Modular Calabi-Yau Threefolds in String Compactifications

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Dedicated to my mother.

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Abstract

All elliptic curves defined over $\mathbb{Q}$ are modular. This is the statement of the modularity theorem that relates arithmetic properties of an elliptic curve to a modular form for a subgroup of $SL(2,\mathbb{Z})$. This modularity has been (and still is) the subject of much research in number theory. At first, one might suspect that this kind of modularity is irrelevant for physics. Nevertheless, we will show in this thesis that modular Calabi-Yau threefolds are distinguished by certain physical processes. Most notably, by the attractor mechanism of IIB supergravity.

We will see that the zeta-function associated to an attractor variety of rank two will factor in a specified manner and that a search for such factorisations leads to an effective strategy for identifying examples of rank two attractor varieties with small $h^{2,1}$. We will also find and study a number of attractor varieties of rank two by taking the *Hadamard product* of two Picard-Fuchs equations of families of elliptic curves. As we will see, in many cases, the Hadamard product admits an involution with fixed points that are attractor points of rank two. We are able to identify the associated modular forms in both types of examples.

In the remainder of this thesis, we will explore the physical implication of modularity. We will show that certain arithmetically interesting quantities such as critical values of the associated L-functions determine the area of the horizon of certain dyonic $\mathcal{N}=2$ black holes. Similarly, we find that topological string free energies, when evaluated at the mirror of a rank two attractor point, may also be expressed in terms of arithmetically interesting quantities.

Statement of Originality

Except where otherwise stated, the work presented in thesis is a product of the following collaborations to which the author contributed substantially:

1. *"A One Parameter Family of Calabi-Yau Manifolds with Attractor Points of Rank Two"* by Philip Candelas, Xenia de la Ossa, Mohamed Elmi and Duco van Straten
   (arXiv:1912.06146)
2. *"Effective Supergravity Actions at Special Arithmetic Points in Calabi-Yau Moduli Spaces"* by Kilian Bönisch, Mohamed Elmi and Albrecht Klemm
   (in preperation) .

Except where otherwise stated, the results presented in Chapters 1 to 5 are from publication 1. They mostly concern free quotients of the Hulek and Verril manifold [1] (see Appendix A) and the associated Picard-Fuchs equation AESZ 34 [2].

Chapters 6 to 9 are mostly concerned with fourth order Picard-Fuchs equations that are constructed as Hadamard products of second order Picard-Fuchs equations. More specifically, many of these operators admit the action of an involution and we study the resulting fixed points. This will appear in publication 2.

Contents

Chapter 1

Introduction

1.1 Introduction

It is often observed that physicists and number theorists study the same objects but in very different settings. A ubiquitous example is an elliptic curve E defined by the cubic

$$y^2 = x^3 + Ax + B; \quad A, B \in \mathbb{Z} \ . \tag{1.1}$$

A number theorists might ask - how many solutions does this cubic have over $\mathbb{Q}$ or a finite field $\mathbb{F}_{p^r}$? While the physicist might suppose that the extra dimensions of string theory take the shape of an elliptic curve and is thus interested in the variety in $\mathbb{CP}^2$. The number theorist and the physicist would use similar techniques to study the elliptic curve. However, because physicists are ultimately interested in solutions over $\mathbb{C}$, the questions they ask tend to be rather different from those of the number theorist. Most notably, this means that the physicist fails to see one of the most fascinating properties of the elliptic curve. Namely, modularity.

A modular form is a holomorphic function f defined on the upper half plane with Fourier expansion

$$f(\tau) = \sum_{n=0}^{\infty} a_n q^n; \quad q = \mathrm{e}^{2\pi i \tau} \tag{1.2}$$

and the defining property

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau); \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}) \tag{1.3}$$

for some $k \in \mathbb{N}$ known as the weight of the modular form and, very often, the coefficients a_n contain interesting arithmetic information.

One may speak about modular forms for subgroups of $SL(2, \mathbb{Z})$ by relaxing the requirement of Equation 1.3 and asking that it only holds for a specific subgroup. We also require that f be regular at certain points on the boundary of the upper half plane known as cusps.

A consequence of the celebrated modularity theorem [3–5] is that there exists a modular form of weight 2 for some subgroup of $SL(2,\mathbb{Z})$ such that the number of solutions of Equation 1.1 over $\mathbb{F}_p\mathbb{P}$ is, for all but finitely many primes, given by

$$|E/\mathbb{F}_p| = p + 1 - a_p \tag{1.4}$$

where a_p is the p^{th} Fourier coefficient of the modular form.

All this, while fascinating and the topic of much research in number theory, is not obviously relevant for the physicist. There are however important exceptions that we will explore in this thesis.

String theory describes the motion of strings in 10, 11 or 12 dimensions whereas we seem to live in $1+3$. The simplest way of reconciling these two facts is to suppose that there are compact extra dimensions that are small enough to have escaped detection thus far. This reduction is known as *compactification* and the geometry of the compact dimensions determines physics in the remaining directions. Calabi-Yau manifolds are ubiquitous in string theory because provide they provide the simplest examples of string compactifications that preserve supersymmetry [6].

When considered as a variety in $\mathbb{CP}^2$, an elliptic curve is a one dimensional Calabi-Yau manifold and, just like elliptic curves, a Calabi-Yau threefold X defined by polynomials over $\mathbb{Q}$ can be modular in the sense that the problem of counting points on X with coordinates in a finite field $\mathbb{F}_{p^r}$ is equivalent to the problem of computing the Fourier coefficients of several modular forms for some subgroup of $SL(2,\mathbb{Z})$.

Unlike elliptic curves, however, a generic Calabi-Yau threefold defined over $\mathbb{Q}$ will not be modular and examples are hard to come by [7]. Given the prominent role that Calabi-Yau manifolds play in string theory, one naturally wonders - are modular Calabi-Yau threefolds distinguished in any way from the point of view of string theory? Surprisingly, the answer is sometimes yes!

As we will see, a modular Calabi-Yau threefold X can arise in the study of four dimensional $\mathcal{N}=2$ black holes that are constructed by compactifying IIB supergravity on X with $D3$ branes wrapping certain 3 dimensional submanifolds of X. It is known that properties of the resulting black hole such as horizon area will depend on the complex structure moduli of X and, since the area of a black hole is a measure of entropy in the limit of large mass [8], it should not depend on continuously varying parameters like the moduli of a Calabi-Yau manifold. The resolution of this apparent paradox goes by the name of the *attractor mechanism* [9].

The preservation of supersymmetry in $1+3$ dimensions requires that the complex structure moduli of the Calabi-Yau manifold X vary with radial distance from the horizon of the black hole in a manner dictated by a set of differential equations [10]. Infinitely far away from the black hole, space-time is flat and the moduli of the Calabi-Yau manifold are unconstrained. As one moves towards the horizon of the black hole, the complex structure moduli of the Calabi-Yau will evolve and approach a point in moduli space (the "attractor point") determined by the charges of the black hole which, in turn, are determined by the homology class $\gamma \in H_3(X, \mathbb{Z})$ wrapped by D3 branes. Moreover, the attractor point is invariant under small perturbations of the moduli at infinity which is the origin of the name "attractor mechanism" which is illustrated in Figure 1.1. If the homology class γ is dual to $\Gamma \in H^3(X, \mathbb{Z})$, the attractor point determines a choice of complex structure such that

$$\Gamma \in H^{3,0} \oplus H^{0,3} \ . \tag{1.5}$$

The connection between the arithmetic geometry of Calabi-Yau manifolds and the attractor mechanism was first noted in the seminal paper of Moore [11, 12] where he studied several examples of attractor varieties. For example, he noted that:

1. If the attractor variety in question is given by $E \times E \times E$ where E is an elliptic curve, then E has complex multiplication.

2. If the attractor variety in question is given by $S \times E$ for some a $K3$ surface S and an elliptic curve E, then the elliptic curve has complex multiplication and the $K3$ surface has Picard rank 20 which implies that

$$H^{2,0} \oplus H^{0,2} \ = \ T_S \otimes \mathbb{C} \tag{1.6}$$

   where $T_S \subset H^2(S, \mathbb{Z})$ (known as the transcendental lattice).

The above examples of attractor varieties have many interesting arithmetic properties that we cannot do justice here and we direct the interested reader to the original papers of Moore. Our goal will be to find exact examples of attractor varieties, study their arithmetic properties and explore what they teach us about string compactifications.
The main result of this thesis rests on the following observation: suppose that $h^{2,1}(X) = 1$ and $\Gamma_1, \Gamma_2 \in H^3(X, \mathbb{Z})$ are dual to linearly independent homology classes that lead to the same attractor point φ_* (as illustrated in in Figure 1.1). We refer to φ_* as a rank two attractor point. We will see that Γ_1 and Γ_2 span a rank two lattice

$$\Lambda \subset H^3(X, \mathbb{Z}) \tag{1.7}$$

such that

$$\Lambda \otimes \mathbb{C} \ = \ H^{3,0} \oplus H^{0,3} \ . \tag{1.8}$$

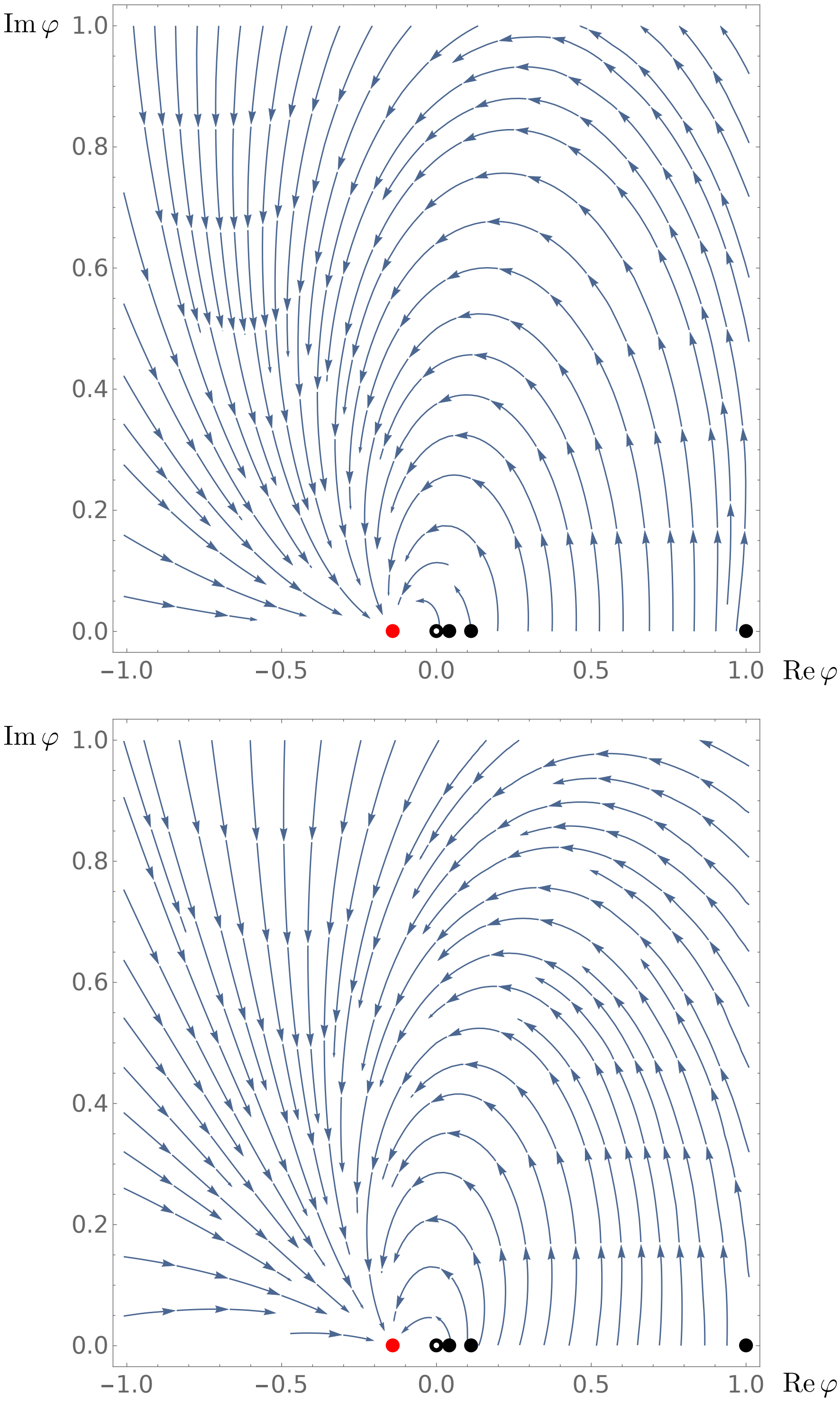

Figure 1.1: *The flow of the complex structure modulus* $\varphi = \varphi(\rho)$ *for two dyonic black hole that arises by compactifying IIB on a Calabi-Yau X (described in Appendix A). The black holes are charged under a* $U(1)^2$ *gauge group and have charges* $Q = (0, 0, 2, 1)$ *(above) and* $Q = (-4, 15, 5, 0)$ *(below) where the two first components are electric charges and the last two magnetic. The point indicated in red is an attractor point of rank two at* $\varphi = -1/7$*. The large complex structure point at* $\varphi = 0$ *is indicated by a hollow black dot while the solid black dots represent conifold singularities at* $\varphi = 1/25$*,* $\varphi = 1/9$ *and* $\varphi = 1$*.*

At the level of Hodge numbers, this is a splitting of $H^3(X, \mathbb{Q})$ of the form

$$(1,1,1,1) \to (1,0,0,1) + (0,1,1,0) \ . \tag{1.9}$$

If the underlying Calabi-Yau variety X_{φ_*} at a rank two attractor point is algebraic (as conjectured by Moore), a splitting of $H^3(X_{\varphi_*}, \mathbb{Q})$ of the form indicated by (1.9) will be visible in the *zeta function* of the Calabi-Yau variety which is a generating function of the number of points of X_{φ_*} when considered as a variety over the finite field $\mathbb{F}_{p^r}$. We will show that this leads to an effective strategy for finding attractor points of rank two.
It was proven in [13] that if X_{φ_*} is defined over $\mathbb{Q}$ as a variety and $H^3(X_{\varphi_*}, \mathbb{Q})$ splits as in (1.9), then X_{φ_*} is modular in the sense that the zeta function of X_{φ_*} will be determined by a pair of weight 2 and 4 modular forms for a subgroup of $SL(2, \mathbb{Z})$. We will identify the associated modular forms in a number of examples and, in the rest of this thesis, explore how this modularity affects physically measurable quantities.

1.1.1 Overview of Thesis

This thesis is organised as follows

2. We begin in Chapter 2 with an overview of the moduli space of Calabi-Yau manifolds. We give a review of special geometry as it plays a prominent role in this thesis. This is then followed by an explanation of the Picard-Fuchs equation and a detailed calculation is presented in a specific example. With all this in place, we review the attractor mechanism and emphasise the difference between attractor points of rank one and two by thinking of rank two attractors as rational points on a Grassmannian (in the Plücker embedding).

3. Chapter 3 is a review of the necessary background in arithmetic geometry. We review zeta-functions associated to a variety and explain that its form is determined the (now proven) Weil conjectures. We then go onto explain that this zeta-function is sometimes determined by modular forms for (subgroups of) $SL(2, \mathbb{Z})$ and provide all the necesary definitions along the way e.g. cusp forms, Hecke operators, etc.

4. One of the main results of this thesis may be found in Chapter 4. It details how the splitting of Hodge structure at a rank two attractor point is visible in the zeta-function as a factorisation for some $\varphi \in \mathbb{F}_p$ where p is a prime. This leads to an effective strategy for finding examples of attractor points of rank two. By searching for algebraic points $\varphi \in \mathbb{C}$ such that their mod p reduction leads to a factorisation of the zeta-function, we find examples of rank two attractors. Moreover, associated to each atttractor point is a pair of modular forms that we identify (we are dealing here with Calabi-Yau manifolds with $h^{2,1} = 1$).

5. It is expected on mathematical grounds that the periods of the holomorphic three form of the underlying smooth Calabi-Yau variety at a rank two attractor are determined by critical L-function values (see, for example, [14]). In this way, we determine the periods and their derivatives at rank two attractor points in Chapter 5. The higher derivatives may be determined by quasi-periods associated to the modular forms at a rank two attractor. This was communicated to us by Bönisch and Klemm [15] shortly before the completion of [16] and we explain how they computed these quasi-periods via the formalism of period polynomials.

6. In Chapter 6, we study fourth order Picard-Fuchs equations constructed from the so-called *Hadamard product* of second order Picard-Fuchs equations. We show that in specific examples that these operators may admit the action of an involution. In

these examples, the fixed points of the involutions are apparent singularities and, at the same time, attractor points of rank two. Two such examples are studied in detail in this chapter.[1]

7. A very similar degeneration of Hodge structure to that of a rank two attractor appears in the study of flux compactifications. We explore these connections in Chapter 7 and study an example of a Hadamard product with an involution where the fixed point is not an attractor point of rank two but may be interpreted as a critical point of a flux superpotential W where $W \neq 0$.

8. In Chapter 8, we compute topological string free energies in several examples. The results of Chapter 5 allow us to evaluate the topological string free energies at all genera at attractor points of rank two.

9. Finally, Chapter 9 is a speculative chapter in which we compare various structures familiar from the study of Seiberg-Witten theory with their analogues in supergravity. For example, we point out that an attractor point of rank two is a point where two mutually non-local dyons become massless. In Seiberg-Witten theory, such a point leads to an Argyres-Douglas superconformal field theory and we speculate on what the analogous theory might be in supergravity.

[1] An apparent singularity is a singularity of the Picard-Fuchs equation where the underlying Calabi-Yau manifold is, nevertheless, smooth.

Chapter 2

Special Geometry and the Attractor Mechanism

2.1 Moduli Space of Calabi-Yau Manifolds

2.1.1 Special Geometry

Much of this thesis will be concerned with the moduli space of Calabi-Yau manifolds so we recall the essential features here. We will follow the review [17] where the reader can find a more detailed account.

Let X be a Calabi-Yau threefold and denote by Ω its nowhere vanishing holomorphic 3-form (unique up to normalisation) and by ω its Kähler form. Yau's theorem states that X admits a unique Ricci-flat Kähler metric in the same cohomology class as ω so we may identify the parameter space of Calabi-Yau manifolds with the space of Ricci-flat metrics on X. We can therefore (redundantly) parametrise the moduli space of X by shifts of the Ricci flat metric (with components G_{mn} where $m, n \in \{1, ..., 6\}$) that preserve the Ricci flat condition

$$R_{mn}(G + \delta G) = 0 \tag{2.1}$$

where R_{mn} are components of the Ricci-tensor. Equation (2.1) and the condition $\nabla^n \delta G_{mn} = 0$ imply that δG_{mn} is a solution of the *Lichnerowicz equation*

$$\nabla^k \nabla_k \delta G_{mn} + 2R_m{}^p{}_n{}^q \delta G_{pq} = 0 \tag{2.2}$$

where $R^m{}_{pnq}$ is the Riemann tensor.

In holomorphic/anti-holomorphic coordinates, components of the metric G_{mn} can be split into those with pure indices ($G_{\mu\nu}$ and $G_{\overline{\mu\nu}}$) and those with mixed indices ($G_{\overline{\mu}\nu}$ and $G_{\mu\overline{\nu}}$). Moreover, it can be shown that the pure terms and the mixed terms of δG_{mn} satisfy the Lichnerowicz equation independently.

Variations of the metric with mixed indices $\delta G_{\mu\overline{\nu}}$ simply define a new Kähler form as is clear from the form of the Kähler form in coordinates

$$\omega = iG_{\mu\overline{\nu}}\, \mathrm{d}x^{\mu} \wedge \mathrm{d}x^{\overline{\nu}}. \tag{2.3}$$

On the other hand, the variations of the metric with pure indices correspond to variations of the complex structure of the Calabi-Yau manifold. This is because, in holomorphic/anti-holomorphic coordinates, the metric is "off diagonal" i.e. $G_{\mu\nu} = 0$. Moreover, any holomorphic change of coordinates preserves the condition $G_{\mu\nu} = 0$ and a non-holomorphic change of coordinates is therefore required to remove the $\delta G_{\mu\nu}$ in the metric. Therefore, $\delta G_{\mu\nu}$ corresponds to a deformation of complex structure.[1]

The dimension of the space of Kähler deformations is given by the Hodge number $h^{1,1}$ because

$$i(G_{\mu\overline{\nu}} + \delta G_{\mu\overline{\nu}})\, \mathrm{d}x^{\mu} \wedge \mathrm{d}x^{\overline{\nu}} \tag{2.4}$$

is harmonic if and only if $\delta G_{\mu\overline{\nu}}$ satisfies the Lichnerowicz equation and there are precisely $h^{1,1}$ linearly independent harmonic $(1,1)$ forms. Note that the space of Kähler deformations is not complex. Only *real* linear combinations of Kähler forms lead to other Kähler forms. Similarly, $\delta G_{\mu\nu}$ satisfies the Lichnerowicz equation if and only if

$$\Omega_{\kappa\lambda}{}^{\overline{\nu}} \delta G_{\overline{\mu}\overline{\nu}}\, \mathrm{d}x^{\kappa} \wedge \mathrm{d}x^{\lambda} \wedge \mathrm{d}x^{\overline{\mu}} \tag{2.5}$$

is harmonic. Moreover, there are precisely $h^{2,1}$ linearly independent harmonic $(2,1)$ forms so the complex structure moduli space of X has *complex* dimension $h^{2,1}$.

There is a natural metric on the space of Ricci-flat metrics that will be of importance in this thesis. If we include a B-field (a real harmonic $(1,1)$ form), a line element in this metric is given by

$$\mathrm{d}s^2 = \frac{1}{4V} \int_X G^{km} G^{ln} \left(\delta G_{kl} \delta G_{mn} + \delta B_{kl} \delta B_{mn} \right) G^{\frac{1}{2}} \mathrm{d}^6 x \tag{2.6}$$

where V is the volume of the Calabi-Yau manifold. If we ignore the B-field, this metric is essentially the first metric that one would write down if asked to write down a metric on the space of metrics and there are many independent ways of arriving at it. In the mathematics literature, it is referred to as the Weil-Petersson metric and, in the context of the moduli space of a two dimensional $\mathcal{N} = 2$ quantum field theory, it can be derived from the so-called $tt^{\star}$ equations [18].

The B-field is included here with the benefit of hindsight because it is in the same super multiplet as the metric G_{mn} which frequently appears in the combination $B + i\omega$. Including

[1] An important subtlety that we are sweeping under the rug is that it is not a-priori obvious that any infinitesimal deformation of complex structure lifts to a global one. However, in this case, it is guaranteed by a theorem of Tian and Todorov [18].

the B-field also has the benefit of complexifying the space of Kähler deformations so that it appears on the same footing as the space of space of complex structure deformations. Without it, we cannot have *mirror symmetry*.

Expanding Equation (2.6) in holomorphic/anti-holomorphic coordinates leads to the line element

$$\begin{aligned} \mathrm{d}s^2 \;=\; & \frac{1}{2V}\int_X G^{\kappa\overline{\mu}}G^{\lambda\overline{\nu}}\left(\delta G_{\kappa\lambda}\delta G_{\overline{\mu\nu}}\right)G^{\frac{1}{2}}\mathrm{d}^6x \\ & + \frac{1}{2V}\int_X G^{\kappa\overline{\mu}}G^{\lambda\overline{\nu}}\left(\delta G_{\kappa\overline{\nu}}\delta g_{\lambda\overline{\mu}} + \delta B_{\kappa\overline{\nu}}\delta B_{\overline{m}\lambda}\right)G^{\frac{1}{2}}\mathrm{d}^6x \;. \end{aligned} \tag{2.7}$$

We see that Equation (2.7) does not mix deformations with pure indices and those with mixed indices (in components, it would be block diagonal). The first integral defines a metric on the moduli space of complex structures $\mathcal{M}_{CS}$ while the second integral defines a metric on the space of complexified Kähler forms $\mathcal{M}_K$. Thus, the total moduli space is, in fact, a product[2]

$$\mathcal{M}_{CS} \times \mathcal{M}_K \;. \tag{2.8}$$

The above factorisation is the first sign of mirror symmetry. It is conjectured that every Calabi-Yau threefold X with $h^{2,1} \geq 1$ has a mirror Calabi-Yau manifold Y where[3]

$$\mathcal{M}_{CS}(X) \simeq \mathcal{M}_K(Y) \qquad \text{and} \qquad \mathcal{M}_{CS}(Y) \simeq \mathcal{M}_K(X) \;. \tag{2.9}$$

The most straightforward consequence of this is that that Hodge numbers $h^{2,1}$ and $h^{1,1}$ of X and Y are interchanged. Physically, a compactification of Type IIB string theory on X leads to identical four dimensional physics as a compactification of Type IIA string theory on Y and it is in this context that mirror symmetry was first discovered.

We will now restrict our attention to the complex structure moduli space $\mathcal{M}_{CS}(X)$. The nowhere vanishing holomorphic 3-form Ω is sensitive to the choice of complex structure and is unique up to an overall scale and can therefore be used to define coordinates $\mathcal{M}_{CS}(X)$. We start by choosing a symplectic basis $\{A^a, B_b\}$ of the torsion free part of $H_3(X,\mathbb{Z})$. By this, we mean that the intersection of A^a and B_b is given by $\delta_a{}^b$ and all other intersections vanish. This generalises the symplectic basis of 1-cycles that one can find on a Riemann surfaces. Dual to this is a basis $\{\alpha_a, \beta^b\}$ of the torsion free part of $H^3(X,\mathbb{Z})$ that satisfies

$$\int_{A^b}\alpha_a \;=\; -\int_{B_a}\beta^b \;=\; \int_{X_\varphi}\alpha_a\wedge\beta^b \;=\; \delta_a{}^b. \tag{2.10}$$

[2]This is true as long as we do not include the boundariesof the moduli space.

[3]We say that Calabi-Yau manifold X is *rigid* if $h^{2,1}(X) = 0$. Clearly, it cannot have a mirror manifold Y in the usual sense as this would imply that $h^{1,1}(Y) = 0$ and the Kähler form always provides a $(1,1)$ cohomology class. Nevertheless, it is possible to find, in some generalised sense, mirror manifolds for rigid Calabi-Yau manifolds [19].

We then expand the holomorphic 3-form as

$$\Omega \;=\; z^a\alpha_a - \mathcal{F}^{(0)}_b(z)\beta^b \;. \tag{2.11}$$

The basis $\{\alpha_a, \beta^b\}$ is insensitive to small variations of complex structure so we expect that the components of Ω parametrise the complex structure moduli space. However, we have $2(h^{2,1}+1)$ components whereas the $\dim_{\mathbb{C}} \mathcal{M}_{CS} = h^{2,1}$. It turns out that half of the components, say z^α provide *projective* coordinates on $\mathcal{M}_{CS}$ and they are projective because Ω is only defined up an overall complex multiple.
As indicated, the remaining components $\mathcal{F}^{(0)}_b$ can be determined in terms of z^a. This follow from the fact that, as a cohomology class,

$$\partial_a\Omega \in H^{3,0} \oplus H^{2,1} \tag{2.12}$$

where the above derivative is with respect to z^a. This implies that

$$\int_X \Omega\wedge\partial_a\Omega = 0 \tag{2.13}$$

which in turn leads to the equation

$$2\mathcal{F}^{(0)}_a = \partial_a\big(z^b\mathcal{F}^{(0)}_b\big) \tag{2.14}$$

that has the solution

$$\mathcal{F}^{(0)}_a = \partial_a\mathcal{F}^{(0)} \tag{2.15}$$

for a function $\mathcal{F}^{(0)}$ that is homogeneous of degree two in z^a. $\mathcal{F}^{(0)}$ is the so-called *holomorphic prepotential* and in, the context of topological string theory, can be identified with B-model free energy at genus 0. We will have more to say about this in Chapter 8.
We refer to $\mathcal{F}^{(0)}$ as a prepotential because it determines a Kähler potential K for the metric on $\mathcal{M}_{CS}$ which is given by

$$K \;= -\log\left(-i\int_X \Omega\wedge\overline{\Omega}\right) \,. \tag{2.16}$$

In coordinates (say t^α on $\mathcal{M}_{CS}$) the resulting Kähler metric is given by

$$g_{\alpha\overline{\beta}} = \frac{\int_X D_\alpha\,\Omega\wedge D_{\overline{\beta}}\,\overline{\Omega}}{\int_X \Omega\wedge\overline{\Omega}} \tag{2.17}$$

and is equivalent to the first term in Equation (2.7) where $D_\alpha = \partial_\alpha + \partial_a K$.
It is straightforward to check that the derivative D_α is, in fact, a covariant derivative for gauge transformations

$$\Omega \to f(\varphi)\,\Omega \tag{2.18}$$

where f is any holomorphic function and this makes the gauge invariance of the metric manifest. Ω should therefore be understood as a section of a line bundle on $\mathcal{M}_{CS}$. Indeed, it is this observation that leads to the choice of K as the natural choice of Kähler potential. More generally, consider a quantity Ψ, which transforms under scale transformations with weight (m, n), by which we mean

$$\Psi \to f^m \bar{f}^n \,\Psi \tag{2.19}$$

e.g. Ω has weight $(1, 0)$ and e^{-K} has weight $(1, 1)$. We may define a covariant derivative for this gauge transformation by

$$\begin{aligned} D_\alpha \Psi \;&=\; \nabla_\alpha \Psi + m\,(\partial_\alpha K)\,\Psi \\ D_{\bar{\beta}} \Psi \;&=\; \nabla_{\bar{\beta}} \Psi + n\,(\partial_{\bar{\beta}} K)\,\Psi \end{aligned} \tag{2.20}$$

where ∇_α is the Levi-Civita connection on $\mathcal{M}_{CS}$. As with any covariant derivative, this ensures that $D_\alpha \Psi$ transforms in a manner parallel to Ψ under a gauge transformation

$$D_\alpha \Psi \to f^m \bar{f}^n \, D_\alpha \Psi \ . \tag{2.21}$$

Note that $\mathrm{e}^{\pm K}$ has weight $(\mp 1, \mp 1)$ and

$$D_\alpha\, \mathrm{e}^{\pm K} \;=\; 0 \quad \text{and} \quad D_{\bar{\beta}}\, \mathrm{e}^{\pm K} \;=\; 0 \ . \tag{2.22}$$

Whereas $\partial_\alpha \Omega \in H^{3,0} \oplus H^{2,1}$ as a cohomology class, it can be shown that the covariant derivative $D_\alpha\, \Omega \in H^{2,1}$ and, in fact, the covariant derivatives $D_\alpha\, \Omega$ generate $H^{2,1}$. Similarly, $D_{\overline{\alpha}}\, \overline{\Omega} \in H^{2,1}$. It is then a standard exercise to derive the special geometry relations

$$D_\alpha \Omega \;=\; \chi_\alpha \qquad\qquad D_\alpha \Omega \;=\; \chi_\alpha \tag{2.23}$$

$$D_\alpha \chi_\beta \;=\; -i\, y_{\alpha\beta\gamma}\, \mathrm{e}^K \bar{\chi}^\gamma \qquad\qquad D_\alpha \chi_\beta \;=\; -y_{\alpha\beta\gamma}\, \widetilde{\chi}^\gamma \tag{2.24}$$

$$D_\alpha \bar{\chi}^\gamma \;=\; \delta_\alpha{}^\gamma\, \overline{\Omega} \qquad\qquad D_\alpha \widetilde{\chi}^\gamma \;=\; \delta_\alpha{}^\gamma\, \widetilde{\Omega} \tag{2.25}$$

$$D_\alpha \overline{\Omega} \;=\; 0 \qquad\qquad D_\alpha \widetilde{\Omega} \;=\; 0\ , \tag{2.26}$$

where, in these relations,

$$\widetilde{\chi}^\gamma \;=\; i\, \mathrm{e}^K g^{\gamma\bar{\beta}} \chi_{\bar{\beta}}\ ; \qquad \widetilde{\Omega} \;=\; i\, \mathrm{e}^K \overline{\Omega}\ ; \qquad y_{\alpha\beta\gamma} \;=\; -\int \Omega \wedge \partial_\alpha \partial_\beta \partial_\gamma \Omega\ . \tag{2.27}$$

Much of the utility of special geometry stems from from the fact that we can compute quantities of interest on $\mathcal{M}_{CS}$ in terms of the periods of Ω. For example, if we collect the periods of Ω into a vector Π defined as

$$\Pi = \begin{pmatrix} \int_{B_b} \Omega \\ \int_{A^a} \Omega \end{pmatrix} = \begin{pmatrix} \mathcal{F}_b^{(0)} \\ z^a \end{pmatrix}, \tag{2.28}$$

it follows from Equation (2.10) that the Kähler potential is simply given by

$$e^{-K} = -i\,\Pi^\dagger \Sigma \Pi \tag{2.29}$$

where

$$\Sigma = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} \tag{2.30}$$

and many other quantities related to the geometry of $\mathcal{M}_{CS}$.

Derivatives with respect to complex structure moduli commute with integrals on X and the metric in Equation (2.17) is, therefore, given by

$$g_{\overline{\alpha}\beta} = -\frac{D_{\overline{\alpha}}\Pi^\dagger \Sigma D_\beta \Pi}{\Pi^\dagger \Sigma \Pi} \tag{2.31}$$

which can then be used to compute, for example, the Christoffel symbols which, on a Kähler manifold, are given by

$$\Gamma^\alpha_{\beta\gamma} = g^{\alpha\overline{\beta}} \partial_\gamma g_{\beta\overline{\beta}}\ . \tag{2.32}$$

As we will see in Chapter 3, the periods of Ω also contain a lot of arithmetically interesting information about X as a variety over $\mathbb{Q}$ and even over finite fields $\mathbb{F}_{p^r}$ for a prime p and $r \in \mathbb{Z}$. For now, we ask - how does one compute the periods Π?

2.1.2 Picard-Fuchs Equations

2.1.2.1 AESZ 34

For ease of exposition, we will illustrate the computation of the periods of Ω for a family of Calabi-Yau manifolds with $h^{2,1} = 1$ that are realised as free quotients of a family first studied by Hulek and Verril [1]. Further details are given in Appendix A. The periods of other Calabi-Yau manifolds studied in this thesis are computed in essentially the same way. Writing down an expression for the nowhere vanishing holomorphic 3-form Ω is usually straightforward for a given Calabi-Yau threefold. Typically, one writes down a three form in some ambient projective/toric variety that restricts to Ω on X [20]. Nevertheless, directly computing all of the integrals in (2.28) is almost always prohibitively difficult. Instead, we make use of the fact the periods satisfy a system of differential equations known as the *Picard-Fuchs equations*.

Let φ be the single complex structure parameter of X. The fact that Π satisfies a differential equation follows from the straightforward observation that

$$\left\{ \Omega,\, \frac{d\Omega}{d\varphi},\, \frac{d^2\Omega}{d\varphi^2},\, \frac{d^3\Omega}{d\varphi^3},\, \frac{d^4\Omega}{d\varphi^4} \right\} \tag{2.33}$$

is linearly dependent in cohomology i.e. some linear combination of Ω and its derivatives is an exact form. Integrating this relation leads to a fourth order ordinary differential equation.

There are a variety of ways of computing the Picard-Fuchs equation. Typically, one computes one of the periods and uses this to find the Picard-Fuchs equation. In the case at hand,

$$\varpi_0(\varphi) = \frac{1}{(2\pi i)^5} \int_C \frac{\mathrm{d}^5 X}{X_1 X_2 X_3 X_4 X_5} \frac{1}{1-\varphi f(X)} \tag{2.34}$$

is a period of Ω where

$$f(X) = (X_1 + X_2 + X_3 + X_4 + X_5)\left(\frac{1}{X_1} + \frac{1}{X_2} + \frac{1}{X_3} + \frac{1}{X_4} + \frac{1}{X_5}\right) \tag{2.35}$$

and the contour C is a product of sufficiently small loops enclosing each of the poles $X_i = 0$ [21].

One may repeatedly differentiate Equation (2.34) with respect to φ and manipulate the integrand until the resulting expression vanishes. A systematic procedure for doing this is known as *Dwork-Griffiths reduction* (see [22] for a review). However, in practice, one computes the Picard-Fuchs equation of one parameter models by first expanding Equation (2.34) for sufficiently small φ. See for example, [23]. The coefficients in this expansion are given by the constant term in $f^n(X)$ and thus,

$$\varpi_0(\varphi) = \sum_{n=0}^{\infty} a_n \varphi^n \tag{2.36}$$

where

$$a_n \;=\; \sum_{p+q+r+s+t=n} \left(\frac{n!}{p!q!r!s!t!}\right)^2 . \tag{2.37}$$

We make the ansatz

$$\mathcal{L} \;= S_4\theta^4 + S_3\theta^3 + S_2\theta^2 + S_1\theta + S_0 \tag{2.38}$$

where $\theta = \varphi\frac{\mathrm{d}}{\mathrm{d}\varphi}$ and $S_i(\varphi)$ are polynomials and then insist that $\mathcal{L}\varpi_0 = 0$. By expanding φ to a sufficiently high order and steadily increasing the order of the polynomials S_i, we eventually find that[4].

$$\begin{aligned}
S_4(\varphi) &= (\varphi-1)(9\varphi-1)(25\varphi-1)\\
S_3(\varphi) &= 2\varphi\left(675\varphi^2-518\varphi+35\right)\\
S_2(\varphi) &= \varphi\left(2925\varphi^2-1580\varphi+63\right)\\
S_1(\varphi) &= 4\varphi\left(675\varphi^2-272\varphi+7\right)\\
S_0(\varphi) &= 5\varphi(180\varphi^2-57\varphi+1).
\end{aligned} \tag{2.39}$$

This resulting operator $\mathcal{L}$ appears as operator number 34 in the AESZ list [2]. Information about AESZ 34 may be conveniently presented in a *Riemann Symbol*

$$\mathcal{P}\left\{\begin{array}{ccccc} 0 & \frac{1}{25} & \frac{1}{9} & 1 & \infty \\ \hline 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 2 & 2 & 2 & 2 \end{array}\ \ \varphi\right\} \tag{2.40}$$

where the top row lists the singularities of the differential equation. These are given by the points $\varphi=0$, $\varphi=\infty$ and the roots of S_4 and typically correspond to the boundaries of the moduli space where the Calabi-Yau manifold becomes singular.[5] Below each singularity are listed four *indices* $\epsilon_i \in \mathbb{R}$ that govern the behaviour of each of the four solution ϖ_i near the corresponding singularity. More precisely, if an index ϵ_i is not repeated, a solution to the Picard-Fuchs equation is given by

$$\sum_{n=0}^{\infty} a_n(\epsilon_i)\varphi^{n+\epsilon_i} \tag{2.41}$$

where the coefficients $a_n(\epsilon_i)$ are determined recursively from the Picard-Fuchs equation. If, on the other hand, an index ϵ_i is repeated, additional solutions may be found by differentiating with respect to ϵ i.e.

$$\frac{\mathrm{d}^j}{\mathrm{d}\epsilon^j}\left(\sum_{n=0}^{\infty} a_n(\epsilon)\varphi^{n+\epsilon_i}\right)\Bigg|_{\epsilon=\epsilon_i}. \tag{2.42}$$

Note that the ambiguity in the choice of basis of solutions to $\mathcal{L}\varpi=0$ is a reflection of the fact that integrating Ω againts any basis of $H^3(X,\mathbb{C})$ will yield four solutions of the Picard-Fuchs equation. In solving the Picard-Fuchs equation, a useful basis is given by (2.41)

[4]Of course, we must then check that this is indeed a Picard-Fuchs equation. In practice, one checks a number of conditions that $\mathcal{L}$ must satisfy such as integrality integrality of the monodromy matrices (as we shall see later in this section) and integrality of the mirror map and the genus 0 and genus 1 Gopakumar-Vafa invariants (as we shall see in Chapter 8)

[5]This is not always the case! It is possible that for some point φ, the Picard-Fuchs equation is singular and, nevertheless, the underlying Calabi-Yau manifold is smooth. We refer to such points as *apparent singularities* and they will feature prominently in Chapter 6.

and (2.42) where we set as many terms as possible equal to zero in the resulting power series. We will refer to this as the *Frobenius basis*.
The singularity at $\varphi = 0$ is a point of *maximal unipotent monodromy* (MUM) and is also referred to as a large complex structure point. This misnomer is a reference to the fact that $\varphi = 0$ is mirror to the infinite volume point at the boundary of the Kähler moduli space of the mirror partner of X. The Frobenius basis around $\varphi = 0$ is given by

$$\begin{aligned}
\varpi_0(\varphi) &= f_0(\varphi) \\
\varpi_1(\varphi) &= f_0(\varphi)\log(\varphi) + f_1(\varphi) \\
\varpi_2(\varphi) &= f_0(\varphi)\log^2(\varphi) + 2f_1(\varphi)\log(\varphi) + f_2(\varphi) \\
\varpi_3(\varphi) &= f_0(\varphi)\log^3(\varphi) + 3f_1(\varphi)\log^2(\varphi) + 3f_2(\varphi)\log(\varphi) + f_3(\varphi)
\end{aligned} \tag{2.43}$$

where the f_j are power series with $f_0(0) = 1$ and $f_j(0) = 0$ for $j \geq 1$. The remaining coefficients of f_j are fixed recursively as

$$\begin{aligned}
f_0(\varphi) &= 1 + 5\varphi + 45\varphi^2 + 545\varphi^3 + 7885\varphi^4 + 127905\varphi^5 + \ldots \\
f_1(\varphi) &= 8\varphi + 100\varphi^2 + \frac{4148}{3}\varphi^3 + \frac{64198}{3}\varphi^4 + \frac{1804058}{5}\varphi^5 + \ldots \\
f_2(\varphi) &= 2\varphi + \frac{197}{2}\varphi^2 + \frac{33637}{18}\varphi^3 + \frac{2402477}{72}\varphi^4 + \frac{121787041}{200}\varphi^5 + \ldots \\
f_3(\varphi) &= -12\varphi - \frac{267}{2}\varphi^2 - \frac{19295}{18}\varphi^3 - \frac{933155}{144}\varphi^4 + \frac{114928799}{6000}\varphi^5 + \ldots
\end{aligned} \tag{2.44}$$

Equations (2.29) and (2.31) require that we compute the periods in an integral symplectic basis of (the torsion free part of) $H_3(X, \mathbb{Z})$. So we need to find a change of basis from (2.43) to a an integral symplectic basis which can be done by computing the monodromy of the solutions around the singularities in Figure 2.1.
The condition that $\{A^a, B_b\} \subset H^3(X, \mathbb{Z})$ forms an integral symplectic basis leaves us the freedom to transform this basis by an element of $\mathrm{Sp}(2h^{1,2}+2, \mathbb{Z})$ and is reflected in the fact that, in an integral symplectic basis of periods, the monodromy matrices are elements of $\mathrm{Sp}(2h^{1,2}+2, \mathbb{Z})$ see e.g. [24]. Thus, in order to compute the periods in an integral symplectic basis, we simply need to find a change of basis in which the monodromy matrices are integral symplectic. An efficient way of doing this takes advantage of mirror symmetry.
There exits a special set of coordinates (the so called *flat coordinates* t^α where $\alpha \in \{1, ..., h^{2,1}(X) = h^{1,1}(Y)\}$) on the complexified Kähler moduli space of the mirror Calabi-Yau Y such that the prepotential on $\mathcal{M}_K(Y)$ near the large volume point $t^\alpha \sim i\infty$ takes the form

$$F_0(t) = -\frac{1}{3!}Y_{\alpha\beta\gamma}t^\alpha t^\beta t^\gamma - \frac{1}{2}Y_{0\beta\gamma}t^\beta t^\gamma - \frac{1}{2}Y_{00\gamma}t^\gamma - \frac{1}{3!}Y_{000} + O(\mathrm{e}^{2\pi i t^\alpha}) \tag{2.45}$$

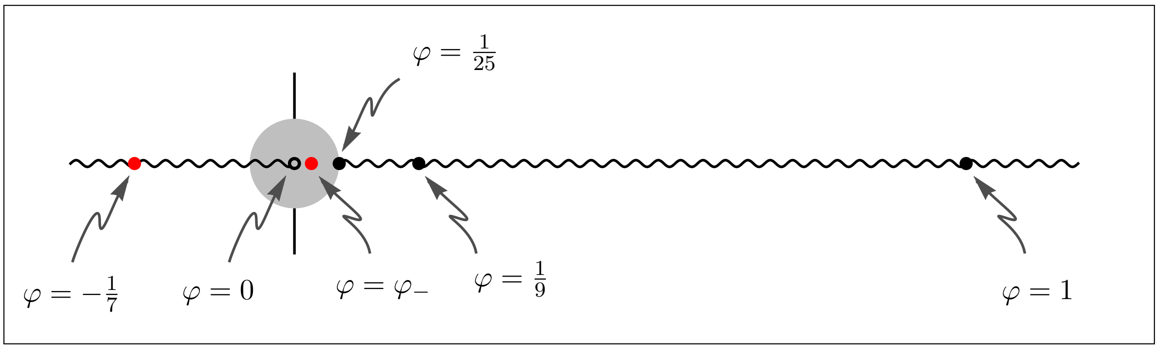

Figure 2.1: *The functions f_j, and so the periods, are defined initially in a disk of radius $\frac{1}{25}$. There is a branch cut on the negative real axis owing to our convention for the definition of the the logarithm. The branch cut that runs out along the positive real axis from $\varphi = \frac{1}{25}$ is due to the singularities of the functions f_j. The two red dots indicate the "attractor points of rank two" at $\varphi = -\frac{1}{7}$ and $\varphi = \varphi_- = 33 - 8\sqrt{17}$ that we will return to in Section 2.2. The large complex structure limit is at $\varphi = 0$, and is marked by a hollow dot, and the black dots indicate (hyper) conifold points. The attractor point at $\varphi = \varphi_+ = 33 + 8\sqrt{17}$ and the conifold point at $\varphi = \infty$ are not shown.*

where

$$\begin{aligned} Y_{\alpha\beta\gamma} &= \int_Y e_\alpha \wedge e_\beta \wedge e_\gamma \\ Y_{0\beta\gamma} &\in \left\{0, \frac{1}{2}\right\} \\ Y_{00\gamma} &= -\frac{1}{12}\int_Y c_2 \wedge e_\gamma \\ Y_{000} &= -3\frac{\zeta(3)}{(2\pi i)^3}\,\chi(Y) \end{aligned} \tag{2.46}$$

where the e_α are a basis for $H^2(Y)$. Note that F_0 is related to $\mathcal{F}^{(0)}$ by the gauge transformation

$$F_0 = \frac{1}{(z^0)^2}\mathcal{F}^{(0)} \tag{2.47}$$

that sets one of the periods equal to 1.

It is perhaps intuitive that the coefficients $Y_{0\beta\gamma}$ should be given by the integral of $c_1 \wedge e_\beta \wedge e_\gamma$ and so vanish. However, this is not quite true. The components can, by choice of basis, be made to take either the value 0 or $\frac{1}{2}$. For the case of one parameter, the rule is simple and depends on whether Y_{111} is even or odd. If Y_{111} is even, then Y_{011} can be taken to vanish, and if Y_{111} is odd, it can be taken to be 1/2. The history of the identification of these terms is a long one. The relation between the prepotential at the large complex structure and the intersection numbers $Y_{\alpha\beta\gamma}$ may be found in [17]. The identification of Y_{000} appears in [24]. The identification of the role of the coefficients $Y_{0\beta\gamma}$ and $Y_{00\gamma}$ may be found in [25]. The advance that sets these observations in context is the Gamma class [26].

Since $\mathcal{M}_K(Y) \simeq \mathcal{M}_{CS}(X)$ and half of the periods provide homogeneous coordinates on $\mathcal{M}_{CS}(X)$, it must be possible to express t^α in terms of the periods. This change of coordinates is known as a mirror map and, for one parameter models near a point of maximal unipotent monodromy (MUM point), a mirror map is given by

$$t(\varphi) = \frac{1}{2\pi i} \frac{\varpi_1(\varphi)}{\varpi_0(\varphi)} \tag{2.48}$$

which undergoes the monodromy $t \to t+1$ around $\varphi = 0$ that corresponds to a shift of the B-field in the complexified Kähler form.[6]

By identifying $z^0(\varphi) = \varpi_0(\varphi)$ and assuming Equation (2.47), it is not too difficult to use the monodromy of (2.43) around φ together with (2.48) and (2.45) to find a change of basis matrix

$$\rho \;=\; \begin{pmatrix} -\frac{1}{3}Y_{000} & -\frac{1}{2}Y_{001} & 0 & \frac{1}{6}Y_{111} \\ -\frac{1}{2}Y_{001} & -\ Y_{011} & -\frac{1}{2}Y_{111} & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & (2\pi i)^{-1} & 0 & 0 \\ 0 & 0 & (2\pi i)^{-2} & 0 \\ 0 & 0 & 0 & (2\pi i)^{-3} \end{pmatrix} \tag{2.49}$$

such that $\Pi = \rho\varpi$ [23].

