On Extended Supersymmetry in Two and Four Dimensions

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Dedication

To my great ancestors
who put their lives on the line
to make our wonderful World possible

Abstract

We study the relationship between gauge theories in two and four dimensions with $\mathcal{N} = 2$ supersymmetry. This includes the duality between their moduli spaces, comparison of Bogomolny-Prassad-Somerfield (BPS) spectra, study of instanton configurations, and other aspects. On they way we use various methods of integrability, conformal filed theories and string theory to achieve our goals.

We start with describing physics of two dimensional $\mathcal{N} = 2$ sigma models, geometry of their target spaces, their BPS spectra, and how they can de derived from four dimensional theories via BPS vortex construction. Two different approaches – gauge and geometric, as tools to study 2d theories, are compared in the light of perturbative as well as nonperturbative aspects of the theories in question.

Then we discuss four dimensional supersymmetric gauge theories in presence of an Omega background – a special deformation used in localization of path integrals of supersymmetric theories. Instead of performing the localization we treat the Omega background physically and study BPS solitons for such theories, albeit the latter already possess less supersymmetry. Theories with $\mathcal{N} = 2$ SUSY in Omega background are conjectured (and proven in special cases) to be dual to nonsupersymmetric conformal field theories in two dimensions by Alday, Gaiotto and Tachikawa (AGT duality). Employing the machinery of the 4d/2d duality combined with powerful methods of integrability we provide a proof of the AGT relation (in the limit where the 2d model in question exists).

In the end we regard heterotic $\mathcal{N} = (0,1)$ and $\mathcal{N} = (0,2)$ sigma models in two dimensions. With fewer supersymmetry one has less control on the nonperturbative dynamics of the theory, however, we get some nice physical understanding of these models at strong coupling by means of the large number of colors approximation.

Contents

List of Figures

Chapter 1

Introduction

Supersymmetric gauge theories provide us with a vast and very mysterious framework in which very powerful physical accomplishments can be made. Since the discovery of supersymmetry in the early 70-th [1, 2, 3, 4], many theoretical physicists focused their efforts on investigating the new quantum field theories with and without gravity in order to uncover the mystery of the "superworld". Throughout the years theorists' attention was migrating from more formal aspects of supersymmetry (SUSY) involving intricate and not less mysterious branches of mathematics to more phenomenologically oriented studies and back. Nowadays SUSY community consists of several thousands of researchers worldwide who are continuously attacking new problems of a constantly growing complexity. In the current manuscript we shall only discuss problems of the first kind as that is where the author's current scientific interests are.

Let us mention, however, that the status of supersymmetry as a branch of theoretical physics may soon completely change its form after new data on SUSY search will start coming from the Large Hadron Collider (LHC) in CERN. Astonishingly the Higgs boson discovery at $m_H \sim 126$ GeV has happened [5, 6] while the author was preparing the current thesis. Experimentalists in Geneva have done (and keep doing) an extraordinary job, and steadily we will know the answers to many questions about how real the supersymmetry is in the nearest future. As of the present day (Summer 2012) the perspectives of finding SUSY at LHC are not very optimistic (see e.g. [7]). In the worst case scenario, when the SUSY will be found not to be present, at least in the form we use to think about it, phenomenological studies in this direction will be virtually over and only abstract and formal aspects of supersymmetry will remain on the frontline of physics.

Thus, in the accord with the current work, supersymmetry is to be treated as a powerful tool in the study of gauge theories at strong coupling. Access to strongly coupled dynamics becomes possible in some sectors of the Hilbert space of the theory which are protected by supersymmetry. Such states are usually called Bogomolny-Prassad-Somerfield (BPS) states [8, 9]. Although BPS sectors of gauge theories with extended supersymmetry are elaborated on to a much better extent than non-BPS sectors, it is the study of the latter which is important for a complete understanding of the theory. However, in SUSY theories with gravity (supergravity or SUGRA) already investigation of BPS objects is nowadays the cutting edge in the field due to much more complicated structure of the space of BPS states. The progress in supergravity is thus much more faint than in field theories.

Remarkably in the past decades supersymmetry proved to be a source of unexpected inspiration to various branches of mathematics. For example, the Seiberg-Witten solution of low energy effective theories in four dimensions with extended SUSY [10, 11] boosted the development of modular forms on the math side, and further development of the ideas of Seiberg and Witten into string theory unveiled even more sophisticated and complex mathematical structures like mock modular forms, etc. The moduli space of BPS states of a gauge theory was found to have a nontrivial structure of domains, separated from each other by walls, each of which possess different BPS spectra. Transition phenomena occur on these walls. Having known the spectrum in one of the domains of the moduli space one may be interested in extending it to other domains. It is made possible thanks to Kontsevich-Soibelman (KS) wall crossing formulae (WCF) which appeared in mathematical literature [12]. Right after that WCF were elaborated in the series of seminal papers by Gaiotto, Moore and Neitzke [13, 14, 15, 16, 17] in connection with various aspects of nonperturbative physics and string theory.

Another example of the interplay between SUSY gauge theories and math can be found in the celebrated $\mathcal{N} = 4$ Super Yang-Mills theory [18] which was extensively studied from various angles. Most notably, the $\mathcal{N} = 4$ theory is related to the Langlands duality [19, 20]. Yet another and very recent study of superconformal quiver gauge theories by employing the Novikov-Shifman-Vainshtein-Zakharov (NSVZ) β-function

[21], contributed to development in understanding of brane tilings [22, 23, 24].

Computations in supersymmetric gauge theories can be greatly simplified by using the power of the path integral localization [25, 26, 27]. Theories with $\mathcal{N} = 2$ SUSY enjoy the fact that their Lagrangians, up to topological terms, can be written as anticommutators involving a nilpotent supercharge (BRST exact). Then one mentions that BRST exact terms have zero vacuum expectation values. Under a certain deformation of the Lagrangian, which does not change physical observables in the theory, one observes that the path integral localizes on the space of the solutions of the self-dual equation, or in the instanton configuration. The space of such solutions of the self-duality equation is finite dimensional which enables us to compute partition functions of $\mathcal{N} = 2$ gauge theories [28].

Many four dimensional gauge theories with $\mathcal{N} = 2$ supersymmetry can be viewed as twisted compactifications of superconformal six dimensional $\mathcal{N} = (0, 2)$ theory [29, 30] on a genus-g Riemann surface $\Sigma_{g,n}$ with n punctures [31]. The result is archived by wrapping an M5 brane around that Riemann surface[1] . One may also study fivebranes with M2 brane defects or surface operators [32] which preserve $(2, 2)$ supersymmetry. Location of the defect on the Riemann surface gives a mapping between the space of parameters of the surface operator and the parameter space of gauge couplings of some 2d twisted superpotential. It brings us to the notion of the 2d theory. Equivalently surface operators can be obtained from BPS strings (vortices) [33, 34, 35] by sending their tension to infinity. Notably the latter construction does not require any higher dimensional brane engineering.

It turns out that $\mathcal{N} = 2$ theories in two and four dimensions share many properties, e.g. BPS spectra of the two sets of theories are in agreement with each other provided that certain constraints on the 4d side are imposed, instanton moduli spaces of 4d gauge theories have nice geometrical properties which can be understood by studying two dimensional exactly solvable models. Relationships of these forms are sometimes referred to as *4d/2d dualities*, however, in each case one has to specify more details on what exactly is described. The 4d/2d correspondence for various $\mathcal{N} = 2$ theories is the core part of the current work. We shall address both string (M) theory as well as pure field theoretical constructions in connection with the $\mathcal{N} = 2$ physics and two and four

[1] Sometimes these theories are called *class S* theories, where S stands for 'six'.

dimensions.[2] Needless to say that two dimensional theories are easier to study than their 4d counterparts.

The current thesis is organized as follows

- In Chapter 2 we discuss $\mathcal{N}=2$ supersymmetry in two dimensions. We study the relationship between gauged linear sigma models ($GL\sigma M$)s and nonlinear sigma models ($NL\sigma M$)s both from the string theory point of view as well as by looking at an explicit BPS vortex construction. Quantum dynamics of several sigma models is investigated in some detail. First we solve the model in the leading order when the number of colors is large (large-N approximation). Then our attention is shifted to the type IIA formalism and the way the above mentioned sigma models can be extracted from there. One finds that in the 't Hooft limit of infinite N both approaches are totally equivalent, however, at finite N the calculations agree only in the BPS sector. Beyond the BPS sector a mismatch is found which calls for further explanation. Finally, we study perturbation theory of these models from various standpoints. The material presented in this Chapter can be partly found in [36].

- In Chapter 3, which is heavily based on [37], we study supersymmetric theories in Omega background and BPS solitons in such theories. Later we shall take advantage of powerful methods of integrability in order to related these theories with some nonsupersymmetric conformal field theories in two dimensions (AGT correspondence). We reconsider string and domain wall central charges in $\mathcal{N}=2$ supersymmetric gauge theories in four dimensions in presence of the Omega background in the Nekrasov-Shatashvili (NS) limit. Existence of these charges entails presence of the corresponding topological defects in the theory – vortices and domain walls. In spirit of the 4d/2d duality we discuss the worldsheet low energy effective theory living on the BPS vortex in $\mathcal{N}=2$ Supersymmetric Quantum Chromodynamics (SQCD). We discuss some aspects of the brane realization of the dualities between various quantum integrable models. A chain of such dualities enables us to check the AGT correspondence in the NS limit.

[2] Counting of supercharges has to be carried out carefully. Indeed, $\mathcal{N}=2$ in four dimensions means 8 supercharges, however, in 2d it's four supercharges. We shall also consider theories with twice less supersymmetry both in 4d and 2d.

- Chapter 4 describes physics of heterotic (chiral) sigma models. First we build a family of heterotic deformations of the O(N) sigma model. These deformations break $(1,1)$ supersymmetry down to $(0,1)$ symmetry. We solve this model at large N. We also find an alternative superfield formulation of the heterotic $\mathbb{CP}^N$ sigma models which was discussed in the literature before. Then we study a heterotic two-dimensional $\mathcal{N}=(0,2)$ gauged non-linear sigma model ($GL\sigma M$) whose target space is a tautological fiber bundle over a projective space. We consider $GL\sigma M$ with N positively and $\widetilde{N} = N_F - N$ negatively charged fields. This model is believed to give a description of the low-energy physics of a non-Abelian semi-local vortex in a four-dimensional $\mathcal{N}=2$ supersymmetric $U(N)$ gauge theory with $N_F > N$ matter hypermultiplets. The supersymmetry in the latter theory is broken down to $\mathcal{N}=1$ by a mass term for the adjoint fields. We solve the model in the large-N approximation and explore a two-dimensional subset of the mass parameter space for which a discrete $\mathbb{Z}_{N-\widetilde{N}}$ symmetry is preserved. Supersymmetry is generically broken, but it is preserved for special values of the masses where a new branch opens up and the model becomes super-conformal.

  The author has published [38, 39] with his collaborators and Chapter 4 is based on these two papers.

- Chapter 5 concludes this manuscript and lists possible direction for the future research. It is more focused on the outlook as each of the above Chapters has its own conclusions.

Note that the current text is not intended to serve as an introduction to supersymmetry. The reader is welcome to use various sources which can be found in abundance in the literature today. Rather we shall only review the necessary parts of the material which will be used for the calculations in the bulk of the thesis.

Chapter 2

Two-Dimensional Sigma Models with $\mathcal{N} = 2$ Supersymmetry

2.1 Introduction

Some time ago it has been observed [33, 40] that the BPS spectrum of the twisted mass-deformed two-dimensional $\mathcal{N} = (2, 2)$ $\mathbb{CP}^{N-1}$ sigma model coincides with that of the four-dimensional $\mathcal{N} = 2$ SU(N) supersymmetric quantum chromodynamics (SQCD) with N massive flavors (in a certain vacuum). This correspondence holds upon identification of the holomorphic parameters of the two theories, e.g. the masses and the strong coupling scales. Similarities between sigma models in two dimensions and gauge theories in four dimensions have been discussed for a long time, since the discovery of asymptotic freedom and instantons in the O(3) sigma model [41, 42]. The observation [33, 40] showed that these similarities go beyond the qualitative level in some supersymmetric theories. The deep reasons for this coincidence were revealed thanks to the discovery of the non-Abelian vortices in the color-flavor locked phase of supersymmetric QCD [35, 43, 44, 34, 45, 46, 47, 48]. The two-dimensional $\mathbb{CP}^{N-1}$ sigma model is nothing other than the low-energy description of the non-Abelian string. Excitations of the non-Abelian string correspond to states of the bulk SQCD which are confined on the strings. In particular, BPS kinks of the $\mathbb{CP}^{N-1}$ model are confined monopoles from the bulk perspective [34]. This observation provides more evidence that the kink spectrum exactly coincides with the monopole spectrum.

The above results were naturally generalized to SU(N) supersymmetric QCD with $N+\tilde{N}$ flavors (i.e. the number of flavors is larger than that of colors). In this case one deals with the so-called *semilocal* [49, 50, 51, 52, 53] non-Abelian strings. Hanany and Tong suggested a world-sheet model for such strings [35] (the HT model[1]) from type-IIA brane considerations. The Hanany–Tong model can be easily formulated as the strong coupling limit of a U(1) gauge theory with N positively charged fields and $\widetilde{N}$ negatively charged fields under this U(1).

While the Hanany–Tong model is exactly the theory considered by Dorey and collaborators, it is *not* the genuine effective theory on the semilocal string world sheet. The program of the field-theoretic honest-to-god derivation started with [54, 55, 56]. Very recently a breakthrough was achieved in [57] with the derivation of the "exact" effective theory on semilocal strings valid in the limit $\log L \to \infty$, where L is an infrared cut-off assumed to be very large.

This exact nonlinear sigma model, to which we will refer to as the zn model, was proven to have a different target-space metric than the HT model (albeit the same topology).

Our task is to explore dynamics of the zn model per se and in comparison with the HT model. It is crucial to explicitly demonstrate that the zn model has the same BPS spectrum as four-dimensional SQCD, as it was noted previously [33, 40] with regards to the HT model. We show that this is indeed the case. Moreover in the 't Hooft limit of infinite N the solutions of both models are identical. However, at finite N the zn and HT models are different in the non-BPS sectors. In particular, they have distinct perturbation theories. We analyze perturbation theory in the zn model and explain in which sense one can use here the notion of a single β function.

We prove that the β functions of the zn model coincide with that of the HT model at one loop. Thanks to supersymmetry, this is enough to show the correspondence of the exact twisted Veneziano-Yankielowicz-type superpotentials which encode the BPS mass formula in terms of the central charges of each state. We conclude that the two models agree in the BPS sectors.

This section is organized as follows. First, in Sec. 2.2 we introduce and compare

[1] The target space of the nonlinear sigma model obtained this way is now noncompact. In mathematics it is mostly known as an $\mathcal{O}(-1)^{\tilde{N}}$ fibration over $\mathbb{CP}^{N-1}$.

two-dimensional sigma models which have recently been discussed in the literature in the context of semilocal strings in SQCD: the zn model [57] and the Hanany-Tong [44, 35] model. In Sec. 2.5 we study the large-N solution, which we use in Sec. 2.5.4 to determine the spectrum of the theory. We present an exact twisted superpotential which encodes the BPS spectrum at finite N in Sec. 2.4. Finally, in Sec. 2.6 we study vacuum manifolds and perturbation theories of these models in the geometric formulation. We summarize and conclude in Sec. 2.7.

2.2 World-Sheet Theory on Non-Abelian Semi-Local Vortices

Non-Abelian semilocal vortex strings (strings for short) are known to be supported by $\mathcal{N}=2$ SQCD with $N_f = N + \widetilde{N}$ massless flavors and the U(N) gauge group [44, 35] provided one introduces a non-vanishing Fayet-Iliopoulos term ξ. Actually, the correct topological object to examine in connection with the semilocal strings is the second homotopy group of the vacuum manifold, which in the present case, is a Grassmannian manifold (defined as follows):

$$\pi_2(\mathcal{M}_{\text{vac}}) = \pi_2\left(\text{Gr}_{N,\tilde{N}}\right) \equiv \pi_2\left(\frac{\text{SU}(N+\widetilde{N})}{\text{SU}(N)\times\text{SU}(\widetilde{N})\times\text{U}(1)}\right) = \mathbb{Z}\,. \qquad (2.2.1)$$

The homotopy group above is the one lying behind the description of lumps in the associated nonlinear sigma-model, which arises as the low-energy limit of the $\mathcal{N}=2$ SQCD. This is the main reason why semilocal strings are similar to lumps [51, 55, 56]. Similarly to lumps, the semilocal strings have power-law behaviors at large distances, and possess new *size* moduli determining their characteristic thickness. Nevertheless, they still retain their nature of strings (flux tubes), which is manifest when we send the size moduli to zero. In this limit we recover just the ANO string, with its exponential behavior [49]. The stringy nature is also justified by the existence of the following non-trivial homotopy group:

$$\pi_1(\text{U}(1)\times\text{SU}(N)/\mathbb{Z}_N) = \mathbb{Z}\,. \qquad (2.2.2)$$

The moduli space of a single semilocal string is a non-compact space of complex dimension $N+\widetilde{N}$ [44, 54, 56]. One can interpret $N-1$ zero modes as parameterizing orientational degrees of freedom of the non-Abelian string,[2] while further $\widetilde{N}$ modes parameterize the size(s) of the semilocal string. Finally, one last parameter is due to translational modes; it is related to the position of the string center on the perpendicular plane. Dynamically the latter moduli is decoupled from the rest. The corresponding dynamics is sterile. In the remainder of the Chapter it will be not mentioned. Then by the moduli space we will understand the $(N + \widetilde{N} - 1)$-dimensional manifold.

A crucial property of semilocal strings is that, in deriving the world-sheet theory, one encounters an infrared divergence of the type

$$\log \frac{L}{|\rho|}\,, \tag{2.2.3}$$

regularized by an infrared (IR) cutoff L. Here ρ is the typical size of a semilocal vortex. The above logarithmic divergence is due to long-range tails of the semilocal string which fall off as *powers* of the distance from the string axis (in the perpendicular plane) rather than exponentially. In the non-Abelian semilocal strings both the size and orientational moduli become logarithmically non-normalizable [54]. A convenient and natural IR regularization, which maintains the BPS nature of the solution[3] can be provided by a mass difference $\Delta m \neq 0$ of the (s)quark masses; then $L \sim 1/|\Delta m|$, so that (2.2.3) becomes

$$\log \frac{1}{|\rho||\Delta m|}\,. \tag{2.2.4}$$

2.2.1 The zn model

These large logarithms account for basically all difficulties in the previous treatments of the semilocal strings. Such divergent terms were calculated e.g. in Refs. [54, 56]. The situation was dramatically reversed in [57]. In this work the problem became an advantage: *all* logarithmic terms were obtained from the bulk-theory description of the semilocal string. Then, one can derive an *exact* world-sheet theory for the semilocal strings in the limit of (2.2.3) or (2.2.4) tending to ∞. The resulting model, which was

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[2] The moduli space of a non-Abelian semilocal string contains indeed a subspace which corresponds to $\mathbb{CP}^{N-1}$ the orientational moduli space of a traditional non-Abelian string.

[3] Alternatively, L can represent a finite length of the string, or a finite volume of the transverse space.

called the zn model, is $\mathcal{N}=(2,2)$ supersymmetric theory with the following action[4]

$$
\begin{aligned}
S_{zn} &= \int d^2x \left\{ \frac{1}{4e^2}F^2_{kl} + \frac{1}{e^2}\,|\partial_k\sigma|^2 + \frac{e^2}{2}\left(|n_i|^2 - r\right)^2 \right. \\
&+ \left. |\partial_k(z_j n_i)|^2 + |\nabla_k n_i|^2 + |m_i - \widetilde{m}_j|^2\,|z_j|^2 |n_i|^2 + |\sigma + m_i|^2\,|n_i|^2 \right\}, \\
& i = 1, ..., N\,, \qquad j = 1, ..., \widetilde{N}\,, \qquad \nabla_k = \partial_k - iA_k\,.
\end{aligned}
\tag{2.2.5}
$$

Here n_i and z_j are the orientational and size moduli fields, respectively, e^2 and r are the gauge coupling and the two-dimensional Fayet-Iliopoulos. In deriving the effective action above from the four-dimensional bulk theory one finds the crucial relationship between four and two dimensional couplings [44, 34]:

$$
r = \frac{4\pi}{g^2_{\rm 4D}}\,. \tag{2.2.6}
$$

Finally, m_i and $\widetilde{m}_j$ are twisted masses.[5] It is assumed that at the very end we take the limit $e \to \infty$. In this limit the gauge field A_k and its superpartners become nondynamical, auxiliary [58, 59] and can be integrated out

$$
A_k = -\frac{i}{2r}(\bar{n}_i \partial_k n_i - n_i \partial_k \bar{n}_i), \qquad \sigma = -\frac{1}{r}\sum_i m_i\,|n_i|^2. \tag{2.2.7}
$$

Moreover, in this limit the term $\left(|n_i|^2 - r\right)^2$ in Eq. (2.2.5) implies the constraint[6]

$$
\sum_i^N |n_i|^2 = r\,. \tag{2.2.8}
$$

The fact that the number of degrees of freedom following from (2.2.5) is correct, namely, $N + \widetilde{N} - 1$, can be seen once we take into account the D-term condition (2.2.8) and, in addition, gauge away a U(1) phase. The global symmetry of the world–sheet theory (2.2.5) is the same as in that of the bulk theory,

$$
{\rm SU}(N) \times {\rm SU}(\widetilde{N}) \times {\rm U}(1)\,, \tag{2.2.9}
$$

which is broken down to ${\rm U}(1)^{N+\tilde{N}-1}$ by the (s)quark mass differences.

4 Here we write down only the bosonic part of the action; we will include fermions in Sec. 2.5.

5 These twisted masses are equal to the four-dimensional complex masses present in the bulk theory.

6 We stress that this constraint is different from that in the Hanany–Tong model, see below.

2.2.2 The HT model

As was already mentioned, non-Abelian semilocal strings were previously studied within a string theory approach based on D-branes by Hanany and Tong (see [60, 35] for the IIB setup and [35] for the IIA setup). In the IIA picture a flux tube is represented by a D2-brane stretched between an NS5 and D4 branes. The effective theory on the world-sheet of the D2-brane, is then given by the strong-coupling limit ($e \to \infty$) of a two-dimensional U(1) gauge theory with N positive and $\widetilde{N}$ negatively charged matter superfields. In components it reads

$$\begin{aligned} S_{\rm HT} &= \int d^2x \left\{ \frac{1}{4e^2} F_{kl}^2 + \frac{1}{e^2} |\partial_k \sigma|^2 + \frac{e^2}{2} \left(|n_i^w|^2 - |z_j^w|^2 - r \right)^2 \right. \\ &+ \left. |\nabla_k n_i^w|^2 + |\widetilde{\nabla}_k z_j^w|^2 + |\sigma + m_i|^2 \, |n_i^w|^2 + |\sigma + \widetilde{m}_j|^2 \left| z_j^w \right|^2 \right\} , \\ & \quad i = 1, ..., N, \qquad j = 1, ..., \widetilde{N} , \\ & \quad \nabla_k = \partial_k - iA_k \, , \qquad \widetilde{\nabla}_k = \partial_k + iA_k \, . \end{aligned} \tag{2.2.10}$$

With respect to the U(1) gauge field A_k the fields n_i^w and z_i^w have charges $+1$ and -1, respectively. We endow these fields with a superscript "w" (weighted) to distinguish them from the n_i and z_j fields which appear in the zn model, see (2.2.5). If only charge $+1$ fields were present, in the limit $e \to \infty$ we would get a conventional twisted-mass deformed $\mathbb{CP}^{N-1}$ model. The Hanany-Tong model can be obtained by the dimensional reduction (from 4D to 2D) of the supersymmetric quantum electrodynamics with N charge 1 and $\widetilde{N}$ charge -1 chiral superfields.

2.3 β function

Let us calculate the one-loop renormalization of the coupling constant r in the zn model (2.2.5). To this end we can limit ourselves to the massless case $m_i = \tilde{m}_j = 0$. Then the action (2.2.5) can be rewritten as

$$S_{\rm zn} = \int d^2x \left\{ \left| \partial_k (z^j n^i) \right|^2 + \left| \nabla_k n^i \right|^2 + iD \left(|n_i|^2 - r_0 \right) \right\} , \tag{2.3.1}$$

where r_0 is a bare coupling constant and the limit $e \to \infty$ is taken. Integration over the auxiliary field D ensures the condition (2.2.8), while the gauge field is given by

$$A_k = -\frac{i}{2|n|^2}\left(\bar{n}_i \partial_k n^i - n^i \partial_k \bar{n}_i\right). \tag{2.3.2}$$

Next, we rearrange the kinetic term by decomposing

$$\partial_k (z^j n^i) = z^j \nabla_k n^i + n^i \widetilde{\nabla}_k z^j\,. \tag{2.3.3}$$

As a result, the action (2.3.1) takes the form

$$\begin{aligned} S_{\rm zn} \;=\; & \int d^2x \left\{ \left|\nabla_k n'^{\,i}\right|^2 + \left|\widetilde{\nabla}_k z'^{\,j}\right|^2 + iD' \left(\left|n'^{\,i}\right|^2 - \left|z'^{\,j}\right|^2 - r_0\right)\right. \\ +\; & \frac{1}{|n'|^2}\left(z'\nabla_k \bar{z}'\right)\left(\bar{n}'\nabla_k n'\right) + \frac{1}{|n'|^2}\left(\bar{z}'\widetilde{\nabla}_k z'\right)\left(n'\widetilde{\nabla}_k \bar{n}'\right) \\ -\; & \left.\frac{1}{2|n'|^2}\left(\partial_k |n'|^2\right)\left(\partial_k |z'|^2\right) - \frac{1}{4|n'|^2}\,\left(\partial_k |z'|^2\right)^2\right\}, \end{aligned} \tag{2.3.4}$$

where we introduced new variables

$$n'^i = \sqrt{1+|z|^2}\, n^i, \qquad z'^j = \sqrt{r_0}\, z^j, \qquad D' = \frac{1}{1+|z|^2}\, D\,, \tag{2.3.5}$$

and the indices i, j are contracted in the brackets, e.g. $(z'\nabla_k \bar{z}') \equiv \left(z'^{\,j}\nabla_k \bar{z}'_j\right)$. In passing from (2.3.1) to (2.3.4) we used the constraint $|n|^2 = r_0$. Solving the equations of motion for the gauge potential A_k in (2.3.4) we find that it is still given by Eq. (2.3.2), as it should, of course.

A disadvantage of formulation (2.3.4) in terms of n' and z' is rather obvious: change of variables (2.3.5) is not holomorphic and, therefore, the metric of the target manifold in (2.3.4) does not explicitly look as a metric of a Kähler manifold. Certainly, we know that the model (2.2.5) is $\mathcal{N} = (2,2)$ supersymmetric and has a Kähler target-space metric in terms of the original fields n, z.

The action (2.3.4) reveals a similarity between the zn model and the HT model (2.2.10). In particular, the first line in (2.3.4) is identical to the massless limit of the HT model (2.2.10) at $e \to \infty$. Moreover, all terms in the second and third lines in (2.3.4) do not contribute at one-loop. Therefore, we conclude that the one-loop renormalization of the coupling constant r is identical in the zn and HT models.

More explicitly, to calculate the one-loop renormalization of r we represent the fields n' and z' in (2.3.4) as sums of classical background fields plus quantum fluctuations,

$$n'^{i} = n_0^i + \delta n^i, \qquad z'^j = z_0^j + \delta z^j. \tag{2.3.6}$$

The renormalization of r can be calculated as that of the linear in D' term in (2.3.4). Let us write the third term in the first line in (2.3.4) as

$$iD' \left(|n_0^i|^2 - |z_0^j|^2 + |\delta n^i|^2 - |\delta z^j|^2 - r_0 \right). \tag{2.3.7}$$

It contributes to the one-loop renormalized coupling r

$$r_{\rm ren} = r_0 - \langle\, |\delta n^i|^2 \,\rangle + \langle\, |\delta z^j|^2 \,\rangle\,, \tag{2.3.8}$$

where $\langle ... \rangle$ stands for vacuum averaging.

Calculating the one-loop tadpole contributions here using canonical propagators of n' and z' fields defined by the first line in (2.3.4) we get

$$r_{\rm ren}(\mu) = r_0 - (N - \widetilde{N}) \int \frac{d^2k}{(2\pi)^2} \frac{1}{k^2} \; = r_0 - \frac{N - \widetilde{N}}{2\pi} \,\log \frac{M}{\mu}, \tag{2.3.9}$$

where M is the ultraviolet cutoff, while μ is the infrared normalization point. The terms proportional to N and $\widetilde{N}$ arise due to loops of n' and z' fields, respectively. Introducing the dynamical scale of the theory Λ,

$$\Lambda \equiv M \exp\left(-\frac{2\pi\, r_0}{N - \widetilde{N}} \right), \tag{2.3.10}$$

we rewrite (2.3.9) as

$$r_{\rm ren}(\mu) = \frac{N - \widetilde{N}}{2\pi} \,\log \frac{\mu}{\Lambda}. \tag{2.3.11}$$

The zn model is asymptotically free at $N > \widetilde{N}$ (which is assumed throughout the Chapter). The one-loop renormalization of its coupling constant is identical to that of the HT model calculated in [59].

The coupling constant r can be complexified by adding a θ term in the theory. The target space in the model at hand is Kählerian but non-Einstein.[7] Therefore, r does not completely specifies the one-loop renormalization group (RG) flow of this theory. We

[7] For an Einstein manifold, the Ricci tensor is proportional to the metric tensor.

will discuss this question in more detail later. Here let us make a statement using the HT model as an example (a similar statement can be formulated for the zn model too). Let us keep the coupling constant e large but finite. Then we have two large parameters of mass dimension one: the ultraviolet cutoff M and e. The normalization point μ is supposed to be $\ll M$. If $\mu \gg e$, then the effective action must be holomorphic in the complexified coupling r, implying that higher loops cannot contribute to the β function in this domain. The one-loop renormalization (2.3.11) is actually exact both, in the zn and HT models for such values of μ. The holomorphicity is lost, generally speaking, when we evolve μ below e, due to emergence of additional structures in the effective Lagrangian, see Sec. 2.6. In this domain the RG flow ceases to be one-loop. However, in the large-N and N_f limit, in the leading order, the one-loop nature is preserved.

2.4 Exact Effective Twisted Superpotentials

The one-loop calculation performed in the previous section can be enhanced by supersymmetry to give exact results, as shown in Refs. [61, 40] in both the regimes $e \ll \mu$ and $e \gg \mu$. To see this, first we recall that the renormalized Fayet-Iliopoulos in two-dimensional $\mathcal{N} = (2,2)$ must be written in terms of a complex twisted superpotential $\widetilde{\mathcal{W}}$ of Veneziano–Yankilowicz type [62, 61], as dictated by supersymmetry:

$$r_{\rm eff} = -\widetilde{\mathcal{W}}'_{\rm eff}(\sigma)\,. \tag{2.4.1}$$

Using the result of the previous section we can write down the following effective twisted superpotential for the zn model in the case of the vanishing masses.

$$\widetilde{\mathcal{W}}_{\rm eff} = -\frac{N-\widetilde{N}}{2\pi}\sigma\left(\log\frac{\sigma}{\Lambda} - 1\right)\,. \tag{2.4.2}$$

The one-loop expression above is exact, thanks to holomorphicity, in the regime $e \ll \mu$. Nevertheless, there are two important observations which makes the potential above a crucial tool for extracting exact results from the theory at all values of the coupling e. First notice that the twisted superpotential above does not depend on the gauge coupling e. This is due to the fact that only the couplings which can be promoted to twisted chiral superfields can appear in $\widetilde{\mathcal{W}}$, and this is certainly not the case for e. The bottom line of this observations is that the all the information which can be extracted from this potential are actually exact, and also valid in the nonlinear sigma model limit when $e \to \infty$. The second observation is that the difference of the values of the twisted superpotential $\widetilde{\mathcal{W}}$ between two vacua gives the central charges and thus the masses of the BPS states of the theory[8]

$$M_{\rm BPS} = |Z| = \Delta\widetilde{\mathcal{W}}\,. \qquad (2.4.3)$$

Notice again that the mass formula written above is exact for all values of e. While it represents a perturbative calculation at small e, it encodes full non-perturbative corrections to the masses of all BPS states in the regime $e \to \infty$.

We wish to emphasize here that (2.4.2) is exact only if applied to the BPS sector of the theory. Once we start looking at perturbations around the vacua given by minimization of the twisted superpotential, formula (2.4.2), or its massive generalization, is of no use. Still, when we treat the model in the large-N approximation, the effective potential

$$V(\sigma) = \left|\widetilde{\mathcal{W}}'_{\rm eff}\right|^2\,, \qquad (2.4.4)$$

give the correct spectrum of the theory. We will address both questions in the next section.

Finally let us note that twisted masses can be introduced in the theory by gauging each U(1) factor in the $\mathrm{U}(1)_f^N$ group by its own gauge field with non-zero σ-component (equal to associated mass) [59]. This leads to the following generalization of the effective twisted superpotential (2.4.2) to the case of non-zero twisted masses:

$$\begin{aligned}\widetilde{\mathcal{W}}_{\rm eff} \;&=\; -\frac{1}{2\pi}\sum_{i=1}^{N}(\sigma+m_i)\left(\log\frac{\sigma+m_i}{\Lambda}-1\right)+\\ &+\;\frac{1}{2\pi}\sum_{j=1}^{\widetilde{N}}(\sigma+\widetilde{m}_j)\left(\log\frac{\sigma+\widetilde{m}_j}{\Lambda}-1\right)\,. \end{aligned}\qquad (2.4.5)$$

Clearly this effective twisted superpotential identically coincides with the one for HT model [59].

This fact together with the matching of the kink spectrum obtained at the classical level in Ref. [57], leads us to claim the matching of the BPS spectra of the zn and HT

[8] Differences between different vacua give the masses of the solitonic states such as kinks. Since $\widetilde{\mathcal{W}}$ is a multi-valued function, it makes sense to take differences between the values of $\widetilde{\mathcal{W}}$ taken between the same vacua but on different Riemann sheets. This will give masses of the perturbative spectrum.

at both semiclassical and quantum levels. As a consequence, the BPS spectrum of the bulk theory coincides with the BPS spectrum of the true effective theory on semilocal vortices, as expected.

2.5 Large-N Solution of the zn Model

In this section we will study the zn model at large N along the lines of Witten's analysis [58]. Namely, we will consider the limit $N \to \infty$, $\widetilde{N} \to \infty$, while the ratio of $\widetilde{N}$ and N is kept fixed. The representations (2.2.10) and (2.3.4) suggest that to the leading order in N the solutions of zn and the HT models are the same. The reason for this is that all terms in the second and third lines in (2.3.4) distinguishing the zn model from the HT model give nonvanishing contributions only at a subleading order in N. Indeed, they can show up in the potential for σ only at the two-loop order and are not reducible to the n' and z' field tadpoles proportional to N or $\widetilde{N}$. Inspection of the SU(N) and SU($\widetilde{N}$) index flow readily reveals that these two- and higher-loop contributions are at most $O(N^0)$ in the large N-limit.

Below we will calculate the effective action for the zn model with twisted masses in the large-N limit. The action of the zn model (2.2.5) in the gauged formulation, with the fermion fields taken into account, is

$$
\begin{aligned}
S_{\rm zn} \;\; = & \;\; \int d^2x \left\{ \frac{1}{4e^2}F^2_{kl} + \frac{1}{2e^2}\, |\partial_k\sigma|^2 + \frac{1}{2e^2}D^2 + \frac{1}{e^2}\,\bar{\lambda}_R\, i\partial_L\,\lambda_R + \frac{1}{e^2}\,\bar{\lambda}_L\, i\partial_R\,\lambda_L \right. \\
+ & \;\; \left|\partial_k(z^j n^i)\right|^2 + \left|\nabla_k n^i\right|^2 + |m_i - \widetilde{m}_j|^2\, |z^j|^2 |n^i|^2 \\
+ & \;\; |\sigma + m_i|^2 \left|n^i\right|^2 + iD\left(|n^i|^2 - r_0\right) \\
+ & \;\; \bar{\xi}_{iR}\, i\nabla_L\, \xi^i_R + \bar{\xi}_{iL}\, i\nabla_R\, \xi^i_L \\
+ & \;\; \Big[i(\,\sigma + m_i)\,\bar{\xi}_{iR}\xi^i_L + i\,\bar{n}_i\,(\lambda_R\xi^i_L - \lambda_L\xi^i_R) + \text{H.c.}\Big] \\
+ & \;\; (\bar{z}_j\bar{\xi}_{iL} + \bar{n}_i\bar{\chi}_{jL})\, i\partial_R(z^j\xi^i_L + n^i\chi^j_L) + (\bar{z}_j\bar{\xi}_{iR} + \bar{n}_i\bar{\chi}_{jR})\, i\partial_L(z^j\xi^i_R + n^i\chi^j_R) \\
+ & \;\; \Big[i(m_i - \widetilde{m}_j)\, \Big(|z^j|^2\bar{\xi}_{iR}\xi^i_L + |n^i|^2\bar{\chi}_{jR}\chi^j_L + \bar{\xi}_{iR}\chi^j_L\bar{z}_j n^i + \bar{\chi}_{jR}\xi^i_L\bar{n}_i z^j \Big) + \text{H.c.}\Big] \\
+ & \;\; \left. \bar{\chi}_{jR}\chi^j_R\bar{\xi}_{iL}\xi^i_L + \bar{\chi}_{jL}\chi^j_L\bar{\xi}_{iR}\xi^i_R + \bar{\chi}_{jL}\chi^j_R\bar{\xi}_{iR}\xi^i_L + \bar{\chi}_{jR}\chi^j_L\bar{\xi}_{iL}\xi^i_R \right\}, \qquad (2.5.1)
\end{aligned}
$$

where the fields A_k, σ, D and $\lambda_{L,R}$ form the gauge supermultiplet, while ξ^i and χ^j are fermion superpartners of n^i and z^j, respectively. Left and right derivatives are defined as

$$
\nabla_L \equiv \nabla_0 - i\nabla_3\,, \qquad \nabla_R \equiv \nabla_0 + i\nabla_3\,. \qquad (2.5.2)
$$

2.5.1 Effective potential at large N

Now we will integrate over the n^i, z^j and ξ^i, χ^j fields and then minimize the resulting effective action with respect to the fields σ and D from the gauge multiplet. This will be done in the saddle point approximation. The large-N limit ensures that the corrections to the saddle point approximation (suppressed by $1/N$) are negligible.

