Lie Algebra of a Derivative Nonlinear Schrödinger Equation

Chau-Chin Wei, Ying-Chuan Yang, H.J. Annegarn*, Ren-Jye Yeh** and Chia-Yu Wang#

Department of Physics, Tsing Hua University, Hsinchu, Taiwan, *WITS-CSIR Schonland Research Centre for Nuclear Sciences, University of the Witwatersrand, Johanesburg, South Africa, ** Medical Systems Group, General Electric Company, 255 Fourier Avenue, Fremont, CA 94539, U.S.A. and #Chung Shan Institute of Science and Technology, Lungtang, Taiwan.

ABSTRACT

A Lie algebra is obtained from the prolongation structure of a derivative nonlinear Schrödinger equation. A similarity solution is obtained through solving the characteristic equation.

1. Introduction

Since the discovery of solitons for nonlinear equation of evolution, a large number of soliton solutions is obtained for nonlinear Schrödinger equation by inverse scattering transform, τ-function theory [1], Hirota's direct method [2], the Bäcklund transform and so on. On the other hand, the similarity solutions for which the nonlinearity has essential consequences are discussed by Redekopp [3]. However, to the best of our knowledge, no Lie group theoretic approach to obtain a similarity solution has been given and no Lie algebra is explicitly shown. The group theoretic approach to the solutions of partial differential equations was initiated by Lie himself in 19th century. Many years later Ovsiannikov [4] and Bluman and Cole [5] extended the theory to discuss the nonlinear equations in hydrodynamics. We followed [4] and [5] closely and applied the theory to a space charge flow [6]. In this contribution, we have obtained Lie operators explicitly by studying the prolongation structure of a derivative nonlinear Schrödinger equation. With Lie algebra obtained we can get a similarity solution through solving the characteristic equations.

2. Lie theory

We consider the following derivative nonlinear Schrödinger equation (DNLSE hereafter)

$$H = iu_t + \beta u_{xx} + i\delta' u^* u u_x + \delta u^* u u = 0 \tag{2.1}$$

where β, δ', δ are real constants, $i = \sqrt{-1}$, $*$ stands for the complex conjugate and the lower suffixes represent partial differentiations. The infinitesimal Lie transformation may be written as follows:

$$u' = u + \varepsilon U(x,t,u) \tag{2.2a}$$

$$x' = x + \varepsilon X(x,t,u) \tag{2.2b}$$

$$t' = t + \varepsilon T(x,t,u) \tag{2.2c}$$

where ε is a small parameter.

Invariance is defined as follows:

i. eq.(2.1) is left invariant when $H'(u',x',t') = 0$ iff $H(u,x,t) = 0$ where $H'(u',x',t')$ is obtained from H if (u,x,t) is replaced by (u',x',t').

ii. the boundary conditions and boundary curves are left invariant.

Assuming that $\int u^*u dx$ is finite and converges to a constant C in the domain of definition R, namely we look for a solution of (2.1) with the following condition:

$$\int_R u^*u dx = C \tag{2.3}$$

The invariance condition becomes

$$\theta(x + \varepsilon X,\ t + \varepsilon T) = \theta(x,t) + \varepsilon U(x,t,\theta) + O(\varepsilon^2) \tag{2.4}$$

where θ is a solution of (2.1).

Expanding the left hand side and equating the O(ε) terms we have

$$X(x,t,\theta)\frac{\partial\theta}{\partial x} + T(x,t,\theta)\frac{\partial\theta}{\partial t} = U(x,t,\theta) \tag{2.5}$$

and the corresponding characteristic equations to eq.(2.5) are in general

$$\frac{dx}{X(x,t,\theta)} = \frac{dt}{T(x,t,\theta)} = \frac{d\theta}{U(x,t,\theta)} \tag{2.6}$$