Equation (2.49) is incredibly useful. Not only does it let us find the periods in an integral symplectic basis and compute the prepotential near $\varphi = 0$, it can also be used to identify the topological data in (2.46) where they are not known. By numerically computing the monodromy matrices around all the singularities of a Picard-Fuchs equation $\mathcal{L}$ and insisting that they are integral symplectic after conjugation by ρ, it is often possible to compute the data in (2.46) up to a finite (usually small) set of possible values. This was used to predict the existence of new one parameter Calabi-Yau manifolds in [27].

An important quantity that may be computed from the periods is the holomorphic Yukawa coupling $y_{\varphi\varphi\varphi}$ defined in (2.27). This may be computed by

$$y_{\varphi\varphi\varphi} \;=\; -\int_X \Omega \wedge \Omega''' \;=\; -\Pi^T \Sigma \Pi''' \tag{2.50}$$

and by differentiating both sides and replacing $\Pi^{(4)}$ with lower derivatives and using identities such as

$$\Pi^T \Sigma \Pi' = 0 \tag{2.51}$$

[6] It is not uncommon to use the mirror map

$$t(\varphi) = \frac{\varpi_1(\varphi)}{\varpi_0(\varphi)}$$

so that the t undergoes monodromy $t \to t + 2\pi i$. This seems to be common in the literature on topological string theory and is the normalisation that we will use in Chapter 8.

we find a differential equation with solution

$$y_{\varphi\varphi\varphi}(\varphi) = \frac{1}{\varphi^3}\exp\left(-\frac{1}{2}\int_X \frac{\mathrm{d}\varphi}{\varphi}\frac{S_3(\varphi)}{S_4(\varphi)}\right) \tag{2.52}$$

which leads to the rational function

$$y_{\varphi\varphi\varphi}(\varphi) = \frac{12\kappa}{(1-25\varphi)(1-9\varphi)(1-\varphi)} \tag{2.53}$$

where the integration constant in the numerator is chosen so that y_{ttt} is equal to the triple intersection number of the mirror of X when $\varphi = 0$ where

$$y_{ttt} = \frac{1}{\varpi_0^2} y_{\varphi\varphi\varphi}\left(\frac{1}{2\pi i}\frac{\mathrm{d}\varphi}{\mathrm{d}t}\right)^3 . \tag{2.54}$$

Note that the factor of ϖ_0^{-2} fixes a gauge since $y_{\varphi\varphi\varphi} \to f^2(\varphi) y_{\varphi\varphi\varphi}$ under a holomorphic gauge transformation $\Omega \to f(\varphi)\Omega$.

The Yukawa coupling is also commonly referred to as C_{ijk}, especially in topological string literature. We will use both in this thesis.

2.1.2.2 Monodromy Matrices

The topological data for the mirror of X is given by

$$\int_Y e_1 \wedge e_1 \wedge e_1 = 12\kappa; \qquad \int_Y c_2 \wedge e_1 = 12\kappa; \qquad \chi(Y) = -8\kappa \tag{2.55}$$

where $\kappa \in \{1, 2\}$ reflects the fact that AESZ 34 is the Picard-Fuchs equation for two possible quotients of the Hulek and Verril manifold discussed in Appendix A. We take $\kappa = 1$ for the $\mathbb{Z}/10\mathbb{Z}$ quotient and $\kappa = 2$ for the $\mathbb{Z}/5\mathbb{Z}$ quotient. This leads to

$$\begin{aligned} Y_{111} &= 12\kappa \\ Y_{011} &= 0 \\ Y_{001} &= -\kappa \\ Y_{000} &= -24\kappa\frac{\zeta(3)}{(2\pi i)^3} \end{aligned} \tag{2.56}$$

that we use to compute the periods in an integral basis Π.

Under monodromy about a singular point $\varphi = \phi$ the integral period vector undergoes a monodromy $\Pi \to M_\phi \Pi$. In an anti-clockwise loop around each singular point, the monodromy matrices are give by

$$M_0 = \begin{pmatrix} 1 & -1 & 3\kappa & 6\kappa \\ 0 & 1 & -6\kappa & -12\kappa \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \qquad M_{\frac{1}{25}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{10}{\kappa} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$M_{\frac{1}{9}} = \begin{pmatrix} -9 & -2 & 2\kappa & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{50}{\kappa} & -\frac{10}{\kappa} & 11 & 0 \\ -\frac{10}{\kappa} & -\frac{2}{\kappa} & 2 & 1 \end{pmatrix} \qquad M_1 = \begin{pmatrix} -39 & -16 & 16\kappa & -24\kappa \\ 60 & 25 & -24\kappa & 36\kappa \\ -\frac{100}{\kappa} & -\frac{40}{\kappa} & 41 & -60 \\ -\frac{40}{\kappa} & -\frac{16}{\kappa} & 16 & -23 \end{pmatrix} \tag{2.57}$$

$$M_\infty = \begin{pmatrix} 31 & 17 & -19\kappa & 42\kappa \\ -60 & -35 & 42\kappa & -96\kappa \\ \frac{60}{\kappa} & \frac{30}{\kappa} & -29 & 60 \\ \frac{30}{\kappa} & \frac{16}{\kappa} & -17 & 37 \end{pmatrix}$$

The monodromy matrix M_0 is readily calculated by hand and the three monodromies corresponding to the conifold points are calculated by numerical integration of the Picard Fuchs equation along loops that encircle the conifold points. This technique can be applied also to the calculation of the monodromy matrix M_∞, but it is easier to note that a contour, as in Figure 2.2, that winds once about each of the singular points can be deformed to a point and this allows us to relate M_∞ to the other matrices.

$$M_\infty = \left(M_0 M_{\frac{1}{25}} M_{\frac{1}{9}} M_1\right)^{-1} .$$

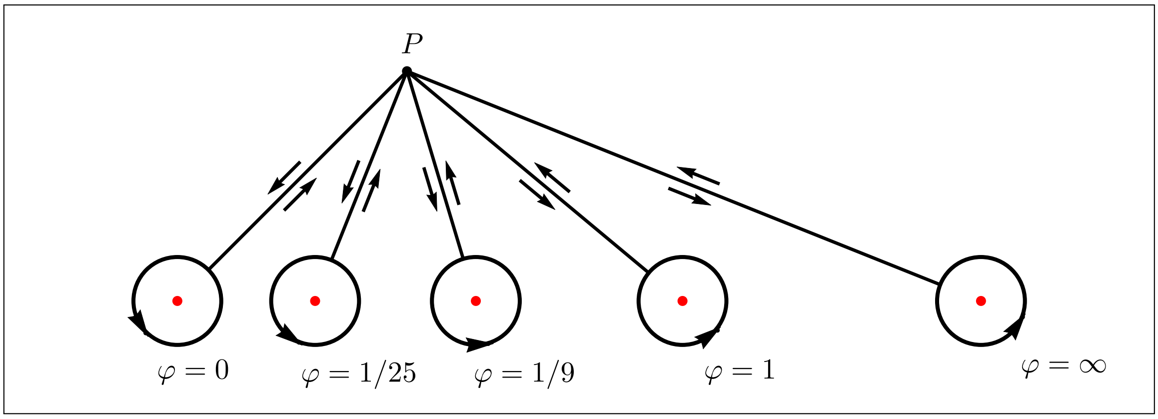

Figure 2.2: *A sketch of a contour, that can be deformed to a point, which shows that the product of all the monodromy matrices, taken in order, is the identity. In the figure, P is a basepoint for the monodromies.*

We refer to points with indices $(0,1,1,2)$ as conifold points because they are typically points at the boundary of moduli space where some S^3/G shrinks to zero volume for some finite groups G i.e. the Calabi-Yau manifold develops a *conical singularity* at such points [28]. By Picard-Lefchets theory, the monodromy around such points is given by

$$M_{\varphi_*} \;=\; \mathbb{1} - c_{\varphi_*}\, w(\Sigma w)^T \;, \tag{2.58}$$

for some integer c_{cnf} and integral vector w [18, 27]. As expected, we find that the monodromy matrices in (2.57) can be decomposed as in Table 2.1.

Monodromy	c_{φ_*}	w^T
$M_{\frac{1}{25}}$	$\frac{10}{\kappa}$	$(0,0,-1,0)$
$M_{\frac{1}{9}}$	$\frac{2}{\kappa}$	$(\kappa,0,5,1)$
M_1	$\frac{4}{\kappa}$	$(2\kappa,-3\kappa,5,2)$

Table 2.1: *The coefficient c_{cnf} and vanishing cycles for the three conifold points of AESZ 34.*

After a symplectic change of basis where the vanishing cycle is now given by $(0,0,-1,0)^T$, the monodromy matrix at a conifold point takes the form

$$M_{\varphi_*} \;=\; \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -c_{\varphi_*} & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \;. \tag{2.59}$$

We will make use of this fact in Chapter 8.
When a single S^3/G shrinks to zero volume, it is believed that $c_{\varphi_*} = |G|$ [27]. More generally, it can happen that the Calabi-Yau manifold develops multiple conical singularities. As is discussed in Appendix A, the conifold point at $\varphi = 1$ is such a point where the Calabi-Yau manifold develops two singularities that are each fixed by a group of order $\frac{2}{\kappa}$. It seems that the effect of this is to double the coefficient $\frac{2}{\kappa}$. As we will see in Chapter 8, the coefficients c_{φ_*} determine the topological string free energy at genus 1 [27, 29].
The integral vectors w at each conifold point determine the homology class of the vanishing S^3/G. That is, there exists a cycle $\gamma \in H_3(X,\mathbb{Z})$ dual to $\Gamma \in H^3(X,\mathbb{Z})$ such that

$$\int_\gamma \Omega \;=\; \int_X \Gamma \wedge \Omega \;=\; w^T \Sigma \Pi \to\; 0 \qquad \text{as } \varphi \to \varphi_*. \tag{2.60}$$

where φ_* is the conifold point. In other words, w is simply a vector of the periods of Γ in the same homology basis $\{A^a, B_b\}$ as the periods Π. As we will see in the following section, a D3 brane wrapping the vanishing cycle will becomes massless at conifold points and, from a string theory point of view, is the origin of the singularity at a conifold point [30].

2.2 The Attractor Mechanism

2.2.1 Review

One may construct four dimensional $\mathcal{N}=2$ dyonic black holes by compactifying IIB supergravity on a Calabi-Yau threefold X with D3 branes wrapping a 3-cycle of X. The black hole will be charged under a $U(1)^{b^3/2}$ gauge group where b^3 is the third Betti number of X and the electric and magnetic charges of the black hole are determined by the precise 3-cycle $\gamma \in H_3(X,\mathbb{Z})$ wrapped by the D3-branes. A review can be found in [11] and we explain the basic features here.

We should point out that, although much of the attractor mechanism is formulated in terms of (co)homology classes, strictly speaking, a D3 brane wraps a *special Lagrangian submanifold* of X and it is not true that any homology class will contain such a submanifold. We will not be too careful about this for most of this thesis and will return to this point in Section 9.

Let φ be the complex structure parameter of a one paramater Calabi-Yau manifold X.[7] Infinitely far from the horizon of the black hole, space-time is flat and the value of φ is unconstrained. However, as one moves towards the horizon of the black hole, φ must evolve in a manner dictated by the *attractor mechanism* first discovered by Ferrara, Kallosh and Strominger [9]. Moreover, the value of φ at the horizon of the black hole is an *attractor point* that (for small enough perturbations) is independent of the value of φ at infinity and is determined only by a choice of $\gamma \in H_3(X,\mathbb{Z})$.

An intuitive justification for this phenomenon is that the area of the horizon of the black hole is sensitive to the value of φ at the horizon. Moreover, the area of the horizon of the black hole is known to be a measure of the entropy in the limit of large mass [8] and cannot depend on continuously varying parameters like φ.

The four dimensional black hole is assumed to be spherically symmetric with a metric of the form

$$\mathrm{d}s^2 \; = -\mathrm{e}^{2U(r)}\mathrm{d}t^2 + \mathrm{e}^{-2U(r)}\mathrm{d}\vec{x}^{\,2},$$

where r is a radial coordinate that is taken to vanish at the horizon. In the supergravity approximation, the preservation of supersymmetry requires that the complex structure of X varies with the radius in a manner governed by differential equations, which are written

[7]The majority of this thesis will be concerned with one parameter Calabi-Yau manifolds where $b^3 = 4$. However, all of the statements about the attractor mechanism made in this section generalise in obvious ways to Calabi-Yau manifolds with $b^3 > 4$.

most simply in terms of a new variable $\rho = \frac{1}{r}$,

$$\begin{aligned} \frac{\mathrm{d}U(\rho)}{\mathrm{d}\rho} &= -\ \mathrm{e}^{U(\rho)}|Z_\gamma(\varphi)|, \\ \frac{\mathrm{d}\varphi(\rho)}{\mathrm{d}\rho} &= -2\mathrm{e}^{U(\rho)} g^{\varphi\bar{\varphi}}\,\partial_{\bar{\varphi}}|Z_\gamma(\varphi)|\ . \end{aligned} \tag{2.61}$$

We use the initial condition $U = 0$ when $\rho = 0$, appropriate to an asymptotically flat space-time. In the above formula, the quantity

$$Z_\gamma(\varphi) \;=\; \mathrm{e}^{K/2}\int_\gamma \Omega \tag{2.62}$$

denotes the *central charge* that appears in the $\mathcal{N} = 2$ SUSY algebra and K denotes the *Kähler potential* of the special geometry metric on moduli space. By a change of variables, these equations can be recast as a *gradient flow* of the function $|Z_\gamma(\varphi)|$ with respect to this metric. In a new variable $\mu = e^{-U}$, Equations (2.61) take the form

$$\begin{aligned} \frac{\mathrm{d}\varphi(\mu)}{\mathrm{d}\mu} &= -g^{\varphi\bar{\varphi}}\,\partial_{\bar{\varphi}}\log|Z_\gamma(\varphi)|^2, \\ \frac{\mathrm{d}\mu(\rho)}{\mathrm{d}\rho} &= \ \ |Z_\gamma(\varphi)| \end{aligned} \tag{2.63}$$

If we pick a symplectic basis $\{A^a, B_b\}$ of the torsion free part of $H_3(X,\mathbb{Z})$, we can write the cycle γ as

$$\gamma \;=\; q_a A^a - p^a B_a \;\in\; H_3(X,\mathbb{Z}) \tag{2.64}$$

and the black hole will have electric and magnetic charges given by the charge vector

$$Q \;=\; \begin{pmatrix} q_a \\ p^b \end{pmatrix}\ . \tag{2.65}$$

The basis $\{A^a, B_b\}$ is only defined up to an $Sp(2h^{2,1}+2,\mathbb{Z})$ transformation which, from a four dimensional perspective, is just an electromagnetic duality transformation of Q. Unless otherwise stated, we will work in an integral symplectic basis defined by Equation (2.49) near a large complex structure point. This fixes a choice of duality frame.

For the basis $\{\alpha_a, \beta^b\}$ of the torsion free part of $H^3(X,\mathbb{Z})$, dual to the symplectic basis $\{A^a, B_b\}$, we have

$$\int_{A^b}\alpha_a \;=-\int_{B_a}\beta^b \;=\; \int_{X_\varphi}\alpha_a\wedge\beta^b \;=\; \delta_a{}^b, \tag{2.66}$$

so that the dual in cohomology of the cycle γ is given by

$$\Gamma \;=\; p^a\alpha_a - q_a\beta^a \tag{2.67}$$

and the central charge can be written as

$$Z_\gamma(\varphi) \;=\; \mathrm{e}^{K/2}\int_X \Gamma\wedge\Omega \;=\; \frac{Q^T\Sigma\Pi}{(-i\Pi^\dagger\Sigma\Pi)^{\frac{1}{2}}}, \tag{2.68}$$

where Π is the vector of periods in an integral symplectic basis and Σ the matrix of the symplectic form on $H^3(X,\mathbb{Z})$. Once Π is known as a function of φ, we can visualise the flow lines for a given Q by plotting the vector field $\left(\mathrm{Re}\,\frac{\mathrm{d}\varphi}{\mathrm{d}\mu}, \mathrm{Im}\,\frac{\mathrm{d}\varphi}{\mathrm{d}\mu}\right)$ as in Figure 2.3.[8]

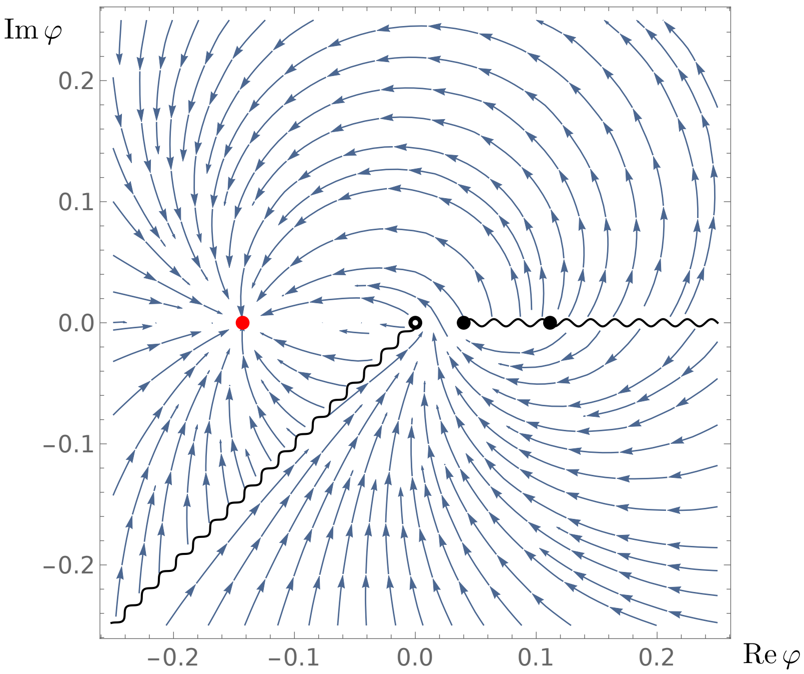

Figure 2.3: *Attractor flow for AESZ 34 with a choice of charge vector $Q = (4, -15, -5, 0)$. The red dot represents the attractor point $\varphi = -1/7$, the hollow black dot is the large complex structure point $\varphi = 0$ and the solid black dots represent the two nearest conifolds at $\varphi = 1/25$ and $\varphi = 1/9$. The flow lines are discontinuous across branch cuts which illustrates the fact that the flow takes place on a Riemann surface that is a multi-sheeted cover of the φ-plane.*

It follows from the gradient nature of Equation (2.61) that, for a given $\gamma \in H_3(X,\mathbb{Z})$, the "end point" $\varphi_* = \varphi_*(\gamma)$ is independent of the starting point φ_∞, at least under small variations of φ_∞ and thus will only depend on the charges Q. This is the origin of the name attractor point. Note however, that due to the multi-valuedness caused by the monodromy around the singular points, the flow really takes place on a Riemann surface covering the φ-plane. This is illustrated in Figure 2.3.

[8] $\Pi(\varphi)$ is computed by solving the Picard-Fuchs equation around a large complex structure point. The resulting expressions have finite radius of convergence and in, in order to generate plots like Figure 2.3, we must analytically continue by numerically solving the Picard-Fuchs equation along a line to points outside this radius of convergence.

It follows from (2.61) that the black hole metric near the horizon is asymptotic to that corresponding to $AdS_2 \times S^2$ and the area of the horizon is given by

$$A \;=\; 4\pi |Z_\gamma(\varphi_*)|^2 \tag{2.69}$$

and this determines the entropy of the black hole in the limit of large charges.
The attractor points have a number of special properties. As already mentioned, attractor points are critical points of the absolute value of the central charge function $|Z_\gamma(\varphi)|$, as can be seen from (2.61).
Since $g^{\varphi\bar{\varphi}}$ and e^U are non-vanishing in the interior of the moduli space, we see from Equation (2.61) that the flow stops when $\partial_{\bar{\varphi}}|Z_\gamma(\varphi)|^2 = 0$. A straightforward computation yields

$$\begin{aligned} \partial_{\bar{\varphi}}|Z_\gamma(\varphi)|^2 = \partial_{\bar{\varphi}}\Big\{ & e^K \Big(\int_X \Gamma\wedge\Omega\Big)\Big(\int_X \Gamma\wedge\overline{\Omega}\Big)\Big\} \\ = \quad & e^K \Big(\int_X \Gamma\wedge\Omega\Big)\Big(\int_X \Gamma\wedge D_{\bar{\varphi}}\overline{\Omega}\Big) \end{aligned} \tag{2.70}$$

where $D_{\bar{\varphi}}\overline{\Omega} = \partial_{\bar{\varphi}}\overline{\Omega} + \partial_{\bar{\varphi}}K\,\overline{\Omega}$. We see that φ_* is an attractor point if either of the above integrals vanishes.
By definition, $\int_X \Gamma\wedge\Omega = 0$ implies that $Z_\gamma(\varphi_*) = 0$. The conifold points descibed in the previous section (Equation (2.60)) are such attractor points. Moreover, we may compute the vanishing cycles explicitly from the monodromy matrices. We will see in Chapter 8 that the fact that D3-branes wrapping the vanishing cycles become massless at this point will lead to singularities in the topological string free energies [30].
If, on the other hand, $Z_\gamma(\varphi_*) \neq 0$, φ_* is an attractor point if and only if

$$\Gamma \;\in\; H^{3,0}\oplus H^{0,3} \;\text{ or equivalently }\; \Gamma^{2,1} \;=\; \Gamma^{1,2} \;=\; 0\;. \tag{2.71}$$

This follows from the fact that, as a cohomology class, $D_\varphi\Omega \in H^{2,1}$ and $\Gamma \in H^3(X,\mathbb{Z})$ is invariant under complex conjugation.
As illustrated in Figure 2.3, we may find the change vectors Γ (or rather, their periods Q) that lead to a given attractor point or the attractor point associated to a given Γ by computing the periods.

2.2.2 Rank One vs. Rank Two Attractors

The condition that (2.71) imposes on φ can be expressed more geometrically in the following way. The space $V(\varphi) = H^{3,0}\oplus H^{0,3}$ is a plane, generated by Ω and $\overline{\Omega}$, in the space $H^3(X,\mathbb{Z})\otimes\mathbb{C} = H^3(X,\mathbb{C})$. The intersection with the real four dimensional space $H^3(X,\mathbb{Z})\otimes\mathbb{R} = H^3(X,\mathbb{R})$ is the 2-plane $V_{\mathbb{R}}(\varphi)$ spanned, over $\mathbb{R}$, by $\operatorname{Re}\Omega$ and $\operatorname{Im}\Omega$. Inside the vector space $H^3(X,\mathbb{R})$, we have the lattice of dual charge vectors $H^3(X,\mathbb{Z})$. This lattice is fixed, but the plane $V_{\mathbb{R}}(\varphi)$ moves with respect to this lattice as φ varies. There are three possibilities:

0. The plane $V_{\mathbb{R}}(\varphi)$ intersects $H^3(X,\mathbb{Z})$ only in 0. This is the generic case and φ is not an attractor point.

1. The intersection $V_{\mathbb{R}}(\varphi) \cap H^3(X,\mathbb{Z})$ is a *lattice line*, i.e. a copy of $\mathbb{Z}$. The point φ is attractor point for any non-zero $\Gamma \in V_{\mathbb{R}}(\varphi) \cap H^3(X,\mathbb{Z})$. In this case φ is an *attractor point of rank one.*

2. The intersection $\Lambda := V_{\mathbb{R}}(\varphi) \cap H^3(X,\mathbb{Z})$ is a *lattice plane*, i.e. a copy of $\mathbb{Z}^2$. In this case one can find two independent charges Γ_1 and Γ_2 in Λ, which have symplectic product $\langle \Gamma_1, \Gamma_2 \rangle \neq 0$. In this case φ is an *attractor point of rank two.*

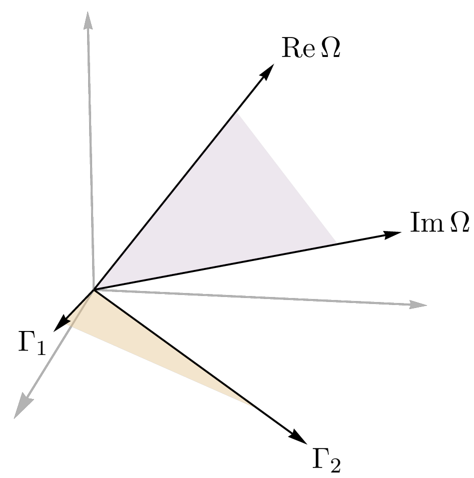

Figure 2.4: *A sketch of the (four dimensional) space $H^3(X,\mathbb{R})$ for generic φ, showing the two planes generated by Re$\,\Omega$ and Im$\,\Omega$ and by charge vectors Γ_1 and Γ_2. As φ varies, the plane generated by Re$\,\Omega$ and Im$\,\Omega$ moves and, when $\varphi = \varphi_*$ is an attractor point of rank two, the two planes coincide.*

As we are dealing with the geometry of 2-planes in a four dimensional vector space, it is natural to formulate equation (2.71) in terms of the Grassmanian $\mathrm{Gr}(2,\mathbb{C}^4)$, which by the Plücker embedding

$$\mathrm{Gr}(2,\mathbb{C}^4) \hookrightarrow \mathbb{CP}^5 \tag{2.72}$$

can be identified with the Plücker quadric. The natural map

$$\varphi \mapsto V(\varphi) = H^{3,0} \oplus H^{0,3} \subset H^3(X,\mathbb{C}) \tag{2.73}$$

from the complex structure moduli space to the Grassmanian can be composed with the Plücker embedding. Since $H^{3,0} \oplus H^{0,3}$ is spanned by the cohomology classes of $\operatorname{Re}\Omega$ and $\operatorname{Im}\Omega$ the resulting map can be identified with the map

$$\varphi \mapsto \mathcal{P} \;=\; \big(\operatorname{Re}\Pi, \operatorname{Im}\Pi\big) \mapsto [\pi_{12},\, \pi_{13},\, \pi_{14},\, \pi_{23},\, \pi_{24},\, \pi_{34}] \in \mathbb{RP}^5 \subset \mathbb{CP}^5 \tag{2.74}$$

where π_{ij} is the minor formed by the i^{th} and j^{th} rows of $\mathcal{P}$. The rows of $\mathcal{P}$ form a basis of $H^{3,0} \oplus H^{0,3}$ and any other basis is related to this one by $\mathcal{P} \mapsto \mathcal{P}g$ for some $g \in \mathrm{GL}(2, \mathbb{C}^4)$ which simply multiplies each π_{ij} by $\det(g)$, so the image in $\mathbb{P}^5(\mathbb{C})$ is left unchanged. One also sees that the map does not depend on the normalization of Ω and that the Grassmannian is given by the Plücker quadric

$$\pi_{12}\,\pi_{34} - \pi_{13}\,\pi_{24} + \pi_{14}\,\pi_{23} \;=\; 0 \tag{2.75}$$

which the moduli space maps into.

The equation (2.71) characterising attractor points is more commonly written as

$$Q \;= -2\mathrm{e}^{K/2}\,\operatorname{Im}\Big(\,\overline{Z}_\gamma(\varphi_*)\;\Pi(\varphi_*)\Big). \tag{2.76}$$

Given $\gamma \in H_3(X, \mathbb{Z})$, one can solve the Picard-Fuchs equation and the attractor equations numerically and find the attractor point $\varphi_*(\gamma)$ that makes γ the $(2,1)$ part and $(1,2)$ part of Γ vanish to high precision. Conversely, at an arbitrary point φ_*, we can solve Eqs. (2.76) for the charges Q for which φ_* would be an attractor point. By a simple computation we find that the charges are given by

$$Q \;=\; \Big(\frac{\pi_{14}}{\pi_{34}}\,p^0 + \frac{\pi_{31}}{\pi_{34}}\,p^1,\; \frac{\pi_{24}}{\pi_{34}}\,p^0 + \frac{\pi_{32}}{\pi_{34}}\,p^1,\; p^0,\; p^1\Big)^T\,. \tag{2.77}$$

However, this charge vector Q will, generically, not be integral.

At a rank one attractor, the first two components of Q are integral for some choice of p_0 and p_1 unique up to an overall scale. In other words, given some Q, the first two components of Equation (2.77) can be treated as two real equations for two real unknowns $\operatorname{Re}\varphi$ and $\operatorname{Im}\varphi$ and this will generically have a solution. Indeed, a numerical computation of the attractor point for a given charge Q will almost always have a solution. Conversely, attractor points are expected to be dense in moduli space [11].

At a rank two attractor, we require that each of the four ratios in Equation (2.77) are rational. In other words, the rank two attractors are precisely the $\mathbb{Q}$-rational points on the moduli space in $\mathrm{Gr}(2, \mathbb{R}^4)$.

Note that the problem of solving Equation (2.77) for rank two attractors is over constrained. Suppose we want to find a value of φ so that $V_{\mathbb{R}}(\varphi)$ is equal to a given lattice. We must make the four ratios of π_{ij} rational but can only vary $\mathrm{Re}\,\varphi$ and $\mathrm{Im}\,\varphi$. This explains the scarcity of rank two attractors on one parameter families.

It is mentioned in [31] that Denef, motivated by the conjectures made by Moore in [11], searched for attractor points of rank two by numerically computing attractor points on the moduli space of the mirror quintic for about 50,000 choices of Q. Unfortunately, this search yielded no convincing candidates.

We will show in Chapter 4 how one may use the arithmetic structure of Calabi-Yau manifolds to search for attractor points of rank two. Moreover, we will show why the existence of rank two attractors on the moduli space of the mirror quintic is unlikely.

2.2.3 On the Conjectures of Moore

Before moving onto rank 2 attractors, we make here a few comments on the attractor conjectures of Moore [31]. We state them here for a one parameter family but they generalise in the obvious way. For a choice of charge vector $\Gamma \in H^3(X, \mathbb{Z})$ that leads to an attractor point φ_*, the attractor conjectures are (loosely stated)

1. The mirror map
$$t(\varphi) \;=\; \frac{1}{2\pi i}\frac{\varpi_1(\varphi)}{\varpi_0(\varphi)} \tag{2.78}$$
when evaluated at φ_* is algebraic i.e. $t(\varphi_*) \in K(\Gamma)$ where $K(\Gamma)$ is some number field determined by the choice of charge vector.

2. The attractor variety X_{φ_*} is algebraic and defined over a number field $\widehat{K}(\Gamma)$.

3. $\widehat{K}(\Gamma)K(\Gamma)$ is a Galois extension of $K(\Gamma)$.

These conjectures were motivated by properties of attractor varieties in $\mathcal{N}=4$ and $\mathcal{N}=8$ theories and were stated in a strong and weak forms for attractor points of rank one and two respectively.

On the first conjecture, we find in Chapter 5 that an attractor point $\varphi_* = -\frac{1}{7}$ on the family of Calabi-Yau manifolds described by AESZ 34 (see Appendix A) that the mirror map is given by
$$t\left(-\tfrac{1}{7}\right) \;=\; \frac{1}{2} + \frac{5i}{28}\frac{L_4(1)\pi}{L_4(2)} \tag{2.79}$$
where L_4 is a weight 4 L-function (defined in the following chapter). While we are not aware of a proof that $L_4(1)$ and $L_4(2)$ are transcendental, they are generally believed to be so and a numerical search for an algebraic approximant to $t\left(-\frac{1}{7}\right)$ yielded no convincing candidates to 2000 decimal places. Similar results are found for all of the exact attractor varieties described in this thesis and probably provide counter examples to the first conjecture.

On the second conjecture, it was pointed out in [31] and later in [16] that this conjecture must be true for attractor points of rank two if the Hodge conjecture is to be believed and that is confirmed in a number of examples in this thesis. With regard to attractor points of rank one, such points can be found numerically for randomly chosen charge vectors. For example, we solve the Picard-Fuchs equation AESZ 34 around $\varphi = 0$ and numerically search for attractor points by minimising $\partial_\varphi |Z_\gamma(\varphi)|^2$.[9] We express the charge vectors Q in the basis defined by Equation (2.49) and find that

- $Q = (-22\kappa, -38\kappa, 18, 35)$ leads to an attractor point at

$$\begin{aligned} \varphi_* = & -0.00125477443304793975886132599699028021404372156353 27\ldots \\ & + i\, 0.0080280433158825837970507069121400674812075037245705\ldots \end{aligned} \tag{2.80}$$

- $Q = (32\kappa, 40\kappa, -32, -41)$ leads to an attractor point at

$$\begin{aligned} \varphi_* = & -0.0064142992866968023645930035500767924843732805215107\ldots \\ & + i\, 0.0029341955082777952708910041626058711332011558385338\ldots \end{aligned} \tag{2.81}$$

- $Q = (91\kappa, -70\kappa, -53, -75)$ leads to an attractor point at

$$\begin{aligned} \varphi_* = & -0.00066180684361111243810944636346899790106554128774 49\ldots \\ & + i\, 0.0044345204990905371983305780465917210921637786165263\ldots \end{aligned} \tag{2.82}$$

where $\kappa = 1$ or $\kappa = 2$ for the $\mathbb{Z}/10\mathbb{Z}$ and $\mathbb{Z}/5\mathbb{Z}$ quotients respectively (see Appendix A for an algebraic description of the associated variety).

We believe the above examples are attractor points of rank one because we compute the "other charge vector" leading to φ_* by using (2.77) and find no convincing rational approximant to 200 decimal places. A numerical search for algebraic approximants to φ_* in the above example also yielded no convincing candidates to 200 decimal places which leads us to conjecture that the above examples are rank one attractors where the underlying variety is not algebraic.

Finally, the third conjecture relies on the algebraicity of the mirror map at attractor points which we do not believe to be true in general.

[9] The minimisation may be done via the two dimensional Newton-Rhapson method. The tricky part here is that the attractor points are generically outside the radius of convergence of the periods around $\varphi = 0$ where we may easily find a choice of integral symplectic basis. We must therefore numerically integrate to a new point in every step of the Newton-Rhapson procedure. The numerical attractor points listed in this section were chosen because they lie inside the radius of convergence.

Chapter 3

Arithmetic of Calabi-Yau Varieties

3.1 Zeta-Function and Weil Conjectures

Any projective variety X defined over $\mathbb{Q}$ can be defined by polynomial equations with integral coefficients. For any prime p we may then ask how many solutions these equations have over $\mathbb{F}_{p^r}$, the field with p^r elements. Let N_{p^r} be this number. These numbers are collected into the generating function

$$\zeta(X/\mathbb{F}_p, T) \;=\; \exp\left(\sum_{r=1}^{\infty} N_{p^r} \frac{T^r}{r}\right) \;,$$

known as the *Artin-Weil Zeta Function.* While this doesn't seem like a very natural object at first sight, it has a number of remarkable properties and its study has driven a great deal of research in arithmetic geometry which culminated in the final proof of the Weil conjectures by Deligne [32, 33]. The Weil conjectures state that, if X is a non-singular projective variety of complex dimension n,

1. *Rationality*
   $\zeta(X/\mathbb{F}_p, T)$ is a rational function of T and takes the form
   $$\zeta(X/\mathbb{F}_p, T) = \frac{P_1(X/\mathbb{F}_p, T)P_3(X/\mathbb{F}_p, T)\dots P_{2n-1}(X/\mathbb{F}_p, T)}{P_0(X/\mathbb{F}_p, T)P_2(X/\mathbb{F}_p, T)\dots P_{2n}(X/\mathbb{F}_p, T)} \tag{3.1}$$
   where $P_i(X/\mathbb{F}_p, T) \in \mathbb{Z}[T]$ is a polynomial of degree b_i where i^{th} Betti number. Moreover, the polynomials associated to the zeroth and top cohomology are given by
   $$P_0(X/\mathbb{F}_p, T) = 1 - T \qquad \text{and} \qquad P_{2n}(X/\mathbb{F}_p, T) = 1 - pT \tag{3.2}$$

2. *Functional Equation/ Poincaré Duality*
   The zeta function satisfies the functional equation
   $$\zeta(X/\mathbb{F}_p, (p^n T)^{-1}) = \pm p^{\frac{n\chi X}{2}} T^{\chi X} \zeta(X/\mathbb{F}_p, T) \tag{3.3}$$
   where χX is the Euler characteristic of X.

3. *Local Riemann Hypothesis*

When the polynomial $P_i(X/\mathbb{F}_p, T)$ is factored over $\mathbb{C}$ as

$$P_i(X/\mathbb{F}_p, T) = \prod_{j=1}^{b_3}(1 - \alpha_{ij}T) \ , \tag{3.4}$$

the coefficients α_{ij} satisfy

$$|\alpha_{ij}| = p^{\frac{i}{2}} \ . \tag{3.5}$$

As indicated by the Weil conjectures, the polynomials $P_i(X/\mathbb{F}_p, T)$ have a cohomological origin and the zeta function contains a lot of topological information about the variety X over $\mathbb{C}$. We pause to explain this in rather greater detail and to recall the basic facts pertaining to the Frobenius map.

For c an integer, recall Fermat's Little Theorem that

$$c^p \ = \ c \mod p \ . \tag{3.6}$$

So if we think of c as a number in $\mathbb{F}_p$ we have $c^p = c$. If however c is in a higher field $\mathbb{F}_{p^r}$ then $c^p \neq c$, in general, since the analogous identity is $c^{p^r} = c$. Now take c_1 and c_2 to be numbers in $\mathbb{F}_{p^r}$, for some r, and note the identity

$$(c_1 + c_2)^p \ = \ c_1^p + c_2^p \ , \tag{3.7}$$

since all the intermediate terms in the binomial expansion are divisible by p.

Suppose now that a variety is defined by a polynomial

$$F(x) \ = \ \sum_m c_m x^m \tag{3.8}$$

where we use a multi-index notation and $x^m = x_1^{m^1} \dots x_n^{m^n}$. Let us further suppose that the coefficients c_m are in $\mathbb{F}_p$, while the coordinates x are in some higher field $\mathbb{F}_{p^r}$. Then we have

$$\begin{aligned} F(x) \ &= \ 0 \\ \Rightarrow \quad F(x)^p \ &= \ 0 \\ \Rightarrow \quad F(x^p) \ &= \ 0 \ . \end{aligned} \tag{3.9}$$

The map $x \to x^p$ is the *Frobenius map*, which we shall denote by Frob. It would be more correct to denote the map by Frob_p, but we shall drop the suffix p in the following. What we have seen is that Frob is an automorphism that every manifold defined over $\mathbb{Q}$ has. The fixed points of the map are of interest. These correspond to the points for which

$$x^p \ = \ x \tag{3.10}$$

and this relation picks out the the points that are defined in $\mathbb{F}_p \subset \mathbb{F}_{p^r}$. So another way to look at N_{p^1} is as the number of fixed points of the Frobenius map; more generally N_{p^k} counts the number of fixed points of Frob^k. It can also be shown that the Frobenius map generates the Galois group of the polynomial (3.8). If suitable cohomology groups are defined, then the action of Frob extends to cohomology and the fixed points can be counted by the Lefschetz fixed point theorem. It was Dwork [34] who proved that the ζ-function is a rational function which decomposes as in (3.1) by showing that the ζ-function is a superdeterminant, though Dwork did not use this term, which decomposes into factors corresponding to the different cohomology groups with

$$P_k(X/\mathbb{F}_p, T) \;=\; \det(1 - T\,\text{Frob}_k^{-1}) \in \mathbb{Z}[T] \;, \qquad \text{Frob}_k :\; H^k(X) \longrightarrow H^k(X) \;, \tag{3.11}$$

where H^k can be any Weil-cohomology, for example ℓ-adic cohomology, $(\ell \neq p)$. In particular, the degree of P_k is equal to the k-th Betti-number b^k of the complex variety defined by X. A textbook account is given in [35] and one in the style of the present work is given in [36], which also gives more detailed references to the original literature.
For the situation of Calabi-Yau threefolds with $h^{21} = 1$ considered here, the zeta function is further constrained and assumes the form

$$\zeta(X/\mathbb{F}_p, T) \;=\; \frac{P_3(X/\mathbb{F}_p, T)}{(1-T)(1-pT)^{h^{11}}(1-p^2T)^{h^{11}}(1-p^3T)} \tag{3.12}$$

The denominator in this expression gives the form of the product $P_0P_2P_4P_6$, while, in the numerator, the factors P_1 and P_5 are trivial, corresponding to the fact that $b^1 = b^5 = 0$, so we are left with P_3 and we henceforth dispense with the suffix. The polynomial P_3 has integer coefficients and is of degree four if the reduction mod p of X is smooth, and we will refer to it as the *Frobenius polynomial*. It is of the form

$$P_3(X/\mathbb{F}_p, T) \;=\; 1 + a_pT + b_ppT^2 + a_pp^3T^2 + p^6T^4 \;, \tag{3.13}$$

and so is determined by two integers a_p and b_p. Although we have suppressed this in the notation, the zeta function will, of course, depend on the parameter φ of the Calabi-Yau. An important point to keep in mind is that, although the Calabi-Yau variety is perfectly well defined for any $\varphi \in \mathbb{C}$ that avoids the singularities, the computation of the zeta function requires that φ be algebraic.

3.2 Computation of Zeta Function

As previously mentioned, the computation of the zeta function of a Calabi-Yau threefold with finite fundamental group (i.e. $b^1 = b^5 = 0$) boils down to the computation of a single polynomial

$$P_3(X/\mathbb{F}_p, T) \ = \ 1 + a_p T + b_p p T^2 + a_p p^3 T^2 + p^6 T^4 \ . \tag{3.14}$$

The coefficients a_p and b_p can, in principle, be determined by directly counting the number of points of X over $\mathbb{F}_{p^r}$, in fact it is sufficient to count points over $\mathbb{F}_p$ and $\mathbb{F}_{p^2}$. This however quickly becomes impractical as p is increased. Sometimes even for small p, it is onerous to count the $\mathbb{F}_{p^r}$-points of a manifold, for example if X is defined as a quotient by a group, since these 'points' are then group-orbits that are defined over $\mathbb{F}_{p^r}$, not the orbits of group-invariant points, and there are frequently orbits without any points, for example.

Fortunately, there are much better ways to compute $P_3(X/\mathbb{F}_p, T)$. It was discovered by Dwork and developed further by Lauder [37] that the ζ-function can be calculated from a p-adic computation of the periods, using the Picard-Fuchs equations. This goes under the name *deformation method.* A more detailed discussion of this fascinating process, pertaining to the ζ-function of one-parameter families of Calabi-Yau manifolds with a point of maximal unipotent monodromy which is taken as expansion point, may be found in [21].

The Frobenius polynomials for the Picard-Fuchs equation AESZ 34 discussed in previous sections has been computed in [21] using the deformation method. The Frobenius polynomials for the prime $p = 19$ are listed in Table 3.1 for $\varphi \in \mathbb{F}_{19}$.

At its core, the deformation method relies on the fact that the Frobenius map can be represented by a matrix constructured from periods that is then expanded p-adically.

The deformation method is especially powerful for one parameter families of Calabi-Yau manifolds where the Picard-Fuchs equation has a MUM (large complex structure) point.[1] The computations are especially quick in these cases and has led to the computatation of the zeta function for many primes and $\varphi \in \mathbb{F}_p$ [21, 39]. For an example with $h^{2,1} > 1$, see [40].

When the Picard-Fuchs equation of a one parameter family has a MUM point (large complex structure) it can be shown that Frob_3^{-1} can be represented by a matrix of the form

$$U(\varphi) = E^{-1}(\varphi^p) U(0) E(\varphi) \tag{3.15}$$

where $\varphi = 0$ at the MUM point and the components of E are given by

$$E_{jk} \ = \ \frac{1}{j!} \vartheta^k \, \varpi_j \big|_{\log(\varphi)=0} \tag{3.16}$$

where ϖ_j are the periods in Frobenius basis that were defined in Equation (2.43) and

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p^2 & 0 \\ p^3\gamma & 0 & 0 & 1 \end{pmatrix} \tag{3.17}$$

[1] Examples of Picard-Fuchs equations for families of Calabi-Yau manifolds without a MUM point can be found in [38].

$p = 19$			
φ	smooth/sing.	singularity	$P_3(X/\mathbb{F}_p, T)$
1	singular	1	$(1-pT)(1-20T+p^3T^2)$
2	smooth		$1+4pT+2pT^2+4p^4T^3+p^6T^4$
3	smooth		$1-8T+242pT^2-8p^3T^3+p^6T^4$
4	smooth		$(1+4pT+p^3T^2)(1-60T+p^3T^2)$
5	smooth		$(1+4pT+p^3T^2)(1-60T+p^3T^2)$
6	smooth		$1+8T-318pT^2+8p^3T^3+p^6T^4$
7	smooth		$1-44T-238pT^2-44p^3T^3+p^6T^4$
8	smooth		$(1-2pT+p^3T^2)(1-80T+p^3T^2)$
9	smooth		$(1+4pT+p^3T^2)(1-160T+p^3T^2)$
10	smooth		$1+12T+562pT^2+12p^3T^3+p^6T^4$
11	smooth		$(1+4pT+p^3T^2)(1-140T+p^3T^2)$
12	smooth		$1+12T+82pT^2+12p^3T^3+p^6T^4$
13	smooth		$1+178T+1082pT^2+178p^3T^3+p^6T^4$
14	smooth		$1+12T-158pT^2+12p^3T^3+p^6T^4$
15	smooth		$1+42T-2p^2T^2+42p^3T^3+p^6T^4$
16	singular	$\frac{1}{25}$	$(1-pT)(1+76T+p^3T^2)$
17	singular	$\frac{1}{9}$	$(1-pT)(1-20T+p^3T^2)$
18	smooth		$1-54T+322pT^2-54p^3T^3+p^6T^4$

Table 3.1: *The P_3-factors for $\varphi \in \mathbb{F}_{19}$. Note the factorisations into two quadrics for the five values $\varphi = 4, 5, 8, 9, 11$.*

for a constant γ that is believed to be given by $\chi_3(Y)\zeta_p(3)/y$ where Y is the mirror manifold of X, y is the triple intersection number of Y and $\zeta_p(3)$ is the p-adic zeta function given by

$$\zeta_p(3) = -\frac{1}{2}\left(\Gamma_p'''(0) - \Gamma_p'(0)^3\right) \tag{3.18}$$

where Γ_p is the p-adic gamma function [21, 39].

Note that, after differentiating we set all of the logarithms equal to zero so that the entries of the matrix E are power series in φ with rational entries.

The coefficients of the Frobenius polynomial in Equation (3.14) are then given by

$$a_p = -\mathrm{Tr}(U) \qquad \text{and} \qquad b_p = \frac{1}{2p}\left(\mathrm{Tr}(U)^2 - \mathrm{Tr}(U^2)\right)\,. \tag{3.19}$$

The computation of $U(\varphi)$ is made easier by the observation that

$$U(\varphi) = \frac{\mathcal{U}(\varphi)}{\Delta(\varphi)} + O(p^4) \tag{3.20}$$

where $\mathcal{U}(\varphi)$ is a matrix of polynomials and $\Delta(\varphi)$ is the *discriminant* given by the coefficient $S_4(\varphi)$ of ϑ^4 in the Picard-Fuchs equation. The computation of $U(\varphi)$ is then simply a matter of evaluating (3.15) at the *Teichmüller* lift of $\varphi \in \mathbb{F}_p$ given by

$$\text{Teich}(\varphi) = \lim_{n\to\infty} \varphi^{p^n} \ . \tag{3.21}$$

Note that the computation of $U(\varphi)$ to a given p-adic order requires the expansion of the periods to some finite order.

It has been observed that, in many cases, factors in $\Delta(\varphi)$ that correspond to conifold points[2] will cancel with the numerator in Equation (3.20) so that $U(\varphi)$ can be evaluated at conifold points. It is this miraculous cancellation that has made possible the efficient computation of the zeta function at conifold points [21, 39]. See, for example, $\varphi \in \{\frac{1}{25}, \frac{1}{9}, 1\}$ in Table 3.1. At conifold points, the Frobenius polynomial degenerates to a third order polynomial and takes the form

$$P_3(X/\mathbb{F}_p, T) = (1 - p\chi_p T)(1 - \alpha_p T + p^3 T^2) \tag{3.22}$$

where $\alpha_p \in \mathbb{Z}$ and $\chi_p = \pm 1$ is a character. Intuitively, this factorisation happens because a one-parameter Calabi-Yau manifold with a conical singularity can often be resolved into a *rigid* Calabi-Yau manifold. That is, one with $h^{2,1} = 0$. This process simply adds new Kähler classes (i.e. it increases b_2) and doesn't affect H^3. Thus, in many cases, we may identify the quadratic factor in (3.22) with the Frobenius polynomial of a rigid Calabi-Yau.

Unfortunately, terms in the denominator of Equation (3.20) will not cancel for other types of singularities of the Picard-Fuchs equation e.g. at K-points that have two repeated indices e.g. $(0, 0, 1, 1)$ or $(1, 1, 2, 2)$. This makes the deformation method difficult to apply at such points.

