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Kinetic Axion $f(R)$ Gravity Phase Space

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Abstract: In this work, we studied the phase space of $f(R)$ gravity in the presence of a misalignment axion, including parity violating Chern–Simons terms. We construct the autonomous dynamical system by using appropriate dimensionless variables and find the cosmological attractors of the phase space, which are basically the fixed points of the autonomous dynamical system. We focus on the R^2 model and the misalignment axion potential near the minimum. We demonstrate that the Chern–Simons terms have no effect on the phase space. We found four distinct, possibly unstable fixed points with physical significance. Specifically, we found two identical de Sitter fixed points, one radiation domination fixed point, and one dark matter dominated fixed point. Thus, in the presence of a kinetic misalignment axion, the vacuum $f(R)$ gravity contains all of the cosmological fixed points that can characterize all of the known evolution eras of our universe.

Keywords: inflation; modified gravity; $f(R)$ gravity; early acceleration; CMB; primordial universe; early universe

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1. Introduction

Currently, the focus of most theoretical physicists is on the inflationary era [1–15], whether it occurred in the first place, and its characteristics. All the current and future gravitational wave experiments [16–26] focus on this fundamental question about the primordial era of our universe. To this end, the stage 4 Cosmic Microwave Background (CMB) experiments [27,28] will probe the *B*-modes of the CMB polarization spectrum. Thus, the essential features of the primordial era will be probed directly via the B-modes, or indirectly via the primordial gravitational wave tensor modes. These gravitational wave experiments along with the CMB stage 4 experiments will shed some light on the evolution of the universe from the primordial era, believed to be controlled by inflation, up to the late stages of the radiation domination era. If the inflationary era is confirmed, the difficult task is to understand which theory controls the inflationary era and what the characteristic features of the universe are during the post inflationary era.

In this work, we concretely examine the phase space of $f(R)$ gravity in the presence of a kinetic misalignment axion, also allowing the axion to have non-trivial Chern–Simons couplings that primordially violate the parity [29–32]. The study of the phase space provides vital information for the existence of cosmological attractor points, mathematically being fixed points of the dynamical system. The nature of the cosmological fixed points is determined by the value of the effective equation of state (EoS) parameter ω_{eff} for these points. Thus, the phase space reveals information about the existence of a physical solution in the context of a specific theory, so there are trajectories that may connect these cosmological fixed points. Since the universe goes through distinct evolutionary eras, it is important for a theoretical model to have these cosmological eras in the form of fixed points in its phase space. Moreover, the fixed points must be unstable in nature in order to have a sequence of distinct cosmological eras. The eras that the universe went through during its evolution are the inflationary era, which is described by a de Sitter fixed point with $\omega_{eff} = -1$, the radiation domination era with $\omega_{eff} = 1/3$, the matter domination era with $\omega_{eff} = 0$, and a late-time acceleration that can be quintessential with $-1 < \omega_{eff} < -1/3$ or even exactly de Sitter with $\omega_{eff} = -1$. The ordinary $f(R)$ gravity possesses the de Sitter fixed points but does not describe all the subsequent eras. Thus, in this paper, we examine the effect of the presence of a kinetic misalignment axion on the $f(R)$ gravity phase space, including the Chern–Simons term. We are interested in an R^2 gravity and the misalignment axion after the axion begins its oscillations near the origin of its potential. We construct an autonomous dynamical system using appropriately chosen dimensionless variables and the field equations. As we will show, the Chern–Simons term does not affect the phase space at all, and it is only affected by the $f(R)$ gravity and the misalignment axion. Remarkably, the kinetic axion has an important effect on the vacuum $f(R)$ gravity, and we report the existence of four distinct fixed points in the phase space, specifically two identical de Sitter fixed points, one radiation domination fixed point, and one matter domination fixed point. Thus, the misalignment axion complements successfully the $f(R)$ gravity, enabling the latter to describe successfully all the evolutionary eras of our universe.