Technically, integrating out the n^i, z^j and ξ^i, χ^j fields in the saddle point boils down to calculating a set of one-loop graphs with the n^i and z^j superfields propagating in loops. As was mentioned, in this section we will obtain the effective potential of the theory as a function of σ and D. Minimization of this potential determines the vacuum structure of the theory. At this stage we can drop the gauge field A_k and its fermion superpartners $\lambda_{L,R}$ in (2.5.1) because they have no vacuum values. If desirable, one can restore the A_k dependence in the final result from gauge invariance, through replacing partial derivatives by covariant.

Since the action (2.5.1) is not quadratic in n^i, z^j and ξ^i, χ^j fields we do the integration in two steps. First, we integrate over n^i and ξ^i. It turns out that the resulting effective action will be quadratic in z^j and χ^j and at the next stage we will be able to integrate out these fields too.

After rescaling the n^i and ξ^i fields similar to that in (2.3.5), namely,

$$n'^i = \sqrt{1+|z|^2}\, n^i, \qquad \xi'^i = \sqrt{1+|z|^2}\, \xi^i \tag{2.5.3}$$

integration over the bosonic fields gives the determinant

$$\prod_i^N \left[\det\left(-\partial_k^2 + \frac{iD}{1+|z|^2} + M_{Bi}^2\right)\right]^{-1}, \tag{2.5.4}$$

while the fermion integration gives

$$\prod_i^N \det\left(-\partial_k^2 + M_{Fi}^2\right), \tag{2.5.5}$$

where M_B^2 and M_F^2 are the following functions:

$$\begin{aligned} M_{Bi}^2(\sigma, z^j, \chi^j) &= \frac{1}{1+|z|^2}\left\{|\sigma+m_i|^2 + |m_i - \widetilde{m}_j|^2\,|z^j|^2\right. \\ &+ \left.|\partial_k z^j|^2 + \bar{\chi}_{jR}\, i\partial_L\, \chi_R^j + \bar{\chi}_{jL}\, i\partial_R\, \chi_L^j + i(m_i - \widetilde{m}_j)\bar{\chi}_{jR}\chi_L^j\right\} \end{aligned} \tag{2.5.6}$$

and

$$\begin{aligned} M_{Fi}^2(\sigma, z^j, \chi^j) &= \frac{1}{(1+|z|^2)^2}\left\{|\sigma+m_i|^2 + \left|(m_i - \widetilde{m}_j)\,|z^j|^2\right|^2\right. \\ &+ (\bar{\sigma}+\bar{m}_i)(m_i - \widetilde{m}_j)\,|z^j|^2 + (\sigma+m_i)(\bar{m}_i - \bar{\widetilde{m}}_j)\,|z^j|^2 \\ &+ \left. i\Big[(\sigma+m_i) + (m_i - \widetilde{m}_j)\,|z^j|^2 \bar{\chi}_{jR}\chi_L^j + \text{H.c.}\Big]\right\}. \end{aligned} \tag{2.5.7}$$

Calculating the determinants (2.5.4) and (2.5.5) gives the effective action as a functional of the fields σ, D, z^j and χ^j,

$$S_{\rm eff}(\sigma, D, z^j, \chi^j) \;=\; \int d^2x \left\{ \frac{1}{4\pi}\sum_{i=1}^{N}\left[\left(M_{Bi}^2 + \frac{iD}{1+|z|^2}\right)\log\frac{M^2}{M_{Bi}^2+\frac{iD}{1+|z|^2}} \right.\right.$$

$$+ \left.\left. \frac{iD}{1+|z|^2} + M_{Fi}^2\,\log\frac{M^2}{M_{Fi}^2} + M_{Bi}^2 - M_{Fi}^2\right] - iDr_0\right\}, \tag{2.5.8}$$

where M is the ultraviolet cut-off scale.

Next, expand the action (2.5.8) in powers of the fields z^j and χ^j. We see that certain terms quadratic in these fields come with an infinitely large logarithmic Z-factors. This is a crucial point. Say, we get kinetic terms of the type

$$\left\{|\partial_k z^j|^2 + \bar\chi_{jR}\, i\partial_L\,\chi_R^j + \bar\chi_{jL}\, i\partial_R\,\chi_L^j\right\}\,\log\frac{M^2}{\mu^2}, \tag{2.5.9}$$

where μ is some infrared scale determined by the value of σ and twisted masses. We absorb this infinite Z-factor redefining the fields z^j and χ^j as

$$z'^j = \sqrt{\frac{N}{4\pi}\,\log\frac{M^2}{\mu^2}}\; z^j, \qquad \chi'^j = \sqrt{\frac{N}{4\pi}\,\log\frac{M^2}{\mu^2}}\;\chi^j. \tag{2.5.10}$$

Now if we re-express the effective action (2.5.8) in terms of new variables, we see that higher powers of the z'^j and χ'^j fields are suppressed by powers of the large logarithm and can be dropped. As a result, the effective action (2.5.8) turns out to be quadratic in the z'^j and χ'^j fields! Thus, we obtain

$$S_{\rm eff}(\sigma, D, z^j, \chi^j)$$

$$= \int d^2x\left\{\frac{1}{4\pi}\sum_{i=1}^{N}\left[\left(|\sigma+m_i|^2 + iD\right)\,\log\frac{M^2}{|\sigma+m_i|^2+iD} + iD\right.\right.$$

$$- \left. |\sigma+m_i|^2\,\log\frac{M^2}{|\sigma+m_i|^2}\right] + |\partial_k z'^j|^2 + \bar\chi'_{jR}\, i\partial_L\,\chi_R'^j + \bar\chi'_{jL}\, i\partial_R\,\chi_L'^j$$

$$- \; iD(r_0 + |z'^j|^2)$$

$$+ \left. |\sigma+\widetilde m_j|^2\,|z'^j|^2 - \left[(\sigma+\widetilde m_j)\bar\chi'_{jR}\chi_L'^j + \text{H.c.}\right]\right\}. \tag{2.5.11}$$

Note, that the sign of the interaction term of z' with D (and $\chi'_{L,R}$ with σ) shows that the z' multiplet has charge -1, as was expected. One can restore the gauge field dependence in (2.5.11) through the substitution

$$\partial_k \to \widetilde{\nabla}_k. \tag{2.5.12}$$

Simultaneously, we will recover terms proportional to $\left(\bar{z}_j \partial_k z^j\right)$ and $\left(\bar{\chi}_{jL}\chi_L^j\right)$, $\left(\bar{\chi}_{jR}\chi_R^j\right)$. The z' and $\chi'_{L,R}$-dependent part of the action (2.5.11) is just the U(1) gauge theory of the z' multiplet with charge -1 plus the FI D-term r_0.

Now, since the action (2.5.11) is quadratic in the fields from the z' multiplet we can integrate out z' and $\chi'_{L,R}$. As a result, we arrive at the effective potential as a function of the fields σ and D

$$\begin{aligned} V_{\text{eff}}(\sigma, D) \;=\;& \frac{1}{4\pi}\sum_{i=1}^{N}\Bigg[\Big(|\sigma+m_i|^2+iD\Big)\,\log\frac{M^2}{|\sigma+m_i|^2+iD}+iD \\ -\;& |\sigma+m_i|^2\,\log\frac{M^2}{|\sigma+m_i|^2}\Bigg] \\ +\;& \frac{1}{4\pi}\sum_{j=1}^{\widetilde{N}}\Bigg[\Big(|\sigma+\widetilde{m}_j|^2-iD\Big)\,\log\frac{M^2}{|\sigma+\widetilde{m}_j|^2-iD} \\ -\;& iD-|\sigma+\widetilde{m}_j|^2\,\log\frac{M^2}{|\sigma+\widetilde{m}_j|^2}\Bigg]-iDr_0\,. \end{aligned} \tag{2.5.13}$$

Using the β function of the theory we can trade the bare coupling r_0 here for the dynamical scale Λ, by writing

$$r_0 = \frac{N-\widetilde{N}}{2\pi}\,\log\frac{M}{\Lambda}. \tag{2.5.14}$$

Substituting this in (2.5.13) we see that the dependence on the ultraviolet cut-off scale

M cancels out, and we get

$$\begin{aligned} V_{\rm eff}(\sigma, D) &= \frac{1}{4\pi}\sum_{i=1}^{N}\Big[-\big(|\sigma+m_i|^2+iD\big)\,\log\frac{|\sigma+m_i|^2+iD}{\Lambda^2}+iD \\ &+ |\sigma+m_i|^2\,\log\frac{|\sigma+m_i|^2}{\Lambda^2}\Big] \\ &+ \frac{1}{4\pi}\sum_{j=1}^{\widetilde{N}}\Big[-\big(|\sigma+\widetilde{m}_j|^2-iD\big)\,\log\frac{|\sigma+\widetilde{m}_j|^2-iD}{\Lambda^2} \\ &- iD+|\sigma+\widetilde{m}_j|^2\,\log\frac{|\sigma+\widetilde{m}_j|^2}{\Lambda^2}\Big]\,. \end{aligned} \tag{2.5.15}$$

This can be viewed as a master formula.

Equation (2.5.15) presents exactly the effective potential which one would obtain from the HT model (2.2.10) by integrating out the n^{wi} and z^{wj} fields at large N and $\widetilde{N}$. As was expected, the large-N solutions of both models coincide.

2.5.2 Switching on vacuum expectation values of n and/or z

Much in the same way as in the HT model, the strong coupling phase with the vanishing vacuum expectation values (VEVs) of both n and z fields occurs in the zn model at $m_i \sim m_j \sim \Lambda$ (we will discuss the vacuum structure of the theory in the large-N approximation in Sec. 2.5.3). At large/small masses the fields n/z develop VEVs and the theory is in the n-Higgs/z-Higgs phase, respectively.

To take into account the possibility of the n and z fields developing VEVs in (2.5.1) we integrate out all n and z fields but one, say, n^1 and z^1, cf. [63]. At the first stage this boils down to adding to (2.5.11) the following term:

$$\int d^2x\,\Big(|\sigma+m_1|^2+iD\Big)\,|n^1|^2. \tag{2.5.16}$$

At the second stage (integrating out z's) we keep intact the terms depending on z'^1 in (2.5.11). This procedure leads us to the following final effective potential, which now depends on the fields σ, D and n^1, z'^1

$$
\begin{aligned}
& V_{\text{eff}}(\sigma, D, n^1, z'^1) \\
= \; & \frac{1}{4\pi}\sum_{i=2}^{N}\left[-\left(\left|\sigma+m_i\right|^2+iD\right)\log\frac{|\sigma+m_i|^2+iD}{\Lambda^2}+iD\right. \\
+ \; & \left.|\sigma+m_i|^2\log\frac{|\sigma+m_i|^2}{\Lambda^2}\right] \\
+ \; & \frac{1}{4\pi}\sum_{j=2}^{\widetilde{N}}\left[-\left(|\sigma+\widetilde{m}_j|^2-iD\right)\log\frac{|\sigma+\widetilde{m}_j|^2-iD}{\Lambda^2}\right. \\
- \; & \left. iD+|\sigma+\widetilde{m}_j|^2\log\frac{|\sigma+\widetilde{m}_j|^2}{\Lambda^2}\right] \\
+ \; & \left(|\sigma+m_1|^2+iD\right)|n^1|^2+\left(|\sigma+\widetilde{m}_1|^2-iD\right)|z'^1|^2. \qquad (2.5.17)
\end{aligned}
$$

Varying the above expression with respect to the fields σ, D, n^1 and z'^1 we derive the vacuum equations of the theory at large N, $\widetilde{N}$.

2.5.3 The vacuum structure

Here we will briefly review the vacuum structure of the the HT and zn models (for a detailed analysis see [39]). Given the fact that Eq. (2.5.17) is the same in both models, so are the solutions.

First we shall consider the case of vanishing expectation values $\langle n^1\rangle$ and $\langle z'^1\rangle$ in Eq. (2.5.17), corresponding to the Coulomb branch of the theory. Then, due to relation (2.4.4), the minima of the effective potential (2.5.17) can be more easily extracted by determining the critical points of $\widetilde{\mathcal{W}}_{\text{eff}}$ (2.4.5). In this way we then derive the following vacuum equation:[9]

$$\prod_{i=1}^{N}(\sigma+m_i)=\Lambda^{N-\widetilde{N}}\prod_{j=1}^{\widetilde{N}}(\sigma+\widetilde{m}_j)\,. \qquad (2.5.18)$$

Now, as was explained in Section 4.1, we choose the twisted masses in such a way that

[9] Note that this equation is valid for any N, not necessarily in the 't Hooft limit.

the $\mathbb{Z}_N$ and $\mathbb{Z}_{\tilde{N}}$ discrete symmetries are preserved, namely,[10]

$$m_k = m\, e^{2\pi i \frac{k}{N}}\,, \quad k = 0, \dots, N-1\,,$$
$$\widetilde{m}_l = \widetilde{m}\, e^{2\pi i \frac{l}{\widetilde{N}}}\,, \quad l = 0, \dots, \widetilde{N}-1\,. \tag{2.5.19}$$

Then, Eq. (2.5.18) takes the following form:

$$\sigma^N + m^N = \Lambda^{N-\widetilde{N}} \left[\sigma^{\widetilde{N}} + \widetilde{m}^{\widetilde{N}}\right]\,. \tag{2.5.20}$$

The above equation obviously has N complex roots (assuming that $\tilde{N} < N$) which can be easily found numerically for any N and $\widetilde{N}$. Interestingly for large N the solutions can be classified. For the future convenience we introduce a new parameter

$$\alpha = \frac{\widetilde{N}}{N}\,, \qquad 0 < \alpha < 1\,. \tag{2.5.21}$$

Then, depending on the relation between α, m, and $\widetilde{m}$, there are two Coulomb branches, which are referred to as $\mathbf{C}m$ and $\mathbf{C}\widetilde{m}$. The roots of Eq. (4.6.36) can be assigned to one of the following three groups:

<u>*m-vacua*</u>: In the domain $\mathbf{C}m$, i.e.

$$\widetilde{m} < \Lambda \left(\frac{m}{\Lambda}\right)^{1/\alpha}\,, \quad m < \Lambda,$$
$$\sigma_{m,l} = \Lambda \left(\frac{m}{\Lambda}\right)^{1/\alpha} e^{2\pi i \frac{l}{\widetilde{N}}}\,, \qquad l = 1, \dots, \widetilde{N}-1\,; \tag{2.5.22}$$

<u>*Λ-vacua*</u>: These vacua exist only in the $\mathbf{C}m$ domain and are located on the circle of radius Λ

$$\sigma_{\Lambda,k} = \Lambda\, e^{2\pi i \frac{k}{N-\widetilde{N}}}\,, \quad k = 0, \dots, N-\widetilde{N}-1\,; \tag{2.5.23}$$

<u>*$\widetilde{m}$-vacua*</u>: In the domain $\mathbf{C}\widetilde{m}$, i.e.

$$\widetilde{m} > \Lambda \left(\frac{m}{\Lambda}\right)^{1/\alpha}\,, \quad \widetilde{m} > \Lambda$$

[10] It is worth noting that a a generic choice of the twisted masses would completely break supersymmetry at the quantum level.

$$\sigma_{\widetilde{m},j} = \Lambda \left(\frac{\widetilde{m}}{\Lambda}\right)^{\alpha} e^{2\pi i \frac{j}{N}} \qquad j = 0, \ldots, N-1\,. \tag{2.5.24}$$

The above expressions are approximate to the leading order in $1/N$. For small N there will be corrections, see Figs. 2.1, 2.2, and 2.3. These figures depict the complex σ plane; the actual vacua that solve Eq. (4.6.36) are located at the centers of the small black nodes in these figures, while the dashed circles drawn for reference have radii given by Eqs. (4.6.39), (2.5.23), and (2.5.24).

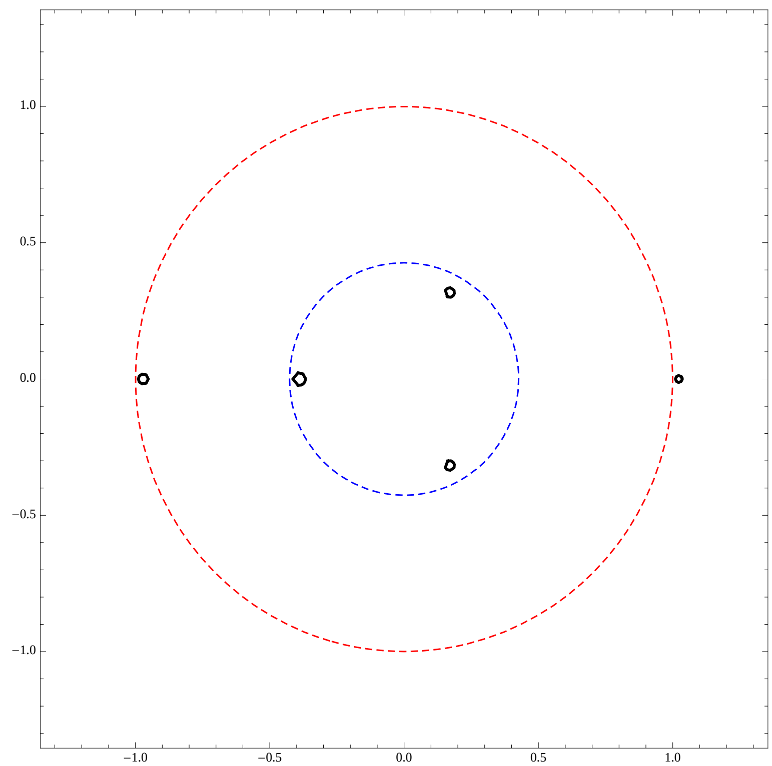

Figure 2.1: Vacua of the HT model for $N = 5, \widetilde{N} = 3$ in the $\mathbf{C}m$ domain. We can see two $(N - \widetilde{N} = 2)$ Λ-vacua near the circle of radius Λ and three $(\widetilde{N} = 3)$ m-vacua near the circle or radius $m^{1/\alpha}$ in units of Λ.

Note also that in the regime

$$\frac{\widetilde{m}}{\Lambda} = \left(\frac{m}{\Lambda}\right)^{1/\alpha}, \tag{2.5.25}$$

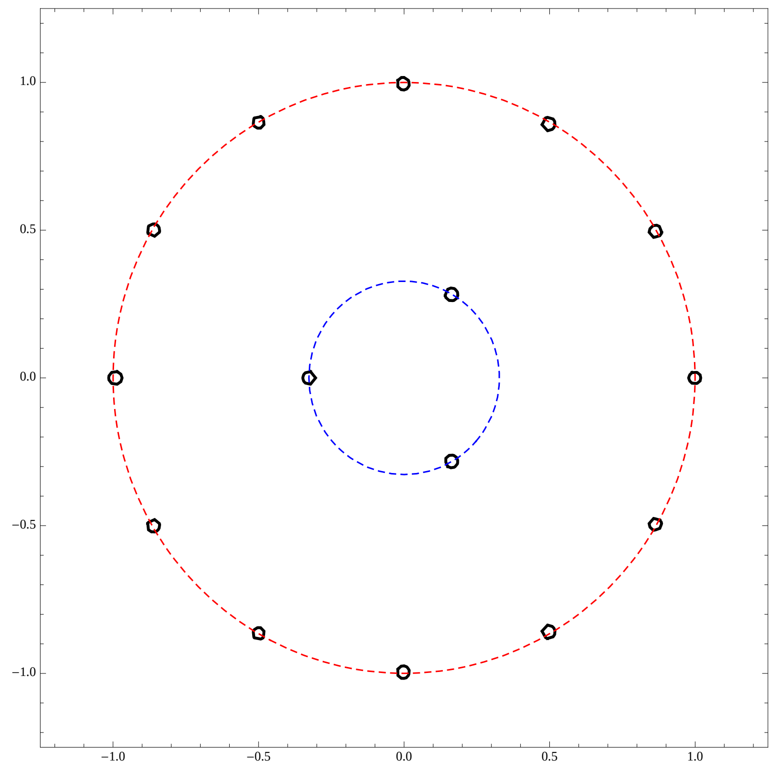

Figure 2.2: Vacua of the HT model for $N = 15, \widetilde{N} = 3$ in the $\mathbf{C}m$ domain. For larger values of N the formulae (4.6.39) and (2.5.23) are getting more precise. Small circle has radius $m^{1/\alpha}$ in units of Λ.

Eq. (4.6.36) degenerates into

$$\sigma^N = \Lambda^{N-\widetilde{N}}\sigma^{\widetilde{N}}\,. \tag{2.5.26}$$

This equation has two sets of solutions,

$$\sigma^{N-\widetilde{N}} = \Lambda^{N-\widetilde{N}}, \quad \sigma = 0\,, \tag{2.5.27}$$

where the former solution gives $N - \widetilde{N}$ massive vacua and the latter applies to the conformal regime.

There are two Higgs branches corresponding to $\langle n^1 \rangle \neq 0$ and $\langle z'^1 \rangle \neq 0$ in (2.5.17). The former exists for $m > \Lambda$ and $m/\Lambda > (\widetilde{m}/\Lambda)^\alpha$ whereas the conditions for the latter are $(m/\Lambda)^{1/\alpha} < \widetilde{m}/\Lambda < 1$. If n^1 or z'^1 develop VEVs we must work with Eq. (2.5.17),

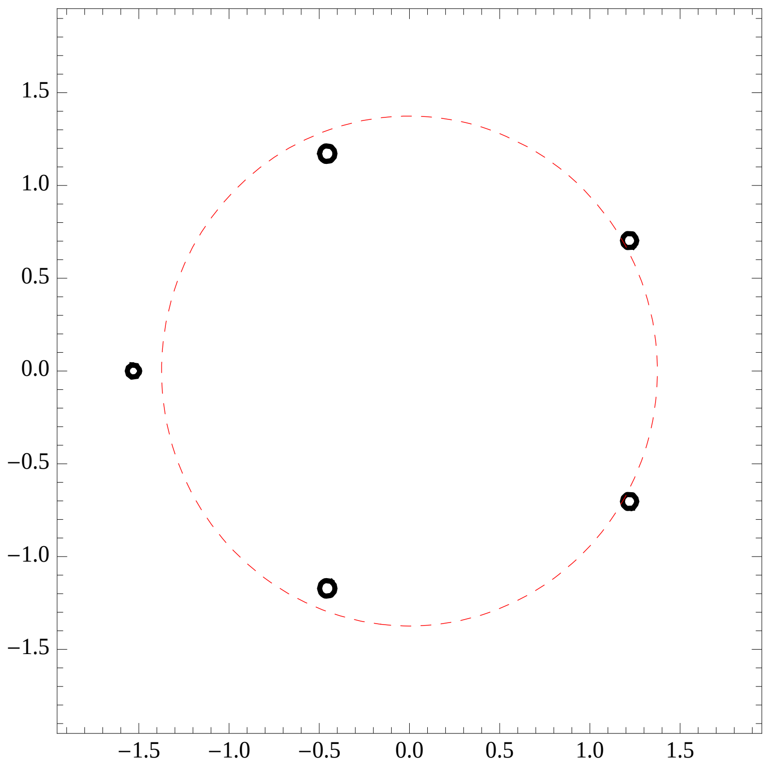

Figure 2.3: Vacua of the HT model for $N = 5, \widetilde{N} = 3$ in the $\mathbf{C}\widetilde{m}$ domain. All vacua localize near the circle of radius $\widetilde{m}^{\alpha}$ in units of Λ.

minimizing V_{eff}. This minimization was done in [39] and we refer the reader to this paper for further details.

2.5.4 Non-BPS spectrum

In Sec. 2.4 we demonstrated that the spectrum of the zn model in the large-N limit coincides with that of the HT model; the latter was discussed in detail in [39]. Here we will calculate the mass of the particles from the vector multiplet V. As was discussed above, there are $N - \widetilde{N}$ Λ-vacua in this model. Let us choose form Eq. (2.5.23) the real vacuum, namely,

$$\sigma_0 = \Lambda \tag{2.5.28}$$

and consider field fluctuations around this vacuum (all Λ-vacua are physically equivalent). The effective action for these fluctuations is

$$
\begin{aligned}
\mathcal{L} &= -\frac{1}{4e_\gamma^2}F_{\mu\nu}^2 + \frac{1}{e_{\sigma\,1}^2}(\partial_\mu \mathfrak{Re}\,\sigma)^2 + \frac{1}{e_{\sigma\,2}^2}(\partial_\mu \mathfrak{Im}\,\sigma)^2 + i\frac{1}{e_\lambda^2}\,\bar{\lambda}\gamma^\mu\nabla_\mu\lambda \\
&+ \; i\mathfrak{Im}(\bar{b}\,\sigma)\,\epsilon_{\mu\nu}F^{\mu\nu} - V_{\text{eff}}(\sigma) - (i\Gamma\bar{\sigma}\bar{\lambda}\lambda + \text{H.c.})\,. \qquad (2.5.29)
\end{aligned}
$$

In the above formula the effective potential $V_{\text{eff}}(\sigma)$ is given by Eq. (2.5.15), while the gauge and scalar couplings can be calculated from the corresponding one-loop Feynman diagrams. The gauge field is coupled to the imaginary part of σ. Figure 4.11 displays the one-loop diagrams which contribute to the mixing. All relevant calculations were carried out in [39]. Here, in addition to these results, we find the mass of the photon from the vector multiplet.

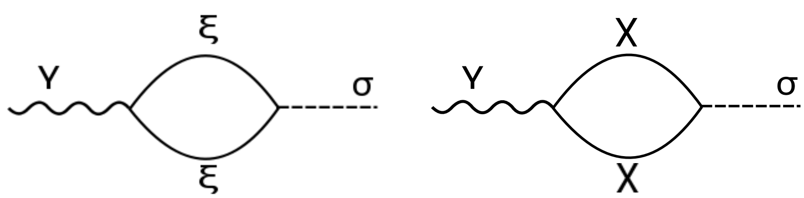

Figure 2.4: One-loop diagrams which contribute to the the photon-scalar anomalous mixing.

Masses. For vanishing twisted masses the one-loop superpotential Eq. (2.5.15) takes the following form:

$$
\begin{aligned}
V_{1-\text{loop}} &= \frac{N}{4\pi}\left(-\left(iD + |\sigma|^2\right)\log\frac{|\sigma|^2 + iD}{\Lambda^2} + |\sigma|^2\log\frac{|\sigma|^2}{\Lambda^2}\right) \\
&- \frac{\widetilde{N}}{4\pi}\left(-\left(iD - |\sigma|^2\right)\log\frac{|\sigma|^2 - iD}{\Lambda^2} - |\sigma|^2\log\frac{|\sigma|^2}{\Lambda^2}\right) \\
&+ \frac{N-\widetilde{N}}{4\pi}iD\,. \qquad (2.5.30)
\end{aligned}
$$

In the case of vanishing twisted masses we can approximately solve the vacuum equation on the Coulomb branch,

$$N\log\frac{|\sigma|^2+iD}{\Lambda^2}-\widetilde{N}\log\frac{|\sigma|^2-iD}{\Lambda^2}=0\,. \tag{2.5.31}$$

Near the vacuum $\sigma=\Lambda$ we expect D to be small. Therefore, we can rewrite the above equation as

$$N\log\left(1+\frac{iD}{|\sigma|^2}\right)-\widetilde{N}\log\left(1-\frac{iD}{|\sigma|^2}\right)+(N-\widetilde{N})\log\frac{|\sigma|^2}{\Lambda^2}=0\,. \tag{2.5.32}$$

Then, Taylor-expanding and denoting

$$d=\frac{iD}{\Lambda^2}\,,\qquad s=\frac{\mathfrak{Re}(\sigma-\Lambda)}{\Lambda}\,, \tag{2.5.33}$$

we get

$$d=-\frac{N-\widetilde{N}}{N+\widetilde{N}}\,s\,. \tag{2.5.34}$$

Equation (2.5.30) can be rewritten in terms of new variables as

$$\begin{aligned} V_{1-loop} &= \frac{N\Lambda^2}{4\pi}\Bigg[-s(\alpha-1)-(s+1)(\alpha+1)\log(s+1) \\ &+ \frac{(2s\alpha+\alpha+1)}{\alpha+1}\log\left(\frac{2s\alpha+\alpha+1}{\alpha+1}\right) \\ &+ \frac{\alpha(2s+\alpha+1)}{\alpha+1}\log\left(\frac{2s}{\alpha+1}+1\right)\Bigg]\,, \end{aligned} \tag{2.5.35}$$

where α is defined in Eq. (2.5.21). Using Eq. (2.5.34) we get, to the second order in s,

$$V_{1-\text{loop}}=\frac{(N-\widetilde{N})^2}{2(N+\widetilde{N})}\frac{s^2\Lambda^2}{4\pi}\,. \tag{2.5.36}$$

Next, we will canonically normalize the kinetic terms in Eq. (2.5.29). In particular, we do a rescaling

$$\mathfrak{Re}\,\sigma\to e_{\sigma\,1}(\mathfrak{Re}\,\sigma)\,.$$

As was shown in [39]

$$e^2_{\sigma\,1}=\frac{4\pi}{(N-\widetilde{N})\Lambda^2}\,. \tag{2.5.37}$$

Therefore, the mass of real part of sigma is

$$m_{\sigma_1} = 2\sqrt{\frac{N-\widetilde{N}}{N+\widetilde{N}}}\Lambda\,. \tag{2.5.38}$$

Note that this expression has $1/N$ corrections since the vacua (2.5.23) are given to the leading order in N. Due to supersymmetry the masses of the photon, fermion and scalar fields are equal,

$$m_\gamma = m_\lambda = m_\sigma\,. \tag{2.5.39}$$

Notice that, as should be obvious from the discussion in Section 2.3, and as we confirmed in this section with an explicit calculation, the full spectra of the zn model and the HT model, including the non-BPS sector, are equivalent at the leading order in the large-N approximation.

2.6 Nonlinear Sigma Model Description and Geometric Renormalization

As was first observed in Ref. [57], the HT and zn models have different metrics on their respective vacua manifolds. In this section we will investigate perturbation theory of both models using a nonlinear sigma model ($NL\sigma M$) description. We will consider in parallel the geometry of the zn and HT models and study their one-loop renormalization in the geometric language. We will also show that the Kähler potential of the HT model reduces to that of the zn model in a *certain limit*.

From $GL\sigma M$ to $NL\sigma M$. Let us first illustrate the main idea with a simple example. We will review here how a vacuum manifold of the $\mathbb{CP}^1$ $NL\sigma M$ emerges from the gauged description of the model in the limit when the gauge coupling(s) are sent to infinity.

The corresponding gauged linear sigma model ($GL\sigma M$) Lagrangian for the $\mathbb{CP}^1$ model in the superfields formalism reads

$$\mathcal{L} = \int d^4\theta \left(\left(|X_1|^2 + |X_2|^2\right) e^V - rV + \frac{1}{e^2}|\Sigma|^2 \right), \tag{2.6.1}$$

where X_1, X_2 are chiral multiplets, V is a twisted vector multiplet with field strength Σ, r is the FI parameter, and e is the gauge coupling. One can see that the following term belongs to the Lagrangian:

$$D(|x_1|^2 + |x_2|^2 - r)\,, \tag{2.6.2}$$

which gives rise to the D-term constraint and it comes from the terms linear in V. Here $x_{1,2}$ are the bottom components of fields $X_{1,2}$. The constraint modulo the U(1) symmetry $(\mathbb{C}^2 - Z)//U(1)$, where Z is the locus of $|x_1|^2 + |x_2|^2 - r$ defines the vacuum target manifold of the model. In this particular case is given by $\mathbb{CP}^1 \simeq S^2$, the two-dimensional sphere of radius r. By making the radius of the sphere very large we go into the flat limit and the target manifold of the model should simply reduce to $\mathbb{C}^1$. However, this statement is not evident from analyzing the D-term constraint (2.6.2). The reason for this is that X_1 and X_2 are not the true coordinates of the vacuum manifold, but their ratio is. Indeed, integrating out V in (2.6.1) we get

$$\mathcal{L} = \int d^4\theta\, r \log\left(|X_1|^2 + |X_2|^2\right)\,. \tag{2.6.3}$$

Now we need to fix the gauge in order to keep only physical degrees of freedom, doing this we obtain the Kähler potential for the $\mathbb{CP}^1$ model

$$K = r\log(1 + |X|^2)\,, \tag{2.6.4}$$

where $X = X_2/X_1$. Let us further do the rescaling $X \to X/\sqrt{r}$ and take the limit $r \to +\infty$. What we get is

$$K = |X|^2\,, \tag{2.6.5}$$

which corresponds to the flat metric on $\mathbb{C}$. Note that one could have considered (2.6.4) and instead of doing the rescaling expand the Kähler potential for fixed r at small values of $|X|^2$ and get the same result. It is, of course, a reflection of the equivalence of rescaling the coordinates and metric. We will compare the HT and zn models later in this section using small field expansion. In the following subsections we will get the vacuum manifolds for the two models in question from their gauged descriptions which have been reviewed in Sec. 2.2.

2.6.1 The *zn* model vs. the HT model

The following Lagrangian describes the zn model [57]

$$\mathcal{L}_{zn} = \int d^4\theta \left(|\mathcal{N}_i|^2 \mathrm{e}^V + |\mathcal{Z}_j|^2|\mathcal{N}_i|^2 - rV + \frac{1}{e^2}|\Sigma|^2\right)\,, \tag{2.6.6}$$

where we use the following chiral superfields

$$\mathcal{N}^i = n^i + \theta^\alpha \xi^i_\alpha + \bar\theta\theta F^i\,, \quad i = 1,\dots,N$$
$$\mathcal{Z}^j = z^j + \theta^\alpha \chi^j_\alpha + \bar\theta\theta \widetilde{F}^j\,, \quad j = 1,\dots,\widetilde{N}\,, \tag{2.6.7}$$

vector field V in the Wess–Zumino gauge ($\theta^1 = \theta^+\,, \theta^2 = \theta^-$ and the same for dotted components, see [61])

$$V = \theta^+\bar\theta^+(A_0 + A_3) + \theta^-\bar\theta^-(A_0 - A_3) + i\sigma\theta^-\bar\theta^+ + i\bar\sigma\theta^+\bar\theta^- \\ + \left(2i\theta^-\theta^+(\bar\theta^-\bar\lambda_- + \bar\theta^+\bar\lambda_+) + \text{H.c.}\right) + \tfrac{1}{2}\theta^4 D\,, \tag{2.6.8}$$

and the twisted chiral field $\Sigma = \mathcal{D}_+\bar{\mathcal{D}}_- V$

$$\Sigma = \sigma + i\theta^+\bar\lambda_+ - i\bar\theta^-\lambda_- + \theta^+\bar\theta^-(D - iF_{01})\,. \tag{2.6.9}$$

Given the above superfield representations one can derive the full action of the zn model in components (2.5.1).

Vacuum manifold of the zn model. Let us proceed with the geometric description of the theory. Taking the limit $e \to \infty$ and integrating out vector superfield V in (2.6.6) we arrive at the following Lagrangian:

$$\mathcal{L}_{zn} = \int d^4\theta\, \left(|\mathcal{Z}_j \mathcal{N}_i|^2 + r\log|\mathcal{N}_i|^2\right)\,. \tag{2.6.10}$$

Similarly to the $\mathbb{CP}^1$ case described above we need to get rid of the unphysical degree of freedom which is present in the above expression. If we define [11]

$$\Phi_i = \frac{\mathcal{N}_i}{\mathcal{N}_N}\,, \quad i = 1,\dots,N-1\,,$$
$$\mathfrak{z}_j = r^{-1/2}\mathcal{N}_N\mathcal{Z}_j\,, \quad j = 1,\dots,\widetilde{N}\,, \tag{2.6.11}$$

we get the following Kähler potential for the zn model:

$$K_{zn} = r|\zeta|^2 + r\log(1 + |\Phi_i|^2)\,, \tag{2.6.12}$$

where

$$|\zeta|^2 \equiv |\mathfrak{z}_j|^2(1 + |\Phi_i|^2)\,. \tag{2.6.13}$$

[11] Assuming $\mathcal{N}_N \neq 0$.

Note that ζ is not a holomorphic variable in any sense. We use the notation (2.6.13) as a shorthand. $|\zeta|^2$ is the only combination involving $\mathfrak{z}_{\mathrm{j}}$'s which is invariant under the global symmetries (2.2.9) of the model. Needless to say, so is any power of $|\zeta|^2$.

The Kähler potential (2.6.12) describes geometry of the vacuum manifold of the zn model in terms of $(N + \widetilde{N} - 1)$ unconstrained complex variables. The global SU(N) is realized nonlinearly much in the same way as in the $\mathbb{CP}^{N-1}$ model while the SU($\widetilde{N}$) symmetry is realized linearly on the $\mathfrak{z}_j$ fields. For $\widetilde{N} = 1$, the Kähler potential (2.6.12) reduces to that describing the *blow-up* of the $\mathbb{C}^N$ space at the origin [64]. In this case we can observe that the SU(N) symmetry becomes manifest and is realized as the isometry of the target space after the following redefinition:

$$|\zeta|^2 = |\Xi_i|^2\,, \quad \Xi_1 = \mathfrak{z}_1\,, \quad \Xi_i = \mathfrak{z}_1 \Phi_i\,, \quad i = 2, \dots, N\,. \tag{2.6.14}$$

In this case the Kähler potential takes the form

$$K_{zn} = r|\Xi_i|^2 + r \log |\Xi_i|^2\,. \tag{2.6.15}$$

It is instructive to reiterate to make explicit all isometries of (2.6.12). For simplicity we put $N = 1$, so that the second part of the action in (2.6.12) is, in fact, that of $\mathbb{CP}^1$. As is well known, $\mathbb{CP}^1$ is invariant under nonhomogenious nonlinear transformations

$$\Phi \to \Phi + \beta + \bar{\beta}\,\Phi^2\,, \qquad \bar{\Phi} \to \bar{\Phi} + \bar{\beta} + \beta\,\bar{\Phi}^2\,, \tag{2.6.16}$$

where β and $\bar{\beta}$ are infinetissimal transformation parameters. This expresses the SU(2)/U(1) invariance of the $\mathbb{CP}^1$ action. Indeed, under these transformations

$$1 + \Phi\bar{\Phi} \to \left(1 + \Phi\bar{\Phi}\right)\left(1 + \beta\bar{\Phi}\right)\left(1 + \bar{\beta}\,\Phi\right) \tag{2.6.17}$$

implying Kähler transformations of $\log\left(1 + |\Phi|^2\right)$ under which the $\mathbb{CP}^1$ action is invariant. Let us supplement (2.6.16) by the following holomorphic transformations of the variables $\mathfrak{z}_j$

$$\mathfrak{z}_j \to \frac{\mathfrak{z}_j}{1 + \bar{\beta}\,\Phi}\,, \qquad \bar{\mathfrak{z}}_j \to \frac{\bar{\mathfrak{z}}_j}{1 + \beta\,\bar{\Phi}}\,. \tag{2.6.18}$$

We immediately confirm that $|\zeta|^2$ is invariant under the combined action of (2.6.16) and (2.6.18). Here it is obvious that this is the only independent invariant of this type. Thus the observed symmetry only allows polynomials in $|\zeta|^2$ in the Kähler potential.

