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Generalizations of the BMS group and results

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Abstract. The ordinary Bondi-Metzner-Sachs (BMS) group B is the common asymptotic symmetry group of all radiating, asymptotically flat, Lorentzian space-times. As such, B is the best candidate for the universal symmetry group of General Relativity. However, in studying quantum gravity, space-times with signatures other than the usual Lorentzian one, and complex space-times, are frequently considered. Generalisations of B appropriate to these other signatures have been defined earlier. In particular, the generalization $B(2,2)$ appropriate to the ultrahyperbolic signature (+,+,-,-) has been described in detail, and the study of its irreducible unitary representations (IRs) has been initiated. The infinite little groups of $B(2,2)$ have been given explicitly but its finite little groups have only been partially described. All the information needed in order to construct the finite little groups is given. Possible connections with gravitational instantons are being put forward.

1. Introduction

The best candidate for the universal symmetry group of General Relativity (G.R), in any signature, is the so called Bondi-Metzner-Sachs (BMS) group. These groups have recently been described [1] for all possible signatures and all possible complex versions of GR as well. The induced irreducible representations (IRs) have also been classified and constructed for complex GR, and, in more detail, for ultrahyperbolic GR [2].

In earlier papers [1, 2, 3, 4] it has been argued that the IRs of the BMS group and of its generalizations in complex space-times as well as in space-times with Euclidean or Ultrahyperbolic signature are what really lie behind the full description of (unconstrained) moduli spaces of gravitational instantons. Kronheimer [5, 6] has given a description of these instanton moduli spaces for *Euclidean* instantons. However, his description only partially describes the moduli spaces, since it still involves *constraints*. Kronheimer does not solve the constraint equations, but it has been argued [1, 4] that IRs of BMS group (in the relevant signature) give an *unconstrained* description of these same moduli spaces.

The representation theory of B(2,2), which is the BMS group in ulatrahyperbolic signature, has been initiated in [2]. It turns out [2] that the problem of constructing the IRs induced from *finite* little groups reduces to a seemingly very simple task; that of classifying all subgroups of the Cartesian product group $\mathrm{C_n} \times \mathrm{C_m}$, where $\mathrm{C_r}$ is the cyclic group of order r, r being finite. Surprisingly, this task is less simple than it may appear at first sight. It turns out that the solution is constructed from the " fundamental cases " $\mathrm{n} = \mathrm{p}^a$, $\mathrm{m} = \mathrm{p}^\beta$, (n,m are powers of the same prime), via the prime decomposition of m and n.

Now, the moduli spaces associated with *instantons* arise from only *one* factor of $\mathrm{C_n} \times \mathrm{C_m}$ ($\mathrm{C_n} \times \mathrm{I}$ for anti-self dual solutions, $\mathrm{I} \times \mathrm{C_m}$ for self dual, I being the identity). The groups

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described here seem [4] to be associated, rather, with the far more general *mixture* of self-dual and anti-self-dual solutions. Furthermore, this will also apply to the IRs for other signatures, or for complex space-times. Thus, in all cases, BMS IRs are likely to be related to *generalized* instantons.

In this paper we restrict attention to B(2,2). In Section 2 a summary of the results obtained so far on the representation theory of B(2,2) group is given. In Section 3 it is shown that the problem of determining the subgroups of $\mathrm{C_n} \times \mathrm{C_m}$ can be reduced to the problem of determining the subgroups of $\mathrm{C}_{\mathrm{p}^\alpha} \times \mathrm{C}_{\mathrm{p}^\beta}$. Then explicit expressions for the generators of the cyclic and non-cyclic subgroups of $\mathrm{C}_{\mathrm{p}^\alpha} \times \mathrm{C}_{\mathrm{p}^\beta}$ are given, and, finally, generators of the subgroups of $\mathrm{C_n} \times \mathrm{C_m}$ are described in detail.