2. Essential Features of Kinetic Axion $f(R)$ Gravity with and without Chern–Simons Corrections

We will consider the kinetic axion $f(R)$ gravity theoretical framework that was developed in [33], by also considering additionally Chern–Simons corrections. The kinetic axion mechanism is based on the misalignment axion theory [34], and it is a variant form of it. The difference between the canonical misalignment axion and the kinetic axion is basically the fact that, in the latter case, the axion has considerably large kinetic energy during the inflationary era, with the primordial $U(1)$ Peccei–Quinn symmetry being broken during the inflationary era. Due to the excess of the kinetic energy at the end of the inflationary era, the axion does not oscillate instantly after it reaches the minimum of its cosine-like potential, but continues its motion toward the local maximum of the potential. During this motion, the background EoS parameter of the axion is described by a stiff value $w = 1$. Accordingly, the axion continues its motion toward the minimum of its potential, where it starts oscillations and thereafter redshifts as cold dark matter. As shown in [33], the combined $f(R)$ gravity with a kinetic axion has some important effects on the inflationary era. Specifically, if the inflationary era is described by an R^2 model, the kinetic axion does not affect the dynamics of the inflationary era directly, but indirectly, by extending and enlarging the duration of it, causing an increase in the e-foldings number. In this paper, we study the phase space of the combined Chern–Simons-corrected R^2 model in the presence of the kinetic axion. We aim to construct an autonomous dynamical system using the resulting field equations, and we will find the fixed points of this autonomous dynamical system and discuss the physical significance of these in some detail. As we will see, the phase space is rich and full of physically significant fixed points, some of which are stable and some of which are potentially unstable. The study of the inflationary dynamics of the Chern–Simons-corrected R^2 gravity in the presence of kinetic axion dark matter will be presented elsewhere [35]. We will start with the most general kinetic axion Chern–Simons-corrected R^2 gravity, the action of which is

$$\mathcal{S} = \int d^4x\sqrt{-g}\left(\frac{f(R)}{2\kappa^2} - \frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - V(\phi) + \frac{\nu(\phi)}{8}R\tilde{R}\right), \tag{1}$$

where R is the Ricci scalar, $\kappa = \frac{1}{M_P}$ with $M_P = 1/\sqrt{8\pi G}$ is the reduced Planck mass, $\frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi$ and $V(\phi)$ denote the kinetic term and the canonical scalar potential of the scalar ϕ, $\nu(\phi)$ denotes the arbitrarily chosen scalar coupling function, and $R\tilde{R} = \eta^{\mu\nu\rho\sigma}R_{\mu\nu}{}^{\alpha\beta}R_{\rho\sigma\alpha\beta}$ is a parity odd (causing parity violation) term, where $R_{\mu\nu\rho\sigma}$ is as usual the Riemann tensor, and $\eta^{\mu\nu\rho\sigma}$ stands for the totally antisymmetric Levi–Civita tensor. Consider the R^2 model,

$$f(R) = R + \frac{R^2}{6M^2}\,, \tag{2}$$

with M being the mass scale, arbitrary for the moment. Consider also a flat Friedmann–Robertson–Walker (FRW) metric of the form

$$ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j\,, \tag{3}$$

with $a(t)$ being the scale factor and δ_{ij} denoting the Kronecker delta for the spatial components of the metric. Due to homogeneity, we assume that the scalar field and in effect all the functions of it, depend solely on the cosmic time. The field equations for the theoretical framework at hand are

$$\frac{3FH^2}{\kappa^2} = \frac{1}{2}\dot{\phi}^2 + V(\phi) + \frac{FR - f}{2\kappa^2} - \frac{3H\dot{F}}{\kappa^2}\,, \tag{4}$$

$$-\frac{2F\dot{H}}{\kappa^2} = \dot{\phi}^2 + \frac{\ddot{F} - H\dot{F}}{\kappa^2}\,, \tag{5}$$

$$\ddot{\phi} + 3H\dot{\phi} + V' = 0\,, \tag{6}$$

where $F = \frac{df}{dR}$, and the prime indicates differentiation with respect to the scalar field. Near the origin of the scalar potential, it can be approximated by a quadratic form as we show later, so the axion energy density redshifts as cold dark matter $\rho = \frac{1}{2}\dot{\phi}^2 + V \sim a^{-3}(t) = (1+z)^3$. As is apparent from the field equations, the Chern–Simons term essentially does not affect the field equations at all. Thus, it is expected that the phase space will not be affected at all. This is a peculiar result, since the Chern–Simons term affects only the tensor perturbations and not the scalar ones. In fact, as will shown in [35], only the tensor-to-scalar ratio and the tensor spectral index are affected by the Chern–Simons term. Thus, we come to the important conclusion that the phase space is probably not affected by the Chern–Simons term, although the dynamics of inflation is. We will show this explicitly for completeness. The stress-energy tensor defined as $T_{\mu\nu} = \frac{1}{4\sqrt{-g}}\frac{\delta(\sqrt{-g}\nu(\phi)R\tilde{R})}{\delta g^{\mu\nu}}$ depends on the Chern–Simons term, and the only non-zero contribution has the following form [36],

$$T_{\mu\nu} = \frac{1}{a}\epsilon_{\mu}{}^{\rho\sigma}\left[(\ddot{\nu} - H\dot{\nu})\dot{C}_{\nu\rho,\sigma} + \dot{\nu}(\ddot{C}_{\nu\rho,\sigma} + 3H\dot{C}_{\nu\rho,\sigma} - \frac{\Box}{a^2}C_{\nu\rho,\sigma})\right] + (\mu \leftrightarrow \nu)\,, \tag{7}$$