Vacuum manifold of the HT model. Using the same notations for the superfields as for the zn model we can formulate the HT model (2.2.10) as the following $GL\sigma M$ ($e \to \infty$):

$$\mathcal{L}_{\rm HT} = \int d^4\theta \, \left(|\mathcal{N}_i|^2 {\rm e}^V + |\mathcal{Z}_j|^2 {\rm e}^{-V} - rV \right) \, . \tag{2.6.19}$$

Using the same change of variables as in (2.6.11), after integrating out V in (2.6.19) we obtain the Kähler potential for the HT model,

$$K_{\rm HT} = \sqrt{r^2 + 4r|\zeta|^2} - r \log \left(r + \sqrt{r^2 + 4r|\zeta|^2} \right) + r \log(1 + |\Phi_i|^2) \, . \tag{2.6.20}$$

For $N = 2$, $\widetilde{N} = 1$, the Kähler potential (2.6.20) describes the so-called Eguchi–Hanson space and was discovered by Calabi [65]. For generic $\widetilde{N}$ the target manifold in question is the $\mathcal{O}(-1)^{\widetilde{N}}$ tautological fiber bundle over $\mathbb{CP}^{N-1}$. For a mathematical derivation of the Kähler potential (2.6.20) see [66].

From the HT model to the zn model. At first sight the zn and HT models look quite different, as much as their Kähler potentials (2.6.12) and (2.6.20). This is indeed the case, but there is a domain of the target space where they reduce to the same model. As we have already mentioned, the target manifold of the HT model is the total space of the $\widetilde{N}$-th power of the tautological bundle over $\mathbb{CP}^{N-1}$. Thus this is a noncompact manifold with $\widetilde{N}$ noncompact directions.

We will now make a more quantitative comparison of the two models. Let us consider Eq. (2.6.20) at small values of $|\zeta|^2$. The result of the small $|\zeta|^2$-expansion depends on the sign of the FI parameter r. Below we will consider both branches.

(i) $r > 0$: For small $|\zeta|^2$ we can Taylor-expand around $|\zeta|^2 = 0$ and observe that the Kähler potential (2.6.20) in the second order in $|\zeta|^2$ takes the form

$$K_{\rm HT} = r|\zeta|^2 + r \log(1 + |\Phi_i|^2) + \mathcal{O}(|\zeta|^4) \, , \tag{2.6.21}$$

This Kähler potential coincides with the one (2.6.12) of the zn model.

(ii) $r < 0$: Small-$|\zeta|^2$ expansion gives the following Kähler potential:

$$K_{\rm HT} = r|\zeta|^2 - r \log(1 + |\tilde{\mathfrak{z}}_j|^2) + \mathcal{O}(|\zeta|^4) \, , \tag{2.6.22}$$

where [12]

$$\widetilde{\mathfrak{z}}_j = \frac{\mathcal{Z}_j}{\mathcal{Z}_{\widetilde{N}}} \quad j = 1, \dots, \widetilde{N} - 1. \tag{2.6.23}$$

This model corresponds to the dual zn sigma model with $\mathbb{CP}^{\widetilde{N}-1}$ as the base manifold. One can rewrite its Kähler potential as follows

$$K_{\widetilde{zn}} = r|\mathcal{N}_i|^2(1 + |\widetilde{\mathfrak{z}}_j|^2) + \widetilde{r}\log(1 + |\widetilde{\mathfrak{z}}_j|^2)\,, \tag{2.6.24}$$

where $\widetilde{r} = -r > 0$. This manifold has N noncompact and $\widetilde{N}$ compact directions. As we will see later, the one-loop β function (or first Chern class of the bundle) will be proportional in this case to $N - \widetilde{N}$. Once we start with the HT model (2.6.20) with $N > \widetilde{N}$, corresponding to the asymptotically free theory, $N - \widetilde{N}$ will be positive, which will entail growth of the FI parameter $\widetilde{r}$ along the RG flow. Thus the dual zn model is not asymptotically free, but rather IR free. In what follows we will only concentrate on the first case $r > 0$.

Thus far we considered small values of $|\zeta|^2$. On the contrary, at large values of $|\zeta|^2$, as can be seen from Eqs. (2.6.12) and (2.6.20), the two models behave differently. As was shown in [57], in this limit the zn model has vanishing scalar curvature, whereas the HT model has not.

One can see from (2.6.21) that in the leading order the HT and zn models have the same Kähler potential,

$$K_{\mathrm{HT}} = K_{zn} + \mathcal{O}(|\zeta|^2)\,. \tag{2.6.25}$$

This observation suggests that at one loop, in the leading order in $|\zeta|^2$ the two models have the same one-loop β functions. Nevertheless, beyond one loop one expects the theories to have different β functions. Moreover, even at one loop for large values of $|\zeta|^2$ the two models get different corrections. We will give explicit expressions later on in this section.

2.6.2 Perturbation theory

For any Kähler nonlinear sigma model with the Kähler metric $g_{i\bar{j}}$ and coupling constant g the Gel-Mann–Low functional (in what follows we shall call it β function for short)

[12] Again, it assumed that $\mathcal{Z}_{\widetilde{N}} \neq 0$.

reads [67, 68, 69, 70, 71]

$$\beta_{i\bar{j}} = a^{(1)} R^{(1)}_{i\bar{j}} + \frac{1}{2r} a^{(2)} R^{(2)}_{i\bar{j}} + \dots\,, \tag{2.6.26}$$

where $a^{(k)}$ are some constants ($k = 1, 2, ...$) and $R^{(k)}$ are operators composed from k-th power of the curvature tensors (see e.g. (2.6.27)). According to the above series a contribution from the nth loop scales as r^{1-n}. For the metric of a general form the first several terms are known. The first two of them are

$$\begin{aligned} R^{(1)}_{i\bar{j}} &= R_{i\bar{j}}\,, \\ R^{(2)}_{i\bar{j}} &= R_{i\bar{k}l\bar{m}} R^{\bar{k}\ \ l\bar{m}}_{\ \bar{j}}\,. \end{aligned} \tag{2.6.27}$$

In supersymmetric sigma models, however, most of the coefficients $a^{(k)}$ from (2.6.26) vanish. For example, in supersymmetric $\mathbb{CP}^{N-1}$ sigma model all terms except the first one in (2.6.26) are zero [72]. The calculation was based on the instanton counting [73] and the coefficients of the β function were expressed in terms of the number of the zero modes.

The common lore in perturbation theory of nonlinear sigma models suggests that for generic Kähler manifolds the theory is nonrenormalizable, as each order in the perturbation series (2.6.26) brings in a new operator, with a different field dependence. For some particular symmetric target manifolds e.g. for the Einstein manifolds, no new structures are produced. The renormalization is merely reduced to a single coupling constant renormalization. It is easy to see that the HT and zn model target spaces are not of this kind and all terms in the series (2.6.26) have different field dependence. However, let us have a closer look the one-loop perturbation theory and see how we can deal with the above mentioned nonrenormalizability.

One-loop renormalization of the Kähler potential in the zn model. For a Kähler manifold with the Kähler potential $K(z_i, \bar{z}_i)$ the metric is given by

$$g_{i\bar{j}} = \partial_i \bar{\partial}_{\bar{j}} K(z_i, \bar{z}_i)\,, \tag{2.6.28}$$

while all other components (such as $g_{ij} = 0$) vanish. The corresponding Ricci tensor is therefore a total derivative and is given by

$$R_{i\bar{j}} = -\partial_i \bar{\partial}_{\bar{j}} \log\det(g_{i\bar{j}})\,. \tag{2.6.29}$$

For Einstein manifolds Ricci tensor is proportional to the metric, therefore

$$-\log\det(g_{i\bar{j}}) = \alpha K(z_i, \bar{z}_i) \tag{2.6.30}$$

up to a Kähler transformation. For instance, for the $\mathbb{CP}^{N-1}$ model the coefficient α in the above formula is equal to N. As we emphasized previously, for the $\mathbb{CP}^{N-1}$ model this result is exact: higher loops do not give any corrections to the β function.

Let us now examine the curvature tensors for the zn model. It turns out that the calculation of the determinant of the metric tensor can be performed exactly for any N and $\widetilde{N}$; the answer is more intricate in the HT model. After some calculations we get[13]

$$-\log\det(g^{(zn)}_{i\bar{j}}) = (N-\widetilde{N})\log(1+|\Phi_i|^2) - (N-1)\log(1+|\zeta|^2)\,. \tag{2.6.31}$$

Let us at this point derive the same quantity for the HT model in order to show how its one-loop result deviates from the one for the zn model. For the HT model a generic formula is harder to get, we therefore focus on an example for, say, $N=2$, $\widetilde{N}=1$. One gets

$$-\log\det(g^{(\mathrm{HT})}_{i\bar{j}}) = \log(1+|\Phi_i|^2) - \log\left(1+\frac{r}{\sqrt{r^2+4r|\zeta|^2}}\right)\,. \tag{2.6.32}$$

This expression obviously gives a different correction to the Kähler potential.

Formula (2.6.31) means that the Kähler potential acquires an infinite correction and becomes

$$K^{(1)}_{zn} = \left(r_0 - \frac{N-\widetilde{N}}{2\pi}\log\frac{M}{\mu}\right)\log(1+|\Phi_i|^2)+|\zeta|^2+\frac{N-1}{2\pi}\log\frac{M}{\mu}\log(1+|\zeta|^2)\,, \tag{2.6.33}$$

where M is the UV cutoff and μ is the normalization scale. We immediately see that the target manifold of the zn model is not of the Einstein type. We can also see that in order to eliminate the divergence in the last term in the above formula one has to introduce a new counterterm.

A side remark. There exist the so-called *quasi-Enstein* manifolds or *Ricci solitons*, for which the following equality takes place:

$$R_{i\bar{j}} = \alpha g_{i\bar{j}} + \partial_i \bar{v}_{\bar{j}} + \bar{\partial}_{\bar{j}} v_i \tag{2.6.34}$$

[13] This result holds up to an additive constant which depends on rSince the Ricci tensor is a total derivative we can allow such a freedom. Certainly we can also change this expression by a Kähler transformation.

for some vector field v. None of the manifolds considered in this work are of this kind. Indeed, one can check that Kählerian structure imposes constraints on the vector field v which are not satisfied in either zn or HT models. Quasi-Einstein Kähler manifolds had been investigated by a number of mathematicians as well as physicists. It was shown by Friedan [74, 75], who carried out a stability analysis of RG equations at one loop, that a fix point of the RG flow has to be a quasi-Einstein manifold. Quasi-Einstein manifolds are quite hard to find explicitly, for most of the known cases their Kähler potentials are known only implicitly and in quadratures (see e.g. [66] and references therein for examples related to our work). However, in the special case of $N = \widetilde{N} = 1$ one can specify the metric explicitly. Its only nonzero component is given by (see [66])

$$g_{RS} = \frac{r}{1+|z|^2}\,, \tag{2.6.35}$$

where z is a coordinate on the target manifold. Note that for $N = \widetilde{N} = 1$ the zn model is trivial: it has $\mathbb{C}$ as its target space, whereas the HT model has a nontrivial metric [14]

$$g_{\rm HT} = \frac{r}{\sqrt{1+|z|^2}}\,. \tag{2.6.36}$$

Based on the arguments given in [74, 75] the HT model in this case should flow to the space with metric (2.6.35). Studying the fixed points of the RG flow in $NL\sigma M$s is an interesting question, but it is beyond the scope of the present thesis. Hence we return to the one-loop renormalization of the zn model.

Renormalization of the FI parameter. The first part of the renormalization procedure is similar to the $\mathbb{CP}^{N-1}$ model. Indeed, we can extract from the first term the coupling constant renormalization

$$r_{\rm ren}(\mu) = r_0 - \frac{N-\widetilde{N}}{2\pi}\log\frac{M}{\mu}\,. \tag{2.6.37}$$

The so-called dimensional transmutation occurs at the scale Λ, when the theory becomes strongly coupled, ($r_{\rm ren}(\Lambda) = 0$),

$$r_0 = \frac{N-\widetilde{N}}{2\pi}\log\frac{M}{\Lambda}\,. \tag{2.6.38}$$

[14] Note that this metric appears on the Higgs branch of the theory when two twisted masses collide (the Argyres–Douglas point) [59]The space is asymptotically $\mathbb{C}/\mathbb{Z}_2$.

Note that this does not happen for $N = \widetilde{N}$, the FI parameter remains unchanged and the theory has an IR conformal fixed point.

It was shown in [66] that the first Chern class of the $\widetilde{N}$-th power of the tautological fiber bundle over $\mathbb{CP}^{N-1}$, or in our notation the target space of the HT model, restricted to the base is given by

$$c_1(M_{\text{HT}})\Big|_{\mathbb{CP}^{N-1}} = (N - \widetilde{N})\left[\omega_{\mathbb{CP}^{N-1}}\right] , \tag{2.6.39}$$

where $\left[\omega_{\mathbb{CP}^{N-1}}\right]$ denotes the Kähler class of $\mathbb{CP}^{N-1}$. In the above calculations this fact is reflected by (2.6.37). Since in the $\mathcal{N} = (2, 2)$ supersymmetric theories the Kähler class is only renormalized at one loop [76, 74], (2.6.37) represents the exact answer for the FI term renormalization. Unfortunately one cannot say much about the exact part of the Kähler form. Generally speaking, it is known to be modified at every order in perturbation theory and its structure is unpredictable unless we carry out an explicit calculation. We will place some argument in the next paragraph about renormalization of such terms at small $|\zeta|^2$.

At this point we can make a connection with the $GL\sigma M$ one-loop computation (2.3.11). We have mentioned earlier that in the $GL\sigma M$ formulation at finite value of the gauge coupling e there are only two divergent one-loop graphs which are regularized by the UV cutoff – the tadpoles emerging from the D-term constraint. The FI renormalization (2.3.11) was obtained after calculating these tadpoles. Equation (2.6.37) confirms this by the corresponding $NL\sigma M$ calculation performed above. One may now ask if we can trace the origin of the remaining terms in the one-loop β function, like the last term in (2.6.31)?

The answer is quite tricky, we will sketch a part of it here. One needs to look more carefully at the perturbation theory at finite e. There will be one-loop (and also higher loop) graphs which will have $\log(\mu/e)$, where μ is the IR cutoff (it appears from propagation of light fields in the loops). After we make a transition from the $GL\sigma M$ to the $NL\sigma M$ by increasing e, we will hit the UV cutoff on the way $e \sim M$. In $NL\sigma M$ we identify $M = e$.

This argument shows us how additional structures, which were not present in the genuine UV domain of the $GL\sigma M$ (i.e. the domain above e) appear in the geometrical renormalization. From the standpoint of the finite-e $GL\sigma M$ they are of the infrared origin.

Below we will analyze the renormalization of the linear term in $|\zeta|^2$ in (2.6.33).

Renormalization of the non-Einstein part. Equation (2.6.31) gives the exact one-loop answer for the β function of the zn sigma model (after applying $\partial_i \bar{\partial}_{\bar{j}}$ to it). Nevertheless it is instructive to understand how the linear term in $|\zeta|^2$ (and higher order terms as well) appear in perturbation theory in geometric formulation. At small $|\zeta|^2$ one can expand the logarithm in the last term in Eq. (2.6.31) to get

$$-\log\det(g_{i\bar{j}}) = (N - \widetilde{N})\log(1 + |\Phi_i|^2) - (N-1)|\zeta|^2 + \mathcal{O}(|\zeta|^4)\,. \qquad (2.6.40)$$

Using (2.6.33) and the coupling renormalization (2.6.37) we obtain for the $|\zeta|^2$ term

$$K^{(1)}_{zn} \supset |\zeta|^2 \left(1 + \frac{1}{r}\frac{N-1}{2\pi}\log\frac{M}{\mu}\right) = Z|\zeta|^2\,. \qquad (2.6.41)$$

Therefore we can absorb this Z factor by redefining $|\zeta|^2 \to |\zeta|^2/Z$. The contribution (2.6.41) arises in the following calculation. Since the general structure of the effective action is already known, we can perform a calculation at any point in the target space. It is convenient to choose the background field $n_i \to 0$ (while, at the same time, $\partial_i n_j \neq 0$). Then, as well-known, the logarithmically divergent contribution comes only from the tadpole graphs of the type depicted in Fig. 2.5. In the one-loop tadpole

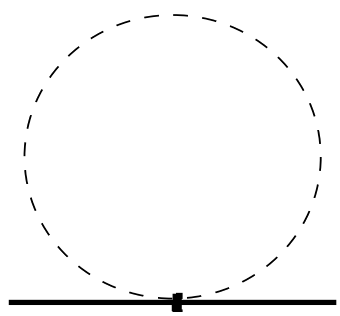

Figure 2.5: Tadpole graphs determining logarithmically divergent contributions to the β function near the origin of the $\mathbb{CP}^{N-1}$ space. Two contributions are considered in the text: (a) the dashed line represents the z_j fields, while the solid line $\partial\bar{n}\partial n$; and (b) the dashed line represents the n_i fields, the solid line $\partial\bar{z}\partial z$.

graphs the contributions of the second and first terms in (2.6.12) in the effective action are completely untangled from each other. The second term produce just the standard $\mathbb{CP}^{N-1}$ renormalization of r,

$$r_{\rm ren}(\mu) = r_0 - \frac{N}{2\pi}\,\log\frac{M}{\mu}\,, \tag{2.6.42}$$

cf. Eq. (2.3.9). Now, let us examine the impact of the first term in (2.6.12). There are two options. We can choose $\partial\bar{n}\partial n$ as the background and let z_j propagate in the loop (option (a) in Fig. 2.5) or vice versa. The first option obviously produces

$$\Delta K^{(1)}_{(a)} = \frac{\widetilde{N}}{2\pi}\,\log\frac{M}{\mu}\,|\Phi_i|^2\,, \tag{2.6.43}$$

which results in the following term in the renormalized Kähler potential

$$\frac{\widetilde{N}}{2\pi}\,\log\frac{M}{\mu}\,\log\left(1+|\Phi_i|^2\right)\,. \tag{2.6.44}$$

The difference in signs compared to (2.6.42) appears from the very beginning. Combining (2.6.42) and (2.6.44) we recover (2.3.9) or (2.6.37). The second option, with the n_i fields are in the loop, leads us to

$$\Delta K^{(1)}_{(b)} = \frac{N-1}{2\pi}\,\log\frac{M}{\mu}\,\sum_{j=1}^{\widetilde{N}}|\mathfrak{z}_j|^2 \tag{2.6.45}$$

which in turn gives

$$\frac{N-1}{2\pi}\,\log\frac{M}{\mu}\,|\zeta|^2\,. \tag{2.6.46}$$

In the course of the RG flow from the UV cut-off M down to μ the first term in the Kähler potential (2.6.12) acquires the following Z factor

$$|\zeta|^2 \to Z|\zeta|^2\,, \qquad Z = 1 + \frac{N-1}{2\pi}\log\frac{M}{\mu}\,. \tag{2.6.47}$$

Thus we can see that at small values of $|\zeta|^2$ the theory can be renormalized at one loop and no counterterm is needed. This is, however, not the case for higher order terms.

2.7 Conclusions

In this Chapter we extensively studied the effective theory on semilocal non-Abelian flux tubes in $\mathcal{N}=2$ SQCD. We continued the developments of [57] where an explicit exact

Lagrangian of the corresponding two-dimensional theory (zn model) was derived in a genuinely field theoretic setup. The analysis we have performed in this work for the zn model has been carried out in parallel with the HT model [35]. The latter was found on semilocal vortices in a D-brane setup. The bottom line of our investigation is that only the BPS sector is correctly reproduced by the HT model; the one-loop β functions of zn and HT models coincide. The one-loop β function exhausts the renormalization of the FI term, which means that the exact twisted superpotentials and the BPS spectra of the two models are the same. This result represents the first proof, carried exclusively in a field theory context, of the correspondence of the BPS spectra between four dimensional $\mathcal{N} = 2$ SQCD and the effective theory on the semilocal vortices therein constructed. We also show that the HT and zn model are equivalent in the large-$N(\widetilde{N})$ approximation.

The physics beyond the BPS sector is however *different*. The difference between the zn and HT models becomes clear when we look at the perturbation theory in the geometric formulation. First of all, the target manifolds of the two models are different, hence their perturbation series do not coincide. We managed to single out a "corner" in the target space of the two models where the metrics look the same at the leading order in the FI parameter (alternatively, in the vicinity of the origin in the noncompact subspace, see Sec. 2.6) for details). However, far from the origin renormalization coefficients are completely different. Speaking geometrically, the zn model is a deformation of the HT model in terms of deforming the sections of the bundle (Eqs. (2.6.31) and (2.6.32) illustrate this).

Contrary to the case of the $\mathbb{CP}^{N-1}$ model, where one-loop renormalization can be completely understood in terms of a single coupling renormalization (the Kähler class, or the FI term), which is also one-loop-exact, this is not the case both in the zn and HT models. It occurs because both target manifolds are non-Einstein, hence the Ricci tensors (which give the one-loop β functions) are not proportional to the metric. Nevertheless, due to nice geometric properties of the fiber bundles (recall that the HT model lives on the total space of the tautological bundle for $\mathbb{CP}^{N-1}$), the first Chern class of this bundle is proportional to the Kähler class with exactly the right coefficient which also appears in the one-loop renormalization of the FI term.

As we discussed in Sec. 2.3 in the $GL\sigma M$ formulation of both zn and HT models there are only two divergent graphs (tadpoles), which contribute to the renormalization of the FI parameter (2.3.9). However, according to the result (2.6.31), in the $NL\sigma M$ formulation the one-loop renormalization consists not only of the FI shift, but also from the wavefunction renormalization of ζ. Moreover, an additional counterterm is needed in order to fully absorb the one-loop divergence. It is interesting if we relate the two perturbation series in any physically meaningful way. The answer to this question may be negative as, generally speaking, perturbations around a $GL\sigma M$ fixed point (small gauge coupling) and $NL\sigma M$ perturbation theory are different. Moreover, the limit $e \to \infty$ leads us away from the perturbative regime of the corresponding $GL\sigma M$. Still, more detailed perturbative analysis of the gauge theory at finite e is required in order to better understand which Feynman graphs contribute to the UV divergences. This is a suggestive topic for the future research.

Chapter 3

From $\mathcal{N} = 2$ 4d Theories to 2d CFTs and Integrability

3.1 Introduction

Recent work by Nekrasov and Shatashvili [77] has initiated the program of quantization of integrable systems by deforming four-dimensional supersymmetric (SUSY) theories these integrable systems are associated with. The relationship between classical integrable systems and $\mathcal{N} = 2$ supersymmetric gauge theories has been extensively studied for a long time [78, 79, 80]. Remarkably the low energy dynamics of $\mathcal{N} = 2$ gauge theories is captured by finite dimensional integrable systems such that the phase space of the latter is related to the instanton moduli space of the former. To be precise, there are two different classical integrable systems involved here. The first one is a holomorphic integrable system of the Hitchin or the spin chain type giving a Seiberg-Witten curve whose Jacobian is mapped onto complex Liouville tori and action variables are identified with order parameters in the gauge theory. The second integrable system is of the Whitham type emerges from the renormalization group (RG) flows, and the very RG equation plays the role of the Hamilton-Jacobi equation written in proper variables.

The quantum integrable systems we are considering in this Chapter can be extracted from four-dimensional theories in Omega background [28] with $\epsilon_1 = \epsilon$, $\epsilon_2 = 0$ (the so called Nekrasov-Shatashvili (NS) limit [81]). Given a prepotential of the 4d theory $\mathcal{F}(a, \epsilon_1, \epsilon_2)$ as a function of the Coulomb branch moduli parameters $\{a\}$ and the Omega deformation parameters $\epsilon_{1,2}$, we can consider in the NS limit an effective 2d theory with the following effective exact twisted superpotential

$$\widetilde{\mathcal{W}}(a,\epsilon) = \lim_{\epsilon_2 \to 0} \frac{\mathcal{F}(a,\epsilon,\epsilon_2)}{\epsilon_2}\,, \tag{3.1.1}$$

where ϵ_1 is replaced by ϵ [77]. The F-terms of the effective Lagrangian effectively become two-dimensional in the NS limit and are described by $\widetilde{\mathcal{W}}(a,\epsilon)$. For small ϵ formula (3.1.1) can be even further simplified

$$\widetilde{\mathcal{W}}(a,\epsilon) = \frac{\mathcal{F}(a)}{\epsilon} + \ldots\,, \tag{3.1.2}$$

where the ellipses denote terms which are regular in ϵ. The above twisted superpotential includes both perturbative and instantonic contributions.

Minimization of superpotential (3.1.1) yields the supersymmetric vacua which, according to the same authors [82, 81] are intimately connected with quantum integrable systems. Indeed, according to Nekrasov and Shatashvili, supersymmetric vacua of an appropriate two (also three and four) dimensional $\mathcal{N}=2$ gauge theory are in one-to-one correspondence with the Bethe roots of a certain integrable system. It is useful to make the Legendre transform and consider the dual superpotential $\widetilde{W_D}(a_D,\epsilon)$ depending on the dual variable a_D. Thus the equation

$$\exp\left(\frac{\partial \widetilde{W_D}(a_D,\epsilon)}{\partial a_D}\right) = 1\,, \tag{3.1.3}$$

can be viewed as a Bethe ansatz equation for some integrable system. This is consistent with the interpretation of the prepotential as an action in the Whitham system. Since a_D is the coordinate variable in the Whitham dynamics, one can recognize the exponent in (3.1.3) as the canonical conjugate momentum.

There also exists a different well known duality between four-dimensional gauge theories and two-dimensional gauge theories (linear sigma models) [33, 34, 35] (see [45] for review, we shall refer to it as *4d/2d duality*). The four-dimensional theory here sits at the root of its baryonic Higgs branch; therefore electric and flavor charges can be combined in a single set of quantum numbers. Together with magnetic charges they form two sets of quantum numbers which parameterize masses of four-dimensional dyons. On the two-dimensional side one has kinks interpolating between different supersymmetric vacua, their masses also depend on two sets of charges – Nöether and topological ones. The statement of the 4d/2d duality in its original formulation [33] is that the two sets perfectly match with each other. Thus the BPS spectrum of an entire four-dimensional theory can be studied using a relatively simple two-dimensional (gauged linear) sigma model with four supercharges. It was later shown by Shifman and Yung [45] that the 4d/2d correspondence is not accidental, the underlying two-dimensional theory in fact could be treated as a low energy effective theory on the worldsheet of a BPS vortex.

The first step in the direction of merging the 4d/2d correspondence and the gauge/quantum integrabilty duality together was put forward by Dorey, Hollowood and Lee [83]. The authors considered all three ingredients at once – four-dimensional $\mathcal{N} = 2$ gauge theory (SQCD with $N_f = 2N$), a two-dimensional $\mathcal{N} = (2, 2)$ gauge linear sigma model (GLSM) (a certain $U(K)$ gauge theory), and an integrable system. Critical ingredient of the duality is that the 4d gauge theory sits at a baryonic root of the Higgs branch, which in Omega background undergoes a deformation, and scalar field VEV gets shifted by an amount proportional to ϵ. Provided the baryonic Higgs root condition is satisfied, the 4d theory is shown to be dual to a given GLSM whose twisted superpotential plays a role of the Yang-Yang function for a $SL(2, \mathbb{R})$ Heisenberg magnet. The four-dimensional superpotential is shown to be equal to the effective 2d twisted superpotential of the form (3.1.1) on shell.

The initial motivation for this research was the identification of the different dualities known for quantum integrable systems in the framework of SUSY gauge theories and their brane realizations. However, first it is necessary to explain the geometrical meaning of degrees of freedom in relevant integrable systems which is a subtle issue. It was clear for a while that these degrees of freedom are described in terms of brane embeddings into the internal space. We refer the reader to [84], where geometrical aspects of dualities between integrable systems are reviewed.

A proper brane content involves surface operators, or equivalently nonabelian strings with large tension. To begin with we revisit the classification of the BPS solitons of different codimensions in the ϵ deformed theory. There is some surprise. It turns out that some BPS states in the deformed theory to the best of our knowledge were overlooked. We study carefully the central charges of the Omega-deformed SUSY algebra and argue that there are new stringy and domain wall type central charges. The key point is that tensions of such strings and domain walls are proportional to the graviphoton field and these defects are absent in the undeformed theory. We shall investigate the string-like object both for the pure supersymmetric Yang-Mills (SYM) theory and for the SQCD. It will be argued that there are some singularities akin to those of cosmic strings. The corresponding BPS equations for such strings will be derived and finiteness of their tension will be discussed. Similarly the domain walls solitons will be found, which are more expected to appear, since due to the Omega deformation the theory has the discrete set of vacuum states, hence domain walls naturally emerge. The monopoles in Omega background which have been already discussed in the literature [85, 86] also require some attention for their proper interpretation.

Turning to the dualities in quantum integrable systems we shall focus on two subjects. First we shall consider the quantum bispectral duality relating two different integrable systems. Classically eigenvalues of the Lax operator in one system get interchanged with coordinates in the second system. Quantum mechanically it means that the single wave function serves for two systems simultaneously when considered as the function of the spectral or coordinate variables. Although this question has not yet been elaborated to the full extent in the literature, by employing the quantum version of the duality we were able to explain the details of the 4d/2d duality in Omega background [83]. Geometrically bispectrality corresponds to a rotation of the brane configuration which represents the 4d gauge theory in question. We consider the relation between Bethe ansatz equations (BAE) for dual integrable systems and briefly discuss their degenerate solutions corresponding to the analogues of Argyres-Douglas points. Using the duality between the families of spin chain models and the Calogero-Ruijsenaars systems we shall identify bispectral pairs in both families. As a byproduct we will show that the Alday-Gaiotto-Tachikawa (AGT) duality [87] in the NS limit can be proved using the chain of dualities involving the bispectral pair.

The Chapter is organized as follows. In the next section we review how the $\mathcal{N}=2$ supersymmetry algebra is affected by the Omega background. We then compute the central charge for a BPS string in pure $\mathcal{N}=2$ Super Yang Mills theory and calculate its tension. In Sec. 3.3 we consider domain walls and monopoles in Omega deformed SYM. Then in Sec. 3.4 we investigate BPS strings in well-studied example of the 4d/2d duality – the supersymmetric QCD. Section 3.5 is devoted to brane constructions and integrable systems associated with $\mathcal{N}=2$ gauge theories and dualities between them. In Sec. 3.6 we show that in the NS limit, the celebrated AGT correspondence [87] can be reduced to the so-called *bispectral* duality between two integrable systems. Finally in Sec. 3.7 we conclude and speculate on further research topics.

3.2 Flux Tubes in Pure Super Yang-Mills Theory

In the standard lore of topological defects in supersymmetric theories, the BPS strings only exist when a gauge group is at least semi-simple, e.g. $U(N)$. A simple reason for this is based on existence of a nontrivial fundamental group of the resulting moduli space due to presence of a $U(1)$ factor. The latter causes a nonzero Fayet-Iliopoulos (FI) term which supports string solutions, and we shall refer them as *FI strings.* In this section, we shall instead consider to a new kind of string-like objects which have not been discussed in the literature before, we shall refer to them as *ϵ-strings.* As we shall later see their tension is proportional to ϵ^2 and construct the relevant classical field configurations. For simplicity we shall only focus on the gauge group $SU(2)$ in this section.

Action. Let us start with the $\mathcal{N}=2$ Super Yang-Mills theory in four dimensions in Omega background. To set the notations, the Lagrangian of the undeformed theory reads

$$\mathcal{L}=\frac{1}{4\pi}\mathfrak{Im}\,\tau\left[\mathrm{Tr}\int d^4\theta\,\bar{\Phi}e^V\Phi\,e^{-V}+\mathrm{Tr}\int d^2\theta\,(W^\alpha)^2\right]\,, \tag{3.2.1}$$

where $\Phi=(\phi,\psi,F)$ is adjoint chiral superfield, $V=(\sigma,\lambda,D)$ is adjoint vector superfield, W^α is its field strength, and $\tau=\frac{4\pi i}{g^2}+\frac{\theta}{2\pi}$ is coupling constant.

Omega deformation. The Omega deformation of a four-dimensional theory like (3.2.1) can be constructed from a six dimensional theory by compactifying the theory on a two-torus with twisted boundary conditions [88, 89]. Torus action on $\mathbb{R}^4$ is given by two matrices $\Omega^m_{a\,n}$ where $m,n=1,2,3,4$ and $a=5,6$ which act by rotations in 12 and 34 planes respectively. In the NS limit matrix Ω_6 vanishes, therefore we shall denote $\Omega=\Omega_5$. Metric on the deformed torus reads

$$G_{AB}dx^Adx^B=Adzd\bar{z}+(dx^m+\Omega^m dz+\bar{\Omega}^m d\bar{z})^2\,, \tag{3.2.2}$$

where $z = x^5 + ix^6$, $\bar{z} = x^5 - ix^6$ and the vector field $\Omega^m = \Omega^m_n x^n$. In the notations of [85] $\Omega^m = (-i\epsilon x^2, i\epsilon x^1, 0, 0)$. In other words vector field $\Omega = i\epsilon\,\partial_\varphi$ is a rotation generator around x_3-axis. Here we denote $\rho = \sqrt{x_1^2 + x_2^2}$. The components of the metric in the limit $A \to 0$ read

$$G_{mn} = \delta_{mn}\,, \quad G_{am} = \Omega_{am}\,, \quad G_{ab} = \delta_{ab} + \Omega^m_a \Omega_{bm}\,. \tag{3.2.3}$$

Upon the dimension reduction, the fifth and sixth components of the gauge field form an adjoint scalar, which undergoes the following deformation due to the Omega background

$$\phi \mapsto \phi - i\Omega^m \nabla_m + \tfrac{i}{2}\Omega^{mn} S_{mn}\,, \tag{3.2.4}$$

where Ω^m, Ω^{mn} were introduced after formula (3.2.2) and S_{mn} is the spin operator for adjoint representation of the gauge group. The latter does not affect the bosonic part of the theory, however it does modify the fermionic part. This issue will be important when we consider the supersymmetry algebra of the theory momentarily. Transformation (3.2.4) itself is not a well-defined change of coordinates, but ϕ enters the Lagrangian in a special way, this shift brings us to a well defined Lagrangian of a modified theory [90]. Another deformation of the theory consists of shifting of the coupling constant, thereby we promote it to a superfield. In the $\mathcal{N} = 2$ superfield language[1] the shift reads as follows

$$\tau \mapsto \tau - \bar{\theta}^m \bar{\theta}^n (\bar{\Omega}_{mn})^\dagger\,, \tag{3.2.5}$$

where $\bar{\theta}^m = (\bar{\sigma}^m)^{\dot{\alpha}I} \bar{\theta}_{\dot{\alpha}I}$ is the twisted Grassmann variable for the diagonal $\mathfrak{su}(2)_{R+\mathcal{R}}$ generators. In components the Lagrangian of the $\mathcal{N} = 2$ SYM after the deformation takes the following form

$$\begin{aligned}\mathcal{L} = {} & \tfrac{1}{4g^2}(F^a_{mn})^2 + \tfrac{1}{g^2}|\nabla_m \phi^a - F^a_{mn}\bar{\Omega}^n|^2 + \tfrac{1}{2g^2}|\phi\tau^a\bar{\phi} - i\nabla_m(\Omega^m\bar{\phi}^a - \bar{\Omega}^m\phi^a) + i\bar{\Omega}^m\Omega^n F^a_{mn}|^2 \\ & + \tfrac{1}{g^2}\bar{\lambda}^{fa}\sigma^m\nabla_m\lambda^a_f - \tfrac{i}{g^2}\lambda^{af}\bar{\phi}\,\tau^a\lambda_f + \tfrac{i}{g^2}\bar{\lambda}^a_f\phi\,\tau^a\bar{\lambda}^f \\ & + \tfrac{1}{g^2}\lambda^{fa}(\bar{\Omega}^m\nabla_m - \tfrac{1}{2}\bar{\Omega}^{mn}\sigma_{mn})\lambda^a_f - \tfrac{1}{g^2}\bar{\lambda}^a_f(\Omega^m\nabla_m - \tfrac{1}{2}\Omega^{mn}\sigma_{mn})\bar{\lambda}^{fa}\,,\end{aligned} \tag{3.2.6}$$

where $f = 1, 2$ denotes the R-symmetry index, and spinor indices are suppressed.

[1] See Shadchin's PhD thesis [89] for details

SUSY transformations. Recall that $\mathcal{N}=2$ supersymmetry algebra in four dimensions has the following form

$$\{Q^I_\alpha, \bar{Q}_{J\,\dot\alpha}\} = 2P_{\alpha\dot\alpha}\delta^I_J + 2Z_{\alpha\dot\alpha}\delta^I_J\,,$$
$$\{Q^I_\alpha, Q^J_\beta\} = \epsilon_{\alpha\beta}\epsilon^{IJ}Z_{\rm mon} + (Z_{\rm d.w.})^{IJ}_{\alpha\beta}\,. \tag{3.2.7}$$

There are three types on central charges: string, monopole and domain wall types. We shall focus on the former in this section leaving monopoles and domain walls to Sec. 3.3.

The full global symmetry of the theory is $SU(2)_L\times SU(2)_R\times SU(2)_{\mathcal{R}}$ (left, right and the R-symmetry). It is broken by the Omega background in the NS limit to $SU(2)_L\times SU(2)_{R+\mathcal{R}}$ by paring the R-symmetry with the right handed $SU(2)$. The supercharges undergo the Donaldson-Witten twist [91]

$$\bar{Q} = \delta^{\dot\alpha}_I\bar{Q}^I_{\dot\alpha}\,,\quad Q_m = (\bar\sigma_m)^{I\alpha}Q_{I\alpha}\,,\quad \bar{Q}_{mn} = (\bar\sigma_{mn})^{\dot\alpha}_I\bar{Q}^I_{\dot\alpha}\,. \tag{3.2.8}$$

These transformations can be inverted as follows

$$Q^I_\alpha = \tfrac{1}{2}(\sigma^m)^I_\alpha Q_m\,,\quad \bar{Q}_{\dot\alpha J} = \tfrac{1}{2}\epsilon_{\dot\alpha J}\bar{Q} + \tfrac{1}{2}(\bar\sigma_{mn})_{\dot\alpha J}\bar{Q}^{mn}\,. \tag{3.2.9}$$

It turns out that a generic Omega background breaks all supersymmetries of the theory (3.2.6) except the BRST charge $\bar{Q}$. Moreover, it can be shown that the Lagrangian (3.2.6) is a $\bar{Q}$-exact expression [28], which makes it possible to compute the partition function of the theory by localization methods.

