Noninteger dimensionality, nonlocal noise and self-decoherence

Subhash Kak
Chapman University, Orange, CA 92866 &
Oklahoma State University, Stillwater, OK 74078
Email: subhash.kak@okstate.edu

Abstract.
This paper argues that noninteger dimensionality creates a hitherto unexplored source of nonlocal noise, called "ND noise" or NDN, that is likely to play a role in measurements at very small distances. New analysis on scale invariant probability distributions arising from noninteger dimensionality is presented. The significance of this when considering performance and noise-correction coding in models of quantum computing is indicated. It is argued that NDN noise will lead to decoherence even without interaction with the environment and it is likely to be relevant also in models of cognition.

Keywords: noninteger dimensionality, nonlocal noise, scale invariance, quantum computing, self-decoherence

1. Introduction

The basic intuition regarding dimensionality appears to be consistent with the convention in science that space is three-dimensional [1]. This is slightly modified in relativity theory that shows that one may consider time on an equal basis with space, giving us a dimensionality of 3+1. But, logically, reality need not be integer-dimensional for there is a mathematical theory of spaces with noninteger dimensions [2].

Noninteger dimensionality is a consequence of optimality of information [3] and as nature is optimal, it should show up both in physical space [4][5], and be associated with scale invariance as is known from the theory of fractals [6]. It is also as an organizing principle for data [7][8] with the optimal dimension of *e*. Based on empirical data from a variety of areas, we know that the Newcomb-Benford (NB) distribution is a commonly occurring scale invariant process [9][10], which we know is related to a wide range of power laws.

Scale-invariance in classical systems is evidenced by the effective use of nonlocal operators to denoise images [11][12][13]. Noninteger dimensionality leads to many scale-invariance properties that are commonly found in natural phenomena [14][15].

In a reductionist or local theory, one must explain the aggregate properties in terms of the accumulative components. But noninteger dimensional systems do not lend themselves to straightforward reductionist analysis and they are often characterized by nonlocal properties. In particular, one will encounter nonlocal noise that may be relevant in many applications.

In a noninteger dimensional system, if the data does not conform to the NB distribution, there will be underlying processes (either from the environment or from the data itself, or both) that will change the statistics to this distribution, which shift may be seen as noise. This noise will tend to change the distribution of the aggregate process towards scale invariancy. We call this noise noninteger dimensionality noise (NDN). In a quantum system, it may be seen as self-decoherence arising from overarching dimensionality conditions.

The objective of the current paper is to present new results related to NDN with possible applications to information communication and engineering. It will be shown that since noninteger dimensionality leads to scale invariance, it will cause the data to tend to move into a scale invariant format, which will thus tend to modify the data. Specifically, the quantum computing data register will have additional noise through some process of self-entanglement.

2. Nonlocality, decoherence and self-similarity

The existence of amplitude-squeezed light, which exhibits intensity noise below the shot noise level, that is now interpreted as an intrinsic property of the light field is like the ND noise described in this paper. Nonlocal aspects of shot noise have been described in the literature [16] [17][18].

In quantum information processing, the term decoherence is used to describe different kinds of noise such as random driving forces from the environment

(Brownian motion), interactions that produce entanglement between the system and the environment, and statistical imprecision in the experimental controls on the system. Noise seen via the lens of decoherence [19][20] is a result of interaction between the quantum system and its environment, in which a preferred set of states, usually called the ‘‘pointer basis’’, determines which observables will receive definite values. In this environment-induced superselection, arbitrary superpositions are dismissed, and the preferred states transition to classical states: they become the definite readings of the apparatus pointer in quantum measurements. But this approach has been criticized for circularity of reasoning [21].

Although noninteger dimensional physical space is *e*-dimensional, our cognitive systems see it projected on a three-dimensional map and likewise so in mathematical theory. As mentioned before, the characteristic of noninteger dimensionality is self-similarity and scale invariance. From the optimization perspective, having more or less than *e* bins leads to reduction in performance, and from the perspective of resources, this implies Pareto optimality.

Pareto efficiency means that it is impossible to make one party better off without making another party worse off. Thus it requires that resources be allocated in the most efficient way possible, and this is captured by power law distributions.