where $\epsilon_{\mu\nu\rho}$ is the 3-D Levi–Civita tensor, and $C_{\mu\nu}$ denotes the tensor perturbations. As can easily be seen, only the off-diagonal terms are affected by the Chern–Simons term, so the field equations are also unaffected. The tensor modes are, however, affected by the Chern–Simons term, as can be seen by their evolution equation,

$$\frac{1}{a^3F}\frac{d}{dt}\left(a^3F\dot{C}_{\mu\nu}\right) - \frac{\Box}{a^2}C_{\mu\nu} - \frac{2}{aF}\epsilon_{(\mu}{}^{\rho\sigma}\left[(\ddot{\nu} - H\dot{\nu})\dot{C}_{\nu)\rho} + \dot{\nu}(\ddot{C}_{\nu)\rho} + 3H\dot{C}_{\nu)\rho} - \frac{\Box}{a^2}C_{\nu)\rho}\right]_{,\sigma} = 0\,, \tag{8}$$

and by making a Fourier transformation,

$$C_{\mu\nu}(t,x) = \sqrt{Vol}\int \frac{d^3k}{(2\pi)^3}\sum_{l} e^{(l)}_{\mu\nu}(k)h_{lk}(t)e^{ik\cdot x}\,, \tag{9}$$

we get

$$\frac{1}{a^3Q_t}\frac{d}{dt}\left(a^3Q_t\dot{h}_{lk}\right) + \frac{k^2}{a^2}h_{lk} = 0\,, \tag{10}$$

where $e^{(l)}_{\mu\nu}$ denotes the circular polarization of the gravitational waves, $Q_t = \frac{F}{\kappa^2} + 2\lambda_l \dot{\nu} \frac{k}{a}$, and λ_l is ± 1 for the right and the left handed polarizations, respectively. Note that the propagation speed of the gravity waves is not affected by the parity-odd Chern–Simons term. Thus, in the presence of the Chern–Simons term, there are two inequivalent tensor perturbations evolving in spacetime. We need to note that the Chern–Simons term does not affect the scalar perturbations at all, and this can be seen explicitly in the field Equations (4)–(6). It does affect though the tensor perturbations, since it explicitly appears in the evolution equation of the tensor perturbations, as can be seen in Equation (8).

We now turn our focus on the dynamics of the axion, since it will be important for the phase space study. In general, as we will show, the presence of this dark matter component will cause differences between the phase space of vacuum $f(R)$ gravity and the kinetic axion $f(R)$ gravity. During inflation, the primordial $U(1)$ symmetry of the axion is broken, and the axion has a large vacuum expectation value $\varphi = \theta_\alpha f_\alpha$, where f_α denotes the axion decay constant, and θ_α is the initial misalignment angle with values in the range $0 < \theta_\alpha < 1$. Note that this angle is a dynamical field, as the axion is, so it is not constant, but it should be considered as an average expectation value of a specific period of time. This is the actual reason why the Chern–Simons term itself is not integrated away from the initial Lagrangian. For the misalignment axion field ϕ, we also have $\dot{\phi} \neq 0$ during the first horizon crossing. After the breaking of the primordial $U(1)$ symmetry, the axion potential is

$$V(\phi) = m_\alpha^2 f_\alpha^2 \left[1 - \cos\left(\frac{\phi}{f_\alpha}\right)\right], \tag{11}$$

and near the origin of the scalar potential, it reads

$$V(\phi) \simeq \frac{1}{2} m_\alpha^2 \phi^2 , \tag{12}$$

which is valid for $\frac{\phi}{f_a} < 1$.

3. Phase Space Analysis of the Chern–Simons-Corrected Kinetic Axion $f(R)$ Gravity

In this section, we thoroughly study the phase space of the Chern–Simons kinetic axion $f(R)$ model. As mentioned in the previous section, due to the fact that the Chern–Simons contributions are in fact absent from the background field equations of motion, the overall phenomenology of the scalar perturbations and the phase space of the model is identical to that in the case of the kinetic axion $f(R)$ gravity, since only the tensor perturbations are affected by the inclusion of the Chern–Simons term. Therefore, if one only studies the phase space of the kinetic axion $f(R)$ gravity, it should coincide with that of the Chern–Simons kinetic axion $f(R)$ model. We will showcase this in some detail.