2. Summary of the representation theory of the Ultrahyperbolic BMS group B(2,2)

The original BMS group B was discovered by Bondi, Metzner and Van der Burg [7] for asymptotically flat space-times which were axisymmetric, and by Sachs [8] for general asymptotically flat space-times, in the usual Lorentzian signature. The group B(2,2) is a different generalised BMS group, namely the one appropriate to the "ultrahyperbolic" signature, and asymptotic flatness in null directions. We now give a condensed summary of the representation theory of B(2,2) given in [2]. Recall that the ultrahyperbolic version of Minkowski space is the vector space R^4 of row vectors with 4 real components, with scalar product defined as follows. Let $x, y \in R^4$ have components x^μ and y^μ respectively, where $\mu = 0, 1, 2, 3$. Define the scalar product $x.y$ between x and y by

$$x.y = x^0y^0 + x^2y^2 - x^1y^1 - x^3y^3. \tag{1}$$

Then the ultrahyperbolic version of Minkowski space, sometimes written $R^{2,2}$, is just R^4 with this scalar product. The "2,2" refers to the two plus and two minus signs in the scalar product.

The group $B(2,2)$ is given by

$$B(2,2) = L^2_e(\mathrm{T}^2) \circledS_T G^2 \tag{2}$$

where the representation T of G^2 on $L^2_e(\mathrm{T}^2)$ which defines this semi-direct product is given by

$$(T(g,h)\alpha)(m,n) = k(m,g)k(n,h)\alpha([mg]\,,[nh]) \tag{3}$$

for $\alpha \in L^2_e(\mathrm{T}^2)$. $G^2 = G \times G$

($G = SL(2,R)$) and $L^2_e(\mathrm{T}^2)$ is the Hilbert space of of all even square integrable functions defined on T^2, where $\mathrm{T}^2 = S^1 \times S^1$ is the 2-torus. Let $Sp = R^2 - 0$ be the set of all nonzero row vectors $x = (x_1, x_2)$ with both components real. Here Sp is the "spin space" appropriate to $\mathcal{N} \subset R^{2,2}$. The null cone $\mathcal{N} \subset R^{2,2}$ is just the set of nonzero vectors with zero length: $\mathcal{N} = \left\{x \in R^{2,2} \mid x \neq 0, \quad x.x = 0\right\}$. We introduce "polar" coordinates into $Sp^2 = R^2 - 0$ as follows. With each vector $x = (x_1, x_2) \in Sp$, we associate the radius $r \equiv |x| \equiv \sqrt{x_1^2 + x_2^2}$, with the square root always taken positive, and the unit length vector $m \equiv [x] \equiv x/\,|x|$ having the same direction as x. Thus we have

$$x = rm, \quad r = |x|\,, \quad m = [x] = x/\,|x|\,. \tag{4}$$

Let $S^1 \subset Sp$ be the set of vectors of unit length in Sp : $S^1 = \{x \in Sp \ \mid \ |x| = 1\}$. Each factor of $\mathrm{T}^2 = S^1 \times S^1$ is given by the last equality. If $(x, y) \in Sp^2$, define the radius and direction of x by equation (4), and the radius t and direction n of y by

$$y = tn, \quad t = |y|\,, \quad n = [y] = y/\,|y|\,. \tag{5}$$

The set of all real valued functions $\alpha : \mathrm{T}^2 \rightarrow R$, $\alpha \in L^2_e(\mathrm{T}^2)$, are *even,* that is, they satisfy the even-ness condition $\alpha(-m,-n) = \alpha(m,n)$. The $k-$factors which appear in (3) are given by $k(m,g) = |mg|$, and similarly, $k(n,h) = |nh|$. Finally, $[mg] = (xg)/\,|xg|$, and, $[nh] = (yh)/\,|yh|$. $L^2_e(\mathrm{T}^2)$ is endowed with the Hilbert topology by using a natural measure on T^2 and G^2 is endowed with the standard topology. In the product topology of $L^2_e(\mathrm{T}^2) \times G^2$, $B(2,2)$ then becomes a topological group. Let $L^2_e(\mathrm{T}^2)'$ be the set of continuous linear functionals on $L^2_e(\mathrm{T}^2)$. As is well known, the topological dual $L^2_e(\mathrm{T}^2)'$ of $L^2_e(\mathrm{T}^2)$ can be identified with $L^2_e(\mathrm{T}^2)$ itself. The action T of G^2 on $L^2_e(\mathrm{T}^2)$ induces a dual action T' of G^2 on $L^2_e(\mathrm{T}^2)'$ by setting, for each $\alpha \in L^2_e(\mathrm{T}^2)$,

