along two of the directions of M_6, corresponding to the co-ordinates s and t. The resulting reduced Killing spinor equations ultimately are, after some further manipulation, equivalent to requiring the existence of certain parallel spinors on N_4. As we shall see, this enables to classify dS_5 backgrounds with extended supersymmetry. First of all, let us introduce the vielbein on M_6

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[3] The other maximally supersymmetric solution which is a warped product dS_5 solution is $AdS_7 \times S^4$, which arises as the near-horizon limit of the standard M5-brane, with transverse space $\mathbb{R}^5$.

$$\tilde{\mathbf{e}}^5 = f^{-2}ds, \quad \tilde{\mathbf{e}}^6 = \frac{1}{\sqrt{k}}\,fdt, \quad \tilde{\mathbf{e}}^I = f^{-2}\,\overset{\circ}{e}{}^I \tag{32}$$

where $\overset{\circ}{e}{}^I = \overset{\circ}{e}{}^I_{\ i}\,dz^i$ and

$$h_{ij} = \delta_{IJ}\,\overset{\circ}{e}{}^I_{\ i}\,\overset{\circ}{e}{}^J_{\ j}. \tag{33}$$

Using (32), equation (17) reads

$$ds^2(M_6) = (\tilde{\mathbf{e}}^5)^2 + (\tilde{\mathbf{e}}^6)^2 + \delta_{IJ}\tilde{\mathbf{e}}^I\tilde{\mathbf{e}}^J. \tag{34}$$

The non-vanishing components of the spin connection on M_6 with respect to the frame (32) are given by

$$\widetilde{\Omega}_{6,65} = f\frac{\partial f}{\partial s} \tag{35}$$

$$\widetilde{\Omega}_{6,6I} = -\frac{1}{2}\widetilde{\Omega}_{5,5I} = f\overset{\circ}{e}_I f \tag{36}$$

$$\widetilde{\Omega}_{I,5J} = 2f\frac{\partial f}{\partial s}\delta_{IJ} \tag{37}$$

$$\widetilde{\Omega}_{I,JK} = f^2\,\overset{\circ}{\Omega}_{I,JK} - 4f\delta_{I[J}\,\overset{\circ}{e}_{K]}f \tag{38}$$

where $\overset{\circ}{\Omega}_{I,JK}$ is the spin connection on N_4.

Using (35)–(38) and (20), the KSE (15) read

$$\frac{\partial\psi}{\partial t} = \frac{1}{2\sqrt{k}}\left(f^2\frac{\partial f}{\partial s}\Gamma^5 + f^2\,\overset{\circ}{\nabla}_I f\Gamma^I\right)\left(\Gamma^6 + \Gamma^{(7)}\right)\psi \tag{39}$$

$$\frac{\partial\psi}{\partial s} = f^{-1}\,\overset{\circ}{\nabla}_I f\Gamma^6\Gamma^5\Gamma^I\left(\Gamma^6 + \Gamma^{(7)}\right)\psi - \frac{1}{2}f^{-1}\frac{\partial f}{\partial s}\Gamma^6\Gamma^{(7)}\psi \tag{40}$$

$$\overset{\circ}{\nabla}_I\psi = f^{-1}\frac{\partial f}{\partial s}\Gamma^6\Gamma_I\Gamma^5\left(\Gamma^6 + \Gamma^{(7)}\right)\psi + f^{-1}\,\overset{\circ}{\nabla}_J f\Gamma^6\Gamma_I{}^J\left(\Gamma^6 + \Gamma^{(7)}\right)\psi$$
$$- \frac{1}{2}f^{-1}\,\overset{\circ}{\nabla}_I f\Gamma^6\Gamma^{(7)}\psi. \tag{41}$$

Since f does not depend on t, equation (39) can be easily integrated, yielding

$$\psi = e^{t\mathcal{X}}\eta, \quad \frac{\partial\eta}{\partial t} = 0 \tag{42}$$

where η is a 32-component Majorana spinor and

$$\mathcal{X} := \frac{1}{2\sqrt{k}}\left(f^2\frac{\partial f}{\partial s}\Gamma^5 + f^2\,\overset{\circ}{\nabla}_I f\Gamma^I\right)\left(\Gamma^6 + \Gamma^{(7)}\right). \tag{43}$$