Firstly, we begin by including perfect matter fluids in the gravitational action, that is,

$$\mathcal{S} = \int d^4x \sqrt{-g} \left(\frac{f(R)}{2\kappa^2} - \frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - V(\phi) + \frac{\nu(\phi)}{8} R\tilde{R} + \mathcal{L}_{matter} \right), \tag{13}$$

where $\mathcal{L}_{matter}$ contains all the information available about relativistic and non-relativistic matter, while the axion dynamics are specified by the scalar field. As a result, the equations of motion are affected by the inclusion of perfect matter fluids as shown below:

$$\frac{3FH^2}{\kappa^2} = \rho + \frac{FR - f}{2\kappa^2} - \frac{3H\dot{F}}{\kappa^2} + \frac{1}{2}\dot{\phi}^2 + V , \tag{14}$$

$$-\frac{2F\dot{H}}{\kappa^2} = \rho + P + \frac{\ddot{F} - H\dot{F}}{\kappa^2} + \dot{\phi}^2 , \tag{15}$$

$$\ddot{\phi} + 3H\dot{\phi} + V' = 0 . \tag{16}$$

In order to perform a consistent and robust phase space analysis, we must first construct an autonomous dynamical system using an appropriate dimensionless variable and manipulating the field equations appropriately. We make use of the fact that the $f(R)$ gravity follows a power-law behavior, that is, $f(R) = R + AR^n$, while the scalar potential is equal to $V(\phi) = V_0(\kappa\phi)^m$. Obviously, we are interested in the cases $n = 2 = m$, where $A = \frac{1}{6M^2}$ and $V_0\kappa^2 = \frac{1}{2}m_\alpha^2$; however, the dynamical system is independent of A and V_0 and is affected only by the exponents. In addition, we define the following dynamical variables [37]:

$$x = \frac{\kappa^2\dot{\phi}^2}{6FH^2}, \; y = \frac{\kappa^2 V}{3FH^2}, \; z = -\frac{\dot{F}}{HF}, \; u = -\frac{f}{6FH^2}, \; v = \frac{R}{6H^2}, \; s = \frac{\dot{V}}{HV}, \; p = \frac{\kappa^2\rho_r}{3FH^2}, \; q = \frac{\kappa^2\rho_m}{3FH^2}, \tag{17}$$

where it should be stated that, due to the presence of the $f(R)$ gravity modifications in the field equations, the variables p and q do not coincide with the usual density parameters Ω_r and Ω_m. By making use of the e-foldings number as a variable instead of cosmic time t, introduced via the differential operator $\frac{d}{dN} = \frac{1}{H}\frac{d}{dt}$, the dynamical variables easily satisfy the following relations:

$$\frac{dx}{dN} = x\left[2\frac{\ddot{\phi}}{H\dot{\phi}} - 2(v-2) + z\right], \tag{18}$$

$$\frac{dy}{dN} = y\left[s + z - 2(v-2)\right], \tag{19}$$

$$\frac{dz}{dN} = z(z+1) + 6x + 4q + 3q + 2(v-2)(1-\frac{z}{2}), \tag{20}$$

$$\frac{du}{dN} = u\left[z - 2(v-2)\right] + \frac{zv^2}{n(u+v)}, \tag{21}$$

$$\frac{dv}{dN} = -v\left[\frac{zv}{n(u+v)} + 2(v-2)\right], \tag{22}$$

$$\frac{ds}{dN} = s\left[\frac{\ddot{\phi}}{H\dot{\phi}} - (v-2) - \frac{s}{m}\right], \tag{23}$$

$$\frac{dp}{dN} = p\left[z - 2v\right], \tag{24}$$

$$\frac{dq}{dN} = q\left[z - 2v + 1\right], \tag{25}$$

where from the Friedmann equation and the Klein-Gordon equation, one can see that the dynamical variables must satisfy the constraints,

$$x + y + z + u + v + p + q = 1, \tag{26}$$

$$\frac{\ddot{\phi}}{H\dot{\phi}} = -3 - \frac{s}{2}\frac{y}{x}. \tag{27}$$

As can be seen, there exist eight dynamical variables that are needed in order to study the phase space. If we focus on the vacuum kinetic axion $f(R)$ model that we are interested in, we can focus only on four dynamical variables. Three derive from the $f(R)$ part, and the final one refers to the kinetic term of the axion. Thus, working with this subsystem, we can focus only on Equations (18)–(22) and demand that

$$x + z + u + v \simeq 1 \qquad \frac{\ddot{\phi}}{H\dot{\phi}} \simeq -3 \qquad \frac{dz}{dN} = z(z+1) + 6x + 3q + 2(v-2)(1-\frac{z}{2}). \tag{28}$$