$$\langle T'(g,h)\zeta, T(g,h)\alpha\rangle = \langle \zeta, \alpha\rangle\,, \tag{6}$$

where $\zeta, \alpha \in \mathcal{H}_e(\mathrm{T}^2)$, $(g,h) \in G^2$ and $\langle \zeta, \alpha\rangle$ is the value of the linear functional ζ on $\alpha \in L^2_e(\mathrm{T}^2)$. It is this dual action T' on $L^2_e(\mathrm{T}^2)'$ which determines the structure of the IRs of $B(2,2)$. The dual action is given by

$$(T'(g,h)\zeta)(m,n) = k^{-3}(m,g)k^{-3}(n,h)\zeta([mg]\,,[nh])\ . \tag{7}$$

Attention is confined to measures on $L^2_e(\mathrm{T}^2)'$ which are concentrated on single orbits of the G^2-action T'. These measures give rise to IRs of $B(2,2)$ which are induced in a sense generalising [9] Mackey's [10] . This induction is materialised as follows. Let $\mathcal{O} \subset L^2_e(\mathrm{T}^2)'$ be any orbit of the dual action T' of G^2 on $L^2_e(\mathrm{T}^2)'$. There is a natural homomorphism $\mathcal{O} \simeq G^2/L_o$ where L_o is the 'little group' of the point $\zeta_o \subset \mathcal{O}$. Let U be a continuous irreducible unitary representation of L_o on a Hilbert space $\mathcal{H}_o$. Every coset space $\mathcal{O}$ can be equipped with a unique class of measures which are quasi-invariant under the action T of G^2. Let μ be any one of these. Let $\mathcal{H} = L^2(\mathcal{O}, \mu, \mathcal{H}_o)$ be the Hilbert space of functions $f : \mathcal{O} \rightarrow \mathcal{H}_o$ which are square integrable with respect to μ. From a given ζ_o and any continuous irreducible unitary representation U of L_o on a Hilbert space $\mathcal{H}_o$ a continuous irreducible unitary representation of $B(2,2)$ on $\mathcal{H}$ can be constructed. The representation is said to be induced from U and ζ_o and is given by

$$\begin{aligned} (\varrho_o f)(\varrho) &= f(\varrho_o^{-1}\varrho), \\ (\alpha f)(\varrho) &= e^{i\langle \varrho\zeta_o, \alpha\rangle} f(\varrho), \end{aligned} \tag{8}$$

where $\varrho, \varrho_o \in G^2$ and $\langle \varrho\zeta_o, \alpha\rangle$ is the scalar product in $L^2_e(\mathrm{T}^2)$. Different points of an orbit $\mathcal{O}$ have conjugate little groups and give rise to equivalent representations of $B(2,2)$. In the product topology of $L^2_e(\mathrm{T}^2) \times G^2$, $B(2,2)$ is not locally compact and as a consequence the problem of determining IRs of $B(2,2)$ arising from strictly ergodic actions T' of G^2 on $L^2_e(\mathrm{T}^2)'$ is hopeless. To conclude, *every* representation of $B(2,2)$ determined uniquely (up to equivalence) via induction by (1) an orbit $\mathcal{O} \in L^2_e(\mathrm{T}^2)'$, (2) a class of equivalent IRs of any little group L_o, is irreducible [11]. It is not known if there are other IRs of $B(2,2)$ emanating from strictly ergodic actions. All the little groups of $B(2,2)$ are compact. The little groups L_o for $B(2,2)$ are the closed subgroups of $K = SO(2) \times SO(2)$ which contain the element $(-I,-I)$. These are (A) K itself, (B) a class of one dimensional not connected Lie groups which are described in detail in [2] , and (C) all finite subgroups containing $(-I,-I)$. The finite subgroups of $K = SO(2) \times SO(2)$ are precisely the subgroups of $\mathrm{C_n} \times \mathrm{C_m}$ where both n and m are finite. These subgroups are not given in [2] and so we proceed now to construct them explicitly.