Another type of singularity that frequently appears and where the deformation method fails is an *apparent singularity* or a *psuedo-singularity*. These are singularities of the Picard-Fuchs equation where the underlying Calabi-Yau manifold is smooth. At first, one might expect that singularities of the Picard-Fuchs equation correspond to singularities of the underlying Calabi-Yau manifold and vice-versa. However this is not true and counter examples can be found, for example, in [23].

Apparent singularities typically have the indices $(0, 1, 3, 4)$ which imply that the periods, the Kähler potential on the moduli space and the resulting metric are all regular at such points. Moreover, the lack of a repeated index means that the periods do not undergo monodromy around such points.

We will see later in this thesis that, although we cannot use the deformation method, we will nevertheless be able to easily compute the zeta function at an apparent singularity in some

[2] A point where the Picard-Fuchs equation has indices $(0, 1, 1, 2)$.

examples. We will use results from Chapter 5 and the fact that some apparent singularities are *modular*.
For now, we recall essential facts on modularity and modular forms.

3.3 Modularity and Factorisations

3.3.1 Modular Forms

3.3.1.1 Definition

Let $\mathbb{H}$ be the upper half plane

$$\mathbb{H} = \{ \tau \in \mathbb{C} \,|\, \mathrm{Im}\, \tau > 0 \} . \tag{3.23}$$

The group $SL(2, \mathbb{R})$ acts on the upper half plane via Mobiüs transformation. That is, if

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{3.24}$$

then, for $\tau \in \mathbb{H}$,

$$\tau \to \gamma\tau \;=\; \frac{a\tau + b}{c\tau + d} . \tag{3.25}$$

This preserves the upper half plane since

$$\mathrm{Im}(\gamma\tau) = \frac{\mathrm{Im}(\tau)}{|c\tau + d|^2} \tag{3.26}$$

and defines a group action because $(\gamma_1\gamma_2)\tau = \gamma_1(\gamma_2\tau)$ for all $\gamma_1, \gamma_2 \in SL(2, \mathbb{R})$.
We say that a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ is a *weakly holomorphic modular form of weight $k \in \mathbb{Z}$ for a subgroup* $\Gamma \subset SL(2, \mathbb{Z})$ if

$$f(\gamma\tau) = (c\tau + d)^k f(\tau) \qquad \forall \gamma \in \Gamma \subset SL(2, \mathbb{Z}) . \tag{3.27}$$

To simplify notation, we will make use of the *slash operator* $|$ defined as

$$(f|_k\gamma)(\tau) = (c\tau + d)^{-k} f(\gamma\tau) \tag{3.28}$$

so that Equation (3.27) can be restated simply as the condition

$$f|_k\gamma = f \tag{3.29}$$

Most of the modular forms that we will meet in this thesis will be for the finite index subgroup of $SL(2, \mathbb{Z})$

$$\Gamma_0(N) \;=\; \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) \mid \gamma \equiv \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \bmod(N) \right\} \tag{3.30}$$

where $N \in \mathbb{N}$ is referred to as the level.

For all $N \in \mathbb{N}$, $\Gamma_0(N)$ contains the element

$$T \;=\; \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tag{3.31}$$

which, when combined with the transformation law in Equation (3.27), implies that a weakly holomorphic modular form is periodic in τ with period 1. In other words, it has a Fourier expansion

$$f(\tau) \;=\; \sum_{n=m}^{\infty} a_n \, q^n \tag{3.32}$$

where $q = \mathrm{e}^{2\pi i \tau}$ and, in fact, explicit examples of modular forms are very often studied via their Fourier expansions.

The action of $SL(2,\mathbb{R})$ can be extended to $\mathbb{Q} \subset \mathbb{C}$ so that we may define the extended upper half plane

$$\overline{\mathbb{H}} \;=\; \mathbb{H} \cup \mathbb{Q} \cup \{\infty\} \tag{3.33}$$

which admits a quotient $\Gamma_0(N)\backslash\overline{\mathbb{H}}$ known as the *modular curve of level N*. The additional points are known as *cusps*.

Note that a weakly holomorphic modular form f defines $f(\tau)\mathrm{d}\tau$ which transforms like a holomorphic differential 1-form on $\Gamma_0(N)\backslash\overline{\mathbb{H}}$ with the exception of cusps and the point at infinity. Holomorphicity of f at infinity is straightforward to check. It translates to the condition that Fourier expansion in Equation (3.32) starts from $m = 0$. Moreover, the point at infinity can be sent to any point $\alpha \in \mathbb{Q}$ via an $\gamma \in SL(2,\mathbb{Z})$ transformation i.e. $\gamma\,(i\infty) = \alpha$. It then follows that f is holomorphic at α if $f|_k\gamma$ is holomomorphic at $i\infty$ and this condition is independent of the choice of γ. Finally, we say that a weakly holomorphic modular form is a *holomorphic modular form*, an *elliptic modular form* or simply a *modular form* if it is holomorphic at all of the cusps.

It is a fundamental fact that the space of holomorphic modular forms of weight k (denoted by $M_k(\Gamma_0(N))$) is a finite dimensional vector space. This means any modular form is uniquely determined by its level, weight and a finite number of Fourier coefficients a_n.

A particularly important subspace of $M_k(\Gamma_0(N))$ is the subspace of *cusp forms* $S_k(\Gamma_0(N))$ where we say that a modular form is a cusp form if it vanishes at all of the cusps. At $i\infty$, this simply translates to the condition that $m > 0$ in Equation (3.32) and likewise for the other cusps after they are transformed to $i\infty$.

We will also meet a modular form for the finite index subgroup $\Gamma_1(N) \subset SL(2,\mathbb{Z})$

$$\Gamma_1(N) = \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \mid \gamma \equiv \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \bmod(N) \right\} . \tag{3.34}$$

Much of the following disccusion for $\Gamma_0(N)$ also applies to $\Gamma_1(N)$ which can be found in standard books on modular forms e.g. [41, 42]. Note that $\Gamma_1(N) \subset \Gamma_0(N)$ implies that

$M(\Gamma_0(N)) \subset M(\Gamma_1(N))$ and a modular form of $\Gamma_1(N)$ is not necessarily a modular form of $\Gamma_0(N)$. In fact, a modular form of $\Gamma_1(N)$ is a modular form of $\Gamma_0(N)$ with a Dirichlet character. Recall that $\chi : \mathbb{Z} \to \mathbb{C}$ is a *Dirichlet character of modulus* N if the following are true

1. $\chi(n) = \chi(n+N) \forall n \in \mathbb{N}$,
2. $\gcd(n,N) > 1 \Longrightarrow \chi(n) = 0$,
3. $\gcd(n,N) = 1 \Longrightarrow \chi(n) \neq 0$,
4. $\chi(mn) = \chi(m)\chi(n) \forall m,n \in \mathbb{Z}$.

We say that f is a *modular form with character* χ if we modify the definition of a modular form by requiring that it transforms as

$$f(\gamma\tau) = \chi(d)(c\tau+d)^k f(\tau) \qquad \forall \gamma \in \Gamma \subset SL(2,\mathbb{Z}) \tag{3.35}$$

instead of the transformation rule in Equation (3.35)
If $M_k(\Gamma_0(N), \chi_i)$ is the space of modular forms, it turns out that

$$M_k(\Gamma_1(N)) = \bigoplus_{\chi_i} M_k(\Gamma_0(N), \chi_i) \tag{3.36}$$

where the sum is over Dirichlet characters χ_i of modulus N .

3.3.1.2 Hecke Eigenforms

In order to make contact with the modular forms that we associate to varieties, we have to choose a special basis of $S_k(\Gamma_0(N))$ consisting of modular forms known as *Hecke eigenforms.* We can then use these basis vectors to define *L-functions* that admit an Euler product. We start by defining the Hecke operators for $n \in \mathbb{N}$ where $\gcd(n,N) = 1$

$$T_n : M_k(\Gamma_0(N)) \to M_k(\Gamma_0(N)) \tag{3.37}$$

which act as[3]

$$f|_k T_N = n^{k-1} \sum_{M \in \Gamma_0(N) \backslash \mathcal{M}_{n,N}} f|_k M \tag{3.38}$$

where

$$\mathcal{M}_{n,N} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2,\mathbb{Z}) |\, \det(\gamma) = n\,,\, c = 0 \bmod N \right\}. \tag{3.39}$$

By using the set of representatives

$$\left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in GL(2,\mathbb{Z}) \,|\, ad = n\,, 0 \le b < d \right\}. \tag{3.40}$$

[3]For general $n \in \mathbb{N}$ see [42].

of $\mathcal{M}_{n,N}$, it is straightforward to check that

$$f|_k T_n(\tau) = \sum_{i=mn}^{\infty} \sum_{r|\gcd(n,i)} r^{k-1} a_{in/r^2} q^i \tag{3.41}$$

Moreover, any two Hecke operators will commute which means that we can find a basis of $M_k(\Gamma_0(N))$ consisting of common eigenvectors for all of the Hecke operators. We will say that a cusp form $f \in S_k(\Gamma_0(N))$ is a *Hecke eigenform* if in addition to being an eigenvector for all of the Hecke operators, it is normalised so that $a_1 = 1$.
The eigenvalues of Hecke eigenforms are simply the Fourier coefficients

$$f|_k T_n = a_n f \tag{3.42}$$

and, thanks to Equation (3.41), the Hecke eigenvalues a_p at primes p determine all of the others.
In addition to the Hecke operators, $M_k(\Gamma_0(N))$ is closed under the action of the *Atkin-Lehner involutions*. For $Q \in \mathbb{N}$ dividing the level N and $\gcd(Q, N/Q) = 1$, the group of Atkin-Lehner involutions is given by

$$\mathcal{W}_Q = \frac{1}{\sqrt{Q}} \begin{pmatrix} Q\mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & Q\mathbb{Z} \end{pmatrix} \cap SL(2,\mathbb{R}) \ . \tag{3.43}$$

Let $W_Q \in \mathcal{W}_Q$. W_Q acts on $f \in M_k(\Gamma_0(N))$ in the usual way as

$$f|_k W_Q \tag{3.44}$$

and commutes with Hecke operators if f is a *new form* that we now define.
Let $N', M \in \mathbb{Z}$ be such that $MN'|N$ and $N' \neq= N$. If $f \in M_k(\Gamma_0(N'))$, then $\tilde{f} \in M_k(\Gamma_0(N))$ where $\tilde{f}(\tau) = f(M\tau)$. We refer to such forms as *old forms* and the rest as *new forms*.
A particularly useful Atkin-Lehner involution is given by the *Fricke involution*

$$\frac{1}{\sqrt{N}} \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix} \tag{3.45}$$

which acts on Hecke forms as

$$f\Big(-\frac{1}{N\tau}\Big) \ = \ \epsilon N^{k/2} \tau^k f(\tau) \tag{3.46}$$

where $\epsilon = \pm 1$ and depends on the particular form f. Note that the Fricke involution plays the role of the "S- transformation"

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{3.47}$$

which is only an element of $\Gamma_0(N)$ when $N = 1$.[4]

For an example of a Hecke eigenform, consider the *modular discriminant* which is the unique element of $S_{12}(SL(2,\mathbb{Z}))$ given by

$$\Delta(\tau) \;=\; q\prod_{n=1}^{\infty}(1-q^n)^{24} \tag{3.48}$$

where the first few coefficients are given by

$$\Delta(\tau) \;=\; q - 24q^2 + 252q^3 - 1472q^4 + 4830q^5 - 6048q^6 - 16744q^7 + \dots \;. \tag{3.49}$$

As expected, we see that the prime Hecke eigenvalues determine all of the others. For example $a_2a_3 = (-24)(252) = -6048 = a_6$ and, more generally,

$$a_m a_n \;=\; \sum_{r|\gcd(m,n)} r^{k-1} a_{mn/r^2} \;. \tag{3.50}$$

The fact that Hecke eigenvalues are multiplicative in the above sense implies that the L-function of Hecke eigenforms can be written as an Euler product.

3.3.1.3 L-Functions

Given a cusp form $f \in S_k(\Gamma_0(N))$ with Fourier expansion as in Equation (3.32), we define the associated L-function by (the analytic continuation of) the Dirichlet series

$$L(f,s) = \sum_{n=1}^{\infty}\frac{a_n}{n^s} \tag{3.51}$$

which converges for $\mathrm{Re}\, s > 1 + \frac{k}{2}$ and can be computed directly from f via the Mellin transform

$$L(f,s) \;=\; \frac{(2\pi)^s}{\Gamma(s)}\int_0^{\infty}\mathrm{d}t\, f(it)t^{s-1} \tag{3.52}$$

where $\Gamma(s)$ is the Gamma function.

For a Hecke eigenform $f \in S_k(\Gamma_0(N))$, we can alternatively define the associated L-function via the Euler product

$$L(f,s) = \prod_{p|N}\frac{1}{1-a_p p^{-s}}\prod_{p\nmid N}\frac{1}{1-a_p p^{-s} + p^{k-1-2s}} \tag{3.53}$$

and the fact that this is equivalent to the Dirichlet series follows from the multiplicative nature of the Hecke eigenvalues.

[4] The Fricke involution is especially useful for computing L-function values quickly. See, for example, §10.5 in [43].

The *Hecke L-function* associated to Hecke eigenforms is part of a class of meromorphic functions that we refer to as L-functions that arise in a myriad of different ways. For example, we will see in the following section that, after a substitution $T \to p^{-s}$, the Frobenius polynomials described in the previous section appear in the Euler product of an L-function. Conjecturally, L-functions satisfy the following properties [44].

1. *Euler Products*

   They can be expressed as an Euler product. For Hecke L-functions, this follows from the multiplicative nature of the Hecke eigenvalues.

2. *Analytic Continuation*

   They can be continued meromorphically to the entire complex plane. For a Hecke L-function, this follows from the functional equation below.

3. *Functional Equation*

   There exists a functional equation that relates the value at s to the value at $k - s$ for some positive integer k. For a Hecke eigenform $f \in S_k(\Gamma_0(N))$, the completion

   $$L^*(f, s) = \left(\frac{2\pi}{\sqrt{N}}\right)^s \Gamma(s) L(f, s) \tag{3.54}$$

   satisfies the functional equation

   $$L^*(s) = \epsilon L^*(k - s) \tag{3.55}$$

   where $\epsilon = \pm 1$ is the sign that appears in the Fricke involution defined in Equation (3.46). A proof of this fact applies the Fricke involution to the integrand of the Mellin tranform in Equation (3.52) [45]. A discussion of the functional equation for modular forms with a character is more involved and can be found in [46].

3.4 Modularity

We explain in this section how some of the factors in the zeta function of certain varieties defined over (a finite extension of) $\mathbb{Q}$ are sometimes determined by a collection of modular forms for some subgroup of $SL(2, \mathbb{Z})$. We refer to such varieties as *modular* and explain what this means by way of the following examples.

3.4.1 Elliptic Curves

As discussed in the introduction of this thesis, an elliptic curve defined over $\mathbb{Q}$ is associated with a modular form and it is a proof of this fact that eventually led to a proof of Fermat's last theorem [3]. By a a rational change of variables, any elliptic curve E defined over $\mathbb{Q}$ may be brought to Weierstrass form

$$zy^2 \;=\; x^3 + Az^2x + Bz^3 \tag{3.56}$$

for some $A, B \in \mathbb{Z}$ where $[x, y, z]$ are homogeneous corrdinates on $\mathbb{CP}^2$ and this can be taken to be the definition of an elliptic curve. Equation (3.56) is more often written in inhomogeneous form where $z = 1$ as the only point on the elliptic curve with $z = 0$ is the "point at infinity" $[0, 1, 0]$. We define the discriminant

$$\Delta(E) \;=\; -16(4A^3 + 27B^2)\ . \tag{3.57}$$

which is just an algebraic definition of the modular discriminant in Equation (3.48).[5] Many definitions of an elliptic curve require that $\Delta(E) \neq 0$ which we assume here.

Equation (3.56) is perfectly well defined as a variety over a finite field $\mathbb{F}_{p^r}$ for some prime p and $r \in \mathbb{N}$ so we may count solutions of Equation (3.56) in $\mathbb{FP}_{p^r}$ and, as mentioned to in the introduction, a consequence of modularity is that

$$|E/\mathbb{F}_p| = p + 1 - a_p \tag{3.58}$$

where a_p is the p^{th} Hecke eigenvalue of a weight 2 modular form for some $\Gamma_0(N)$. In order to state the modularity theorem more carefully, we introduce the *Hasse-Weil L-function*. First, note that the Hodge diamond of an elliptic curve over $\mathbb{C}$ is given by

$$h^{pq}(E) \;=\; \begin{matrix} & 1 & \\ 1 & & 1 \\ & 1 & \end{matrix} \tag{3.59}$$

and that the Weil conjectures imply that

$$\zeta(E/\mathbb{F}_p, T) \;=\; \frac{1 - a_pT + pT^2}{(1-T)(1-pT)}\ . \tag{3.60}$$

From the discussions of Section 3.1, we recognise that the only non-trivial term in Equation (3.60) is the Frobenius polynomial $P_3(E/\mathbb{F}_p, T)$ which is a second order polynomial because $b^1(E) = 2$. A statement of the modularity theorem is that the coefficients a_p appearing in (3.60) are, for all but finitely many primes the Hecke eigenvalues of a weight 2 modular form for some $\Gamma_0(N)$ which implies the formula in Equation (3.58). A slightly more sophisticated statement of the modularity theorem uses L-functions.

[5] Associated to every elliptic curve E over $\mathbb{C}$ is a parameter τ computed as a ratio of the two periods of the holomorphic 1-form on E. The two periods can be thought of as homogeneous coordinates on the moduli space of elliptic curves which is exactly analogous to Equation (2.11) and the fact that some of the periods of the holomorphic 3-form on a Calabi-Yau threefold provide homogeneous coordinates on the moduli space of complex structures.

A *Hasse-Weil L-function* can be associated to any of the Frobenius polynomials appearing in the zeta function and is defined as

$$L^{(i)}(E,s) = \prod_{p \text{ a good prime}} \frac{1}{P_i(E/\mathbb{F}_p, p^{-s})} \tag{3.61}$$

where *primes of bad reduction* p are those that lead to singular varieties when the original variety over $\mathbb{Q}$ is reduced mod p. For an elliptic curve over $\mathbb{Q}$, the bad primes are simply those that divide the discriminant 3.57.[6].

While any Frobenius polynomial can be used to define a Hasse-Weil L-function, only those associated to the middle cohomology of an elliptic curve lead to a non-trivial L-function. Up to a finite number of Euler factors, the Hasse-Weil L-functions associated to zeroth and second cohomology are given by $\zeta(s)$ and $\zeta(s-1)$ respectively where ζ is the Riemann zeta function.[7] Thus, whenever we refer to the L-function of an elliptic curve we will mean the Hasse-Weil L-function associated to middle cohomology that is, furthermore, completed to

$$L(E,s) = \prod_{\text{bad } p} \frac{1}{1-\epsilon(p)p^{-s}} \prod_{\text{good } p} \frac{1}{1-a_p p^{-s} + p^{1-2s}} \tag{3.62}$$

where $\epsilon(p) \in \{-1, 0, 1\}$ depending on the particular prime (see §3 of [47]).

As with all L-functons, we expect that $L(E,s)$ satisfies all of the conditions discussed at the end of Section 3.3.1.3. That is, it can be analytically continued to a meromorphic function on the complex plane and it satisfies a functional equation. This is typically difficult to prove for a given L-function and remains conjectural in many cases. However, for the Hasse-Weil L-function of an elliptic curve defined over $\mathbb{Q}$, these properties follow from the Modularity Theorem.

Modularity Theorem

If E is an elliptic curve defined over $\mathbb{Q}$, then

$$L(E,s) = L(f,s) \tag{3.63}$$

where f is a Hecke eigenform form of weight two for the finite index subgroup $\Gamma_0(N) \subset SL(2,\mathbb{Z})$ of level N. Moreover, N is only divisible by the primes of bad reduction.

3.4.2 Rigid Calabi-Yau Threefolds

The next example of a modular Calabi-Yau manifold that we will consider is a rigid Calabi-Yau threefold X defined over $\mathbb{Q}$. Recall that this means that the Hodge diamond of X is

[6] Note the substitution of $T \to p^{-s}$ in the Frobenius polynomial

[7] It is straightforward to check that the Riemann zeta function is, in fact, the Hasse-Weil L-function of a point. Note that all primes are primes of good reduction for a point.

given by

$$h^{pq}(X) \;=\; \begin{matrix} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1} & & 0 & \\ 1 & & 0 & & 0 & & 1 \\ & 0 & & h^{1,1} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{matrix}\;. \tag{3.64}$$

where $h^{1,1} \geq 1$.

The zeta function of a Calabi-Yau threefold is given by Equation (3.12) and, in the rigid case,

$$P_3(X/\mathbb{F}_p, T) = (1 - a_p T + p^3 T^2) \tag{3.65}$$

which is a degree two polynomial because $b^2(X) = 2$.

Just like elliptic curves, there is only one non-trivial Frobenius polynomial for a rigid Calabi-Yau manifold and, more importantly, it has been proven that rigid Calabi-Yau threefolds over $\mathbb{Q}$ are modular. That is, for good primes p, the coefficients a_p appearing in Equation (3.65) are the p^{th} Hecke eigenvalues of a cusp form of weight 4 [13].

Note that, since $h^{2,1}(X) = 0$ for a rigid Calabi-Yau threefold, X admits no complex structure deformations. Thus, there is no Picard-Fuchs equation and a computation of the periods requires the explicit evaluation of period integrals which can be done in simple examples as in [14]. The lack of a Picard-Fuchs also means that the deformation method for computing the zeta is unavailable and one is forced to resort to point counting in order to identify the associated weight four form. Examples of rigid Calabi-Yau threefolds and their associated modular forms can be found in the thesis of Meyer [7].

3.4.3 One-Parameter Calabi-Yau Threefolds

Let X be a Calabi-Yau threefold with $h^{2,1}(X) = 1$. We have seen that the Frobenius polynomial associated to middle cohomology is given by

$$P_3(X/\mathbb{F}_p, T) \;=\; 1 + a_p T + b_p p T^2 + a_p p^3 T^2 + p^6 T^4 \tag{3.66}$$

and, given the results of previous sections, one naturally wonders - are the coefficients a_p and b_p associated with a modular form? This is still an open question. It is believed that the coefficients a_p and b_p are associated to a *Siegel modular form* [40]. A Siegel modular form of weight k is a holomorphic function on the *Siegel upper half space*

$$\mathbb{H}_g \;=\; \{\, \tau \in M_{g\times g}(\mathbb{C}) \,|\, \tau = \tau^T \,,\, \det \operatorname{Im} \tau > 0 \,\} \tag{3.67}$$

that satisfies

$$f(\gamma\tau) \;=\; \det(C\tau + D)^k f(\tau) \qquad \forall \gamma \in Sp(2g, \mathbb{Z}) \tag{3.68}$$

where

$$\gamma \;=\; \begin{pmatrix} A & B \\ C & D \end{pmatrix} \qquad \text{and} \qquad \gamma\tau \;=\; (A\tau+B)(C\tau+D)^{-1} \; . \tag{3.69}$$

Note that $Sp(2,\mathbb{Z}) = SL(2,\mathbb{Z})$ and, when the *genus* g is equal to 1, a Siegel modular form is a weakly holomorphic modular form. For $g \geq 2$, the analogue of holomorphicity at infinity is automatically fulfilled. This is known as the *Koecher Principle.*

Although the theory is much more involved, analogues of $\Gamma_0(N)$, Hecke operators and L-functions can all be defined for Siegel modular forms [48]. For example, for a Siegel modular form f of genus $g = 2$ and weight k, we may define the *spinor zeta function* which is given by

$$\zeta(f,p,T) \;=\; 1 - \lambda_p T + (\lambda_p^2 - \lambda_{p^2} - p^{2k-4})T^2 - \lambda_p p^{2k-3} T^3 + p^{4k-6} T^4 \tag{3.70}$$

where λ_p and λ_{p^2} are Hecke eigenvalues. Note that, when $k = 3$, the spinor zeta function has the same form as $P_3(X/\mathbb{F}_p,T)$ which suggests that a_p and b_p can be expressed in terms of the Hecke eigenvalues of a weight 3, genus 2 Siegel modular form.

An important distinction between the $g = 1$ case and $g \geq 2$ is that, in the latter case, the Hecke eigenvalues are not directly related to Fourier coefficients of f. Furthermore, the $g = 1$ case has been extensively studied and databases such as the *L-Functions and Modular Forms Database* (LMFDB) [49] are available online that can be used to quickly identify a modular form from its Fourier coefficients. There are also implementations in MAGMA [50] and PARI/GP [51] for computing spaces of modular forms for $SL(2,\mathbb{Z})$. This makes the process of identifying the modular form associated to a given variety relatively straightforward when $g = 1$ but impractical when $g \geq 2$.

Whereas a generic Calabi-Yau variety defined over $\mathbb{Q}$ is believed to be Siegel modular, there are examples of Calabi-Yau threefolds that are simply modular. By this we mean that $P_3(X/\mathbb{F}_p,T)$ factors over $\mathbb{Z}$ into two quadrics where each quadratic term is determined by a modular form of $SL(2,\mathbb{Z})$.[8] In order to understand how such Calabi-Yau factorisations might arise, recall that the Frobenius polynomial $P_k(X/\mathbb{F}_p,T)$ is a charectistic polynomial of the inverse of the Frobenius map

$$\text{Frob}_3: \; H^3(X) \longrightarrow H^3(X) \; , \tag{3.71}$$

where H^3 can be any Weil-cohomology. For example, suppose that $H^3(X,\mathbb{Q})$ factors over $\mathbb{Q}$ as

$$H^3(X,\mathbb{Q}) \;=\; \Lambda_{\mathbb{Q}} \oplus \Lambda_{\mathbb{Q}}^{\perp} \tag{3.72}$$

[8] If X is defined over $\mathbb{Q}$, $P_3(X/\mathbb{F}_p,T)$ will factor for all but finitely many primes whereas, if X is defined over some finite extension $\mathbb{Q}(\,a_1,\alpha_2,..\alpha_n)$, $P_3(X/\mathbb{F}_p,T)$ will factor for the primes such that all of the α_i have good reduction mod p. We will see examples of both in the following chapter.

where $\Lambda_{\mathbb{Q}}$ and $\Lambda_{\mathbb{Q}}^{\perp}$ are each two dimensional vector spaces over $\mathbb{Q}$ with Hodge numbers $(3,0)+(0,3)$ and $(2,1)+(1,2)$ respectively (this is the case for attractor varieties of rank two when $h^{2,1}=1$). Then, after a change of basis over $\mathbb{Q}$, Frob_3 will be block diagonal and its characteristic polynomial must factor into two quadrics. Since $H^3(X,\mathbb{Q})$ looks like the sum of $H^3(E\times\mathbb{P}^1,\mathbb{Q})$ and $H^3(X_{\mathrm{rigid}},\mathbb{Q})$ for an elliptic curve E and a rigid Calabi-Yau threefold X_{rigid} (see Section 4.2), we expect that the Frobenius polynomial will, for all but finitely many primes, factor as

$$P_3(X/\mathbb{F}_p,T) \;=\; (1-\alpha pT+p^3T^2)(1-\beta T+p^3T^2)\;. \tag{3.73}$$

where α_p and β_p are the Fourier coeffcients of eigenforms in $S_2(\Gamma_0(N_1))$ and $S_4(\Gamma_0(N_2))$ for some $N_1,N_2\in\mathbb{N}$.

The above can be stated more formally in term of *Galois representations* which in turn define Artin L-functions (see [45, 52] for definitions). For the case under consideration, the relevant Galois representation is given by

$$\rho:\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\to\mathrm{GL}_4(\mathbb{Q}_\ell) \tag{3.74}$$

that maps a Frobenius element at p to the matrix Frob for some prime $\ell\neq p$. When $H^3(X)$ splits as in (3.72), ρ is reducible and we are left with two 2 dimensional Galois representations

$$\rho_j:\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\to\mathrm{GL}_2(\mathbb{Q}_\ell)\;,\qquad j\in\{1,2\} \tag{3.75}$$

which is the subject of *Serre's conjecture*[53, 54]. This asserts that such representations are attached to modular forms of specific weight and conductor and can as such be seen as a generalisation of the Taniyama-Weil conjecture. A proof of Serre's conjecture [55–58] led Gouvêa and Yui [13] to prove the modularity of rigid Calabi-Yau threefolds. Moreover, in the same paper, they prove that splittings of the form in Equation (3.72) are also modular and the coefficients α_p and β_p are Fourier coefficients of cusp forms of weight 2 and 4 for some congruence groups $\Gamma_0(N_1)$ and $\Gamma_0(N_2)$. In other words, rank two attractors over $\mathbb{Q}$ are modular.

If the rank two attractor variety X is *not* defined over $\mathbb{Q}$ but over some number field $\mathbb{K}$, the situation is more complicated, as we are then dealing with representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{K})$. But the Chebotarëv density theorem [59] implies that in such cases one still has such a splitting of $P_3(X/\mathbb{F}_p,T)$ for infinitely many and in fact a positive fraction of primes p. In the case of totally real fields one in general expects Hilbert modular forms.[9]

[9] A Hilbert modular form is the analogue of an elliptic modular form where the integers $\mathbb{Z}$ are replaced with the ring of integers $\mathcal{O}_K$ of a number field K. If K has degree n as an extension of $\mathbb{Q}$, a Hilbert modular form is a defined on the n-fold product of the upper half place $\mathbb{H}^n$. A group element $\gamma\in SL(2,\mathcal{O}_K)$ acts on

However, in the cases we encounter in this thesis, we find classical modular forms for $\Gamma_1(N)$. Before searching for rank two attractors, we mention that examples of modular Calabi-Yau threefolds can be found by computing the zeta-function of a one-parameter family of Calabi-Yau manifolds at a conifold point defined over $\mathbb{Q}$ i.e. $\varphi \in \mathbb{Q}$. If, instead, $\varphi \in \mathbb{K}$, one generally finds Hilbert modular forms instead [39].

$\tau = (\tau_1, \tau_2, \ldots, \tau_n) \in \mathbb{H}^n$ as

$$\gamma\tau = (\sigma_1(\gamma)\tau_1, \sigma_2(\gamma)\tau_2, \ldots, \sigma_n(\gamma)\tau_n) \tag{3.76}$$

where the σ_i are the embeddings of K into $\mathbb{C}$. We say that a holomorphic function $f : \mathbb{H}^n \to \mathbb{C}$ is a *Hilbert modular form* of weight $(k_1, \ldots, k_n)$ if

$$f(\gamma\tau) = \prod_{i=1}^{n} \big(\sigma_i(c)\tau + \sigma_i(d)\big)^{k_i} f(\tau) \qquad \forall\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \ SL(2, \mathcal{O}_K) \tag{3.77}$$

As is the case with Siegel modular forms, holomorphicity at cusps is guaranteed and there is no need to include it in the definition of a Hilbert modular form [48].

Chapter 4

Search for Rank Two Attractors

4.1 Persistent Factorisations

We consider a 1-parameter family X_φ of Calabi-Yau threefolds with $h^{2,1} = 1$, defined by a polynomial equation

$$P(x, \varphi) \;=\; 0 \tag{4.1}$$

with integral coefficients. In the light of the discussion of Chapter 3, a strategy for finding rank two attractor points φ_* is now quite clear: we compute the Frobenius polynomial

$$P_3(X/\mathbb{F}_p, T) \;=\; 1 + a_p T + b_p p T^2 + a_p p^3 T^2 + p^6 T^4 \tag{4.2}$$

for many p and φ and look for *persistent factorisations* into a product of two quadratic factors. By this we mean that the factorisations occur whenever φ is the root of some algebraic equation $G(\varphi)$ defined over $\mathbb{Q}$, without any reference to a particular prime.
Using the deformation method, the quantities $P_3(X/\mathbb{F}_p, T)$ were calculated, in [21], for $\varphi = 1, ..., p-1$ for the 500 values $p = 5, ..., 3467$, for the Hulek-Verril manifolds described in Appendix A with Picard-Equation AESZ 34..
For example for $p = 19$ we have Table 3.1 and we see that $P_3(X/\mathbb{F}_p, T)$ factors in the form indicated for the five values $\varphi = 4, 5, 8, 9, 11$. At the conifold points $P_3(X/\mathbb{F}_p, T)$ degenerates to a cubic, and factorises into a linear factor and a quadric. These cases are also very interesting, not least because they also exhibit modular behaviour and can be thought of as corresponding to massless black holes. We will not however pursue the factorisations due to the conifolds here.
We do not want to assert that every factorisation of the form (3.73) corresponds to a rank two attractor point. However, there is a form of converse statement that we do expect. Let us suppose that, as conjectured by Moore [11], the rank two attractor points are algebraic, in the sense that there is a polynomial $G(\varphi)$ with rational (so integer) coefficients, whose roots are the rank two attractor points. If this is so, then it makes sense to reduce $G(\varphi)$

mod p and the roots will exist in $\mathbb{F}_p$ for some, and in fact for infinitely many, p. For these p we expect $P_3(X/\mathbb{F}_p, T)$ to factorise. By assuming that there is a single polynomial $G(\varphi)$, whose roots are the rank two attractor points, we are implicitly assuming, not only that the rank two attractor points are algebraic, but also that there are finitely many such points. These comments are made for the case that there is one parameter. If there are more parameters, we would expect the rank two attractor points to lie on algebraic submanifolds of the parameter space. We thus look for a single polynomial

$$G(\varphi) \;=\; c_n\varphi^n + c_{n-1}\varphi^{n-1} + \ldots + c_1\varphi + c_0 \tag{4.3}$$

with roots at the attractor points of rank two.

The crudest summary of the tables produced in [21] is to count how many times $P_3(X/\mathbb{F}_p, T)$ factorises in the indicated way for each prime p. We have just seen that for $p = 19$ it factorises 5 times. This leads to the two plots in Figure 4.1. The first gives the data for the manifold AESZ34, while the second gives the analogous data for the mirror of the quintic threefold and is presented for comparison. Clearly $P_3(X/\mathbb{F}_p, T)$ for AESZ34 factorises much more often than for the mirror quintic. Notice also that while for the mirror quintic there are many primes for which $P_3(X/\mathbb{F}_p, T)$ does not factorise, for AESZ34 the polynomial $P_3(X/\mathbb{F}_p, T)$ factorises at least once for each p. This suggests that, for AESZ34, the polynomial $G(\varphi)$ has a linear factor, since a linear equation

$$c_1\varphi + c_0 \;=\; 0 \tag{4.4}$$

has a solution mod p for all p, apart from primes that divide c_1.

By looking first at the primes for which $P_3(X/\mathbb{F}_p, T)$ factorises precisely once, and using a variant of the Chinese Remainder Theorem, or by simply performing a computer search over integers c_0 and c_1, we find that (apart from the case $p = 7$) the polynomial $P_3(X/\mathbb{F}_p, T)$ always factorises when

$$\varphi \;= -1/7\;.$$

[In $\mathbb{F}_p$, $\varphi = -\,1/7$ is the integer that satisfies the relation $7\varphi + 1 = 0$. For $p = 19$, for example, we have $7 \times 8 = -\,1$, so $-1/7 = 8$ in $\mathbb{F}_{19}$ and this indeed is one of the values for which factorisation of the desired form occurs in Table 3.1.]

It is easy to check that, considered as a point of $\mathbb{C}$, $\varphi = -\,\frac{1}{7}$ is indeed a rank two attractor point. By this, we mean that we solve the Picard-Fuchs equation around $\varphi = 0$ and, by numerical integration, evaluate it at $\varphi = -\frac{1}{7}$ to 1000 decimal places. We then check that, the ratios in Equation (2.77) are rational to this precision. We will later, in Chapter 5, propose identities between the periods at $\varphi = -\frac{1}{7}$ and critical L-values that will also be verified to at least 1000 decimal places. Although not a proof, these observations leave

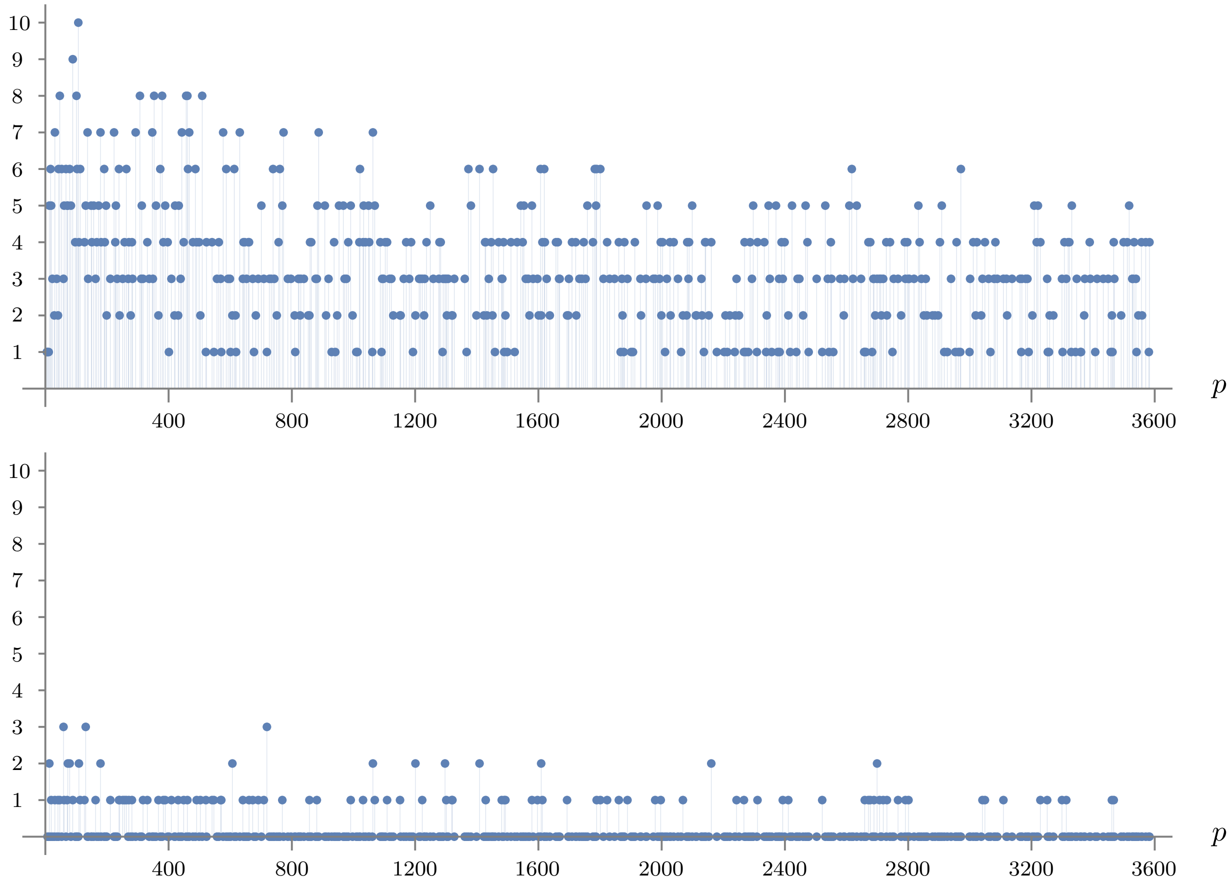

Figure 4.1: *The upper plot shows the number of factorisations into two quadrics as φ varies over over the smooth values in $\mathbb{F}_p$, $7 \leq p \leq 3583$, for the manifold AESZ34. For comparison, the lower plot provides the same information for the mirror of the quintic which explains why it is difficult to find rank two attractor points on this family.*

little doubt that $\varphi = -\frac{1}{7}$ is indeed a rank two attractor. The flow towards the attractor point $\varphi = -\frac{1}{7}$ is illustrated in the introduction in the introduction in Figure 1.1. See also Figure 4.2 and 4.3 in this section.

Encouraged by finding a linear factor of $G(\varphi)$, we search for a quadratic factor

$$c_2\varphi^2 + c_1\varphi + c_0 \;=\; 0$$

and find that $P_3(X/\mathbb{F}_p, T)$ always factorises when $\varphi^2 - 66\,\varphi + 1 = 0$ and so when

$$\varphi \;=\; \varphi_\pm \;=\; 33 \pm 8\sqrt{17}$$

exists in $\mathbb{F}_p$. This occurs when 17 is a square mod p, and so, by quadratic reciprocity, when p is a square mod 17.

[Pursuing our example for $p = 19$, note that $17 = 6^2$ in $\mathbb{F}_{19}$ so $\varphi_\pm = 4,\, 5$ and the desired factorisations also occur for these values of φ in Table 3.1.]

Again, if we take $\varphi_\pm$ to be points in $\mathbb{C}$, then it is straightforward to check numerically that these values correspond to rank two attractor points. These flow plots are presented in Figure 4.2 and Figure 4.3.

The tables of the Frobenius polynomials $P_3(X/\mathbb{F}_p, T)$ contain much more information than that shown in Figure 4.1. For example, let us consider the coefficients α and β for the attractor points, as p varies. For $\varphi = -\frac{1}{7}$ we list primes $5 \le p \le 137$. While for $\varphi = 33\pm 8\sqrt{17}$ we list primes $5 \le p \le 349$ such that 17 is a square mod p. A first remark is that $P_3(X/\mathbb{F}_p, T)$ is the same for $\varphi = \varphi_\pm$ so we need only present a single table for these parameter values.

For $\varphi = -\,1/7$ we observe that the α's are the p^{th} coefficients of a weight 2 modular form, with LMFDB designation **14.2.a.a** for the group $\Gamma_0(14)$. The coefficients β are similarly the p^{th} coefficients of a weight four modular form, with designation **14.4.a.a**, also for $\Gamma_0(14)$.

Although the appearence of modular forms here was anticipated on mathematical grounds, their appearance is rather mysterious to a physicist and raises the question - "What role, if any, are these modular forms playing physically?". We will see in Chapte 5 that certain values of the L-function associated to these modular forms determine the periods of the holomorphic 3-form of the Calabi-Yau X over $\mathbb{C}$ which, in turn, determines the area of black holes, topological string free energies, etc. We will speculate further in the following chapters.

For the modular forms for $\Gamma_0(14)$, that we need, the weight 2 form admits a representation in terms of the Dedekind η-function

$$\begin{aligned} f_{\mathbf{14.2.a.a}}(\tau) &= \eta(\tau)\eta(2\tau)\eta(7\tau)\eta(14\tau) \\ &= q - q^2 - 2q^3 + q^4 + 2q^6 + q^7 - q^8 + q^9 - 2q^{12} - 4q^{13} - q^{14} \\ &\qquad + q^{16} + 6q^{17} - q^{18} + 2q^{19} + \dots\ . \end{aligned} \tag{4.5}$$

For the weight 4 form we do not know of an analogous expression, however the LMFDB provides the expansion

$$\begin{aligned} f_{\mathbf{14.4.a.a}} = \ & q - 2q^2 + 8q^3 + 4q^4 - 14q^5 - 16q^6 - 7q^7 - 8q^8 + 37q^9 + 28q^{10} - 28q^{11} + \\ & 32q^{12} + 18q^{13} + 14q^{14} - 112q^{15} + 16q^{16} + 74q^{17} - 74q^{18} + 80q^{19} + \dots\ . \end{aligned}$$

For $\varphi = 33\pm 8\sqrt{17}$, with the exception of $p = 17$, the correspondence is for primes such that 17 is a square mod p. For these primes, the α's are the p^{th} coefficients of the weight two modular form, with designation **34.2.b.a** and the β's are the p^{th} coefficients the weight 4 modular form **34.4.b.a**, both for the congruence subgroup $\Gamma_1(34)$.

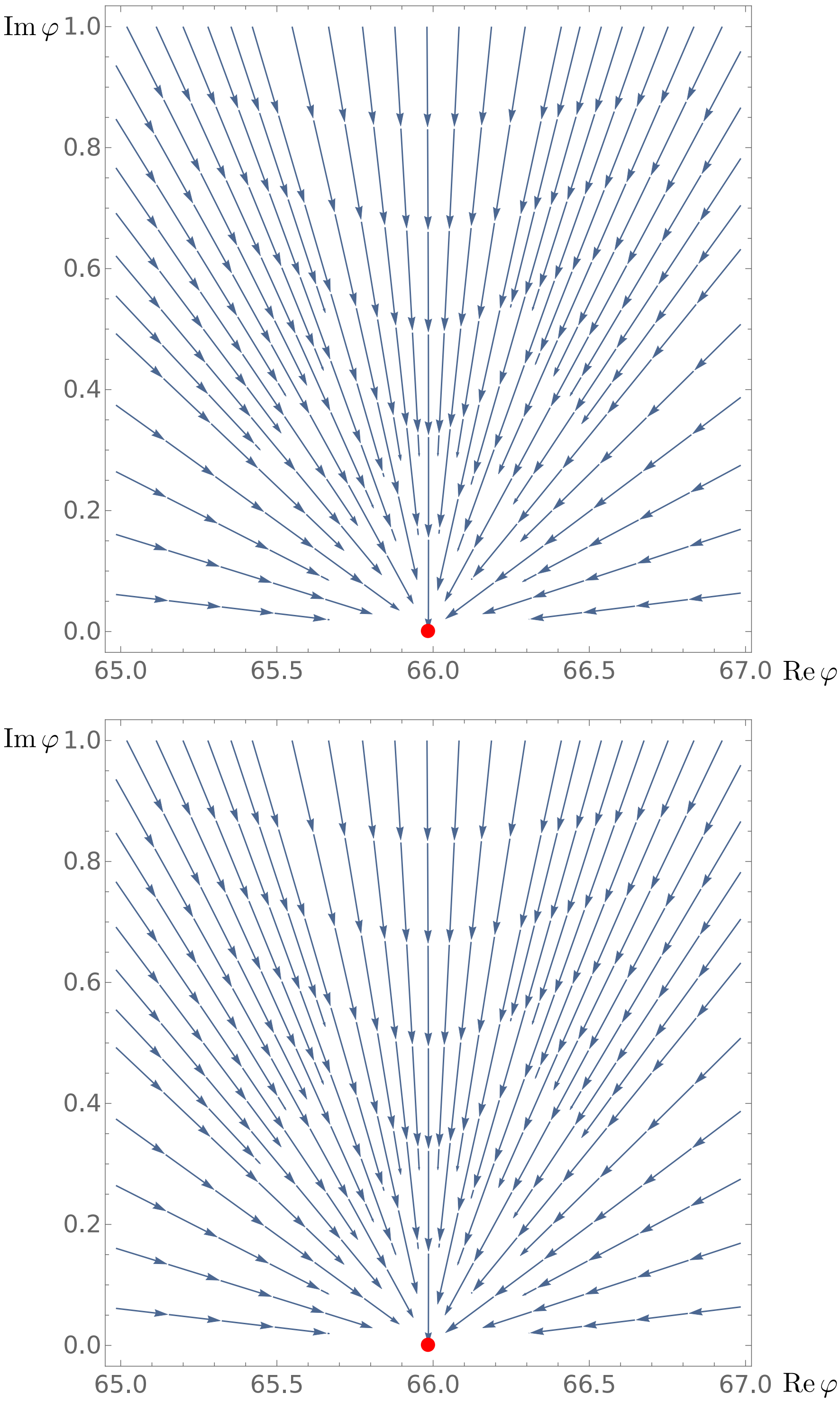

Figure 4.2: *The flows for* $\varphi = \varphi(\rho)$ *for the charges* $Q = (4, -9, 7, 4)$ *(above) and* $Q = (4, -30, -30, -5)$ *(below) leading to attractor point at* $\varphi = 33 + 8\sqrt{17}$*. These plots look very similar because* $33 + 8\sqrt{17}$ *is far from any singularities and the periods are very slowly varying.*

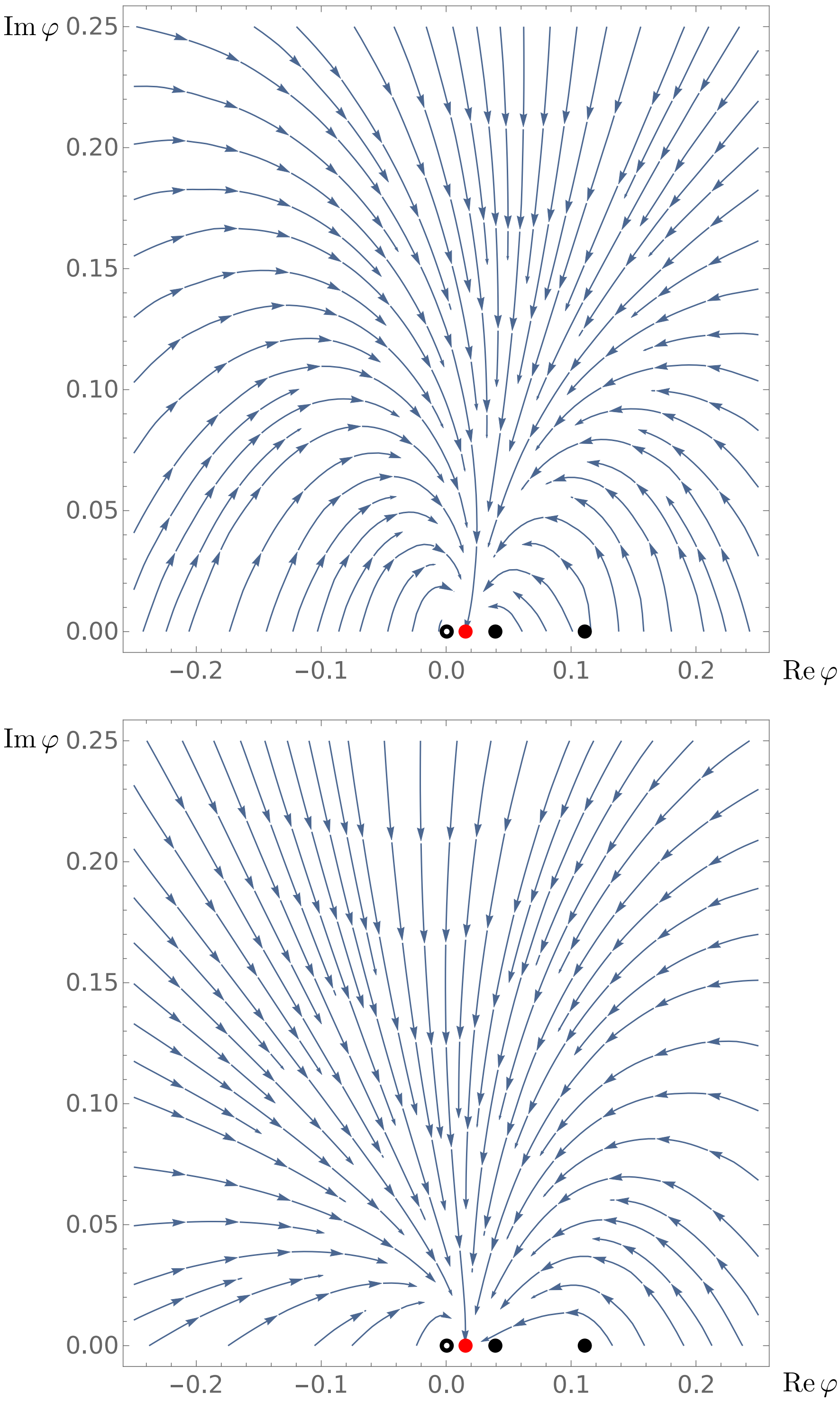

Figure 4.3: *The flows for* $\varphi = \varphi(\rho)$ *for the charges* $Q = (-2, 0, 0, 5)$ *(above) and* $Q = (0, 3, 1, 0)$ *(below) leading to attractor point at* $\varphi = 33 - 8\sqrt{17}$*. The point of maximal unipotent monodromy at* $\varphi = 0$ *is indicated by a hollow black dot while the solid black dots represent conifold singularities.*

For $\varphi = 33 \pm 8\sqrt{17}$, the α's appear as the coefficients of q^p in the q-expansion of the weight 2 modular form for $\Gamma_1(34)$ with LMFDB designation **34.2.b.a** and Fourier expansion

$$f_{\mathbf{34.2.b.a}} = q - q^2 + 2i\sqrt{2}q^3 + q^4 - 2i\sqrt{2}q^5 - 2i\sqrt{2}q^6 - q^8 - 5q^9 + 2i\sqrt{2}q^{10} - 2i\sqrt{2}q^{11} + 2i\sqrt{2}q^{12} + 2q^{13} + 8q^{15} + q^{16} - \big(3 - 2i\sqrt{2}\big)q^{17} + 5q^{18} - 4q^{19} + \dots \,. \tag{4.6}$$

The β's appear as the the coefficients of q^p of the weight 4 modular form for $\Gamma_1(34)$ with LMFDB designation **34.4.b.a** and Fourier expansion

$$f_{\mathbf{34.4.b.a}} = q - 2q^2 + 2iq^3 + 4q^4 + 8iq^5 - 4iq^6 + 34iq^7 - 8q^8 + 23q^9 - 16iq^{10} - 30iq^{11} + 8iq^{12} - 42q^{13} - 68iq^{14} - 16q^{15} + 16q^{16} + (17 - 68i)q^{17} - 46q^{18} + 60q^{19} + \dots \,. \tag{4.7}$$

At first sight, these last two q-series are surprising since the coefficients are not all integers. However, the coefficients we need to compare with the α's and β's are those of terms q^p for primes such that 17 is a square mod p, and for these the coefficients *are* integers. The coefficients in these expansions that are not integral are complex so there is a choice that has been made in defining the forms $f_{\mathbf{34.2.b.a}}$ and $f_{\mathbf{34.4.b.a}}$ above, since the complex conjugates of these forms are also modular forms of the same weight for $\Gamma_1(34)$.