This can be done without a loss of generality, as we focus only on the dominant terms. In turn, the inclusion of the remaining dynamical variables will not spoil the analysis; rather, it will result in the inclusion of additional fixed points in the phase space. This can easily be inferred from the fact that Equations (19) and (23)–(25) can become zero for $y = 0$, $s = 0$, $p = 0$, and $q = 0$; thus, they do not affect the rest of the differential equations. Studying this subspace implies that the total number of fixed points for $n = 2$ is 3, as shown in Table 1. Here, it becomes abundantly clear that there exists a matter domination fixed point, a radiation domination fixed point, and two de Sitter fixed points; however, the two de Sitter fixed points are identical. Out of the these fixed points, only the matter dominated fixed point is a saddle, while the rest are in fact non-hyperbolic fixed points. All of them are physical and always exist. We can also further have a concrete idea on the phase space by checking the trajectories focusing on the case $n = 2$. In Figure 1, we present the plot for the trajectories $u(N)$ and $v(N)$ as functions of the e-foldings number, for the first 10 e-foldings. We used many different sets of initial conditions for all the dynamical variables, but the physical picture remains the same in all cases. The physical fixed points are reached quite fast as can be seen, even before the first 10 e-foldings. For the plot of Figure 1, we used the initial conditions $x(0) = -8$, $z(0) = 5$, $u(0) = 1.5$, and $v(0) = 0.1$, but as mentioned, the result is robust and does not change by changing the initial conditions. In Figure 1, the red curve represents the variable $u(N)$, which approaches the fixed point $P_2 = (0, 0, -1, 2)$ value of $u(N) = -1$. Accordingly, the blue curve represents the variable $v(N)$, which approaches the fixed point $P_2 = (0, 0, -1, 2)$ value of $v(N) = 2$. An apparent comment is that this result holds true irrespective of the initial conditions chosen, and it is noticeable that the fixed point is reached quite fast. Unfortunately, it is not easy to see the degeneracy of the double de Sitter fixed point in the phase space trajectory plot. A deeper investigation is required, where the center manifolds are studied. Such a study though exceeds by far the purposes of this work. Furthermore, the fixed point $P_2 = (0, 0, -1, 2)$ seems stable, but there is a zero eigenvalue, which means that the fixed point might be unstable, a fact that can be verified only numerically. Thus, we can claim that the de Sitter fixed point is not entirely stable.

Table 1. Fixed points for the kinetic axion R^2 vacuum model with the kinetic axion scalar potential.

Fixed Point	(x,z,u,v)	Eigenvalues	Stability	q	ω_{eff}
P_1	(0,−4,5,0)	(−6,−5,4,0)	Non Hyperbolic	1	$\frac{1}{3}$
P_2	(0,0,−1,2)	(0,−6,−3,0)	Non Hyperbolic	−1	−1
P_3	$(-\frac{9}{4},3,-\frac{1}{4},\frac{1}{2})$	$(6,3,3,-\frac{3}{2})$	Saddle	$\frac{1}{2}$	0

Including a potential does not alter the already known fixed points; it simply predicts an additional fixed point describing cold dark matter that differs only in the variable s. Therefore, it does not influence the Friedmann constraint; rather, it affects the evolution of the dynamical variable y and $\dot{\phi}$. Based on the phase space analysis, the most important feature that must be mentioned is that the vacuum R^2 kinetic axion model possesses all the cosmologically interesting fixed points. In fact, in the phase space, there is a potentially unstable de Sitter fixed point, an unstable radiation domination fixed point, and a matter domination fixed point. The instability is due to the existence of a zero eigenvalue for the corresponding fixed point, with the other two eigenvalues being negative. Thus, the trajectories in the phase space are attracted to the fixed point, but eventually should be repelled from it. A deeper analysis of this can reveal this physical picture, by analyzing the center manifolds of the model, but this exceeds the aims of this work. We therefore suspect that the fixed point is possibly unstable, but it is certainly not entirely stable. The physical picture obtained is quite interesting, since it is apparent that the phase space may contain trajectories that may connect four important cosmological eras for our universe, starting from the unstable de Sitter fixed point, which describes the inflationary era with a possible exit due to the instability caused by the zero eigenvalue, continuing to the unstable radiation era, followed by an unstable matter domination era, and lastly ending up in an unstable final de Sitter era. Thus, this model can potentially describe early- and late-time acceleration, with all the intermediate eras potentially able to drive the evolution. This picture seems quite attractive, but needs to be confirmed numerically. This requires the analysis of stable manifolds, which exceeds, however, the scope of this work, which was to demonstrate the existence of these physical fixed points.