If X is a random variable with a Pareto (Type I) distribution, then the probability that X is greater than some number x, that is the survival function, is given by

$$\Pr(X > x) = \begin{cases} \left(\frac{x_m}{x}\right)^{\alpha} & x \geq x_m \\ 1 & x < x_m \end{cases} \qquad (1)$$

The x_m and α that are both positive, are the scale and shape parameters and when used to model wealth α is known as the Pareto index. Clearly, the probability density function associated with it is:

$$f_X(x) = \begin{cases} \frac{\alpha {x_m}^{\alpha}}{x^{\alpha+1}} & x \geq x_m \\ 0 & x < x_m \end{cases} \qquad (2)$$

The density function is inversely proportional to *x,* with a power of $-\alpha - 1$.

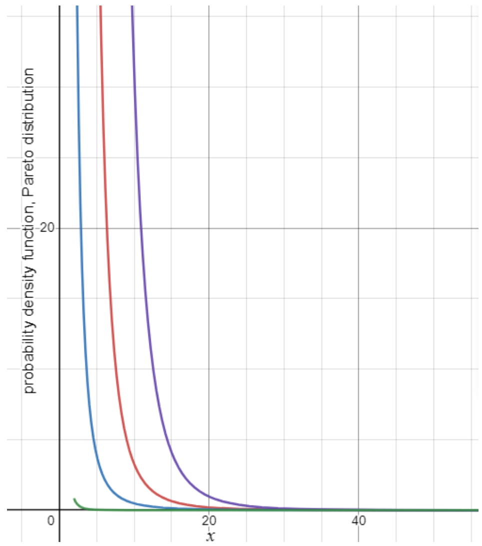

Figure 1. Four plots of Pareto distribution $x_m = 2$ and α=2.4, 2, 3, 4

The shape parameter shifts the plot.

3. Scale invariance

Let X be a scale-invariant continuous random variable with probability density function $f(x)$ and cumulative density function $F(x)$. Let the lower bound u be X. Then scale invariance means that for any $a > 0$,

$$P(u < X < x) = P(u < \frac{1}{a}X < x) = P(au < X < ax) \tag{3}$$

Hence $F(ax) - F(au) = F(x) - F(u)$. Differentiating with respect to x, we see that $af(ax) = f(x)$, so

$$f(ax) = \frac{1}{a}f(x) \tag{4}$$

Now we consider two distributions that present good approximations to (4).

3.1. Newcomb-Benford (NB) distribution

Let us consider

$$f(x) = ln\left(\frac{x+1}{x}\right) \tag{5}$$

Using (4), we have that $f(ax) = \frac{1}{a}ln\left(\frac{x+1}{x}\right) = ln\left(\frac{x+1}{x}\right)^{1/a} = ln(1+1/x)^{1/a}$

(6)

This may be simplified by the use of the binomial theorem to:

$$f(ax) = \ln\left(1 + \frac{1}{ax} + O(ax)^{-2}\right)$$

By leaving out the third term between the parentheses above, we may write

$$f(ax) \approx \ln\left(1 + \frac{1}{ax}\right) \tag{7}$$

When we consider the discrete case of (5) and make the range finite, we must replace the natural log with the logarithm equal to the number of discrete points, and this constitutes the NB distribution.

According to the NB probability function, if the counting process is uniformly distributed over the range {1, …, S}, with random values of S, then the sum of a large number of these will satisfy the NB Law, where the leading digit n (n ∈ {1, ..., r - 1}) for number to the base r, r ≥ 2, occurs with probability as a logarithmic function:

$$P_{NB}(n) = \ log_r\left(1 \ + \frac{1}{n}\right) \tag{8}$$

Figure 2 shows a plot of this distribution. This may be written as:

$$P_{NB}(n) = \frac{1}{\ln(r)}\sum_1^{\infty}\frac{1}{k(n+1)^k} =$$

$$\frac{1}{\ln(r)}\left\{\frac{1}{n+1} + \frac{1}{2(n+1)^2} + \frac{1}{3(n+1)^3} + \frac{1}{4(n+1)^4} + \cdots\right\} \tag{9}$$

This may be approximated by

$$P_{NB}(n) \propto (n+1)^{-\gamma}, \text{ where } \gamma \text{ is positive constant} \quad (10)$$

which is inverse power law quite like (1).