[Returning, once again, to the case $p = 19$, notice that the coefficients of q^{19} in the modular forms above are -4 and 60 and that these are the α and β coefficients that appear for $\varphi = 4$, 5 in Table 3.1.]

The covering of $X_{-\frac{1}{7}}$ was conjectured by Meyer to be modular in [7] on the basis of point counting and our results seem to confrm his predicted modular form. The modularity of $X_{33\pm 8\sqrt{17}}$ seems to be new.

An important point to keep in mind is that, although the deformation method quickly computes the zeta-function at conifold points due to miraculous cancellation alluded to in Section 3.2, the same cancellation does not take place at other singularities of the Picard-Fuchs equation and we do not compute the zeta-function there. This is important because, as we shall see in Chapter 6, there are examples of apparent singularities (singularities of the Picard-Fuchs equation where the underlying Calabi-Yau manifold is smooth) that are also attractor points of rank two and the method described in this chapter would miss these points.

$\varphi = -\frac{1}{7}$		
p	α	β
5	0	-14
7		
11	0	-28
13	-4	18
17	6	74
19	2	80
23	0	-112
29	-6	190
31	-4	72
37	2	-346
41	6	162
43	8	-412
47	-12	24
53	6	318
59	-6	-200
61	8	-198
67	-4	-716
71	0	392
73	2	538
79	8	240
83	-6	-1072
89	-6	810
97	-10	1354
101	0	-1358
103	-4	-832
107	12	444
109	2	1870
113	6	1378
127	-16	1944
131	18	-848
137	18	-2966

$\varphi = 33 \pm 8\sqrt{17}$		
p	α	β
13	2	-42
17	-6	34
19	-4	60
43	-4	508
47	0	-136
53	6	318
59	12	300
67	-4	-676
83	-12	-1132
89	6	-350
101	-6	-1218
103	8	8
127	-16	-1216
137	-18	1954
149	6	-1010
151	8	-968
157	14	1654
179	12	-980
191	0	952
223	-16	-712
229	-22	5230
239	0	2040
251	-12	-5868
257	6	-4646
263	24	-6472
271	-16	8312
281	18	-518
293	6	-6402
307	20	-3516
331	-4	2892
349	-34	5270

Table 4.1: *The* (α, β)*-coefficients for the attractor points* $\varphi = -\frac{1}{7}$ *and* $\varphi = 33 \pm 8\sqrt{17}$.

4.2 Speculations on Geometric Origin of the Splitting

The calculations of this chapter and the identities of Chapter 5 provide overwhelming evidence for a splitting of $H^3(X_{\varphi_*})$ into a sum of two 2-dimensional pieces when $\varphi_* = \{\frac{1}{7}, 33 - 8\sqrt{17}, 33 + 8\sqrt{17}\}$. One naturally wonders if there is a geometric explanation for the splitting. Once identified, it should lead to a rigorous proof of our observations on the splitting of the Frobenius polynomials and the expression of periods in terms of L-values. We propose a few geometric explanations of the observed splitting.

One of the simplest explanations for the splitting would be that X has self-map ι such that ι^* splits $H^3(X_{\varphi_*}, \mathbb{Z})$ as

$$H^3(X_{\varphi_*}, \mathbb{Z}) \;=\; \Lambda_+ \oplus \Lambda_- \tag{4.8}$$

where $\Lambda_\pm \otimes \mathbb{Q}$ are eigenspaces of ι^* where ι^* has a positive (negative) eigenvalue on Λ_+ (Λ_-) and, furthermore, Λ_+ has Hodge numbers $(3,0)+(0,3)$ whereas Λ_- has Hodge numbers $(2,1)+(1,2)$. Such a transformation might arise from a symmetry of the family X_φ, for which φ_* is a fixed point, but we have been unable to find such a map and the Picard-Fuchs equation AESZ 34 is not symmetric in any obvious way.

In [60], a very non-trivial example of a map (defined over $\mathbb{Q}(\sqrt{2})$) that splits H^3 as in Equation (4.8) was found for a certain Calabi-Yau threefold (defined over $\mathbb{Q}$), which then led to a proof of Hilbert modularity for that particular variety.[1]

An alternative scenario that was used by Hulek and Verrill to explain the modularity of some non-rigid Calabi-Yau manifolds [61] is the existence of an elliptic ruled surface Y (i.e. a family of $\mathbb{P}^1$s over an elliptic curve E) and a birational map

$$Y \to X \tag{4.9}$$

such that the induced map

$$H^3(X, \mathbb{C}) \to H^3(Y, \mathbb{C}) \tag{4.10}$$

is surjective.

For example, suppose $Y = E \times \mathbb{P}^1$. By the Künneth formula,

$$H^3(E \times \mathbb{P}^1, \mathbb{Z}) \;=\; H^1(E, \mathbb{Z}) \otimes H^2(\mathbb{P}^1, \mathbb{Z})\ . \tag{4.11}$$

which has Hodge numbers $(2,1)+(1,2)$.[2] From the point of view of the attractor mechanism, this lattice is orthogonal to the "charge lattice" which has Hodge numbers (3,0)+(0,3). Hulek and Verill show that the existence of Y would lead to a factorisation of the zeta function where a degree two factor is modular and associated to the weight two form of E which is exactly what we have found.

[1] Examples of similarly symmetric Picard-Fuchs equations are found in Chapter 6 where the involution splits H^3 as in Equation (4.8) at certain fixed points.

[2] Essentially because $H^2(\mathbb{P}^1, \mathbb{Z}) \otimes \mathbb{R}$ is generated by the volume form $\mathbb{P}^1$ which is a $(1,1)$ form.

Chapter 5

Periods at Rank Two Attractor Points

5.1 L-Function Values and Periods of Calabi-Yau Manifolds

Let E be an elliptic curve over $\mathbb{Q}$ and f the weight two form of $\Gamma_0(N)$ associated to E. An alternative statement of the modularity theorem is that there exists a surjective holomorphic map

$$X_0(N) \to E \tag{5.1}$$

from the modular curve of level N to E [45]. Moreover, the holomorphic 1-form of E is pulled back to some multiple of $2\pi i f \mathrm{d}\tau$ (see [62] and [63] for instructive examples).

Recall that the L-function associated to the modular form f can be computed by a Mellin transform

$$L(s) \;=\; \frac{(2\pi)^s}{\Gamma(s)} \int_0^\infty \mathrm{d}y\, f(iy) y^{s-1} \;. \tag{5.2}$$

and consider, for example the modular curve at level 1. Evaluating the L-function at $s = 1$ gives

$$L(1) \;=\; -2\pi i \int_0^{i\infty} \mathrm{d}\tau f(\tau) \tag{5.3}$$

and, given the modular parametrisation in Equation (5.1) and the fact that 0 is identified with $i\infty$ by an $SL(2,\mathbb{Z})$ transformation, we see that $L(1)$ is a period of $f\mathrm{d}\tau$ on a 1-cycle and is therefore computable by computing the period of the holomorphic 1-form on E.[1] By moving the contour of integration, we see that this is true at any level N.

Let $\gamma(s) = (2\pi/\sqrt{N})^s\Gamma(s)$ which appears in the reflection formula for a weight k modular form. We say that an integer s_0 is a *critical point* if neither $\gamma(s)$ nor $\gamma(k-s)$ has a pole there. In other words, the critical points for a weight w-modular form of $\Gamma_0(N)$ are $\{1, 2, ..., k-1\}$. We refer to the value of a Hecke L-function at a critical point as a *critical L-value*.

[1]Strictly speaking, this integral diverges but this is not really a problem. We can split Equation (5.2) at $N^{-1/2}$ and use the Fricke involution to rewrite the integral from 0 to $N^{-1/2}$ as integral from $N^{-1/2}$ to ∞.

From the above discussion, we see that the periods of an elliptic curve compute the critical L-values of the associated eigenform. It turns out that analogous statements can be made about modular Calabi-Yau threefolds.

Just as cusp forms of weight two for $\Gamma_0(N)$ can be identified with holomorphic one-forms on the modular curve $X_0(N)$, which is the moduli space of elliptic curves with a subgroup of order N. The union of these elliptic curves makes up the elliptic modular surface $\mathcal{E} \to X_0(N)$ and weight three modular forms for $\Gamma_0(N)$ can be identified with holomorphic two forms on $\mathcal{E}$. More generally, a weight k cusp form for $\Gamma_0(N)$ gives a $(k-1)$-form on the *Kuga-Sato variety* $\mathcal{E}^{(k-2)}$, defined as the $k-2$ fold fibre product of elliptic surface $\mathcal{E} \to X_0(N)$.

It is believed, based on the Tate conjecture, that there is a correspondence (an algebraic cycle on the product of the two varieties) between the Kugo-Sato variety $\mathcal{E}^{(2)}$ and X [7]. Since critical L-values are periods of the corresponding 3-form on $\mathcal{E}^{(2)}$, we expect relations between the periods of a rank two attractor variety X and critical L-values of the associated weight 4 form. This will be the topic of this chapter.

The correspondences are expected to exist also for all rigid Calabi-Yau threefolds, but only in very few cases have these been found explicitly. For an overview of known cases we refer to the thesis of C. Meyer [7]. Expressions for the periods of a rigid Calabi-Yau in terms of the associated critical L-values can be found in [14].

Before moving on to the results of this chapter, we should mention that, based on known relations between periods of modular varieties and the associated critical L-values like the ones discussed above, Deligne formulated a very general conjecture relating critical L-values and period integrals of certain cohomology theories (the so-called *critical motives*) [64]. More precisely, he gave a prescription for taking specific minors of the associated period matrix and conjectured that they are, up to a rational multiple, equal to critical L-values of an associated L-function. A review of Deligne's conjecture can be found in [65, 66] where Deligne's conjecture is verified for the examples announced in [16] and reviewed in this chapter.

5.1.1 AESZ 34 at $\varphi = -1/7$

We have seen that the the two quadratic factors of $P_3(X/\mathbb{F}_p, T)$ when $\varphi = -\frac{1}{7}$ are related to Hecke eigenforms for $\Gamma_0(14)$ of weight 2 and weight 4 with LMFDB designations **14.2.a.a** and **14.4.a.a** respectively.

To distinguish the L-functions of the two forms, we will denote the weight 4 L-funtcion by L_4 and the weight two L-function by L_2. Since the functional equation relates $L(s)$ to $L(k-s)$, we will only consider the critical values $L_4(1)$ and $L_4(2)$ for the weight 4 form. A weight two L function has critical value $L(1)$. They are given by

$$L_4(1) = 0.67496319716994177129269568273091339919322842904407\ldots \tag{5.4}$$

$$L_4(2) = 0.91930674266912115653914356907939249680895763199044\ldots \tag{5.5}$$

and

$$L_2(1) = 0.33022365934448053902826194612283487754045234078189\ldots\ . \tag{5.6}$$

The accuracy given is sufficient to check the simpler relations that follow, however, unless otherwise stated, our numerical calculations are performed with an accuracy of at least 1000 figures.

We numerically integrate the integral symplectic periods Π defined by Equation (2.49) and evaluate them at at $\varphi = -\frac{1}{7}$. We then look for relations between the periods and the critical L-values and find

$$\Pi\left(-\tfrac{1}{7}\right) = i\frac{L_4(1)}{4\pi}\begin{pmatrix} 8\kappa \\ -30\kappa \\ 0 \\ 5 \end{pmatrix} + \frac{7}{2}\frac{L_4(2)}{\pi^2}\begin{pmatrix} 0 \\ 0 \\ 2 \\ 1 \end{pmatrix}\ . \tag{5.7}$$

We conjecture that that this is an exact expression.[2]

It is instructive to consider Equation (5.7) in the Frobenius basis defined in Equation (2.43). Because any imaginary term in $\varpi_j(-\frac{1}{7})$ comes from the fact that $\log\varphi = \log|\varphi| + i\pi$ on the negative real axis, the imaginary parts of $\varpi_j(-\frac{1}{7})$ are determined by the real parts. The observation that $\varphi = -\frac{1}{7}$ is an attractor point of rank two then boils down to the condition that the real parts of the periods satisfy the linear relations

$$\begin{aligned} 0 &= 3\mathrm{Re}\,\varpi_2(-\tfrac{1}{7}) + \pi^2\,\varpi_0(-\tfrac{1}{7}) \\ 0 &= 5\Big(\mathrm{Re}\,\varpi_3(-\tfrac{1}{7}) - 4\zeta(3)\varpi_0(-\tfrac{1}{7})\Big) + 11\pi^2\,\mathrm{Re}\,\varpi_1(-\tfrac{1}{7})\ . \end{aligned} \tag{5.8}$$

Note that, in the above relations, $\varpi_3(-\frac{1}{7})$ is always accompanied by a term proportional to $\zeta(3)\varpi_0(-\frac{1}{7})$. This follows from the $\zeta(3)$ that appears in the expansion of the prepotential near a MUM point (see (2.49)) and the assumption that $L_4(1)$, $L_4(2)$ and $\zeta(3)$ are algebraically independent (even after multiplying by the appropriate powers of π).

The relation to L-values in Frobenius basis is given by

$$0 = 5L_4(1) + 2\,\mathrm{Re}\,\varpi_1\left(-\frac{1}{7}\right) \quad \text{and} \quad 0 = 7L_4(2) - \pi^2\varpi_0\left(-\frac{1}{7}\right)\ . \tag{5.9}$$

[2]The constant κ determines the Calabi-Yau we are considering where $\kappa = 1$ for the $\mathbb{Z}/10\mathbb{Z}$ quotient and $\kappa = 2$ for the $\mathbb{Z}/5\mathbb{Z}$ quotient.

Now that we have seen the critical L-values of the weight 4 form determine the periods at a rank two attractor point - what about the critical L-value of the weight 2 form? A natural place to look would be in the periods of

$$[D_\varphi\Omega] \in H^{2,1}(X) \ . \tag{5.10}$$

We have seen that the $(2,1)+(1,2)$ part of $H^3(X)$ is associated with a weight 2 form so we might expect to see critical L-values in the expression for $D_\varphi\Pi$. This is indeed the case and we find

$$D_\varphi\Pi\left(-\tfrac{1}{7}\right) \;=\; \frac{3{\cdot}7^2}{2^5\pi^2}\frac{iL_2(1)}{v^\perp}\left\{\begin{pmatrix}-5\kappa\\10\kappa\\-5\\-3\end{pmatrix} - \tau^\perp\begin{pmatrix}-7\kappa\\14\kappa\\-10\\-5\end{pmatrix}\right\} \tag{5.11}$$

where

$$\tau^\perp \;=\; \frac{1}{2} + iv^\perp \tag{5.12}$$

and

$$v^\perp \;=\; 0.37369955695472976699767292752499463211766555651682\ldots \ . \tag{5.13}$$

To understand the significance of $\tau^\perp$, note that the components of $D_\varphi\Pi$ lie on a sublattice of $\mathbb{C}$. We search for minimal generators of this lattice and note that they span a lattice with parameter $\tau^\perp$. We find that this is a lattice wih rational j-invariant given by

$$j(\tau^\perp) \;=\; \left(\frac{215}{28}\right)^3 .$$

LMFDB contains only one elliptic curve defined over $\mathbb{Q}$ with this j-invariant and with the form **14.2.a.a** as its associated weight 2 eigenform. This curve can be defined by the equation

$$y^2 + xy + y \;=\; x^3 + 4x - 6 \ . \tag{5.14}$$

and is indeed the modular curve $X_0(14)$ itself!

In computing the periods and its covariant derivatives, a useful sanity check is provided by the following special geometry identities

$$\int D_\varphi\Omega\wedge\Omega \;=\; 0\,; \quad \int D_\varphi^2\Omega\wedge\Omega \;=\; 0\,; \quad \int \Omega\wedge\Omega''' \;=\; -y \ \text{ and } \int D_\varphi^2\Omega\wedge D\Omega \;=\; -y\,, \tag{5.15}$$

which translate into

$$D_\varphi\Pi^T\Sigma\Pi \;=\; 0\,,\quad D_\varphi^2\Pi^T\Sigma\Pi \;=\; 0\,;\quad \Pi^T\Sigma\Pi''' \;=\; -y \ \text{ and } \ D_\varphi^2\Pi^T\Sigma D_\varphi\Pi \;=\; -y\ . \tag{5.16}$$

Once $\Pi(-\frac{1}{7})$ and $D_\varphi\Pi(-\frac{1}{7})$ are known exactly, it is not too difficult to evaluate various geometric quantities of interest on the complex structure moduli space such as the metric at $\varphi = -\frac{1}{7}$. We collect some of these quantities in Table 5.1.

e^{-K}	K'	K''	$g_{\varphi\bar{\varphi}}$	y	$\Gamma' + \Gamma^2$
$\frac{7^2\kappa}{2\pi^3}L_4(1)L_4(2)$	$-\frac{5\cdot 7}{2^3}$	$-\frac{5\cdot 7^3}{2^6}$	$\frac{3^2 7^3 L_2(1)^2}{2^8 \pi v^{\perp} L_4(1) L_4(2)}$	$-\frac{3\cdot 7^6\,\kappa}{2^{10}(2\pi i)^3}$	$\frac{7}{2^7}(412\Gamma - 1197)$

Table 5.1: *A few geometric quantities evaluated at $\varphi = -\frac{1}{7}$. In this table, Γ denotes the Christoffel symbol $\Gamma^{\varphi}_{\varphi\varphi}$.*

Note that the integral vectors in the expressions $\Pi(-\frac{1}{7})$ and $D_\varphi\Pi(-\frac{1}{7})$ are elements of Λ and $\Lambda^\perp$ respectively which are defined as

$$H^3(X,\mathbb{Z}) \supset \Lambda \oplus \Lambda^\perp \tag{5.17}$$

where Λ and $\Lambda^\perp$ have Hodge numbers $(3,0)+(0,3)$ and $(2,1)+(1,2)$ respectively.
The integral vectors in $\Pi(-\frac{1}{7})$ define a lattice that can be made finer by noting that

$$(8\kappa, -30\kappa, 0, 5) - (0,0,2,1) \;=\; 2\,(4\kappa, -15\kappa, -1,\, 2) \tag{5.18}$$

Similarly, the integral vectors in $D_\varphi\Pi(-\frac{1}{7})$ which define the lattice $\Lambda^\perp$ are as fine as we can make it. These generators are collected in Table 5.2.

Λ	$\Lambda^\perp$
$(4\kappa, -15\kappa, -5, 0)$, $(0, 0, 2, 1)$	$(3\kappa, -6\kappa, 0, 1)$, $(\kappa, -2\kappa, -5, -1)$

Table 5.2: *Generators for the lattices Λ and $\Lambda^\perp$ for the attractor point at $\varphi = -\frac{1}{7}$.*

Since Ω, $D_\varphi\Omega$, $D_{\overline{\varphi}}\overline{\Omega}$ and $\overline{\Omega}$ give the Hodge decomposition of $H^3(X,\mathbb{C})$, by computing $\Pi\left(-\frac{1}{7}\right)$ and $D_\varphi\Pi\left(-\frac{1}{7}\right)$, we have found a (conjecturally) exact expression for the period matrix[3] at the rank two attractor point!
We note that the period matrix can be made block diagonal by choosing a basis of (the torsion free part of) $H_3(X,\mathbb{Z})$ consisting of cycles dual to the four generators in Table 5.2. However, since $\Lambda\oplus\Lambda^\perp$ has index $7^2\kappa^2$ within $H^3(X,\mathbb{Z})$, there is no $Sp(4,\mathbb{Z})$ transformation that will bring you into this basis.

[3] By this, we mean the matrix with columns Π, $D_\varphi\Pi$, $D_{\overline{\varphi}}\overline{\Pi}$ and $\overline{\Pi}$.

Coming back to physics, the lattice Λ is the lattice of electric and magnetic $U(1)^2$ charges of a black hole with attractor point $\varphi = -\frac{1}{7}$. An element in Λ is of the form

$$Q_{k\ell} \;=\; k\,(4\kappa, -15\kappa, -5, 0) + \ell\,(0, 0, 2, 1) \tag{5.19}$$

for some $k, \ell \in \mathbb{Z}$. Equation (2.69) can now be used to find that the black hole with charge $Q_{k\ell}$ will have horizon area given by

$$\frac{A\big(-\frac{1}{7}\big)}{4\pi} \;=\; \frac{(5k-2\ell)^2}{8}\left(\frac{\pi L_4(1)}{L_4(2)}\right) + \frac{49k^2}{2}\left(\frac{\pi L_4(1)}{L_4(2)}\right)^{-1}~. \tag{5.20}$$

We can rewrite (5.7) in terms of the basis vectors of the finer lattice and, in this way, we see that, up to an $SL(2,\mathbb{Z})$ transformation, the lattice has parameter

$$\tau \;=\; -\frac{1}{2} + i v_* \quad \text{with} \quad v_* \;=\; 7\,\frac{L_4(2)}{\pi L_4(1)}~. \tag{5.21}$$

The area of the black hole can be rewritten in a simpler form in terms of v_*

$$A\big(-\tfrac{1}{7}\big) \;=\; 14\pi\left\{k^2 v_* + \left(\ell - \frac{5k}{2}\right)^2 \frac{1}{v_*}\right\}~. \tag{5.22}$$

The parameter τ is a ratio of periods and the periods are, as we have seen, $\mathbb{Q}$-linear in the two quantities $(2\pi i)^{-1}L_4(1)$ and $(2\pi i)^{-2}L_4(2)$. So it is inevitable that τ should be a fractional linear function $(av_* + b)/(cv_* + d)$ of the ratio we have called v_*. For the τ we have chosen, this is just a linear function, but an $\mathrm{SL}(2,\mathbb{Z})$ transform of this would yield a a fractional linear function, in general. The special geometry coordinate t is also a ratio of periods, so has this same general form. In fact we see from (5.7) that

$$t_* \;=\; \frac{1}{2} + \frac{5i}{4v_*}~, \tag{5.23}$$

where we have written $t\big(-\frac{1}{7}\big) = t_*$.

5.1.2 AESZ 34 at $\varphi_\pm = 33 \pm 8\sqrt{17}$

Unsurprisingly, $\sqrt{17}$ appears frequently in this section. Thus, in order to simplify the expressions that follow, we define the algebraic integers (see Appendix B for a quick run through of arithmetic in $\mathbb{Q}(\sqrt{17})$)

$$\epsilon_\pm \;=\; 4 \pm \sqrt{17} \quad \text{and} \quad \delta_\pm \;=\; \frac{3 \pm \sqrt{17}}{2}~. \tag{5.24}$$

The relevant L-functions at $\varphi = 33 \pm 8\sqrt{17}$ have LMFDB designations **34.2.b.a** and **34.4.b.a**. As in the previous section, we denote the corresponding weight-j L-function by $L_j(s)$. These functions are complex but we can concentrate on the real parts since the imaginary parts are simply related to these. We set

$$L_j(s) \;=\; \lambda_j(s) + i\mu_j(s) \tag{5.25}$$

and note that the real parts take the following values at their critical points

$$\begin{aligned} \lambda_4(1) \;&=\; 0.61300748403501690756896255581360559790853555213198\ldots \\ \lambda_4(2) \;&=\; 0.72053904959503349611018739597922735350251006854978\ldots \end{aligned} \tag{5.26}$$

and

$$\lambda_2(1) \;=\; 0.51696098116017249777442349444758176009873137273013\ldots\;. \tag{5.27}$$

At the critical values, the imaginary parts of the L-functions are determined in terms of the real parts up to a sign. This choice follows from the choice of sign in the square root in the Fourier expansion of the weight two form **34.2.b.a** and the weight four forms **34.4.33.a**. With the choices in (4.6) and (4.7), we find that

$$\mu_4(1) \;=\; \left(\frac{1+\sqrt{17}}{4}\right)^3 \lambda_4(1)\;, \qquad \mu_4(2) \;=-\left(\frac{1-\sqrt{17}}{4}\right)\lambda_4(2) \tag{5.28}$$

and

$$\mu_2(1) \;=-\left(\frac{3-\sqrt{17}}{2\sqrt{2}}\right)\lambda_2(1)\;. \tag{5.29}$$

The coefficients in the first two relations are numbers in $\mathbb{Q}(\sqrt{17})$ but the coefficient in the third relation is a number in the quartic extension $\mathbb{Q}(\sqrt{17},\sqrt{2})$.
Just as was the case for $\varphi = -\frac{1}{7}$, we can determine the period matrix at $\varphi = \varphi_\pm$ in terms of L-function values and a single new modular parameter.
By computing the periods and L-function values numerically and comparing them, we find that

$$\begin{aligned} \Pi(\varphi_+) \;&=-\frac{i}{2^5\sqrt{17}\pi}\;\;\delta_-^3\,\lambda_4(1)\begin{pmatrix}-4\kappa\\ 30\kappa\\ 30\\ 5\end{pmatrix} - \frac{\sqrt{17}}{2^3\pi^2}\,\epsilon_-^2\,\delta_+\,\lambda_4(2)\begin{pmatrix}4\kappa\\ -9\kappa\\ 7\\ 4\end{pmatrix} \\ \Pi(\varphi_-) \;&=\; \frac{i}{2^5\sqrt{17}\pi}\,\epsilon_+^3\delta_-^3\,\lambda_4(1)\begin{pmatrix}2\kappa\\ 0\\ 0\\ -5\end{pmatrix} + \frac{\sqrt{17}}{2^3\pi^2}\,\epsilon_+\delta_+\,\lambda_4(2)\begin{pmatrix}0\\ 3\kappa\\ 1\\ 0\end{pmatrix} \end{aligned} \tag{5.30}$$

which we conjecture is an exact expression. Note that the periods at $\varphi = \varphi_\pm$ have the same form as those at $\varphi = -\frac{1}{7}$.

We also find that

$$
\begin{aligned}
D_\varphi \Pi(\varphi_+) \;&=\; -\frac{3}{2^5\pi^2}\,\epsilon_-^4\,\lambda_2(1)\left\{\begin{pmatrix} 9\kappa \\ -16\kappa \\ 20 \\ 9 \end{pmatrix} - \frac{1}{\tau_+^\perp}\begin{pmatrix} 15\kappa \\ -36\kappa \\ 15 \\ 11 \end{pmatrix}\right\} \\
D_\varphi \Pi(\varphi_-) \;&=-\frac{3}{2^6\pi^2}\,\epsilon_+^4\delta_-\,\lambda_2(1)\left\{\begin{pmatrix} 0 \\ -2\kappa \\ 5 \\ 0 \end{pmatrix} + \tau_-^\perp\begin{pmatrix} 3\kappa \\ 0 \\ 0 \\ 1 \end{pmatrix}\right\} .
\end{aligned}
\tag{5.31}
$$

where $v_\pm^\perp$ have the numerical values

$$
\begin{aligned}
v_+^\perp \;&=\; 1.96968944535175054904797169828645169138345314175 17\ldots \\
v_-^\perp \;&=\; 1.01538849422165459167627298688254098649388778807 31\ldots
\end{aligned}
\tag{5.32}
$$

and

$$
\tau_\pm^\perp \;=\; i v_\pm^\perp\ . \tag{5.33}
$$

We find that the two constants $\tau_\pm^\perp$ are related by

$$
\tau_+\tau_- \;=\; -2\ . \tag{5.34}
$$

As before, we note that the four components of $D_\varphi\Pi(\varphi_\pm)$ are valued in a lattice in $\mathbb{C}$ with parameter $\tau_\pm^\perp$ and j-invariant

$$
j(\tau_\pm^\perp) \;=\; \frac{1}{2^4}\epsilon_\pm^4\delta_\mp^2\big(2\mp\sqrt{17}\big)^3\big(14\mp 5\sqrt{17}\big)^3\ , \tag{5.35}
$$

This suggests that there exist conjugate elliptic curve defined over $\mathbb{Q}(\sqrt{17})$ associated with the weight 2 eigenform $f_{\mathbf{34.2.b.a}} \in S_4(\Gamma_1(34))$. A search of the LMFDB reveals the curves $\mathcal{E}_\pm$ listed as 4.1–a8 and defined by the equations

$$
\mathcal{E}_\pm\ : \qquad y^2 + xy + \delta_\pm\, y \;=\; x^3 + \epsilon_\pm\delta_\mp^2\, x^2 - \delta_\mp\, x - \epsilon_\pm\delta_-^2\ , \tag{5.36}
$$

Thus, we interpret the relation (5.34) as the action of the Galois group on the moduli of $\mathcal{E}_\pm$. We cannot resist reproducing a sketch of these curves in Figure 5.1 .

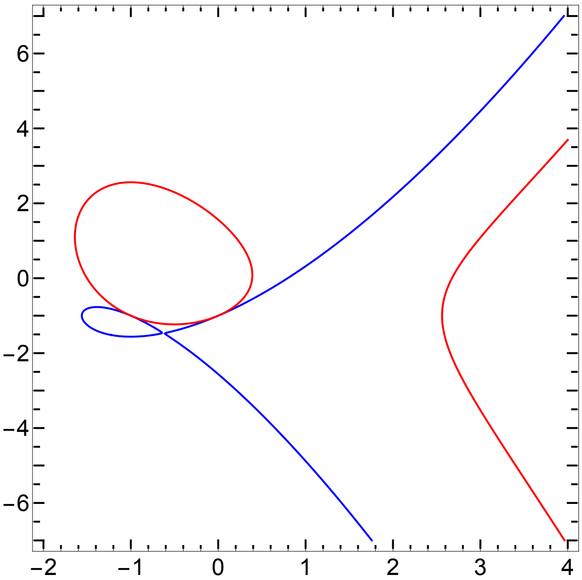

Figure 5.1: *The elliptic curves $\mathcal{E}_\pm$. The curve $\mathcal{E}_+$ is shown in blue while $\mathcal{E}_-$ is shown in red. Despite appearances, the curve $\mathcal{E}_+$ is smooth. Over the reals, this curve has two components and there is a gap where, at this scale, the curve appears to have self intersection.*

A few geometric quantities that can be calculated exactly from the periods are collected in Table 5.3.

	e^{-K}	K'	K''
φ_+	$\frac{17\kappa}{2^6\pi^3}\epsilon_-^2\delta_-^2\lambda_4(1)\lambda_4(2)$	$\frac{5}{2^3\sqrt{17}}\epsilon_-^2(2+\sqrt{17})$	$-\frac{5}{2^6.17}\epsilon_-^4(135+16\sqrt{17})$
φ_-	$\frac{17\kappa}{2^6\pi^3}\epsilon_+^4\delta_-^2\lambda_4(1)\lambda_4(2)$	$-\frac{5}{2^3\sqrt{17}}\epsilon_+^2(2-\sqrt{17})$	$-\frac{5}{2^6.17}\epsilon_+^4(135-16\sqrt{17})$
	$g_{\varphi\bar\varphi}$	y	$\Gamma'+\Gamma^2$
φ_+	$\frac{9}{2^6\pi}\epsilon_-^6\delta_+^2\frac{\lambda_2(1)^2v_-^\perp}{\lambda_4(1)\lambda_4(2)}$	$\frac{3\kappa}{2^{11}(2\pi i)^3}\epsilon_-^{10}\delta_-$	$\frac{\epsilon_-\delta_-}{2^6\sqrt{17}}(2-\sqrt{17})(8-\sqrt{17})(21+2\sqrt{17})\,\Gamma_+ + \frac{\epsilon_-^3\delta_-}{2^8\cdot17}(9-4\sqrt{17})(206+21\sqrt{17})$
φ_-	$\frac{9}{2^5\pi}\epsilon_+^4\frac{\lambda_2(1)^2v_-^\perp}{\lambda_4(1)\lambda_4(2)}$	$\frac{3\kappa}{2^{11}(2\pi i)^3}\epsilon_+^{10}\delta_+$	$-\frac{\epsilon_+\delta_+}{2^6\sqrt{17}}(2+\sqrt{17})(8+\sqrt{17})(21-2\sqrt{17})\,\Gamma_- + \frac{\epsilon_+^3\delta_+}{2^8\cdot17}(9+4\sqrt{17})(206-21\sqrt{17})$

Table 5.3: *A few geometric quantities evaluated at φ_+ and φ_-. In this table $\Gamma_\pm$ denotes the Christoffel symbols $\Gamma^\varphi_{\varphi\varphi}(\varphi_\pm)$.*

We can identify the generators of $\Lambda_\pm$ and $\Lambda_\pm^\perp$ at $\varphi_\pm$ from Equations (5.30) and (5.31). They are listed in Table 5.4.

	$\Lambda_{\pm}$	$\Lambda_{\pm}^{\perp}$
φ_+	$(4\kappa, -9\kappa, 7, 4),\ (4\kappa, -30\kappa, -30, -5)$	$(6\kappa,\ -20\kappa\ -5, 2),\ (3\kappa, 4\kappa, 25, 7)$
φ_-	$(-2\kappa, 0, 0, 5),\ (0, 3\kappa, 1, 0)$	$(0, -2\kappa, 5, 0),\ (3\kappa, 0, 0, 1)$

Table 5.4: *Generators for the lattices $\Lambda_{\pm}$ and $\Lambda_{\pm}^{\perp}$*

As before, we note that the period matrix can be made block diagonal by choosing a basis of (the torsion free part of) $H_3(X, \mathbb{Z})$ consisting of cycles dual to the four generators in Table 5.4. However, there is no $Sp(4, \mathbb{Z})$ transformation that takes you into this basis since the lattices $\Lambda_+ \oplus \Lambda_+^{\perp}$ and $\Lambda_- \oplus \Lambda_-^{\perp}$ in $H^3(X_{\varphi_\pm}, \mathbb{Z})$ both have index $17^2\kappa^2$.
By taking combinations of generators of $\Lambda_{\pm}$ with coefficients k and ℓ as our charge vector, we can calculate the area of the horizon of the black hole

$$\frac{A(\varphi_\pm)}{4\pi} \;=\; \frac{k^2}{32}(9+\sqrt{17})\left(\frac{\pi\lambda_4(1)}{\lambda_4(2)}\right) + \frac{(17\ell)^2}{8}(9-\sqrt{17})\left(\frac{\pi\lambda_4(1)}{\lambda_4(2)}\right)^{-1} . \tag{5.37}$$

It is a surprising fact that the black holes associated with φ_- and φ_+ have the same horizon areas. This is related to the fact that the expressions for the periods $\Pi(\varphi_\pm)$ in (5.30) are remarkably similar. The coefficients multiplying the charge vectors are related by a factor of $-\epsilon_+^3$. The parameters for these two lattices are therefore the same. Let us denote this parameter by τ and write

$$\tau \;=\; iv\ ; \quad \text{with} \quad v \;=\; \frac{17}{4}(9-\sqrt{17})\,\frac{\lambda_4(2)}{\pi\lambda_4(1)} \;=\; \frac{17}{2}\epsilon_-^2\delta_-^4\,\frac{\lambda_4(2)}{\pi\lambda_4(1)} . \tag{5.38}$$

We find that the area can, analogously to the case of the attractor point at $\varphi = -1/7$, be written very succinctly in terms of v

$$A(\varphi_\pm) \;=\; 34\pi\left(\frac{k^2}{v} + \ell^2 v\right) . \tag{5.39}$$

5.1.3 Identifying Higher Derivatives

In the preceding sections, we chose to work with the covariant derivaties of Π instead of the ordinary derivatives. We do this for two reasons: the first is that we obtain cleaner expressions. This is due to the fact that Ω takes values in $H^{3,0}$ and, owing to special geometry relations, $D_\varphi\Omega$ takes values purely in $H^{2,1}$, while $\partial_\varphi\Omega$ takes values in $H^{3,0} \oplus H^{2,1}$. It follows that the periods of Ω can be expressed purely in terms of weight four L-values and the periods of $D_\varphi\Omega$ only depend on weight two L-values and the modulus of the relevant elliptic curve. Had we instead computed the periods of $\partial_\varphi\Omega$, we would have found that they mix the weight two L-values with the weight four L-values. The second reason covers for our ignorance; had we calculated the partial derivatives, or even the covariant derivatives beyond those shown in the table, we would come across unidentified numbers. That is, numbers we are unable to express in terms of L-values or any other presumably transcendental number that can compute independently of Π. This happens first in evaluating $\partial_\varphi^2\Pi_j$. We can apportion the blame for this in various ways. We find that we need six numbers in order to evaluate any derivative of $\partial_\varphi^2\Pi_j$ at a rank two attractor point. Whereas, we have at our disposal only four. Namely, $(2\pi i)^{-1}L_4(1)$, $(2\pi i)^{-2}L_4(2)$, $(2\pi i)^{-2}L_2(1)$ and the modulus of the relevant elliptic curve. There are two numbers that we are unable to identify and, at $\varphi = -\frac{1}{7}$, we can take these to be

$$\begin{aligned} \partial_{\bar\varphi}\partial_\varphi{}^2 K(-\tfrac{1}{7}) &= 13.39575666237991448474040454080284935049142 56\ldots \\ \partial_\varphi^3 K(-\tfrac{1}{7}) &= -345.296197568387252384535830788469867726435775\ldots \end{aligned} \tag{5.40}$$

Similarly, at φ_+, we can take the unidentified numbers to be

$$\begin{aligned} \partial_{\bar\varphi}\partial_\varphi{}^2 K(\varphi_+) &= -2.112480928128536598319217958868136916857913 40\ldots \times 10^{-8} \\ \partial_\varphi^3 K(\varphi_+) &= 6.412991577460653039633428801773164395517925 91\ldots \times 10^{-6}\ . \end{aligned} \tag{5.41}$$

Finally, at φ_-, the unknown numbers can be taken to be

$$\begin{aligned} \partial_{\bar\varphi}\partial_\varphi{}^2 K(\varphi_-) &= -9401.327202723069828967614139540831536264164 9\ldots \\ \partial_\varphi^3 K(\varphi_-) &= 170631.685809372752493637298347668593555721135\ldots\ . \end{aligned} \tag{5.42}$$

Given $\partial_{\bar\varphi}\partial_\varphi{}^2 K$ and $\partial_\varphi^3 K$ at a rank two attractor point, we can identify all the second and third derivatives at that point. All the higher derivatives are then fixed by invoking the Picard-Fuchs equation. We could then, for example, identify all the coefficients in an expansion of the periods about the rank two attractor points. This leads to the expansion of topological string free energies in Chapter 8.

In the final stages of preparing [16], we were informed by Bönisch and Klemm [15] that they were able to express the second and third derivatives at $\varphi = -\frac{1}{7}$, and so the unrecognised numbers above, in terms of periods and quasi-periods of the associated weight two and weight four forms. Thus, we are able to expand the periods in a neighbourhood of a rank two attractor point which, in turn leads to an expansion of the genus g topological string free energy around rank two attractor points. This will be explored in Chapter 8. First, we explain a formalism for computing periods of modular forms.

5.2 Period Polynomials

5.2.1 Definitions

Period polynomials provide a formalism for computing integrals of modular forms on the upper half plane and modular curves that was used by Bönisch and Klemm [15] to determine the second and third derivatives of the periods of the holomorphic three form at a rank two attractor (in addition to the periods and the first derivative). As previously discussed, this determines all the higher derivatives via the Picard-Fuchs equation and may be used to expand the periods in a neighbourhood of a rank two attractor. Their result is an extension of unpublished work by Klemm, Scheidegger and Zagier [67] and a discussion of how it relates to AESZ 34 may be found in the masters thesis of Bönisch [63]. We summarise the essential points below.

In order to define period polynomials, we must first define Eichler integrals. As in previous chapters, we use the slash operator on modular forms which, for a smooth function $f : \mathbb{H} \to \mathbb{C}$ and any $\gamma \in SL(2,\mathbb{R})$, is given by

$$(f|_n\gamma)(\tau) = (c\tau + d)^{-n} f(\gamma\tau) \tag{5.43}$$

If $f \in S_k(\Gamma_0(N))$ we say that $\widetilde{f}$ is an Eichler integral of f if

$$\left(\frac{1}{2\pi i}\frac{\mathrm{d}}{\mathrm{d}\tau}\right)^{k-1} \widetilde{f} \;=\; f \tag{5.44}$$

which is unique up to some the addition of some

$$p_{\tau_0} \in V_{k-2}(\mathbb{C}) \;=\; \{p \in \mathbb{C}[\tau] \mid \deg p \le k-2\} \ . \tag{5.45}$$

By differentiating under the integral sign, we see that we may represent an Eichler integral as

$$\widetilde{f}(\tau) = \frac{(2\pi i)^{k-1}}{\Gamma(k-1)} \int_{\tau_0}^{\tau} (\tau - z)^{k-2} f(z)\mathrm{d}z \tag{5.46}$$

for some choice of basepoint $\tau_0 \in \mathbb{H}$. We define the associated period polynomial for $\gamma \in \Gamma_0(N)$ as

$$r_f(\gamma) \;=\; \widetilde{f}|_{2-k}(\gamma - 1) \;=\; \frac{(2\pi i)^{k-1}}{\Gamma(k-1)} \int_{\tau_0}^{\gamma^{-1}\tau_0} (\tau - z)^{k-2} f(z)\mathrm{d}z \ . \tag{5.47}$$

In other words, to a cusp form $f \in S_k(\Gamma_0(N))$, we assing a map

$$r_f : \Gamma_0(N) \to V_{k-2}(\mathbb{C}) \tag{5.48}$$

which is uniquely determined by the polynomials it assigns to the finite set of generators of $\Gamma_0(N)$. This follows from the observation

$$r_f(\gamma_1\gamma_2) \;=\; r_{\widetilde{f}}(\gamma_1)|_{2-k}\gamma_2 + r_{\widetilde{f}}(\gamma_2) \ . \tag{5.49}$$

We see that a cusp form $f \in S_k(\Gamma_0(N))$ determines a finite collection of polynomials in $V_{k-2}(\mathbb{C})$ so one naturally wonders - can we determine a cusp form $f \in S_k(\Gamma_0(N))$ from a finite collection of polynomials? This turns out to be possible and it motivates the definition of parabolic cohomology. For starters, we define a set of cocyles as

$$Z^1(\Gamma_0(N)) \;=\; \{r : \Gamma_0(N) \to V_{k-2}(\mathbb{C}) \,|\, r(\gamma_1\gamma_2) = r(\gamma_1)|_{2-k}\gamma_2 + r(\gamma_2)\} \tag{5.50}$$

and a set of coboundaries as

$$B^1(\Gamma_0(N)) \;=\; \{r : \Gamma_0(N) \to V_{k-2}(\mathbb{C}) \,|\, r(\gamma) = p|_{2-k}(\gamma - 1) \text{ where } p \in V_{k-2}(\mathbb{C})\} \tag{5.51}$$

which is motivated by the fact that the Eichler integral is only defined up to the addition of $p \in V_{k-2}(\mathbb{C})$.
For a fixed $\gamma \in \Gamma_0$, one can check that there exists a choice of base point τ_0 such that

$$r_f(\gamma) \in V_{k-2}(\mathbb{C})|_{2-k}(\gamma - 1)\ . \tag{5.52}$$

Since a change of base point is equivalent to the addition of some element of $p \in V_{k-2}(\mathbb{C})$, we may represent $r_{\tilde{f}}$ as a cohomology class by a parabolic cycle which is defined as

$$Z^1_{\text{par}}(\Gamma_0(N)) \;=\; \{r \in Z^1(\Gamma_0(N)) \,|\, |\text{Tr}\,\gamma| = 2 \implies r(\gamma) \in V_{k-2}(\mathbb{C})|_{2-k}(\gamma - 1)\}\ . \tag{5.53}$$

As indicated above, it turns out to be enough to impose (5.52) for $\gamma \in \Gamma_0(N)$ such that $|\text{Tr}\,\gamma| = 2$. Finally, we come to a theorem of Eichler [68] which states

$$S_k(\Gamma_0(N)) \oplus \overline{S_k(\Gamma_0(N))} \cong H^1_{\text{par}}(\Gamma_0(N)) \tag{5.54}$$

where one of the $S_k(\Gamma_0(N))$ terms is the image of

$$f \mapsto r_f \tag{5.55}$$

in cohomology and the other is its complex conjugate.