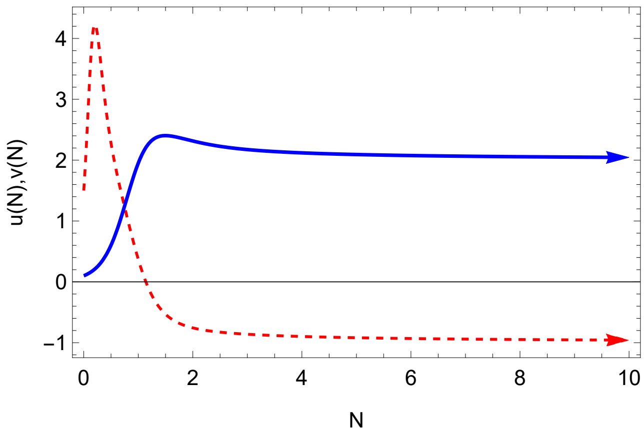

Figure 1. The numerical solutions $u(N)$ and $v(N)$ for the dynamical system (18)–(25), for the initial conditions $x(0) = -8$, $z(0) = 5$, $u(0) = 1.5$, and $v(0) = 0.1$ and for $n = 2$. The red curve represents the variable $u(N)$, which approaches the fixed point $P_2 = (0,0,-1,2)$ value of $u(N) = -1$. Accordingly, the blue curve represents the variable $v(N)$, which approaches the fixed point $P_2 = (0,0,-1,2)$ value of $v(N) = 2$.

Finally, including perfect matter fluids does not increase the number of fixed points in the matter and radiation eras as one may expect; on the contrary, their presence in the equations of motion predict the appearance of two additional fixed points: one that describes neither acceleration nor deceleration as $\ddot{a} = 0$ with an effective EoS parameter $\omega_{eff} = -\frac{1}{3}$ and another that describes quintessential late-time acceleration since the effective EoS parameter is in this case $\omega_{eff} = -\frac{1}{2} < -\frac{1}{3}$ (see Table 2). Overall, the kinetic axion model can potentially describe the evolution of the universe smoothly, even if the axion mass is infinitesimal and the overall phase space is indeed quite richer compared to the vacuum $f(R)$ gravity. Obviously, the same applies to the kinetic axion $f(R)$ model in the presence of a Chern–Simons parity-odd term since the background equations remain unaffected, and only tensor perturbations are influenced by such a term. In a sense, even though the inclusion of perfect matter fluids in the axion-free vacuum $f(R)$ gravity case does not result in the generation of matter and radiation fixed points, the inclusion of the kinetic axion can result in the appearance of a matter fixed point, which makes perfect sense considering that the axion perfectly describes the matter of the dark sector of the universe and thus has the most dominant contribution on non-relativistic matter. In addition, the de-Sitter fixed point that emerges is, in a sense, degenerate. This is because, for a power-law $f(R)$ model, while there is a total of four fixed points, two of them coincide in the limit $n \to 2$. If a different exponent was selected, then an additional fixed point that describes acceleration can be produced.

Table 2. Fixed points for the kinetic axion R^2 model in the presence of perfect matter fluids.

Fixed Point	(x,y,z,u,v,p,q)	Eigenvalues	Stability	q	ω_{eff}
P_1	(0,0,−4,5,0,0,0)	(−6,−5,−4,4,−3,0,0)	Non Hyperbolic	1	$\frac{1}{3}$
P_2	(0,0,0,−1,2,0,0)	(0,0,−6,−4,−3,−3,0)	Non Hyperbolic	−1	−1
P_3	$(-\frac{9}{4},0,3,-\frac{1}{4},\frac{1}{2},0,0)$	$(6,3,3,3,2,-\frac{3}{2},6)$	Saddle	$\frac{1}{2}$	0
P_4	$(0,0,2,-\frac{1}{2},1,-\frac{3}{2},0)$	$(4,4,-\sqrt{6},\sqrt{6},-2,2,1)$	Saddle	0	$-\frac{1}{3}$
P_5	$(0,0,\frac{3}{2},-\frac{5}{8},\frac{5}{4},0,-\frac{9}{8})$	$(-3,3,3,\frac{3}{8}(-1-\sqrt{41}),\frac{3}{8}(-1+\sqrt{41}),\frac{3}{2},-1)$	Saddle	$-\frac{1}{4}$	$-\frac{1}{2}$

4. Conclusions

We studied in detail the phase space of $f(R)$ gravity in the presence of a kinetic misalignments axion, including parity violating Chern–Simons terms. We used appropriately chosen dimensionless variables, and in conjunction with the field equations, we constructed an autonomous dynamical system. We focused on the R^2 model and the axion while it undergoes its oscillations around the minimum of its potential. We found four unstable cosmological fixed points with physical significance. The latter is determined by the value of the total EoS parameter at the fixed points. The fixed points we found in the phase space in the form of unstable fixed points are two de Sitter fixed points with $\omega_{eff} = -1$, one radiation domination fixed point with $\omega_{eff} = 1/3$, and one matter domination fixed point with $\omega_{eff} = 0$. The two de Sitter points correspond to the early- and late-time acceleration eras. This is remarkable since ordinary $f(R)$ gravity possesses only the de Sitter fixed points but does not describe all the subsequent cosmological eras. The existence of the fixed points indicates that there might be trajectories in the phase space that may connect these fixed points. It is also worth mentioning that the parity violating term does not affect the phase space at all, since it only affects the evolution of the tensor perturbations and not the background field equations. Thus, with this work, we highlighted the importance of the coexistence of some modified gravity controlling the inflationary era with some coherent non-thermal dark matter scalar field.