Notice that $\mathcal{X}^2 = 0$, thus $e^{t\mathcal{X}} = 1 + t\mathcal{X}$ and (42) yields

$$\psi = \eta + \frac{t}{2\sqrt{k}}\left(f^2\frac{\partial f}{\partial s}\Gamma^5 + f^2\,\overset{\circ}{\nabla}_I f\Gamma^I\right)\left(\Gamma^6 + \Gamma^{(7)}\right)\eta. \tag{44}$$

We can rewrite (40), (41) and (44) covariantly on $\mathbb{R} \times N_4$ as follows

$$D_\alpha\psi = f^{-1}D_\beta f\Gamma^6\Gamma_\alpha{}^\beta\left(\Gamma^6 + \Gamma^{(7)}\right)\psi - \frac{1}{2}f^{-1}D_\alpha f\Gamma^6\Gamma^{(7)}\psi \tag{45}$$

and

$$\psi = \eta + \frac{t}{2\sqrt{k}}f^2 D_\alpha f\Gamma^\alpha\left(\Gamma^6 + \Gamma^{(7)}\right)\eta \tag{46}$$

where D is the Levi-Civita connection on $\mathbb{R} \times N_4$ and α, β are frame indices on $\mathbb{R} \times N_4$. Inserting (46) into (45), the vanishing of the t-independent terms and the terms linear in t yields

$$D_\alpha \eta = f^{-1} D_\beta f \Gamma^6 \Gamma_\alpha{}^\beta (\Gamma^6 + \Gamma^{(7)}) \eta - \frac{1}{2} f^{-1} D_\alpha f \Gamma^6 \Gamma^{(7)} \eta \tag{47}$$

and

$$\mathcal{H}_{\alpha\beta} \Gamma^\beta (\Gamma^6 + \Gamma^{(7)}) \eta = 0 \tag{48}$$

respectively, where we have defined

$$\mathcal{H}_{\alpha\beta} := f^2 D_\alpha D_\beta f + 3f D_\alpha f D_\beta f - 2f(Df)^2 \delta_{\alpha\beta}. \tag{49}$$

Two distinct cases must be considered, depending on whether $\mathcal{H}_{\alpha\beta}$ vanishes or not.

3.1. $\mathcal{H}_{\alpha\beta} = 0$

If $\mathcal{H}_{\alpha\beta} = 0$, then (48) is automatically satisfied and (49) implies

$$f^2 D_\alpha D_\beta f + 3f D_\alpha f D_\beta f - 2f(Df)^2 \delta_{\alpha\beta} = 0. \tag{50}$$

Decomposing (50) on N_4, we find (26)–(28), hence $\widetilde{\nabla} G = 0$ and equations (29)–(31) hold. In the following, it is convenient to write η as

$$\eta = \eta^+ + \eta^- \tag{51}$$

where

$$\Gamma^6 \Gamma^{(7)} \eta^\pm = \pm \eta^\pm. \tag{52}$$

Using (51) and (52), equation (47) yields

$$D_\alpha \eta^+ = 2f^{-1} D_\beta f \Gamma_\alpha{}^\beta \eta^+ - \frac{1}{2} f^{-1} D_\alpha f \eta^+ \tag{53}$$

and

$$D_\alpha \eta^- = \frac{1}{2} f^{-1} D_\alpha f \eta^-. \tag{54}$$

Let us perform two different conformal transformations on the spinors η_+ and η_- as follows

$$\hat{\eta}^+ = f^{\frac{1}{2}} \eta^+, \quad \hat{\eta}^- = f^{-\frac{1}{2}} \eta^-. \tag{55}$$

Implementing (55) into (53) and (54), we find

$$D_\alpha \hat{\eta}^+ = 2f^{-1} D_\beta f \Gamma_\alpha{}^\beta \hat{\eta}^+ \tag{56}$$

and

$$D_\alpha \hat{\eta}^- = 0 \tag{57}$$

respectively. Equation (57) implies that

$$\hat{\eta}^- = \sigma^- \tag{58}$$

where σ^- is a 32-component Majorana spinor independent of s which is covariantly constant on N_4, i.e.

$$\overset{\circ}{\nabla}_I \sigma^- = 0. \tag{59}$$

Let us now analyze (56). The five-component of (56) yields

$$\frac{\partial \hat{\eta}^+}{\partial s} = \frac{1}{2(c^2 s^2 + P(z))} \, \overset{\circ}{\nabla}_I P \Gamma^1 \Gamma^I \hat{\eta}^+ \tag{60}$$

where we have implemented (29). We shall explicitly integrate this condition by setting

$$\hat{\eta}^+ = \exp\left(u(s,z) \, \overset{\circ}{\nabla}_I P \Gamma^1 \Gamma^I \right) \phi^+, \qquad \frac{\partial \phi^+}{\partial s} = 0. \tag{61}$$