5.2.2 Hecke Operators and Periods of Modular Forms

As discussed in Chapter 3.3, there are a variety of interesting operators that act on the space of cusp forms $S_k(\Gamma_0(N))$ such as Hecke operators, Atkin-Lehner involutions, etc and the isomorphism in (5.54) suggests that it is possible to describe the action of these operators on period polynomials.
In order to define the action of a Hecke operator $T_n : S_k(\Gamma_0(N)) \to S_k(\Gamma_0(N))$ on a period polyomial $r_f(\gamma) \in V_{k-2}(\mathbb{C})$, we first fix a set of representatives of M_i of $\Gamma_0(N)\mathcal{M}_{n,m}$ and $M_{\pi_\gamma(i)}$ where $\pi_\gamma(i)$ is a permutation such that

$$M_i\gamma \;=\; \gamma_i M_{\pi_\gamma(i)} \tag{5.56}$$

where $\gamma_i \in \Gamma_0(N)$. We use this to define

$$r_f|_{2-k}T_N \;=\; \sum_{i=1}^{\sigma_1(n)} r_f(\gamma_i)|_{2-k}M_{\pi_\gamma(i)} \tag{5.57}$$

which agrees with the action of the Hecke operators in the sense that

$$r_f|_{2-k}T_N \;=\; r_{f|_kT_n}\;. \tag{5.58}$$

Similarly, Atkin-Lehner involutions W_Q act on period polynomials via $r_{\widetilde{f}} \to r_{\widetilde{f}}|_{2-k}W_Q$ which translates to

$$r_{f|_kW_Q} \;=\; r_{f|_kW_Q}(W_Q\gamma W_Q^{-1})|_{2-k}W_Q \tag{5.59}$$

We may also define the action complex conjugation on period polynomials via

$$\overline{r}_{\overline{f}} = (-1)^{k+1}r_f|_{2-k}\epsilon \tag{5.60}$$

where $\epsilon = \mathrm{diag}(-1,1)$. This action commutes with the action of Hecke operators and may be used to decompose a given period polynomial r_f according to its eigenvalues as

$$r_f \;=\; r_f^+ + r_f^- \tag{5.61}$$

By insisting that a given period polynomial has the same eigenvalues as a Hecke eigenform f under the action of Hecke operators, it is possible to find a set of representative period polynomials for a given eigenform f. Moreover, these representatives can be chosen so that

$$r_f(\gamma) = \omega^+ r_f^+(\gamma) + \omega^- r_f^-(\gamma) \tag{5.62}$$

where $r_f^{\pm}(\gamma) \in V_{k-2}(\mathbb{Q})\forall\gamma \in \Gamma_0(N)$ (it is enough to check this for the generators of $\Gamma_0(N)$) [67]. The complex numbers ω^+ and ω^- are known as *periods of modular forms* and in many cases are equal to a rational multiple of critical L-values (up to factors of $2\pi i$).
Finally, we point out that the formalism of period polynomials also applies to certain meromorphic modular forms (say F) with the property that the integral

$$\int_{\tau_0}^{\tau} (\tau - z)^{k-2} F(z)\mathrm{d}z \tag{5.63}$$

is independent of the path of integration The formalism of period polynomials works for meromorphic forms in much the same way as it does for holomorphic modular forms forms and for a given eigenform f, we may find a meromorphic form F with period polynomial

$$r_F(\gamma) = \eta^+ r_f^+(\gamma) + \eta^- r_f^-(\gamma) \tag{5.64}$$

where the complex numbers $\eta^{\pm}$ are periods of F and $r_f^{\pm}(\gamma) \in V_{k-2}(\mathbb{Q})$ are the same period polynomials as for the holomorphic eigenform f. The complex numbers $\eta^{\pm}$ are known as *quasi-periods* of f.

Note that the normalisation of both $\omega^{\pm}$ and $\eta^{\pm}$ are ambiguous up to a rational multiple. Moreover, since adding some multiple of f to F does not change its Hecke eigenvalue, F is defined only up to the addition of some multiple of f. Similarly, $\eta^{\pm}$ is only defined up to the addition of $\omega^{\pm}$. For a suitable choice of F, the periods satisfy the Legendre relation

$$\omega^{+}\eta^{-} - \omega^{-}\eta^{+} \in (2\pi i)^{k-1}\mathbb{Q} \tag{5.65}$$

A more detailed explanation of period polynomials and the associated periods may be found in [63, 67].

5.2.3 Periods at $\varphi = -1/7$

We list below the transition matrix T as it is found in the thesis of Bönisch [63] where $\Pi = T\varpi^{(-\frac{1}{7})}$ where $\varpi^{(-\frac{1}{7})}$ is a vector of periods in Frobenius basis around $\varphi = -\frac{1}{7}$. The interested reader may find more detail and the transition matrices to the irrational attractors in [63]. This determines the periods at $\varphi = -\frac{1}{7}$ along with its first, second and third derivatives. All the higher derivatives are then determined by the Picard-Fuchs equation.

$$T = \begin{pmatrix} -8\kappa & 0 & 343\kappa & 147\kappa \\ 0 & 0 & -686\kappa & -294\kappa \\ 30\kappa & 4 & 490 & 0 \\ -5 & 2 & 245 & 49 \end{pmatrix} \begin{pmatrix} \frac{\omega_4^+}{(2\pi i)^3} & \frac{\eta_4^+}{(2\pi i)^3} & 0 & 0 \\ \frac{\omega_4^-}{(2\pi i)^3} & \frac{\eta_4^-}{(2\pi i)^3} & 0 & 0 \\ 0 & 0 & \frac{\omega_2^+}{(2\pi i)^2} & \frac{\eta_2^+}{(2\pi i)^2} \\ 0 & 0 & \frac{\omega_2^-}{(2\pi i)^2} & \frac{\eta_2^-}{(2\pi i)^2} \end{pmatrix} \begin{pmatrix} -\frac{1}{6} & -\frac{35}{48} & -\frac{245}{64} & 0 \\ 0 & 0 & 0 & -\frac{1715}{98304} \\ 0 & \frac{1}{8} & 0 & -\frac{15337}{3072} \\ 0 & 0 & \frac{21}{1024} & \frac{7987}{32768} \end{pmatrix} \tag{5.66}$$

where $\Pi(\varphi) = T\varpi^{(-\frac{1}{7})}(\varphi)$ where $\varpi^{(-\frac{1}{7})}$ is a vector of periods in Frobenius basis around $\varphi = -\frac{1}{7}$. The periods $\omega_2^{\pm}$ and $\omega_4^{\pm}$ are determined by the modular forms $f_{\mathbf{14.2.a.a}} \in S_2(\Gamma_0(14))$ and $f_{\mathbf{14.4.a.a}} \in S_4(\Gamma_0(14))$ as in previous sections and have the numerical values

$$\begin{aligned} \omega_2^+ &= 0.99067097803344161708478583836850463262135702234 56\ldots \\ \omega_2^- &= i\,1.32549123968248671433496629202941907167283153777 8\ldots \\ \omega_4^+ &= 79.9394368963416219591281158865263007534884560872 39\ldots \\ \omega_4^- &= i\,242.599333969849377044966505399978209279764512854 4\ldots\,. \end{aligned} \tag{5.67}$$

Similarly, the quasi-periods $\eta_2^{\pm}$ and $\eta_4^{\pm}$ take the numerical values

$$\begin{aligned} \eta_2^+ &= 44.0832622333172101044404274986889695670162700272 21\ldots \\ \eta_2^- &= i\,71.6669308938313236762636752104501910317921421731 2\ldots \\ \eta_4^+ &= 99735.2580408276270951937529729909199803781241033 03\ldots \\ \eta_4^- &= i\,300486.016364598410131072369821489034298020691199 7\ldots\,. \end{aligned} \tag{5.68}$$

and are determined by the weight 2 and 4 meromorphic forms

$$F_2 \; = \; \frac{G_2}{(f_{\mathbf{14.2.a.a}})^2} + \frac{181}{3} f_{\mathbf{14.2.a.a}} \tag{5.69}$$

and

$$F_4 \; = \; \frac{G_4}{(f_{\mathbf{14.2.a.a}})^4} + 1267 f_{\mathbf{14.4.a.a}} \; . \tag{5.70}$$

respectively where $G_2 \in S_6(\Gamma_0(14))$ and $G_4 \in S_{12}(\Gamma_0(14))$ which are uniquely determined by the Fourier expansions

$$\begin{aligned} G_2 &= q - 2q^2 - 3q^3 + 53q^4 + 107q^5 - 210q^6 + 49q^7 + 117q^8 + \dots \\ G_4 &= 8q^2 - 35q^3 - 4q^4 + 198q^5 + 734q^6 + 2062q^7 + 1442q^8 + 9873q^9 + 35118q^{10} \\ &\quad - 56083q^{11} + 27856q^{12} - 182362q^{13} - 51976q^{14} - 368969q^{15} + 83904q^{16} - 68498q^{17} \\ &\quad + 288580q^{18} 430179q^{19} + 1741480q^{20} + \dots \; . \end{aligned} \tag{5.71}$$

Note that neither G_2 nor G_4 is an eigenform.

The periods and quasi-periods satisfy the Legendre relations

$$\omega_4^+ \eta_4^- - \omega_4^- \eta_4^+ \; = \; \left(\frac{3528}{5}\right)(2\pi i)^3 \qquad \text{and} \qquad \omega_2^+ \eta_2^- - \omega_2^- \eta_2^+ \; = \; 2(2\pi i) \; . \tag{5.72}$$

It is straightforward to confirm that

$$\begin{aligned} \frac{\omega_4^+}{(2\pi i)^3} &= -3 \frac{L_4(1)}{2\pi i} \\ \frac{\omega_4^-}{(2\pi i)^3} &= 42 \frac{L_4(2)}{(2\pi i)^2} \; . \end{aligned} \tag{5.73}$$

Similarly,

$$\begin{aligned} \frac{\omega_2^+}{(2\pi i)^2} &= 4 \frac{L_2(1)}{(2\pi i)^2} \\ \frac{\omega_2^-}{(2\pi i)^2} &= -\frac{3}{2iv^\perp} \frac{L_2(1)}{(2\pi i)^2} \; . \end{aligned} \tag{5.74}$$

In other words, (5.66) implies the identities for $\Pi\left(-\frac{1}{7}\right)$ and $D_\varphi \Pi\left(-\frac{1}{7}\right)$ found in Section 5.1.1. In addition, thanks to the quasi-periods, we now find that

$$\begin{aligned} \partial_{\bar\varphi} \partial_\varphi^2 K(-\tfrac{1}{7}) &= \frac{151263\pi^2}{1024} \left(\frac{3\omega_2^+ \eta_2^- + 3\omega_2^- \eta_2^+ - 160\omega 2_+ \omega_2^-}{\omega_4^+ \omega_4^-} \right) \\ \partial_\varphi^3 K(-\tfrac{1}{7}) &= \frac{111475}{256} - \frac{5145}{16384} \left(\frac{\omega_4^+ \eta_4^- + \omega_4^- \eta_4^+}{\omega_4^+ \omega_4^-} \right) . \end{aligned} \tag{5.75}$$

Chapter 6

Rank Two Attractor as Fixed Point of Involution

In this chapter, we will describe a class of Picard-Fuchs operators of order four constructed by taking certain products of Picard-Fuchs operators of order two. These are the so-called *Hadamard products* of Picard-Fuchs equations that we explain below. A more detailed discussion of the geometric interpretation of a Hadamard product can be found in [69].

6.1 Hadamard Products

In their effort to find new fourth order differential operators of "Calabi-Yau type", Almkvist and Zudilin (on the advice of van Straten) began to consider the *Hadamard Product* of certain second order operators in [70].[1]

Suppose $\mathcal{L}_X$ and $\mathcal{L}_Y$ are Picard-Fuchs operators with holomorphic solutions

$$\varpi_X(\varphi) \;=\; \sum_{n=0}^{\infty} a_n \varphi^n \qquad \text{and} \qquad \varpi_Y(\varphi) \;=\; \sum_{n=0}^{\infty} b_n \varphi^n \tag{6.1}$$

respectively. A Hadamard product $\mathcal{L}_X * \mathcal{L}_Y$ is then an operator that annihilates

$$(\varpi_X * \varpi_Y)(\varphi) \;=\; \sum_{n=1}^{\infty} a_n b_n \varphi^n. \tag{6.2}$$

Many of the operators in the AESZ list [2] are of this type. More generally, if ϖ_X and ϖ_Y satisfy a differential equation of finite order with polynomal coefficients, then $\varpi_X * \varpi_Y$ will also satisfy a differential equation of finite order with polynomial coefficients. Unfortunately, there is no general algorithm for computing $\mathcal{L}_X * \mathcal{L}_Y$ and there might be many

[1] By this, we mean that the operator satisfies a number of conditions that we expect the Picard-Fuchs equation of Calabi-Yau manifold to satisfy. For example, it must have a point of maximal unipotent monodromy with an integral mirror map and (up to an overall rational scale) integral Gromov-Witten invariants at genus 0. See [71] for a list of conditions.

operators annihilating $\varpi_X * \varpi_Y$. However, in practice, there is often a natural choice that is straightforward to compute. One simply makes an ansatz

$$\mathcal{L}_X * \mathcal{L}_Y \;=\; S_4\theta^4 + S_3\theta^3 + S_2\theta^2 + S_1\theta + S_0, \tag{6.3}$$

where $\theta = \varphi\frac{\mathrm{d}}{\mathrm{d}\varphi}$ and S_i are polynomials in φ. The coefficients in the polynomials S_i are then determined by requiring that the coefficients of φ in

$$(\mathcal{L}_X * \mathcal{L}_Y)(\varpi_X * \varpi_Y)(\varphi) \tag{6.4}$$

all vanish. By expanding (6.4) to a sufficiently high order and then incrementally increasing the degree of the polynomials S_i, we eventually find a solution. In all of the examples described below, there is a unique operator (up to some overall multiple) of minimal degree in the polynomials S_i. This is the operator that we will refer to as the Hadamard product $\mathcal{L}_X * \mathcal{L}_Y$.

When $\mathcal{L}_X$ and $\mathcal{L}_Y$ are each the Picard-Fuchs operator of some family of hypersurfaces, a simple geometric interpretation of $\mathcal{L}_X * \mathcal{L}_Y$ can be found in the thesis of Samol [72] that we now describe. More generally, the Hadamard product describes the *join* of two varieties as is explained in [69].

Suppose $\mathcal{L}_{E_1}$ and $\mathcal{L}_{E_2}$ describe the variation of Hodge structure of families of elliptic curves

$$E_1 \to \mathbb{P}^1\backslash\Sigma_1 \qquad \text{and} \qquad E_2 \to \mathbb{P}^1\backslash\Sigma_2 \tag{6.5}$$

where Σ_1 and Σ_2 are discriminant loci and E_1 and E_2 are defined by the Laurent polynomials

$$F(x) - s = 0 \qquad \text{and} \qquad G(y) - t = 0 \tag{6.6}$$

where $s \in \mathbb{P}^1\backslash\Sigma_1$ and $t \in \mathbb{P}^1\backslash\Sigma_2$ are parameters of each family.

We write $E_1(s)$ and $E_2(t)$ for the fibres and construct the family

$$E_1 * E_2 \to \mathbb{P}^1\backslash\Sigma_1\Sigma_2 \tag{6.7}$$

where $\Sigma_1\Sigma_2$ is the set consisting of 0, ∞ and the pairwise products of all the singularities Σ_1 and Σ_2 that are not 0 or ∞. The fibre $(E_1 * E_2)(\varphi)$ is the threefold defined by the Laurent polynomials

$$\begin{aligned} 0 &= F(x) - s \\ 0 &= G(y) - t \\ 0 &= st - \varphi \end{aligned} \tag{6.8}$$

where $\varphi \in \mathbb{P}^1\backslash\Sigma_1\Sigma_2$.

The relevance of the above construction is that $(\varpi_{E_1} * \varpi_{E_2})(\varphi)$ is a period of $(E_1 * E_2)(\varphi)$. To see this suppose that

$$\varpi_{E_1}(s) = \int_{T(\gamma_1)} \frac{\omega_1}{F(x) - s} = \sum_{m=0}^{\infty} a_m s^m \tag{6.9}$$

and

$$\varpi_{E_2}(t) = \int_{T(\gamma_2)} \frac{\omega_2}{G(y) - t} = \sum_{n=0}^{\infty} b_n t^n \tag{6.10}$$

where γ_1 and γ_2 are 1-cycles on $E_1(s)$ and $E_2(t)$ respectively, T is the Leray coboundary map that assigns to each 1-cycle a tube around it in some ambient space and ω_i are differential forms on that ambient space. If $S^1 \times S^1$ is a small tube around $st - \varphi = 0$, we can compute

$$\begin{aligned} (\varpi_{E_1} * \varpi_{E_2})(\varphi) &= \frac{1}{(2\pi i)^2} \int_{T(\gamma_1)\times T(\gamma_2)\times S^1\times S^1} \frac{\omega_1 \wedge \omega_2 \wedge \mathrm{d}s \wedge \mathrm{d}t}{(F(x)-s)(G(y)-t)(st-\varphi)} \\ &= \frac{1}{(2\pi i)^2} \sum_{m,n=0}^{\infty} a_m b_n \int_{S^1\times S^1} \frac{\mathrm{d}s \wedge \mathrm{d}t}{st - \varphi} \\ &= \sum_{n=0}^{\infty} a_n b_n \varphi^n \end{aligned} \tag{6.11}$$

for small enough φ.

It is not generally true that $(E_1 * E_2)(\varphi)$ can be resolved to a one-parameter Calabi-Yau manifold. Indeed, in some of the following examples, we will be able to rule this out. Nevertheless, these examples will provide many interesting examples of Picard-Fuchs equations with attractor points of rank two and two examples (AESZ 100 and AESZ 101) are known to be the Picard-Fuchs equations of Calabi-Yau threefolds.

Smooth Calabi-Yau models for Hadamard products of some second order Picard-Fuchs equations and an algorithm for finding them can be found in [73].

6.2 Involutions and Apparent Singularities

There are many interesting examples of second order Picard-Fuchs operators that lead to Picard-Fuchs equations of "Calabi-Yau Type". We will focus on ten operators described in [70] that we know lead to fourth order operators of "Calabi-Yau Type". These are listed in Appendix C. We use the naming convention of [70] and list the operators as $(a) - (j)$.
An observation that is important for what follows is that all of the operators listed in Appendix C admit the action of an involution. Consider for example operator (a) given by

$$\mathcal{L}_{(a)} \; = \; (1+\varphi)(-1+8\varphi)\,\theta^2 + \varphi(7+16\varphi)\,\theta + 2\varphi(1+4\varphi) \tag{6.12}$$

where $\theta = \varphi\frac{\mathrm{d}}{\mathrm{d}\varphi}$ which has the Riemann symbol

$$\mathcal{P}\left\{\begin{array}{cccc} -1 & 0 & \frac{1}{8} & \infty \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right\} . \tag{6.13}$$

It is straightforward to check that, if $\varpi(\varphi)$ is a solution of $\mathcal{L}_{(a)}$ then so is $\frac{1}{8\varphi}\varpi(\frac{1}{8\varphi})$ and, since the scaling of the periods is just a gauge transformation, this suggests that the family of elliptic curves described by $\mathcal{L}_{(a)}$ is invavriant under the involution

$$\varphi \to \frac{1}{8\varphi} \; . \tag{6.14}$$

Now suppose that that the two families of elliptic curves in (6.5) admit such an involution i.e.

$$E_1(s) \;\cong\; E_1\!\left(\frac{1}{\alpha_1 s}\right) \qquad \text{and} \qquad E_2(t) \;\cong\; E_2\!\left(\frac{1}{\alpha_2 t}\right) \tag{6.15}$$

for some $\alpha_1, \alpha_2 \in \mathbb{Q}$ and $s \in \mathbb{P}^1\backslash\Sigma_1$ and $t \in \mathbb{P}^1\backslash\Sigma_2$. This would lead to yet another involution

$$(E_1 * E_2)(\varphi) \to (E_1 * E_2)\!\left(\frac{1}{\alpha_1\alpha_2\varphi}\right) \tag{6.16}$$

which applies an involution to each of the elliptic curves at the same time. This would be visible in the Picard-Fuchs equation $\mathcal{L}_{E_1} * \mathcal{L}_{E_2}$.
Suppose φ_* is one of the fixed points of (6.16) and suppose further that $(E_1 * E_2)(\varphi)$ can be resolved to a smooth Calabi-Yau manifold $(\widetilde{E_1 * E_2})(\varphi)$with $h^{2,1} = 1$. This would imply the existence of a linear map

$$A \;:\; H^3((\widetilde{E_1 * E_2})(\varphi_*), \mathbb{Z}) \to H^3((\widetilde{E_1 * E_2})(\varphi_*), \mathbb{Z}) \; . \tag{6.17}$$

As we shall see, the map A can be computed from the Picard-Fuchs equation in a straightforward manner. Moreover, we shall see examples where A splts $H^3((\widetilde{E_1 * E_2})(\varphi_*), \mathbb{Q})$ into two eigenspaces with positive and negative eigenvalues with Hodge numbers $(3,0)+(0,3)$

and $(2,1)+(1,2)$ parts of H^3 respectively. Thus, making the smooth fixed point φ_* a rank two attractor point.

The results of [39] suggest that conifold points defined over real quadratic extensions of $\mathbb{Q}$ are associated with Hilbert modular forms of parallel weight 4. Although, no examples were identified, it is conjectured that the conifolds at imaginary quadratic values of φ are associated with *Bianchi modular forms*.[2] To avoid such subtleties at the fixed points, we restrict our attention to the Hadamard products with a fixed point at a rational value value of the parameter $\varphi \in \mathbb{Q}$. It is only at these points that the results of [13] guarantee modularity.

For simplicity, we will restrict our attention further yet to the Hadamard squares. The results of Appendix D show that only two of the operators in Appendix C can square to the Picard-Fuchs operator of a family of Calabi-Yau manifolds with $h^{2,1}=1$. Namely, $(a)*(a)$ and $(b)*(b)$ which lead to operators AESZ 100 and AESZ 101. Both operators are known to be the Picard-Fuchs equations of families of smooth Calabi-Yau threefolds [75].

Remarkably, even though we rule out in Appendix D the possibility that the Hadamard squares of operators $(c)-(j)$ describe one-parameter Calabi-Yau manifolds, the methods of the the following section seem to apply equally well to these opeartors. Not only do we find attractor points, we are also able to identify the associated modular forms.

[2] The definition of a Bianchi modular form is more involved and may be found in [74] or the LMFDB [49].

6.3 Examples

6.3.1 AESZ 100

As previously mentioned, the Hadamard square $\mathcal{L}_{(a)} * \mathcal{L}_{(a)}$ appears as operator 100 in the AESZ list [2]. It is given by

$$\mathcal{L}_{(a)} * \mathcal{L}_{(a)} = S_4\theta^4 + S_3\theta^3 + S_2\theta^2 + S_1\theta + S_0, \tag{6.18}$$

where

$$\begin{aligned}
S_4(\varphi) &= (-1+\varphi)(-1+8\varphi)^2(1+8\varphi)(-1+64\varphi) \\
S_3(\varphi) &= 2\varphi(-1+8\varphi)(49+912\varphi-5184\varphi^2+8192\varphi^3) \\
S_2(\varphi) &= \varphi(-77-2904\varphi+26688\varphi^2-113152\varphi^3+196608\varphi^4) \\
S_1(\varphi) &= 4\varphi(-7-512\varphi+2880\varphi^2-15872\varphi^3+32768\varphi^4 \\
S_0(\varphi) &= 4\varphi(-1-120\varphi+448\varphi^2-3584\varphi^3+8192\varphi^4).
\end{aligned} \tag{6.19}$$

where $\theta = \varphi\frac{\mathrm{d}}{\mathrm{d}\varphi}$ which leads to the Riemann symbol

$$\mathcal{P}\left\{\begin{array}{cccccc} -\frac{1}{8} & 0 & \frac{1}{64} & \frac{1}{8} & 1 & \infty \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 3 & 1 & 1 \\ 2 & 0 & 2 & 4 & 2 & 1 \end{array}\right\}. \tag{6.20}$$

Note that, with the exception of $\frac{1}{8}$, the singularities of this operator are 0, ∞ and the pairwise products of those in (6.13). Moreover, as expected, it is invariant under the involution

$$\varphi \to \frac{1}{8^2\varphi}\,. \tag{6.21}$$

This operator has two large complex structure points at 0 and ∞ that are swapped by the involution. Such operators have been studied before in the literature and we will have more to say on them in Section 6.4.

For our purposes, The most interesting singularity of $\mathcal{L}_{(a)} * \mathcal{L}_{(a)}$ is the singularity at the fixed point $\varphi = \frac{1}{8}$. It has indices $(0, 1, 3, 4)$ which leads to trivial monodromy around this point and the vanishing of the Yukawa coupling. This is indicative of an apparent singularity and $\varphi = \frac{1}{8}$ is indeed such a singularity.

A Calabi-Yau manifold Y mirror to a Calabi-Yau manifold X with AESZ 100 as the associated Picard-Fuchs equation can be constructed by taking a free $\mathbb{Z}/2\mathbb{Z}$ quotient of a complete intersection in a toric variety. We direct the reader to [75] for more details.

The topological data of Y is given by

$$\int_Y e_1 \wedge e_1 \wedge e_1 = 18; \qquad \int_Y c_2 \wedge e_1 = 36; \qquad \chi(Y) = -36 \tag{6.22}$$

which we use to compute the periods in an integral symplectic basis $\Pi = \rho\varpi$ where ρ is given by Equation (2.49). We numerically compute the monodromy matrices around each singularity and find

$$M_{-\frac{1}{8}} = \begin{pmatrix} -11 & -6 & 18 & -18 \\ 12 & 7 & -18 & 18 \\ -8 & -4 & 13 & -12 \\ -4 & -2 & 6 & -5 \end{pmatrix} \qquad M_0 = \begin{pmatrix} 1 & -1 & 6 & 9 \\ 0 & 1 & -9 & -18 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$M_{\frac{1}{64}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad M_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -12 & 1 & 0 & 18 \\ -8 & 0 & 1 & 12 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{6.23}$$

$$M_\infty = \begin{pmatrix} -11 & -5 & 12 & -9 \\ -72 & -29 & 63 & -18 \\ -54 & -22 & 49 & -18 \\ -4 & -2 & 5 & -5 \end{pmatrix} .$$

Recall that the monodromy matrices around each conifold point is of the form

$$M_{\varphi_*} = \mathbb{1} - c_{\varphi_*}\, w(\Sigma w)^T \ , \tag{6.24}$$

for some integer c_{φ_*} and a primitive vanishing cycle w [18, 27]. We collect c_{φ_*} and the vanishing cycle for each conifold point in Equation (6.1).

Monodromy	c_{φ_*}	w^T
$M_{-\frac{1}{8}}$	2	$(-3, 3, -2, -1)$
$M_{\frac{1}{64}}$	2	$(0, 0, -1, 0)$
M_1	2	$(0, -3, -2, 0)$

Table 6.1: *The coefficient c_{cnf} and vanishing cycles for the three conifold points of AESZ 100.*

We have seen that if $\Pi(\varphi)$ is a solution of the Picard-Fuchs equation, then so is $\frac{1}{8^2\varphi}\Pi\big(\frac{1}{8^2\varphi}\big)$. More precisely, we numerically find that

$$\frac{1}{8^2\varphi}\Pi\left(\frac{1}{8^2\varphi}\right) = A\Pi(\varphi) \qquad \text{where} \qquad A = \frac{1}{16}\begin{pmatrix} -4 & 0 & 0 & 12 \\ 6 & 4 & -12 & 0 \\ 0 & 1 & -4 & 6 \\ -1 & 0 & 0 & 4 \end{pmatrix} \in \frac{1}{8}Sp(4,\mathbb{Q}) \; . \tag{6.25}$$

A has eigenvalues $\left(\frac{1}{8}, \frac{1}{8}, -\frac{1}{8}, -\frac{1}{8}\right)$ and, since $\varphi = \frac{1}{8}$ is fixed by the involution, A splits the (co)homology of the underlying Calabi-Yau manifold X into positive and negative eigenspaces. In other words, at $\varphi = \frac{1}{8}$, we may define two rank two sublattices

$$\Lambda_+ \oplus \Lambda_- \subset H^3(X, \mathbb{Z}) \tag{6.26}$$

generated by the eigenvectors of A. These eigenvectors are listed in Table 6.2.

Λ_+	Λ_-
$(2, -6, 0, 1)$, $(0, 6, 1, 0)$	$(6, -6, 0, 1)$, $(0, -2, 1, 0)$

Table 6.2: *Generators of $\Lambda_\pm$ at $\varphi = \frac{1}{8}$ for AESZ 100.*

We confirm that the complex structure at $\varphi = \frac{1}{8}$ is such that

$$\Lambda_+ \otimes \mathbb{C} = H^{3,0} \oplus H^{0,3} \qquad \text{and} \qquad \Lambda_- \otimes \mathbb{C} = H^{2,1} \oplus H^{1,2} \tag{6.27}$$

by numerically establishing that

$$\int_X \Gamma_+ \wedge D_\varphi \Omega \;=\; Q_+^T \Sigma D_\varphi \Pi \;=\; 0 \qquad \text{and} \qquad \int_X \Gamma_- \wedge \Omega \;=\; Q_-^T \Sigma \Pi \;=\; 0 \tag{6.28}$$

where $\Gamma_\pm$ is either of the generators of $\Lambda_\pm$, $Q_\pm$ is a vector of its periods in an integral symplectic basis and Σ is the standard symplectic matrix.

One might also be interested in the matrix A in Frobenius basis around $\varphi = 0$ which is given by

$$a \;=\; r\left\{\frac{1}{96}\begin{pmatrix} -6 & 36 & -108 & 0 \\ 0 & 6 & 0 & -36 \\ -1 & 0 & 6 & -12 \\ 0 & -3 & 0 & -6 \end{pmatrix} + \frac{1}{2}\frac{\zeta(3)}{(2\pi i)^3}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 9 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 9 & -27 & 0 \end{pmatrix}\right\} r^{-1} \tag{6.29}$$

where $r \;=\; \text{diag}\{1, (2\pi i), (2\pi i)^2, (2\pi i)^3\}$. Of course, the origin of the $\zeta(3)$ is the constant term in the prepotential around $\varphi = 0$ (equivalently, $\varphi = \infty$).

As promised, we have shown that the apparent singularity at the fixed point of (6.21) is an attractor point of rank two. The next question one might as is - what are the associated eigenforms at this point?

Recall that the method used in Chapter 4 to find the associated modular forms is not available to us at apparent singularities since the deformation method fails at such points. This is where the results of Chapter 5 come in handy. We can instead find the associated modular forms by using the following method:

1. Numerically compute $\mathrm{Re}\,\Pi\big(\frac{1}{8}\big)$ to a high precision (alternatively, $\mathrm{Im}\,\Pi\big(\frac{1}{8}\big)$ would also work).
2. Numerically compute the periods ω_4^+ (or ω_4^-) for all newforms in $S_4(\Gamma_0(N))$ up to some sufficiently high N.
3. Finally, look for a rational equivalence between the periods of X and the periods of the modular form.

The same procedure can be used with $D_\varphi\Pi\big(\frac{1}{8}\big)$ to find the associated weight 2 form.
In this way, one finds that the weight 2 form is given by the form with LMFDB label $f_{\mathbf{14.2.a.a}} \in S_2(\Gamma_0(14))$ which has the Fourier expansion

$$\begin{aligned} f_{\mathbf{14.2.a.a}}(\tau) &= \eta(\tau)\eta(2\tau)\eta(7\tau)\eta(14\tau) \\ &= q - q^2 - 2q^3 + q^4 + 2q^6 + q^7 - q^8 + q^9 - 2q^{12} - 4q^{13} - q^{14} \\ &\quad + q^{16} + 6q^{17} - q^{18} + 2q^{19} + \dots\ . \end{aligned} \tag{6.30}$$

Note that this is the same weight 2 form that appears in the L-function of AESZ 34 at $\varphi = -\frac{1}{7}$. The Tate conjecture would predict a correspondence between the the two varieties which, by a result of Batyrev, cannot be induced by a birational map [7, 76].
Similarly, the associated weight 4 form at $\varphi = \frac{1}{8}$ is found to be $f_{\mathbf{14.4.a.b}} \in S_4(\Gamma_0(14))$. with Fourier expansion

$$\begin{aligned} f_{\mathbf{14.4.a.b}} &= q + 2q^2 - 2q^3 + 4q^4 - 12q^5 - 4q^6 + 7q^7 + 8q^8 - 23q^9 - 24q^{10} \\ &\quad + 48q^{11} - 8q^{12} + 56q^{13} + 14q^{14} \dots\ . \end{aligned} \tag{6.31}$$

The periods at $\varphi = \frac{1}{8}$ are determined by the transition matrix

$$T = \begin{pmatrix} 3 & 0 & 0 & -3 \\ 0 & -30 & -3 & 0 \\ 3 & -10 & -3 & -3 \\ 3 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \frac{\omega_4^+}{(2\pi i)^3} & \frac{\eta_4^+}{(2\pi i)^3} & 0 & 0 \\ \frac{\omega_4^-}{(2\pi i)^3} & \frac{\eta_4^-}{(2\pi i)^3} & 0 & 0 \\ 0 & 0 & \frac{\omega_2^+}{(2\pi i)^2} & \frac{\eta_2^+}{(2\pi i)^2} \\ 0 & 0 & \frac{\omega_2^-}{(2\pi i)^2} & \frac{\eta_2^-}{(2\pi i)^2} \end{pmatrix} \begin{pmatrix} 7 & -28 & -1024 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 512 \\ 0 & 0 & -1 & 16 \end{pmatrix} \tag{6.32}$$

where $\Pi = T\varpi^{(\frac{1}{8})}$ where $\varpi^{(\frac{1}{8})}$ is a vector of periods in Frobenius basis around $\varphi = \frac{1}{8}$. The periods of the weight 2 and 4 eigenforms are given by

$$\begin{aligned} \omega_2^+ &= 15.850735648535065873356573413896074121941712357 53\ldots \\ \omega_2^- &= i\,21.207859834919787429359460672470705146765304604 4\ldots \\ \omega_4^+ &= 4.5950754724061461607299066510128875630064145461 89\ldots \\ \omega_4^- &= i\,2.8555905676472073409308836015078677468484060274 7\ldots . \end{aligned} \tag{6.33}$$

The quasi-periods are

$$\begin{aligned} \eta_2^+ &= -\,984.2635438769480587569945211913758522590677048 59\ldots \\ \eta_2^- &= -\,i\,1250.647144585537612387309268388328408576517182 8\ldots \\ \eta_4^+ &= \;\;31833.966351755446665885419851838688309921889981 44\ldots \\ \eta_4^- &= \;\;i\,19847.54951071924701952180637356541681914344482 42 \end{aligned} \tag{6.34}$$

and are determined by the following weight 2 and weight 4 meromorphic forms. The weight 2 forms are given by

$$F_2 \;=\; \frac{G_2}{(f_{\mathbf{14.2.a.a}})^2} - \frac{523}{3} f_{\mathbf{14.2.a.a}} \tag{6.35}$$

where $G_2 \in S_6(\Gamma_0(14))$ (not an eigenform) and has Fourier expansion

$$G_2 \;=\; q - 2q^2 - 3q^3 + 53q^4 + 107q^5 - 210q^6 + 49q^7 + 117q^8 + \ldots \tag{6.36}$$

Similarly, the weight 4 meromorphic form is given by

$$F_4 \;=\; \frac{G_4}{(f_{\mathbf{14.4.a.b}})^4} + \frac{2663}{3} f_{\mathbf{14.4.a.b}} \tag{6.37}$$

where $G_4 \in S_{12}(\Gamma_0(14))$ (not an eigenform) which has the Fourier expansion

$$\begin{aligned} G_4 &= q^3 - 4q^4 - 2q^5 + 62q^6 + 36q^7 - 8q^8 - 567q^9 + 226q^{10} - 1249q^{11} + 1528q^{12} \\ &\quad + 3520q^{13} - 3168q^{14} + 27990q^{15} + 13920q^{16} + 27280q^{17} - 169560q^{18} \\ &\quad - 256150q^{19} + 264320q^{20} + \ldots \end{aligned} \tag{6.38}$$

Both G_2 and G_4 are uniquely determined by the given Fourier expansions.

6.3.2 AESZ 101

The Hadamard square $\mathcal{L}_{(b)} * \mathcal{L}_{(b)}$ appears as operator 101 in the AESZ list [2] and works much the same way as AESZ 100. It is given by

$$\mathcal{L}_{(b)} * \mathcal{L}_{(b)} \;=\; S_4\theta^4 + S_3\theta^3 + S_2\theta^2 + S_1\theta + S_0, \tag{6.39}$$

where

$$\begin{aligned} S_4(\varphi) &= (-1+\varphi)^2(1+\varphi)(1-123\varphi+\varphi^2) \\ S_3(\varphi) &= 2\varphi(-1+\varphi)(121+244\varphi-125\varphi^2+2\varphi^3) \\ S_2(\varphi) &= \varphi(-187-787\varphi+689\varphi^2-205\varphi^3+6\varphi^4) \\ S_1(\varphi) &= 2\varphi(-33-277\varphi+105\varphi^2-39\varphi^3+2\varphi^4) \\ S_0(\varphi) &= \varphi(-9-124\varphi+12\varphi^2-12\varphi^3+\varphi^4). \end{aligned} \tag{6.40}$$

which leads to the Riemann symbol

$$\mathcal{P}\left\{\begin{array}{cccccc} -1 & 0 & \lambda_- & 1 & \lambda_+ & \infty \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 3 & 1 & 1 \\ 2 & 0 & 2 & 4 & 2 & 1 \end{array}\right\} \tag{6.41}$$

where $1-123\lambda_\pm+\lambda_\pm^2=0$

As expected, up to a gauge transformation, this operator is invariant under

$$\varphi \to \frac{1}{\varphi} \tag{6.42}$$

and, just like AESZ 100, we find that there is an apparent singularity at $\varphi = 1$ that is fixed by the involution.

A Calabi-Yau manifold Y mirror to a one-parameter Calabi-Yau manifold X with AESZ 101 as the associated Picard-Fuchs equation can be found in [75, 77**?**]. The Calabi-Yau manifold Y is, rather unusually, constructed as the intersection of two Grassmannians $\mathrm{Gr}(2,\mathbb{C}^5)$ and $\Psi(\mathrm{Gr}(2,\mathbb{C}^5))$ in $\mathbb{P}^9$ where the first is embedded via the standard Plücker embedding and the second is embedded via the standard Plücker embedding followed by a rotation $\Psi \in PGL(10,\mathbb{C})$.

The topological data of Y is given by

$$\int_Y e_1\wedge e_1\wedge e_1 = 25; \qquad \int_Y c_2\wedge e_1 = 70; \qquad \chi(Y) = -100 \tag{6.43}$$

which we use in conjunction with Equation (2.49) to fix a basis of periods in an integral symplectic basis $\Pi = \rho\varpi$. This leads to the following monodromy matrices

$$
M_{-1} \;=\; \begin{pmatrix} -19 & -10 & 50 & -20 \\ 8 & 5 & -20 & 8 \\ -8 & -4 & 21 & -8 \\ -4 & -2 & 10 & -3 \end{pmatrix} \qquad M_{0} \;=\; \begin{pmatrix} 1 & -1 & 10 & 13 \\ 0 & 1 & -12 & -25 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}
$$

$$
M_{\lambda_-} \;=\; \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad M_{\lambda_+} \;=\; \begin{pmatrix} 1 & 0 & 0 & 0 \\ -60 & 1 & 0 & 225 \\ -16 & 0 & 1 & 60 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{6.44}
$$

$$
M_{\infty} \;=\; \begin{pmatrix} -19 & -9 & 40 & -8 \\ -252 & -95 & 392 & 208 \\ -75 & -29 & 121 & 52 \\ -4 & -2 & 9 & -3 \end{pmatrix}
$$

As always, the monodromy matrices at a conifold point φ_* are of the form

$$
M_{\varphi_*} \;=\; \mathbb{1} - c_{\varphi_*}\, w(\Sigma w)^T \tag{6.45}
$$

and we list the vanishing cycles and the constants c_{cnf} in Table 6.3.

Monodromy	c_{φ_*}	w^T
M_{-1}	2	$(-5, 2, -2, -1)$
M_{λ_-}	1	$(0, 0, -1, 0)$
M_{λ_+}	1	$(0, -15, -4, 0)$

Table 6.3: *The coefficient c_{φ_*} and vanishing cycles for the three conifold points of AESZ 101.*

It's straightforward to check that, if $\Pi(\varphi)$ is a solution of AESZ 101, then so is $\frac{1}{\varphi}\Pi(\frac{1}{\varphi})$. We numerally find that

$$
\frac{1}{\varphi}\Pi\left(\frac{1}{\varphi}\right) = A\Pi(\varphi) \qquad \text{where} \qquad A = \begin{pmatrix} -4 & 0 & 0 & 15 \\ 8 & 4 & -15 & 0 \\ 0 & 1 & -4 & 8 \\ -1 & 0 & 0 & 4 \end{pmatrix} \in \; Sp(4,\mathbb{Z}) \; . \tag{6.46}
$$

As with AESZ 101, we find that A has eigenvalues $(1, 1, -1, -1)$ and its eigenvectors span two lattices

$$
\Lambda_+ \oplus \Lambda_- \subset H^3(X, \mathbb{Z}) \; . \tag{6.47}
$$

Λ_+	Λ_-
$(3,-8,0,1),\ (0,5,1,0)$	$(5,-6,0,1),\ (0,3,1,0)$

Table 6.4: *Generators of $\Lambda_\pm$ at $\varphi = 1$ for AESZ 101*

As before,

$$\Lambda_+ \otimes \mathbb{C} \;=\; H^{3,0} \oplus H^{0,3} \qquad \text{and} \qquad \Lambda_- \otimes \mathbb{C} \;=\; H^{2,1} \oplus H^{1,2} \tag{6.48}$$

which makes $\varphi = 1$ into a rank two attractor point. We confirm this numerically by checking that

$$\int_X \Gamma_+ \wedge D_\varphi \Omega \;=\; Q_+^T \Sigma D_\varphi \Pi \;=\; 0 \qquad \text{and} \qquad \int_X \Gamma_- \wedge \Omega \;=\; Q_-^T \Sigma \Pi \;=\; 0 \tag{6.49}$$

where $\Gamma_\pm$ is either of the generators of $\Lambda_\pm$, $Q_\pm$ is a vector of its periods in an integral symplectic basis and Σ is the standard symplectic form.

We may alternatively express the matrix A in Frobenius basis around $\varphi = 0$ where it is given

$$a \;=\; r \left\{ \frac{1}{72} \begin{pmatrix} -78 & 540 & -900 & 0 \\ 0 & 78 & 0 & -300 \\ 1 & 0 & 78 & -180 \\ 0 & 3 & 0 & -78 \end{pmatrix} + \frac{\zeta(3)}{(2\pi i)^3} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 100 & 0 & 0 & 0 \\ 60 & 0 & 0 & 0 \\ 0 & 180 & -300 & 0 \end{pmatrix} \right\} r^{-1} \tag{6.50}$$

where $r \;=\; \mathrm{diag}\big\{1, (2\pi i), (2\pi i)^2, (2\pi i)^3\big\}$.

The modular forms associated to AESZ 101 at $\varphi = 1$ are found as in the previous section. The weight two form is given by $f_{\mathbf{11.2.a.a}} \in S_2(\Gamma_0(11))$ which has the Fourier expansion

$$\begin{aligned} f_{\mathbf{11.2.a.a}} &= (\eta(\tau)\eta(11\tau))^2 \\ &= q - 2q^2 - q^3 + 2q^4 + q^5 + 2q^6 - 2q^7 - 2q^9 - 2q^{10} + q^{11} - 2q^{12} \\ &\quad + 4q^{13} + 4q^{14} - q^{15} - 4q^{16} - 2q^{17} + 4q^{18} + 2q^{20} \ldots \end{aligned} \tag{6.51}$$

and the weight four form by $f_{\mathbf{22.4.a.a}} \in S_4(\Gamma_0(22))$ which has the Fourier expansion

$$\begin{aligned} f_{\mathbf{22.4.a.a}} &= q - 2q^2 - 7q^3 + 4q^4 - 19q^5 + 14q^6 + 14q^7 - 8q^8 + 22q^9 + 38q^{10} \\ &\quad + 11q^{11} - 28q^{12} - 72q^{13} - 28q^{14} + 133q^{15} + 16q^{16} \ldots \;. \end{aligned} \tag{6.52}$$

The periods at $\varphi = 1$ are determined by the transition matrix

$$T = \begin{pmatrix} -30 & 0 & 0 & -10 \\ -5 & -15 & -15 & -1 \\ -15 & -3 & -5 & -5 \\ -10 & 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} \frac{\omega_4^+}{(2\pi i)^3} & \frac{\eta_4^+}{(2\pi i)^3} & 0 & 0 \\ \frac{\omega_4^-}{(2\pi i)^3} & \frac{\eta_4^-}{(2\pi i)^3} & 0 & 0 \\ 0 & 0 & \frac{\omega_2^+}{(2\pi i)^2} & \frac{\eta_2^+}{(2\pi i)^2} \\ 0 & 0 & \frac{\omega_2^-}{(2\pi i)^2} & \frac{\eta_2^-}{(2\pi i)^2} \end{pmatrix} \begin{pmatrix} -44 & 22 & 13 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 2 \end{pmatrix} \tag{6.53}$$

where $\Pi = T\varpi^{(1)}$ where $\varpi^{(1)}$ is a vector of periods in Frobenius basis around $\varphi = 1$. As before, the ηs are quasi-periods of the weight 2 and weight 4 eigenforms are given by

$$\begin{aligned}
\omega_2^+ &= 0.31730232606988835542219865418863682630487306045 7\ldots \\
\omega_2^- &= i\,0.72940830846924761466544480645183762857962171447\ldots \\
\omega_4^+ &= 0.13175318425123555843767530323006108673505942484 4\ldots \\
\omega_4^- &= -\,i\,0.67739511732404764443595027436686347112472656259\ldots\,.
\end{aligned} \tag{6.54}$$

The quasi-periods are

$$\begin{aligned}
\eta_2^+ &= -\,0.30319560665609101261251555174047427480056329125 2\ldots \\
\eta_2^- &= -\,i\,0.67993298063152016069149144226081406170404998351\ldots \\
\eta_4^+ &= -\,1.46736637982346562735312655224883982903493829379 4\ldots \\
\eta_4^- &= i\,7.55044894980641039495122020900582015008904522110\ldots
\end{aligned} \tag{6.55}$$

and are determined by the following weight 2 and 4 meromorphic modular forms. The weight 2 meromorphic form is given by

$$F_2 \;=\; \frac{G_2}{(f_{\mathbf{11.2.a.a}})^2} - 256 f_{\mathbf{11.2.a.a}} \tag{6.56}$$

where $G_2 \in S_6(G_0(11))$ and has the Fourier series

$$G_2 \;=\; q - 4q^2 + 2q^3 + 124q^4 - 108q^5 + \ldots\,. \tag{6.57}$$

Similarly, the weight 4 form is given by

$$F_4 \;=\; \frac{G_4}{f_{\mathbf{22.4.a.a}}} - \left(\frac{785401561593748}{75}\right) f_{\mathbf{22.4.a.a}} \tag{6.58}$$

where $G_4 \in S_{12}(\Gamma_0(22))$ and has the Fourier expansion

$$\begin{aligned}
G_4 = \;& 1134375000q^2 - 14528538750q^3 + 66315810000q^4 - 90656851833q^5 \\
& - 286034343812q^6 + 1210781539991q^7 - 847420889504q^8 \\
& - 2410906349573q^9 + 3480388351476q^{10} + 6089744740600q^{11} \\
& - 27995206075536q^{12} + 68707026652165q^{13} - 139387829232872q^{14} \\
& - 1328437689094q^{15} - 94503893805632q^{16} - 241747522986465q^{17} \\
& + 115563143492980q^{18} + 131765699331457q^{19} + 83866568325488q^{20} \\
& + 1202088080627425q^{21} + 426282017749000q^{22} + 49476287896244q^{23} \\
& + 718406736310112q^{24} - 106160601223783q^{25} - 1790216489836136q^{26} \\
& - 861004586174224q^{27} - 754073394866016q^{28} + 1915582654539556q^{29} \\
& - 409038140502692q^{30} - 5881170946193952q^{31} + \ldots\,.
\end{aligned} \tag{6.59}$$

Both G_2 and G_4 are uniquely determined by the above Fourier expansions.

6.4 Physical Interpretation of Involuton

This section will explain how the involutions that led to the rank two attractors described in previous sections may be understood from a physics/string theory point of view and will be more speculative than the preceding sections.

AESZ 100 and 101 both have the unusual property that they have two large complex structure points. One at $\varphi = 0$ and the other at $\varphi = \infty$. Such operators have been studied previously in the literature.[3] Most notably, in a paper by Rødland [78] where he constructs two Calabi-Yau manifolds the first of which is given by a complete intersection in the Grassmannian $\mathrm{Gr}(2, \mathbb{C}^7)$ defined by 7 generic linear equations in the Plücker coordinates. The other Calabi-Yau manifold is defined as the intersection of the *Pfaffian variety* $\mathrm{Pf}(\bigwedge^2 \mathbb{C}^7)$ with a generic $\mathbb{CP}^6$ in $\mathbb{CP}^{20}$.[4] Both Calabi-Yau threefolds have Hodge numbers $(h^{2,1}, h^{1,1}) = (50, 1)$ and they are *not* birationally equivalent. Nevertheless, Rødland conjectured that they are mirror to Calabi-Yau manifolds that are related by complex structure deformations and, as evidence, he showed that the associated Picard-Fuchs equation has two large complex structure points each related to the large volume point of one of the two Calabi-Yau manifolds.