Finally, we will briefly discuss the potential implications of our findings for observational cosmology or experimental tests. With our phase space study, we proved that the Chern–Simons $f(R)$ gravity model is able to produce all of the cosmological eras of our universe. This is important and serves as a first step toward the phenomenological viability of the model. More concrete calculations are now needed in order to quantitatively study the inflationary and other eras implications of this work, and this is deferred to a future work.

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References

1. Guth, A.H. The Inflationary Universe: A Possible Solution to the Horizon and Flatness Problems. *Phys. Rev. D* **1981**, *23*, 347. [CrossRef]
2. Linde, A.D. Chaotic Inflation. *Phys. Lett. B* **1983**, *129*, 177–181. [CrossRef]
3. Starobinsky, A.A. Dynamics of Phase Transition in the New Inflationary Universe Scenario and Generation of Perturbations. *Phys. Lett. B* **1982**, *117*, 175–178. [CrossRef]
4. Linde, A.D. Inflationary Cosmology. *arXiv* **2008**, arXiv:0705.0164.
5. Gorbunov, D.S.; Rubakov, V.A. *Introduction to the Theory of the Early Universe: Cosmological Perturbations and Inflationary Theory*; World Scientific: Hackensack, CA, USA, 2011; 489p.
6. Lyth, D.H.; Riotto, A. Particle physics models of inflation and the cosmological density perturbation. *Phys. Rept.* **1999**, *314*, 1–146. [CrossRef]
7. Linde, A.D. Initial Conditions For Inflation. *Phys. Lett.* **1985**, *162B*, 281. [CrossRef]
8. Albrecht, A.; Steinhardt, P.J. Cosmology for Grand Unified Theories with Radiatively Induced Symmetry Breaking. *Phys. Rev. Lett.* **1982**, *48*, 1220. [CrossRef]
9. Sasaki, M.; Stewart, E.D. A General analytic formula for the spectral index of the density perturbations produced during inflation. *Prog. Theor. Phys.* **1996**, *95*, 71–78. [CrossRef]
10. Turok, N. A critical review of inflation. *Class. Quant. Grav.* **2002**, *19*, 3449. [CrossRef]
11. Linde, A.D. Inflation and string cosmology. *Prog. Theor. Phys. Suppl.* **2006**, *163*, 295. [CrossRef]
12. Kachru, S.; Kallosh, R.; Linde, A.D.; Maldacena, J.M.; McAllister, L.P.; Trivedi, S.P. Towards inflation in string theory. *JCAP* **2003**, *0310*, 013. [CrossRef]
13. Brandenberger, R. Initial Conditions for Inflation—A Short Review. *arXiv* **2016**, arXiv:1601.01918.
14. Bamba, K.; Odintsov, S.D. Inflationary cosmology in modified gravity theories. *Symmetry* **2015**, *7*, 220–240. [CrossRef]
15. Martin, J.; Ringeval, C.; Vennin, V. Encyclopaedia Inflationaris. *Phys. Dark Univ.* **2014**, *5–6*, 75–235. [CrossRef]
16. Arzoumanian, Z.; Baker, P.T.; Blumer, H.; Bécsy, B.; Brazier, A.; Brook, P.R.; Burke-Spolaor, S.; Chatterjee, S.; Chen, S.; Cordes, J.M.; et al. The NANOGrav 12.5 yr Data Set: Search for an Isotropic Stochastic Gravitational-wave Background. *Astrophys. J. Lett.* **2020**, *905*, L34. [CrossRef]
17. Agazie, G.; Anumarlapudi, A.; Archibald, A.M.; Arzoumanian, Z.; Baker, P.T.; Bécsy, B.; Blecha, L.; Brazier, A.; Brook, P.R.; Burke-Spolaor, S.; et al. The NANOGrav 15 yr Data Set: Evidence for a Gravitational-wave Background. *Astrophys. J. Lett.* **2023**, *951*, L8. [CrossRef]
18. Hild, S.; Abernathy, M.; Acernese, F.; Amaro-Seoane, P.; Andersson, N.; Arun, K.; Barone, F.; Barr, B.; Barsuglia, M.; Beker, M.; et al. Sensitivity Studies for Third-Generation Gravitational Wave Observatories. *Class. Quant. Grav.* **2011**, 28, 094013. [CrossRef]