It is more or less clear that the obstacle to supersymmetry is due to the spin operator terms $\frac{1}{2}\Omega^{mn}\sigma_{mn}$ in the fermionic sector. The theory thus has to be further deformed to gain more supersymmetry. To understand what we need to do, let us look at the supersymmetry transformations of (3.2.6) with the problematic spin operators omitted, and see what additional terms do we need to introduce. One has [85] the following under variations

$$\delta\phi = \zeta^I_\alpha(\lambda^\alpha_I - \Omega^m(\sigma_m)^{\alpha\dot\alpha}\bar\lambda_{I\dot\alpha}) + \bar\zeta^I_{\dot\alpha}\Omega^m(\bar\sigma_m)^{\alpha\dot\alpha}\lambda_{I\alpha}\,,$$
$$\delta\lambda_{I\alpha} = \zeta_{I\beta}((\sigma^{mn})^\beta_\alpha F_{mn} + i[\phi,\bar\phi]\delta^\beta_\alpha + \nabla_m(\bar\Omega^m\phi - \Omega^m\bar\phi)\delta^\beta_\alpha)$$
$$+\,\bar\zeta_{I\dot\beta}(\sigma^m)^{\dot\beta}_\alpha(\nabla_m\phi - F_{mn}\Omega^n)\,. \tag{3.2.10}$$

However, since the action is not supersymmetric, the above transformations do not leave the Lagrangian invariant. As suggested in [88] and later extended in [85], one has to turn on R-symmetry Wilson lines properly to restore partial supersymmetries. Thus, in the NS limit one has to add

$$-\bar{\mathcal{A}}_I^J \lambda^I \lambda_J - \mathcal{A}_I^J \bar{\lambda}^I \bar{\lambda}_J \,, \tag{3.2.11}$$

where

$$\mathcal{A}_I^J = -\tfrac{1}{2}\bar{\Omega}_{mn}(\bar{\sigma}^{mn})_J^I \,, \tag{3.2.12}$$

to the Lagrangian. One can also treat the above terms as emerging from a superpotential which we add to the theory, and we shall speculate more on this in Sec. 3.3. In the NS limit, when $\epsilon_2 = 0$, the supersymmetry of the theory is enhanced to $\mathcal{N} = (2, 2)$ [85] and is generated by the following supercharges

$$Q_1, Q_2, \bar{Q}_{13}, \bar{Q}_{14} \,. \tag{3.2.13}$$

Using the inverted transformation (B.3.3) we conclude that in the original formulation of SUSY algebra (B.3.1) the following generators are included into $\mathcal{N} = (2, 2)$ subalgebra

$$Q_{12},\ Q_{21},\ \bar{Q}_{\dot{1}2},\ \bar{Q}_{\dot{2}1} \,. \tag{3.2.14}$$

In the remaining part of the section we shall investigate 1/2 BPS object – a string which is annihilated by the above charges.

String central charge and string tension. The supercurrent for the Omega deformed SYM theory was computed in [85]. Its Euclidean time component has the following form (assuming static configuration, $B_3 \neq 0$, others components of F_{mn} vanish)

$$\begin{aligned} J^4_{I\alpha} = &\frac{1}{g^2}\left((-i[\phi, \bar{\phi}] + (\phi\bar{\Omega}^n - \bar{\phi}\Omega^n)\nabla_n)\sigma^4_{\alpha\dot{\alpha}} + \tilde{F}_{4n}\sigma^n_{\alpha\dot{\alpha}}\right)\bar{\lambda}^{\dot{\alpha}}_I \\ &+ \frac{2\sqrt{2}}{g^2}(\sigma^{4n})^\beta_\alpha(-\nabla_n\phi + F_{np}\bar{\Omega}^p)\lambda_{I\beta} \end{aligned} \tag{3.2.15}$$

Let us find the string central charge current. Performing standard variation of the above supercurrent we get[2]

$$\delta_{\zeta^I_\alpha}\bar{J}^{4J}_{\dot{\alpha}} = 2\sigma^4_{\alpha\dot{\alpha}}\delta^J_I\mathcal{P}_4 + \partial_m\left((\phi^a\bar{\Omega}^m - \bar{\phi}^a\Omega^m)B^a_3\right)\sigma^3_{\alpha\dot{\alpha}}\delta^J_I \,, \tag{3.2.16}$$

[2] Technically there is another contribution from the R-current [92], which contributes to the $(\frac{1}{2}, \frac{1}{2})$ central charge.

where $\mathcal{P}_4$ is Hamiltonian of the system. Note that there is an additional contribution to the above string charge current which is bilinear in fermions of the form $\partial_m(\Omega^m\bar{\lambda}\lambda)$. For classical analysis, where all fermionic fields can be put to zero, this contribution can be omitted. We see that there is a correction which represents the string central charge, specifically the correction takes the following form

$$\zeta_3 = \tfrac{1}{2g^2}\partial_m\left((\phi^a\bar{\Omega}^m - \bar{\phi}^a\Omega^m)B_3^a\right)\sigma^3_{\alpha\dot{\alpha}}\delta^{IJ}\,, \tag{3.2.17}$$

where $\rho^2 = x_1^2 + x_2^2$ is the transversal coordinate to the string. If ϵ is real then

$$\zeta_3 = \tfrac{1}{g^2}\partial_\varphi(\Re e\,\epsilon\bar{\phi}^a B_3^a)\,. \tag{3.2.18}$$

The central charge is given by

$$\begin{aligned} Z_{\text{string}} &= \int d^3x\,\zeta_3 = \frac{1}{g^2}\int dz\int d\rho\,\rho\int\limits_0^{2\pi} d\varphi\,\partial_\varphi(\Re e(\epsilon\bar{\phi}^a)B_3^a) \\ &= \frac{1}{g^2}\int dz\int d\rho\,\rho B_3^a\,\Re e(\epsilon\bar{\phi}^a)\Big|_0^{2\pi}\,. \end{aligned} \tag{3.2.19}$$

We can immediately see that multi-valuedness of ϕ as a function of the azimuthal angle is required in order to make the central charge nonzero. The tension of the string solution under consideration (let's call them *ϵ-strings*) is therefore given by

$$T = \frac{1}{g^2}\int\limits_0^\infty d\rho\,\rho B_3^a\,\Re e(\epsilon\bar{\phi}^a)\Big|_0^{2\pi}\,. \tag{3.2.20}$$

Assuming that

$$\phi(\rho,\varphi) = \phi(\rho)e^{i\alpha\varphi}\,, \tag{3.2.21}$$

where α is an constant, we arrive to

$$T = \frac{1}{g^2}\int\limits_0^\infty d\rho\,\rho\,\Re e\left(\epsilon B_3^a\bar{\phi}^a(e^{-2\pi i\alpha} - 1)\right)\,. \tag{3.2.22}$$

The above expression for the tension of ϵ-string only makes sense if it is finite. In order to establish that one has to solve BPS equations in order to find the profile functions for ϕ and B_3 as function of the radial coordinate ρ.

BPS equations. Let us now find the BPS equations which describe such a string. Once supersymmetry algebra is understood (3.2.14), we can focus on the bosonic part of the action

$$\mathcal{L} = \frac{1}{4g^2}F_{mn}^2 + \frac{1}{g^2}|\nabla_m\phi - F_{mn}\bar{\Omega}^n|^2 + \frac{1}{2g^2}|\phi\tau^a\bar{\phi} - i\nabla_m(\Omega^m\bar{\phi}^a - \bar{\Omega}^m\phi^a)|^2\,. \qquad (3.2.23)$$

Note that in the NS limit $\bar{\Omega}^m\Omega^n F^a_{mn}$ identically vanishes. Let us now do the Bogomolny completion, as the supersymmetry algebra suggests

$$\begin{aligned}\mathcal{L} = &\frac{1}{2g^2}|B_3^a + \phi\tau^a\bar{\phi} - i\nabla_m(\Omega^m\bar{\phi}^a - \bar{\Omega}^m\phi^a)|^2 + \frac{1}{2g^2}|\nabla_1\phi^a + i\nabla_2\phi^a - (\Omega_2 - i\Omega_1)B_3^a|^2 \\ &+ \frac{1}{2g^2}\partial_m(B_3^a(\Omega^m\bar{\phi}^a - \bar{\Omega}^m\phi^a)) \geq \frac{1}{2g^2}\partial_m(B_3^a(\Omega^m\bar{\phi}^a - \bar{\Omega}^m\phi^a))\,. \qquad (3.2.24)\end{aligned}$$

The above inequality is saturated provided that the following BPS equations are satisfied

$$\begin{aligned}B_3^a + \bar{\phi}\tau^a\phi - i\nabla_m(\Omega^m\bar{\phi}^a - \bar{\Omega}^m\phi^a) = 0\,, \\ \nabla_1\phi^a + i\nabla_2\phi^a - (\Omega_2 - i\Omega_1)B_3^a = 0\,. \qquad (3.2.25)\end{aligned}$$

One can also check that the above BPS equations are consistent with the $\mathcal{N} = (2,2)$ supersymmetry algebra. Indeed, by looking at the gluino variation in (3.2.10) we need to set to zero all expressions which enter the right hand side together with ζ_{11}, ζ_{22} and their complex conjugates. Contributions proportional to $\zeta_{12}\,, \zeta_{21}$ and conjugated terms vanish automatically due to the BPS condition. By doing so one arrives at equations (3.2.25).

Sometimes it is more convenient to switch to the complex coordinates

$$w = x + iy\,, \quad \bar{w} = x - iy\,, \qquad (3.2.26)$$

then the BPS equations (3.2.25) take the following form

$$\begin{aligned}\partial_{\bar{w}}A_w^a - \partial_w A_{\bar{w}}^a + \epsilon^{abc}\phi^b\bar{\phi}^c - i(w\nabla_{\bar{w}} - \bar{w}\nabla_w)(\epsilon\bar{\phi}^a - \bar{\epsilon}\phi^a) = 0\,, \\ \partial_{\bar{w}}\phi^a - \epsilon^{abc}A_{\bar{w}}^b\phi^c + \epsilon w(\partial_{\bar{w}}A_w^a - \partial_w A_{\bar{w}}^a) = 0\,. \qquad (3.2.27)\end{aligned}$$

The above equations can be used in study of the effective two-dimensional theory living on the ϵ-string.

Solution of BPS equations and vortex tension. Let us proceed with the solution of (3.2.25). We will look for a background solution when all fields are aligned along the Cartan subalgebra of the gauge algebra. Thus color superscript will always be $a = 3$, in the rest of the section we shall omit it. One has the following

$$B_3 - i\partial_m(\Omega^m\bar{\phi} - \bar{\Omega}^m\phi) = 0\,,$$
$$\partial_1\phi + i\partial_2\phi - (\Omega_2 - i\Omega_1)B_3 = 0\,. \tag{3.2.28}$$

Decomposing $\phi = \phi_1 + i\phi_2$, $\epsilon = e_1 + ie_2$ into real and imaginary parts we obtain

$$B_3 = 2\partial_\varphi(e_1\phi_1 + e_2\phi_2)\,, \tag{3.2.29}$$

and two first order equations on ϕ_1 and ϕ_2. After some simple manipulations one gets

$$\Delta\phi_1 + 4e_2\partial_\varphi(e_1\phi_1 + e_2\phi_2) = 0\,,$$
$$\Delta\phi_2 - 4e_1\partial_\varphi(e_1\phi_1 + e_2\phi_2) = 0\,, \tag{3.2.30}$$

which, after adding these equations with proper coefficients, implies that $e_1\phi_1 + e_2\phi_2$ is a harmonic function. However, at this point boundary conditions of the solution remain unclear as we need to have $\phi(2\pi) \neq \phi(0)$ in order to gain finite tension (3.2.22). In order to make the problem mathematically precise we can make the following trick. The phase difference of $e^{2\pi i\alpha}$ will be identified with the deficit angle of a cone which is obtained by gluing $\varphi = 0$ ray with $\varphi = 2\pi$ one.

Solution of Laplace equation on a cone is formally given by a series of positive and negative powers of ρ with angle dependent coefficients. The latter are normally expressed in terms of ellipsoidal harmonics. Since we are interested in normalizable solutions we only leave negative powers of the radial coordinate in the series. The solution will however be divergent at the origin. The dependence on phase α is then hidden in the harmonic coefficients. We shall refrain from giving more details here since we are not investigating any dynamics on ϵ-strings in this paper. Rather we provide the evidence of existence of ϵ-strings and finiteness of their tension.

Let us now look at the vortex tension (3.2.22). Using (3.2.29) we conclude that

$$T = \frac{1}{2g^2}\int d^2x\,\mathrm{Tr}\,B^2 = \frac{1}{g^2}\int\limits_0^{2\pi} d\varphi \int\limits_0^{\infty} d\rho\,\rho\,\partial_\varphi(\Re e(\epsilon\bar{\phi}^a)B_3^a)\,. \tag{3.2.31}$$

As we argued above the integral over the radial coordinate diverges, however it does not make the tension infinite. To see this let us regularize the radial integral on its lower limit by putting a cutoff at some small value of $\rho = \rho_0$. After integrating the full angular derivative we see that the contribution to the integral at the lower limit cancel each other as $\rho_0 \to 0$. Thus we are left with the contribution from large ρ. As we are not specifying the full solution of the BPS equations we shall not evaluate integral (3.2.31) here. However, from dimensional ground we anticipate

$$T = A(\alpha)|\epsilon|^2\,, \tag{3.2.32}$$

where A is a constant. Therefore we find that the tension of ϵ-strings is quadratic in ϵ, and it is only nonzero for fractional winding numbers.

To conclude this section let us make a few general comments concerning ϵ-strings. First, at large values of the graviphoton field the string tension is large and we can safely consider it as semiclassical object and the string could serve as the new type of the surface operator. In this case one can use the standard technique to get the worldvolume theory. We shall discuss the worldvolume theory elsewhere. The second point to be mentioned is some analogue with the string in the noncommutative gauge theory found by Gross and Nekrasov [93]. They have discovered that the Dirac string attached to the monopole in the noncommutative theory becomes observable and its tension is proportional to the noncommutativity parameter. Since using the chain of dualities [94, 95, 96] the ϵ parameter can be traded to the noncommutativity in the internal space one could look for more close relation between two types of strings. Finally, one could ask for the brane realization of the ϵ-string. Certainly it can not be identified as D2 brane similar to the FI string since we can not reproduce the tension with such brane realization. Hence the most natural candidate is the properly embedded D4 brane. We hope to discuss the details of the brane realization of ϵ-strings elsewhere.

3.3 Monopoles and Domain Walls

We have discussed color flux tubes in Omega background in Sec. 3.2. Here we shall address two other types of topological defects we often encounter in supersymmetric theories – monopoles and domain walls.

Central charge. Recall from (B.3.1) that domain walls and monopoles saturate holomorphic central charges in supersymmetry algebra. Symmetric combination of these charges give domain wall piece, whereas an antisymmetric one contributes to monopoles. Analogously to a string charge density we have evaluated in (3.2.16), we can proceed with the monopole and domain wall. The former calculation has been performed in [85], and the latter gives

$$\delta_{\zeta_{I\beta}} J^4_{J\alpha} = \tfrac{1}{g^2}\delta_{IJ}(\sigma^4)_{\alpha\dot{\alpha}}(\sigma^3)^{\dot{\alpha}}_{\beta}\partial_3(\phi^a\nabla_n(\phi^a\bar{\Omega}^n - \bar{\phi}^a\Omega^n))\,. \tag{3.3.1}$$

We have included here only the bosonic contribution to the supercurrent which is relevant for classical analysis. The full expression will also contain bilinear term in fermions $\partial_m(\Omega^m\lambda\lambda) + H.c.$ which will be manifest for quantum calculations [97].

Structurally (3.3.1) is very reminiscent of the string current (3.2.16). As in the string charge case, a field dependent FI term appears

$$\xi^a = \tfrac{1}{g^2}\nabla_m(\Omega^m\bar{\phi}^a - \bar{\Omega}^m\phi^a)\,, \tag{3.3.2}$$

leading to the following expression of the domain wall tension

$$T = \tfrac{1}{g^2}\xi^a(\phi^a_{+\infty} - \phi^a_{-\infty})\,, \tag{3.3.3}$$

which can be viewed as a non-Abelian generalization of the standard calculation in theories with superpotentials. Let us now have a closer look on the BPS monopoles and domain walls appearing in $\mathcal{N}=2$ SYM theory, again for simplicity we shall consider $SU(2)$ gauge group here.

BPS equations and monopole solution. Ito et al [85, 86] have investigated BPS monopole solution of $\mathcal{N}=2$ SYM with gauge group $SU(2)$ in Omega background in the NS limit. It reads

$$B^a_i - \nabla_i\phi^a + i\epsilon_{ijk}\Omega^j B^{ak} = 0\,, \tag{3.3.4}$$

or in components

$$\begin{aligned} B^a_3 - \nabla_3\phi^a - \epsilon\, x^m B^a_m &= 0\,,\\ B^a_m - \nabla_m\phi^a + \epsilon\, x_m B^a_3 &= 0\,, \end{aligned} \tag{3.3.5}$$

where $m = 1, 2$. The solution of these equations is given in [85]. The authors' conclusion is that the monopole's mass is not changed, the magnetic field strength has the same form as the one in the undeformed case for $\epsilon = 0$. However, there is a correction of the scalar field profile. In the singular gauge (when only $\phi^3 \neq 0$) the solution reads

$$\phi^3(\rho, z) = \sqrt{v^2 H(\rho, z)^2 + \epsilon^2 + \frac{2vz\epsilon H(\rho, z)}{\sqrt{\rho^2 + z^2}} + G(\rho, z)^2}\,, \tag{3.3.6}$$

where function

$$\begin{aligned} G(\rho, z)^2 = &\frac{2\rho^2 vz\epsilon F(\rho, z)H(\rho, z)}{\rho^2 + z^2} + \frac{\rho^2 z^2 \epsilon^2 F(\rho, z)^2}{\rho^2 + z^2} - \frac{2\rho^2 \epsilon^2 F(\rho, z)}{\rho^2 + z^2} \\ &- \frac{2\rho^2 vz\epsilon F(\rho, z)H(\rho, z)}{(\rho^2 + z^2)^{3/2}} + \frac{\rho^4 \epsilon^2 F(\rho, z)^2}{(\rho^2 + z^2)^2} \end{aligned} \tag{3.3.7}$$

vanishes at $z \to \pm\infty$. Functions F and H are taken from the 't-Hooft-Polyakov monopole solution [9, 8]

$$H(\rho, z) = \coth v\sqrt{\rho^2 + z^2} + \frac{1}{v\sqrt{\rho^2 + z^2}}\,, \quad F(\rho, z) = 1 - \frac{v\sqrt{\rho^2 + z^2}}{\sinh v\sqrt{\rho^2 + z^2}}\,. \tag{3.3.8}$$

Note that scalar field ϕ does not go to its vacuum value v any longer as it does for $\epsilon = 0$, but rather interpolates between $\phi_{+\infty} = v + \epsilon$ to $\phi_{-\infty} = v - \epsilon$ at plus and minus z-infinity respectively. This suggests us that maybe 1/2 BPS monopole is not a proper interpretation of the above solution and more structures can be involved.

Before we go further, let us mention an useful symmetry of equations (3.3.5). In the above analysis axial symmetry was assumed such that both ϕ and B fields depended only on ρ and z. We can also introduce azimuthal angle φ in the game by giving the scalar field a phase

$$\phi \mapsto \phi\, e^{i\alpha\varphi}\,. \tag{3.3.9}$$

In order to preserve (3.3.5) the magnetic field strength also acquires a phase and its azimuthal component B^a_φ gets generated due to $\nabla_m \phi$ term in the second equation. Provided that such a configuration is chosen, the FI field reads

$$\xi^a = \tfrac{i}{g^2}\alpha(\bar{\epsilon}\phi^a + \epsilon\bar{\phi}^a)\,. \tag{3.3.10}$$

Then the corresponding domain wall's tension (3.3.3) becomes

$$T = \tfrac{2}{g^2}\epsilon \mathrm{Tr}\phi^2\,, \tag{3.3.11}$$

for imaginary ϕ and real ϵ. Since (3.3.6) still solves equations (3.3.5) we can compute the tension for the solution in hand. One gets

$$T = \tfrac{2}{g^2}\epsilon\left((v+\epsilon)^2 - (v-\epsilon)^2\right) = \tfrac{8}{g^2}v\epsilon^2\,. \tag{3.3.12}$$

Let us now see how to construct monopoles and domain walls in $SU(2)$ SYM theory in the NS Omega background.

Monopole on a domain wall. We shall perform a slightly different Bogomolny completion of the action (3.2.23) than the authors of [85, 86]

$$\begin{aligned}\mathcal{L} = &\tfrac{1}{2g^2}\left|B_3^a - i\nabla_m(\Omega^m\bar{\phi}^a - \bar{\Omega}^m\phi^a) + \nabla_3\phi^a + i\epsilon_{3jk}\Omega^j B^{ak}\right|^2 \\ &+ \tfrac{1}{2g^2}\left|B_1^a + iB_2^a + (\nabla_1 + i\nabla_2)\phi^a - (\Omega_2 - i\Omega_1)B_3^a\right|^2 \\ &+ \tfrac{1}{g^2}\partial_m\left(B_3^a(\Omega^m\bar{\phi}^a - \bar{\Omega}^m\phi^a)\right) + \tfrac{1}{g^2}\partial_3(\phi^a\nabla_m(\Omega^m\bar{\phi}^a - \bar{\Omega}^m\phi^a)) - \tfrac{1}{g^2}\partial_i(B_i^a\phi^a)\,.\end{aligned} \tag{3.3.13}$$

Three terms in the third line above correspond to strings, domain walls and monopoles respectively. Existence of the first two types of solitons solely relies on the non-trivial field dependent FI term (3.3.2)

$$\xi^a = \tfrac{1}{g^2}\nabla_m(\Omega^m\bar{\phi}^a - \bar{\Omega}^m\phi^a)\,. \tag{3.3.14}$$

BPS equations follow from (3.3.13) immediately

$$\begin{aligned}B_3^a + \nabla_3\phi^a + i\epsilon_{3jk}\Omega^j B^{ak} - i\nabla_m(\Omega^m\bar{\phi}^a - \bar{\Omega}^m\phi^a) &= 0\,,\\ B_1^a + iB_2^a + (\nabla_1 + i\nabla_2)\phi^a - (\Omega_2 - i\Omega_1)B_3^a &= 0\,.\end{aligned} \tag{3.3.15}$$

We can now observe that if ϵ and ϕ^a are real valued, as they were chosen to be in [85, 86], then the latter equation above splits into two, one for the real part, one for the imaginary part of its l.h.s. Also the field dependent FI term vanishes. We immediately identify them as the last two equations of (3.3.5). However, if a different ansatz is chosen, when either ϵ or ϕ^a or both have imaginary part, the FI parameter (3.3.2) kicks in, and equations (3.3.15) no longer decouple. Their solution is probably more complicated than reported in [86], and will be reported elsewhere.

Boojums. At this point let us also mention that the setup we have just described also admits strings provided that the parameter α in (3.3.9) is non integer, otherwise the string central charge vanishes, indeed it follows from (3.2.22). Thus the solution above describes a BPS monopole on a domain wall, and, if α is not integer, a more complicated *boojum* construction [45] which is a junction of a string, domain wall and a monopole Fig. 3.1; it is a 1/4 BPS configuration. Remarkably, all three structures coexist together and emerge together from Omega deformation. It appears to be impossible, as far as our analysis suggests, to find, say, only domain wall without a monopole, or vice versa – they always come in pairs. Strings, however, as we discussed in Sec. 3.2, can exist on their on provided that α is a non integer.

Interestingly, the object we have just described – a boojum, can be placed on ends of ϵ-strings, the same way as monopoles in the Higgs phase of $\mathcal{N} = 2$ theory can have non-Abelian flux tubes emerging from them [45]. Such a string is depicted in Fig. 3.1

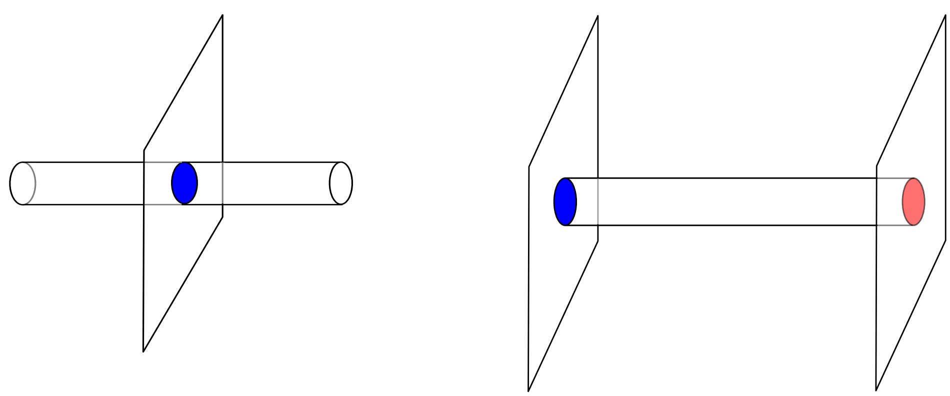

Figure 3.1: Left: Boojum as a monopole-string-domain wall junction. The string is infinite and is stretched along the z-axis. Right: ϵ-string ending at monopoles located on two parallel domain walls in xy-plane can be viewed as the superposition of two boojums. The string does not continue through the domain walls to the outer area, since the scalar field, main building block of the ϵ-string, vanishes outside of the domain walls.

Relationship to coupled 4d/2d systems. In [98, 16] coupled 4d/2d systems were studied. An example of such a system is a four-dimensional gauge theory with a surface operator insertion. The 4d theory is considered to be in the Coulomb branch, a 2d theory lives on the surface defect and both systems are coupled. Remarkably, both 2d (kinks) and 4d (monopoles, dyons) BPS states can be found in such systems and the authors of [16] managed to derive the full 2d/4d wall crossing formula. Bound states of monopoles on surface defects are present in the theory, since the 4d theory is at the Coulomb branch, its magnetic field has a spherically symmetric pattern, unlike a Higgs monopole whose field lines are trapped to a vortex. These two pictures – Higgsed monopole an a vortex and a Polyakov-'t Hooft monopole on a surface defect Fig. 3.2 may represent two different limiting configurations of a more generic setup, which involves more sophisticated 2d/4d dynamics. Keeping the calculations we have done in this section, we may hope that 4d theories in Omega background may be reasonable candidates for such a theory. It would be interesting to investigate the solution of BPS equations (3.3.15) more closely and study different values of the deformation parameter ϵ. At large ϵ the surface operator limit emerges and Gaiotto et al story [16] may also arise.

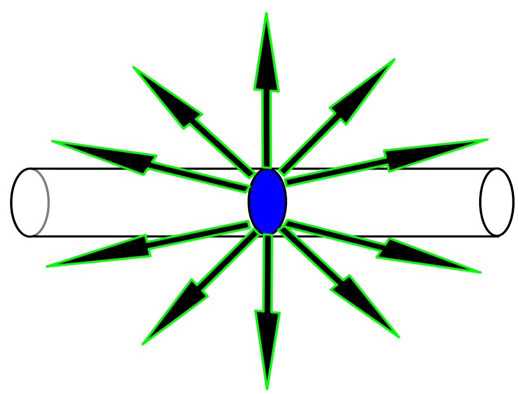

Figure 3.2: Polyakov-'t Hooft monopole on a surface defect.

3.4 $\mathcal{N} = 2$ SQCD in Omega Background

The 4d/2d duality was initially formulated for $\mathcal{N} = 2$ supersymmetric QCD [33], with gauge group $U(N)$ with $N \leq N_f \leq 2N$ flavors. Physical explanation of the duality [35, 34] (see also [99]) relies on existence of BPS flux tubes (FI strings) – solitonic solutions of the theory. Below we demonstrate how these solutions can be constructed and what are the resulting BPS equations for Omega-deformed theory, which as we know, preserves only $\mathcal{N} = (2,2)$ part of supersymmetry (4 out of 8 supercharges). Similarly to the pure SYM discussed in the previous section, the BPS vortex in question will invariant under the unbroken part of the SUSY algebra, therefore it will still remain to be a BPS configuration.

3.4.1 Chen-Dorey-Hollowood-Lee duality

Here we shall review main aspects of the duality unveiled in [83, 100]. The authors have proposed and proved [3] that in NS limit, four-dimensional $U(N)$ SQCD with N fundamental hypermultiplets of masses $m_1, \ldots m_N$ together with N antifundamental hypermultiplets of masses $\tilde{m}_1, \ldots \tilde{m}_N$ and coupling constant τ is dual to the two-dimensional $U(K)$ GLSM with N chiral fundamentals of twisted masses $M_1, \ldots, M_N$ together with N chiral antifundamentals of twisted masses $\tilde{M}_1, \ldots, \tilde{M}_N$ and coupling constant $\hat{\tau}$. The above statement holds provided that the four-dimensional theory is considered in the Higgs branch defined by the condition

$$\phi_a = m_a - n_a \epsilon \,, \tag{3.4.1}$$

for some $\mathbb{Z}_N$ vector n_a, rank of the gauge group of the 2d GLSM is given by

$$K = \sum_{a=1}^{N} (n_a - 1) = \sum_{a=1}^{N} \hat{n}_a \,, \tag{3.4.2}$$

masses of the 4d theory and twisted masses of the 2d theory are related to each other in the following way

$$M_a = m_a - \tfrac{3}{2}\epsilon \,, \quad \widetilde{M}_a = \widetilde{m}_a + \tfrac{1}{2}\epsilon \,, \tag{3.4.3}$$

and coupling constants obey

$$\hat{\tau} = \tau + \tfrac{1}{2}(N+1) \,. \tag{3.4.4}$$

Quantitatively the CDHL duality states that in NS limit, the chiral rings of the 4d and 2d theories are isomorphic and we can relate effective twisted superpotential (3.1.1)

[3] We change the notations of [83] to make them consistent with our notations.

from the 4d gauge theory with the effective twisted superpotential of the corresponding 2d GLSM as follows

$$\widetilde{\mathcal{W}}(\phi_a = m_a - n_a\epsilon) - \widetilde{\mathcal{W}}(\phi_a = m_a - \epsilon) = \widetilde{\mathcal{W}}^{2d}_{\text{eff}}(\hat{n}_a)\,. \tag{3.4.5}$$

This means that the sets of stationary points (vacua) of the two superpotentials are isomorphic and the above equality holds in the corresponding vacua. The equality (3.4.5) has been proven by computing the Nekrasov partition function on the Higgs branch of the theory (3.4.1), and deriving the effective twisted superpotential then matching it with the 2d superpotential on shell [83, 100]. Let us mention a related contribution [101], where moduli space of vortices was shown to be a submanifold of the instanton moduli space of the Omega deformed 4d theory.

Below we shall provide some further supports for this duality based on the study of non-Abelian BPS vortices in the Omega deformed four-dimensional theory. This will be done by identifying the classical 2d theory living on the vortex along the course of the Shifman-Yung program [45].

3.4.2 Constructing non-Abelian vortices $N_f = N$

Let us now construct the action of the $\mathcal{N} = 2$ SQCD in the NS Omega background in four dimensions. Although in [83, 100] the superconformal $N_f = 2N$ case was considered, from the viewpoint of the non-Abelian vortices it is more instructive to start with the left boundary of the stability window $N_f = N$.

Action. Let us begin again with the undeformed SQCD Lagrangian in four dimensions with $N = N_f$

$$\begin{aligned}\mathcal{L} = \frac{1}{4\pi}\text{Tr}\left[\mathfrak{Im}\,\tau\left(\int d^4\theta\,\bar{\Phi}\,e^V\Phi\,e^{-V} + \int d^2\theta\,(W^\alpha)^2\right)\right]\\ + \int d^4\theta\left(\bar{Q}_i e^V Q^i + \widetilde{Q}_i e^{-V}\overline{\widetilde{Q^i}}\right) + \int d^2\theta\left(\widetilde{Q}_i\Phi Q^i + m^i_j\widetilde{Q}_i Q^j + \text{H.c.}\right)\,,\end{aligned} \tag{3.4.6}$$

where V is an adjoint $\mathcal{N} = 1$ vector superfield, $\Phi = (\phi, \lambda, D)$ is an adjoint $SU(N) \subset U(N)$ chiral superfield, quark superfields $Q_i = (q_i, \psi_i, F_i)$ and $\tilde{Q}_i = (\tilde{q}_i, \tilde{\psi}_i, \tilde{F}_i)$ are transformed in $\mathbf{N}$ and $\bar{\mathbf{N}}$ of the $SU(N)$ and as $\mathbf{N}_f$ and $\bar{\mathbf{N}}_f$ under global flavor $SU(N_f)$ respectively; m^j_i is quark mass matrix and $\tau = \frac{4\pi i}{g^2} + \frac{\theta}{2\pi}$ is coupling constant.

Omega Deformation. Our task now is to construct the Omega deformed theory. As in the pure SYM case the deformation can naturally be understood in terms of the six dimensional $\mathcal{N}=1$ theory [88]. It is convenient to use dual frame description $G_{AB}=e^{(c)}_A e^{(c)}_B$. The components of sixbeins read

$$e^{(m)}_n=\delta^m_n\,,\quad e^{(m)}_a=\Omega^m_a\,,\quad e^{(a)}_m=0\,,\quad e^{(a)}_b=\delta^a_b\,. \tag{3.4.7}$$

Using the above equation we can rewrite the kinetic term for squarks

$$e^{(B)}_a\nabla_{(B)}q=\nabla_a q-i\Omega^m_a\nabla_m q\,. \tag{3.4.8}$$

Thus we have

$$|\nabla_A q|^2=|\nabla_m q|^2+|(\phi-i\Omega^m\nabla_m)q|^2\,, \tag{3.4.9}$$

and analogously the kinetic term for anti squarks. The bosonic part of the action after quark masses m_i and $\widetilde{m}_i$ are included reads

$$\begin{aligned}\mathcal{L}=&\ \tfrac{1}{4g^2}F^2_{mn}+\tfrac{1}{g^2}|\nabla_m\phi-F_{mn}\bar{\Omega}^n|^2+\tfrac{1}{2g^2}|\phi\tau^a\bar{\phi}-i\nabla_m(\Omega^m\bar{\phi}^a-\bar{\Omega}^m\phi^a)+g^2(\bar{q}\tau^a q-\widetilde{q}\tau^a\bar{\widetilde{q}})|^2\\&+\tfrac{1}{2}|\nabla_m q|^2+\tfrac{1}{2}|\nabla_m\widetilde{q}|^2+\tfrac{1}{2}|(\phi-m_i-i\Omega^m\nabla_m)q_i|^2+\tfrac{1}{2}|(\phi-\widetilde{m}_i-i\Omega^m\nabla_m)\widetilde{q}_i|^2\\&+2g^2|\widetilde{q}\tau^a q|^2+\tfrac{g^2}{2}|\widetilde{q}_i q_i-N\xi_{FI}|^2+\tfrac{g^2}{8}(|q|^2-|\widetilde{q}|^2)^2\,,\end{aligned} \tag{3.4.10}$$

where we have included Fayet-Iliopoulos term ξ_{FI}. This theory has $U(N)_c\times SU(N)_f$ global color and flavor symmetry group.

Supersymmetry transformations. In what follows it is convenient to package squarks and anti-squarks into a single vector $q^{if}=(q^i,\ \bar{\widetilde{q}}^i)$, where $f=1$ for squarks and $f=2$ for antisquarks. Supersymmetry acts on the fields in the following way[4]

$$\begin{aligned}\delta\phi^a&=-\sqrt{2}\zeta^{\alpha f}(\lambda^a_{\alpha f}-\Omega^{\alpha\dot{\alpha}}\bar{\lambda}^a_{\dot{\alpha}f})+\bar{\zeta}^f_{\dot{\alpha}}\Omega^{\alpha\dot{\alpha}}\lambda^a_{f\alpha}\,,\\\delta\lambda^{af}_\alpha&=-\zeta^{\beta f}F^a_{\alpha\beta}+i\zeta^g_\alpha \mathrm{D}^{af}_g+i\sqrt{2}\bar{\zeta}^{\dot{\alpha}f}\left(\nabla_{\alpha\dot{\alpha}}\phi^a-F^a_{\alpha\dot{\alpha}\beta\dot{\beta}}\Omega^{\beta\dot{\beta}}\right)\,,\\\delta q^{if}&=\sqrt{2}\zeta^{\alpha f}\psi^i_\alpha+i\sqrt{2}\bar{\zeta}^f_{\dot{\alpha}}\bar{\widetilde{\psi}}^{i\dot{\alpha}}\,,\\\delta\psi^i_\alpha&=-i\sqrt{2}\bar{\zeta}^{\dot{\alpha}}_f\nabla_{\alpha\dot{\alpha}}q^{if}+2i\zeta^f_\alpha\left(\bar{\phi}_a(\tau^a)^i_j q^j_f-i\Omega^{\beta\dot{\beta}}\nabla_{\beta\dot{\beta}}q^i_f\right)\,,\end{aligned} \tag{3.4.11}$$

[4] We use chiral notation here.

where $i = 1, \ldots, N$ runs through fundamental representation, $a = 1, \ldots, N^2$ runs through the adjoint representation of $U(N)$, $f, g = 1, 2$ denote $SU(2)$ R-symmetry index, and the D-term contribution in the first line above has the following form

$$\mathrm{D}_g^{a\,f} = -\bar{\phi}\tau^a\phi\delta_g^f - g^2\left(\bar{q}_g\tau^a q^f - \Xi_g^{a\,f}\right) , \tag{3.4.12}$$

where the generalized FI-term reads

$$\Xi_g^{a\,f} = \tfrac{i}{g^2}\nabla_{\alpha\dot{\alpha}}(\bar{\Omega}^{\alpha\dot{\alpha}}\phi^a - \Omega^{\alpha\dot{\alpha}}\bar{\phi}^a)\delta_g^f + \xi_{FI\,g}^f\delta_{N^2}^a . \tag{3.4.13}$$

The first term we have already seen in the previous section, similar story here – it is generated by the Omega background. The second contribution to Ξ is the standard FI term. Here, as in [102], we formally kept the FI parameter $\xi_{FI\,g}^f$ as a triplet. Usually only diagonal part is left over, as it simplifies the calculations, however, it is absolutely unnecessary. As we can see, generalized FI parameter (3.4.13) is a sum of the two terms, the field dependent FI term, which appears due to Omega deformation, and the conventional $U(1)$ FI term, which is normally considered in supersymmetric QCD. As we know [103], presence of the latter does not affect the supersymmetry of the theory, however, it's broken to the $(2,2)$ SUSY due to the former. Similar to the pure SYM case, considered in the previous section, generators (3.2.14) form the supersymmetry algebra. We shall use this information to extract the BPS equations later.