Now in order to find out which infinitesimal transformation can be admitted we need also to study the invariance of the differential operators H and calculate how derivatives transform. It must be stressed here that once the transformations of the basic coordinates (x,t,u) are known, the rest of the transformations of higher order derivatives are determined accordingly. This basic fact is usually known as a prolongation (or extension). After straightforward but lengthy calculations we obtained the following results,

$$X(x,t) = \kappa + \mu t + \gamma xt + \nu x \tag{2.7a}$$

$$T(t) = \alpha + 2\nu t + \gamma t^2 \tag{2.7b}$$

$$U(x,t,u) = u[- \gamma\{t/2 - ix^2/4\beta\} + (ix/2\beta)\mu + \lambda + i\nu c_1\delta + i\gamma c_2\delta] \quad (2.7c)$$

where κ, α, λ, ν, γ, μ are 6 arbitrary parameters and c_1, c_2 are constants. The Lie operators corresponding to 6 parameters are:

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial t}, \quad X_3 = u\frac{\partial}{\partial u}$$

$$X_4 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + ic_1\delta' u\frac{\partial}{\partial u}$$

$$X_5 = t^2\frac{\partial}{\partial t} + xt\frac{\partial}{\partial x} + \left\{-\left(\frac{t}{2} - \frac{ix^2}{4\beta}\right) + ic_2\delta\right\}u\frac{\partial}{\partial u}$$

$$X_6 = t\frac{\partial}{\partial x} + \frac{ix}{2\beta}u\frac{\partial}{\partial u} \quad (2.8)$$

These operators form a Lie algebra and its group table is shown below. It is noted that this group has a strong resemblance of the classical group of the (linear) heat equation [7]. Indeed if we put $\delta' = \delta = 0$, eq.(2.1) is a heat equation.

	X_1	X_2	X_3	X_4	X_5	X_6
X_1	0	0	0	X_1	X_6	$\frac{i}{2\beta}X_3$
X_2	0	0	0	$2X_2$	$X_4 - (ic_1\delta+\frac{1}{2})X_3$	X_1
X_3	0	0	0	$ic_1\delta' X_3$	0	0
X_4	$-X_1$	$-2X_2$	$-ic_1\delta' X_3$	0	$2X_5 + 2ic_2\delta X_3$	X_6
X_5	$-X_6$	$-X_4 + (ic_1\delta' + \frac{1}{2})X_3$	0	$-2X_5 - 2ic_2\delta X_3$	0	0
X_6	$-\frac{i}{2\beta}X_3$	$-X_1$	0	$-X_6$	0	0

3. Similarity Solution

Substituting eqs.(2.7a,b,c) into eq.(2.6), the characteristic equation for DNLSE has the form

$$\frac{dx}{X(x,t)} = \frac{dt}{T(t)} = \frac{d\theta}{U(x,t,\theta)} \qquad (3.1)$$

where X/T is independent of u. Thus we can obtain the similarity solution of the form

$$u(x,t) = F(x,t,\ \eta, f(\eta\ )) \qquad (3.2)$$

for the solution θ. The similarity variable η is an integal of the first equality in eq.(3.1) and

$$\eta(x,t) = \text{constant} \qquad (3.3)$$

defines path curves (similarity curve) in (x,t)-space. The dependence of F on η involves an arbitrary function $f(\eta\ )$ which is the solution to some ordinary differential equation obtained by substituting eq.(3.2) into eq.(2.1). Now we proceed to obtain the explicit form of eq.(3.2). Path curve is obtained by integrating the first equality of eq.(3.1) which may be rewritten as

$$dx/dt = \{\ \kappa + \ \mu t + x(\gamma t + \nu)\}/\{\alpha + 2\ \nu t + \gamma t^2\} \qquad (3.4)$$

Elementary calculus gives the following four cases:

(i) $\nu^2 - \alpha\gamma \neq 0$, $\gamma \neq 0$ (ii) $\nu^2 - \alpha\gamma = 0$, $\gamma \neq 0$

(iii) $\nu^2 - \alpha\gamma = 0$, $\nu = \gamma = 0$, $\alpha \neq 0$ (iv) $\alpha = \nu = \gamma = 0$.