3. Construction of the subgroups of $\mathrm{C_n} \times \mathrm{C_m}$.

The following proposition shows that the problem of finding the subgroups of $\mathrm{C_n} \times \mathrm{C_m}$ is reduced to the problem of finding the subgroups of $\mathrm{C_{p^\alpha}} \times \mathrm{C_{p^\beta}}$.

Proposition 1 Let $\mathrm{C_n} \times \mathrm{C_m}$ be the direct product of the cyclic groups of finite order $\mathrm{C_n}$ and $\mathrm{C_m}$. Let $\mathrm{n} = \mathrm{p_1^{r_1}} \cdot \mathrm{p_2^{r_2}} \cdots \mathrm{p_s^{r_s}}$ and $\mathrm{m} = \mathrm{p_1^{t_1}} \cdot \mathrm{p_2^{t_2}} \cdots \mathrm{p_s^{t_s}}$ be the prime decomposition of the integers n and m, i.e., p_i , i=1,2,...,s , are distinct prime numbers and r_i , t_i are non-negative integers. Any subgroup of $C_n \times C_m$ has the form

$$C_{q_1^{\lambda_1}} \times C_{q_2^{\lambda_2}} \times \ldots \times C_{q_\sigma^{\lambda_\sigma}} , \tag{9}$$

i.e., is a direct product where the numbers $q_1, q_2, ..., q_\sigma$ are prime and each one of them appears at most twice. For any q_j, $j = 1, 2, ..., \sigma$, there exists a p_i, $i = 1, 2, ..., s$, so that $q_j = p_i$. When q_j appears once $\lambda_j \in [1, \max(r_i, t_i)]$. When q_j occurs twice, say $q_j = q_{j+k}$, then one of the indices λ_j , λ_{j+k} belongs to $[1, r_i]$ and the other one belongs to $[1, t_i]$. For every subgroup of $C_n \times C_m$ the expression (9) is unique.

The group $C_{p^a} \times C_{p^\beta}$ is a finite abelian group and therefore its rank is higher than the rank of any of its subgroups. Consequently, the subgroups of $C_{p^a} \times C_{p^\beta}$ have either one or two generators. The following two propositions give explicit expressions for the generators of the cyclic subgroups of $C_{p^a} \times C_{p^\beta}$, whereas, the theorem which follows gives explicit expressions for the generators of the non-cyclic subgroups of $C_{p^a} \times C_{p^\beta}$.

Proposition 2 *Let* p *be a prime number and let* a *and* β *be positive integers. Let* C_{p^a} *and* C_{p^β} *be cyclic groups of order* p^a *and* p^β *respectively. When* $1 \leq k \leq \min(a, \beta)$ *the direct product* $C_{p^a} \times C_{p^\beta}$ *has* $p^k + p^{k-1}$ *cyclic subgroups of order* p^k*. The generators of these subgroups are given by*

(i)

$$(x^{rp^{a-k}}, y^{p^{\beta-k}}) , \quad r \in \{0, 1, 2, ..., p^k - 1\} , \tag{10}$$

and ,

(ii)

$$(x^{p^{a-k}}, y^{\rho p^{\beta-k+1}}) , \qquad \rho \in \{0, 1, ..., p^{k-1} - 1\} , \tag{11}$$

where x *and* y *are generators of the groups* C_{p^a} *and* C_{p^β} *respectively. The parameters* r*, which takes values in the set* $\{0, 1, ..., p^k - 1\}$*, and* ρ*, which takes values in the set* $\{0, p, 2p, ..., (p^{k-1} - 1)p\}$*, parameterize the distinct* $p^k + p^{k-1}$ *groups .*

Proposition 3 *Let* p *be a prime number and let* a *be a non-negative integer and let* β *be a positive integer. Let* C_{p^a} *and* C_{p^β} *be cyclic groups of order* p^a *and* p^β *respectively. The direct product* $C_{p^a} \times C_{p^\beta}$ *has* p^a *cyclic subgroups of order* p^k*, where* $a < k \leq \beta$*. The generators of these subgroups are the following*

$$(x^j, y^{\beta-k}) , \quad j \in \{0, 1, 2, ..., p^a - 1\} , \tag{12}$$

where x *and* y *are generators of the groups* C_{p^a} *and* C_{p^β} *respectively. The parameter* j*, which takes values in the set* $\{0, 1, ..., p^a - 1\}$*, parametrises the groups .*

In the following theorem the generators of the non-cyclic subgroups of $C_{p^a} \times C_{p^\beta}$ are given.