When the number consists of several digits, the same law applies with n replaced by the appropriate number. As the NB law is scale invariant, if numbers in the data set are rescaled to another base, the probabilities become adjusted for the new base. For example, if numbers are represented to base 4, the first digit probabilities will be

$$P(1) = log_4(2) = 0.5;\ P(2) = log_4(3/2) = 0.292;\ and\ P(3) = log_4(4/3) = 0.2075$$

and so half the random numbers will begin with the digit 1.

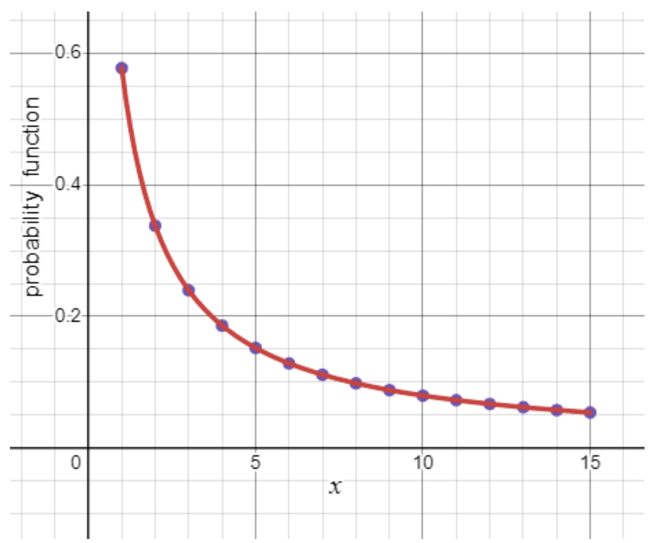

Figure 2. The NB distribution for $r = 16$

The relation between the Pareto and the Newcomb-Benford (NB) distribution has been widely investigated [22][23]. It is fascinating that NB is nearly true for the dimensionless constants of nature [24], and the reason behind it may be the noninteger dimensionality of the universe as well as the intrinsic dimensionality of data.

3.2. The Power Law

Since probability in a scale invariant system goes down with the variable as shown in (4), we now consider the power law function:

$$f(x) = cx^{-\alpha} \quad (11)$$

where c is a suitable constant.

Then

$$f(ax) = c\, a^{-\alpha}x^{-\alpha} \quad (12)$$

This may be rewritten as

$$f(ax) = \frac{c}{a^{\alpha}}x^{-\alpha} \quad (13)$$

Thus if $\frac{1}{a^{\alpha}} \approx \frac{1}{a}$, then the condition $f(ax) = \frac{1}{a}f(x)$ is satisfied and the distribution will be scale invariant.

Generalizing to the discrete case, we define as scale-invariant any property that satisfies the power-law expression:

$$f(\lambda n) = \lambda^{b} f(n) \quad (14)$$

Example 1.

Consider the power law distribution from (11) for the discrete variable *n*:

$$p(n) = cn^{-\alpha} \quad (15)$$

We wish to see how it compares to the NB distribution in its ability to represent scale invariance.

If $\alpha = 0.87$, a number chosen to optimize the fit with respect to the NB distribution, we get the plots of Figure 3:

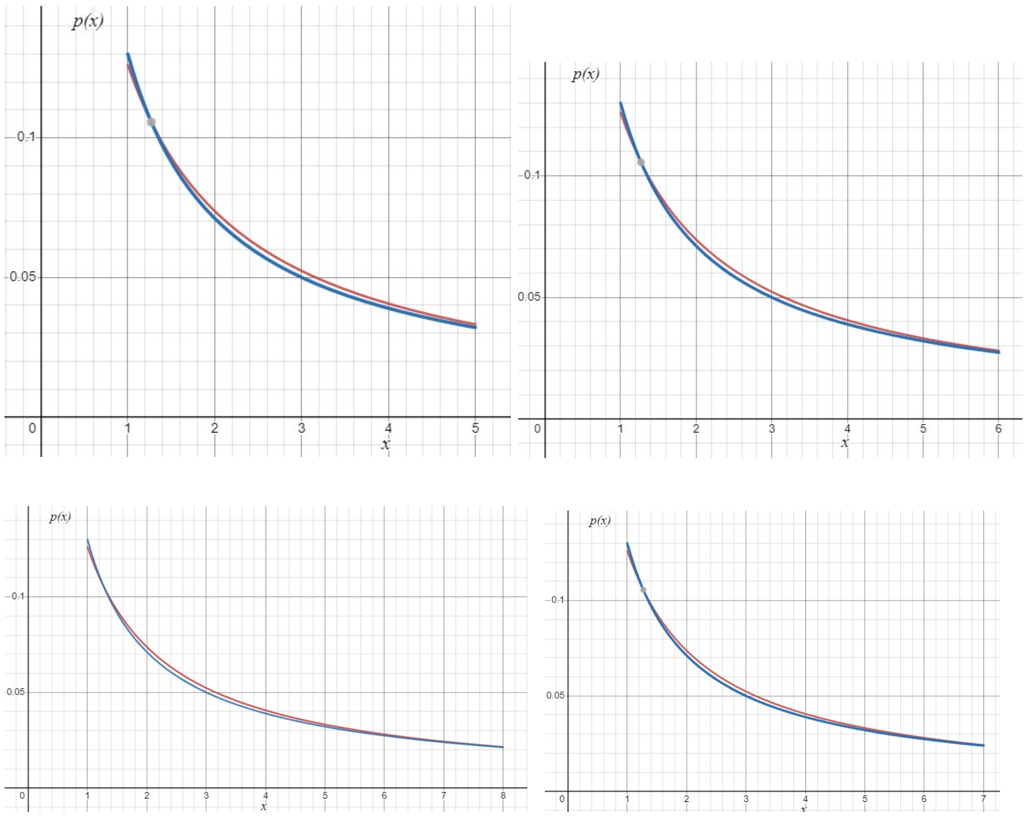

Figure 3. NB ($log_x\left(1 + \frac{1}{n}\right)$) and power law $x^{-0.87}$distributions for x ranging from 5 through 7

The fit between the NB distribution and the power law $x^{-0.87}$is excellent.

Now we consider whether the specific value of $\alpha = 0.87$ is an artifact of the size of the data. To examine, this note that this value of α is close to *e*/3, which is the characteristic number if we consider one-dimensional data (corresponding to *e*-dimensionality for three-dimensional space).

$$g(x) = cx^{-e/3} \qquad (16)$$

We first compare it to the data for NB distribution for the first digit phenomenon for digits 1 through 9 as in Table 1.

Table 1. NB distribution versus $cx^{-e/3}$

number	1	2	3	4	5	6	7	8	9
NB	0.301	0.176	0.125	0.097	0.079	0.067	0.058	0.051	0.046
$cx^{-e/3}$	0.325	0.173	0.120	0.093	0.076	0.064	0.056	0.050	0.044

Figure 4 compares the two distributions for a larger sized data set with 18 classes and we find that the correspondence between the two is excellent.

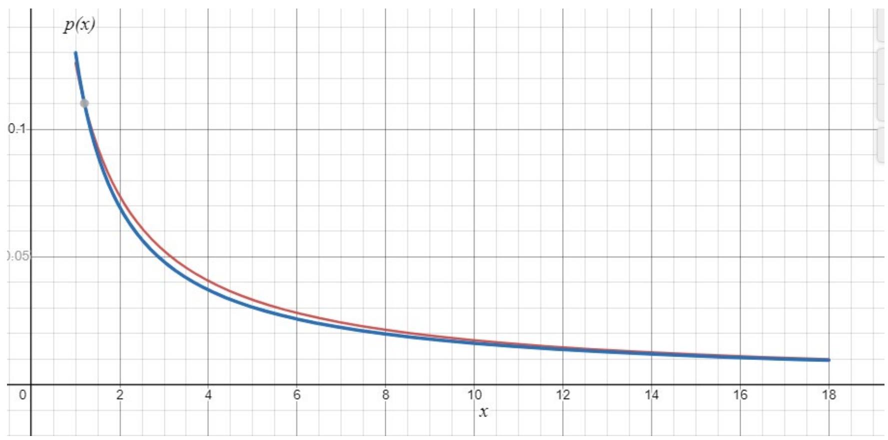

Figure 4. NB ($log_x\left(1 + \frac{1}{n}\right)$) and power law $x^{-e/3}$distributions for *x* over (1,18)

In a noninteger dimensional system, the power of the noise will depend on the distance between the distribution arising out of known physical processes and the self-similar intrinsic distribution. The noise will be greater if the distance is larger. This noise will effect the system even if it is isolated.