Motivated by the conjecture of Rødland, Hori and Tong went on to show that the two manifolds of Rødland arise as two different phases of the same *non-abelian gauged linear sigma model* (GLSM) which, in particular, implies that their mirrors are related by complex structure deformations [79].

GLSMs are a powerful tool used by physicists to study Calabi-Yau manifolds.[5] The idea is as follows: since non-linear sigma models (NLSMs) with a Calabi-Yau target space are difficult to study, one instead studies a two dimensional gauge theory (the GLSM) that flows to the NLSM under renormalisation group flow (i.e. the GLSM reduces to the NLSM at low energies). From the GLSM point of view, the Calabi-Yau manifold X is the moduli space of vacua of the GLSM in a certain "phase" which depends on parameters appearing in the Lagrangian of the gauge theory. For example, the Kähler parameter of X can be identified with the Fayet–Iliopoulos (FI) term r of a $U(1)$ gauge group (possibly a subgroup of the full gauge group). In the $r >> 0$ limit, the moduli space of vacua is determined classically as the space of solutions to the equations of motion modulo gauge transformations. The advance of [79] is that the authors managed to understand quantum corrections in the

[3]Many operators with multiple points of maximal unipotent monodromy can be found in the AESZ list [2].

[4]If $A \in \bigwedge^2 \mathbb{C}^7$ is viewed as a 7×7 anti-symmetric matrix, then $\mathrm{Pf}(\bigwedge^2 \mathbb{C}^7)$ is the locus where $\mathrm{rank}(A) \leq 6$ in $\mathbb{CP}^{20}$.

[5]We cannot hope to give a complete list of references here. Instead, we direct the reader to the seminal paper of Witten [80] and the monograph [18].

$r << 0$ limit well enough to determine the moduli space of vacua in this regime. More precisely, they showed that the moduli space of vacua of a certain $U(2)$ GLSM is given by the Grassmannian Calabi-Yau of Rødland in the $r >> 0$ phase and by the Pfaffian Calabi-Yau in the $r << 0$ phase. Thus, confirming the conjecture of Rødland.

Now, coming back to our examples, GLSMs for AESZ 100 and AESZ 101 have been constructed and can be found in [75] and [75, 77] respectively. Just like the Rødland manifolds, the two large complex structure points of AESZ 100 and AESZ 101 can be identified with the $r >> 0$ and $r << 0$ phases of a GLSM. The difference now being that the two phases lead to the same Calabi-Yau manifold. Moreover, the authors of [75, 77] construct dual GLSMs along the lines of [81] such that the $r >> 0$ and $r << 0$ phases are swapped. We naturally conjecture that the $\varphi \to \frac{1}{8^2\varphi}$ and $\varphi \to \frac{1}{\varphi}$ involutions of AESZ 100 and 101 are a consequence of this GLSM duality.

From the gauge theory point of view, the "self-dual" GLSMs are at attractor points of rank two and it would be interesting to explore this connection further. A detailed analysis would require a precise understanding of the phase transition between the $r >> 0$ and $r << 0$ phases that we hope to return to elsewhere.

Another, seemingly unrelated area where involutions similar to those of AESZ 100 and 101 have appeared is in the study of flux compactifications. In this setting, the rank two attractors of AESZ 100 and AESZ 101 are interpreted as flux compactifications with vanishing superpotential W and discrete symmetries like the involution discussed in this chapter have appeared in [82]. We elaborate further on this in Chapter 7.

6.5 An Example of Cynk, Schütt and van Straten

Before ending this chapter, we should point out that a splitting similar to the one described above has appeared in [60] where the authors show that a certain "double-octic" Calabi-Yau threefold is Hilbert modular. This Calabi-Yau manifold belongs to a family with Picard-Fuchs equation

$$\mathcal{P}\left\{\begin{array}{cccccc} -2 & -1 & -\frac{1}{2} & 0 & 1 & \infty \\ \hline 0 & 0 & 0 & 0 & 0 & 1/2 \\ 1 & 1/2 & 1 & 1/2 & 1 & 1/2 \\ 1 & 1/2 & 3 & 1/2 & 1 & 3/2 \\ 2 & 0 & 4 & 1 & 2 & 3/2 \end{array} \;\varphi\right\} \tag{6.60}$$

and the authors of [60] note that, although the family is not symmetric in any obvious way, the Picard-Fuchs equation is, nevertheless, invariant under $\varphi \to -1-\varphi$ which leaves $-\frac{1}{2}$ fixed. This Picard-Fuchs equation has no point of maximal unipotent monodromy so mirror symmetry cannot be used to fix a canonical choice of integer symplectic basis. This makes working with such operators more difficult.

The lack of a large complex structure point is not an insurmountable obstacle and the methods of previous sections should apply here as well. However, the authors of [60] prove modularity of $X_{-\frac{1}{2}}$ by finding an explicit self map of the underlying Calabi-Yau manifold that is defined over $\mathbb{Q}(\sqrt{2})$ that splits $H^3(X_{-\frac{1}{2}}, \mathbb{Q})$ over $\mathbb{Q}(\sqrt{2})$ into two parts with Hodge numbers $(3,0)+(0,3)$ and $(2,1)+(1,2)$. Because the splitting is over an extension of $\mathbb{Q}$, the results of [13] are not applicable and, indeed, the associated modular form at this point is not a modular form of $\Gamma_0(N)$. Instead, the authors find a Hilbert modular form of weight $(4,2)$ defined over $\mathbb{Q}(\sqrt{2})$.

We again note that the point $-\frac{1}{2}$ has indices $(0,1,3,4)$ which implies trivial monodromy and the vanishing of the Yukawa coupling. As suggested by the indices, this point is indeed an apparent singularity where the underlying Calabi-Yau smooth. Now, one might be tempted to conjecture that such points are always attractor points of rank two. However, the $(4,1)$-manifold studied in [23] has an apparent singularity at $\varphi = -\frac{1}{18}$ and, based on our computations, it does not appear to be an attractor point of rank two. Of course, the Picard-Fuchs equation of the $(4,1)$-manifold is not symmetric in any obvious way so there is no reason to expect a splitting at $\varphi = -\frac{1}{18}$.

Chapter 7

Flux Compactifications

An attractor point of rank two is a point in complex structure moduli space of a Calabi-Yau threefold X where the Hodge structure splits over $\mathbb{Q}$ as

$$(1, h^{2,1}, h^{2,1}, 1) \to (1, 0, 0, 1) + (0, h^{2,1}, h^{2,1}, 0) \ . \tag{7.1}$$

A very similar splitting can be found in the context of IIB compactifications on a Calabi-Yau manifold X with flux. In this context, a point in moduli space that admits a *flux vacuum with vanishing superpotential* is a point where the Hodge structure of X splits over $\mathbb{Q}$ as

$$(1, h^{2,1}, h^{2,1}, 1) \to (0, 1, 1, 0) + (1, h^{2,1} - 1, h^{2,1} - 1, 1) \ . \tag{7.2}$$

We see that, when $h^{2,1} = 1$, a flux vacuum with vanishing superpotential is the same thing as a rank two attractor (as pointed out in [31]). Thus, all of the examples found in previous sections can be interpreted in this way. However, when $h^{2,1} > 1$, the latter condition is less constrained than that of a rank two attractor.

The flux compactification point of view is interesting because it lets us view the elliptic curve associated to the $(0, 1, 1, 0)$ part of $H^3(X)$ (and thus the weight 2 eigenform) as a compactification manifold on the same footing as X [16]. We will explain this below in more detail but the essential point is that, via the usual F-theory chain of dualities, a compactification of IIB on X is equivalent to a compactification of M-theory on $X \times E$ where E is the ellipitic curve associated to the weight 2 form.

We saw in the preceeding sections that, when $h^{2,1} = 1$, we expect the slitting (7.3) to be visible in the Frobenius polynomials $P_3(X/\mathbb{F}_p, T)$ as a factorisation into quadrics for infinitely many p. By computing the Frobenius polynomials for all smooth values of the parameter $\varphi \in \mathbb{F}_p^*$ (and singular values, where possible), and looking for persistent factorisations into quadrics, we were able to identify rank two attractor points . As shown in [83], the same method can be used to search for splittings of the form (7.3) for Calabi-Yau manifolds with $h^{2,1} > 1$ where we expect the Frobenius polynomial to factor into a degree 2 polynomial and a degree $2h^{2,1}$ polynomial (which may or may not factor further). More precisely, the authors of [83] studied a degree 8 hypersurface in the weighted projective space $\mathbb{P}^4(1,1,2,2,2)$ [84]. This is mirror to a Calabi-Yau manifold with $h^{2,1} = 2$ with parameters (ψ, ϕ) and, for small primes, the associated sixth order Frobenius polynomials have been computed in [40]. By reading off the coefficient of in the degree 2 factor of the Frobenius polynomial, the authors of [83] were able to identify the weight 2 eigenforms of $\Gamma_0(N)$ associated to the two dimensional piece of (7.3).

The flux vacua prespective is also useful because it accommodates other kinds of splittings. For example, we will see later in this chapter, an example of a splitting of the form

$$(1,1,1,1) \to (1,0,1,0) + (0,1,0,1) \ . \tag{7.3}$$

Such splittings are impossible over $\mathbb{Q}$ and, indeed, the splitting is only visible over the Gaussian rational $\mathbb{Q}(i)$. This is interpreted as a point in moduli space admitting flux vacua with non-vanishing superpotentials.

The modularity of flux compactification once again raises questions about which we can only speculate for the moment. For example - what physical role, if any, is the weight 2 eigenform playing in flux vacua with vanishing superpotentials? While, it could have been predicted on mathematical grounds, it is rather mysterious to a physicist.

For now, let us review the basics of flux compactifications.

7.1 Review

Type IIB supergravity contains the scalar fields C_0 and ϕ known as the axion and dilaton respectively (the latter determines the string coupling $g_s = e^{\phi}$) that are commonly packaged into the so-called axio-dilaton

$$\tau \ = \ C_0 + i\mathrm{e}^{-i\phi} \ . \tag{7.4}$$

It also contains 3-form fluxes e.g. the Ramond-Ramond (RR) flux F_3 and the Neveu Schwarz - Neveu Schwarz (NSNS) flux H_3 (the "curvature" of the B-field) that are commonly packaged into the complex 3-form

$$G_3 \ = \ F_3 - \tau H_3 \ . \tag{7.5}$$

The utility of this presentation is that the the equations of motion of IIB supergravity are invariant under the transformation

$$\tau \to \frac{a\tau + b}{c\tau + d} \qquad \text{and} \qquad G_3 \to \frac{G_3}{c\tau + d} \tag{7.6}$$

where[1]

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \qquad (7.7)$$

As the transformation law (7.7) suggests, τ may indeed be interpreted as the complex structure parameter of an elliptic curve E. This is the subject of F-theory and there are a few ways of interpreting this elliptic curve (see [85] for a review). For our purposes, the M-theory interpretation will be the most useful. That is, 11-dimensional supergravity on $\mathbb{R}^{1,2} \times X \times E$ leads to IIA on $\mathbb{R}^{1,2} \times X \times S^1$ when one of the 1-cycles of E is shrunken to zero radius. Shrinking the remaining S^1 and then T-dualising it leads to an uncompact direction and we are left with IIB on $\mathbb{R}^{1,3} \times X$. The chain of dualities can also be applied fibrewise so that instead of $X \times E$, we have an elliptically fibred Calabi-Yau fourfold. By reversing this chain of dualities, an axio-dilaton in IIB that varies over a base X (not necessarily Calabi-Yau) is identified with the complex structure parameter of the elliptic fibre over X. Coming back to flux compactifications - a compactification of IIB on a Calabi-Yau threefold X will be depend on a Kähler potential

$$K \;=\; -\log\big(-i(\tau - \overline{\tau})\big) - \log\left(-i\int_X \Omega \wedge \overline{\Omega}\right) \qquad (7.8)$$

for the complex structure moduli of X and the axio-dilaton τ. More importantly, the fluxes F_3 and H_3 generate a superpotential for the moduli

$$W \;=\; \int_X G_3 \wedge \Omega \; . \qquad (7.9)$$

This is phenomenologically very important because the moduli of X lead to massless scalar fields in the four dimensional effective theory. Since we have been unable to detect such fields (the Higgs boson is the only scalar in the standard model), any attempt to obtain the standard model from a Calabi-Yau compactification must have a mechanism for fixing these moduli. This is the subject of moduli stabilisation and we, once again, direct the interested reader to the review [85].

The preservation of supersymmetry in four dimensions fixes the axio-dilaton and the complex structure moduli to the solutions of

$$D_\tau W \;=\; 0 \qquad \text{and} \qquad D_\alpha W \qquad (7.10)$$

where D_α is the covariant derivative on the complex structure moduli space of X that we met in Section 2.1.1 and $D_\tau = \partial_\tau + \partial_\tau K$. Charge quantisation and the equations of motion require that[2]

$$F_3 \in H^3(X,\mathbb{Z}) \qquad \text{and} \qquad H_3 \in H^3(X,\mathbb{Z}) \; . \qquad (7.11)$$

[1]Strictly speaking the equations of motion are invariant under $SL(2,\mathbb{R})$ but, quantum mechanically, quantisation of F_3 and H_3 charge breaks this down to $SL(2,\mathbb{Z})$.

[2]We have dropped a factor of $(2\pi)^2\alpha'$ to clean up notation.

It is now easy to check that (7.10) is equivalent to the condition

$$G_3 \in H^{2,1}(X) \oplus H^{0,3}(X) \tag{7.12}$$

where we have used the fact that F_3 and G_3 are real, $\tau \notin \mathbb{R}$ for finite string coupling and the fact that the covariant derivatives $D_\alpha \Omega$ generate $H^{2,1}(X)$ as cohomology classes.
It is also straightforward to check that, if $W = 0$, then $D_\tau W = 0$ implies that

$$G_3 \in H^{2,1}(X) \tag{7.13}$$

In other words, F_3 and H_3 generate a sublattice $\Lambda^\perp \subset H^3(X, \mathbb{Z})$ with Hodge numbers $(2,1) + (1,2)$. Thus, a flux compactification with $W = 0$ is just a rank two attractor if $h^{2,1} = 1$ and we find the charge lattice $\Lambda \subset H^3(X, \mathbb{Z})$ by taking the complement of $\Lambda^\perp$ with respect to the intersection product.
Solutions of (7.12) with $W = 0$ lead to four dimensional Minkowski vacua while those with $W \neq 0$ may be used to construct supersymmetryic AdS_4 vacua.
The flux compactification point of view also explains some of the structure we observed in Chapter 5. For example, recall that, at the rank two attractor $\varphi = -\frac{1}{7}$ of AESZ 34, the periods of $D_\varphi \Omega$ are given by

$$D_\varphi \Pi\left(-\tfrac{1}{7}\right) \;=\; \frac{3{\cdot}7^2}{2^5\pi^2}\,\frac{iL_2(1)}{v^\perp}\left\{\begin{pmatrix} -5\kappa \\ 10\kappa \\ -5 \\ -3 \end{pmatrix} - \tau^\perp \begin{pmatrix} -7\kappa \\ 14\kappa \\ -10 \\ -5 \end{pmatrix}\right\} \tag{7.14}$$

Since $F_3 - \tau H_3 \in H^{2,1}$ and $H^{2,1}$ is one dimensional, we now see that, if $\varphi = -\frac{1}{7}$ is viewed as a point admitting a flux vacuum with $W = 0$, the integral vectors in (7.14) can be identified with the periods F_3 and H_3 and $\tau^\perp$ can be identified with the vacuum expectation value of the axio-dilaton.
It is very satisfying to see that the axio-dilaton in the flux vacuum at $\varphi = -\frac{1}{7}$ is given by $\tau^\perp$ which we identified as the complex structure modulus of of the elliptic curve

$$E:\; y^2 + xy + y \;=\; x^3 + 4x - 6 \tag{7.15}$$

in Chapter 5. If we didn't know about F-theory, we would be forced to invent it in order to explain the above relationship.
The flux vacua point of view also provides an alternative viewpoint on some of the results of Chapter 6. Recall that we found examples of Picard-Fuchs equations that admitting involutions

$$\varphi \to \frac{1}{\alpha^2 \varphi} \tag{7.16}$$

for some $\alpha \in \mathbb{Z}$ which acted on the periods Π in an integral symplectic basis as

$$\frac{1}{\alpha^2\varphi}\Pi\left(\frac{1}{\alpha^2\varphi}\right) \;=\; A\Pi(\varphi) \tag{7.17}$$

where $A \in GL(4,\mathbb{Q})$ is projectively symplectic i.e. $\alpha A^T\Sigma A = \Sigma$ and has eigenvectors $\left(\frac{1}{\alpha}, \frac{1}{\alpha}, -\frac{1}{\alpha}, -\frac{1}{\alpha}\right)$. Such symmetries have previously appeared in the literature on flux vacua where they are sometimes interpreted as discrete R-symmetries [82, 86]. The involution A acts on the superpotential

$$W\left(\frac{1}{\alpha^2\varphi}\right) \;=\; G_3^T\Sigma\Pi\left(\frac{1}{\alpha^2\varphi}\right) \;=\; \alpha^2\varphi(AG_3)^T\Sigma\Pi(\varphi) \tag{7.18}$$

where we have abused notation slightly by denoting the 3-form and its periods by G_3. If G_3 is chosen as one of the negative eigenvectors, we see that

$$W\left(\frac{1}{\alpha^2\varphi}\right) \;=\; -\alpha\varphi W(\varphi)\ . \tag{7.19}$$

Thus, the superpotential vanishes at the fixed point at $\varphi = \frac{1}{\alpha}$.
Although a flux vacuum with vanishing superpotential only requires a splitting over $\mathbb{Q}$ of the form

$$(1, h^{2,1}, h^{2,1}, 1) \to (1, h^{2,1} - 1, h^{2,1} - 1, 1) + (0, 1, 1, 0)\ , \tag{7.20}$$

there are known examples of Calabi-Yau varieties where H^3 splits over $\mathbb{Q}$ as

$$(1, h^{2,1}, h^{2,1}, 1) \to (1, 0, 0, 1) + h^{2,1} \times (0, 1, 1, 0)\ . \tag{7.21}$$

In other words, a single complex structure modulus may support $h^{2,1}$ independent choices of F_3 and H_3 that lead to flux compactification with vanishing superpotential. In such examples, each 2 dimensional piece with Hodge numbers $(0, 1, 1, 0)$ is associated to a weight 2 eigenform of $\Gamma_0(N)$. The covering of the Hulek-Verill manifold at $\varphi = -\frac{1}{7}$ has $h^{2,1} = 5$ and is precisely of this type. Each $(0, 1, 1, 0)$ factor supports a flux vacuum with vanishing superpotential and is associated to an elliptic curve/weight 2 eigenform. It would be interesting to understand how, in general, the different factors are related both from an arithmetic perspective and a string theory perspective.
Before moving onto an example, we should mention an issue that we have swept under the rug. That is, the RR fluxes and NSNS fluxes generate a contribution to the total D3 charge of the form

$$N_{\text{flux}} \;=\; \frac{1}{2\pi i}\int_X F_3 \wedge H_3\ . \tag{7.22}$$

Since the total D3 charge on a compact manifold must vanish, N_{flux} must be cancelled by a source of negative D3 charge. This is provided by the so-called orientifold planes.

7.2 AESZ 3 (An Example With $W \neq 0$)

AESZ 3 is a the Picard-Fuchs operator

$$\theta^4 - 2^8\varphi\big(\theta + \tfrac{1}{2}\big)^4 \tag{7.23}$$

where $\theta = \varphi\frac{\mathrm{d}}{\mathrm{d}\varphi}$ and it has the Riemann symbol

$$\mathcal{P}\left\{\begin{array}{ccc} 0 & \frac{1}{2^8} & \infty \\ \hline 0 & 0 & 1/2 \\ 0 & 1 & 1/2 \\ 0 & 1 & 1/2 \\ 0 & 2 & 1/2 \end{array}\right\}. \tag{7.24}$$

A holomorphic solution of AESZ 3 around $\varphi = 0$ is proportional to

$$\varpi(\varphi) \;=\; \sum_{n=0}^{\infty}\binom{2n}{n}^4\varphi^n \tag{7.25}$$

from which we see that the AESZ 3 is the Hadamard product of four copies of the first order operator

$$\theta - 2^2\varphi\big(\theta + \tfrac{1}{2}\big) \tag{7.26}$$

which has the solution

$$\frac{1}{\sqrt{1 - 2^2\varphi}} \;=\; \sum_{n=0}^{\infty}\binom{2n}{n}\varphi^n\ . \tag{7.27}$$

This may be interpreted as the periods of a family of varieties mirror to a degree 2 hypersurfaces in $\mathbb{P}^1$ i.e. a 0 dimensional Calabi-Yau variety [87].

AESZ 3 may be also be thought of as the Hadamard square of the Hadamard square

$$\theta^2 - 2^4\varphi\big(\theta + \tfrac{1}{2}\big)^2 \tag{7.28}$$

with Riemann symbol

$$\mathcal{P}\left\{\begin{array}{ccc} 0 & \frac{1}{2^4} & \infty \\ \hline 0 & 0 & 1/2 \\ 0 & 0 & 1/2 \end{array}\right\}\ . \tag{7.29}$$

Alternatively, AESZ 3 is the Hadamard product of (7.26) and the Hadamard cube

$$\theta^3 - 2^6\varphi\big(\theta + \tfrac{1}{2}\big)^3 \tag{7.30}$$

with the Riemann symbol

$$\mathcal{P}\left\{\begin{array}{ccc} 0 & \frac{1}{2^6} & \infty \\ \hline 0 & 0 & 1/2 \\ 0 & 1/2 & 1/2 \\ 0 & 1 & 1/2 \end{array}\right\}\ . \tag{7.31}$$

As one might surmise from the fact that it appears early in the AESZ list [2], AESZ 3 was one of the earliest known fourth order Picard-Fuchs equations. It is of hypergeometric type[3] and, as an A-model, describes the intersection of four quadrics in $\mathbb{P}^7$. Generically, this is a smooth Calabi-Yau threefold with the topological data

$$\int e_1 \wedge e_1 \wedge e_1 \ = 16; \qquad \int c_2 \wedge e_1 \ = 64; \qquad \chi \ = -128 \ . \tag{7.32}$$

Much is known the intersection of four quadrics in $\mathbb{P}^7$. For example, the topological string free energies have been determined in [88] to a high genus. For now, we focus on the fact that it is a Hadamard product.

[3]Like the mirror quintic, this refers to the fact the solutions of the Picard-Fuchs equations are given by the hypergeometric functions ${}_4F_3$.

7.2.1 Involution and Fixed Point

AESZ 3 is a Hadamard product like the operators we studied in Chapter 6 so one naturally wonders - does it admit an involution and do we observe any splitting at its fixed point? By rewriting (7.23) in terms of $\frac{1}{2^8\varphi}$, we see that the solutions around $\varphi = 0$ and $\varphi = \infty$ are essentially the same (e.g. all the coefficients in the power series are identical) just as observed for AESZ 100 and 101. However, there is a crucial difference here. The solutions around $\varphi = 0$ and $\varphi = \infty$ are not related by a holomorphic gauge transformation because of the indices $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ at $\varphi = \infty$.
By a change of variables, AESZ 3 defines another operator $\mathcal{L}'$ with Riemann symbol

$$\mathcal{P}\left\{\begin{array}{cccccc} -\frac{1}{2^4} & 0 & \frac{1}{2^4} & \infty & & \\ \hline 0 & 0 & 0 & 1 & & \\ 1 & 0 & 1 & 1 & ; & \sqrt{\varphi} \\ 1 & 0 & 1 & 1 & & \\ 2 & 0 & 2 & 1 & & \end{array}\right\} \tag{7.33}$$

which admits an involution

$$\sqrt{\varphi} \to \frac{1}{2^8\sqrt{\varphi}}. \tag{7.34}$$

This fixes the points $\sqrt{\varphi} = \pm\frac{1}{2^4}$ which are then identified in AESZ 3 and lead to the conifold points at $\varphi = \frac{1}{2^8}$. The involution also swaps the points $\sqrt{\varphi} = \pm\frac{i}{2^4}$ which are then identified in AESZ 3 and lead to the point $\varphi = -\frac{1}{2^8}$.
Alternatively, $\mathcal{L}'$ is also invariant under the involution

$$\sqrt{\varphi} \to -\frac{1}{2^8\sqrt{\varphi}}\ . \tag{7.35}$$

This time, $\sqrt{\varphi} = \pm\frac{i}{2^4}$ are fixed points while $\sqrt{\varphi} = \pm\frac{1}{2^4}$ are exchanged. We find that this involution acts on the periods as (in Frobenius basis around $\pm\frac{i}{2^8}$)

$$\begin{pmatrix} \frac{\pm i}{16} & -1 & \mp 16i & 256 \\ 0 & \mp\frac{i}{16} & 2 & \pm 48i \\ 0 & 0 & \pm\frac{i}{16} & -3 \\ 0 & 0 & 0 & \mp\frac{i}{16} \end{pmatrix}$$

which has eigenvalues $\left(\frac{i}{16}, \frac{i}{16}, -\frac{i}{16}, -\frac{i}{16}\right)$ and suggests that the rational cohomology described by AESZ 3 splits over $\mathbb{Q}(i)$. This indeed turns out to be the case.

7.2.2 Periods and Splitting of $H^3(X, \mathbb{Q}(i))$

By using Equation (2.49) and the topological data in (7.32), we find the periods Π in an integral symplectic basis around $\varphi = 0$. By numerical analytic conituation, we find that

$$\Pi\left(-\frac{1}{2^8}\right) = \alpha Q_1 + \beta Q_2 \quad \text{and} \quad D_\varphi\Pi\left(-\frac{1}{2^8}\right) = \gamma\overline{Q}_1 + \delta\overline{Q}_2 \tag{7.36}$$

where

$$Q_1 \;=\; \begin{pmatrix} -4 \\ 0 \\ -2 \\ -1 \end{pmatrix} + i \begin{pmatrix} 4 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \text{and} \qquad Q_2 \;=\; \begin{pmatrix} 0 \\ 8 \\ 2 \\ 1 \end{pmatrix} + i \begin{pmatrix} 0 \\ -8 \\ 0 \\ 1 \end{pmatrix} \tag{7.37}$$

and

$$\begin{aligned} \alpha &= 0.3937059793128100299309225578026374633623336225758967 90\ldots \\ \beta &= 0.8692849371864182033505489668001055029415510689423918 20\ldots \\ \gamma &= 39.3299813643404394908938815081529145925251910409640437 7\ldots \\ \delta &= 13.9081074442901919741992008126195023898603960956359186 3\ldots \end{aligned} \tag{7.38}$$

These numbers satisfy the quadratic relation

$$\alpha\gamma - \beta\gamma - \alpha\delta + 2\beta\delta \;=\; 0$$

which follows from the condition

$$\int_X \Omega \wedge D_\varphi \Omega \;=\; \Pi^T \Sigma D_\varphi \Pi \;=\; 0\;. \tag{7.39}$$

It's straightforward to check that

$$Q_1^\dagger \Sigma Q_1 \;=\; 16i; \qquad Q_1^\dagger \Sigma Q_2 \;=\; -16i; \qquad Q_2^\dagger \Sigma Q_2 \;=\; 32i \tag{7.40}$$

and, aside from their conjugates and transposes, all other intersections vanish. This is useful because we may write that Q_1 as a linear combination of $\Pi(-2^{-8})$, $D_\varphi \Pi(-2^{-8})$, $D_{\overline{\varphi}}\overline{\Pi}(-2^{-8})$ and $\overline{\Pi}(-2^{-8})$. The above identities imply that Q_1 is only a linear combination of $\Pi(-2^{-8})$ and $D_{\overline{\varphi}}\overline{\Pi}(-2^{-8})$ and, therefore, has Hodge numbers $(3,0)+(1,2)$ as a cohomology class. Similarly, we find that Q_2 also has the Hodge numbers $(3,0)+(1,2)$.
In other words, $H^3(X,\mathbb{Q}(i))$ splits as

$$(1,1,1,1) \to (1,0,1,0) \oplus (0,1,0,1).$$

Note that such a splitting is impossible over $\mathbb{Q}$ since each term must be invariant under complex conjugation.

7.2.3 Flux Vacua With $W \neq 0$

As previously mentioned, in order to construct supersymmetric flux vacua, we must first find a point where the the superpotential W satisfies $D_\varphi W = 0$ and $D_\tau W = 0$. This condition is equivalent to

$$F_3 - \tau H_3 \in H^{2,1} \oplus H^{0,3} \tag{7.41}$$

and, from previous discussions, we see that both $\overline{Q}_1$ and $\overline{Q}_2$ satisfy this condition if $\tau = i$.

Because $\overline{Q}_1$ and $\overline{Q}_2$ have non-vanishing $(0,3)$ part at $\varphi = -\frac{1}{2^8}$, the associated superpotentials are non-vanishing and are given by

$$\begin{aligned} W_1\left(-\frac{1}{2^8}\right) &= \overline{Q}_1^T \Sigma \Pi \left(-\frac{1}{2^8}\right) &=& \quad 16i(\alpha-\beta) \\ W_2\left(-\frac{1}{2^8}\right) &= \overline{Q}_2^T \Sigma \Pi \left(-\frac{1}{2^8}\right) &=& -16(\alpha-2\beta) \end{aligned} \tag{7.42}$$

7.2.4 Splitting of $H^4(X \times T^2, \mathbb{Q})$

From the point of view of flux compactifications, it is natural to consider an M-theory compactification on $X \times T^2$ where T^2 is a torus with complex structure parameter τ. The same argument at a rank two attractor reveals that T^2 is the elliptic curve associated to the weight four form. This might be helpful in understanding the modularity of AESZ 3 at $\varphi = -\frac{1}{2^8}$ as well.

Let $\Gamma_1, \Gamma_2 \in H^3(X, \mathbb{Z}(i))$ be the cohomology classes with periods given by Q_1 and Q_2 respectively and let $\Lambda \subset H^3(X, \mathbb{Z}(i))$ be the lattice that they generate. We've seen that Λ has Hodge numbers $(1,0,1,0)$ and that

$$\Lambda \oplus \overline{\Lambda} \subset H^3(X, \mathbb{Z}[i]). \tag{7.43}$$

Now consider what this looks like in $H^4(X \times T^2)$. By the Künneth formula, we see that

$$H^3(X, \mathbb{Z}[i]) \otimes H^1(T^2, \mathbb{Z}[i]) \subset H^4(X \times T^2, \mathbb{Z}[i]). \tag{7.44}$$

We see that the piece of $H^3(X, \mathbb{Z}[i]) \otimes H^1(T^2, \mathbb{Z}[i])$ with Hodge numbers $(0,2,0,2,0)$ is given by

$$\Lambda \otimes [\mathrm{d}\overline{z}] \;\oplus\; \overline{\Lambda} \otimes [\mathrm{d}z] \tag{7.45}$$

which is generated by $\Gamma_i \otimes [\mathrm{d}\overline{z}]$ and $\overline{\Gamma}_i \otimes [\mathrm{d}z]$. We may decompose $\mathrm{d}z$ as

$$\mathrm{d}z = \mathrm{d}x + i\mathrm{d}y \tag{7.46}$$

where $[\mathrm{d}x], [\mathrm{d}y] \in H^3(T^2, \mathbb{Z})$ and see that

$$\begin{aligned} \frac{1}{2}\Big(\Gamma_i \otimes [\mathrm{d}\overline{z}] + \overline{\Gamma}_i \otimes [\mathrm{d}z]\Big) &= \operatorname{Re}\Gamma_i \otimes [\mathrm{d}x] + \operatorname{Im}\Gamma_i \otimes [\mathrm{d}y] \in H^4(X \times T^2, \mathbb{Z}) \\ \frac{i}{2}\Big(\Gamma_i \otimes [\mathrm{d}\overline{z}] - \overline{\Gamma}_i \otimes [\mathrm{d}z]\Big) &= \mathrm{Re}\Gamma_i \otimes [\mathrm{d}y] - \mathrm{Im}\Gamma_i \otimes [\mathrm{d}x] \;\in H^4(X \times T^2, \mathbb{Z}) \end{aligned} \tag{7.47}$$

We see that $\big(\operatorname{Re}\Gamma_1 \otimes [\mathrm{d}x] + \operatorname{Im}\Gamma_1 \otimes [\mathrm{d}y]\big)$ and $\big(\operatorname{Re}\Gamma_2 \otimes [\mathrm{d}x] + \operatorname{Im}\Gamma_2 \otimes [\mathrm{d}y]\big)$ generate a rank two lattice in $H^4(X \times T^2, \mathbb{Z})$ with Hodge numbers $(0,1,0,1,0)$ and the same is true for $\big(\operatorname{Re}\Gamma_1 \otimes [\mathrm{d}y] - \operatorname{Im}\Gamma_1 \otimes [\mathrm{d}x]\big)$ and $\big(\operatorname{Re}\Gamma_2 \otimes [\mathrm{d}y] - \operatorname{Im}\Gamma_2 \otimes [\mathrm{d}x]\big)$ as well.

In other words, there is a piece of $H^4(X \times T^2, \mathbb{Q})$ that splits over $\mathbb{Q}$ as

$$(0,2,0,2,0) \to (0,1,0,1,0) + (0,1,0,1,0)$$

and we expect this to be visible in the Frobenius polynomial of $H^4(X \times T^2)$. This may be helpful in identifying any modular forms at $\varphi = -\frac{1}{2^8}$.

The holomorphic 4-form of $X \times T^2$ is simply $\Omega_4 = \Omega_3 \wedge \mathrm{d}z$ so if we can identify the numbers that appear in the periods of Ω_4, we should be able to identify the numbers α, β, γ and δ that appear in the periods of Ω_3.

Chapter 8

Topological String Theory

The periods of a Calabi-Yau manifold play a prominent role in many string theory constructions. For example, as we shall see, they are the building blocks of topological string free energies. We saw in the preceding chapters that we are able to evaluate the periods (and the derivatives) of the holomorphic three form exactly at attractor points of rank two which means that we should be able to do the same for topological string free energies at all genera. This will be the subject of this chapter.

8.1 A and B Models

We start by reviewing the essentials of topological string theory. A detailed review may be found in the monograph [18]. Computational details and the construction of the propagators are reviewed in the lectures on Alim [89].

Just like type II string theory, topological string theory starts with a non-linear sigma model with $(2,2)$ supersymmetry and we assume that the target space is given by a Calabi-Yau manifold. The theory has $U(1)_A \times U(1)_V$ R-symmetry and (in Euclidean signature) $U(1)_L$ Lorentz symmetry. The axial R-symmetry acts on the SUSY generators as

$$\begin{aligned} [F_A, Q_\pm] &= \mp Q_\pm \\ [F_A, \overline{Q}_\pm] &= \pm \overline{Q}_\pm \end{aligned} \tag{8.1}$$

whereas the vector R-symmetry acts as

$$\begin{aligned} [F_V, Q_\pm] &= -Q_\pm \\ [F_V, \overline{Q}_\pm] &= \overline{Q}_\pm \ . \end{aligned} \tag{8.2}$$

Similarly, the $U(1)_L$ Lorentz symmetry group acts on the SUSY generators as

$$\begin{aligned} [F_L, Q_\pm] &= \mp Q_\pm \\ [F_L, \overline{Q}_\pm] &= \mp \overline{Q}_\pm \ . \end{aligned} \tag{8.3}$$

One may define a related quantum field theory by keeping the Lagrangian fixed but instead assuming that the diagonal subgroup of $U_{V/A}(1) \times U(1)_L$ is the Lorentz group and can define the theory on a curved Riemann surface by gauging this Lorentz group. We say that the new theory is a *topological twist* of the original theory. The twist with the vector $U_V(1)$ is known as the A-twist whereas the twist with the axial $U_A(1)$ is known as the B-twist.
The A and B twists lead to topological theories in the sense that they define the nilpotent charges

$$Q_A = \overline{Q}_+ + Q_- \tag{8.4}$$

and

$$Q_B = \overline{Q}_+ + \overline{Q}_- \tag{8.5}$$

which are both scalars in the twisted Lorentz group. The new energy momentum tensor (with respect to the new Lorentz group) is given by

$$T_{\mu\nu} \;=\; \{Q_{A/B}, G^{A/B}_{\mu\nu}\} \tag{8.6}$$

where $G^{A/B}_{\mu\nu}$ is some fermionic tensor. The fact that the energy momentum tensor is Q-exact implies that the path integral and all correlation functions are invariant under infinitesimal variations of the world-sheet metric. It is in this sense that the A and B twists are topological theories. $Q_{A/B}$ is essentially playing the role of a BRST operator in these theories. This leads to the definition of a *chiral ring* as the set of Q-closed operators

$$\{Q_{A/B}, \mathcal{O}\} = 0 \tag{8.7}$$

modulo the set of Q-exact operators.
By studying the transformation properties of the various fields under Q_A, one may check that the chiral ring of the A-model is generated by the world-sheet spinors ψ^i_- and $\overline{\psi}^{\bar{i}}_+$ (see [89]) where i is a target space index and that the generic element of the chiral ring is given by

$$\omega_{i_1,\ldots,i_p\bar{j}_1,\ldots,\bar{j}_q}(\phi^k)\psi^{i_1}_- \ldots \psi^{i_p}_- \overline{\psi}^{\bar{j}_1}_+ \ldots \overline{\psi}^{\bar{j}_q}_+ \tag{8.8}$$

where ϕ^k is the k$^{\text{th}}$ component of the sigma-model map from the world-sheet to the target space and the components ω are functions of ϕ^k such that the entire combination is Q closed. Since ψ^i_- and $\overline{\psi}^{\bar{j}}_+$ anti-commute, there is a linear map that send (8.8) to the differential form

$$\omega_{i_1,\ldots,i_p\bar{j}_1,\ldots,\bar{j}_q}(\phi^k)\mathrm{d}z^{i_1} \wedge \ldots \wedge \mathrm{d}z^{i_p} \wedge \mathrm{d}\bar{z}^{\bar{j}_1} \wedge \ldots \wedge \mathrm{d}\bar{z}^{\bar{j}_q} \;. \tag{8.9}$$

Moreover, the supercharge Q_A is mapped to the exterior derivative d and Q_A-exact operators are mapped to exact forms. Thus, the chiral ring is simply the de Rham cohomology ring and (8.8) represents a (p, q) cohomology class.

It follows from a localisation computation that correlation functions of chiral ring operators in the A-twisted sigma model depend on the complexified Kähler parameter t of the target space (see [18]). Generically, the dependence on t appears in chiral ring correlation functions through world-sheet instanton corrections.

The axial and vector R-symmetries remain symmetries of the twisted theories which leads to selection rules that non-vanishing correlation functions must satisfy. More precisely, a correlation function of s operators each corresponding to a (p_i, q_i) form must satisfy

$$\sum_{i=1}^{s} p_i \;=\; \sum_{i=1}^{s} q_i \;=\; 3(1-g) \tag{8.10}$$

where g is the genus of the world-sheet.

The treatment of the B-twisted Calabi-Yau sigma model proceeds in much the same way. This time, the chiral ring is isomorphic to

$$\bigoplus_{p,q=0}^{3} H^{0,p}(X, \wedge^q TX) \tag{8.11}$$

and the Q_B operator is mapped to the Dolbeault operator $\overline{\partial}$ and the R-symmetries lead to the selection rules

$$\sum_{i=1}^{s} p_i \;=\; \sum_{i=1}^{s} q_i \;=\; 3 \tag{8.12}$$

at genus 0 and

$$\sum_{i=1}^{s} p_i \;=\; \sum_{i=1}^{s} q_i \;=\; 0 \tag{8.13}$$

at genus 1. Moreover, just like in (8.10), the selection rules can never be satisfied if $g \geq 2$. It follows from the so-called tt^* equations that correlation functions in the B-twisted sigma model are determined by the complex structure of the target space [18].

So far, we have defined the A/B-twisted non-linear sigma models with Calabi-Yau target spaces which are are quantum field theories and not string theories. In order to promote them to string theories, we need to integrate over world-sheet metrics in the path integral. Since we are dealing with topological quantum field theories, one might be tempted to simply introduce a string coupling and sum over the different correlation functions at all genera (perhaps introducing factors for the volume of the moduli space at genus g). However, this turns out to be not so fruitful. For example, correlation functions of chiral operators are trivial at genus $g \geq 2$ for both the A-twisted and B-twisted sigma models. A more careful treatment of the coupling to world-sheet gravity treats the twisted theories as bosonic string theories where $Q_{A/B}$ plays the role of the BRST operator. In this approach, the genus $g \geq 2$ free energies are non-trivial because the path integral picks up contributions from the genus

$g \leq 1$ free energies at the boundary of the moduli space where the world-sheet metric degenerates and the Riemann surface becomes pinched. This brings us to the subject of BCOV recursion.

8.2 BCOV Holomorphic Anomaly Equations

In [90, 91], Bersadsky, Cecotti, Ooguri and Vafa (BCOV) show how, although one might naively expect topological string free-energies to depend holomorphically on the parameters of topological string theory, they nevertheless pick up a non-holomorphic dependence through the so-called holomorphic anomaly. We will review here what is necessary for what follows and direct the interested reader to the seminal papers of BCOV. Once again, this subject is treated in detail in the monograph [18].

We have already met the genus 0 topological string free energy in Chapter 2. The B-model free energy at genus 1 on a Calabi-Yau manifold X was determined in [90] and is given by

$$\mathcal{F}^{(1)} \;=\; \log\left\{ g^{-\frac{1}{2}} \mathrm{e}^{\frac{K}{2}\left(3+h^{2,1}(X)+\frac{\chi(X)}{12}\right)} |f^{(1)}|^2 \right\} \tag{8.14}$$

where g is the determinent of the metric on complex structure moduli space and f_1 is the so-called holomorphic ambiguity at genus 1 which vanishes at the various singularities of the moduli space. More precisely, if the moduli space is singular at $\varphi = \varphi_i$, then

$$f^{(1)}(\varphi) \;=\; (\varphi-\varphi_1)^{r_1}\ldots(\varphi-\varphi_k)^{r_k} \tag{8.15}$$

where the exponents r_i are determined by the particular singularity at φ_i. At conifold points, they may be determined from the monodromy of the periods around that point. Note that $\mathcal{F}^{(1)}$ is non-holomorphic because of the Kähler potential K (which also determines the metric).

The genus 1 B-model free energy on a Calabi-Yau manifold X is related to the A-model free energy on the mirror Calabi-Yau Y via a mirror map and a gauge transormation. For example, for one parameter models, a mirror map is given by

$$t(\varphi) \;=\; \frac{\varpi_1(\varphi)}{\varpi_0(\varphi)} \tag{8.16}$$

where ϖ_i are the Frobenius basis around the large complex structure point at $\varphi = 0$.[1] More generally, if $\mathcal{F}^{(g)}$ is the B-model free energy at genus g, then the holomorphic genus g A-model free energy is determined by mirror symmetry and is given by[2]

$$F_g \;=\; \lim_{\bar{t}\to-\infty} \varpi_0^{2g-2}\mathcal{F}^{(g)}\;. \tag{8.17}$$

[1] Note that this choice of complexfied Kähler parameter undergoes monodromy $t \to t+2\pi i$ instead of $t \to t+1$. This seems to be the more common choice in literature on topological string theory so we will stick with this convention in this chapter.

[2] The holomorphic limit is found by sending $\bar{t}$ to the large volume point. In our choice of t coordinate, this is when $t \to -\infty$ which is mirror to the large complex structure point at $\varphi = 0$.

The holomorphic F_g is the physical topological string free energy in the sense that it appears in the effective 4 dimensional supergravity action (see Section 8.3). Moreover, it admits a q-expansion at large volume that determines instanton numbers genus g.

Perhaps the most significant advance of BCOV [91] is the recursion formula

$$\partial_{\overline{k}}\mathcal{F}^{(g)} = \frac{1}{2}\overline{C}_{\overline{i}}^{jk}\left\{D_iD_j\mathcal{F}^{(g-1)} + \sum_{r=1}^{g-1} D_j\mathcal{F}^{(g-1)}D_i\mathcal{F}^{(r)}\right\} \tag{8.18}$$

where D_i is the covariant derivative introduced in Chapter 2 and

$$\overline{C}_{\overline{i}}^{jk} = \overline{C_{ijk}}g^{i\overline{i}}g^{j\overline{j}}\mathrm{e}^{2K} \tag{8.19}$$

where C_{ijk} is the Yukawa couplling (see Section 2.1.2). This formula recursively determines $\mathcal{F}^{(g)}$ up to a holomorphic function $f^{(g)}$ that we refer to as the *holomorphic ambiguity*. The determination of the holomorphic ambiguity is the major conceptual obstacle in determining the topological string free energies at all genera. Each term on the right hand side of (8.18) corresponds to a degeneration of a genus g Riemann surface into a genus $g-1$ Riemann surface or into a genus r and genus $g-r$ Riemann surface that meet at a point.

In practice, integrating the right hand side of (8.18) can become tricky. The approach taken by BCOV in [91] was to realise that $D_{i_1}D_{i_2}\ldots D_{i_n}\mathcal{F}^{(g)}$ can be expressed in terms of a finite number of functions (propagators) and $D_{i_1}D_{i_2}\ldots D_{i_n}\mathcal{F}^{(g)}$ can be determined by certain Feynman rules. This approach was later refined by Yamaguchi and Yau [92]. The propagators are are the functions S^{ij}, S^i and S where i, j and k are directions in the complex structure moduli space. They satisfy[3]

$$C_{ijk}S^{kl} = \delta_i^l K_j + \delta_j^l K_i - \Gamma_{ij}^l + s_{ij}^l \tag{8.20}$$

and

$$\begin{aligned}
\partial_i S^{jk} &= C_{imn}S^{mj}S^{nk} + \partial_j^i S^k + \partial_i^k S^j - s_{im}^j S^{mk} - s_{im}^k S^{mj} + h_i^{jk} \\
\partial_i S^j &= C_{imn}S^{mj}S^n + 2\delta_i^j S - s_{im}^j S^m - h_{ik}S^{kj} + h_i^j \\
\partial_i S &= \frac{1}{2}C_{imn}S^m S^n - h_{ij}S^j + h_i \\
\partial_i K_j &= K_iK_j - C_{ijn}S^{mn}K_m + s_{ij}^m K_m - C_{ijk}S^k + h_{ij}
\end{aligned} \tag{8.21}$$

where s_{ij}^k, h_i^{jk}, h_j^i, h_i and h_{ij} are holomorphic functions. For one parameter models, three can be chosen arbitrarily and the remaining two fixed by the above conditions.

If z_i are coordinates on the complex structure moduli space, then[4]

$$D_{i_1}D_{i_2}\ldots D_{i_n}\mathcal{F}^{(g)} \in \mathbb{Q}\big(z_1, z_2, \ldots, z_{h^{2,1}(X)}\big)[S^{jk}, S^k, S, K_j]\ . \tag{8.22}$$

[3] Note that we are using the conventions of [89] which differ slightly from those of [91].

[4] That the propagators are multiplied by rational functions with *rational coefficients* follows from the fact that the power series appearing in the expansion of the periods near the large complex structure point can be chosen to have rational coefficients.

Moreover, $\mathcal{F}^{(g)}$ (without any derivatives) does not depend on K_j. This follows from (8.21) and the fact that

$$\partial_i \mathcal{F}^{(1)} \;=\; \frac{1}{2} C_{ijk} S^{jk} - \left(\frac{\chi(Y)}{24} - 1\right) K_i + \partial_i f^{(1)} \ . \tag{8.23}$$

The BCOV recursion formula may be rewritten by first rewriting the right hand side of (8.18) in terms of the propagators and K_j and then replacing any propagator derivatives with propagators by using (8.21). Finally, by collecting powers of K_j, we find that the BCOV recursion formula is equivalent to the following set of equations

$$\begin{aligned}
\frac{\partial \mathcal{F}^{(g)}}{\partial S^{ij}} &= \frac{1}{2}\partial_i\left(\partial'_j \mathcal{F}^{(g-1)}\right) + \frac{1}{2}\left(C_{ijl}S^{lk} - s^k_{ij}\right)\partial'_k \mathcal{F}^{(g-1)} + \frac{1}{2}\left(C_{ijk}S^k - h_{ij}\right)c_{g-1} \\
&\quad + \frac{1}{2}\sum_{h=1}^{g-1} \partial'_i \mathcal{F}^{(h)} \partial'_j \mathcal{F}^{(g-h)} \\
\frac{\partial \mathcal{F}^{(g)}}{\partial S^{i}} &= (2g-3)\partial'_i \mathcal{F}^{(g-1)} + \sum_{h=1}^{g-1} c_h \partial'_i \mathcal{F}^{(g-h)} \\
\frac{\partial \mathcal{F}^{(g)}}{\partial S} &= (2g-3)c_{g-1} + \sum_{h=1}^{g-1} c_h c_{g-h}
\end{aligned} \tag{8.24}$$

where

$$c_g \;=\; \begin{cases} \frac{\chi(Y)}{24} - 1 & \text{if } g = 1 \\ (2g-2)\mathcal{F}^{(g)} & \text{if } g > 1 \end{cases} \tag{8.25}$$

and

$$\partial'_i \mathcal{F}^{(g)} \;=\; \begin{cases} \partial_i \mathcal{F}^{(g)} + \left(\frac{\chi(Y)}{24} - 1\right) K_i & \text{if } g = 1 \\ \partial_i \mathcal{F}^{(g)} & \text{if } g > 1 \end{cases} . \tag{8.26}$$

It is shown in [92] that $D_{i_1} D_{i_2} \ldots D_{i_n} \mathcal{F}^{(g)}$ is a polynomial of degree of $3g-3+n$ where K_i, S^{ij}, S^i and S are of degree 1, 1, 2 and 3 respectively. Knowing this, it is straightforward (with the aid of a computer) to make an ansatz for $\mathcal{F}^{(g)}$ that determines the left hand side of (8.24). Comparing this with the known right hand side determines $\mathcal{F}^{(g)}$ recursively up to the rational function $f^{(g)}$.