Classical vacua. Vacua of the theory can be chosen similarly to the undeformed SQCD. Indeed, after identifying

$$\phi^a = m^a \quad \tilde{q}_{ai} = \bar{q}_{ai} = \sqrt{\xi_{FI}}\delta_{ia} , \tag{3.4.14}$$

we find that the potential in (3.4.10) vanishes. Alternatively one could have put $\tilde{q}^i = 0$ and work only with squarks q^i (see [45] for details). This vacuum is invariant under the color-flavor rotations

$$G_{c+f}: \quad \phi \mapsto U_c^{-1}\phi\, U_f\,, \quad q_a^i \mapsto (U_c^{-1})_a^b\, q_b^j\, U_{f\,j}\,, \tag{3.4.15}$$

where $U_c = U_f \in G_{c+f}$. In the most generic case, when masses m^a are arbitrary, one has

$$G_{c+f} = S(U(n_1) \times \cdots \times U(n_k))\,, \quad \sum_{j=1}^{k} n_j = N\,, \tag{3.4.16}$$

where $S(\dots)$ stands for the stabilizer. If all masses are different then $G_{c+f} = S\left(U(1)^N\right) = U(1)^{N-1}$. The pattern of the symmetry breaking depends on the relationship between the masses m_i and the FI parameter ξ_{FI}. In what follows we shall assume $\xi \gg m_i^2$ for any i and all the masses to be of the same order, thus in our case we have the following breaking

$$U(N)_c \times SU(N)_f \xrightarrow{\sqrt{\xi}} U(1) \times SU(N)_{c+f} \xrightarrow{m_i} U(1) \times G_{c+f}\,. \tag{3.4.17}$$

Note that $U(1)$ factor in the above formula will be very important in constructing vortices, which we shall now do. A vortex configuration will further break the above symmetry in a nontrivial way – the above mentioned $U(1)$ will be coupled to the generators of the Cartan subalgebra of G_{c+f}. By using the extra $U(1)$ symmetry we can also change the second equality relation in (3.4.14), and this is what exactly is done in the DHL paper. For the convenience of the calculations of [83] squarks have charge $-3/2$ and anti-squarks have charge $+1/2$ with respect to this symmetry. In the current paper it is more convenient to keep the condition (3.4.14) on the vortex solution as well.

Vortex configuration. In order to find non-Abelian BPS strings we need to organize winding around the z-axis. Thus we allow one of the flavors, say q^N, to depend on the azimuthal angle $e^{i\hat{n}\varphi}q^N$, where $\hat{n}$ is an integer. Algebraically it corresponds to breaking the symmetry of (3.4.17) down to $U(1)_{\text{diag}} \times SU(N-1)$, where the first $U(1)_{\text{diag}}$ factor is the diagonal subalgebra of the $U(1)$ from (3.4.17) and the $N-1$'st Cartan generator of the G_{c+f}. Then for the two terms in the second line of (3.4.10) read

$$V \supset \left|(\phi^i_j - m_N\delta^i_j + \hat{n}\epsilon\delta^i_j + \epsilon\rho(A_\varphi)^i_j)q^{Nj}\right|^2\,. \tag{3.4.18}$$

It vanishes provided that the expression in the parentheses above is equal to zero. So we put

$$\phi^N_N = m_N - \hat{n}\epsilon - \epsilon\rho(A_\varphi)^N_N\,. \tag{3.4.19}$$

It can certainly be generalized to the case where more squark fields have angular dependences

$$\phi^a = m^a - \hat{n}^a\epsilon - i\Omega^m A^a_m\,, \tag{3.4.20}$$

where $\hat{n}^a$ is an integer valued vector, which is intended to count the number of flux quanta which flow through the vortex. We see that the above classical vacuum equation related the adjoint scalar and the gauge field.

Vortex BPS equations. While studying a 1/2-BPS object we work with the half of supersymmetry algebra which acts trivially on it. In the case at hand this algebra is generated by (3.2.14). Remarkably it coincides with the BPS subalgebra of the non-Abelian vortex considered by Shifman and Yung [45]. Thus even in the Omega deformed background in the NS limit, the vortex configuration we are considering in this section will remain 1/2-BPS.

Performing Bogomol'ny completion of the action (3.4.10) we get the following energy density

$$\begin{aligned}\mathcal{L} = \tfrac{1}{2g^2}\left|(B_3^a)^2 + g^2(\bar{q}\tau^a q - \Xi^a)\right|^2 + \tfrac{i}{g^2}\left|(\nabla_1 + i\nabla_2)\phi^a - (\Omega_2 - i\Omega_1)B_3^a\right|^2 \\ + |(\nabla_1 + i\nabla_2)q|^2 + N\,\xi_{FI}B_3^N + \tfrac{1}{g^2}\partial_m(\Omega^m\bar{\phi}^a - \bar{\Omega}^m\phi^a)B_3^a\,. \qquad (3.4.21)\end{aligned}$$

Here we assumed that the adjoint scalar and gauge field are only aligned along the Cartan subalgebra of the gauge Lie algebra. The last two terms in the second line of the above expression are total derivatives, but due to a different reason: the former is the Abelian field strength, which gives circulation of the gauge field after removing one integration, the latter involves ∂_φ derivative and is of the same kind as (3.2.16). We can see that the Lagrangian (3.4.10) under the constraint (3.4.20) and color-flavor locked condition $\bar{q}^i = \widetilde{q}^i$ takes almost exactly the same form as for the undeformed case considered by Shifman and Yung [45]. It means that the BPS construction for the vortex will also be almost exactly the same. The only difference is that adjoint scalar ϕ^a will have a nontrivial profile defined by the magnetic field and the Omega background. The corresponding BPS equations read

$$\begin{aligned}B_3^a + g^2(\bar{q}_i\tau^a q^i - \Xi^a) = 0\,, \\ (\nabla_1 + i\nabla_2)q^i = 0\,, \\ (\nabla_1 + i\nabla_2)\phi^a - (\Omega_2 - i\Omega_1)B_3^a = 0\,, \qquad (3.4.22)\end{aligned}$$

where, again as in (3.4.13), the color index $a = 1, \ldots, N^2$ runs through all $U(N)$ generators. For convenience we can split up $U(1)$ and $SU(N)$ parts and rewrite the first equation above using the definition of the generalized FI term (3.4.13)

$$\begin{aligned}B_3 + g^2(|q|^2 - \xi_{FI} - \xi_{nFI}^N) = 0\,, \\ B_3^a + g^2(\bar{q}_i\tau^a q^i - \xi_{nFI}^a) = 0\,, \qquad (3.4.23)\end{aligned}$$

where we have denoted

$$\xi^a_{nFI} = \frac{i}{g^2}\nabla_{\alpha\dot{\alpha}}(\bar{\Omega}^{\alpha\dot{\alpha}}\phi^a - \Omega^{\alpha\dot{\alpha}}\bar{\phi}^a) = \frac{1}{g^2}\partial_\varphi(\bar{\epsilon}\phi^a + \epsilon\bar{\phi}^a)\,, \quad a = 1, \ldots, N\,, \tag{3.4.24}$$

the non-Abeilan FI field. In its absence equations (3.4.23) and second equation in (3.4.22) exactly reproduce the BPS set considered in [45]; hence the solutions for the profile functions can be extracted from there directly. Thus the modification to the BPS vortex equations in the Omega background consist of introducing ξ^a_{nFI} (which for some configurations can vanish) and the nontrivial profile for the adjoint scalar dictated by the third equation in (3.4.22).

Asymptotic behavior of solutions. Let us for the moment assume that ϕ^a is invariant under rotations around the z-axis, in other words ξ^a_{nFI} vanishes. From the analysis of the previous section we conclude that it happens when ϕ does not depend on the azimuthal angle φ. Then, we know the solution for the magnetic field in all color directions, since it is exactly the same as in [45]. In particular, far away from the vortex, the gauge field exhibits $1/\rho$ behavior. This makes the quantization condition (3.4.20) physical and well defined. Indeed, it tells us that the adjoint scalar at large ρ approaches its vacuum value

$$\phi^a_{vac} = m^a - \epsilon(n^a + k^a)\,, \tag{3.4.25}$$

where k^a is a $\mathbb{Z}_N$-valued vector of winding numbers of along the different Cartan color directions. We still need to figure out what k^a is in terms of n^a. The reasoning for that comes from the following physical requirement – string tension (energy per unit length) should be finite. Indeed as in the undeformed case, the conclusion comes from the requirement that $|\nabla_m q|^2$ terms are finite.

$$\int_0^{+\infty} d\rho\, \rho\, |\nabla_m q^i|^2\,. \tag{3.4.26}$$

This was archived by a proper asymptotic behavior of the azimuthal component of gauge field A_φ such that the integrand above could decay fast enough. Indeed, let's say $q \sim e^{in\varphi}q(\rho)$, thus the integral becomes

$$\int_0^{+\infty} d\rho\, \frac{1}{\rho}\, |(in - iA_\varphi\rho)\, q^i|^2\,, \tag{3.4.27}$$

so $A_\varphi \to n/\rho$ at large ρ. In other words, A_φ should be proportional to the number of flux quanta which flow through the vortex. So, given (3.4.25) we can easily figure out that $k^a = -n^a$ and ϕ tends to its undeformed value m^a at large radial distances. Thus we conclude that the adjoint scalar interpolates between

$$\phi^a = m^a - n^a \epsilon \tag{3.4.28}$$

at $\rho = 0$ and

$$\phi^a = m^a \tag{3.4.29}$$

at $\rho = \infty$, see Fig. 3.3.

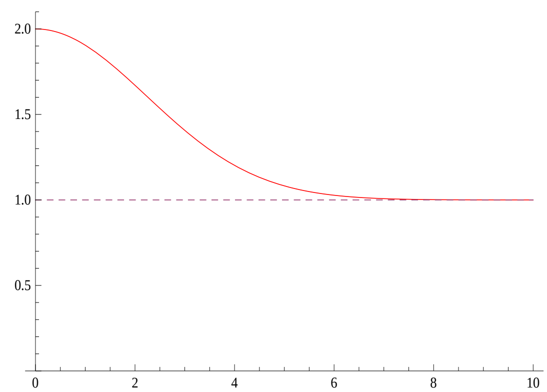

Figure 3.3: Scalar field interpolating between two different minimum values of the potential

Now we understand why in the left hand side of (3.4.5) involves the difference of the superpotential in two points – they are two different minima of the effective potential for ϕ in four-dimensional theory, and the vortex can now be viewed as a *kink* which interpolates between these minima! Evidently (3.4.5) is only applicable for nonzero ϵ. Note that the superpotential in the left hand side of (3.4.5) is evaluated at values of ϕ which are shifted by a unit of ϵ. As we shall explain below it happens because of an additional $U(1)$ twist of (anti)squark fields.

Let us mention that a completely different situation occurs if ϕ acquires nontrivial topology in the spirit of the previous section. Then BPS equations (3.4.22) do not decouple any longer. We expect a significant change in the asymptotic behavior of the solutions in that case. Investigations in this direction will be reported elsewhere.

3.4.3 Constructing non-Abelian vortices $N < N_f \leq 2N$

Let us now address semilocal vortices [54]. In order to be more generic we shall keep $\widetilde{N} = N_f - N$ generic inside the conformal window. The vacuum condition is generalized as follows

$$\phi = m\,, \quad q_a^i = \begin{cases} \sqrt{\xi_{FI}}\delta_a^i\,, & 1 < i < N \\ 0\,, & i \geq N \geq 0\,. \end{cases} \tag{3.4.30}$$

Symmetry breaking pattern is similar to (3.4.17), only there is a residual global symmetry left due to the additional quark fields

$$G_{c+f} = SU(N)_{c+f} \times SU(\widetilde{N})\,. \tag{3.4.31}$$

Most recent review of the semilocal vortex constructions can be found in [57]. The structure of the BPS equations (3.4.22) will not change, only the flavor index will now range $i = 1, \ldots, N + \widetilde{N}$. On the level of the low energy effective action the vortex theory will be modified by adding $\tilde{N}$ so-called *size* moduli. Kinetic terms of size moduli bring logarithmic divergence to the energy density of the effective theory, therefore one has to introduce an infrared cutoff.

Vortex moduli space. The moduli space of the simple vortex (when only one color-flavor direction has a single winding, say $q_N \sim e^{i\varphi}$) is given by the quotient

$$\mathcal{M} = \frac{U(1) \times SU(N)_{c+f}}{SU(N-1) \times U(1)} = U(1) \times \mathbb{P}^{N-1}\,, \tag{3.4.32}$$

is the well know complex projective space coupled to translations in the 12 plane. The situation becomes more complicated when multiple windings are allowed and in more color directions. These issues have beed addressed in [104]. Generally speaking the full metric on the moduli space is not known and is hard to compute. For the current Chapter we shall leave this problem behind and dwell on more convenient GLSM description of the effective 2d theory, which in turn was used in [83]. It is believed that the UV description of such a sigma model (GLSM) is the $U(K)$ gauge theory with K given in (3.4.2). This can easily been concluded from the brane construction of [83] as there are K D2 branes in total which coincide in 03 directions.

The effective worldsheet theory and the 4d/2d correspondence. Once a BPS string is identified, one can study effective dynamics on its worldsheet. It has now become a standard lore, we refer the reader to [99] where this procedure is elaborated in great details. Most importantly, the explicit derivation of the worldsheet effective theory provides a *proof* of the 4d/2d correspondence first outlined in [33], in the class of theories it was constructed, and enables us to match the parameters of the two theories.

Two dimensional degrees of freedom are constructed in [99] explicitly from the solutions of vortex BPS equations. Fermions are added to the action by analyzing zero modes, which are generated by nontrivial elements of SUSY algebra, in this case Q_{11}, Q_{22} and their conjugates. Such a prescription provided us with the full $(2,2)$ supersymmetric worldsheet effective action of the sigma model.

Given a solution of the BPS equation, like (3.4.22), one can make a global rotation using the isometry of the vortex moduli space. Each field present in the theory, say F as an $U(N)$ matrix transforms as

$$F \mapsto U^{-1} F\, U\,, \tag{3.4.33}$$

where $U \in G_{c+f}$ is an element of the isometry group. The 2d degrees of freedom we are looking for are contained in the matrix U, a proper ansatz can help us to extract them from it. For example, spatial components of the gauge field contributes with

$$(A_m)^i_j = n^{ia}\bar{n}^a_j\, \epsilon_{mn} x^m f(\rho)\,, \quad m,n = 1,2\,, \quad a = 1,\ldots,K\,, \tag{3.4.34}$$

where $f(\rho)$ is some yet to be determined profile function of the radial coordinate. In order to study the dynamics on the string worldsheet matrices n^{ia} are promoted to functions which depend on t, x_3 and after plugging the above formula and other fields of the theory in terms of n^{ia}s and unknown profile functions into the four-dimensional action one arrives to a two-dimensional $U(K)$ theory as a functional of the new fields n in (3.4.34). This procedure becomes quite elaborate for composite vortices, and a more generic moduli matrix approach [105] is used. It allows to conveniently package non-Abelian 2d fields in a moduli matrix and derive an effective 2d nonlinear sigma model (NLSM). Calculations along these lines have been presented in many sources, see [45] for review, for the latest complete calculations, which also applies to semilocal vortices, see [57].

We will not present the full derivation of the worldsheet effective theory, partly because the derivation has already been done in [57], however for without adjoint matter. Let us first assume that the rank of the gauge group (3.4.2) of the 2d theory is unity, so no adjoint multiplet appears. Then [57] provides a complete derivation of the worldsheet effective theory.

According to the 4d/2d duality quark masses of the 4d theory are mapped onto the twisted masses of the 2d theory. As we have already mentioned earlier, the authors of [33] have a slightly different Higgs branch condition than (3.4.28). Instead they used $\phi^a = m^a - n^a\epsilon$ (recall that $n^a = \hat{n}^a + 1$) which differs from (3.4.28) by a shift of ϕ by one unit of ϵ. Relations (3.4.28) and (3.4.29) are obtained when q and $\bar{\tilde{q}}$ have the same phase (3.4.14). However, this can be easily changed by choosing the phase $e^{i5\varphi/2}$ for the squarks and $e^{i\varphi/2}$ for the antisquarks. Having done so we can see that (3.4.3) formula is reproduced. From the field theory perspective the choice of phases seems to be random, and by tuning the phases of the (anti)fundamental fields appropriately one could change the Higgs branch condition. This, in turn, fixes the relationship between the 4d masses and 2d twisted masses. However, in the exact computations of [83, 100] involving Nekrasov partition function all pieces of data are important and the correspondence only takes place provided that all (3.4.1),(3.4.2),(3.4.3),(3.4.4) are satisfied. Therefore the instanton calculation only picks a single representative out of the continuous family of parameters. A better understanding of this fact is certainly necessary.

Now let us look at higher ranks of the 2d theory's gauge group, so more than one n_a's are turned on. As me mentioned earlier, it leads to having an adjoint field in the 2d Lagrangian. However, unless special efforts are taken, this field appears to be massless as no mass term appears from the flavor part of the action. The new ingredient of the theory in the Omega background is that the four-dimensional adjoint scalar is no longer frozen to its VEV, but represents a nontrivial background, and its fluctuations do contribute to the effective action. Indeed, $|\nabla_m\phi - \bar{\Omega}^n F_{mn}|^2$ in (3.4.10) provides some new data to the 2d theory. Fluctuations of ϕ and A_m will combine together to produce kinetic and massive terms. We refer the reader to [57] and its appendices, where the calculations we are discussing are reported in great details. We can merely take their calculations and generalize to non-Abelian matter content n^{ia}. Thus when

ϵ deformation is turned off, and only F^2_{mn} and squark terms are present, no explicit mass term is generated. However, in the deformed version of the theory the cross term in $|\nabla_m \phi - \bar{\Omega}^n F_{mn}|^2$ sources it. One can further speculate that the mass of n field is proportional to ϵ. The exact proportionality coefficient can be computed from the normalization integral, after the integrating along the transversal plane. We refrain from doing it here leaving it to further contributions, which could be done with more effective methods, like moduli matrix.

GLSM description. As usual, a GLSM description of the theory is more effective for computations. The 2d theory which is dual to the 4d SQCD in the NS Omega background with $N_f = N + \widetilde{N}$ quarks is given by the following Lagrangian provided that (3.4.1)-(3.4.4) hold

$$\mathcal{L} = \mathrm{Tr}\left[\int d^4\theta \left(\frac{1}{2e^2}|\Sigma|^2 + \bar{\Phi}\, e^V \Phi\, e^{-V}\right) + \int d^2\tilde{\theta}\, \left(\tau\Sigma + H.c.\right)\right] \\ + \int d^4\theta \left(\sum_{i=1}^{N} \bar{X}_i e^V\, X^i + \sum_{i=1}^{\widetilde{N}} \bar{Y}_i e^{-V}\, Y^i\right), \tag{3.4.35}$$

where the trace is taken over the adjoint representation of $U(K)$ gauge group, Φ is adjoint chiral multiplet, and Σ is field strength for 2d vector superfield V. The second line in the above Lagrangian represents the twisted F-terms of the theory. There are $N+\widetilde{N}+1$ twisted mass parameters turned on including $N+\widetilde{N}$ masses for X and Y fields together with the twisted mass for the adjoint scalar Φ, which according to [83, 100] equals to ϵ. In the limit $e \to \infty$ the gauge field becomes non dynamical, and we can integrate it out. In this limit we can recover the geometry of the NLSM's target space, which naturally appears in the derivation of the low energy theory.

In order to get the effective twisted superpotential in the right hand side of (3.4.5) we need to integrate out X's, Y's and Φ's in (3.4.35). When $N_f = 2N_c$ the theory is superconformal, the coupling does not run and no dynamical scale is generated.

$$\widetilde{\mathcal{W}}^{2d}_{\text{eff}}(\lambda) = \epsilon \sum_{a=1}^{K}\sum_{i=1}^{N} f\left(\frac{\lambda_a - M_i}{\epsilon}\right) - \epsilon \sum_{a=1}^{K}\sum_{i=1}^{N} f\left(\frac{\lambda_a - \widetilde{M}_i}{\epsilon}\right) \\ + \epsilon \sum_{a,b=1}^{K} f\left(\frac{\lambda_a - \lambda_b - \epsilon}{\epsilon}\right) + 2\pi i \hat{\tau} \sum_{a=1}^{K} \lambda_a\,, \tag{3.4.36}$$

where $f(x) = x(\log x - 1)$. Note the change of the coupling constant to $\hat{\tau}$ compared to (3.4.35). Minimizing the above superpotential we arrive to the ground state equations

$$\prod_{l=1}^{N} \frac{\lambda_j - M_l}{\lambda_j - \widetilde{M}_l} = e^{2\pi i \hat{\tau}} \prod_{k\neq j}^{K} \frac{\lambda_j - \lambda_k - \epsilon}{\lambda_j - \lambda_k + \epsilon}\,, \qquad (3.4.37)$$

which coincide with Bethe ansatz equations for the twisted anisotropic Heisenberg $SL(2,\mathbb{R})$ magnet. This observation quantifies the so-called *Bethe/gauge correspondence* for the $\mathcal{N} = 2$ SQCD.

Theories with $\widetilde{N} < N$ can be obtained from the conformal theory by sending some masses to infinity and renormalizing the coupling constant. Dynamically generated scale Λ_{QCD} will then appear.

A note on the $\mathcal{N} = 2^*$ theory. Recently a similar to DHL and CDHL study of the $\mathcal{N} = 2^*$ theory in five dimensions appeared in the literature [106], and a duality with a three dimensional integrable system was discussed. Although the calculations involving the Nekrasov partition function look very similar to [83, 100], there is a technical difference: the Higgs branch condition of the 5d theory looks similar to (3.4.1), however, there is no shift by N in the rank of the 3d gauge group like in (3.4.2). Clearly, is occurs because fundamental matter in [83, 100] and adjoint matter in [106] contribute differently to the Nekrasov partition function; nevertheless physical understanding of the second duality remains to be uncovered. Since there are no BPS vortices in $\mathcal{N} = 2^*$ theory, one cannot apply the method we used in the current section.

3.5 Brane Constructions and Dualities in Integrable Systems

Solutions to the Bethe ansatz equations mentioned above correspond to the ground states in the world volume theory on the non-abelian strings realized as D2 branes, hence it is desirable to translate the full powerful machinery of the integrable systems into the brane language. In this section we focus on the particular issue namely the realization of known dualities between quantum integrable systems using brane language.

We shall first review the Hanany-Witten type IIA brane construction which yields the $\mathcal{N} = 2$ SQCD and integrable systems related to it – the XXX spin chain and

Gaudin model together with the dualities these models are involved in. Employing the Gaudin/XXX duality we will be able to give a vortex interpretation of the AGT duality in the next section, where the XXX model appears on the $\mathcal{N}=2$ theory side and the Gaudin model naturally arises in study of Liouville CFT. Here we shall make some preparations to that study. In addition to that the Gaudin/XXX duality will be examined by studying the simplest examples of Argyres-Douglas type points and wall crossing phenomena in presence of the Omega background. At the end we shall discuss yet another duality between spin chains and Calogero-Moser systems.

3.5.1 Dualities from the Hanany-Witten brane construction

Brane configuration for the $\mathcal{N}=2$ SQCD employs the Hanany-Witten construction [60]. As it was shown in[83] and further explained here in Sec. 3.4, in presence of Omega background the Higgs branch condition gets deformed (3.4.1). Hence the positions of the flavor D4 branes are shifted by $n_a\epsilon$ for each color (see bottom picture in Fig. 3.4, 3.5). It contains two NS5, N D4 branes which are stretched between the two NS5 branes and two sets of semi infinite D4's which are attached to these NS5's. All D4 branes occupy 01236 directions, NS5's lie in 012345 directions.

	0	1	2	3	4	5	6	7	8	9
NS5	x	x	x	x	x	x				
D4	x	x	x	x			x			
D2	x			x				x		

Under geometric transition the brane configuration described in [83, 100] interpolate between the 4d theory and the 2d theory. The latter can be obtained by moving the right NS5 brane in the 7th direction and emerging D2 branes (037) which are stretched between this NS5 and D4's (see Fig. 3.5). The value of x_7 gives tension of D2 strings which is equal to ϵ in our construction.

The rank of the gauge group of the two-dimensional GLSM is given by summing up all the D2 branes $K=\sum_i \hat{n}_i$, where, we remind, $\hat{n}_i = n_i - 1$. The low energy dynamics of the two-dimensional theory is given by the effective twisted superpotential and the

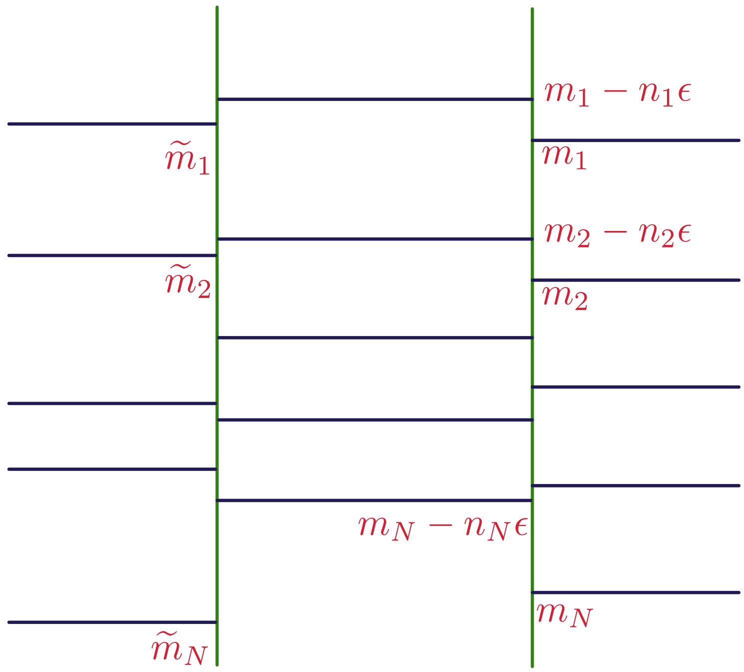

Figure 3.4: Type IIA brane picture. Positions of semi infinite D4 branes in 45 plane is given by $m_i - n_i\epsilon$, where $i = 1, \ldots, N$. In the deformed configuration the Higgs phase of the theory is given by the condition $a_i = m_i - n_i\epsilon$.

following ground state equations

$$\prod_{l=1}^{N} \frac{\lambda_j - M_l}{\lambda_j - \widetilde{M}_l} = q \prod_{k \neq j}^{K} \frac{\lambda_j - \lambda_k - \epsilon}{\lambda_j - \lambda_k + \epsilon}\,, \tag{3.5.1}$$

which is the Bethe ansatz equations for the anisotropic $SL(2)$ spin chain. Note that for generic 2d masses M_a and $\widetilde{M}_a$ at each spins at each site $a = 1, \ldots, N$ have different representations. Indeed, in order to match each term in the left hand side of (3.5.1) with phases of anisotropic chain

$$\prod_{l=1}^{N} \frac{\lambda_i - \theta_a + S_a\epsilon}{\lambda_i - \theta_a - S_a\epsilon}\,, \tag{3.5.2}$$

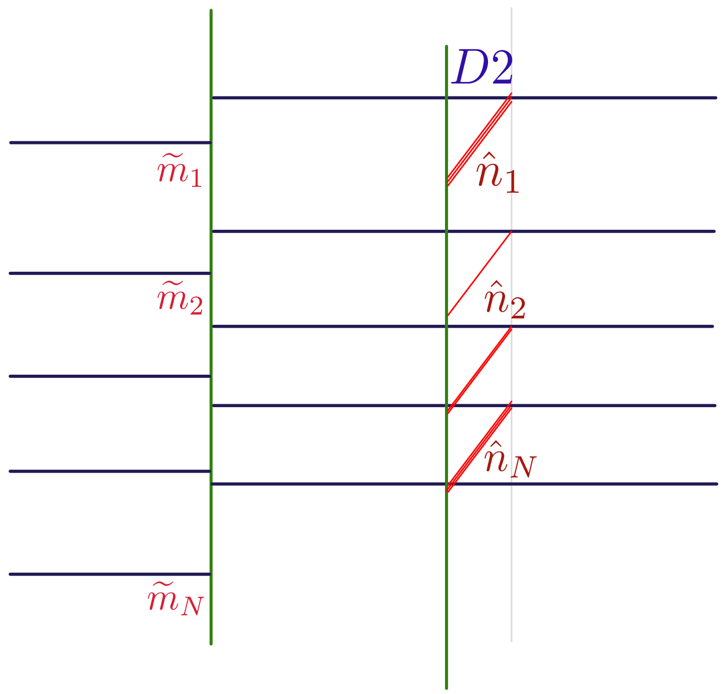

Figure 3.5: Vortex strings as D2 branes stretched in the 7th direction.

where ν_a are anisotropies and S_a are spins[5] , one identifies [81]

$$M_a = \theta_a - S_a\epsilon\,, \quad \widetilde{M}_a = \theta_a + S_a\epsilon\,. \tag{3.5.3}$$

3.5.2 The Gaudin/XXX duality

It is known that the Gaudin model [107] enjoys several dualities.[6] First we recall the duality introduced at the classical level in [108]. It relates the rational Gaudin model with SL(N) group at M sites and SL(M) group at N sites. The positions of marked points z_i on the sphere corresponding to the inhomogenities and the diagonal element

[5] For example, $S_a = -1/2$ gives the $SL(2)$ chain
[6] Some details about the Gaudin model are given in App. C.1.

of the twist matrix get interchanged. At the classical level the spectral curves and the action differentials are equivalent. At the quantum level the Bethe ansatz equations reflect this symmetry at the level of spectra.

Let us explain this symmetry in the brane picture. Let us first remind ourselves the similar symmetry in the Toda system discussed in [80]. It the Toda case this symmetry merely implies the equivalence of 2×2 and $N \times N$ Lax operator representations which can be explained as the 90 degrees rotation of the viewpoint of the brane picture. In the first representation the gauge group is connected to NS5 branes, while in the second case it is defined by the number of D4 branes in the IIA picture.

If we add the fundamental matter and consider the conformal case there are additional data which have to be matched via the duality. In the 2×2 representation the $SL(2)$ twist matrix emerges which reflects the positions of NS5 branes in the 6-10 plane Fig. 3.6.

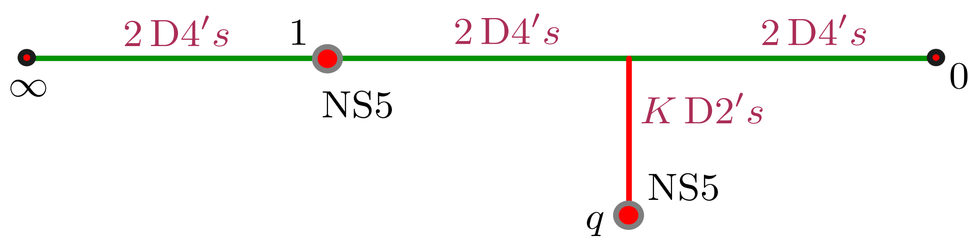

Figure 3.6: $(6 + i10, 7)$ section of the HW brane construction (view from "below").

The masses of the fundamentals provide the inhomogenities at the corresponding lattice sites. Upon the 90 degrees rotation similar to the Toda case the two sets of data get interchanged.

The duality between a pair of rational Gaudin models can be generalized to a similar duality between a trigonometric Gaudin model and a XXX spin chain via the so-called

$\mathfrak{gl}(M)/\mathfrak{gl}(N)$ duality [109]. For $M = N = 2$ Bethe ansatz equations read as follows[7]

$$\frac{\mathcal{M}_1 - \mathcal{M}_2 - \epsilon}{t_i} + \sum_{b=1}^{2} \frac{\nu_b \epsilon}{t_i - z_b} - \sum_{\substack{j=1\\ j\neq i}}^{\kappa_2} \frac{2\epsilon}{t_i - t_j} = 0, \quad i = 1, \ldots, \kappa_2\,, \tag{3.5.4}$$

for trigonometric Gaudin, and

$$\prod_{a=1}^{2} \frac{\lambda_i + \mathcal{M}_a}{\lambda_i + \mathcal{M}_a + \kappa_a \epsilon} = \frac{z_2}{z_1} \prod_{\substack{j=1\\ j\neq i}}^{\nu_2} \frac{\lambda_i - \lambda_j - \epsilon}{\lambda_i - \lambda_j + \epsilon}\,, \quad i = 1, \ldots, \nu_2\,, \tag{3.5.5}$$

for the $SL(2)$ XXX chain. The Mukhin-Tarasov-Varchenko (MTV) duality [109] states that (3.5.4) as set of equations with respect to $t_1, \ldots t_{\kappa_2}$ has isomorphic space of orbits of solutions with the one of (3.5.5) as set w.r.t. $\lambda_1, \ldots, \lambda_{\nu_2}$ provided that

$$\kappa_2 + \kappa_2 = \nu_1 + \nu_2\,. \tag{3.5.6}$$

Parameters $\mathcal{M}_{1,2}$ and $z_{1,2}$ are generic. We can now recognize (3.5.1) in (3.5.5) with

$$M_a = -\mathcal{M}_a, \quad \widetilde{M}_a = -\mathcal{M}_a - \kappa_a \epsilon, \quad K = \nu_2, \quad N = 2, \quad z_1 = 1, \quad z_2 = q\,, \tag{3.5.7}$$

and parameters $\kappa_{1,2}$ and ν_1 will be specified later. Also it will be more useful for us to use the 4d masses instead of the 2d ones. We can then rewrite set of MTV dual equations (3.5.4), (3.5.5) as follows

$$\frac{-m_1 + m_2 - \epsilon}{t_i} + \frac{\nu_1 \epsilon}{t_i - z_1} + \frac{K\epsilon}{t_i - z_2} = \sum_{\substack{j=1\\ j\neq i}}^{\kappa_2} \frac{2\epsilon}{t_i - t_j}\,,$$
$$\prod_{a=1}^{2} \frac{\lambda_i - m_a + \frac{3}{2}\epsilon}{\lambda_i - \widetilde{m}_a - \frac{1}{2}\epsilon} = \frac{z_2}{z_1} \prod_{\substack{j=1\\ j\neq i}}^{K} \frac{\lambda_i - \lambda_j - \epsilon}{\lambda_i - \lambda_j + \epsilon}\,. \tag{3.5.8}$$

Thus we can see that twists z_1, z_2, corresponding to the positions of the NS5 branes in 6-10 plane Fig. 3.6, and masses of the fundamentals m_1, m_2 interchange their roles upon the duality. We see that matching to the BAE corresponding to $U(2)$, $N_f = 4$ SQCD shows that the strange nonequal mass shifts to the fundamentals and antifundamentals (3.4.3) have now clear interpretation within the duality. Namely, the number of the

[7] We have adopted the notation and made some change of variable compared to [109].

Gaudin Bethe roots yields the asymmetry between the fundamental and antifundamental masses. Also Gaudin spins match with the number of Bethe roots at the XXX side. Later in the next section we shall use these spins in order to make the AGT duality manifest.

Let us emphasize that the Hamiltonian of the Gaudin model is nothing but the r.h.s. of the Knizhnik-Zamolodchikov (KZ) equation [110] on the sphere with $L+3$ marked points z_i [111]

$$b^2\frac{d\Psi(z_i)}{dz_i} = \mathcal{H}_{Gaud}\Psi(z_i)\,, \quad i=1,\dots,L\,, \tag{3.5.9}$$

where b is some constant. In the next section, when we will discuss Liouville theory on the same Riemann surface, we shall specify its value.

One could also introduce the so called dynamical operators with respect to boundary conditions [109]. Under the bispectral duality transformations the Gaudin KZ operator and the dynamical operators get interchanged as well. The number of marked points in the $N\times N$ representation of the Lax operator corresponds to the number of NS5 branes involved in the gauge theory brane construction.

3.5.3 Bispectral duality and Argyres-Douglas points

Classically, the bispectral duality just states that two systems have almost the same (differ by the simple factor) spectral curves and action differentials. At the quantum level the situation is more subtle. Since naively the bispectral duality connects the systems with different degrees of freedom, one should be able to analyze the phenomena of merging of two degrees of freedom into the single one. Below the simplest example shall be considered.

Let us now consider (3.5.4) again, this time we identify $\mathcal{M}_1 = -\mathcal{M}_2 = l\epsilon$. Then one has

$$\frac{2l-1}{t_i} + \frac{\nu_1}{t_i - z_1} + \frac{\nu_2}{t_i - z_2} - \sum_{\substack{b=1\\ b\neq a}}^{\kappa_2} \frac{2}{t_i - t_j} = 0, \quad i = 1,\dots,\kappa_2, \tag{3.5.10}$$

for the Gaudin model and

$$\frac{s_a - l - \epsilon}{s_a - l - \epsilon - \kappa_1\epsilon}\,\frac{s_a + l - \epsilon}{s_a + l - \epsilon - \kappa_2}\,\frac{z_1}{z_2}\prod_{\substack{b=1\\ b\neq a}}^{\nu_2}\frac{s_a - s_b - \epsilon}{s_a - s_b + \epsilon} = 1, \quad a = 1,\dots,\nu_2 \tag{3.5.11}$$

for the XXX model. Integers κ_a, ν_a satisfy the relation $\kappa_1 + \kappa_2 = \nu_1 + \nu_2$. One of the results in [109] is the precise correspondence of the orbits of solutions to the Bethe equations under the group of permutations of variables (permutations of $t_1, ..., t_{\kappa_2}$ for the Gaudin model and of $s_1, ..., s_{\nu_2}$ for the XXX model). At first glance such correspondence seems to be quite weak as it does not establish a direct connection between the roots of both systems and does not allow one to simplify one set of equations having known the solution of the other. But it preserves one important feature of the XXX model, namely the degeneration locus which could be called a little bit loosely "quantum Argyres-Douglas (AD) points" [112].

The "classical" Argyres-Douglas manifold is the locus in the moduli space of the theory where different vacua merge together. In the "quantum" case we have one more parameter from Omega deformation ϵ and the AD manifold involves this additional coordinate in the parameter space. The rest is the same and AD manifold corresponds to the coalessing vacua. In this subsection we shall normalize $\epsilon = 1$.