We discuss case (i) in detail and other cases are the obvious consequences of (i). For case (i), eq.(3.4) can be integrated as

$$\eta = \{x - (At + B)\}/\sqrt{\gamma t^2 + 2\nu t + \alpha} \qquad (3.5)$$

where

$$A = (\gamma + \mu\nu)/(\alpha\delta - \nu^2) \quad (3.6a)$$

$$B = (\alpha\mu - \nu)/(\alpha\gamma - \nu^2) \quad (3.6b).$$

In order to simplify the algebraic calculation we transform

$$t = \tilde{t}/\sqrt{\gamma} - \nu/\gamma \quad (3.7a)$$

$$x = \tilde{x}/\sqrt{\gamma} - \mu/\gamma \quad (3.7b)$$

and define

$$b^2 = (\nu^2 - \alpha\gamma)/\gamma \quad (3.8a)$$

$$V = (\mu\nu - \kappa\gamma)/(\nu^2 - \alpha\gamma) \quad (3.8b)$$

and rewrite $\tilde{x} \to x$, $\tilde{t} \to t$ to simplify eq.(3.4) and eq.(3.5). Then the last equality of eq.(3.1) is reduced to

$$\frac{d\theta}{\theta} = \frac{i\eta}{4}\,dt + \frac{(\frac{\nu}{2} - \frac{\mu^2 i}{4\beta\gamma} + \lambda + i\nu c_1\delta + i\gamma c_2\delta)}{(t^2 - b^2)}\,dt - \frac{\sqrt{\gamma}}{2}\frac{t}{(t^2-b^2)}\,dt + \frac{iV^2}{4\beta}\frac{t^2dt}{(t^2-b^2)} + \frac{iV}{2\beta}\frac{t\,dt}{\sqrt{(t^2-b^2)}} \quad (3.9)$$

Finally the similarity solution with similarity variable η is obtained by integrating eq.(3.9) as

$$\theta = f(\eta)(t^2-b^2)^{-\frac{\sqrt{\gamma}}{4}}\left(\frac{t-b}{t+b}\right)^{\rho}\exp\{\frac{i}{4\beta}(\eta^2+V^2)t + \frac{iV\eta}{2\beta}\sqrt{t^2-b^2}\} \quad (3.10)$$

where

$$\rho = (\tfrac{\nu}{2}) - (\mu^2\, i/4\beta\gamma) + \lambda + i\nu c_1\delta + i\gamma c_2\delta + \frac{iV^2b^2}{4\beta} \quad (3.11a)$$

$$\eta = (x - Vt)/\sqrt{(t^2-b^2)} \quad (3.11b)$$

In order to determine $f(\eta)$, we substitute eq.(3.10) into eq.(2.1) to have an ordinary differential equation

$$\frac{d^2 f}{d\eta^2} + [\Omega + \frac{b^2\eta^2}{4\beta^2}]f = 0 \tag{3.12}$$

where

$$\Omega = i(\sqrt{\gamma} + 1)/(2\beta) + 2i\rho b/\beta \tag{3.12a}$$

The solution $f(\eta)$ can be expressed in terms of parabolic cylinder function.

Acknowledgement

The work is supported by the National Science Council of China (NSC76-0208-M007-75)

References.

1. Michio Jimbo and Tetsuji Miwa, Publ. RIMS. Kyoto Univ. 19, 943 (1983)
2. A. Nakamura and H.H. Chen, Journ. Phys. Soc. Japan 49, 813 (1980)
3. L.G. Redekopp, Studies in Applied Math. 63, 185 (1980)
4. L.V. Ovsiannikov, "Group Analysis of Differential Equations" Academic Press, New York (1982)
5. G.W. Bluman and J.D. Cole, "Similarity Methods for Differential Equations" Springer-Verlag, New York (1974)
6. Chau-Chin Wei, C.Y. Wang, P.J. Chang and J.C. Wu, Scaling Law of a Space Charge Flow, XIIIth International Colloquium on Group Theoretical Methods in Physics. W.W. Zachary, editor, World