Consider the example of AESZ 100, regularity of $\mathcal{F}^{(g)}$ at $\varphi = \infty$ and the boundary conditions at the various conifold points (see below) constrain the holomorphic ambiguity to be

$$f^{(g)}(\varphi) \;=\; \sum_{n=0}^{2g-2} a_n \varphi^n + \frac{\sum_{n=0}^{2g-3} b_n \varphi^n}{(\varphi + \frac{1}{8})^{2g-2}} + \frac{\sum_{n=0}^{2g-3} c_n \varphi^n}{(\varphi - \frac{1}{64})^{2g-2}} + \frac{\sum_{n=0}^{2g-3} d_n \varphi^n}{(\varphi - \frac{1}{8})^{2g-2}} + \frac{\sum_{n=0}^{2g-3} e_n \varphi^n}{(\varphi - 1)^{2g-2}} \ . \tag{8.27}$$

The major conceptual obstacle to determining $\mathcal{F}^{(g)}$ is the computation of the unknown coefficients in the above expression. It was shown in [88] how one may fix these coefficients up to some high degree (but not inifnite) genus by studying the behaviour of F_g at the various boundaries. We will turn to this in the next section but first, note that in defining the holomorphic F_g, we must compute the holomorphic limit $\bar{t} \to -\infty$. For a one-parameter model with parameter φ, this simply amounts to making the replacements

$$\Gamma^{\varphi}_{\varphi\varphi} \to \frac{\partial\varphi}{\partial t}\frac{\partial^2 t}{\partial^2\varphi} \quad \text{and} \quad K \to -\log\varpi_0 \tag{8.28}$$

where t is a mirror map around a particular singular point and ϖ_0 is the holomorphic period with lowest exponent around that particular singular point. This leads to the holomorphic propagators that we note by a caligraphic S e.g. $S^{\varphi\varphi} \to \mathcal{S}^{\varphi\varphi}$.

8.3 Boundary Conditions and Mirror Maps

It is often the case that, at the boundary of moduli space, some cycle will collapse to zero volume. For example, at a conifold point, an S^3/G submanifold will collapse to zero volume where G is some finite group. This results in a singularity of the Calabi-Yau manifold which is reflected in a singularity of the topological string free energy F_g.

From the four dimensional space-time point of view, the moduli t^i are the vacuum expectation values of scalar fields that we also denote by t^i. Schematically, they appear in the effective four dimensional action obtained by compactifying IIA on a Calabi -Yau manifold as

$$\int \mathrm{d}^4x\, F_g(t^i)\, R_+^2\, F_+^{2g-2} + \ldots \tag{8.29}$$

where R_+ is the self dual part of the Riemann tensor and F_+ is the self dual part of the field strength of a $U(1)$ gauge field (the graviphoton).

At the boundary of the moduli space where a p-cycle γ collapses to zero volume, a Dp brane wrapping γ will become massless. However, the derivation of the four dimensional effective action assumes that the mass of a Dp brane wrapping γ is very large so that it can be safely integrated out of the effective four dimensional action. This inconsistency is the physical origin of the singularity of F_g at boundaries of the moduli space.

For example, consider IIB compactified on a Calabi-Yau manifold X. The singularity at a conifold point may be cured by including extra massless hypermultiplets in the effective action and the singularity of F_g is recovered when they are integrated out [30, 93]. Since the number of massless hypermultiplets in the effective action is counted by the second Betti number $b^2(X)$, the addition of extra hypermultiplets is realised geometrically by the addition of extra 2-cycles (equivalently, exceptional divisors) that resolve the conical singularity [94]. This provides a string-theoretic description of topology change and is commonly referred to as a *conifold transition* [95, 96]

We see that, understanding the singularity of F_g at the boundaries of moduli space is equivalent to understanding the effect of integrating out a charged scalar field coupled to a constant $U(1)$ field strength in the effective action. This is essentially a problem that was first considered by Schwinger and a discussion can be found in [18]. By understanding the Dp branes that become massless at different boundaries of moduli space and how, the form of F_g has been determined to varying extents at certain boundaries.

8.3.1 Large Volume/Complex Structure Point

Consider IIA on a Calabi-Yau manifold Y near the large volume point in the complexified Kähler moduli space $\mathcal{M}_K(Y)$. As previously mentioned, this is mirror to a point of maximal unipotent monodromy of the periods on the complex structure moduli space $\mathcal{M}_{CS}(X)$. Moreover, mirror symmetry teaches us that the Dp branes of IIA (even p) are equivalent to the Dp branes of IIB (odd p) and, more precisely, the masses of Dp branes are given by

$$\begin{pmatrix} m_{D6} \\ m_{D4} \\ m_{D0} \\ m_{D2} \end{pmatrix} = \mathrm{e}^{K/2}\Pi \tag{8.30}$$

where Π is the vector of periods of X near the large complex structure point and K is the Kähler potential on $\mathcal{M}_{CS}(X)$. Note that each term on the right hand side is the central charge computing the charge of a D3 branes (see Equations (2.62) and (2.68)).

We see from Equations (2.43) and (2.49) that

$$\Pi(\varphi) \sim \begin{pmatrix} \log^3(\varphi) \\ \log^2(\varphi) \\ 1 \\ \log(\varphi) \end{pmatrix} \tag{8.31}$$

which, when combined with the fact that $\mathrm{e}^{-K} = -i\Pi^\dagger\Sigma\Pi$ implies that m_{D0} and m_{D2} vanish at the large volume point.

We saw in the preceeding section that F_g may be computed via mirror symmetry. In particular, near a large complex structure point on a one-parameter Calabi-Yau manifold, the mirror map may be chosen as

$$t = \frac{\varpi_1(\varphi)}{\varpi_0(\varphi)}. \tag{8.32}$$

By a Schwinger type computation, Gopakumar and Vafa computed the effect of vanishing m_{D0} and m_{D2} at large volume on F_g which led to the celebrated *Gopakumar-Vafa formula*

[5]

$$\sum_{g=0}^{\infty}\lambda^{2g-2}F_g(t) \;=\; \frac{c(t)}{\lambda^2}+\ell(t)+\sum_{g=0}^{\infty}\sum_{\beta\in H_2(Y,\mathbb{Z})}\sum_{k=1}^{\infty}\frac{n^g_\beta}{k}\left(2\sin\frac{k\lambda}{2}\right)^{2g-2}q^{\beta m} \tag{8.33}$$

where $q=\mathrm{e}^t$ and $c(t)$ and $l(t)$ are cubic and linear polynomials in t respectively that appear determine F_0 and F_1 at large volume (see below and Equation (2.45)).[6] The integers n^g_β are the values of an index that counts the number of M2 branes wrapping a curve of genus g in the homology class $\beta\in H_2(Y,\mathbb{Z})$ which, roughly speaking, is the number of holomorphic curves of genus g in the homology class β. These BPS numbers were first computed at genus 0 for the quintic in [24], at genus 1 in [90], genus 2 in [91] and to genus 20 in [88]. In principle, the method of [88] can be used to fix the holomorphic ambiguity up to genus 51 at which point one runs out of boundary conditions.

By expanding both sides of the Gopakumar-Vafa formula and comparing powers of λ, we can compute the expansions of F_g around the large volume point up to a constant term. The leading term of $F_0(t)$ was determined in [24], of $F_1(t)$ in [90] and at higher genera in [91, 98] and is given by

$$\lim_{t\to-\infty}F_g(t) \;=\; (-1)^g\frac{\chi}{2}\frac{|B_{2g}B_{2g-2}|}{2g(2g-2)(2g-2)!}\;;\qquad g\geq 2 \tag{8.34}$$

For one paramter models, the expansions up to genus 5 are given by

$$\begin{aligned}
F_0'''(t) &= \left(\int_Y e_1\wedge e_1\wedge e_1\right)+\sum_{d=1}^{\infty}n^0_d\,\mathrm{Li}_3\big(q^d\big)\\
F_1(t) &= \left(\frac{1}{24}\int_Y c_2\wedge e_1\right)t+\sum_{d=1}^{\infty}\left(\frac{n^0_d}{12}+n^1_d\right)\mathrm{Li}_1\big(q^d\big)\\
F_2(t) &= \frac{\chi}{5760}+\sum_{d=1}^{\infty}\left(\frac{n^0_d}{240}+n^2_d\right)\mathrm{Li}_{-}1\big(q^d\big)\\
F_3(t) &= -\frac{\chi}{1451520}+\sum_{d=1}^{\infty}\left(\frac{n^0_d}{6048}-\frac{n^2_d}{12}+n^3_d\right)\mathrm{Li}_{-3}\big(q^d\big)\\
F_4(t) &= \frac{\chi}{87091200}+\sum_{d=1}^{\infty}\left(\frac{n^0_d}{172800}+\frac{n^2_d}{360}-\frac{n^3_d}{6}+n^4_d\right)\mathrm{Li}_{-5}\big(q^d\big)\\
F_5(t) &= -\frac{\chi}{2554675200}+\sum_{d=1}^{\infty}\left(\frac{n^0_d}{5322240}-\frac{n^2_d}{20160}+\frac{n^3_d}{80}-\frac{n^4_d}{4}+n^5_d\right)\mathrm{Li}_{-7}\big(q^d\big)
\end{aligned} \tag{8.35}$$

[5] As is the case with many perturbative expansions one meets in quantum field theory, the Gopakumar-Vafa formula is an asymptotic series in λ. While the expansions of $F_g(t)$ in q lead to series with finite radius of convergence, the sum over λ diverges factorially. See, for example, [97].

[6] Note that the t in Equation (2.45) differs from this by a factor of $2\pi i$.

where $\chi = \chi(Y)$, c_2 is the second Chern class of Y, e_1 is the generator of $H^2(Y, \mathbb{Z})$ and $\mathrm{Li}_s(z)$ is the polylogarithm function

$$\mathrm{Li}_s(z) \;=\; \sum_{k=1}^{\infty} \frac{z^k}{k^s}\;. \tag{8.36}$$

Although it is generally difficult to compute the BPS numbers n_d^g directly, it can be done in simple cases and each known n_d^g can then be used as a boundary condition to fix one of the coefficients in the holomorphic ambiguity.

It is known that, for a fixed degree d, there exists a a curve of maximal genus among the irreducible curves of degree d in Y. This implies that, for a fixed d, $n_d^{g_{\max}} = 0$ for some $g_{\max} \in \mathbb{N}$. We will assume that if $n_d^{g_0} = 0$, then $n_d^g = 0$ for all $g > g_0$ which leads to additional boundary conditions that we can use to fix the holomorphic ambiguity [88].

8.3.2 Conifold Point

We say that a point $\varphi_{\text{con}} \in \mathcal{M}_{CS}(X)$ is a *conifold point* if (after a gauge transformation) the indices are of the form $(0, 1, 1, 2)$.

As discussed in Section 2.1.2, at a conifold point φ_*, the monodromy matrix M around φ_* in an integral symplectic basis takes the form

$$M_{\varphi_*} \;=\; \mathbb{1} - c_{\varphi_*}\, w (\Sigma w)^T \tag{8.37}$$

where w is a vector of integers that represents the homology class of the vanishing cycle S^3/G at φ_* and c_{φ_*} is an integer. In cases, when a single S^3/G vanishes at the conifold point, $c_{\varphi_*} = |G|$.

The Frobenius basis of periods around a conifold point c_{φ_*} is given by

$$\begin{aligned}
\varpi_0^{(\varphi_*)}(\varphi) \;&=\; f_0(\varphi)\\
\varpi_1^{(\varphi_*)}(\varphi) \;&=\; f_1(\varphi)\\
\varpi_2^{(\varphi_*)}(\varphi) \;&=\; f_2(\varphi)\,\log(\varphi - \varphi_*) + f_3(\varphi)\\
\varpi_3^{(\varphi_*)}(\varphi) \;&=\; f_4(\varphi)
\end{aligned} \tag{8.38}$$

where the f_j are power series and, by a change of basis, we set as many coefficients equal to zero as possible. For example, AESZ 100 has a conifold point at $\varphi_* = \frac{1}{64}$ where the

Frobenius basis is defined by

$$
\begin{aligned}
f_0(\varphi) &= 1+\frac{17760256}{3969}(\varphi-\varphi_*)^3+\frac{37829476352}{83349}(\varphi-\varphi_*)^4+\ldots \\
f_1(\varphi) &= (\varphi-\varphi_*)-\frac{160}{3}(\varphi-\varphi_*)^2+\frac{231424}{81}(\varphi-\varphi_*)^3+\ldots \\
f_2(\varphi) &= -\frac{3570176}{11907}(\varphi-\varphi_*)^2+\frac{7937302528}{250047}(\varphi-\varphi_*)^3+\ldots \\
f_3(\varphi) &= (\varphi-\varphi_*)^2-\frac{608}{7}(\varphi-\varphi_*)^3+\frac{71810560}{11907}(\varphi-\varphi_*)^4+\ldots\ .
\end{aligned}
\tag{8.39}
$$

It was shown in [88] that a mirror map in the neighbourhood of a conifold point can be defined by

$$
t_{\varphi_*} \;=\; k_{\varphi_*}\frac{\varpi_1^{(\varphi_*)}(\varphi)}{\varpi_0^{(\varphi_*)}(\varphi)} \tag{8.40}
$$

where, this time, the ϖ's are the periods in Frobenius basis around the conifold point φ_* and k_{φ_*} is some normalisation that must be determined at every confiold point.

Each singular points of the Picard-Fuchs equation appears in the genus 1 holomorphic ambiguity $f^{(g)}$. For a conifold point, it is known that this takes the form

$$
f^{(1)}(\varphi) \;=\; \ldots(\varphi-\varphi_*)^{-\frac{c_{\varphi_*}}{6}}\ldots \tag{8.41}
$$

where c_{φ_*} is the coefficient that appears in the monodromy matrices in Picard-Lefscehtz form (see Equation (2.59)) [27].

By a Schwinger type computation, it was shown in [88] that F_g near a conifold point takes the form

$$
F_g(t_{\varphi_*}) = c_{\varphi_*}\frac{|B_{2g}|}{2g(2g-2)t_{\varphi_*}^{2g-2}}+O(t_{\varphi_*}^0) \tag{8.42}
$$

where B_{2g} is a Bernoulli number. Note that there are no subleading poles at $t_{\varphi_*}=0$. This is known as the *gap condition* and by computing the periods around a conifold point, expanding F_g in t_{φ_*} and comparing powers of t_{φ_*} with the above expression one obtains $2g-3$ linear equations that can be used to determine some of the coefficients in $f^{(g)}$.

8.3.3 Apparent Singularities

Topological string theory on Calabi-Yau manifolds where the Picard-Fuchs equation has an apparent singularity have been studied previously in the literature [99, 100]. In both papers, the apparent singularity has been treated on the same footing as conifold points and a mirror map around the apparent singularity φ_* was conjectured to be

$$
t_{\varphi_*} \;=\; \frac{\varpi_1^{(\varphi_*)}(\varphi)}{\varpi_0^{(\varphi_*)}(\varphi)}\ . \tag{8.43}
$$

where $\varpi^{(\varphi_*)}$ is the vector of periods in Frobenius basis around the conifold point.
It was found in [99, 100] that the apparent singularity does *not* appear as a factor of $f^{(1)}$. However, it *does* appear as a pole in $f^{(g)}$ when $g > 1$. We may treat the regularity of F_g at an apparent singualrity as a gap condition where the leading singularity also vanishes. This leads to $2g - 2$ linear equations that we may use to fix the holomorphic ambiguity.
Formally, an apparent singularity looks like a conifold point where $c_{\varphi_*} = 0$. Since c_{φ_*} is known to compute the difference of the number of additional hyper and vector multiplets that one must introduce in the four dimensional effective action in order to resolve the singularity, we naturally conjecture that an equal number of additional hyper and vector multiplets appear at the attractor points of AESZ 100 and 101. We will elaborate further in Chapter 9.

8.4 AESZ 34

We saw in previous sections that AESZ 34 describes an A-model with the topological data

$$\int_Y e_1 \wedge e_1 \wedge e_1 = 12\kappa; \qquad \int_Y c_2 \wedge e_1 = 12\kappa; \qquad \chi(Y) = -8\kappa \tag{8.44}$$

where $\kappa = 1$ or $\kappa = 2$ for the $\mathbb{Z}/10\mathbb{Z}$ quotients respectively (see Appendix A).
Either by analysing the geometry in Appendix A or by computing the monodromy matrices as in 2.1, we find that the genus 1 holomorphic ambiguity is given by

$$f^{(1)}(\varphi) = \left(\varphi - \tfrac{1}{25}\right)^{-\frac{10}{6\kappa}} \left(\varphi - \tfrac{1}{9}\right)^{-\frac{2}{6\kappa}} (\varphi - 1)^{-\frac{4}{6\kappa}} \ . \tag{8.45}$$

Finally, by computing F_g via mirror symmetry and comparing with (8.35), we determine the instanton numbers in Table 8.4.

k	n_d^0	n_d^1
1	12κ	$20 - 10\kappa$
2	24κ	$102 - 30\kappa$
3	112κ	$1180 - 438\kappa$
4	624κ	$12096 - 4428\kappa$
5	4200κ	$133780 - 48938\kappa$
6	31408κ	$1511730 - 550266\kappa$
7	258168κ	$17647076 - 6407530\kappa$
8	2269848κ	$210201644 - 76161400\kappa$
9	21011260κ	$2545255572 - 920643442\kappa$
10	202527600κ	$31212421126 - 11273118446\kappa$
11	2017537884κ	$386727907536 - 139494386712\kappa$
12	20654747200κ	$4832555488984 - 1741106040676\kappa$
13	216372489804κ	$60820504439296 - 21890039477888\kappa$
14	2311525544064κ	$770125991800110 - 276916193102934\kappa$
15	25115533695300κ	$9802710122549832 - 3521744606381596\kappa$
16	276942939016224κ	$125345358831091796 - 44996106417473728\kappa$
17	3093639869100240κ	$1609189343845395964 - 577237489764357422\kappa$
18	34957447938066952κ	$20732103878422556262 - 7431797271319182118\kappa$
19	399082284262216044κ	$267947664660167267360 - 95989385991015664456\kappa$
20	$4598143339631725920\kappa$	$3472847998674908410256 - 1243366526895209656540\kappa$

Table 8.1: *BPS numbers for AESZ 34 at genus 0 and genus 1 up to degree 20. We take the constant κ to be equal to 1 for the $\mathbb{Z}/10\mathbb{Z}$ quotient and 2 for the $\mathbb{Z}/5\mathbb{Z}$ quotient.*

Unfortunately, we do not find enough boundary conditions to determine $f^{(2)}$ exactly for AESZ 34. Nevertheless, it is useful to consider the propagators. We may fix a choice of propagators by choosing[7]

$$s^{\varphi}_{\varphi\varphi}(\varphi) \;=\; -\frac{3}{\varphi}\;; \qquad h_{\varphi\varphi}(\varphi) \;=\; \frac{1}{\varphi^2}\;; \qquad h^{\varphi}_{\varphi}(\varphi) \;=\; 0\;. \tag{8.46}$$

By expanding the periods around $\varphi = 0$ and solving (8.21), we find that

$$h_{\varphi}(\varphi) \;=\; \frac{1}{24}(1+5\varphi-26\varphi^2)\;; \qquad h^{\varphi\varphi}_{\varphi}(\varphi) \;=\; \frac{1}{6}(\varphi-14\varphi^2+13\varphi^3)\;. \tag{8.47}$$

With this choice of propagators, we may evaluate the periods and their derivatives at the rank two attractor at $\varphi = -\frac{1}{7}$ with (5.66) and find that they lead to the (non-holomorphic)

[7] This particular choice ensures regularity at $\varphi = \infty$.

propagators

$$\begin{aligned} S^{\varphi\varphi}(-\tfrac{1}{7}) &= -\frac{1024}{16807} + \frac{8}{16807}\left(\frac{\eta_2^-}{\omega_2^-} + \frac{\eta_2^+}{\omega_2^+}\right) \\ S^{\varphi}(-\tfrac{1}{7}) &= -\frac{1984}{7203} + \frac{5}{2401}\left(\frac{\eta_2^-}{\omega_2^-} + \frac{\eta_2^+}{\omega_2^+}\right) \\ S(-\tfrac{1}{7}) &= -\frac{44}{147} - \frac{25}{5488}\left(\frac{\eta_2^-}{\omega_2^-} + \frac{\eta_2^+}{\omega_2^+}\right) + \frac{5}{10976}\left(\frac{\eta_4^-}{\omega_4^-} + \frac{\eta_4^+}{\omega_4^+}\right) \end{aligned} \tag{8.48}$$

which, when combined with the fact that $K_\varphi\left(-\tfrac{1}{7}\right) = -\tfrac{35}{8}$ leads to the result

$$D_\varphi^n \mathcal{F}^{(g)}\left(-\tfrac{1}{7}\right) \in \mathbb{Q}\left[\left(\frac{\eta_2^-}{\omega_2^-} + \frac{\eta_2^+}{\omega_2^+}\right), \left(\frac{\eta_4^-}{\omega_4^-} + \frac{\eta_4^+}{\omega_4^+}\right)\right] \tag{8.49}$$

$\forall n \in \mathbb{N}, \forall g > 1$.

Alternatively, we may consider the holomorphic propagators $\mathcal{S}^{\varphi\varphi}$, $\mathcal{S}^{\varphi}$ $\mathcal{S}$ at the attractor point by making the replacements in (8.28). We find that

$$\begin{aligned} \mathcal{S}^{\varphi\varphi}\left(-\tfrac{1}{7}\right) &= -\frac{1024}{16807} + \frac{16}{16807}\left(\frac{2\eta_2^-\omega_4^- + 25\eta_2^+\omega_4^+}{2\omega_2^-\omega_4^- + 25\omega_2^+\omega_4^+}\right) \\ \mathcal{S}^{\varphi}\left(-\tfrac{1}{7}\right) &= -\frac{1984}{7203} + \frac{480}{343}\left(\frac{\pi^2}{2\omega_2^-\omega_4^- + 25\omega_2^+\omega_4^+}\right) + \frac{10}{2401}\left(\frac{2\eta_2^-\omega_4^- + 25\eta_2^+\omega_4^+}{2\omega_2^-\omega_4^- + 25\omega_2^+\omega_4^+}\right) \\ \mathcal{S}\left(-\tfrac{1}{7}\right) &= -\frac{44}{147} - \frac{300}{49}\left(\frac{\pi^2}{2\omega_2^-\omega_4^- + 25\omega_2^+\omega_4^+}\right) + \frac{5}{5488}\left(\frac{2\eta_4^-\omega_2^- + 25\eta_4^+\omega_2^+}{2\omega_2^-\omega_4^- + 25\omega_2^+\omega_4^+}\right) \\ &\quad + \frac{25}{2744}\left(\frac{2\eta_2^-\omega_4^- + 25\eta_2^+\omega_4^+}{2\omega_2^-\omega_4^- + 25\omega_2^+\omega_4^+}\right). \end{aligned} \tag{8.50}$$

Similarly, the holomorphic connection evaluates to

$$\mathcal{K}_\varphi\left(-\tfrac{1}{7}\right) = -\frac{35}{8} + \frac{735}{4}\frac{\pi\omega_2^+}{\omega_4^-} \tag{8.51}$$

If we denote by $D_\varphi^n \mathcal{F}_{\text{holo}}^{(g)}$ the quantity $D_\varphi^n \mathcal{F}^{(g)}$ but computed with the holomorphic propagators, we find that

$$\begin{aligned} D_\varphi^n \mathcal{F}_{\text{holo}}^{(g)}\left(-\tfrac{1}{7}\right) \in \mathbb{Q}\Bigg[&\frac{\pi\omega_2^+}{\omega_4^-}, \left(\frac{\pi^2}{2\omega_2^-\omega_4^- + 25\omega_2^+\omega_4^+}\right), \left(\frac{2\eta_2^-\omega_4^- + 25\eta_2^+\omega_4^+}{2\omega_2^-\omega_4^- + 25\omega_2^+\omega_4^+}\right), \\ &\left(\frac{2\eta_4^-\omega_2^- + 25\eta_4^+\omega_2^+}{2\omega_2^-\omega_4^- + 25\omega_2^+\omega_4^+}\right) \Bigg] \end{aligned} \tag{8.52}$$

$\forall n \in \mathbb{N}, \forall g > 1$. Note that there is some freedom in choosing the normalisation of the period polynomials (see Section 5.2). We may very well have chosen a normalisation that removes the overall numerical factors in the various transcendental factors e.g. $2\omega_2^-\omega_4^- + 25\omega_2^+\omega_4^+ \to \omega_2^-\omega_4^- + \omega_2^+\omega_4^+$ and similar for the other factors.

8.5 AESZ 100

Our choice of propagators for AESZ 100 is fixed by

$$s^{\varphi}_{\varphi\varphi}(\varphi) \;=\; 0 \qquad h_{\varphi\varphi}(\varphi) \;=\; 0 \qquad h^{\varphi}_{\varphi}(\varphi) \;=\; 0 \tag{8.53}$$

which imply that

$$\begin{aligned} h^{\varphi\varphi}_{\varphi}(\varphi) \;&=\; \frac{\varphi(1-142\varphi-1824\varphi^2+13184\varphi^3-20480\varphi^4)}{18(1-8\varphi)^2} \\ h_{\varphi}(\varphi) \;&=\; \frac{1+120\varphi-448\varphi^2+3584\varphi^3-8192\varphi^4}{9(1-8\varphi)^3} \;. \end{aligned} \tag{8.54}$$

Using the exact expressions for the periods and their derivatives implied by (6.32), we find that the propagators and K_{φ} at $\varphi = \frac{1}{8}$ are given by

$$\begin{aligned} \lim_{\varphi\to 1/8}(1-8\varphi)S^{\varphi\varphi}(\varphi) \;&=\; -\frac{49}{2304} \\ \lim_{\varphi\to 1/8}(1-8\varphi)S^{\varphi}(\varphi) \;&=\; \frac{7}{288} \\ \lim_{\varphi\to 1/8}(1-8\varphi)^2\, S(\varphi) \;&=\; -\frac{7}{72} \\ K_{\varphi}\left(\tfrac{1}{8}\right) \;&=\; 4 \;. \end{aligned} \tag{8.55}$$

Which implies that there exists a choice of holomorphic anomaly $f^{(g)}$ such that

$$\mathcal{F}^{(g)}\left(\tfrac{1}{8}\right) \in \mathbb{Q} \;. \tag{8.56}$$

We also find that the holomorphic propagators and the connection defined by(8.28) evaluate to

$$\begin{aligned} \lim_{\varphi\to 1/8}(1-8\varphi)\mathcal{S}^{\varphi\varphi}(\varphi) \;&=\; -\frac{49}{2304} \\ \lim_{\varphi\to 1/8}(1-8\varphi)\mathcal{S}^{\varphi}(\varphi) \;&=\; \frac{7}{288} \\ \lim_{\varphi\to 1/8}(1-8\varphi)^2\, \mathcal{S}(\varphi) \;&=\; -\frac{7}{72} \\ \mathcal{K}_{\varphi}\left(\tfrac{1}{8}\right) \;&=\; 4-\frac{6\pi i}{70}\left(\frac{\omega_2^{+}+\omega_2-}{21\omega_4^{+}-70\omega_4^{-}}\right) \;. \end{aligned} \tag{8.57}$$

Finally, regularity at $\varphi = \infty$ and the boundary conditions discussed in 8.3.2 constrain the holomorphic ambiguity to be

$$f^{(g)}(\varphi) \;=\; \sum_{n=0}^{2g-2} a_n\varphi^n + \frac{\sum_{n=0}^{2g-3} b_n\varphi^n}{(\varphi+1)^{2g-2}} + \frac{\sum_{n=0}^{2g-3} c_n\varphi^n}{(\varphi-1)^{2g-2}} + \frac{\sum_{n=0}^{4g-5} d_n\varphi^n}{(1-123\varphi+\varphi^2)^{2g-2}} \;. \tag{8.58}$$

The parameter space described by AESZ 100 is symmetric under

$$\varphi \to \frac{1}{8^2\varphi} \tag{8.59}$$

which swaps two large complex structure points. Thus, by comparing the expansions of F_g around either of the corresponding large volume points, we may fix half of the unknown constants in (8.58). The rest of the unknown constants are determined via the boundary conditions discussed in Section 8.3.2 and 8.3.3. In this way, we determine the BPS invariants listed in Table 8.2.

We find that the mirror maps at the various conifolds are normalised by

$$k_{-\frac{1}{8}} \; = \; \frac{3}{32} \qquad\qquad k_{\frac{1}{64}} \; = \; -\frac{3}{512} \qquad\qquad k_1 \; = \; -3 \; . \tag{8.60}$$

8.6 AESZ 101

Our choice of propagators for AESZ 100 is fixed by

$$s^{\varphi}_{\varphi\varphi}(\varphi) \; = \; 0 \qquad h_{\varphi\varphi}(\varphi) \; = \; 0 \qquad h^{\varphi}_{\varphi}(\varphi) \; = \; 0 \tag{8.61}$$

which imply that

$$\begin{aligned} h^{\varphi\varphi}_{\varphi}(\varphi) \; &= \; \frac{\varphi(1-310\varphi-488\varphi^2+318\varphi^3-5\varphi^4)}{25(1-\varphi)^2} \\ h_{\varphi}(\varphi) \; &= \; \frac{9+124\varphi-12\varphi^2+12\varphi^3-\varphi^4}{50(1-\varphi)^3} \; . \end{aligned} \tag{8.62}$$

Using the exact expressions for the periods and their derivatives implied by (6.53), we find that the propagators and K_φ at $\varphi = 1$ are given by

$$\begin{aligned} \lim_{\varphi\to 1}(1-\varphi)S^{\varphi\varphi}(\varphi) \; &= \; -\frac{48}{25} \\ \lim_{\varphi\to 1}(1-\varphi)S^{\varphi}(\varphi) \; &= \; \frac{66}{25} \\ \lim_{\varphi\to 1}(1-\varphi)^2\, S(\varphi) \; &= \; -\frac{33}{25} \\ K_\varphi(1) \; &= \; \frac{1}{2} \; . \end{aligned} \tag{8.63}$$

Similarly, we find that the holomorphic propagators and the connection defined by (8.28) evaluate to

$$\begin{aligned} \lim_{\varphi\to 1}(1-\varphi)\mathcal{S}^{\varphi\varphi}(\varphi) \; &= \; -\frac{48}{25} \\ \lim_{\varphi\to 1}(1-\varphi)\mathcal{S}^{\varphi}(\varphi) \; &= \; \frac{66}{25} \\ \lim_{\varphi\to 1}(1-\varphi)^2\, \mathcal{S}(\varphi) \; &= \; -\frac{33}{25} \\ \mathcal{K}_\varphi\left(\tfrac{1}{8}\right) \; &= \; \frac{1}{2} + \frac{5\pi i}{66}\left(\frac{\omega_2^- + \omega_2^+}{\omega_4^- + 5\omega_4^+}\right) \; . \end{aligned} \tag{8.64}$$

Regularity at $\varphi = \infty$ and the boundary conditions discussed in 8.3.2 constrain the holomorphic ambiguity to be

$$f^{(g)}(\varphi) \;=\; \sum_{n=0}^{2g-2} a_n \varphi^n + \frac{\sum_{n=0}^{2g-3} b_n \varphi^n}{(\varphi+1)^{2g-2}} + \frac{\sum_{n=0}^{2g-3} c_n \varphi^n}{(\varphi-1)^{2g-2}} + \frac{\sum_{n=0}^{4g-5} d_n \varphi^n}{(1-123\varphi+\varphi^2)^{2g-2}} \tag{8.65}$$

and half of the unknown constants may be fixed by expanding $\mathcal{F}^{(g)}$ in terms of φ and $\frac{1}{\varphi}$. Since AESZ 101 describes a parameter space that is invariant under

$$\varphi \to \frac{1}{\varphi} \tag{8.66}$$

and both $\varphi = 0$ and $\varphi = \infty$ are large complex structure points, we may compare the expansions of F_g at both of the corresponding large volume points and, in this way fix half of unknown constants in the holomorphic ambiguities.

The rest of the unknown constants are determined via the boundary conditions discussed in Section 8.3.2 and 8.3.3. In this way, we determine the BPS invariants listed in Table 8.3. We find that the mirror maps at the various conifold points are normalised by

$$k_{-1} \;=\; \sqrt{5} \qquad \text{and} \qquad k_{\frac{1}{2}\left(123\pm 55\sqrt{5}\right)} \;=\; -1525 \mp 682\sqrt{5}\;. \tag{8.67}$$

d	n_d^0	n_d^1	n_d^2	n_d^3	n_d^4	n_d^5
1	90	0	0	0	0	0
2	756	45	0	0	0	0
3	8172	1960	0	0	0	0
4	134964	70776	90	0	0	0
5	2778318	2544264	31140	0	0	0
6	65740572	94352372	4046904	4122	6	0
7	1702308690	3569404140	365230836	4450320	-270	0
8	47183857812	137275587216	26900570880	1034960382	3880494	-1467
9	1377747076626	5346613351848	1760385501288	147431456748	2724211620	3364200
10	41922975354048	210390516483147	106642635820020	16164914503311	761143561164	8168186250
11	1318935927692700	8348782881586536	6120980347377978	1500366229669836	138260551529052	4306924384848
12	42654898384516620	333629087155587020	337643438080659678	123992265991476510	19342969185761592	1252626855044238

Table 8.2: *BPS numbers for AESZ 100 up to genus 5 and degree 12*

d	n_d^0	n_d^1	n_d^2	n_d^3	n_d^4	n_d^5
1	325	0	0	0	0	0
2	3200	0	0	0	0	0
3	66250	0	0	0	0	0
4	1985000	325	0	0	0	0
5	73034875	109822	0	0	0	0
6	3070310300	19018900	650	0	0	0
7	141603560675	2367994150	1829200	0	0	0
8	6990803723200	247337794400	938148600	72650	0	0
9	363591194115575	23368078640700	253848387875	287055600	-975	0
10	19705196405545000	2075562931566226	48865015050900	225293359750	68762800	500
11	1104153966524594850	177059059777938850	7643658178867550	90644383230350	193715083675	11987950
12	63598129792406485600	14692505162202526525	1041954995886347300	25018039373344450	168891045608450	177174891400

Table 8.3: *BPS numbers for AESZ 101 up to genus 5 and degree 12*

8.7 Normalisation of Mirror Map at Conifold Points

In the process of computing topological string free-energies, we have made the following observation. Let Π be a vector of periods in an integral symplectic basis around the large complex structure point $\varphi = 0$ such that the monodromy of Π around the conifold point φ_* is in Picard-Lefschetz form (see Equation (2.59)). One may always find a symplectic change of basis such that this is the case. Let T be the transition matrix

$$\Pi \ = \ T\varpi^{(\varphi_*)} \tag{8.68}$$

where $\varpi^{(\varphi_*)}$ is a vector of periods in Frobenius basis around the conifold point. We have observed that

$$\Pi_1(\varphi) \ = \ \frac{k_{\varphi_*}^{-1}}{2\pi i}\varpi_1^{(\varphi_*)}(\varphi) \ . \tag{8.69}$$

We have confirmed this for AESZ 100, AESZ 101, the mirror of the bi-cubic and the the mirror quintic. In other words, we may normalise the mirror map by numerically computing the transition matrix T and looking for the only non-vanishing term on the top row of T. In all the cases we have considered, k_{φ_*} is an algebraic number that we were able to identify from its decimal expansion.[8] To the best of our knowledge, this observation has not appeared previously in the literature.

8.8 Black hole microstates

We saw in Chapter 5 that the periods of a Calabi-Yau manifold X may be expressed in terms of critical L-values at a rank two attractor point. This has the physical consequence that the area of the associated four dimensional black hole is determined by these L-values. For example, for AESZ 34 at $\varphi = -\frac{1}{7}$ we found that

$$\frac{A\big(-\frac{1}{7}\big)}{4\pi} \ = \ \frac{(5k-2\ell)^2}{8}\left(\frac{\pi L_4(1)}{L_4(2)}\right) + \frac{49k^2}{2}\left(\frac{\pi L_4(1)}{L_4(2)}\right)^{-1} . \tag{8.70}$$

where L_4 is the L-function associated to the weight 4 form and $kQ_1 + lQ_2$ where $k, l \in \mathbb{Z}$ is the charge vector of a black hole with area A and $\varphi = -\frac{1}{7}$ as its associated attractor point. Since the area of a black hole is a measure of entropy and since the coefficients of modular forms frequently appear in counting problems, this strongly suggests that the modular forms that we find at rank two attractor points are somehow counting the microstates.

The periods of a Calabi-Yau manifold also compute topological string free energies and we saw examples of rank two attractors where the topological string free energies evaluate to

[8] For hypergeometric models, this number is known to be related to the square root of the triple intersection number [67].

numbers valued in $\mathbb{Q}$ extended by certain (presumably) transcendental numbers that are determined by the associated modular forms. This is related to the previous observation about the area of a black hole by a conjecture of Ooguri, Strominger and Vafa (OSV) which gives a formula for the number of microstates in terms of the all-genus topological string free energy. We hope that a direct computation of the number of microstates of the black holes assoiated to rank two attractors will shed light on the precise role (if any) that the modular forms are playing in counting of microstates.

Finally, we point out that the BPS numbers computed in Tables 8.2 and 8.3, in fact, compute the microstates of a black hole. This is a result from [101] where Katz, Klemm and Vafa study five dimensional $\mathcal{N} = 2$ black holes that arise from wrapping an M2 brane on a curve in a Calabi-Yau threefold Y. More precisely, if the M2 brane wraps a curve in homology d times the generator of $H^2(Y, \mathbb{Z})$ (we are assuming that $h^{1,1}(Y) = 1$ and ignoring torsion) and the black hole has spin m, then the microstates are computed by

$$\Omega(d, m) \;=\; \sum_{r=0}^{g_{\max}(d)} n_d^r \binom{2r+2}{r+1+m} \tag{8.71}$$

where $g_{max}(d)$ is the maximal genus of a curve in homology class d which ensures that the right hand side is a finite sum. Thus, the instanton numbers in Tables 8.2 and 8.3 can be used to compute the microstates of a five dimensional black hole up to degrees $d = 7$ and $d = 9$ for AESZ 100 and 101 respectively.

Chapter 9

Seiberg-Witten Theory Analogies

We will describe some ongoing work in this section and, although much of what follows will be rather speculative, it will describe some of the intuition that motivated the investigations of previous chapters so it would be a shame not to include it here.

An important observation is that the special geometry constructions that have featured prominently in this thesis bear a striking resemblance to constructions in the Seiberg-Witten theory of $\mathcal{N} = 2$ supersymmetric gauge theories. This is not an accident and Seiberg-Witten theory can often be recovered from string theory on a Calabi-Yau manifold. See, for example, [102–104].

In compactifications of IIB on a Calabi-Yau manifold X, the vacuum expectation values of scalars in vector multiplets (known as the *Coulomb branch*) are identified with coordinates on the complex structure moduli space $\mathcal{M}_{CS}(X)$. We have seen that $\mathcal{M}_{CS}(X)$ is a special Kähler manifold and it turns out that much of this structure can be found in $\mathcal{N} = 2$ gauge theories where the Coulomb branch is a *rigid special Kähler* manifold. The Coulomb branch of $\mathcal{N} = 2$ gauge theories can be identified with the base of a family of curves where the generic fibre C is known as a *Seiberg-Witten curve*. The analgoue of the holomorphic 3-form of X is played by the *Seiberg-Witten differential* and, just like in the study of mirror symmetry, the periods of the Seiberg-Witten differential may be computed by solving a differential equation. These periods in turn can be used to compute a prepotential that determines the effective Lagrangian of the gauge theory at low energies and, just like in the study of mirror symmetry, the prepotential depends on certain rational "instanton numbers" that are very difficult to compute directly.

The $U(1)^{b^3/2}$ charges of a dyon are given by $\gamma \in H_3(X, \mathbb{Z})$ on the IIB side whereas, in Seiberg-Witten theory, the $U(1)^r$ charges of a dyon are given by $\gamma \in H_1(C, \mathbb{Z})$.[1]

[1] The integer r is the rank of a gauge group G (e.g. $SU(3)$) and we are considering a $U(1)^r$ subgroup because a generic point on the Coulomb branch corresponds to a choice of vacuum expectation values for the scalars in the vector multiplets that completely break G down to a maximal torus (e.g. $U(1) \times U(1)$).

We have seen that, on the IIB side, the mass of a BPS state with charge $\gamma \in H_3(X,\mathbb{Z})$ is given by $|Z_\gamma(\varphi)|$ where Z_γ is a central charge given by

$$Z_\gamma(\varphi) \;=\; \mathrm{e}^{\frac{K}{2}} \int_\gamma \Omega(\varphi) \; . \tag{9.1}$$

Similarly, in Seiberg-Witten theory, the mass of a BPS state with charge $\gamma \in H_1(C,\mathbb{Z})$ is equal to $|\mathcal{Z}_\gamma(u)|$ where $\mathcal{Z}$ is a central charge given by

$$\mathcal{Z}_\gamma(u) \;=\; \int_\gamma \lambda(u) \; . \tag{9.2}$$

Typically, some dyon will becomes massless at a singularity of the Coulomb branch and it might happen that at a singularity u_*, two *mutually non-local* dyons become massless. By mutually non-local, we mean that two charges $\gamma_1, \gamma_2 \in H_1(C,\mathbb{Z})$ satisfy

$$\langle \gamma_1, \gamma_2 \rangle \neq 0 \tag{9.3}$$

where $\langle\;,\;\rangle$ is the intersection product on $H_1(C,\mathbb{Z})$. In particular, this implies that there is no duality frame in which γ_1 and γ_2 are both purely electric (A-cycles) or purely magnetic (B-cycles) and this rules out a Lagrangian description. The condition of masslessness is simply the condition that

$$\mathcal{Z}_{\gamma_1}(u_*) \;=\; \mathcal{Z}_{\gamma_2}(u_*) \;=\; 0 \; . \tag{9.4}$$

It is known that if two mutually non-local dyons become massless at some point u_*, then the underlying quantum field theory at that point is an interacting, non-Lagrangian *superconformal field theory* (SCFT) that is commonly reffered to as an Argyres-Douglas theory [105, 106].

Now to the first bit of speculation. We note that the rank two attractor points φ_* that we found on families of Calabi-Yau manifolds with $h^{2,1} = 1$ are the exact analogues of Argyres-Douglas points. Suppose X is a rank two attractor variety i.e.

$$\Lambda \oplus \Lambda^\perp \subset H^3(X,\mathbb{Z}) \tag{9.5}$$

where

$$\Lambda \otimes \mathbb{C} \;=\; H^{3,0} \oplus H^{0,3} \qquad \text{and} \qquad \Lambda^\perp \otimes \mathbb{C} \;=\; H^{2,1} \oplus H^{1,2} \; . \tag{9.6}$$

It is straightforward to check in specific examples that the generators of $\Lambda^\perp$ are mutually non-local i.e. they have non-trivial intersection (see Tables 5.2, 5.4, 6.2 and 6.4) . For example, the generators of $\Lambda^\perp$ at the rank two attractor $\varphi = 1$ of AESZ 101 are in an integral symplectic basis given by

$$Q_1 \;=\; \begin{pmatrix} 5 \\ -6 \\ 0 \\ 1 \end{pmatrix} \qquad \text{and} \qquad Q_2 \;=\; \begin{pmatrix} 0 \\ -3 \\ 1 \\ 0 \end{pmatrix} \; . \tag{9.7}$$

They are mutually non-local because

$$\int_X \Gamma_1 \wedge \Gamma_2 \;=\; Q_1^T \Sigma Q_2 \;=\; 2 \tag{9.8}$$

and they have vanishing mass

$$|Z_{\gamma_i}(\varphi)| \;=\; \mathrm{e}^{K/2}\left| \int_X \Gamma_i \wedge \Omega \;\right| \;=\; \mathrm{e}^{K/2}|Q_i^T \Sigma \Pi| \;=\; 0 \tag{9.9}$$

because Γ_1 and Γ_2 have Hodge numbers $(2,1)+(1,2)$ at the attractor point.

Based on the similarity between rank two attractor points and Argyres-Douglas theories, one might conjecture that a compactification of IIB on a rank two attractor variety X with $h^{2,1}=1$ leads to a conformal supergravity in four dimensions. Some support for this conjecture is given by the observation that the topological string free energies associated to AESZ 100 and AESZ 101 have holomorphic ambiguities with poles at the attractor point. For example, for $g \geq 2$, the holomorphic ambiguities of AESZ 101 are given by

$$f^{(g)}(\varphi) \;=\; \sum_{n=0}^{2g-2} a_n \varphi^n + \frac{\sum_{n=0}^{2g-3} b_n \varphi^n}{(\varphi+1)^{2g-2}} + \frac{\sum_{n=0}^{2g-3} c_n \varphi^n}{(\varphi-1)^{2g-2}} + \frac{\sum_{n=0}^{4g-5} d_n \varphi^n}{(1-123\varphi+\varphi^2)^{2g-2}} \tag{9.10}$$

where the pole at $\varphi=1$ is distinguished by fact that it is exactly cancelled in the physical free energy F_g. Aside from the pole at $\varphi=1$, it is known that each of the remaining poles is the result of a D3 brane that become massless at that point (see discussion of Section 8.3). Since we can explicitly identify homology classes with vanishing central charge at $\varphi=1$, it is natural to conjecture that the pole at $\varphi \;=\; 1$ arises in exactly the same way.[2]

Of course, it is important to identify the actual submanifolds wrapped by the D3 branes instead of just dealing with (co)homology classes as we have doing been for much of this thesis. This brings us to our second point.

The existence of the BPS black holes described in previous sections requires that the associated D3 brane takes the shape of a *special Lagrangian submanifold* L. If L is embedded in X via $\iota : L \hookrightarrow X$, we say that it is special Lagrangian if

$$\iota^*\omega \;=\; 0 \qquad\quad \text{and} \qquad\quad \iota^*\left\{\mathrm{Im}\,\mathrm{e}^{-i\theta}\Omega\right\} \;=\; 0 \tag{9.11}$$

where ω and Ω are the Kähler and holomorphic 3-forms of X respectively and θ is a real number that is included because of the ambiguous normalisation of Ω.

[2]The attractor points of AESZ 34 are *not* singularities of the Picard-Fuchs equation. The associated holomorphic ambiguities are, therefore, regular at these points and we remain agnostic about whether we should expect a conformal supergravity. Similarly, there are known examples of apparent singularities where we have searched for an have been unable to find integral vanishing cycles by numerical computation e.g. the (4,1) manifold [23]. We remain agnostic about such points as well. AESZ 100 and 101 are distinguished by the fact that they have apparent singularities that are also rank 2 attractors and, thus, we are able to identify integral vanishing cycles.

Finding specific special Lagrangian submanifolds of X is typically a very difficult problem. For example, consider that D2 branes wrap holomorphic curves in X and, until [24], the counting of such curves at genus 0 on the quintic was difficult for all but the smallest degrees. Nevertheless, since AESZ 100 and AESZ 101 are Hadamard products, there is some hope of directly finding the relevant special Lagrangian 3-cycles. This uses the fact that a local models for AESZ 100 and 101 are provided by fibre product of families of elliptic curves.