4. Quantum system

In a quantum register, one loads only one kind of qubits, say $|0\rangle$ before they are operated upon by the gates of the hardware. The contents of the register after the data has gone through a sufficiently larger number of gates will be either $|0\rangle$, $|1\rangle$, or a qubit that is in a superposition state [25][26][27].

If we consider several qubits together then we can map them to numbers and test them against the Newcomb-Benford or the Power Law distributions. If the quantum gate transformations and hardware noise lead to a distribution at any intermediate stage that varies from the NB distribution that will create an internal noise that will be in addition to the ones that have been considered in the literature. Since quantum error models do not take this possibility into account, the implications of this for the effectiveness of quantum error-correction systems needs to be investigated.

Note further that "The chosen quantum noise model has a drastic impact on the performance of quantum algorithms. Hence, one must be sure that the assumptions on the noise present in a physical system are appropriate. Additionally, the effect of the quantum noise is largely determined by the nature of the quantum algorithm being performed." [28]

The problem of self-decoherence in a quantum system has also been investigated from a traditional perspective [29][30] but that does not consider the question of scale invariance statistics that complicate the matter.

It is normally assumed that if a quantum system were to be perfectly isolated, it would maintain coherence. In this paper we presented arguments that negates this view. The noninteger dimensionality noise described in this paper is an overarching information theory based phenomenon that will complicate the problem of error correction in any model of quantum computing.

It has been argued that noninteger dimensionality is relevant in cognitive models [31], which is also related to quantum models [32][33][34], NDN could have implications for cognitive science. This needs to be investigated further.

5. Conclusions

This paper argued that noninteger dimensionality creates a hitherto unexplored source of nonlocal noise, called ND noise or NDN, that is likely to play a role in measurements at very small distances. New analysis on scale invariant probability distributions arising from noninteger dimensionality was presented. The significance of this when considering performance and noise-correction coding in models of quantum computing was indicated. It was argued that NDN noise will lead to decoherence even without interaction with the environment and it is likely to be relevant also in models of cognition.

6. References

1. Barrow, J.D. Dimensionality. Philosophical Transactions of the Royal Society A 310, 337 – 346 (1983); https://doi.org/10.1098/rsta.1983.0095
2. Stillinger, F.H. Axiomatic basis for space with noninteger dimension. Journal of Mathematical Physics 18, 1224- 1234 (1977)
3. Kak, S. Information theory and dimensionality of space. Scientific Reports 10, 20733 (2020). https://doi.org/10.1038/s41598-020-77855-9
4. Kak, S. Information-theoretic view of the gravitational constant in Dirac’s large numbers hypothesis. Indian J. of Physics 97, 503-507 (2023); https://link.springer.com/article/10.1007/s12648-022-02431-y
5. Kak, S. Evolutionary stages in a noninteger dimensional universe. Indian J Phys (2023). https://doi.org/10.1007/s12648-023-02653-8
6. Falconer, K. Fractal Geometry. Wiley (2003)
7. Kak, S. New classes of regular symmetric fractals. Circuits Syst. Signal Process. 41, 4149-4159 (2022). https://doi.org/10.1007/s00034-022-01966-z
8. Kak, S. The intrinsic dimensionality of data. Circuits Syst. Signal Process. 40, 2599–2607 (2021); https://doi.org/10.1007/s00034-020-01583-8
9. Hürlimann, W. Generalizing Benford's law using power laws. International Journal of Mathematics and Mathematical Sciences (2009); https://doi.org/10.1155/2009/970284
10. Kak, S. The iterated Newcomb-Benford distribution for structured systems. Int. J. Appl. Comput. Math 8, 51 (2022); https://doi.org/10.1007/s40819-022-01251-2
11. Buades, T. et al., A note on multi-image denoising, in Proceedings of the 2009 International Workshop on Local and Non-Local Approximation in Image Processing, Tuusula, Finland, pp. 1-15 (2009)
12. Li, Z. et al. Adaptive nonlocal means filtering based on local noise level for CT denoising. Med Phys. 41(1), 011908 (2014); doi: 10.1118/1.4851635. PMID: 24387516.
13. Manjón, J.V. et al. Adaptive non-local means denoising of MR images with spatially varying noise levels. J Magn Reson Imaging 31(1),192-203 (2010); doi: 10.1002/jmri.22003. PMID: 20027588.
14. Kak, S. Fractal Dimensions in Circular and Spiral Phenomena. TechRxiv. (2022); https://doi.org/10.36227/techrxiv.21766706.v1
15. Kak, S. Weighted linear fractals. TechRxiv (2022). https://doi.org/10.36227/techrxiv.21554634.v1
16. Surgailis, D., & Woyczynski, W. Burgers' equation by non-local shot noise data. Journal of Applied Probability, 31(A), 351-362. (1994) doi:10.2307/3214968
17. Weng, Q. et al. Imaging of nonlocal hot-electron energy dissipation via shot noise. Science 360, 775-780 (2018); doi: 10.1126/science.aam9
18. Neder, I., Marquardt, F., Heiblum, M. et al. Controlled dephasing of electrons by non-gaussian shot noise. Nature Phys 3, 534–537 (2007). https://doi.org/10.1038/nphys627