On substituting this into (60), one obtains

$$\frac{\partial u}{\partial s} = \frac{1}{2(c^2 s^2 + P(z))}. \tag{62}$$

Notice that P is positive by means of (30). Hence, a solution to equation (62) is given by

$$u(s,z) = \frac{1}{2c\sqrt{P(z)}} \arctan\left(\frac{cs}{\sqrt{P(z)}} \right). \tag{63}$$

Moreover, equation (61) implies

$$\begin{aligned}
\hat{\eta}^+ = \bigg\{ &\cos\left(2c\sqrt{P(z)} u(s,z) \right) \mathbb{I} \\
&+ \frac{1}{2c\sqrt{P(z)}} \sin\left(2c\sqrt{P(z)} u(s,z) \right) \overset{\circ}{\nabla}_I P \Gamma^1 \Gamma^I \bigg\} \phi^+.
\end{aligned} \tag{64}$$

Substituting (63) into (64), we get

$$\hat{\eta}^+ = \frac{1}{\sqrt{c^2 s^2 + P(z)}} \left(\sqrt{P(z)} \mathbb{I} + \frac{s}{2\sqrt{P(z)}} \overset{\circ}{\nabla}_I P \Gamma^1 \Gamma^I \right) \phi^+. \tag{65}$$

The I-component of (56) is given by

$$\overset{\circ}{\nabla}_I \hat{\eta}^+ = \frac{c^2 s}{c^2 s^2 + P(z)} \Gamma_I \Gamma^1 \hat{\eta}^+ + \frac{1}{2(c^2 s^2 + P(z))} \overset{\circ}{\nabla}_J P \Gamma_I{}^J \hat{\eta}^+. \tag{66}$$

Substituting (65) into (66) and using (31), we get

$$\overset{\circ}{\nabla}_I \sigma^+ = 0 \tag{67}$$

where σ^+ is given by

$$\phi^+ = \frac{1}{\sqrt{P(z)}} \Gamma^I \overset{\circ}{\nabla}_I P \sigma^+. \tag{68}$$

Substituting (68) in (65), we obtain

$$\hat{\eta}^+ = \frac{1}{\sqrt{c^2 s^2 + P(z)}} \left(\Gamma^I \overset{\circ}{\nabla}_I P + 2c^2 s \Gamma^5 \right) \sigma^+. \tag{69}$$

Moreover, (46) is equivalent to

$$\psi = \eta^+ + \eta^- + \frac{1}{\sqrt{2}k} t (c^2 s^2 + P(z))^{-\frac{1}{4}} \left(2c^2 s \Gamma^5 + \Gamma^I \overset{\circ}{\nabla}_I P \right) \Gamma^6 \eta^+ \tag{70}$$

where we have used (51) and (52). Using (55), (58) and (69), equation (70) yields

$$\psi = 2^{-\frac{1}{4}}\left(c^2 s^2 + P(z)\right)^{-\frac{5}{8}}\left(2c^2 s\Gamma^5 + \Gamma^I \overset{\circ}{\nabla}_I P\right)\sigma^+ + 2^{\frac{1}{4}}\left(c^2 s^2 + P(z)\right)^{\frac{1}{8}}\sigma^-$$

$$- \frac{2^{\frac{5}{4}}c^2}{\sqrt{k}}t\left(c^2 s^2 + P(z)\right)^{\frac{1}{8}}\Gamma^6 \sigma^+. \tag{71}$$

Defining $\check{\sigma}^+ := 2^{-\frac{1}{4}}\sigma^+$ and $\check{\sigma}^- := 2^{\frac{1}{4}}\sigma^-$ and dropping the check for simplicity, (71) is equivalent to

$$\psi = \left(c^2 s^2 + P(z)\right)^{-\frac{5}{8}}\left(2c^2 s\Gamma^5 + \Gamma^I \overset{\circ}{\nabla}_I P\right)\sigma^+ + \left(c^2 s^2 + P(z)\right)^{\frac{1}{8}}\sigma^-$$

$$- 2c^2\sqrt{\frac{2}{k}}\,t\left(c^2 s^2 + P(z)\right)^{\frac{1}{8}}\Gamma^6 \sigma^+. \tag{72}$$