Since the solution to the BA equation correspond to the vacuum state "quantum" AD point corresponds to the appearance of multiple roots. We shall consider a simple case $\kappa_1 = \nu_2 = 2$, $\kappa_2 = \nu_1 = 1$ as an example and find the AD manifolds for both models. Bethe ansatz equations the XXX chain (3.5.11) then read

$$\begin{aligned} \frac{s_1 - l - 1}{s_1 - l - 3} \frac{s_1 + l - 1}{s_1 + l - 2} \frac{s_1 - s_2 - 1}{s_1 - s_2 + 1} &= q \,, \\ \frac{s_2 - l - 1}{s_2 - l - 3} \frac{s_2 + l - 1}{s_2 + l - 2} \frac{s_2 - s_1 - 1}{s_2 - s_1 + 1} &= q \,, \end{aligned} \tag{3.5.12}$$

and the Gaudin system is described by a single Bethe equation. In general, the XXX BAE contain a number of degenerate solutions, such that for some i, j the roots coincide $s_i = s_j$. The vacua of the theory correspond to the non-degenerate solutions of the system [83]. Obviously every solution to the degenerate system is a solution to the full BAE system. This property can be used to lower the degree of the Bethe equations. In our case the degenerate solution is $s_1 = s_2 = s$, so from (3.5.12) we obtain

$$\frac{s - l - 1}{s - l - 3} \frac{s + l - 1}{s + l - 2} = -q. \tag{3.5.13}$$

We can now solving the first equation of (3.5.12) with respect to s_2 and substitute this solution into the second equation to obtain a polynomial roots of which solve the XXX

BAE system. In order to eliminate the degenerate roots we merely need to divide this polynomial by (3.5.13). To find the AD manifold we calculate the discriminant of the reduced polynomial

$$\begin{aligned} D_1(l,q) = 4(1-q)^6 q^2 \left(4l^2q^2 - 8l^2q + 4l^2 - 4lq + 4l + 8q + 1\right)^2 \\ \left(4l^2q^4 - 32l^2q^3 + 56l^2q^2 - 32l^2q + 4l^2 + 4lq^4 - 36lq^3 \right. \\ \left. + 28lq^2 + 4lq + q^4 - 18q^3 + 17q^2 - 8q\right). \end{aligned} \tag{3.5.14}$$

However, the set $D_1 = 0$ still contains extra roots. Although the equation was divided by the degenerate one, the discriminant still captures the cases when the roots of the reduced system coincide with the roots of the degenerate system. In order to exclude such cases discriminant D_1 (3.5.14) must be divided by the resultant of the reduced and degenerate polynomials. This resultant turns out to be precisely the polynomial in the last parentheses in (3.5.14). Thus the resulting AD set is described by the zero locus of the following polynomial (we do not discuss trivial cases when $q = 0$ and $q = 1$)

$$D(l,q) = 4l^2q^2 - 8l^2q + 4l^2 - 4lq + 4l + 8q + 1\,, \tag{3.5.15}$$

which is precisely the discriminant of the Gaudin equation written in a polynomial form!

The above example describes the way to calculate the quantum AD set for the XXX BAE. First, we identify the degenerate subset and divide all equations by the corresponding polynomials. Then we find the discriminant of the reduced polynomial and divide it by all possible resultants with the degenerate polynomials. The same procedure can be done for the Gaudin system. The resulting polynomial describes the set of the quantum AD points and coincides for XXX and Gaudin systems. Certainly we have considered the simplest example and this issue deserves a separate study.

3.5.4 Walls of marginal stability

One more issue we shall briefly discuss concerns the interpretation of the walls of marginal stability in the Omega deformed theory in the language of the quantum integrable system. The wall can be described in terms of the superpotential however the Yang-Yang function in the quantum integrable system has the interpretation of the twisted superpotential as well. Hence we could formulate the problem of finding the walls of marginal stability in terms of the YY function for the spin chain or Bethe ansatz equations.

The ϵ-deformed theory has exact effective twisted superpotential (3.4.36). XXX Bethe ansatz equations (3.4.37) specify the positions of the vacua. Kinks in the two-dimensional theory which interpolate between these vacua possess two kinds of charges: Noether charges $M_l, \widetilde{M}_l$ or 2d twisted masses (3.5.3) and the topological charges which are given by the difference of the vacuum values of the superpotential (3.4.36) evaluated at the vacua the kink is hopping between. A wall of marginal stability is the solution to the following kinematical condition of the decay of the kink into its constituents

$$\mathfrak{Im}\frac{Z_{\text{top}}}{Z_{\text{Nöther}}} = \frac{\widetilde{\mathcal{W}}(\lambda^{(1)}_{vac}) - \widetilde{\mathcal{W}}(\lambda^{(2)}_{vac})}{M_1 - M_2} = 0\,. \tag{3.5.16}$$

We follow the procedure described in [113, 114]. The idea of the analysis is that the vacua and the vacuum values of the superpotential should be continuous functions of all parameters across the wall of marginal stability. We investigate the system depending on a single complex parameter θ and make all the deformation parameters to be equally spaced on a circle

$$\theta_l = \theta \exp\left(\frac{2\pi i l}{N}\right), \tag{3.5.17}$$

where θ_l are the spin chain impurities and contribute to 2d masses (3.5.3). Note that the above $\mathbb{Z}_L$ choice of θ's is a simplification as it certainly cannot be done for generic masses. However, as it argued in [113, 114], it is generic enough for the study of wall crossing phenomena in 2d.

We can now interpret the vacuum solutions to the BAE equations $\left\{ \lambda^{(i)}_{vac}\Big|_{\theta} \right\}$ not as separate functions, but rather as branches of some continuous function $\lambda_{vac}(\theta)$ on the complex plane. The number of branches ℓ is of course finite and coincide with the degree of the BAE system as a system of polynomial equations. The superpotential is the continuous function depending on $\lambda_{vac}(\theta)$. It has infinitely many branches but the sequence of branches has a certain periodicity of order ℓ.

The supersymmetric $\mathbb{P}^{N-1}$ theory considered in [113, 114, 115] has infinitely many walls of marginal stability. The vacuum as the function of the twisted mass parameter M_0

$$M_l = M_0 \exp\left(\frac{2\pi i l}{N}\right) \tag{3.5.18}$$

has N branches and the branches of the superpotential differ by the quantity proportional to m_0

$$\widetilde{\mathcal{W}}_{vac}^{(n)}(M_0) = \exp\left(\frac{2\pi i n}{N}\right)\left(\widetilde{\mathcal{W}}_{vac}^{(0)}(M_0) - 2\pi i n M_0\right). \tag{3.5.19}$$

The condition for the kink interpolating between nth and $(n+1)$st vacua to merge with a state of Nöther charge $m_l - m_0$ on the wall of marginal stability reads

$$\mathfrak{Re}\left(\frac{\widetilde{\mathcal{W}}_{vac}^{(0)}}{M_l} - \frac{2\pi i n \exp\left(\frac{2\pi i}{N}\right)}{\exp\left(\frac{2\pi i l}{N}\right) - 1}\right) = 0\,. \tag{3.5.20}$$

The above equation describes N walls which are roughly logarithmic spirals with infinitely many branches.

In the case of the Omega deformed theory the situation changes a little bit. In order to depict the walls of marginal stability graphically in general case, one needs to solve the system of BAE. We take the case $N = 3$, $K = 1$ in (3.5.1) as the simplest example. The BAE system is

$$\frac{(\lambda + S\epsilon)^3 - \theta^3}{(\lambda - S\epsilon)^3 - \theta^3} = q. \tag{3.5.21}$$

The solution has three branches. The difference between nth and $(n+3)$rd branches of the superpotential equals $4\pi i$. The walls of marginal stability are depicted in Fig. 3.7. In the limit $\epsilon \to 0$ the XXX system passes into the Gaudin system. The walls of marginal stability can be plotted for the Gaudin superpotential too. They possess the main features of the walls of marginal stability for the XXX system. The superpotential (3.4.36) at small ϵ becomes

$$\widetilde{\mathcal{W}}_{\text{Gaudin}}(\lambda_j) = G\sum_{j=1}^{K}\lambda_j - 2\sum_{j=1}^{K}\sum_{l=1}^{N} s_l \log\left(\lambda_j - \theta_l\right) - \sum_{j\neq k}^{N}\log\left(\lambda_j - \lambda_k\right)\,, \tag{3.5.22}$$

where G is external field. In the case $N = 3$, $K = 1$ the solution to the corresponding Gaudin equation has three branches. The difference between nth and $(n+3)$rd branches of the superpotential remains $4\pi i$ as in the XXX case. The corresponding walls of marginal stability are drawn in Fig. 3.8.

3.5.5 On the spin chain/Calogero duality

One may wonder if the bispectral duality between two different types of the spin chains has any relation to a similar duality discussed in the context of the Calogero–Ruijsenaars

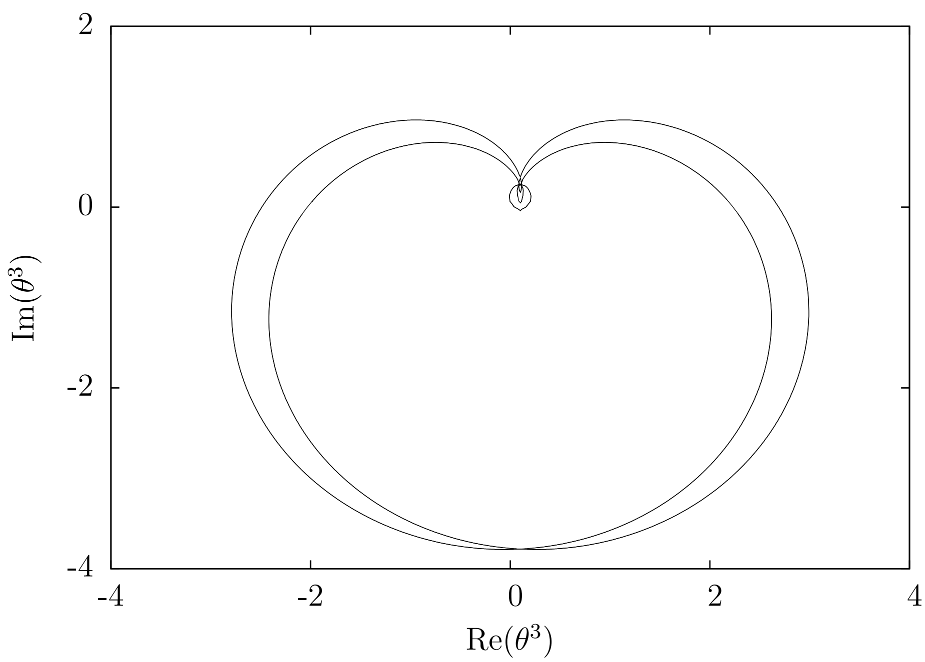

Figure 3.7: The walls of marginal stability for the XXX superpotential on the θ^3 plane. The parameters are: $q = -1$, $S = \frac{1}{2}$, $\epsilon = 1$. The AD point is at $\theta^3 = \frac{i}{4}$.

family of the integrable systems. The answer turns out to be positive. The prototype of this duality has been discovered in [116], where the correspondence between the zero locus of the Toda Hamiltonians in the phase space and the manifold associated with the quantum model has been discovered. This correspondence has been generalized for the rational Calogero model [117, 118, 119] at fixed coupling which turns out to be dual in the same sense to the rational Gaudin model [120].

The identification of the parameters goes as follows. The inhomogenities in the Gaudin model z_i are identified with the coordinates of Calogero model x_i, while the eigenvalues of the Gaudin Hamiltonians H_i are identified with the Calogero momenta. The inhomogenities are Poisson dual to the positions of the marked points, hence their identification with the Poisson pair in the Calogero model is natural. The duality above can be extended from the zero locus at the Calogero side to the arbitrary values of the

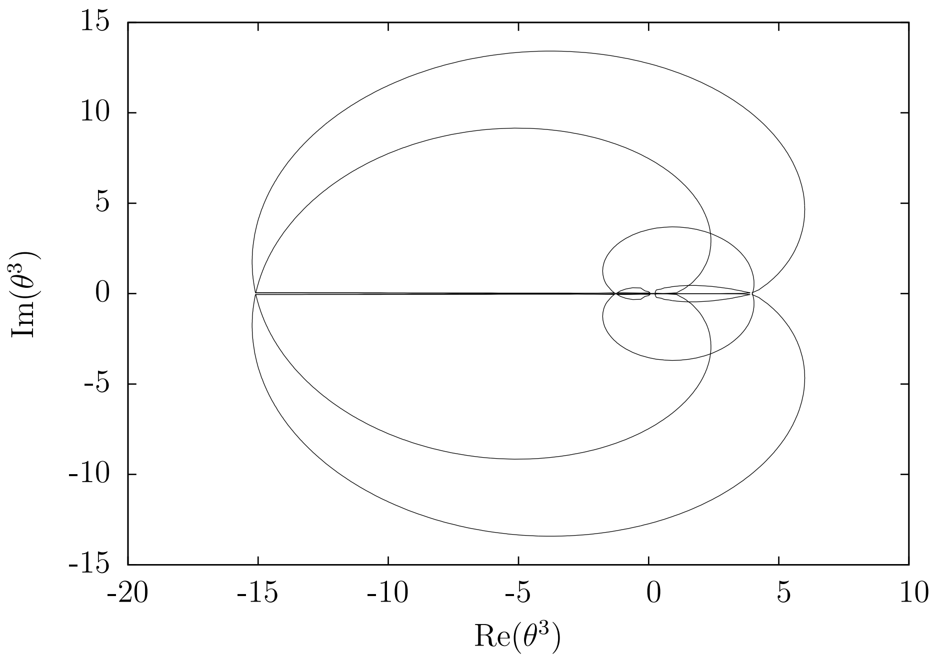

Figure 3.8: The walls of marginal stability for the Gaudin superpotential on the θ^3 plane. The parameters are: $G = 1$, $S = \frac{1}{2}$. The AD point is at $\theta^3 = 4$.

classical Calogero Hamiltonians [121]

$$\mathcal{H}_{\mathrm{Cal},k} = \sigma_k(\lambda)\,, \tag{3.5.23}$$

where σ_k is the k-th symmetric power of the Lax connection eigenvalues. It turns out that the Lax eigenvalues at the Calogero side are mapped onto the eigenvalues of the twist matrix at the spin chain side.

As we have discussed above the rational Gaudin model enjoys the marked points/twist duality and in some sense is selfdual. Its bispectral dual – the rational Calogero model is selfdual as well in the same sense. The bispectral duality can be generalized to the trigonometric and relativistic cases [122, 123, 124, 125]. Thus the trigonometric Calogero-Moser model is known to be dual to the rational Ruijsenaars-Schneider model [126, 124].[8] .The quantum version of this duality has been elaborated in [131]. Recall

[8] See also [127, 128, 129, 130]

that in Sec. 3.5.2 we discussed another bispectrally dual pair between the trigonometric Gaudin and the XXX models [132]. We can now see that the bispectrality at the Calogero–Ruijsenaars side matches perfectly with the duality at the Gaudin–XXX side. Note that this duality has the clear interpretation in terms of the Chern-Simons theory with inserted Wilson lines and its Yang-Mills degenerations [133]. The relationship between the two dualities is summarized in Fig. 3.9

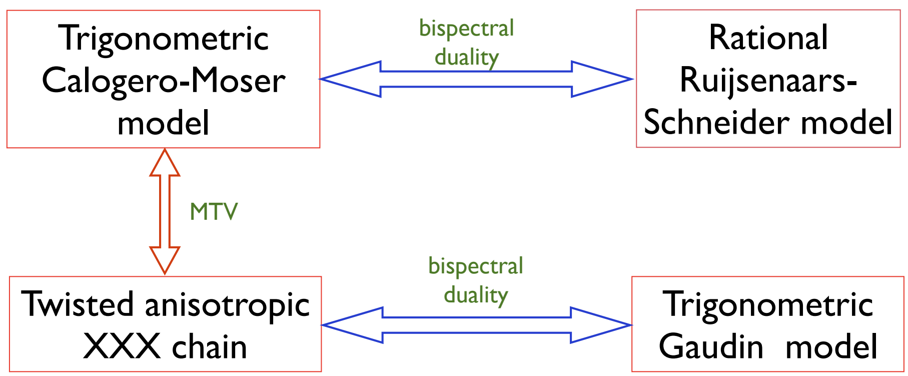

Figure 3.9: A pair of bispectral dualities mapped onto each other

It would be interesting to make the next step and consider the selfdual trigonometric Ruijsenaars model at the Calogero side of the correspondence. Its dual on the spin chain side is expected to be the XXZ chain which, according to the correspondence considered above, should enjoy some kind of bispectral selfduality. Another important issue concerns the generalization of these dualities to the elliptic integrable models. Not much is known about the self-dual elliptic model yet (see, however, [133, 134, 84, 135, 136])

3.6 The AGT Correspondence in the NS Limit

The Alday-Gaiotto-Tachikawa duality [87] relates a conformal block of the Liouville CFT on a Riemann surface of genus g and n punctures with the instanton part of the Nekrasov partition function of a quiver gauge theory naturally associated with this Riemann surface. Furthermore it concludes that the n-point correlator in the Liouville theory on this Riemann surface can be evaluated as an integral of the square of the absolute value of the full Nekrasov partition function.

Having done the above analysis on spectral duality between integrable systems, we arrive at an interesting observation which envisages the AGT correspondence for the Liouville theory with large central charge on S^2 with four punctures and $U(2)$ SQCD with four flavors. It has already been addressed in the literature earlier [137] where a Liouville conformal block at large c simply becomes a hypergeometric function of the conformal dimensions and the instanton number [138] was used. Then the authors figured out that only chiral terms in the Nekrasov partition function will contribute, either ϵ_1 or ϵ_2 have to vanishes; it enabled them to identify each multi-instanton contribution with corresponding terms of the hypergeometric function's expansion. The proof is rather formal and it will be desirable to have a more physical rationale to it. The current section is intended to fulfill this goal.

In order to see how the AGT relation comes about, in what follows we shall relate both the gauge theory and the Liouville theory to a pair of integrable systems which enjoy a certain duality between them. The roadmap we shall use to guide us through this section is presented in Fig. 3.10.

Starting from the 4d gauge theory on the top right of Fig. 3.10 we shall use the results of Sec. 3.4 and [100] in order to relate the 4d theory with the corresponding 2d GLSM. As it was shown in [100], the equivalence was established *to all orders in the instanton parameter*. Thus, instead of working with each summand of the instanton partition function, as all known proofs of the AGT [137, 139, 140] did so far, we shall look at the entire expression and the effective twisted superpotential (3.1.1) which follows from it. Following this line of thought we provide a physical rationale for the AGT correspondence in the NS limit, namely, modulo certain dualities between two given integrable systems, it is reduced to the 4d/2d duality in the NS limit [100]. The latter exists due to non-Abelian semilocal BPS vortices which we have discussed in Sec. 3.4, also vortices exist provided that the corresponding FI parameter is turned on, hence we uncover why an extra $U(1)$ factor on the gauge theory side is important.

The quantum duality, together with its brane interpretation, was discussed earlier in Sec. 3.5.2. Now we shall start with the left column of Fig. 3.10 by reminding ourselves how the Gaudin model is related to Liouville conformal blocks, and later on, by means

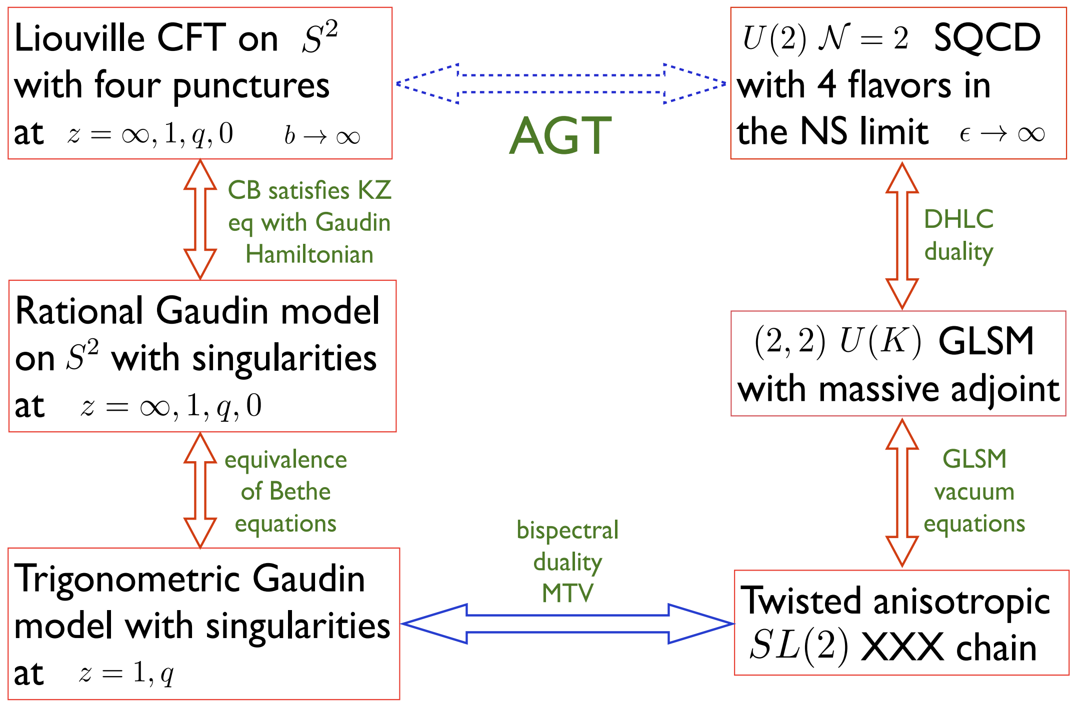

Figure 3.10: Roadmap of the AGT duality in the NS limit. The main statement (top horizontal line) is obtained by the chain of dualities between various integrable systems.

of the bispectral duality, we shall connect the story to the Heisenberg $SL(2)$ chain and to the 4d gauge theory.[9]

3.6.1 Liouville theory and rational Gaudin model

Recall that the Liouville theory has central charge

$$c = 1 + 6Q^2\,, \qquad Q = b + \frac{1}{b}\,, \tag{3.6.1}$$

and in the classical limit $b \to \infty$ so $Q \to \infty$ as well. Let us now consider a conformal block $\mathcal{F}_{\alpha_0}^{\mu_0}{}_{\alpha}^{\mu_1}{}_{\alpha_1}(q)$ of the Virasoro algebra with central charge $c \to \infty$ with the four

[9] Some remarks on the role of bispectral duality between the trigonometric Gaudin model and the XXX chain on the classical level can be found in [141, 142]

primary operators of dimensions

$$\Delta_1=\alpha_0(Q-\alpha_0)\,,\quad \Delta_2=\mu_0(Q-\mu_0)\,,\quad \Delta_3=\mu_1(Q-\mu_1)\,,\quad \Delta_4=\alpha_1(Q-\alpha_1)\,,\tag{3.6.2}$$

inserted at points $\infty, 1, q, 0$ respectively on the S^2 with an intermediate s-channel state of dimension $\Delta=\alpha(Q-\alpha)$. In the above formula

$$\alpha_0=\tfrac{1}{2}Q+\widetilde{\mu}_0\,,\quad \alpha=\tfrac{1}{2}Q+a\,,\quad \alpha_1=\tfrac{1}{2}Q+\widetilde{\mu}_1\,,\tag{3.6.3}$$

where a is the $SU(2)$ Coulomb branch coordinate. In the above formulae the mass parameters represent the following linear combinations of the SQCD quark masses $m_{1,2,3,4}$

$$\mu_0=\tfrac{1}{2}(m_1+m_2),\quad \widetilde{\mu}_0=\tfrac{1}{2}(m_1-m_2),\quad \mu_1=\tfrac{1}{2}(m_3+m_4),\quad \widetilde{\mu}_1=\tfrac{1}{2}(m_3-m_4)\,.\tag{3.6.4}$$

There is an obvious notational conflict with [87], where μ's and m's are interchanged compared to our work. We had to switch the notations in order to be consistent with Sec. 3.4, were m's are used for the quark masses. As far as the rest of the notations are concerned, they will be in agreement with [87]. Note that in Sec. 3.4 we treated all the four flavors as fundamental hypermultiplets, however, in [87] as well as in [83] two of them, with masses m_3 and m_4 are considered to be fundamental and two others, with masses m_1 and m_2 to be antifundamental. For the purposes of Sec. 3.4 this turned out to be a mild difference and we were able to relate the 4d and 2d theories by studying the vortex effective theory. Also from the GLSM perspective it was natural to distinguish fundamental and antifundamental fields. In this section we have to be more careful about this issue as contributions from the fundamental and anti-fundamental multiplets to the Nekrasov partition at finite ϵ are different.

Note that all conformal dimensions (3.6.2) diverge at least linearly with b, however, as we shall later see, in order to match the Liouville CFT with the four-dimensional theory in this limit, the dimensions will diverge quadratically and proper regularization is needed. Teschner in [143] have identified effective twisted superpotential (3.1.1)[10] with the NS limit of a Liouville conformal block on the sphere as well as the proper regularization of the conformal dimensions. Conformal block $\Psi(z_i)$ as the function of punctures' locations was found to satisfy the KZ equation (3.5.9) for the dual WZNW

[10] According to the NS dictionary this is also a Yang-Yang function

model with level k and $b^2 = -(k+2)^{-1}$

$$-\frac{1}{k+2}\frac{d\Psi(z_i)}{dz_i} = \mathcal{H}_{Gaud}\Psi(z_i)\,, \quad i = 1,\ldots,L\,, \tag{3.6.5}$$

where $\mathcal{H}_{\rm Gaud}$ is the Hamiltonian of the rational Gaudin model[11] . Thus the large b limit corresponds to taking $k \to -2$. The conformal dimensions of chiral primary operators get rescaled and become

$$\delta_i = -\frac{\Delta_i}{b^2}\,, \tag{3.6.6}$$

as $b \to \infty$. For S^2 with four punctures at $\infty, 1, q$ and 0 respectively from (??) and (3.6.17) we obtain

$$\begin{aligned}
\delta_1 &= \left(\frac{\widetilde{\mu}_0}{b} - \frac{1}{2}\right)\left(\frac{\widetilde{\mu}_0}{b} + \frac{1}{2}\right)\,,\\
\delta_2 &= \left(\frac{\mu_0}{b} - 1\right)\frac{\mu_0}{b}\,,\\
\delta_3 &= \left(\frac{\mu_1}{b} - 1\right)\frac{\mu_1}{b}\,,\\
\delta_4 &= \left(\frac{\widetilde{\mu}_1}{b} - \frac{1}{2}\right)\left(\frac{\widetilde{\mu}_1}{b} + \frac{1}{2}\right)\,,
\end{aligned} \tag{3.6.7}$$

as $b \to \infty$. Our next step is to allow the mass parameters μ_a and $\widetilde{\mu}_a$ scale with b upon identification with the 4d theory.

Equivalently one can also probe Liouville conformal blocks with surface operator insertions [32, 145], those conformal blocks satisfy Gaudin eigenvalue problem in the NS limit[12] .

3.6.2 $\mathcal{N} = 2$ SQCD in the NS Omega background

On the 4d gauge theory side, we compute the Nekrasov partition function for the 4d $\mathcal{N} = 2$ SQCD with mass parameters $\mu_0, \widetilde{\mu}_0, \mu_1, \widetilde{\mu}_1$ whose instanton part is

$$\mathcal{Z}_{\rm inst}(a, \mu_0, \widetilde{\mu}_0, \mu_1, \widetilde{\mu}_1) = (1-q)^{2\mu_0(Q-\mu_1)}\mathcal{F}_{\alpha_0\ \alpha\ \alpha_1}^{\ \mu_0\ \mu_1}(q)\,, \tag{3.6.8}$$

where $\alpha = \frac{1}{2}Q - a$ and a is the $SU(2)$ Coulomb modulus. For a generic Omega background the AGT dictionary says the deformation parameters are related to the 2d

[11] See also [144] where the matrix model approach to Hitchin systems in connection with the Liouville CFT was constructed.

[12] See also [146] where a systematic study of 4d gauge theories with surface operators in Omega-background was done.

Liouville theory via

$$b^2 = \frac{\epsilon_1}{\epsilon_2}\,, \quad \hbar^2 = \epsilon_1 \epsilon_2\,. \tag{3.6.9}$$

The usual NS limit $\epsilon_2 \to 0$ corresponds to $b \to \infty$ and ϵ_1 is kept fixed, then the Liouville theory becomes classical as $\hbar \to 0$.[13] In this section, we will be rather interested in the quantum regime of the Liouville theory so we shall allow $\epsilon_1 \to \infty$ such that $\hbar$ is kept fixed. It is clear that by a proper tuning of ϵ_1 and ϵ_2 one can obtain any desired value of the Planck constant.

As we have already discussed above, in the NS limit a more appropriate object to study is not the Nekrasov partition function but the effective twisted superpotential (3.1.1). As it was shown in [83] that this superpotential also emerges from the $(2,2)$ GLSM which we have described in Sec. 3.4.

The DHL paper has done a perturbative calculation in the instanton number q in order to establish their 4d/2d duality (3.4.5) and the proof to all orders was further established in [100]. CDHL showed that in the NS limit the Nekrasov partition function can be represented as an integral over a finite set of variables and can be explicitly evaluated, and the saddle point condition is shown to be equivalent to the Bethe ansatz equations for the $SL(2)$ XXX chain. One may ask immediately why the vortices are relevant, indeed they only exist in a Higgs branch of the four-dimensional theory, whereas the AGT statement relates Liouville momenta with Coulomb branch coordinates. In order to understand this, let us recall that at zero value of the FI term the Higgs branch touches the Coulomb branch, and as it was pointed out in [83], by making a proper limit in the relation[14]

$$a_a = m_{2+a} - n_a \epsilon\,, \quad a = 1, 2\,, \tag{3.6.10}$$

one may recover *any* point of the Coulomb branch of the $U(2)$ SQCD. Indeed, as $\epsilon \to 0$ the Higgs lattice becomes more and more dense filling the Coulomb branch in that limit. However, for what we are doing here, the opposite $\epsilon \to \infty$ limit is relevant, as it is required by the connection to the Liouville theory. Still we want to be able to cover any point on the Coulomb branch, so one has to scale the fundamental masses m_a with ϵ as well in order to keep combination (3.6.10) finite. So at any given Liouville momentum we only need to sit at a certain point on a Coulomb branch and the Higgs branch root

[13] Note that $\hbar$ is not a Planck constant per se, rather it signifies that $\mathcal{Z} \sim e^{-H/\hbar}$.

[14] From now on we shall work with the $U(2)$ SQCD with 4 flavors.

has all information we need about that point. Recall that the anti-fundamental masses and, correspondingly μ_0 and $\widetilde{\mu}_0$ are not affected by (3.6.10) and therefore do not scale with ϵ.

We now make an observation that the ground state equations for the $(2,2)$ GLSM (3.5.1) (or second equation in (3.5.8) where $\widetilde{m}_{1,2}$ are now denoted as $m_{1,2}$ (antifundamental) and $m_{1,2}$ became $m_{3,4}$ (fundamental) respectively[15] .)

$$\prod_{a=1}^{2} \frac{\lambda_i - m_{2+a} + \frac{3}{2}\epsilon}{\lambda_i - m_a - \frac{1}{2}\epsilon} = q \prod_{\substack{j=1 \\ j\neq i}}^{K} \frac{\lambda_i - \lambda_j - \epsilon}{\lambda_i - \lambda_j + \epsilon}\,, \tag{3.6.11}$$

can be written as the second equation from the MTV dual pair (3.5.8). In order to see this we need to employ (3.6.10) and substitute m_3 and m_4 into the numerators of the let hand side of (3.6.11). Then we take the limit of large ϵ keeping in mind that rapidities λ_i also scale with ϵ. Neither Coulomb moduli a_a nor the antifundamental masses $m_{1,2}$ enjoy this scaling, so they will drop out from the equations. We then arrive to (3.5.8) where $z_2/z_1 = q$ and

$$n_a = \kappa_a + 2\,. \tag{3.6.12}$$

3.6.3 The duality

Now let us start connecting the story with the Liouville. By means of the bispectral duality these equations are mapped onto (3.5.4) yielding the trigonometric Gaudin model from the Heisenberg chain. Note that (3.5.4) depends only on two points z_1 and z_2 corresponding to the locations of the NS5 branes in 6-10 plane in Fig. 3.4. However, the Liouville conformal block depends on four operators sitting at $\infty, 1, q, 0$. Interestingly we can mention that trigonometric Gaudin Bethe equations (3.5.4) when only z_1 and z_2 punctures are involved can be treated as a *rational* $\mathfrak{sl}(2)$ Gaudin Bethe equations on S^2 with all four punctures included. Indeed,

$$\sum_{b=0}^{3} \frac{\nu_b \epsilon}{t_i - z_b} - \sum_{\substack{j=1 \\ j\neq i}}^{\kappa_2} \frac{2\epsilon}{t_i - t_j} = 0\,, \tag{3.6.13}$$

[15] The obvious notational conflict occurs here. In spite of this we still keep the notations of this section and Sec. 3.4 as they are since both are natural in where they stand. We hope this issue will not confuse the reader.

where $z_{0,1,2,3} = \{\infty, 1, q, 0\}$, is equivalent to (3.5.4) with

$$\epsilon\nu_2 = K, \quad \epsilon\nu_3 = m_3 - m_4 - \epsilon = 2\widetilde{\mu}_1 - \epsilon\,, \tag{3.6.14}$$

being spins of the $\mathfrak{sl}(2)$ representations sitting at points q and 0. Specification of ν_0 is not important, as the corresponding contribution drops out from the equation since $z_0 = \infty$. Also, as we have already mentioned in (3.6.12) there is an exact matching between the number of the Gaudin Bethe roots with the parameters n^a of the Higgs branch. In other words, all sectors of the trigonometric Gaudin model Hilbert space parameterized by number of Bethe roots (excitations over the Bethe vacuum), by means of the bispectral duality, are mapped onto various points of the Higgs branch lattice $\{n_a\}$ of the four-dimensional theory. Finally, the value of ν_1 can be found from (3.5.6) and (3.6.12). We conclude that $\nu_1 = -2$, which formally corresponds to the spin -1 representation for $z_1 = 1$

Note that one should also take out the $U(1)$ factor from the $U(2)$ gauge group, as it does not have an analogue in the Liouville theory. Imposing it on the $U(2)$ Coulomb moduli a_1, a_2 with the help of (3.6.10) we get

$$m_3 + m_4 - (n_1 + n_2)\epsilon = 0\,, \tag{3.6.15}$$

or, using the Liouville mass parameters, one gets

$$\frac{\mu_1}{\epsilon} = \frac{n_1 + n_2}{2}\,. \tag{3.6.16}$$

The $U(1)$ condition balances the count of the parameters on both sides of the correspondence as in order to match $\mathfrak{sl}(2)$ spin at $z_4 = 0$ we used only one antifundamental mass parameter (which is related to the fundamental one).

Again, from the gauge theory perspective we are interested in keeping Coulomb branch parameters in (3.6.10) finite while masses μ_0 and μ_1 and ϵ are sent to infinity. The rescaled conformal dimensions (3.6.7) upon identification $b = \epsilon$ and by using (3.6.16) therefore read

$$\delta_0 = -\frac{1}{4}\,, \quad \delta_1 = 0\,, \quad \delta_2 = \frac{K}{2}\left(\frac{K}{2} + 1\right)\,, \quad \delta_3 = \gamma_0(\gamma_0 + 1)\,, \tag{3.6.17}$$

where

$$\gamma_0 = -\frac{\hat{n}_1}{2} + \frac{\hat{n}_2}{2} - \frac{1}{2} = \frac{\nu_3}{2}\,, \tag{3.6.18}$$

and we recall $K = \hat{n}_1 + \hat{n}_2$ is the number of the D2 branes stretched between the NS5 brane at $z_2 = q$ and the D4 branes. We can also see that in such limit the $SU(2)$ Coulomb coordinate and the anti-fundamental masses have dropped from the formulae. In the last two terms of (3.6.17) we recognize $\mathfrak{sl}(2)$ quadratic Casimir eigenvalues on representations of spins $\frac{1}{2}K$ and $\frac{\nu_3}{2}$ respectively. We can see from (3.6.14) and (3.6.17) that the matching occurs at these points. Vanishing eigenvalues δ_1 confirms the fact that $\nu_1 = -2$ corresponds to the spin -1 representation. Spin of the representation at $z_0 = \infty$ is formally equal to $-\frac{1}{2}$.

Here is the summary table of the correspondence between the objects we have discussed in this section in addition to the standard AGT dictionary

Liouville conformal block at $b \to \infty$ on S^2 with four punctures	$U(2)\,, N_f = 4$ SQCD instanton partition function in the NS limit
Rational Gaudin model from KZ equation on conformal blocks	$SL(2)$ spin chain from the ground state equation for the 2d GLSM dual to 4d theory
Punctures' positions z_2/z_1	Instanton number q
$\mathfrak{sl}(2)$ spin at $z_2 = q$	$U(1)$ condition, number of D2 branes emerging at NS5 brane at $z_2 = q$
Conformal dimensions of chiral operators at points $z_2 = q$, $z_3 = 0$	Quadratic $\mathfrak{sl}(2)$ Casimir eigenvalues on spin $\frac{1}{2}\hat{n}_1 + \frac{1}{2}\hat{n}_2$ and $-\frac{1}{2}\hat{n}_1 + \frac{1}{2}\hat{n}_2 - \frac{1}{2}$ representations
Gaudin Hilbert space sectors with different number κ_a of Bethe roots	Higgs branch lattice $\{n_a\}$

3.6.4 Generalization to $SU(2)$ linear quivers

One can easily generalize the above construction to the Liouville theory on S^2 with $L+3$ punctures. A natural quiver gauge theory associated to this Riemann surface has L $SU(2)$ gauge nodes with Coulomb moduli a_i successively connected together. Liouville conformal block of $L+3$ operators located at points

$$\infty,\, 1,\, q_1,\, q_1 q_2,\, \ldots,\, q_1 q_2 \ldots q_L,\, 0\,, \tag{3.6.19}$$

with the following scaling dimensions

$$\alpha_0(Q-\alpha_0)\,, \quad \mu_0(Q-\mu_0)\,,\ldots, \quad m_L(Q-m_L)\,, \quad \alpha_{L+1}(Q-\alpha_{L+1})\,, \tag{3.6.20}$$

respectively glued by operators of dimensions $\alpha_i(Q-\alpha_i)$ in the intermediate s-channels.