Theorem 1 *Let* p *be a prime number and let* k , l , a , β *be integers which satisfy* $0 < k < l \leq \beta$ *and* $a \leq \beta$*. Let* $C_{p^a} \times C_{p^\beta}$ *denote the direct product of the cyclic groups* C_{p^a} *and* C_{p^β} *and let* $C_{p^k} \times C_{p^l}$ *denote the direct product of the cyclic groups* C_{p^k} *and* C_{p^l}*. Then*

(i) When $0 < k < l \leq a \leq \beta$ *the group* $C_{p^a} \times C_{p^\beta}$ *has* $p^{l-k} + p^{l-k-1}$ *subgroups which are isomorphic to the group* $C_{p^k} \times C_{p^l}$ *. From these subgroups,* p^{l-k} *are generated by the elements*

$$(x^{rp^{a-l}}, y^{p^{\beta-l}}) , \ (x^{p^{a-k}}, I), \quad \text{where} \ r \in \{0, 1, 2, ..., p^{l-k} - 1\},$$

and the remaining p^{l-k-1} *subgroups are generated by the elements*

$$(x^{p^{a-l}}, y^{\rho p^{\beta-l+1}}) , \ (I, y^{p^{\beta-k}}), \quad \text{where} \ \rho \in \{0, 1, 2, ..., p^{l-k-1} - 1\}.$$

(ii) When $0 < \mathrm{k} \leq a < \mathrm{l} \leq \beta$ *the group* $\mathrm{C}_{\mathrm{p}^{a}} \times \mathrm{C}_{\mathrm{p}^{\beta}}$ *has* $\mathrm{p}^{a-\mathrm{k}}$ *subgroups which are isomorphic to the group* $\mathrm{C}_{\mathrm{p}^{\mathrm{k}}} \times \mathrm{C}_{\mathrm{p}^{\mathrm{l}}}$ *. These* $\mathrm{p}^{a-\mathrm{k}}$ *subgroups are generated by the elements*

$$(x^{\mathrm{j}}, y^{\mathrm{p}^{\beta-\mathrm{l}}}) \ , \ (x^{\mathrm{p}^{a-\mathrm{k}}}, \mathrm{I}), \qquad \text{where} \ \ \mathrm{j} \in \{0, 1, 2, ..., \mathrm{p}^{a-\mathrm{k}} - 1\}.$$

The task of writing explicit expressions for the generators of the subgroups of $\mathrm{C_n} \times \mathrm{C_m}$ is facilitated by the use of the set $S_{\mathcal{P}}$ of the permutations of s pairs of numbers $(\mathrm{p}_1^{a_1}, \mathrm{p}_1^{\beta_1}), (\mathrm{p}_2^{a_2}, \mathrm{p}_2^{\beta_2}), ..., (\mathrm{p}_\mathrm{s}^{a_\mathrm{s}}, \mathrm{p}_\mathrm{s}^{\beta_\mathrm{s}})$. Let $\mathcal{P} \in S_{\mathcal{P}}$. By $\mathcal{P}(\mathrm{p}_\mathrm{i}^{a_\mathrm{i}}, \mathrm{p}_\mathrm{i}^{\beta_\mathrm{i}}) = (p_\mathrm{j}^{\mathrm{a_j}}, p_\mathrm{j}^{\mathrm{b_j}})$, $\quad \mathrm{i}, \mathrm{j} = 1, 2, ..., \mathrm{s}$, we denote that the permutation $\mathcal{P}$ moves the pair $(\mathrm{p}_\mathrm{i}^{a_\mathrm{i}}, \mathrm{p}_\mathrm{i}^{\beta_\mathrm{i}})$ at the i^{th} position, to the j^{th} position. Obviously, $\mathrm{p_i} \equiv p_\mathrm{j}$, $a_\mathrm{i} \equiv \mathrm{a_j}$, and, $\beta_\mathrm{i} \equiv \mathrm{b_j}$. By using Propositions 2, 3 and Theorem 1 we can give explicit expressions for the generators of the subgroups of $\mathrm{C_n} \times \mathrm{C_m}$. The results are stated in the Theorem which follows.