Let us count the number of supersymmetries preserved by these backgrounds. To this end, define

$$\mathcal{S}^{\pm} := \mathrm{span}\{\sigma^{\pm}\} \tag{73}$$

where $\sigma^{\pm}$ satisfy $\Gamma^6\Gamma^{(7)}\sigma^{\pm} = \pm\sigma^{\pm}$ and

$$\overset{\circ}{\nabla}_I \sigma^{\pm} = 0. \tag{74}$$

Notice that if $\sigma^{\pm} \in \mathcal{S}^{\pm}$, then $\Gamma_{\mu\nu}\sigma^{\pm} \in \mathcal{S}^{\pm}$, where μ, ν denote the de-Sitter directions. Hence, using the argument of section 2.2 of [15], it follows that

$$\dim \mathcal{S}^{\pm} = 8k^{\pm}, \quad k^{\pm} = 1, 2. \tag{75}$$

Moreover, if $\sigma^+ \in \mathcal{S}^+$, then $\Gamma_{\mu}\sigma^+ \in \mathcal{S}^-$, hence

$$\dim \mathcal{S}^+ = \dim \mathcal{S}^- \tag{76}$$

that is $k^+ = k^- := k$. Using (72), (75) and (76), it follows that the number of supersymmetries is

$$N = \dim \mathcal{S}^+ + \dim \mathcal{S}^- = 16k \quad k = 1, 2. \tag{77}$$

Notice that equation (77) implies that in this class of solutions there are no backgrounds preserving exactly $N = 24$ supersymmetries.

To summarize, for this class of solutions, the metric is given by

$$ds^2(M_{11}) = H^{-\frac{1}{3}}ds^2(\mathbb{R}^{1,5}) + H^{\frac{2}{3}}ds^2(\mathbb{R} \times N_4) \tag{78}$$

where N_4 is a hyperKähler manifold and H satisfies (50), that is

$$-3HD_{\alpha}D_{\beta}H + 5D_{\alpha}HD_{\beta}H - (DH)^2\delta_{\alpha\beta} = 0. \tag{79}$$

where the functions f and H are related by $f = H^{-\frac{1}{6}}$. In particular, (79) implies that H is harmonic on $\mathbb{R} \times N_4$, (22). Moreover, the four-form is given by (23) and is covariantly constant with respect to the Levi-Civita connection of 11-dimensional spacetime. In this case, it follows that the geometry is that of a generalized $M5$-brane configuration, for which the transverse space is $\mathbb{R} \times N_4$, and the harmonic function H on $\mathbb{R} \times N_4$ is determined in terms of a hyper-Kähler potential P on N_4 via

$$H = \frac{1}{8}\left(c^2 s^2 + P(z)\right)^{-\frac{3}{2}} \tag{80}$$

where P satisfies (31).

3.2. $\mathcal{H}_{\alpha\beta} \neq 0$

If $\mathcal{H}_{\alpha\beta} \neq 0$, then (48) yields

$$(\Gamma^6 + \Gamma^{(7)})\eta = 0 \tag{81}$$

hence $\eta = \eta^-$. Using (81), (47) simplifies to

$$D_\alpha \chi^- = 0, \tag{82}$$

where we have defined $\chi^- := f^{-\frac{1}{2}}\eta^-$. Equation (82) implies that

$$\chi^- = \sigma^- \tag{83}$$

where σ^- is a spinor independent of s which satisfies

$$\overset{\circ}{\nabla}_I \sigma^- = 0. \tag{84}$$

Using (83), equation (46) yields

$$\psi = f^{\frac{1}{2}}\sigma^-. \tag{85}$$

Equations (75) and (85) imply that the number of supersymmetries preserved by these backgrounds is

$$N = 8k^-, \quad k^- = 1, 2. \tag{86}$$

For this class of solutions, the metric tensor is given by

$$ds^2(M_{11}) = H^{-\frac{1}{3}}ds^2(\mathbb{R}^{1,5}) + H^{\frac{2}{3}}ds^2(\mathbb{R} \times N_4) \tag{87}$$

where N_4 is a hyperKähler manifold and H is harmonic on $\mathbb{R} \times N_4$. i.e. it satisfies (22). Moreover, the four-form is given by (23).