NS limit of such quiver theory has been elaborated in [100]. Brane interpretation of the bispectral duality is very useful in this case. The corresponding Hanany-Witten picture view from "below" (in $(6+i10)-7$ space) is shown in Fig.3.11. Quiver

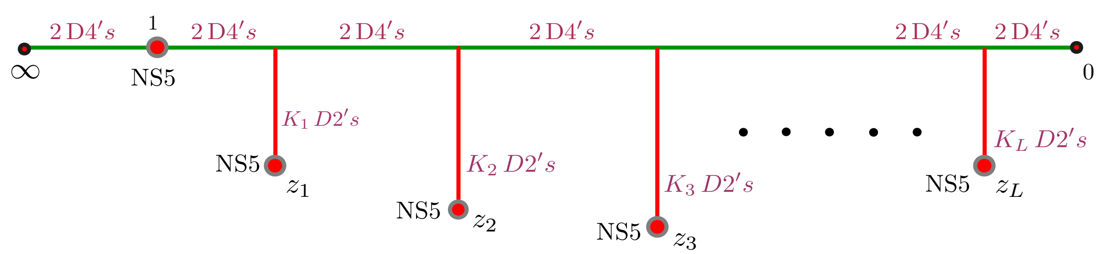

Figure 3.11: $(6+i10)-7$ slice of the CDHL quiver construction. Vertical lines correspond to stacks of $D2$ branes.

theories have bifundamental matter with masses $\mu_k^{(p)}$, so the Higgs branch conditions get changed

$$a_a^{(p)} = m_a^{(p)} + n_a^{(p)}\epsilon + \sum_{k=1}^{L}\mu_k^{(p)}\,. \tag{3.6.21}$$

Using the above relation we can express conformal dimensions (3.6.20) in terms of Coulomb branch coordinates, quantization parameters and bifundamental masses. Performing the rescaling analogous to (3.6.7) we conclude that operators located at

$$z_1 = q_1, \quad z_2 = q_1q_2, \quad z_3 = q_1q_2q_3, \quad \ldots, \quad z_L = q_1\ldots q_L \tag{3.6.22}$$

have dimensions

$$\frac{K_1}{2}\left(\frac{K_1}{2}+1\right), \quad \ldots, \quad \frac{K_L}{2}\left(\frac{K_L}{2}+1\right)\,, \tag{3.6.23}$$

where $K_i = \hat{n}_1^{(i)} + \hat{n}_2^{(i)}$, $i=1,\ldots,L$, corresponding to the $\mathfrak{sl}(2)$ Casimir eigenvalues on representations of spins $\frac{1}{2}K_1\,\ldots,\frac{1}{2}K_L$. Spins sitting at each point z_i (multiplied by 2) correspond to the total number of D2 branes stretched between the i'th NS5 brane and both D4 branes. Our construction has to be supplemented by L $U(1)$ conditions similar to (3.6.16) for each gauge group.

The full treatment of the AGT duality for linear quivers requires the construction of the bispectral dual to the twisted anisotropic $SL(L,\mathbb{R})$ chain, which emerges from it [100]. We shall postpone this analysis for the future work.

3.7 Conclusions

In this Chapter we have investigated BPS solitons – strings, monopoles and domain walls in $\mathcal{N} = 2$ four-dimensional gauge theories in Omega background with the Nekrasov-Shatashvili limit imposed. We derived the central charges for these solitons from the supersymmetry algebra and observed that string and domain wall charges are proportional to the external graviphoton field present in Omega background. At large values of ϵ, string and domain wall tensions are large which makes semiclassical considerations legitimate. Existence of a BPS string in pure SYM implies fractional windings for the adjoint scalar and hence presence of a conical singularity. We have presented arguments that its tension is nevertheless finite. Moreover we have argued that BPS monopoles found in [86] are actually located on domain walls interpolating between two different vacua of the theory. The next step along this road will be to analyze the complete set of the moduli space of the corresponding solutions and worldvolume theories of BPS solitons. Also one may expect more surprises in study of wall crossing phenomena for such BPS objects. We postpone the discussion on these issues for a separate publication.

There is an interesting question regarding the behavior of solitonic BPS states at small ϵ. In the pure SYM strings and domain walls become highly quantum objects and could potentially condense. However it is not clear if such condensation of the extended defects should be taken into account. In SQCD at large value of the FI term in the large ϵ limit the standard nonabelian string is recovered. Note that small ϵ limit corresponds to semiclassical regime from the integrability viewpoint, hence the potential condensation of BPS solitons also deserves further investigation in the Hamilton-Jacobi framework.

An interplay between quantum integrability and Omega deformed gauged theories allowed us to apply some known dualities in the integrable systems literature. Degrees of freedom in an integrable systems are interpreted as coordinates of the corresponding brane positions since all dualities can be reformulated in brane language. The integrability being just the reflection of the symmetry of the brane geometry involved plays a role of a consistency condition for the whole construction. We have shown that different dualities connecting Gaudin, XXX and Calogero type models can be explained in terms of brane geometry both at classical and at quantum levels.

As a byproduct of the vortex construction and the dualities between integrable systems we were able to reconstruct the AGT correspondence in the NS limit. The Liouville CFT has infinite central charge and all scaling dimensions needed to be regularized. In this Chapter we discussed the Liouville theory on S^2 with $L+3$ punctures, which was shown to be dual to linear quiver gauge theories with corresponding matter content. Our construction certainly has to be extended to other known AGT dual pairs, e.g. a torus with multiple punctures, etc.

There are many questions which certainly deserve additional study. Surely, more complicated defects involving strings, domain walls and monopoles have to be explored. It would be interesting to investigate similar defects in five and six dimensions and with the complete Omega deformation beyond the NS limit.

Chapter 4

Heterotic Sigma Models

4.1 Introduction

For many years two-dimensional $\mathbb{CP}^{N-1}$ sigma-models have been providing extremely useful insights into physics of four-dimensional non-Abelian gauge theories. One of the most important features the two types of theories share is the non-perturbative generation of a mass gap [58, 63]. The connection has been tightened up thanks to recent results in gauge theories with extended supersymmetry. First it has been proven that the $\mathcal{N} = (2,2)$ extension of the two-dimensional $\mathbb{CP}^{N-1}$ sigma-model has the same spectrum of massive BPS states as the $\mathcal{N} = 2$ four-dimensional $SU(N)$ gauge theory with N hypermultiplets, provided that the parameters of the two theories are identified in a proper way [40, 33]. Remarkably, the correspondence holds at both the classical and quantum levels. The physical reason behind that was unclear until it was realized that the correct two-dimensional model arises naturally as an effective theory on string-like solitons existing in the four-dimensional bulk theory. The key point was the discovery of non-Abelian vortices [44, 43, 147, 148], which posses internal degrees of freedom with non-trivial dynamics. The fluctuations of the fields around the vortex configuration can be thought of as the original particles confined to the world-sheet of the vortex, due to the Higgs screening [35, 34].

In an attempt to further study the relationship between the theories (in a set-up which may be closer to the real QCD) one introduces mass terms which decouple the adjoint scalar fields [149] present in $\mathcal{N} = 2$ theories. Having done that, one breaks supersymmetry down to $\mathcal{N} = 1$. SUSY breaking terms correspond to a very interesting deformation of the vortex world-sheet theory which gives rise to a particular type of $\mathcal{N} = (0, 2)$ $\mathbb{CP}^{N-1}$ sigma-model called "heterotic" [150, 151, 152].

Two-dimensional sigma-models with $(0, 2)$ supersymmetry have been considered some time ago [61, 153][1] . More recently, the new interest in the heterotic $\mathbb{CP}^N$ sigma model was induced by the fact that this model naturally emerges as an effective theory of the moduli on the world-sheet of the non-Abelian flux tubes [147] which are present in four dimensional SU(N) Yang-Mills theories [44, 35, 34, 43, 148](for a review see [45, 48, 46]). The non-Abelian vortices for arbitrary gauge group in particular for SO(N) and USp(N) were constructed in [155, 156], while the generalization for the higher winding numbers was done in [104]. Edalati and Tong suggested [150] that the world sheet theory is a heterotic $\mathcal{N} = (0, 2)$ theory. It was proven to be correct in the papers [151, 157], where the heterotic model was explicitly obtained from the $\mathcal{N} = 2$ SYM bulk theory deformed by the mass terms of the adjoint fields. Since then the different aspects of the $\mathbb{CP}^N$ sigma model were considered in great details [63, 158, 152, 159, 160].

Historically Polyakov introduced the O(N) bosonic sigma model and showed that it is asymptotically free [42]. This model was solved exactly for $N = 3$ in Zamolodchikov's paper [161]. The supersymmetric generalization of the O(3) sigma model was constructed by Witten in [162]. He also developed a technique for solving the $\mathbb{CP}^{N-1}$ sigma model at large N for both supersymmetric and non-supersymmetric theories [58]. The common feature of the sigma models (without a mass), apart from being asymptotically free, is that due to quantum effects the spontaneous symmetry breaking disappears, and the symmetry gets restored. This fact is reflected by the absence of massless (not sterile) particles in the theory.

We would like to start this Chapter with the study of heterotic deformation of a supersymmetric O(N) sigma model before going into the complex projective spaces. The undeformed theory contains N scalar and N spinor real-valued fields. The bosonic fields are confined on a $(N-1)$-sphere. For generic values of N only $(1, 1)$ supersymmetry is present in the model, while for $N = 3$ the model is equivalent to the supersymmetric $\mathbb{CP}^1$ sigma model, which actually possesses $\mathcal{N} = (2, 2)$ supersymmetry.[2] The O(3) sigma

[1] More references about study of superspaces, renormalization of heterotic sigma models can be found in [154]

[2] Due to the Kähler structure of the target manifold, which is S^2 in this case, the extra supercurrent

model is also distinguished among the other O(N)s due to the presence of instantons, which are absent for $N \neq 3$.

Although, to our knowledge, no bulk theory has been constructed for the heterotic O(N) supersymmetric sigma model [3] – it is interesting in its own right.

Given these obvious differences between SU(N) and O(N) models, one may expect to see different physical properties in the large N limit. In the current Chapter we find the spectrum of both $(1,1)$ and $(0,1)$ supersymmetric O(N) sigma models and observe that it is very much reminiscent of the SU(N) models. The only major difference between the two models is the number of vacua – it is always two in the orthogonal case and N in the unitary case (for the $\mathbb{CP}^{N-1}$ model).

The heterotic $\mathbb{CP}^{N-1}$ sigma-model was first analyzed in [164], then it was solved in the large-N approximation in [158, 159]. The model shows a rich set of phenomena like spontaneous supersymmetry breaking and transitions between Higgs and Coulomb/confining phases. Again, the two-dimensional sigma-models have proven to capture important properties of the corresponding four-dimensional bulk theories [164, 165].

We shall investigate a particular extension of the $\mathbb{CP}^{N-1}$ sigma-model which can be obtained by gauging N positively charged fields. Considering additional $\widetilde{N} = N_F - N$ matter multiplets with negative charge, we obtain what is called a “weighted” $\mathbb{CP}^{N_F-1}$ (or $\mathbb{WCP}^{N_F-1}$) sigma-model.[4] The target space $\mathbb{WCP}^{N_F-1}$ contains $\mathbb{CP}^{N-1}$ as a subspace. The crucial point is that the weighted projective space is not compact. The model was proposed in [44] as the low-energy description of non-Abelian semi-local vortices. Semi-local vortices appear in gauge theories when large global symmetries are present [52]. These symmetries are usually realized as flavor symmetries by introducing additional matter fields. The main feature of these vortices is the existence of a new set of degenerate solutions with arbitrary size [53, 54, 166]. In fact, this property makes semi-local vortices quite similar to instantons and lumps [50, 56, 167, 168]. Employing

emerges lifting $(1,1)$ supersymmetry up to $(2,2)$.

[3] However, there is an example [163] when the O(3) sigma model emerges as an effective theory on the world-sheet of the string in the $\mathcal{N} = 1^*$ supersymmetric SU(2) gauge theory. But as we know the $N = 3$ case is special.

[4] The notation is borrowed from a previous work of one of the authors [56], where it was used in connection with the moduli space of semi-local vortices. ln the context of algebraic geometry, where these spaces are well studied, they are more correctly referred to as $\mathcal{O}(-1)^{\widetilde{N}}$ line bundles over $\mathbb{CP}^{N-1}$.

a D-brane construction, the authors of [150] found the unique heterotic deformation to this model which could arise when the symmetry breaking term in the bulk theory is turned on. Motivated by this, we use the large-N techniques exploited in [58, 158, 159] to solve the model and understand its physics.

In the correspondence between the supersymmetric QCD and the $\mathcal{N} = (2,2)$ sigma-model mentioned above, the complex masses of the hypermultiplets in the former theory coincide with the *twisted* masses of the latter theory. In the model addressed in the current Chapter, we introduce N twisted masses m_i for each positively charged field and additional $\widetilde{N}$ twisted masses μ_j for each negatively charged field. As is known from the $\mathbb{CP}^{N-1}$ sigma-model, the values of the twisted masses control the phases of the theory. Indeed, if the masses are much bigger than the dynamically generated scale Λ, the theory is essentially classical, whereas quantum effects become significant for $m_i \lesssim \Lambda$. Due to larger variety of twisted masses, the phase diagram of the theory is quite complicated. We shall consider a particular choice of the masses which preserves a discrete symmetry, by appropriately putting them on two circles of radii m and μ. We thus focus on the determination of the phase diagram of the model in terms of these two parameters.

The supersymmetric $\mathbb{CP}^{N-1}$ sigma-model is known to have an exact "twisted" superpotential [169, 62, 40] which is similar to the Veneziano–Yankielowicz superpotential [170]. It can be straightforwardly generalized to the weighted sigma-model [62, 59, 33]. The exact superpotential depends only on the twisted chiral superfield containing the gauge multiplet and twisted masses of the theory. Once the superpotential is known one can in principle determine the full BPS spectrum, including the vacua of the theory for any N and $\tilde{N}$. However, if we break half of the supersymmetries by introducing the heterotic deformation, we cannot rely on the existence of an exact superpotential anymore, and we have to dwell on a more robust technique of solving quantum theories at strong coupling, like the large-N approximation.

This Chapter is organized as follows. In Sec. 4.2 we discuss the $(1,1)$ supersymmetric $\mathrm{O}(N)$ sigma model and introduce its heterotic deformation first in terms of superfields, then in components. In Sec. 4.3 we find the vacua of $(0,1)$ heterotic $\mathrm{O}(N)$ model and derive the effective potential. Sec. 4.4 is devoted to the investigation of the spectrum of the model in question, we give explicit formulas for the masses and the couplings at different values of the deformation parameter. Then we turn our attention to complex projective spaces in Sec. 4.5. First we introduce the heterotic model on $\mathbb{WCP}^{N-1}$ and discuss its quantum aspects. Then in Sec. 4.6 we present the master set of equations which gives the vacuum expectation values of all the fields. We solve them exactly in the massless case, while we give approximate analytical solutions and numerical evaluations in various regimes for non-zero masses. In Sec. 4.7 we discuss the one-loop low-energy effective action which describes excitations above the vacua found earlier. Sec. 4.8 contains conclusions and discussions.

4.2 Supersymmetric O(N) sigma model and its heterotic deformation

The bosonic O(N) sigma model with coupling constant g_0 can be formulated as follows. The dynamics of N real-valued scalar fields, subject to the constraint

$$(n^i)^2 = 1, \tag{4.2.1}$$

is governed by the action

$$S = \frac{1}{4g_0^2} \int d^2x \, \partial_\mu n^i \partial^\mu n^i \, . \tag{4.2.2}$$

The constraint (4.2.1) means that the isovector field $n^i \, , i = \overline{1, N}$ is confined on a unit $(N-1)$-sphere. The coupling constant g_0 in (4.2.2) is a bare one. Considering additional fermionic degrees of freedom one can easily supersymmetrize the model [162]. Supersymmetry is obvious when the action is written in terms of the superfields [5]

$$\mathcal{L} \;=\; \frac{1}{4} \int d^2\theta \, \varepsilon_{\alpha\beta} \mathcal{D}_\beta \mathcal{N}^i \mathcal{D}_\alpha \mathcal{N}_i = 2\partial_L n^i \partial_R n^i + i\psi_L^i \partial_R \psi_L^i + i\psi_R^i \partial_L \psi_R^i + \frac{1}{2}(F^i)^2, \tag{4.2.3}$$

where the the so-called isovector superfield has the following components

$$\mathcal{N}^i = n^i + \bar\theta \psi^i + \tfrac{1}{2} \bar\theta\theta F^i \, , \qquad i = 1, \ldots, N \, , \tag{4.2.4}$$

with a generalization of the bosonic constraint (4.2.1)

$$\mathcal{N}^2 = r_0, \tag{4.2.5}$$

[5] Our notations can be found in Appendix A.2.

where coupling $r_0 = g_0^{-2}$. In (4.2.4) ψ is a Majorana two-component spinor together with θ in (4.2.3). All components of the isovector superfield N^i (4.2.4) are real-valued. We have rescaled the fields in such a way that the coupling constant appears in the constraint (4.2.5) rather than in the action (4.2.3). Taking into account that

$$\mathcal{N}^2 = n^i n^i + 2\bar{\theta}\psi^i n^i + \bar{\theta}\theta\left(F^i n^i - \tfrac{1}{2}\bar{\psi}^i\psi^i\right), \tag{4.2.6}$$

one can write the relations for the components of the superfields $\mathcal{N}^i$ in the following form

$$\begin{aligned} n^2 &= r_0\,, \\ n^i\psi_i^\alpha &= 0\,, \\ F^i n^i - \tfrac{1}{2}\bar{\psi}^i\psi^i &= 0. \end{aligned} \tag{4.2.7}$$

The usual way to take into account the constraint is to introduce a Lagrange multiplier. For the case at hand the latter is the following superfield

$$\mathcal{S} = \sigma + \bar{\theta}\lambda + \tfrac{1}{2}\bar{\theta}\theta D. \tag{4.2.8}$$

Again, as in (4.2.4) all the components of the superfield $\mathcal{S}$ are real-valued. Therefore the action (4.2.3)can be rewritten in the following form

$$\mathcal{L} = \int d^2\theta\,\left[\frac{1}{4}\,\varepsilon_{\alpha\beta}\mathcal{D}_\beta\mathcal{N}^i\mathcal{D}_\alpha\mathcal{N}_i + \frac{1}{4e_0^2}\,\varepsilon_{\alpha\beta}\mathcal{D}_\beta\mathcal{S}\,\mathcal{D}_\alpha\mathcal{S} + \frac{i}{2}\mathcal{S}\left(\mathcal{N}^2 - r_0\right)\right], \tag{4.2.9}$$

where the limit for the coupling constant $e_0^2 \to \infty$ is implied. Hence the auxiliary fields are not dynamical for the time being, however, in Sec. 4.7 it will be shown that the coupling constant gets renormalized, therefore, providing non vanishing kinetic terms for the auxiliary fields. Those will be used in investigating the mass spectra of the theory.

Heteroric deformation. The model (4.2.9) is $\mathcal{N} = (1,1)$ supersymmetric, namely it is invariant under both left-handed and right-handed transformations. Now we are going to deform it by adding an extra left-handed fermion mixing with the initial ones, obviously breaking the $(1,1)$ supersymmetry down to $(0,1)$. Using the language of the superfields, we add the new term which contains only left-handed fermion and an auxiliary field

$$\Delta\mathcal{L} = \int d^2\theta\ \left[\tfrac{1}{4}\,\varepsilon_{\alpha\beta}\mathcal{D}_\beta\mathcal{B}\,\mathcal{D}_\alpha\mathcal{B} - i\gamma\mathcal{S}\mathcal{B}\right]\,, \tag{4.2.10}$$

where the chiral superfield $\mathcal{B}$ has the form

$$\mathcal{B} = \bar{\theta}\zeta + \tfrac{1}{2}\bar{\theta}\theta G\,, \quad \zeta = \begin{pmatrix} 0 \\ \zeta_L \end{pmatrix}\,. \tag{4.2.11}$$

It can be checked by a direct calculation that the above expression is indeed a superfield only with respect to the following left-handed transformations

$$\delta\zeta_L = \varepsilon_L G\,, \quad \delta G = -2i\varepsilon_L\partial_R\zeta_L\,. \tag{4.2.12}$$

It is clear that the transformations involving ε_R do not preserve the form of $\mathcal{B}$.

In the expression (4.2.11) the first term is the kinetic term for the left-handed field $\mathcal{B}$ (4.2.11), γ is the real-valued parameter of the deformation, and the latter term has explicit dependence on the Lagrange multiplier field $\mathcal{S}$. Combined together with (4.2.9) the new constraint on the isovector superfield reads

$$\mathcal{N}^i\mathcal{N}_i = r_0 + 2\gamma\mathcal{B}, \tag{4.2.13}$$

which only changes the latter two constraints from (4.2.7), namely

$$\begin{aligned} \psi_L^i n^i &= \gamma\zeta_L\,, \\ F^i n^i - i\psi_L^i\psi_R^i &= \gamma G\,, \end{aligned} \tag{4.2.14}$$

leaving the first constraint and the constraint for the right-handed component of ψ_R intact. The full Lagrangian is given by the expression

$$\begin{aligned} \mathcal{L} &= 2\partial_L n^i\partial_R n^i + i\psi_L^i\partial_R\psi_L^i + i\psi_R^i\partial_L\psi_R^i + \tfrac{1}{2}F^{i2} \\ &+ i\zeta_L\partial_R\zeta_L + \tfrac{1}{2}G^2 + i\gamma\lambda_R\zeta_L + \gamma G\sigma \\ &- \sigma\left(F^i n^i - i\psi_L^i\psi_R^i\right) - \tfrac{1}{2}D\left(n^i n^i - r_0\right) - i\lambda_R\psi_L^i n^i + i\lambda_L\psi_R^i n^i\,. \end{aligned} \tag{4.2.15}$$

Integration over the auxiliary fields F^i and G yields

$$\begin{aligned} \mathcal{L} = 2\partial_L n^i\partial_R n^i + i\psi_L^i\partial_R\psi_L^i + i\psi_R^i\partial_L\psi_R^i \\ + i\zeta_L\partial_R\zeta_L - i\lambda_R(\psi_L^i n_i - \gamma\zeta_L) + i\lambda_L\psi_R^i n^i \\ -\tfrac{1}{2}\gamma^2\sigma^2 - \tfrac{1}{2}\left(D + \sigma^2\right)n^i n^i + \tfrac{1}{2}Dr_0 + i\sigma\psi_L^i\psi_R^i\,. \end{aligned} \tag{4.2.16}$$

4.3 Effective Potential and Vacua

Given (4.2.16) we first wish to find the vacua of the theory similarly to [58] where the nonsupersymmetric $U(N)$ sigma model was solved in the large-N limit. Integrating out n^i and ψ^i fields and expressing the result in terms of the dynamical scale Λ related to the bare coupling constant by

$$r_0 = \frac{N}{4\pi}\log\frac{M_{UV}^2}{\Lambda^2}, \tag{4.3.1}$$

with M_{UV} being the ultraviolet cutoff, we obtain the following effective potential provided the rest of the fermionic fields are put to zero

$$V_{eff} = \frac{N}{8\pi}\left[D\log\frac{\Lambda^2}{D+\sigma^2} + \sigma^2\log\frac{\sigma^2}{\sigma^2+D} + D + u\sigma^2\right], \tag{4.3.2}$$

where we have introduced a new deformation parameter

$$u = \frac{4\pi\gamma^2}{N}. \tag{4.3.3}$$

Minimizing the potential with respect to D and σ one finds the vacua of the theory

$$\begin{aligned} \sigma_0 &= \pm\Lambda \mathrm{e}^{-\frac{u}{2}}, \\ D &= \Lambda^2 - \sigma^2. \end{aligned} \tag{4.3.4}$$

Thus, there are two different vacua for any finite parameter u. One can integrate out the D field (4.3.2) and obtain the potential which depends only on σ

$$V_{eff} = \frac{N}{8\pi}\left[\Lambda^2 + \sigma^2\left(\log\frac{\sigma^2}{\Lambda^2} - 1 + u\right)\right]. \tag{4.3.5}$$

Vacuum Energy. Plugging the vacuum solution (4.3.4) into (4.3.2) we calculate the vacuum energy

$$\mathcal{E}_{vac} = \frac{N}{8\pi}\Lambda^2\left(1 - \mathrm{e}^{-u}\right). \tag{4.3.6}$$

From the expression above it is obvious that the supersymmetry is spontaneously broken for $u \neq 0$. At small u the vacuum energy behaves linearly with u or quadratically with the deformation parameter γ

$$\mathcal{E}_{vac} \sim \frac{N}{8\pi}u\Lambda^2 = \frac{\gamma^2}{2}\Lambda^2. \tag{4.3.7}$$

At large u the vacuum energy scales linearly with N

$$\mathcal{E}_{vac} \sim \frac{N}{8\pi}\Lambda^2. \tag{4.3.8}$$

4.4 Spectrum of the O(N) Model

The next goal is to find the spectrum of the theory. To do so we are to obtain the one-loop effective action. The most straightforward way to calculate the action is to consider small fluctuations of the fields around the vacuum and to use what is called the long wave approximation. As a result we get the one-loop effective action in the large-N approximation for the fields σ, λ and ζ

$$\mathcal{L}_{eff} = \frac{1}{2e_\sigma^2}\left(\partial_\mu\sigma\right)^2 + \frac{i}{2e_\lambda^2}\bar{\lambda}\gamma^\mu\partial_\mu\lambda - V_{eff}(\sigma) + i\zeta_L\partial_R\zeta_L + \frac{1}{2}\Gamma\sigma\bar{\lambda}\lambda + i\gamma\lambda_R\zeta_L, \quad (4.4.1)$$

where e_σ and e_λ are the coupling constants that define the wave function renormalization of the σ and λ fields correspondingly, and Γ is induced the Yukawa coupling of the λ and σ. The wave function renormalization is easily calculated in the limit of small momenta. The diagrams contributing to e_σ are shown in Fig. 4.1 The actual calculation yields

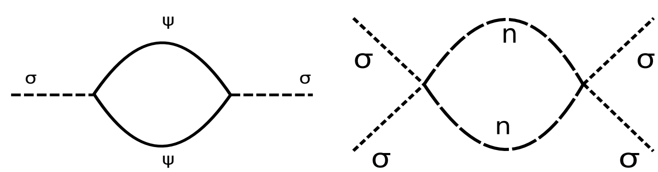

Figure 4.1: Feynman diagrams contributing to the wave function renormalization of σ.

$$\frac{1}{e_\sigma^2} = \frac{N}{8\pi}\left(\frac{2}{3}\frac{\sigma_0^2}{(\sigma_0^2+D)^2} + \frac{1}{3}\frac{1}{\sigma_0^2}\right) = \frac{N}{24\pi}\frac{\mathrm{e}^u}{\Lambda^2}\left[1+2\mathrm{e}^{-2u}\right]. \quad (4.4.2)$$

Similarly, the renormalization of λ is given by the diagram in Fig. 4.2.

$$\frac{1}{e_\lambda^2} = \frac{N}{4\pi}\left(\frac{1}{D} - \frac{\sigma_0^2}{D^2}\log\frac{\sigma_0^2+D}{\sigma_0^2}\right) = \frac{N}{4\pi}\frac{1}{\Lambda^2}\frac{1-\mathrm{e}^{-u}(1+u)}{(1-\mathrm{e}^{-u})^2}. \quad (4.4.3)$$

Finally, the Yukawa coupling Γ can be found from either the triangular graph (see Fig. 4.3), or, equivalently, as the masses renormalization from the diagram in Fig. 4.2,

$$\Gamma = \frac{N}{4\pi}\frac{1}{D}\log\frac{\sigma_0^2+D}{\sigma_0^2} = \frac{N}{4\pi}\frac{1}{\Lambda^2}\frac{u}{1-\mathrm{e}^{-u}}. \quad (4.4.4)$$

It is worth noting that although the fields σ and λ were introduced as auxiliary dummy fields in (4.2.10), they become dynamical after integrating out the fields n and ψ. It

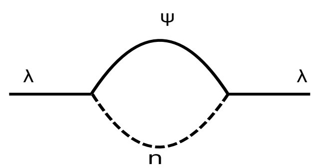

Figure 4.2: The wave function renormalization for λ.

is clear that in the limit $D \to 0$ or $u \to 0$, which corresponds to the restoration of the supersymmetry, the coupling constants e_σ and e_λ coincide

$$\frac{1}{e_\sigma^2} = \frac{1}{e_\lambda^2} = \frac{N}{8\pi}\frac{1}{\sigma_0^2}. \tag{4.4.5}$$

Now we can turn to actual calculation of the mass spectrum. It is obvious that the

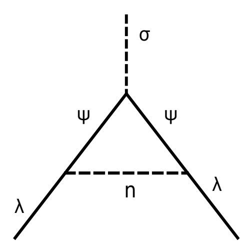

Figure 4.3: The induced Yukawa vertex.

n and ψ fields acquire mass due to the VEV of D and σ, namely

$$\begin{aligned} m_\psi^2 &= \sigma_0^2, \\ m_n^2 &= \sigma_0^2 + D. \end{aligned} \tag{4.4.6}$$

From the effective Lagrangian (4.4.1) one can easily find the expressions for the masses of the auxiliary field σ

$$m_\sigma = \Lambda\sqrt{6}\frac{e^{u/2}}{\sqrt{1+\frac{1}{2}e^{2u}}}. \tag{4.4.7}$$

At nonzero values of the heterotic deformation parameter mixing between ζ_L, λ_L and λ_R occurs. In order to find the mass states, one needs to transform the fermion mass matrix to the canonical form. However, it is clear that there is a massless mode since only the left fermion was introduced. To see this, one has to find the solution of the characteristic polynomial, corresponding to the fermion mass matrix

$$m\left(m^2-\gamma^2 e_\lambda^2-\sigma_0^2 e_\lambda^4\Gamma^2\right)=0, \qquad (4.4.8)$$

which indeed has zero solution for any u. There is also another solution of the characteristic polynomial. Summarizing, we find the following masses of the fermions

$$\begin{aligned} m_F &= 0, \\ m_F &= \sqrt{\gamma^2 e_\lambda^2+\sigma_0^2 e_\lambda^4\Gamma^2}=2\Lambda\frac{\sqrt{u\,(e^u-1)}}{e^u-1-u}\,\sinh\frac{u}{2}. \end{aligned} \qquad (4.4.9)$$

4.4.1 Spectrum of masses at small u

First, when the supersymmetry is unbroken $u=0$, $D=0$ the masses of superpartners coincide. For the fields n^i and ψ^i it become

$$m_{\psi_{L,R}}=m_n=\Lambda, \qquad (4.4.10)$$

while for the fermion $\lambda_{L,R}$ and boson σ we have

$$m_{\lambda_{L,R}}=m_\sigma=2\Lambda\,. \qquad (4.4.11)$$

The ζ_L field is decoupled from other fields, it is sterile and massless.

4.4.2 Spectrum of masses at large u

When the parameter of heterotic deformation gets bigger, the splitting between the masses in our theory becomes more dramatic. For the n^i and ψ^i particles they are

$$m_n=\sqrt{D+\sigma^2}=\Lambda\,, \quad m_\psi=\Lambda \mathrm{e}^{-\frac{u}{2}}\,, \qquad (4.4.12)$$

thus the fermions become much lighter than bosons. The couplings behave differently as well. The coupling for σ is

$$\frac{1}{g_\sigma^2}=\frac{N}{4\pi}\frac{\mathrm{e}^u}{6\Lambda^2}\,, \qquad (4.4.13)$$

and for λ it becomes

$$\frac{1}{g_\lambda^2}=\frac{N}{4\pi}\frac{1}{\Lambda^2}\,. \qquad (4.4.14)$$

The mass of the σ field goes exponentially to zero for large u

$$m_\sigma = 2\sqrt{3}\Lambda e^{-u/2}. \tag{4.4.15}$$

The fermion mass matrix has a zero eigenvalue corresponding to the now massless left component of the field λ_L, while the mixture of the fields ζ_L and λ_R produce the mass term

$$m_{\lambda_R,\zeta_L} = \Lambda\sqrt{u}. \tag{4.4.16}$$

We see that being equal in the limit $u \to 0$ the masses of σ and λ now become essentially different. For arbitrary value of the deformation parameter the ratio of the fermion matrix eigenvalue and the mass of σ is plotted in Fig. 4.4. Although the masses

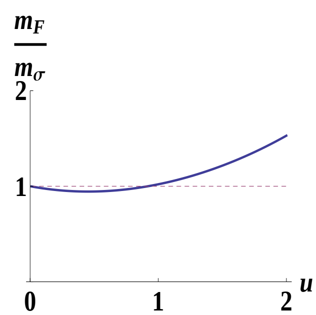

Figure 4.4: The ratio of the fermion mass matrix eigenvalue to the mass of σ as a function deformation parameter u.

become equal not only for $u = 0$, there is no restoration of the supersymmetry. First, there is still a Goldstino and second, the vacuum energy (4.3.7) is not equal to zero at that point.

4.5 Heterotic $\mathcal{N} = (0, 2)$ $\mathbb{CP}^{N-1}$ Sigma Model

Here we formulate the sigma-model in gauged approach and discuss its moduli space. First we consider the $(2, 2)$ sypersymmetric model and then introduce the heterotic deformation.

4.5.1 $\mathcal{N} = (2,2)$ weighted non-linear sigma-model

Let us start by introducing the undeformed $\mathcal{N} = (2,2)$ weighted sigma-model[6] $\mathbb{WCP}^{N_F-1}$. A detailed discussion of these models can be found in [61], where a relationship with Landau-Ginzsburg models is considered.[7] The same models can be studied in the mirror representation [172, 173, 174]. The model can be built out of N positively charged fields n_i, $\widetilde{N}$ negatively charged fields ρ_j and a non-dynamical auxiliary field. The full Lagrangian, including the fermionic superpartners can be written in a superfield formalism which make supersymmetry manifest (see Sec. B.1). The Lagrangian (B.4.4) has the following component expansion

$$\begin{aligned}
\mathcal{L}_{\mathbb{WCP}^{N_F-1}} &= |\nabla_\mu n_i|^2 + |\nabla_\mu \rho_j|^2 - |\sigma|^2|n_i|^2 - |\sigma|^2|\rho_j|^2 - D\left(|n_i|^2 - |\rho_j|^2 - r_0\right) \\
&+ i\bar{\xi}_{L,i}\nabla_R\xi^i_L + i\bar{\xi}_{R,i}\nabla_L\xi^i_R + i\bar{\eta}_{L,j}\nabla_R\eta^j_L + i\bar{\eta}_{R,j}\nabla_L\eta^j_R + \\
&+ \left[i\bar{n}_i\left(\lambda_L\xi^i_R - \lambda_R\xi^i_L\right) - i\sigma\bar{\xi}_{R,i}\xi^i_L - i\bar{\rho}_j\left(\lambda_L\eta^j_R - \lambda_R\eta^j_L\right) + i\sigma\bar{\eta}^j_R\eta^j_L + \text{H.c.}\right] ,
\end{aligned} \tag{4.5.1}$$

where the covariant derivatives are given by

$$\nabla_\mu n_i = (\partial_\mu - iA_\mu)n_i, \quad \nabla_\mu\rho_j = (\partial_\mu + iA_\mu)\rho_j\,. \tag{4.5.2}$$

The fields A_μ, σ, $\lambda_{L,R}$ and D all belong to the same $\mathcal{N} = 2$ supermultiplet, they are non-dynamical, and can be integrated out using their equations of motion. However, as we shall see later, in strongly coupled phases these auxiliary fields do become dynamical and describe particles in the low energy effective theory.

The model has a unique parameter which determines the strength of the interactions, the two-dimensional Fayet-Iliopoulos term r_0 [175]. Classically, the model has a continuous set of vacua determined by the vacuum equation

$$\sum_{i=0}^{N-1}|n_i|^2 - \sum_{j=0}^{\tilde{N}-1}|\rho_j|^2 = r_0\,. \tag{4.5.3}$$

---

[6] Many gauged sigma-models which are studied in the literature, including this one, follow from a very generic construction developed by Distler and Kachru [171].

[7] For an alternative superfield formulation of the model see Sec. B.4

The first and the most important quantum effect is the generation of a dynamical scale Λ through dimensional transmutation. In fact, the Fayet-Iliopoulos term gets renormalized, flowing with respect to the energy scale ϵ through the following one loop expressions

$$r(\epsilon) = r_0 - \frac{N-\widetilde{N}}{4\pi}\log\left(\frac{M_{UV}^2}{\epsilon^2}\right) \equiv -\frac{N-\widetilde{N}}{4\pi}\log\left(\frac{\Lambda^2}{\epsilon^2}\right). \tag{4.5.4}$$

The theory is thus asymptotically free for $N > \widetilde{N}$. From the expression above we can also guess that for $N = \widetilde{N}$ we have super-conformal theory, and this is indeed the case [61].

Actually, thanks to supersymmetry, (4.5.4) is exact in perturbation theory because of the vanishing of higher order contributions. Furthermore, integrating out the matter fields in the functional integral we can find an exact superpotential for the field σ [61, 169, 62, 59]

$$W(\sigma) = \frac{N-\widetilde{N}}{4\pi}\sigma\left(\log\left(\frac{\sigma}{\Lambda}\right)-1\right). \tag{4.5.5}$$

This superpotential includes all the non-perturbative instantonic contributions to the functional integral. At the classical level the theory has two $U(1)$ R-symmetries, $U(1)_R \times U(1)_V$. The first one is an axial symmetry, under which σ has charge $+2$. This symmetry is anomalous and is broken down to $\mathbb{Z}_{2N-2\widetilde{N}}$ by the one-loop corrections. By minimization of the superpotential (4.5.5) we find $N-\widetilde{N}$ massive vacua. We will discuss in more details the vacuum structure of the theory in Sec. 4.6.

4.5.2 $\mathcal{N} = (0,2)$ weighted sigma-model: heterotic deformation

As is well-known from early studies of two-dimensional supersymmetric sigma-models [176], there is no smooth $\mathcal{N} = (0,2)$ deformation of the $\mathcal{N} = (2,2)$ $\mathbb{CP}^{N-1}$ sigma-model[8] . On the other hand, it is possible to have deformation of the $\mathbb{C}\times\mathbb{CP}^{N-1}$ model, which is the relevant effective theory emerging in when studying the non-Abelian vortices (the $\mathbb{C}$ factor describes the translation modes of the vortex). From the additional $\mathbb{C}$ piece, one can keep only a right-handed fermion, while the scalar and left-handed fermionic

[8] See [150, 177] for a discussion of this issue in a context related to non-Abelian vortices

super-partners is free. A similar situation occurs for the weighted sigma-model[9] . As a result we consider the following Lagrangian

$$\mathcal{L}^{het}_{\mathbb{WCP}^{N_F-1}} \;=\; \mathcal{L}_{\mathbb{WCP}^{N_F-1}} + \tfrac{i}{2}\bar{\zeta}_R \partial_L \zeta_R - 2|\omega|^2|\sigma|^2 - [i\omega\lambda_L\zeta_R + \text{H.c.}]\ . \qquad (4.5.6)$$

The heterotic coupling ω is introduced by means of an additional right-handed fermion ζ_R. Obviously the modification dramatically changes the physics of the sigma-model at hand. For example, the Witten index is modified from $N - \widetilde{N}$ to zero as in the $\mathbb{CP}^{N-1}$ case. This observation is indeed consistent with supersymmetry breaking [158, 178] occurring in the model.