Consider AESZ 101. It is equal to the Hadamard square $(b) * (b)$ where (b) is a second order Picard-Fuchs operator (see Appendix C) that computes the periods of a family of elliptic curves

$$E \to \mathbb{P}^1 \backslash \Delta \tag{9.12}$$

where $\Delta = \left\{\frac{1}{2}(-11-5\sqrt{5}), 0, \frac{1}{2}(-11+5\sqrt{5}), \infty\right\}$. Since solutions of AESZ 101 compute the periods of the holomorphic 3-form on the threefold $\mathcal{E}^{(2)}(\varphi)$ that we define as the fibre of

$$\mathcal{E}^{(2)} \to \mathbb{P}^1 \backslash \Delta^2 \tag{9.13}$$

over φ where $\Delta^2 = \left\{0, \frac{1}{2}(123-55\sqrt{5}), 1, \frac{1}{2}(123+55\sqrt{5}),\ \infty\right\}$ (see Section 6.1), we look for a real invariant 3 dimensional submanifold L of $\mathcal{E}^{(2)}(1)$ that are dual to the cohomology classes in $\Lambda^{\perp}$ and Λ.

An invariant 3-submanifold of $\mathcal{E}^{(2)}(1)$ can be constructured by first choosing a closed loop $\gamma_0 \subset E(s)$ that is mapped to another closed loop $A\gamma_0 \subset E(\frac{1}{s})$ under the involution of E. The union of the 2-cycles $\gamma_0 \times A\gamma_0 \subset E(s) \times E(\frac{1}{s})$ over an open invariant path on $\mathbb{P}^1 \backslash \Delta$ is an invariant 3-submanifold L of $\mathcal{E}^{(2)}(1)$. Now suppose that the invariant path in $\mathbb{P}^1 \backslash \Delta$ runs between two points $s_0, \widetilde{s}_0 \in \Delta$. If γ_0 is the vanishing cycle at s_0, then $A\gamma_0$ must be the vanishing cycle at $\widetilde{s}_0$ and we might expect that L can be compactified to a 3-cycle.

To visualise L, it is helpful to imagine a famliy of circles of varying radius over an open interval where the radius tends to zero at either end of the interval. The union of these circles is a 2-sphere with the north and south pole missing. The construction of L proceeds in essentially the same way.

For example, let L be the union of the 2 cycles $A\gamma_0 \times \gamma_0$ over the positive imaginary axis where γ_0 is the vanishing cycle at $s = 0$. We may check a non-trivial necessary condition for L to be special Lagrangian by considering a 1-form Υ on $\mathbb{P}^1 \backslash \Delta$ defined below. We find it helpful to use the notation

$$\mathcal{Z}_\gamma(s) \;=\; \int_\gamma \mathrm{d}z(s) \;=\; Q^T \sigma \varpi(s) \tag{9.14}$$

where ϖ is a vector of periods of (b) defined in (9.19) and Q is a vector of periods of Γ which is dual to γ.

A slight modification of Equation (6.11) which computes periods of the holomorphic 3-form on $\mathcal{E}^{(2)}(1)$ leads us to the 1-form Υ on $\mathbb{P}^1\backslash\Delta$

$$\begin{aligned}\Upsilon &= \frac{1}{(2\pi i)^2}\int_{T(A\gamma_0)\times T(\gamma_0)\times S^1}\frac{\eta_1\wedge\eta_2\wedge \mathrm{d}s\wedge \mathrm{d}t}{(F(x)-s)(G(y)-t)(st-1)} \\ &= \frac{1}{(2\pi i)}\mathcal{Z}_{A\gamma_0}(s)\mathcal{Z}_{\gamma_0}\left(\frac{1}{s}\right)\frac{\mathrm{d}s}{s}\end{aligned} \tag{9.15}$$

where we have used the fact that $\mathcal{Z}_{\gamma_0}$ is the holomorphic period around $s=0$. A necessary condition that Υ must satisfy is

$$\operatorname{Im} \mathrm{e}^{-i\theta}\Upsilon|_{i\mathbb{R}_{>0}} = 0 \tag{9.16}$$

for some $\theta\in\mathbb{R}$. We show this by studying solutions of the Picard-Fuchs equation (b) and the associated involution.

The operator (b) has the Riemann symbol

$$\mathcal{P}\left\{\begin{array}{cccc c} \frac{1}{2}(-11-5\sqrt{5}) & 0 & \frac{1}{2}(-11+5\sqrt{5}) & \infty & \\ \hline 0 & 0 & 0 & 1 & s \\ 0 & 0 & 0 & 1 & \end{array}\right\} \tag{9.17}$$

and is invariant under the involution

$$s\to -\frac{1}{s}\ . \tag{9.18}$$

An integral symplectic basis for a second order operator is straightforward to find. Around $s=0$, a basis of solutions is given by

$$\begin{aligned}\varpi_0(\varphi) &= f_0(\varphi) \\ \varpi_1(\varphi) &= \frac{1}{2\pi i}\Big(f_0(\varphi)\,\log(\varphi)+f_1(\varphi)\Big)\end{aligned} \tag{9.19}$$

where $f_0(0)=1$ and $f_1(0)=0$. The remaining coefficients are fixed recursively and the first few terms are given by

$$\begin{aligned}f_0(\varphi) &= 1+3\varphi+19\varphi^2+147\varphi^3+1251\varphi^4+11253\varphi^5+104959\varphi^6+\ldots \\ f_1(\varphi) &= 5\varphi+\frac{75}{2}\varphi^2+\frac{1855}{6}\varphi^3+\frac{10875}{4}\varphi^4+\frac{299387}{12}\varphi^5+\frac{943397}{4}\varphi^6+\ldots.\end{aligned} \tag{9.20}$$

By numerical analytic continuation of the periods ϖ, we find that the involution (9.18) acts on the periods as

$$-\frac{1}{s}\varpi\left(-\frac{1}{s}\right) = A\,\varpi(s) \tag{9.21}$$

where

$$A = \begin{pmatrix}-2 & 5\\ -1 & 2\end{pmatrix}\ \in SL(2,\mathbb{Z}) \tag{9.22}$$

The matrix A has eigenvalues $+i$ and $-i$ with eigenvectors

$$Q_+ \;=\; \begin{pmatrix} 2-i \\ 1 \end{pmatrix} \qquad \text{and} \qquad Q_- \;=\; \begin{pmatrix} 2+i \\ 1 \end{pmatrix} \tag{9.23}$$

respectively. $Q_\pm$ are the periods of cohomology classes that we denote by $\Gamma_\pm \in H^1(E,\mathbb{Z}[i])$ which are dual to homology classes $\gamma_\pm \in H_1(E,\mathbb{Z}[i])$. The points $s = \pm i$ are fixed under the involution (9.18) so they are the analogues of the rank two attractors of AESZ 101 but this time on a family of elliptic curves. We find that the complex structure at $s = i$ is such that the cohomology classes Γ_+ and Γ_- have Hodge numbers $(1,0)$ and $(0,1)$ respectively which we confirm numerically by checking that

$$\int_{\gamma_+} \mathrm{d}z \;=\; \int_E \Gamma_+ \wedge \mathrm{d}z \;=\; Q_+^T \sigma \varpi \;=\; 0 \tag{9.24}$$

and

$$\int_{\gamma_-} \mathrm{d}\overline{z} \;=\; \int_E \Gamma_- \wedge \mathrm{d}z \;=\; Q_-^T \sigma \overline{\varpi} \;=\; 0 \tag{9.25}$$

where $\mathrm{d}z$ is the holomorphic 1-form of E and σ is the 2×2 standard symplectic form.
By Picard-Lefschetz theory, we see that the vanishing cycle γ_0 at $s = 0$ has the components

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \;=\; \left(\frac{1}{2} - i\right) Q_+ + \left(\frac{1}{2} + i\right) Q_- \tag{9.26}$$

in the same basis of solutions as (9.19)
It follows from Equation (9.21) that

$$\mathcal{Z}_{\gamma_\pm}(s) \;=\; \mp i \left(\frac{1}{s}\right) \mathcal{Z}_{\gamma_\pm}\left(-\frac{1}{s}\right) . \tag{9.27}$$

which implies that

$$\mathcal{Z}_{A\gamma_0}(s) \;=\; \frac{1}{s} \mathcal{Z}_{\gamma_0}\left(-\frac{1}{s}\right) . \tag{9.28}$$

The above identity and the change of variables $\widetilde{s} = -\frac{1}{s}$ lead to the expression

$$\Upsilon \;=\; \frac{1}{2\pi i} \mathcal{Z}_{\gamma_0}(\widetilde{s}) \mathcal{Z}_{A\gamma_0}(-\widetilde{s}) \mathrm{d}\widetilde{s} . \tag{9.29}$$

We now restrict Υ to the real imaginary axis i.e. $\widetilde{s} = iy$ where $y \in \mathbb{R}$ and find that

$$\Upsilon \;=\; \frac{1}{2\pi} |\mathcal{Z}_{\gamma_0}(iy)|^2 \mathrm{d}y . \tag{9.30}$$

Note that we have used the fact that $\mathcal{Z}_{\gamma_0}$ is a real function. Thus,

$$\operatorname{Im} \Upsilon|_{i\mathbb{R}_{>0}} = 0 . \tag{9.31}$$

Paths on $\mathbb{P}^1$ like the one described above in the definition of L have appeared before in the string theory literature in attempts to understand Seiberg-Witten theory from a stringy perspective [107]. In the context of field theories of *class S*, such paths are known as *spectral networks* [108].[3]

Finally, at the risk of becoming overly speculative, we point out that $\mathcal{E}^{(2)}(1)$ is very nearly the Kugo-Sato variety described in Section 4.2. Since a weight 4 cusp form $f \in S_4(\Gamma_0(N))$ determines a holomorphic 3-form on $\mathcal{E}^{(2)}(1)$, this raises the intriguing possibility that a special Lagrangian submanifold may be determined by a condition on f. It would be interesting to better understand this condition and the resulting special Lagrangian. For example, one might compute open string instanton numbers associated to L.[4]

Open string instanton numbers have been computed for the real quintic (a special Lagrangian submanifold) inside a quintic threefold by Walcher in [109]. His method involved finding solutions of an extension of the Picard-Fuchs equation and it would be interesting to see if these methods apply here and if the weight 4 eigenform plays any role in the counting of open string instanton numbers. At the very least, we hope to find identities like those found in Chapter 8 in the context of closed strings. This is the subject of ongoing work that we hope to return to elsewhere.

[3]Theories of class S are four dimensional $\mathcal{N} = 2$ quantum field theories that arise from the compactification of the world volume theory of an M5 brane on a Riemann surface.

[4]The closed string instanton numbers computed in Chapter 8 count, in some sense, the number of holomorphic maps from a genus g Riemann surface (the world-sheet of a closed string) to a curve in a given homology class in a Calabi-Yau manifold X. The generalisation to open strings instead computes the the number of holomorphic maps from a genus g Riemann surface with boundary (the world-sheet of an open string) into a Calabi-Yau manifold where the boundary of the Riemann surface is mapped to a special Lagrangian L. More precisely, the open string instanton numbers compute the number of such maps into a given relative homology class $H_2(X, L; \mathbb{Z})$.

Appendix A

A Quotient of a Hulek-Verrill Manifold (AESZ 34)

A.1 Geometry

Hulek and Verrill in [1] consider a family of Calabi-Yau manifolds that are birational to a variety defined on $\mathbb{T} = \mathbb{P}^4 \setminus \{X_1X_2X_3X_4X_5 = 0\}$ by the equation

$$(X_1 + X_2 + X_3 + X_4 + X_5)\left(\frac{\mu_1}{X_1} + \frac{\mu_2}{X_2} + \frac{\mu_3}{X_3} + \frac{\mu_4}{X_4} + \frac{\mu_5}{X_5}\right) = \mu_6 \ . \tag{A.1}$$

For generic parameters $\mu_1, ..., \mu_6$, the variety $X^\sharp$ that is defined by this equation is smooth on $\mathbb{T}$, however there are 30 nodes where a subset of the coordinates X_j vanish. Three nodes lie on each of ten surfaces. The singularities can be simultaneously resolved by blowing up each of these these ten surfaces yielding a smooth Calabi-Yau manifold $\widehat{X}$.

A multiplication of the coefficients μ_j, $j = 1, ..., 6$ in (A.1) by a common scale has no effect, so superficially this equation defines a five parameter family of manifolds. The equation defines a reflexive polyhedron, in the sense of Batyrev. Analysis of the polyhedron and the resolution just described reveals that the superficial count of complex structure parameters is in fact correct and that the Hodge numbers for a generic member of the family are given by

$$h^{pq}\big(\widehat{X}\big) \;=\; \begin{matrix} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 45 & & 0 & \\ 1 & & 5 & & 5 & & 1 \\ & 0 & & 45 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{matrix} \ . \tag{A.2}$$

Thus $\chi\big(\widehat{X}\big) = 2(h^{11} - h^{21}) = 80$.

We consider now a 1-parameter subfamily where $\mu_j = 1, \ j = 1, \ldots, 5$ and $\mu_6 = 1/\varphi$ then the manifold admits a symmetry isomorphic to $\mathbb{Z}/10\mathbb{Z}$ with generator

$$X_i \ \to \ \frac{1}{X_{i+1}} \ , \tag{A.3}$$

with the indices understood mod 5. This symmetry is fixed point free if $\varphi \notin \{1, \frac{1}{9}, \frac{1}{25}, \infty\}$. This is easy to see for points of the singular variety $X^\sharp$. For the resolution $\widehat{X}$, we note that, since it is a resolution, there is a projection $\widehat{X} \to X^\sharp$ and so a fixed point of $\widehat{X}$ would project to a fixed point of $X^\sharp$, and these do not exist unless φ takes one of the values $\{1, \frac{1}{9}, \frac{1}{25}, \infty\}$. Taking the quotient by either the $\mathbb{Z}/10\mathbb{Z}$, or the $\mathbb{Z}/5\mathbb{Z}$ subgroup with generator $X_i \to X_{i+2}$, yields a family of smooth manifolds, that we shall denote by X, with one complex structure parameter and the following Hodge numbers:

$$h^{pq}(X) \;=\; \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 4\kappa{+}1 & & 0 & \\ 1 & & 1 & & 1 & & 1 \\ & 0 & & 4\kappa{+}1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}\;, \tag{A.4}$$

where $\kappa = 1$ or 2 if the quotient is by the group of order 10 or 5 respectively.

We wish to describe the singular members of the family X_φ and how the symmetries act on these in somewhat greater detail. We will restrict attention to points X_j in $\mathbb{T}$ since the discussion of the points not in $\mathbb{T}$ is part of the story of how the non-compact manifold described by (A.1) is compactified so as to yield a Calabi-Yau manifold.

A first remark is that the manifold defined by (A.1) can be regarded as arising from two linear equations in six variables

$$\sum_{i=1}^{6} X_i \;=\; 0 \quad\text{and}\quad \sum_{i=1}^{6} \frac{\mu_i}{X_i} \;=\; 0\;, \tag{A.5}$$

since eliminating X_6 between these two equations returns us to (A.1).

Let $P(X)$ denote the defining equation

$$P(X) \;=\; \left(\sum_{i=1}^{5} X_i\right)\left(\sum_{i=1}^{5} \frac{1}{X_i}\right) - \frac{1}{\varphi}\;.$$

The partial derivatives of P vanish at a singularity, yielding the conditions

$$\left(\sum_{i=1}^{5} \frac{1}{X_i}\right) - \frac{1}{X_j^2}\left(\sum_{i=1}^{5} X_i\right) \;=\; 0\;, \tag{A.6}$$

for $j = 1, ..., 5$. It follows, since we are assuming that X_j does not vanish, that if either $\sum_{i=1}^{5} X_i$ or $\sum_{i=1}^{5} \frac{1}{X_i}$ vanish, then both sums vanish. This can only happen when $\varphi = \infty$, but in this case (A.6) provides no further constraints so the singular set is a two-dimensional surface described by the equations

$$\sum_{i=1}^{5} X_i \;=\; 0 \quad\text{and}\quad \sum_{i=1}^{5} \frac{1}{X_i} \;=\; 0\;, \tag{A.7}$$

analogous to (A.5) but with five variables, instead of six. This being so, we expect the singular set to be a K3 surface.

If now neither sum in (A.6) vanishes, then the X_j^2 are all equal and by choice of scale can all be set to unity. Let us suppose that r of the X_j take the value -1 and the remaining $5-r$ take the value 1. So, up to permutation of the coordinates, the singular points are given by

$$X_j \;=\; (\underbrace{1,\ldots,1}_{5-r},\underbrace{-1,\ldots,-1}_{r})\;, \tag{A.8}$$

and we may assume that $r = 0, 1$ or 2. Such a point lies on the manifold with $\varphi = (5-2r)^{-2}$.

$\underline{r = 0}$

In this case $\varphi = 1/25$ and there is one singular point $X_j = (1, 1, 1, 1, 1)$. This point is fixed by both the $\mathbb{Z}/5\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$ symmetry generators and so gives rise to a single point that is fixed by either $\mathbb{Z}/10\mathbb{Z}$ or $\mathbb{Z}/5\mathbb{Z}$ on the respective quotient manifolds.

$\underline{r = 1}$

This case corresponds to $\varphi = 1/9$ and $X_j = (1, 1, 1, 1, -1)$, up to cyclic permutation. These five points are fixed by the $\mathbb{Z}/2\mathbb{Z}$ generator and give rise to a single point on the $\mathbb{Z}/5\mathbb{Z}$ quotient and a single point that is fixed by $\mathbb{Z}/2\mathbb{Z}$ on the $\mathbb{Z}/10\mathbb{Z}$ quotient.

$\underline{r = 2}$

The last case corresponds to $\varphi = 1$. Now there are ten points, which are the cyclic permutations of $(1, 1, 1, -1, -1)$ and $(1, 1, -1, 1, -1)$. These points give rise to two points in the $\mathbb{Z}/5\mathbb{Z}$ quotient and two points fixed by a $\mathbb{Z}/2\mathbb{Z}$ action, in the $\mathbb{Z}/10\mathbb{Z}$ quotient.

It is easy to see that a point that is fixed by an element of $\mathbb{Z}/10\mathbb{Z}$ must be fixed by either, or both of, the $\mathbb{Z}/2\mathbb{Z}$ or the $\mathbb{Z}/5\mathbb{Z}$ generators. A point fixed by the $\mathbb{Z}/2\mathbb{Z}$ generator is, up to permutation, of the form (A.8). So these coincide with the singular points of the $\varphi = 1/25,\ 1/9,\ 1$ manifolds and have the effect of turning the conifold singularities into hyperconifold singularities. The fixed point $(1, 1, 1, 1, 1)$ is also fixed by the $\mathbb{Z}/5\mathbb{Z}$ generator. The other fixed points of the $\mathbb{Z}/5\mathbb{Z}$ generator are the four points

$$X_j \;=\; \zeta^{jk}\;; \quad k \;=\; 1,\, 2,\, 3,\, 4\;, \tag{A.9}$$

where ζ is a nontrivial fifth root of unity. For such a point we have

$$\sum_j X_j = \sum_j 1/X_j = 0, \tag{A.10}$$

so these lie in the singular surface of the $\varphi = \infty$ manifold.

The order of the group that fixes each of the singularities at each of the conifold points will determine the holomorphic ambiguity in the genus 1 topological string free energy in

Chapter 8. These integers may alternatively be computed from the monodromy matrices around each of the conifold point as described in Section 2.1.2.2.

The mirror family of Calabi-Yau manifolds may be found in the appendix of [16] which should prove useful in attempts to directly verify the genus 0 and genus 1 instanton numbers computed in Chapter 8. A direct computation of the number of genus 2 curves should provide the additional boundary conditions required to fix the holomorphic ambiguity at genus 2.

A.2 Feynman Diagrams and Lattice Walks

It is interesting that the Hulek-Verrill manifold with five complex structure parameters (before taking the quotient) has appeared prominently in other, seemingly unrelated, contexts. One of these is in the study of field theory amplitudes, principally in relation to the banana or sunrise graphs. An example, with four loops, is shown in Figure A.1.

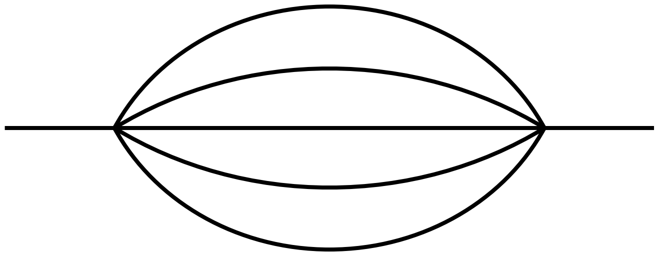

Figure A.1: *The four-loop banana graph that is related to the Hulek-Verrill manifold.*

This is a Feynman diagram for a scalar field with momentum p flowing through the diagram and the internal lines refer to particles of mass m_i, $i = 1, \ldots, 5$. Denoting the maximally cut diagram in two dimensions by $F(p^2)$ and with the identifications

$$\mu_i \;=\; m_i^2 \quad \text{and} \quad p^2 \;=\; \frac{1}{\varphi}\,, \tag{A.11}$$

it has been observed that $p^2 F(p^2)$ is a period for the five parameter Hulek-Verrill manifold defined by the $n = 5$ case of the equation

$$\left(\sum_{i=1}^{n} X_i\right)\left(\sum_{i=1}^{n} \frac{\mu_i}{X_i}\right) \;=\; \frac{1}{\varphi}\,. \tag{A.12}$$

Note however that this equation is often written with coordinates related to those here by the transformation $X_i \to 1/X_i$. In the case that all the masses are equal, the quantity $p^2 F(p^2)$ is a period for the quotient manifold. There is a considerable literature on this subject, to which we cannot do justice. The expository article of Vanhove [110] and references cited therein can serve as an introduction.

The identities of Chapter 5 imply that for certain values of external momenta, certain maximally cut Feynman diagrams evaluate to linear combinations of critical L-values. However, it's not clear if these values of the external momentum are distinguished in from a field theory perspective. On a related note, it would be interesting to try and interpret the attractor mechanism from the point of view of Feynman diagrams. In this case, complex structure moduli are identified with external momenta and one naturally wonders if this dictionary can be expanded. For example, what would play the role of the "charge lattice" $H^3(X, \mathbb{Z})$? This is the subject of ongoing investigation.

Another seemingly unrelated area where the Hulek and Verril manifold has appeared is in the study of lattice walks. The fundamental periods of many Calabi-Yau manifolds have an interpretation as generating functions for the numbers of lattice walks, with the n^{th} coefficient a_n being the number of lattice walks that return tthe origin after n steps. The lattice in question being the lattice generated by the monomials of the defining equation. For the Hulek-Verrill manifold these considerations apply and the fundamental period is generating function for walks in the A_4 lattice. The Hulek-Verrill manifold fits into a closely related sequence of manifolds that correspond to taking $n = 3, 4, 5, \ldots$ in (A.12). Verrill [111] examined this sequence and noted, for the case of the K3 manifold, corresponding to $n = 4$, that the fundamental period is the generating function for lattice walks in the A_3 lattice.

The study of lattice walks and of Feynman diagrams such as the banana graph leads naturally to integrals of products of Bessel functions, so the Hulek-Verrill manifold has appeared also in this context, see for example [112].

Appendix B

Arithmetic in $\mathbb{Q}(\sqrt{17})$

In disccusing the attractor points $33 \pm 8\sqrt{17}$, where possible, we have tried to simplify complicated expressions involving $\sqrt{17}$ by writing them in terms of simple algebraic integers. Although much of this section applies to other quadratic extensions of $\mathbb{Q}$, we will focus on $\mathbb{Q}(\sqrt{17})$ which is the field of numbers of the form

$$t = r + s\sqrt{17} \; ; \quad r, s \in \mathbb{Q} \; . \tag{B.1}$$

The conjugate of t, denoted by $\bar{t}$ is the number

$$\bar{t} = r - s\sqrt{17} \; . \tag{B.2}$$

For the avoidance of doubt: in this subsection, the quantity t bears no relation to the coordinate of special geometry.
An integer in a field $\mathbb{K}$ is a number $x \in \mathbb{K}$ that is a root of an irreducible monic polynomial with coefficients in $\mathbb{Z}$. Thus, for example, the rational integers, as well as numbers such as $\sqrt{17}$ and $(1+\sqrt{17})/2$, are integers of $\mathbb{Q}(\sqrt{17})$, since they satisfy the respective equations

$$\begin{aligned} x - m &= 0 \; ; \quad m \in \mathbb{Z} \; , \\ x^2 - 17 &= 0 \; , \\ x^2 - x - 4 &= 0 \; . \end{aligned} \tag{B.3}$$

If x is an integer, then so is $-x$, and one can show that the sum and product of two integers is again an integer. It follows from the foregoing that the integers of $\mathbb{Q}(\sqrt{17})$ are of the form

$$a + b\sqrt{17} \; ; a, b \in \mathbb{Z} \quad \text{and} \quad \frac{a + b\sqrt{17}}{2} \text{ , if } a \text{ and } b \text{ are both odd integers.} \tag{B.4}$$

A number $t \in \mathbb{Q}(\sqrt{17})$ of the form (B.1) has a *norm* $\mathcal{N}(t)$

$$\mathcal{N}(t) = t\bar{t} = r^2 - 17s^2 \; . \tag{B.5}$$

The term norm is universally used in this context, even though it is somewhat a misnomer, since $\mathcal{N}(t)$ is not necessarily positive. It has however the property that $\mathcal{N}(yz) = \mathcal{N}(y)\mathcal{N}(z)$, for all $y, z \in \mathbb{Q}(\sqrt{17})$. Moreover, $\mathcal{N}(x) \in \mathbb{Z}$ if x is an integer of the field.

An integer, whose inverse is also an integer, is a *unit* and the set of all units form a group. A unit necessarily has norm ± 1. For $\mathbb{Q}(\sqrt{17})$ the unit group is infinite and is generated by $4 + \sqrt{17}$ and we have

$$\mathcal{N}(4 \pm \sqrt{17}) \;=\; -1\ . \tag{B.6}$$

The conjugate satisfies $4 - \sqrt{17} = -\,(4 + \sqrt{17})^{-1}$ and so also generates.

The attractor points $33 \pm 8\sqrt{17}$ are units, so are powers of the generator. In fact

$$33 \pm 8\sqrt{17} \;=\; (4 \pm \sqrt{17})^2\ . \tag{B.7}$$

The existence of units complicates the process of factorizing integers. In general, for a field $\mathbb{Q}(\sqrt{d})$, the factorisation of integers, even leaving aside multiplication by units, is not unique. However, for $\mathbb{Q}(\sqrt{17})$, it is unique, up to multiplication by units. Given an integer x, that is not a unit, we can ask if it can be factored into a product $x = yz$ of integers, neither of which is a unit. If x cannot be factored, in this way, then x is a *prime* of the field. Since $\mathcal{N}(x) = \mathcal{N}(y)\mathcal{N}(z)$ the integer x can only factor if $\mathcal{N}(x)$ factors as a rational integer. In particular, if $\mathcal{N}(x)$ is a rational prime, then x is a prime of the field. Note that some of the rational primes factor and so are not primes of the field. For example

$$2 \;=\; -\left(\frac{3+\sqrt{17}}{2}\right)\left(\frac{3-\sqrt{17}}{2}\right) \quad \text{and} \quad 17 \;=\; \left(\sqrt{17}\right)^2\ . \tag{B.8}$$

We will often factorise integers in the following, in order both to save space, particularly in tables, and to show that otherwise inscrutable numbers are the products of a small number of primes with small norm. The numbers $4 \pm \sqrt{17}$ and $(3 \pm \sqrt{17})/2$, the latter being the prime with the smallest absolute value of the norm, so somewhat analogous to 2, are ubiquitous in expressions, so we will often write

$$4 \pm \sqrt{17} \;=\; \epsilon_\pm \quad \text{and} \quad \frac{3 \pm \sqrt{17}}{2} \;=\; \delta_\pm\ . \tag{B.9}$$

As an illustration of the utility of this consider a relation that we will meet shortly

$$j(\tau_+^\perp) \;=\; \frac{1}{2^4}\epsilon_+^4\delta_-^2\left(2-\sqrt{17}\right)^3\left(14-5\sqrt{17}\right)^3\ . \tag{B.10}$$

If expanded, the right hand side becomes the somewhat more inscrutable number

$$\frac{1}{32}\left(3832069 + 915957\sqrt{17}\right)\ . \tag{B.11}$$

Appendix C

Second Order Operators

List of some second order Picard-Fuchs operators that appered in [70] and the coefficients a_n that appear in the holomorphic period around $\varphi = 0$. We note that all of these operators are invariant under an involution that exchanges $\varphi = 0$ and $\varphi = \infty$ and, by taking Hadamard products, we find operators with attractor points of rank two.

(a)	
Operator	$(1+\varphi)(-1+8\varphi)\ \theta^2 + \varphi(7+16\varphi)\ \theta + 2\varphi(1+4\varphi)$
Riemann Symbol	$\mathcal{P}\left\{\begin{array}{cccc} -1 & 0 & \frac{1}{8} & \infty \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{array}\ s\right\}$
a_n	$\sum_{k=0}^{n} \binom{n}{k}^3$
Involution	$s \to -\frac{1}{8s}$

(b)	
Operator	$(1-11\varphi-\varphi^2)\ \theta^2-\varphi(11+2\varphi)\ \theta-\varphi(3+\varphi)$
Riemann Symbol	$\mathcal{P}\left\{\begin{array}{cccc} 0 & \frac{1}{2}(-11+5\sqrt{5}) & \frac{1}{2}(-11-5\sqrt{5}) & \infty \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{array}\ \varphi\right\}$
a_n	$\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}$
Involution	$s\to-\frac{1}{s}$

(c)	
Operator	$(1-\varphi)(1-9\varphi)\ \theta^2-2\varphi(5-9\varphi)\ \theta-3\varphi(1-3\varphi)$
Riemann Symbol	$\mathcal{P}\left\{\begin{array}{cccc} 0 & \frac{1}{9} & 1 & \infty \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{array}\ \varphi\right\}$
a_n	$\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{k}$
Involution	$s\to\frac{1}{9s}$

(d)	
Operator	$(1-4\varphi)(1-8\varphi)\ \theta^2-4\varphi(3-16\varphi)\ \theta-4\varphi(1-8\varphi)$
Riemann Symbol	$\mathcal{P}\left\{\begin{array}{cccc} 0 & \frac{1}{8} & \frac{1}{4} & \infty \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{array}\ s\right\}$
a_n	$\sum_{k=0}^{n}\binom{n}{k}\binom{2k}{k}\binom{2n-2k}{n-k}$
Involution	$s\to\frac{1}{32s}$

(e)	
Operator	$(1-16\varphi)^2\ \theta^2 - 32\varphi(1-16\varphi)\ \theta - 4\varphi(3-64\varphi)$
Riemann Symbol	$\mathcal{P}\left\{\begin{array}{ccc} 0 & \frac{1}{16} & \infty \\ \hline 0 & -\frac{1}{2} & 1 \\ 0 & -\frac{1}{2} & 1 \end{array}\ s\right\}$
a_n	$\sum_{k=0}^{n} 4^{n-k}\binom{2k}{k}^2\binom{2n-2k}{n-k}$
Involution	$s \to \frac{1}{256s}$

(f)	
Operator	$(1-9\varphi+27\varphi^2)\ \theta^2 - 9\varphi(1-6\varphi)\ \theta - 3\varphi(1-9\varphi)$
Riemann Symbol	$\mathcal{P}\left\{\begin{array}{cccc} 0 & \frac{1}{18}(3-i\sqrt{3}) & \frac{1}{18}(3+i\sqrt{3}) & \infty \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{array}\ \varphi\right\}$
a_n	$\sum_{k=0}^{n}(-1)^k 3^{n-3k}\binom{n}{3k}\frac{(3k)!}{k!^3}$
Involution	$s \to \frac{1}{27s}$

(g)	
Operator	$(1-8\varphi)(1-9\varphi)\ \theta^2 - \varphi(17-144\varphi)\ \theta - 6\varphi(1-12\varphi)$
Riemann Symbol	$\mathcal{P}\left\{\begin{array}{cccc} 0 & \frac{1}{9} & \frac{1}{8} & \infty \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{array}\ s\right\}$
a_n	no closed form expression known
Involution	$s \to \frac{1}{72s}$

(h)	
Operator	$(1-27\varphi)^2\ \theta^2 - 54\varphi(1-27\varphi)\ \theta - 3\varphi(7-243\varphi)$
Riemann Symbol	$\mathcal{P}\left\{\begin{array}{ccc\|c} 0 & \frac{1}{27} & \infty & \\ \hline 0 & -\frac{1}{3} & 1 & s \\ 0 & -\frac{2}{3} & 1 & \end{array}\right\}$
a_n	$27^n \sum_{k=0}^{n} (-1)^k \binom{-2/3}{k} \binom{-1/3}{n-k}^2$
Involution	$s \to \frac{1}{729s}$

(i)	
Operator	$(1-64\varphi)^2\ \theta^2 - 128\varphi(1-64\varphi)\ \theta - 4\varphi(13-1024\varphi)$
Riemann Symbol	$\mathcal{P}\left\{\begin{array}{ccc\|c} 0 & \frac{1}{27} & \infty & \\ \hline 0 & -\frac{1}{4} & 1 & \varphi \\ 0 & -\frac{3}{4} & 1 & \end{array}\right\}$
a_n	$64^n \sum_{k=0}^{n} (-1)^k \binom{-3/4}{k} \binom{-1/4}{n-k}^2$
Involution	$s \to \frac{1}{4096s}$

(j)	
Operator	$(1-432\varphi)^2\ \theta^2 - 864\varphi(1-432\varphi)\ \theta - 12\varphi(31-15552\varphi)$
Riemann Symbol	$\mathcal{P}\left\{\begin{array}{ccc\|c} 0 & \frac{1}{432} & \infty & \\ \hline 0 & -\frac{1}{6} & 1 & \varphi \\ 0 & -\frac{5}{6} & 1 & \end{array}\right\}$
a_n	$432^n \sum_{k=0}^{n} (-1)^k \binom{-5/6}{k} \binom{-1/6}{n-k}^2$
Involution	$s \to \frac{1}{186624s}$

Appendix D

Hadamard Squares

We list below the Hadamard squares of the second order differential operators in Appendix C. Each second order operator admits the action of an involution

$$s \to \frac{1}{\alpha s} \tag{D.1}$$

for some $\alpha \in \mathbb{Z}$ and which means that the Hadamard square is invariant under the involution

$$\varphi \to \frac{1}{\alpha^2 \varphi} \,. \tag{D.2}$$

More precisely, if the periods in Frobenius basis around $\varphi = 0$ are given by ϖ (see Equations (2.43)), then

$$\frac{1}{\alpha^2 \varphi} \varpi \left(\frac{1}{\alpha^2 \varphi} \right) \; = \; a \varpi \left(\varphi \right) \tag{D.3}$$

where a is a 4×4 matrix with eigenvalues $\big(+ \frac{1}{\alpha}, + \frac{1}{\alpha}, - \frac{1}{\alpha}, - \frac{1}{\alpha} \big)$ which satisfies

$$\big(r^{-1} a r \big)_{ij} \in \mathbb{Q} + \frac{\zeta(3)}{(2\pi i)^3} \mathbb{Q} \tag{D.4}$$

where $r = \mathrm{diag}\big(1, (2\pi i)^{-1}, (2\pi i)^{-2}, (2\pi i)^{-3}\big)$. As we saw in Chapter 6, this follows from the fact that the matrix a is projectively symplectic when expressed in an integral symplectic basis. The origin of $\frac{\zeta(3)}{(2\pi i)^3}$ is therefore the constant term in the prepotential near $\varphi = 0$. Unlike the case of AESZ 100 and 101, there is no natural choice of integral symplectic basis for the remaining operators so we leave a in Frobenius basis.

In all of the following examples, the fixed point of the involution (D.2) is an attractor point of rank two. Moreover, there is always a fixed point with indices $(0, 1, 3, 4)$. The trivial monodromy around this point implies that the fixed point is a an apparent singularity i.e. a singularity of the Picard-Fuchs equation where the underlying Calabi-Yau manifold is nevertheless smooth.

We saw in Chapter 6 that the Hadamard squares AESZ 100 and 101 describe A-models on explicitly known Calabi-Yau manifolds. This cannot be the case for the remaining

Hadamard squares! We may use the matrix ρ in Equation (2.49) to change bases to a supposed integral symplectic basis. Doing so and numerically computing the monodromy matrices and insisting that they be integral symplectic leads to restrictions on the allowed A-model topological data (second Chern class, Euler characteristic and triple intersection numbers). In particular, they should have no imaginary part. In this way, we find that, aside from AESZ 100 and 101, all of the remaining Hadamard squares describe A-models with positive Euler characteristic and thus cannot describe one-parameter models. Nevertheless, all of the analysis of Chapter 6 seems to apply. We fix some arbitrary choice of integral symplectic basis and see that, at the fixed points, the Hodge structure splits into positive and negative eigenspaces of a with Hodge numbers $(3,0)+(0,3)$ and $(2,1)+(1,2)$ respectively. Moreover, by comparing the components of Π and $D_\varphi \Pi$ with numerical L-values for all weight 4 and 2 modular forms of low level, we identify the associated modular forms at these points.[1] In the following examples, the eigenforms are listed via their LMFDB designations [49].

Aside from AESZ 100 and 101, some of the remaining Hadamard squares have something other than a conifold point as the nearest singularity which can be seen from the Riemann symbol (a conifold point has the indices (0,1,1,2)). Whereas, one typically expects a conifold point at the nearest singularity [70]. It would be interestingo to try and understand these operators better. A natural conjecture that one might make is that they describe one-parameter families inside the moduli space of a Calabi-Yau manifold with $h^{2,1} > 1$.

[1]Strictly speaking, we should use arbitrary periods of modular forms because the weight 2 critical L-value is expected to vanish in some cases (by the conjecture of Birch and Swinnerton-Dyer). However, all of the following examples have analytic rank equal to zero.

(a)∗(a)	
Operator	AESZ 100 [2]
Riemann Symbol	$\mathcal{P}\left\{\begin{array}{cccccc} -\frac{1}{8} & 0 & \frac{1}{64} & \frac{1}{8} & 1 & \infty \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 3 & 1 & 1 \\ 2 & 0 & 2 & 4 & 2 & 1 \end{array} \;\varphi\right\}$
Involution	$\varphi \to \frac{1}{8^2\varphi}$
Weight 2 form	$f_{\mathbf{14.2.a.a}} \in S_2(\Gamma_0(14))$
Weight 4 form	$f_{\mathbf{14.4.a.b}} \in S_4(\Gamma_0(14))$
$r^{-1}ar$	$\frac{1}{96}\begin{pmatrix} -6 & 36 & -108 & 0 \\ 0 & 6 & 0 & -36 \\ -1 & 0 & 6 & -12 \\ 0 & -3 & 0 & -6 \end{pmatrix} + \frac{1}{2}\frac{\zeta(3)}{(2\pi i)^3}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 9 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 9 & -27 & 0 \end{pmatrix}$

(b)∗(b)	
Operator	AESZ 101 [2]
Riemann Symbol	$\mathcal{P}\left\{\begin{array}{cccccc} -1 & 0 & \frac{1}{2}(123-55\sqrt{5}) & 1 & \frac{1}{2}(123+55\sqrt{5}) & \infty \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 3 & 1 & 1 \\ 2 & 0 & 2 & 4 & 2 & 1 \end{array} \;\varphi\right\}$
Involution	$\varphi \to \frac{1}{\varphi}$
Weight 2 form	$f_{\mathbf{11.2.a.a}} \in S_2(\Gamma_0(11))$
Weight 4 form	$f_{\mathbf{22.4.a.a}} \in S_4(\Gamma_0(22))$
$r^{-1}ar$	$\frac{1}{72}\begin{pmatrix} -78 & 540 & -900 & 0 \\ 0 & 78 & 0 & -300 \\ 1 & 0 & 78 & -180 \\ 0 & 3 & 0 & -78 \end{pmatrix} + \frac{\zeta(3)}{(2\pi i)^3}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 100 & 0 & 0 & 0 \\ 60 & 0 & 0 & 0 \\ 0 & 180 & -300 & 0 \end{pmatrix}$

(c)∗(c)	
Operator	AESZ 103 [2]
Riemann Symbol	$\mathcal{P}\left\{\begin{array}{cccccc} -\frac{1}{9} & 0 & \frac{1}{81} & \frac{1}{9} & 1 & \infty \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 3 & 0 & 1 & 1 & 1 & 1 \\ 4 & 0 & 2 & 2 & 2 & 1 \end{array}\ \varphi\right\}$
Involution	$\varphi \to \frac{1}{9^2\varphi}$
Weight 2 form	$f_{\mathbf{90.2.a.c}} \in S_2(\Gamma_0(90))$
Weight 4 form	$f_{\mathbf{180.4.a.d}} \in S_4(\Gamma_0(180))$
$r^{-1}ar$	$\frac{1}{18}\begin{pmatrix} 4 & -12 & 12 & 0 \\ 2 & -4 & 0 & 4 \\ 1 & 0 & -4 & 4 \\ 0 & 3 & -6 & 4 \end{pmatrix} + \frac{1}{9}\frac{\zeta(3)}{(2\pi i)^3}\begin{pmatrix} 0 & 0 & 0 & 0 \\ -16 & 0 & 0 & 0 \\ -16 & 0 & 0 & 0 \\ 0 & -48 & 48 & 0 \end{pmatrix}$

(d)∗(d)	
Operator	AESZ 107 [2]
Riemann Symbol	$\mathcal{P}\left\{\begin{array}{cccccc} -\frac{1}{32} & 0 & \frac{1}{64} & \frac{1}{32} & \frac{1}{16} & \infty \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 3 & 0 & 1 & 1 & 1 & 1 \\ 4 & 0 & 2 & 2 & 2 & 1 \end{array}\ \varphi\right\}$
Involution	$\varphi \to \frac{1}{32^2\varphi}$
Weight 2 form	$f_{\mathbf{48.2.a.a}} \in S_2(\Gamma_0(48))$
Weight 4 form	$f_{\mathbf{48.4.a.c}} \in S_4(\Gamma_0(48))$
$r^{-1}ar$	$\frac{1}{288}\begin{pmatrix} 15 & -36 & 36 & 0 \\ 9 & -15 & 0 & 12 \\ 5 & 0 & -15 & 12 \\ 0 & 15 & -27 & 15 \end{pmatrix} + \frac{1}{8}\frac{\zeta(3)}{(2\pi i)^3}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 5 & 0 & 0 & 0 \\ 5 & 0 & 0 & 0 \\ 0 & 15 & -15 & 0 \end{pmatrix}$

(e)∗(e)	
Operator	AESZ 115 [2]
Riemann Symbol	$\mathcal{P}\left\{\begin{array}{cccc} -\frac{1}{256} & 0 & \frac{1}{256} & \infty \\ \hline 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 3 & 0 & 0 & 1 \\ 4 & 0 & 0 & 1 \end{array} \varphi \right\}$
Involution	$\varphi \to \frac{1}{256^2 \varphi}$
Weight 2 form	$f_{\mathbf{32.2.a.a}} \in S_2(\Gamma_0(32))$
Weight 4 form	$f_{\mathbf{32.4.a.c}} \in S_4(\Gamma_0(32))$
$r^{-1}ar$	$\frac{1}{4608}\begin{pmatrix} 24 & -36 & 36 & 0 \\ 18 & -24 & 0 & 12 \\ 11 & 0 & -24 & 12 \\ 0 & 33 & -54 & 24 \end{pmatrix} + \frac{1}{16}\frac{\zeta(3)}{(2\pi i)^3}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 5 & 0 & 0 & 0 \\ 5 & 0 & 0 & 0 \\ 0 & 15 & -15 & 0 \end{pmatrix}$

(f)∗(f)	
Operator	AESZ 165 [2]
Riemann Symbol	$\mathcal{P}\left\{\begin{array}{cccccc} -\frac{1}{27} & 0 & \frac{1}{54}(1-i\sqrt{3}) & \frac{1}{54}(1+i\sqrt{3}) & \frac{1}{27} & \infty \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 3 & 0 & 1 & 1 & 1 & 1 \\ 4 & 0 & 2 & 2 & 2 & 1 \end{array} \varphi \right\}$
Involution	$\varphi \to \frac{1}{27^2 \varphi}$
Weight 2 form	$f_{\mathbf{27.2.a.a}} \in S_2(\Gamma_0(27))$
Weight 4 form	$f_{\mathbf{54.4.a.d}} \in S_4(\Gamma_0(54))$
$r^{-1}ar$	$\frac{1}{648}\begin{pmatrix} 42 & -108 & 108 & 0 \\ 24 & -42 & 0 & 36 \\ 13 & 0 & -42 & 36 \\ 0 & 39 & -72 & 42 \end{pmatrix} + \frac{1}{9}\frac{\zeta(3)}{(2\pi i)^3}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 \\ 0 & 12 & -12 & 0 \end{pmatrix}$

(g)∗(g)	
Operator	AESZ 144 [2]
Riemann Symbol	$\mathcal{P}\left\{\begin{array}{cccccc} -\frac{1}{72} & 0 & \frac{1}{81} & \frac{1}{72} & \frac{1}{64} & \infty \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 3 & 0 & 1 & 1 & 1 & 1 \\ 4 & 0 & 2 & 2 & 2 & 1 \end{array} \quad \varphi\right\}$
Involution	$\varphi \to \frac{1}{72^2\varphi}$
Weight 2 form	$f_{\mathbf{306.2.a.c}} \in S_2(\Gamma_0(306))$
Weight 4 form	$f_{\mathbf{306.4.a.c}} \in S_4(\Gamma_0(306))$
$r^{-1}ar$	$\frac{1}{864}\begin{pmatrix} 18 & -36 & 36 & 0 \\ 12 & -18 & 0 & 12 \\ 7 & 0 & -18 & 12 \\ 0 & 21 & -36 & 18 \end{pmatrix} + \frac{1}{18}\frac{\zeta(3)}{(2\pi i)^3}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 11 & 0 & 0 & 0 \\ 11 & 0 & 0 & 0 \\ 0 & 33 & -33 & 0 \end{pmatrix}$

(h)∗(h)	
Operator	AESZ 145 [2]
Riemann Symbol	$\mathcal{P}\left\{\begin{array}{cccc} -\frac{1}{729} & 0 & \frac{1}{729} & \infty \\ \hline 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 3 & 0 & -1/3 & 1 \\ 4 & 0 & 1/3 & 1 \end{array} \quad \varphi\right\}$
Involution	$\varphi \to \frac{1}{729^2\varphi}$
Weight 2 form	$f_{\mathbf{54.2.a.b}} \in S_2(\Gamma_0(54))$
Weight 4 form	$f_{\mathbf{108.4.a.d}} \in S_4(\Gamma_0(108))$
$r^{-1}ar$	$\frac{1}{5832}\begin{pmatrix} 10 & -12 & 12 & 0 \\ 8 & -10 & 0 & 4 \\ 5 & 0 & -10 & 4 \\ 0 & 15 & -24 & 10 \end{pmatrix} + \frac{1}{27}\frac{\zeta(3)}{(2\pi i)^3}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 \\ 0 & 12 & -12 & 0 \end{pmatrix}$

(i)∗(i)	
Operator	AESZ 155 [2]
Riemann Symbol	$\mathcal{P}\left\{\begin{array}{cccc} -\frac{1}{4096} & 0 & \frac{1}{4096} & \infty \\ \hline 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 3 & 0 & -1/2 & 1 \\ 4 & 0 & 1/2 & 1 \end{array} \varphi\right\}$
Involution	$\varphi \to \frac{1}{4096^2\varphi}$
Weight 2 form	$f_{\mathbf{128.2.a.b}} \in S_2(\Gamma_0(128))$
Weight 4 form	$f_{\mathbf{128.4.a.a}} \in S_4(\Gamma_0(128))$
$r^{-1}ar$	$\frac{1}{147456}\begin{pmatrix} 42 & -36 & 36 & 0 \\ 36 & -42 & 0 & 12 \\ 23 & 0 & -42 & 12 \\ 0 & 69 & -108 & 42 \end{pmatrix} + \frac{1}{512}\frac{\zeta(3)}{(2\pi i)^3}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 19 & 0 & 0 & 0 \\ 19 & 0 & 0 & 0 \\ 0 & 57 & -57 & 0 \end{pmatrix}$

(j)∗(j)	
Operator	AESZ 166 [2]
Riemann Symbol	$\mathcal{P}\left\{\begin{array}{cccc} -\frac{1}{186624} & 0 & \frac{1}{186624} & \infty \\ \hline 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 3 & 0 & -2/3 & 1 \\ 4 & 0 & 2/3 & 1 \end{array} \varphi\right\}$
Involution	$\varphi \to \frac{1}{186624^2\varphi}$
Weight 2 form	$f_{\mathbf{864.2.a.g}} \in S_2(\Gamma_0(864))$
Weight 4 form	$f_{\mathbf{864.4.a.b}} \in S_4(\Gamma_0(864))$
$r^{-1}ar$	$\frac{1}{13436928}\begin{pmatrix} 78 & -36 & 36 & 0 \\ 72 & -78 & 0 & 12 \\ 47 & 0 & -78 & 12 \\ 0 & 141 & -216 & 78 \end{pmatrix} + \frac{1}{46656}\frac{\zeta(3)}{(2\pi i)^3}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 61 & 0 & 0 & 0 \\ 61 & 0 & 0 & 0 \\ 0 & 183 & -183 & 0 \end{pmatrix}$

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