Theorem 2 *Let* $\mathcal{C}$ *be a subgroup of* $\mathcal{C}$ *of* $\mathrm{C_n} \times \mathrm{C_m}$*. We distinguish two cases :*

(i) When $\mathcal{C}$ *is non-cyclic then it can be written in a highly non unique way as a direct product of two cyclic groups* $\mathcal{C}_1$ and $\mathcal{C}_2$ *,i.e.,*

$$\mathcal{C} = \mathcal{C}_1 \times \mathcal{C}_2 \ , \tag{13}$$

whose orders are not relatively prime. A legitimate choice for the generators g_1 *and* g_2 *of* $\mathcal{C}_1$ *and* $\mathcal{C}_2$ *is the following :*

$$g_1 = (x^{\mathcal{A}_1}, y^{\mathcal{B}_1}), \tag{14}$$

where, x *is a generator of* $\mathrm{C_n}$ *,* y *is a generator of* $\mathrm{C_m}$ *,*

$$\frac{\mathcal{A}_1}{\mathrm{n}} = \sum_{i=1}^{\nu} \mathrm{r}_i p_i^{-\mathrm{k}_i} + \sum_{i=\nu+1}^{\nu+\chi} p_i^{-\mathrm{k}_i} + \sum_{i=\nu+\chi+1}^{\nu+\chi+\tau} \mathrm{j}_i / p_i^{\mathrm{a}_i} + \sum_{i=\nu+\chi+\tau+1}^{\nu+\chi+\tau+\psi} p_i^{-\mathrm{k}_i} + \sum_{i=\nu+\chi+\tau+\psi+1}^{\nu+\chi+\tau+\psi+\sigma} \mathrm{r}_i p_i^{-\mathrm{k}_i}$$
$$+ \sum_{i=\nu+\chi+\tau+\psi+\sigma+1}^{\nu+\chi+\tau+\psi+\sigma+\theta} p_i^{-\mathrm{k}_i} + \sum_{i=\nu+\chi+\tau+\psi+\sigma+\theta+1}^{\nu+\chi+\tau+\psi+\sigma+\theta+\phi} \mathrm{t}_i / p_i^{\mathrm{a}_i} + \sum_{i=\nu+\chi+\tau+\psi+\sigma+\theta+\phi+1}^{\nu+\chi+\tau+\psi+\sigma+\theta+\phi+\xi} p_i^{-\mathrm{k}_i}, \ \text{and}, \tag{15}$$

$$\frac{\mathcal{B}_1}{\mathrm{m}} = \sum_{i=1}^{\nu} p_i^{-\mathrm{k}_i} + \sum_{i=\nu+1}^{\nu+\chi} \rho_i p_i^{-\mathrm{k}_i+1} + \sum_{i=\nu+\chi+1}^{\nu+\chi+\tau} p_i^{-\mathrm{k}_i} + \sum_{i=\nu+\chi+\tau+1}^{\nu+\chi+\tau+\psi} \mathrm{j}_i / p_i^{\mathrm{b}_i} + \sum_{i=\nu+\chi+\tau+\psi+1}^{\nu+\chi+\tau+\psi+\sigma} p_i^{-\mathrm{k}_i}$$
$$+ \sum_{i=\nu+\chi+\tau+\psi+\sigma+1}^{\nu+\chi+\tau+\psi+\sigma+\theta} \rho_i p_i^{-\mathrm{k}_i+1} + \sum_{i=\nu+\chi+\tau+\psi+\sigma+\theta+1}^{\nu+\chi+\tau+\psi+\sigma+\theta+\phi} p_i^{-\mathrm{k}_i} + \sum_{i=\nu+\chi+\tau+\psi+\sigma+\theta+\phi+1}^{\nu+\chi+\tau+\psi+\sigma+\theta+\phi+\xi} \mathrm{t}_i / p_i^{\mathrm{b}_i} \ . \tag{16}$$