Notice that equation (86) implies that $N = 24$ is excluded in this class of solutions as well, hence there are no $D = 11$ warped dS_5 backgrounds preserving exactly $N = 24$ supersymmetries. Moreover, the case $N = 8$ has already been analyzed in [15]. Hence, we are left to consider $N = 16$. In this case, (86) implies that there are 16 linearly independent negative chirality spinors σ^- which are covariantly constant on N_4. Using (76), it follows that there are also 16 positive chirality spinors σ^+ which are covariantly constant on N_4. Hence there are 32 linearly independent covariantly constant spinors on N_4. This implies $N_4 = \mathbb{R}^4$, and (87), (23) and (22) yield

$$
\begin{aligned}
ds^2(M_{11}) &= H^{-\frac{1}{3}}ds^2(\mathbb{R}^{1,5}) + H^{\frac{2}{3}}ds^2(\mathbb{R}^5) \\
F &= \star_5 dH \\
\Box_5 H &= 0.
\end{aligned}
\tag{88}
$$

The configuration (88) corresponds to the standard M5-brane, which indeed preserves 16 supersymmetries in the bulk [17].

4. Conclusion

In this work, we have fully classified the warped product dS_5 backgrounds in $D = 11$ supergravity with enhanced supersymmetry. It is known from [15] that supersymmetric warped product backgrounds must preserve $N = 8k$ supersymmetries for $k = 1, 2, 3, 4$. As the exactly $N = 8$ and $N = 32$ solutions were considered in [15], we remark that the analysis here completes the classification, by including the $N = 16$ and $N = 24$ cases. The classification is complete, as we have obtained necessary and sufficient conditions on all parallel spinors preserving $N > 8$

supersymmetry, by integrating up the covariant constancy condition along the directions s and t on M_6, which were shown to exist for all supersymmetric warped product dS_5 solutions in [15]. The resulting conditions are sufficiently strong, when rewritten in terms of covariantly constant spinors on a hyper-Kähler manifold, to exclude the case of $N = 24$ supersymmetry, and also to reduce the $N = 16$ case to two possible sub-cases, found in the previous section and listed below.

The summary of the conditions on the warped product dS_5 backgrounds corresponding to the different possible proportions of supersymmetry which we have established is as follows:

- $N = 8$: the $N = 8$ solutions were classified in [15], and the results for this case are summarized in section 2. The geometries are generalized $M5$-brane solutions, for which the transverse space is $\mathbb{R} \times N_4$, where N_4 is a hyper-Kähler manifold.
- $N = 16$: there are two possibilities for $N = 16$ supersymmetry, corresponding to whether $\mathcal{H}_{\alpha\beta} = 0$, where $\mathcal{H}_{\alpha\beta}$ is defined in (49).
  * If $\mathcal{H}_{\alpha\beta} = 0$ then the $N = 16$ solutions have the property that the four-form F is covariantly constant with respect to the 11-dimensional Levi-Civita connection and is given by (23). Such solutions have been discussed in section 5.4 of [15]; here we establish that these solutions actually have enhanced supersymmetry. The geometry corresponds to a generalized M5-brane configuration (78), with transverse space $\mathbb{R} \times N_4$ for which the harmonic function H on $\mathbb{R} \times N_4$ is determined in terms of a hyper-Kähler potential P on N_4 via (80).
  * If however, $\mathcal{H}_{\alpha\beta} \neq 0$ then we find that the bosonic fields are given by (88). This configuration corresponds to a stack of M5-branes with transverse space $\mathbb{R}^5$ [17].
- $N = 24$: there are no warped product dS_5 solutions preserving exactly $N = 24$ supersymmetries.
- $N = 32$: this case has been considered in [15] and it has been shown that the only possibilities are $\mathbb{R}^{1,10}$ with vanishing four-form, or the maximally supersymmetric $AdS_7 \times S^4$ solution.

We remark that the result that there are no warped product dS_5 solutions preserving exactly $N = 24$ supersymmetries is somewhat analogous to a similar non-existence theorem for $N = 24$ warped product AdS_5 solutions established in [16], where it was also noted that this is consistent with the results of [29]. In particular it is known that there are no exactly $\mathcal{N} = 3$ theories which admit exactly marginal deformations, and in particular there are no such theories defined perturbatively in terms of a Lagrangian, though non-Lagrangian theories are permitted [30]. It would be interesting to understand the absence of $N = 24$ warped product dS_5 solutions in the context of any proposed dS_5/CFT_4 duality.

Data availability statement

No additional research data beyond the data presented and cited in this work are needed to validate the research findings in this work.

All data that support the findings of this study are included within the article (and any supplementary files).

Acknowledgment

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