19. Baker, J.; Bellovary, J.; Bender, P.L.; Berti, E.; Caldwell, R.; Camp, J.; Conklin, J.W.; Cornish, N.; Cutler, C.; DeRosa, R.; et al. The Laser Interferometer Space Antenna: Unveiling the Millihertz Gravitational Wave Sky. *arXiv* **2019**, arXiv:1907.06482.
20. Smith, T.L.; Caldwell, R. LISA for Cosmologists: Calculating the Signal-to-Noise Ratio for Stochastic and Deterministic Sources. *Phys. Rev. D* **2019**, *100*, 104055. [CrossRef]
21. Crowder, J.; Cornish, N.J. Beyond LISA: Exploring future gravitational wave missions. *Phys. Rev. D* **2005**, *72*, 083005. [CrossRef]
22. Smith, T.L.; Caldwell, R. Sensitivity to a Frequency-Dependent Circular Polarization in an Isotropic Stochastic Gravitational Wave Background. *Phys. Rev. D* **2017**, *95*, 044036. [CrossRef]
23. Seto, N.; Kawamura, S.; Nakamura, T. Possibility of direct measurement of the acceleration of the universe using 0.1-Hz band laser interferometer gravitational wave antenna in space. *Phys. Rev. Lett.* **2001**, *87*, 221103. [CrossRef]
24. Kawamura, S.; Ando, M.; Seto, N.; Sato, S.; Musha, M.; Kawano, I.; Yokoyama, J.; Tanaka, T.; Ioka, K.; Akutsu, T.; et al. Current status of space gravitational wave antenna DECIGO and B-DECIGO. *arXiv* **2020**, arXiv:2006.13545.
25. Weltman, A.; Bull, P.; Camera, S.; Kelley, K.; Padmanabhan, H.; Pritchard, J.; Raccanelli, A.; Riemer-Sørensen, S.; Shao, L.; Andrianomena, S.; et al. Fundamental physics with the Square Kilometre Array. *Publ. Astron. Soc. Austral.* **2020**, 37, e002. [CrossRef]
26. Auclair, P.; Bacon, D.; Baker, T.; Barreiro, T.; Bartolo, N.; Belgacem, E.; Bellomo, N.; Ben-Dayan, I.; Bertacca, D.; Besancon, M.; et al. Cosmology with the Laser Interferometer Space Antenna. *arXiv* **2023**, arXiv:2204.05434.
27. Abazajian, K.N.; Adshead, P.; Ahmed, Z.; Allen, S.W.; Alonso, D.; Arnold, K.S.; Baccigalupi, C.; Bartlett, J.G.; Battaglia, N.; Benson, B.A.; et al. CMB-S4 Science Book, First Edition. *arXiv* **2016**, arXiv:1610.02743.
28. Abitbol, M.H.; Adachi, S.; Ade, P.; Aguirre, J.; Ahmed, Z.; Aiola, S.; Ali, A.; Alonso, D.; Alvarez, M.A.; Arnold, K.; et al. The Simons Observatory: Astro2020 Decadal Project Whitepaper. *Bull. Am. Astron. Soc.* **2019**, 51, 147.
29. Nishizawa, A.; Kobayashi, T. Parity-violating gravity and GW170817. *Phys. Rev. D* **2018**, *98*, 124018. [CrossRef]
30. Wagle, P.; Yunes, N.; Garfinkle, D.; Bieri, L. Hair loss in parity violating gravity. *Class. Quant. Grav.* **2019**, *36*, 115004. [CrossRef]
31. Yagi, K.; Yunes, N.; Tanaka, T. Gravitational Waves from Quasi-Circular Black Hole Binaries in Dynamical Chern–Simons Gravity. *Phys. Rev. Lett.* **2012**, *109*, 251105. [CrossRef]
32. Odintsov, S.D.; Oikonomou, V.K. Chirality of gravitational waves in Chern–Simons f(R) gravity cosmology. *Phys. Rev. D* **2022**, *105*, 104054. [CrossRef]
33. Oikonomou, V.K. Kinetic axion F(R) gravity inflation. *Phys. Rev. D* **2022**, *106*, 044041. [CrossRef]
34. Marsh, D.J.E. Axion Cosmology. *Phys. Rept.* **2016**, *643*, 1–79. [CrossRef]

35. Oikonomou, V.K. Kinetic Axion Chern Simons Inflationary Dynamics. 2023, Unpublished work.
36. Hwang, J.; Noh, H. Classical evolution and quantum generation in generalized gravity theories including string corrections and tachyons: Unified analyses. *Phys. Rev. D* **2005**, *71*, 063536. [CrossRef]
37. Shah, P.; Samanta, G.C. Stability analysis for cosmological models in f(R) gravity using dynamical system analysis. *Eur. Phys. J. C* **2019**, *79*, 414. [CrossRef]