$$g_2 = (x^{\mathcal{A}_2}, y^{\mathcal{B}_2}), \quad \text{where}, \tag{17}$$

$$\frac{\mathcal{A}_2}{\mathrm{n}} = \sum_{i=\nu+\chi+\tau+\psi+1}^{\nu+\chi+\tau+\psi+\sigma} p_i^{-\mathrm{l}_i} + \sum_{i=\nu+\chi+\tau+\psi+\sigma+\theta+1}^{\nu+\chi+\tau+\psi+\sigma+\theta+\phi} p_i^{-\mathrm{l}_i} \quad \text{and}, \tag{18}$$

$$\frac{\mathcal{B}_2}{\mathrm{m}} = \sum_{i=\nu+\chi+\tau+\psi+\sigma+1}^{\nu+\chi+\tau+\psi+\sigma+\theta} p_i^{-\mathrm{l}_i} + \sum_{i=\nu+\chi+\tau+\psi+\sigma+\theta+\phi+1}^{\nu+\chi+\tau+\psi+\sigma+\theta+\phi+\xi} p_i^{-\mathrm{l}_i} \quad . \tag{19}$$

(ii) When $\mathcal{C}$ is cyclic then it is generated by

$$g = (x^{\mathcal{A}}, y^{\mathcal{B}}), \tag{20}$$

$$\text{where,} \quad \mathcal{A} = \mathcal{A}_1 \quad \text{and} \quad \mathcal{B} = \mathcal{B}_1 \quad \text{when} \quad \sigma = \theta = \phi = \xi = 0. \tag{21}$$

The non-negative integers $\nu, \chi, \tau, \psi, \sigma, \theta, \phi, \xi$ *are such that* $\nu + \chi + \tau + \psi + \sigma + \theta + \phi + \xi \leq \mathrm{s}$ *. When $\mathcal{C}$ is cyclic then $\sigma = \theta = \phi = \xi = 0$. When $\mathcal{C}$ is non-cyclic then at least one of the $\sigma, \theta, \phi, \xi$ must be non-zero. Moreover,* $(p_j^{a_j}, p_j^{b_j}) = \mathcal{P}(p_i^{a_i}, p_i^{\beta_i})$, $\mathrm{i}, \mathrm{j} = 1, ..., \mathrm{s}$, *for some permutation $\mathcal{P}$ of the s pairs of numbers* $(p_1^{a_1}, p_1^{\beta_1}), (p_2^{a_2}, p_2^{\beta_2}), ..., (p_s^{a_s}, p_s^{\beta_s})$ *. Furthermore, the allowed values of the other indices are easily deduced from Propositions 2 , 3 and Theorem 1.*

By using the previous results we can give explicit expressions for the little groups of of B(2,2) by isolating those subgroups of $\mathrm{C_n} \times \mathrm{C_m}$ which contain the element $(-I, -I)$. Details and proofs of the aforementioned propositions and theorems as well as an explicit construction of the irreducibles of the group B(2,2) will be given elsewhere. Ultrahyperbolic General Relativity has not been well studied ([15] contains a non group theoretic approach). Subgroups of $\mathrm{C_n} \times \mathrm{C_m}$ not only do they appear as little groups of B(2,2) but they also appear [1] as little groups of the Complex BMS group CB and of the Euclidean BMS group EB. In particular, in the case of EB, the large number of little groups which are subgroups of $\mathrm{C_n} \times \mathrm{C_m}$ and which lie not just in one factor of $\mathrm{C_n} \times \mathrm{C_m}$ but they ‘ sit across ’ both factors of $\mathrm{C_n} \times \mathrm{C_m}$, strongly suggests [4] that the gravitational multi-instantons of Gibbons and Hawking [16] represent only a very small number of solutions of a class of solutions whose more general members are mixtures of self-dual and anti-self dual solutions. In the Euclidean case the existence of the aforementioned mixtures has been suggested in a different context by Hooft G ’H [17]. The physical content and significance for low-energy quantum gravity of these odd-looking mixtures [4] - flat manifolds whose certain points are identified at the neighbourhood of infinity - is an open problem in all cases.

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