19. Zurek, W. H. Decoherence and the transition from quantum to classical. Physics Today 44 (October), 36–44 (1991)
20. Zurek, W. H. Preferred sets of states, predictability, classicality and environment-induced decoherence. In J. J. Halliwell, J. Perez-Mercader, & W. H. Zurek (Eds.), Physical origins of time asymmetry (pp. 175–207). Cambridge : Cambridge University Press (1994)
21. Kastner, R. E. 'Einselection' of pointer observables: the new H-theorem? Studies in History and Philosophy of Modern Physics. 48, 56–58 (2014)
22. Miller, S.J, Nigrini, M.J. Order statistics and Benford's law. International Journal of Mathematics and Mathematical Sciences, (2008); https://doi.org/10.1155/2008/382948
23. Ihor O. Bohachevsky, Mark E. Johnson & Myron L. Stein (2006) A derivation of Benford's law. American Journal of Mathematical and Management Sciences 26, 3-4, 355-370 (2006); DOI: 10.1080/01966324.2006.10737678
24. Kak, S. and Kafatos, M. Black holes, disk structures & cosmological implications in e-dimensional space. Physics Essays. 35, 345-355, 2022; https://doi.org/10.4006/0836-1398-35.4.345
25. Terhal, B. M. Quantum error correction for quantum memories. Reviews of Modern Physic, 87(2), 307 (2015)
26. Roffe, J. Quantum error correction: an introductory guide, Contemporary Physics 60, 226-245 (2019); DOI: 10.1080/00107514.2019.1667078
27. Kak, S. Encoding circuit, entropy and error correction. TechRxiv (2021); https://doi.org/10.36227/techrxiv.14478369.v1
28. Resch, S., Karpuzcu, U.R. Benchmarking quantum computers and the impact of quantum noise. ACM Computing Surveys 54, 7, 1–35 (2021); https://doi.org/10.1145/3464420
29. Castagnino, M., Lombardi, O. Self-induced decoherence: a new approach. Studies in History and Philosophy of Modern Physics 35, 73–107 (2004)
30. Castagnino, M., Lombardi, O. The self-induced approach to decoherence in cosmology. International Journal of Theoretical Physics 42, 1281-1299 (2003)
31. Kak, S. An information principle based on partitions for cognitive data. Journal of Artificial Intelligence and Consciousness, 10, 1-14, 2023. https://www.worldscientific.com/doi/10.1142/S2705078522500138
32. Kak, S. Number of autonomous cognitive agents in a neural network. Journal of Artificial Intelligence and Consciousness 9, 227-240, 2022; https://doi.org/10.1142/S2705078522500023
33. Bruza, P.D., Wang, Z., Busemeyer, J.R. Quantum cognition: a new theoretical approach to psychology. Trends in cognitive Sciences 19, 383-393 (2015)
34. Kak, S. No-go theorems on machine consciousness. TechRxiv (2023). https://doi.org/10.36227/techrxiv.22580872.v1