Adding the twisted masses. Twisted masses can be easily introduced into the model by first gauging the $U(1)^{N_F-1}$ independent flavor symmetries and then setting to zero all the fields in the additional twisted multiplets but not the lowest components [59]. The resulting Lagrangian takes the following form

$$\begin{aligned}
\mathcal{L}^{het}_{\mathbb{WCP}^{N_F-1}} = {}& |\nabla_\mu n_i|^2 + |\nabla_\mu \rho_j|^2 + i\bar{\xi}_{L,\,i}\nabla_R \xi^i_L + i\bar{\xi}_{R,\,i}\nabla_L \xi^i_R + i\bar{\eta}_{L,\,j}\nabla_R \eta^j_L + i\bar{\eta}_{R,\,j}\nabla_L \eta^j_R \\
& - \sum_{i=0}^{N-1} |\sigma - m_i|^2 |n_i|^2 - \sum_{j=0}^{\tilde{N}-1} |\sigma - \mu_j|^2 |\rho_j|^2 - D\left(|n_i|^2 - |\rho_j|^2 - r_0\right) \\
& + \left[i\bar{n}_i \left(\lambda_L \xi^i_R - \lambda_R \xi^i_L\right) - i \sum_{i=0}^{N-1} (\sigma - m_i)\, \bar{\xi}_{R,\,i}\xi^i_L + \text{H.c.} \right] \\
& + \left[-i\bar{\rho}_j \left(\lambda_L \eta^j_R - \lambda_R \eta^j_L\right) + i \sum_{j=0}^{\tilde{N}-1} (\sigma - \mu_j)\, \bar{\eta}_{R,\,j}\eta^j_L + \text{H.c.} \right] \\
& + \tfrac{i}{2}\bar{\zeta}_R \partial_L \zeta_R - [i\omega\lambda_L\zeta_R + \text{H.c.}] - 2|\omega|^2|\sigma|^2\,. \qquad (4.5.7)
\end{aligned}$$

For zero values of the twisted masses there is a $U(1)$ R-symmetry under which the fermions $\xi^i_R, \eta^j_R, \lambda_R$ ($\xi^i_L, \eta^j_L, \lambda_L$) have charge $+1(-1)$, whereas σ has charge $+2$. A generic choice of the masses m_i and μ_j breaks this symmetry completely. Instead,

[9] In fact, it is possible to introduce $\mathcal{N} = (0, 2)$ deformations of the weighted sigma-model without introducing any new degrees of freedom, or $\mathbb{C}$ factors. However, all the possible deformations different from the one considered in the text do not arise in the context of non-Abelian vortices. Nevertheless, it may be interesting to study the effects of such deformations. For more details on this aspect, see [150].

we make the following choice for the masses

$$m_k = m\, e^{2\pi i \frac{k}{N}}\,, \quad k = 0, \ldots, N-1\,,$$
$$\mu_l = \mu\, e^{2\pi i \frac{l}{\tilde{N}}}\,, \quad l = 0, \ldots, \tilde{N}-1\,. \tag{4.5.8}$$

For the further convenience we define a new constant $\alpha = \widetilde{N}/N$. Notice that in the $N \to \infty$ limit, the masses are distributed uniformly on circles with radii $|m|$ and $|\mu|$ correspondingly. We consider m and μ to be real. There are particular choices of α which are interesting because they leave some residual discrete symmetry on the classical level. In particular, if N and $\tilde{N}$ have $N - \widetilde{N}$ as a common divisor, a discrete $\mathbb{Z}_{N-\widetilde{N}}$ symmetry is preserved[10] . As we shall later see in Sec. 4.6, in quantum theory VEV of σ breaks this symmetry, however, for certain values of the twisted masses (4.5.8) $\langle\sigma\rangle = 0$ and the symmetry gets restored.

4.6 Large-N Solution of the $\mathbb{WCP}^{N-1}$ Model

In this section we solve the model in the large-N approximation, closely following the analysis of [158, 159]. Since the $n^i\,, \rho^j\,, \xi^i\,, \eta^j$ fields appear in the action quadratically, we can perform the Gaussian integration over these fields. We integrate over all but the following four fields $(n^0, \rho^0, \xi^0, \eta^0)$. The scalar fields (n^0, ρ^0) will represent the helpful set of the order parameters defining various phases of the theory.

The Gaussian integration leads to the following determinants

$$\prod_{i=1}^{N-1}\left[\frac{\det\big((\partial_k + iA_k)^2 + D + |\sigma - m_i|^2\big)}{\det\left((\partial_k + iA_k)^2 + |\sigma - m_i|^2\right)}\right]\prod_{j=1}^{\widetilde{N}-1}\left[\frac{\det\big((\partial_k - iA_k)^2 - D + |\sigma - \mu_j|^2\big)}{\det\left((\partial_k - iA_k)^2 + |\sigma - \mu_j|^2\right)}\right]. \tag{4.6.1}$$

The large-N approximation is technically equivalent to a one-loop calculation of the above determinants, where we can also drop the gauge fields [58]. The result gives an

[10] This symmetry is a combination of the flavor and R symmetry.

effective potential for the σ field[11]

$$V_{1-loop} = \frac{1}{4\pi}\sum_{i=1}^{N-1}\left(-\left(D+|\sigma-m_i|^2\right)\log\frac{|\sigma-m_i|^2+D}{\Lambda^2}+|\sigma-m_i|^2\log\frac{|\sigma-m_i|^2}{\Lambda^2}\right)$$
$$-\frac{1}{4\pi}\sum_{j=1}^{\tilde{N}-1}\left(-\left(D-|\sigma-\mu_j|^2\right)\log\frac{|\sigma-\mu_j|^2-D}{\Lambda^2}-|\sigma-\mu_j|^2\log\frac{|\sigma-\mu_j|^2}{\Lambda^2}\right)$$
$$+\frac{N-\tilde{N}}{4\pi}D\,. \tag{4.6.2}$$

To get the above result we have again traded the UV cut-off for the scale Λ. Including the pieces already present at the classical level we get the expression for the effective potential

$$V_{eff} = V_{1-loop} + \left(|\sigma-m_0|^2+D\right)|n_0|^2 + \left(|\sigma-\mu_0|^2-D\right)|\rho_0|^2 + \frac{uN}{4\pi}|\sigma|^2\,, \tag{4.6.3}$$

where we set $u = 8\pi|\omega|^2/N$.

Vacuum equations. The extremization[12] of this potential with respect to n_0 and ρ_0, D and σ gives us the master set of equations which determines the vacuum structure of the theory

$$\left(|\sigma-m_0|^2+D\right)n_0 = 0\,,\quad \left(|\sigma-\mu_0|^2-D\right)\rho_0 = 0\,, \tag{4.6.4}$$
$$\frac{1}{4\pi}\sum_{i=1}^{N-1}\log\frac{|\sigma-m_i|^2+D}{\Lambda^2} - \frac{1}{4\pi}\sum_{j=1}^{\tilde{N}-1}\log\frac{|\sigma-\mu_j|^2-D}{\Lambda^2} = |n_0|^2-|\rho_0|^2\,,$$
$$\frac{1}{4\pi}\sum_{i=1}^{N-1}(\sigma-m_i)\log\frac{|\sigma-m_i|^2+D}{|\sigma-m_i|^2} + \frac{1}{4\pi}\sum_{j=1}^{\tilde{N}-1}(\sigma-\mu_j)\log\frac{|\sigma-\mu_j|^2-D}{|\sigma-\mu_j|^2} =$$
$$= (\sigma-m_0)|n_0|^2 + (\sigma-\mu_0)|\rho_0|^2 + \frac{uN}{4\pi}\sigma\,. \tag{4.6.5}$$

The second equation above gives us the renormalized coupling constant

$$r = |n_0|^2 - |\rho_0|^2\,. \tag{4.6.6}$$

[11] For a discussion of the relationship between the Large-N potential and the exact $\mathcal{N}=(2,2)$ superpotential (4.5.5) see [159]. It is indeed possible to reconstruct a full exact potential like (4.5.5) from this expression, by noticing that the large-N expression must give, at the first linear order in D, the following term: $D(W'(\sigma)+h.c)$. We thank A. Vainshtein for this observation.

[12] The solution of the vacuum equations for D gives V_{eff} a maximum rather than a minimum. This fact, being usual in supersymmetric gauge theories, is consistent since the D field is not dynamical. We get a true minimum with respect to the σ field.

In the next section we shall solve the weighted heterotic $\mathbb{CP}^{N-1}$ model in the large-N approximation. First we address the massless case, and then work out the more involved model with twisted masses.

4.6.1 Massless case

Let us warm-up with the problem when all twisted mass are zero. We will be able to investigate more easily all the features which will be also present in the massive case. The potential (4.6.3) takes much simpler form now[13]

$$\begin{aligned} V_{eff} = & \frac{N}{4\pi}\left(D\log\frac{\Lambda^2}{|\sigma|^2+D} + |\sigma|^2\log\frac{|\sigma^2|}{|\sigma|^2+D}\right) \\ & - \frac{\tilde{N}}{4\pi}\left(D\log\frac{\Lambda^2}{|\sigma|^2-D} - |\sigma|^2\log\frac{|\sigma^2|}{|\sigma|^2-D}\right) \\ & + \frac{N-\tilde{N}}{4\pi}D + \frac{uN}{4\pi}|\sigma|^2\,, \end{aligned} \tag{4.6.7}$$

from which the corresponding vacuum equations follow

$$\begin{aligned} \log\frac{|\sigma|^2+D}{\Lambda^2} - \alpha\log\frac{|\sigma|^2-D}{\Lambda^2} = 0\,, \\ \sigma\log\left(1+\frac{D}{|\sigma|^2}\right) + \sigma\alpha\log\left(1-\frac{D}{|\sigma|^2}\right) = u\sigma\,. \end{aligned} \tag{4.6.8}$$

Let us rewrite them in a more compact form

$$\begin{aligned} (1+x)(1-x)^{\alpha} = e^{u}\,, \\ (1-x)^{\frac{\alpha}{1-\alpha}}(1+x)^{\frac{1}{\alpha-1}} = s\,, \end{aligned} \tag{4.6.9}$$

where we introduced the following dimensionless parameters

$$s \equiv |\sigma|^2/\Lambda^2, \quad x \equiv D/|\sigma|^2\,. \tag{4.6.10}$$

Let us first discuss the undeformed case ($u=0$). From the first equation of (4.6.9) we get $x=0$, and from the second $s=1$, thus

$$|\sigma_\Lambda| = \Lambda, \quad D = 0\,. \tag{4.6.11}$$

[13] Notice that in this case we have integrated out all the fields.

The vanishing of the VEV of D implies unbroken supersymmetry. The VEV of σ, on the contrary, lies on a circle[14] . We can compare this result with the exact $\mathcal{N}=(2,2)$ solution at finite-N by minimizing the potential (4.5.5), from which we get the vacuum equation

$$\sigma^{N-\tilde{N}} = \Lambda^{N-\tilde{N}}\,. \tag{4.6.12}$$

There are $N-\widetilde{N}$ vacua characterized by the vacuum expectation value of σ

$$\sigma_{\Lambda,k} = \Lambda e^{2\pi i \frac{k}{N-\tilde{N}}}\,. \tag{4.6.13}$$

We can see that in the large-N_F limit the number of vacua becomes infinite and uniformly distributed on the circle.

Let us now turn on the heterotic deformation. The first equation from (4.6.9) gives us x, the second one can be used to find both D and σ in the vacuum configuration. The r.h.s of the first equation has an upper bound. There is thus a critical value u_{crit} for the heterotic deformation such that there are no solutions for larger u. Maximization of this term gives

$$u_{crit} = \log\left[\frac{2(2\alpha)^{\alpha}}{(1+\alpha)^{\alpha+1}}\right] \quad \text{at} \quad x_{crit} = \frac{1-\alpha}{1+\alpha}, \quad s_{crit} = \frac{1}{2}(1+\alpha)\alpha^{\frac{\alpha}{1-\alpha}}\,. \tag{4.6.14}$$

The numerical solution of equations (4.6.9) is presented in Fig. 4.5. Note that x is always smaller than unity. This is consistent with the fact that larger values of D ($D > |\sigma^2|$) would imply imaginary masses for the scalar particles, as it can easily be seen from (4.5.1).

The disappearance of the solutions which minimize the energy becomes clear after we look at the plots of the effective potential at different values of the heterotic deformation parameter. Using (4.6.8) we can find the auxiliary field D and substitute it into the effective potential (4.6.7), which we can now plot in Fig. 4.6 as a function of σ.

Conformal sector. The existence of a critical value for u forces us to search for a new vacuum solution other than those given by (4.6.11). Indeed, Fig. 4.6 clearly shows

[14] This is a natural result if we keep only the leading terms in the large-N approximation. Separating vacua into a discrete set should be possible, in principle, by considering sub-leading corrections to the potential.

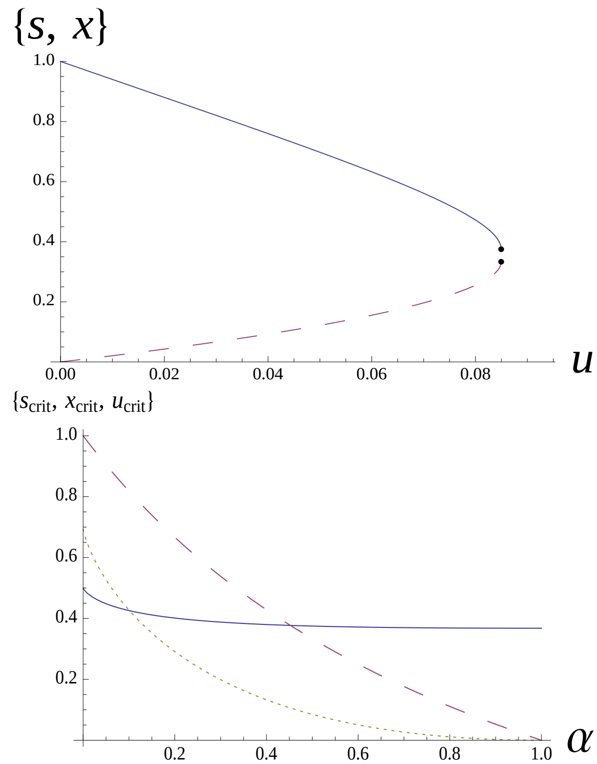

Figure 4.5: On the left plot, values of s and x are shown as a function of u, for $\alpha = 0.5$. On the right plot, the critical values as functions of α are shown. s is solid-blue, x dashed-red and u dotted-yellow.

a new vacuum located at $\sigma = 0$ which survives at nonzero heterotic deformations. For arbitrary value of u equations (4.6.9) admit the following solution

$$|\sigma_0| = 0, \quad D = 0\,. \tag{4.6.15}$$

This solution formally exists for the $\mathbb{CP}^{N-1}$ sigma-model as well, but in that case it must be discarded. As can be seen in Fig. 4.6, it represents a maximum, rather than a minimum. Strictly speaking, the effective potential cannot be trusted for $\sigma = 0$, where some degrees of freedom become massless. The existence of massless kinks is ensured once we interpret the vacuum at $\sigma = 0$ as a degenerate point where $\widetilde{N}$ vacua coalesce. We will check this explicitly in the next section, where we will resolve the $\widetilde{N}$ vacua by the introduction of twisted masses [33, 40, 61]. This sector of the theory is described by

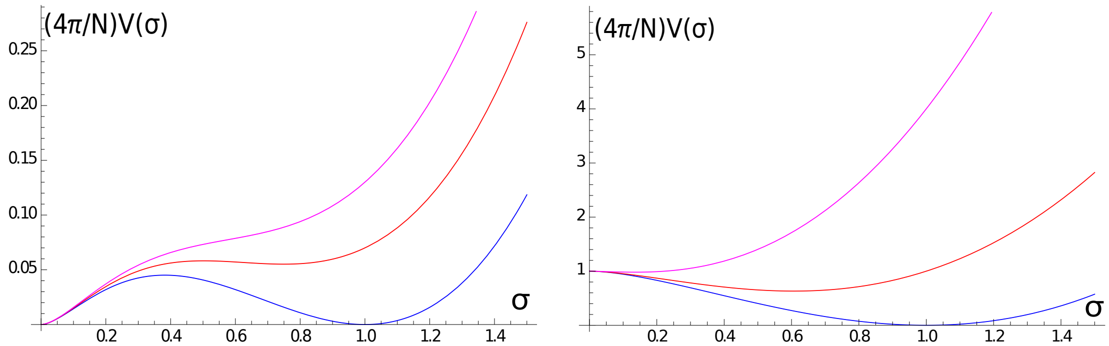

Figure 4.6: On the left: one-loop effective potential for the weighted $\mathbb{WCP}^{N_F-1}$ sigma-model, $u = 0, 0.07, 0.13$ (from the lowest to the highest curve) and $\alpha = 0.5$ as functions of σ in units of Λ. The lower lying plot corresponds to the unbroken SUSY ($u = 0$) where the vacuum energy of both zero vacuum and Λ vacuum is equal to zero. When we enhance the heterotic deformation the vacuum energy of the Λ vacuum becomes nonzero (the vacuum becomes metastable), whereas it always vanishes for the zero vacuum. At some value of the deformation parameter, u_{crit} the metastable vacuum ceases to exist. On the right: potential for the ordinary $\mathbb{CP}^{N-1}$ sigma-model, $u = 0, 1, 4$. The vacuum value of σ approaches zero for large u, but there is no loos of vacua, as soon as the deformation is kept finite.

a super-conformal field theory. This was first conjectured in [33, 179], by analogy with the four dimensional case: coalescence of vacua in two-dimensional theories corresponds to the degeneration of Seiberg-Witten curves [10, 11] at the so-called Argyres-Douglas points of four-dimensional theories[15], where the appearance of massless, mutually non-local degrees of freedom gives rise to an interacting super-conformal field theory [112, 180]. This expectation was confirmed in [181], where it was shown that the two-dimensional theory flows to an interacting super-conformal fixed point, identified as an A_{N-1} minimal model [182, 183, 184], as $\sigma \to 0$[16]. Notice that we can trust both the large-N effective potential and the exact twisted superpotential for arbitrarily small σ as soon as we interpret them as valid at energy scales ϵ much smaller than the masses of the hypermultiplets $\epsilon \ll |m_{hyp}| \sim |\sigma|$ (see (4.5.1)). The divergences of both potentials arise because of infrared instabilities due to the developing of massless states, as described

[15] The four-dimensional curve $y^2 = f(x, \Lambda)$ is given in terms of the two-dimensional superpotential: $y^2 = (\partial W(x)/\partial x)^2$. Using (4.5.5) we have $y^2 = x^{2\tilde{N}}(x^{N-\tilde{N}} - \Lambda^{N-\tilde{N}})^2$, which has $\tilde{N}$ degenerate singularities.

[16] [181] deals with the case of complete degeneration of the vacua: $y^2 = x^{2N}$. The qualitative aspects of that analysis hold in our case as well.

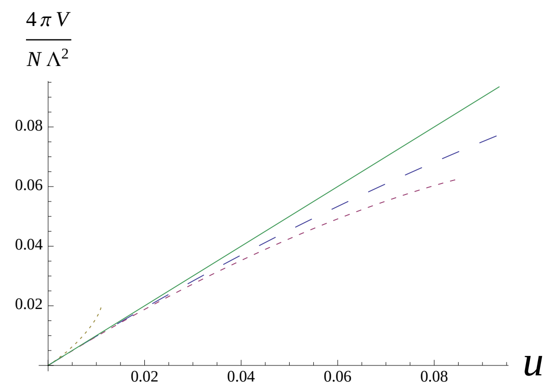

Figure 4.7: Numerical solutions for the vacuum energy of the massive vacua for various values of $\alpha = .2, .5, .8$, from the lowest to the highest curve. The solid line is the line given by (4.6.18).

above.

If we assume continuity of physical quantities in the limit $\sigma \to 0$, the result of this section (see Fig. 4.6) implies that this super-conformal sector is not lifted by the heterotic deformation. Furthermore, supersymmetry is not broken for $\sigma = 0$. The massive vacua discussed in this section become metastable when we turn on u, and disappear as we increase the heterotic deformation above the critical value u_{crit}.

Let us conclude this section by analytically solving (4.6.9) for small values of u. As we see from Fig. 4.5 this also implies small x. One thus has from (4.6.9)

$$(1+x)(1-\alpha x) = 1+u\,,$$
$$\left(1-\frac{\alpha}{1-\alpha}x\right)\left(1+\frac{1}{\alpha-1}x\right) = s\,, \tag{4.6.16}$$

which gives

$$x \approx \frac{u}{1-\alpha}\,, \quad s \approx 1-\frac{1+\alpha}{(1-\alpha)^2}u\,. \tag{4.6.17}$$

Substituting (4.6.17) in the expression for the effective potential (**??**) we find the the following expression for the vacuum energy

$$V_{eff} = \frac{N}{4\pi}u\Lambda^2\,, \tag{4.6.18}$$

which is to be compared with the numerical solution in Fig. 4.7. Notice that the small u limit does not depend on the value of α.

4.6.2 Massive case

We shall now determine the vacuum structure of the massive model in terms of the two-dimensional space of parameters m and μ. With a quick inspection to the first line of (4.6.4)

$$\left(|\sigma - m_0|^2 + D\right) n_0 = 0\,, \quad \left(|\sigma - \mu_0|^2 - D\right) \rho_0 = 0\,, \tag{4.6.19}$$

we can easily identify three branches of solutions, which correspond to three different phases of the theory

Hn : Higgs phase with non-zero VEV for n^i

$$\rho_0 = 0\,, \quad D = -\left|\sigma - m_0\right|^2\,, \tag{4.6.20}$$

Hρ : Higgs phase with non-zero VEV for ρ^j

$$n_0 = 0 \quad D = \left|\sigma - \mu_0\right|^2\,, \tag{4.6.21}$$

C : Coulomb phase

$$n_0 = \rho_0 = 0\,. \tag{4.6.22}$$

Recall that the renormalized coupling is given by (4.6.6), $r = |n_0|^2 - |\rho_0|^2$. Thus, the **H**n phase is characterized by a positive coupling, while in the **H**ρ phase the renormalized Fayet-Iliopoulos term is negative. In the **C** phase $r = 0$. We will determine the appearance of these phases in the $m - \mu$ plane, by starting with the undeformed case.

Undeformed case

The $\mathcal{N} = (2,2)$ sigma-model is solved by virtue of the exact superpotential. As we have mentioned in the introduction, in the current Chapter we shall work with the large-N approximation as it can be used both for the $(2,2)$ and $(0,2)$ models.

We anticipate here the discussion of this section by proposing the phase diagram in Fig. 4.8. Below we list the vacua solutions in each domain of the phase diagram.

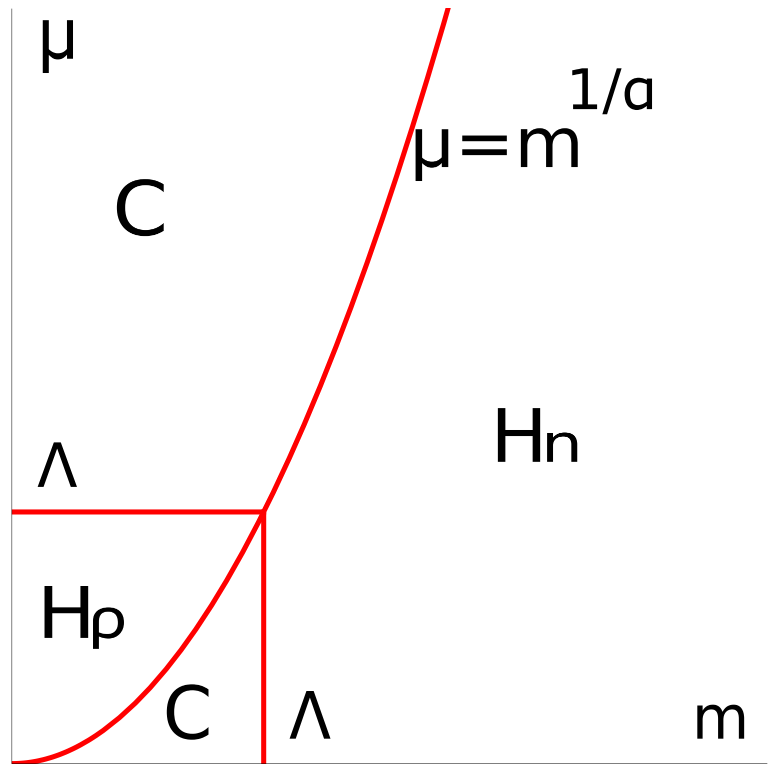

Figure 4.8: Phase Diagram of the weighted (2, 2) $\mathbb{CP}^{N-1}$ model in the large-N approach. There are four domains with different VEVs for σ: two Higgs branches $\mathbf{H}\rho$ and $\mathbf{H}n$, and two Coulomb branches $\mathbf{C}$. In the Coulomb phase $\mathbf{C}$ $r = 0$. The curve $\mu/\Lambda = (m/\Lambda)^{1/\alpha}$ together with horizontal and vertical lines starting from $\mu = \Lambda$ and $m = \Lambda$ respectively separates the $\mathbf{C}$ phases from the Higgs phases. In $\mathbf{H}n$ $r > 0$ and in $\mathbf{H}\rho$ $r < 0$. On the super-conformal line $\mu/\Lambda = (m/\Lambda)^{1/\alpha}$ a new branch described by a super-conformal theory opens up.

H*n* phase. The unbroken supersymmetry of the undeformed model implies $D = 0$ for all the phases. From (4.6.20) we thus find

$$\sigma = m_0 \,, \quad \rho_0 = 0, \quad D = 0 \,. \tag{4.6.23}$$

From the second line of (4.6.4) we determine the coupling constant

$$r = |n_0|^2 = \frac{1}{4\pi} \sum_{i=1}^{N-1} \log \frac{|m_0 - m_i|^2}{\Lambda^2} - \frac{1}{4\pi} \sum_{j=1}^{\tilde{N}-1} \log \frac{|m_0 - \mu_j|^2}{\Lambda^2} \geq 0 \,. \tag{4.6.24}$$

The sums in the expression above can be exactly calculated in the large-N limit as shown in [152]

$$r = \begin{cases} \frac{N-\tilde{N}}{2\pi}\log\frac{m}{\Lambda}\,, & \mu < m \\ \\ \frac{N}{2\pi}\log\frac{m}{\Lambda} - \frac{\tilde{N}}{2\pi}\log\frac{\mu}{\Lambda}\,, & \mu > m\,. \end{cases} \tag{4.6.25}$$

By asking for r to be positive, we obtain the following conditions for the existence of the $\mathbf{H}n$ phase

$$\mathbf{H}n: \quad \begin{cases} m > \Lambda, & \mu < m \\ \\ \frac{m}{\Lambda} > \left(\frac{\mu}{\Lambda}\right)^{\alpha}, & \mu > m\,. \end{cases} \tag{4.6.26}$$

Hρ phase. In this phase we use (4.6.21) to find

$$\sigma = \mu_0\,, \quad n_0 = 0, \quad D = 0\,, \tag{4.6.27}$$

and the coupling constant

$$r = \begin{cases} \frac{N-\tilde{N}}{2\pi}\log\frac{\mu}{\Lambda}\,, & \mu > m \\ \\ \frac{N}{2\pi}\log\frac{m}{\Lambda} - \frac{\tilde{N}}{2\pi}\log\frac{\mu}{\Lambda}\,, & \mu < m \end{cases} \tag{4.6.28}$$

Negativity of r now implies the following conditions for the existence of the $\mathbf{H}\rho$ phase

$$\mathbf{H}\rho: \quad \left(\frac{m}{\Lambda}\right)^{1/\alpha} < \frac{\mu}{\Lambda} < 1\,. \tag{4.6.29}$$

The renormalized coupling constant vanishes, as expected, along the boundaries of the Higgs phases. As we will explain later the curve defined as

$$\frac{m}{\Lambda} = \left(\frac{\mu}{\Lambda}\right)^{\alpha}, \tag{4.6.30}$$

is of particular interest. Notice that the renormalized coupling in both Higgs regimes scales with N. Let us mention that a more natural coupling constant would be

$$\ell = \frac{1}{\lambda} = \frac{4\pi r}{N}, \tag{4.6.31}$$

which is reminiscent of the 't-Hooft coupling constant which naturally appears in large-N gauge theories.

C phases. The Coulomb phase exists in the regions where $|n_0| = |\rho_0| = 0$. There are two distinct regions, $\mathbf{C}\mu$ and $\mathbf{C}m$, which complete the phase diagram as shown in Fig. 4.8. With the first and the third equations of (4.6.4) being satisfied automatically for $u = D = 0$, the second one gives

$$\prod_{i=1}^{N-1} |\sigma - m_i|^2 = \Lambda^{N-\tilde{N}} \prod_{j=1}^{\tilde{N}-1} |\sigma - \mu_j|^2 \,. \tag{4.6.32}$$

Note that each part of this equation is real for the complex variable σ. This implies a continuous set of solutions which are located on a closed line. Again, this is the effect of the large-N approximation, where an infinite number of vacua is continuously distributed on a curve. The solution in the leading approximation is qualitatively different in the two $\mathbf{C}$ regions. In the $\mathbf{C}\mu$ region the vacua sit on a single circle

$$|\sigma_\mu| = \Lambda \left(\frac{\mu}{\Lambda}\right)^{\alpha} , \qquad \mu > \Lambda \left(\frac{m}{\Lambda}\right)^{1/\alpha} , \quad \mu > \Lambda \,. \tag{4.6.33}$$

In the $\mathbf{C}m$ region the vacua split between two separate circles

$$\begin{aligned} &|\sigma_m| = \Lambda \left(\tfrac{m}{\Lambda}\right)^{1/\alpha} , \\ & \qquad\qquad\qquad\qquad \mu < \Lambda \left(\frac{m}{\Lambda}\right)^{1/\alpha} , \quad m < \Lambda \,. \\ &|\sigma_\Lambda| = \Lambda \end{aligned} \tag{4.6.34}$$

In order to resolve the vacua into a discrete set we can compare the result above with the exact one given by the $\mathcal{N} = (2,2)$ superpotential. Vacua are the solutions of the equation [33]

$$\prod_{i=1}^{N-1} (\sigma - m_i) = \Lambda^{N-\tilde{N}} \prod_{j=1}^{\tilde{N}-1} (\sigma - \mu_j) \,, \tag{4.6.35}$$

which, making use of Eq. (4.5.8), can be rewritten as the following:

$$\sigma^N - m^N = \Lambda^{N-\tilde{N}} (\sigma^{\tilde{N}} - \mu^{\tilde{N}}) \,. \tag{4.6.36}$$

It is exact even for small N, but in the large-N approximation one obtains three groups of solutions: N "μ-vacua" in the $\mathbf{C}\mu$ region

$$\sigma_{\mu,j} = \Lambda \left(\frac{\mu}{\Lambda}\right)^{\alpha} e^{2\pi i \frac{j}{N}} \qquad j = 0, \ldots, N-1 \,, \tag{4.6.37}$$

while in the $\mathbf{C}m$ we have $N-\tilde{N}$ "Λ-vacua"

$$\sigma_{\Lambda,k} = \Lambda\, e^{2\pi i \frac{k}{N-\tilde{N}}}\,, \quad k = 0,\ldots,N-\tilde{N}-1\,, \tag{4.6.38}$$

and $\tilde{N}$ "m-vacua"

$$\sigma_{m,l} = \Lambda\left(\frac{m}{\Lambda}\right)^{1/\alpha} e^{2\pi i \frac{l}{\tilde{N}}}\,, \qquad l = 1,\ldots,\widetilde{N}-1\,. \tag{4.6.39}$$

Super-conformal line. The following special situation occurs on the line

$$\frac{\mu}{\Lambda} = \left(\frac{m}{\Lambda}\right)^{1/\alpha}\,, \tag{4.6.40}$$

where (4.6.36) degenerates to

$$\sigma^N = \Lambda^{N-\tilde{N}}\sigma^{\tilde{N}}\,. \tag{4.6.41}$$

This equation has two sets of solutions

$$\sigma^{N-\tilde{N}} = \Lambda^{N-\tilde{N}}, \quad \sigma = 0\,, \tag{4.6.42}$$

where the latter solution applies to the conformal regime. Recall that we had a similar situation in the massless case. For this particular configuration we obtain $N-\widetilde{N}$ massive vacua and a sector where σ vanishes. The same considerations made for the massless case apply in the massive case on the whole super-conformal line.

Superconformal vortices and 4d/2d duality at glance. Notice also that the Seiberg-Witten curve of the corresponding four-dimensional theory

$$y^2 = x^N - m^N - \Lambda^{N-\widetilde{N}}(x^{\widetilde{N}} - \mu^{\widetilde{N}})^2\,, \tag{4.6.43}$$

provided that (4.6.40) holds, is reduced to

$$y^2 = \left(x^N - x^{\widetilde{N}}\Lambda^{N-\widetilde{N}}\right)^2\,, \tag{4.6.44}$$

i.e. it has the same form along the whole super-conformal line. As it was mentioned in [181], the the above equation can be written as

$$y^2 = \left(\frac{\partial\widetilde{\mathcal{W}}(x)}{\partial x}\right)^2\,, \tag{4.6.45}$$

where we expand the effective twisted superpotential $\widetilde{\mathcal{W}}$ near the vacuum. The above relation can be viewed as the shortest way to formulate the 4d/2d duality. Indeed, from the Seiberg-Witten curve at the superconformal point (4.6.40) one can read off the twisted superpotential immediately together with scaling dimensions of chiral primary operators of the 2d SCFT.

Small heterotic deformation

We shall now introduce the heterotic deformation in the model. Let us first study corrections to the vacuum expectation values of our fields for small u.

H$\boldsymbol{n}$ phase. We can easily solve the first and second equations of (4.6.4) for D and $|n_0|^2$. The third line is thus an equation for σ

$$\sum_{i=1}^{N-1}(\sigma - m_i)\log\left(1 - \frac{|\sigma - m_0|^2}{|\sigma - m_i|^2}\right) + \sum_{j=1}^{\tilde{N}-1}(\sigma - \mu_j)\log\left(1 + \frac{|\sigma - m_0|^2}{|\sigma - \mu_j|^2}\right) + \\ -(\sigma - m_0)\, r = uN\sigma\,. \tag{4.6.46}$$

We shall now expand the equation above in terms of small deviations from the undeformed case

$$\sigma = m_0 + \delta\sigma, \quad D = 0 + \delta D, \quad r = r_0 + \delta r\,, \tag{4.6.47}$$

which is a consistent procedure when u is small. Equation (4.6.46) gives the following result

$$\delta\sigma = -m\frac{uN}{r_0}\,, \tag{4.6.48}$$

where r_0 is given by (4.6.25). The correction to D can be easily found to be

$$\delta D = -|\delta\sigma|^2\,. \tag{4.6.49}$$

Finally we write the expression for the correction to the renormalized coupling constant

$$\delta r = \frac{N}{2\pi}\delta\sigma\left(\frac{1}{N}\sum_{i=1}^{N-1}\frac{2\,\mathrm{Re}(m_0 - m_i)}{|m_0 - m_i|^2} - \alpha\frac{1}{\tilde{N}}\sum_{j=1}^{\tilde{N}-1}\frac{2\,\mathrm{Re}(m_0 - \mu_j)}{|m_0 - \mu_j|^2}\right) \\ = -\frac{Nu}{2\pi r_0}\left(1 - \alpha f\left(\frac{\mu}{m}\right)\right)\,, \tag{4.6.50}$$

where

$$f(\beta) = \begin{cases} 2\,, & \beta < 1 \\ 1\,, & \beta = 1 \\ 0\,, & \beta > 1\,. \end{cases} \tag{4.6.51}$$

The last equality holds in the large-N limit. Notice that all the corrections contain a $1/r_0$ factor. They all diverge as we approach the Coulomb phase boundary, when $r \to 0$. In this region our approximation fails. Nonetheless, in the boundary region with the Coulomb phase we have $\mu > m$ and the correction is negative, thereby reducing the value of r. We can argue that the $\mathbf{H}n$ phase gets shrunk. This expectation will be confirmed further in the study of the large u case.

H$\boldsymbol{\rho}$ phase. We can proceed analogously to the $\mathbf{H}n$ phase, obtaining

$$\delta\sigma = \mu \frac{uN}{r_0}\,, \tag{4.6.52}$$

whereas the correction to the coupling reads

$$\delta r = \frac{Nu}{2\pi r_0}\left(f\left(\frac{m}{\mu}\right) - \alpha\right)\,. \tag{4.6.53}$$

The same comments holds for this phase. In particular, the correction near the boundary with the Coulomb phase $\mu < m$ is positive, thus it is plausible that the second $\mathbf{H}\rho$ region is also reduced.

C phase. In this phase both n_0 and ρ_0 vanish. We only need (4.6.4) to determine the correction to the VEV of σ. The second equation of (4.6.4) now is

$$\prod_{i=1}^{N-1}(|\sigma - m_i|^2 + D) = \Lambda^{N-\tilde{N}} \prod_{j=1}^{\tilde{N}-1}(|\sigma - \mu_j|^2 - D)\,, \tag{4.6.54}$$

while the third one gives

$$\begin{aligned} &\sum_{i=1}^{N-1}(\sigma - m_i)\log\left(1 + \frac{D}{|\sigma - m_i|^2}\right) \\ &+ \sum_{j=1}^{\tilde{N}-1}(\sigma - \mu_j)\log\left(1 - \frac{D}{|\sigma - \mu_j|^2}\right) \quad = \quad Nu\sigma\,. \end{aligned} \tag{4.6.55}$$

We look again for the solution of the form $\sigma = \sigma_0 + \delta\sigma$. From (4.6.55) we get

$$\begin{aligned} \delta D &= u|\sigma_0|^2\,, && m < |\sigma_0| < \mu\,, \\ \delta D &= -\frac{u}{\alpha}|\sigma_0|^2\,, && \mu < |\sigma_0| < m\,, \\ \delta D &= \frac{u}{1-\alpha}|\sigma_0|^2\,, && \mu, m < |\sigma_0|\,. \end{aligned} \tag{4.6.56}$$

From (4.6.54) we can find the correction to σ_0. Expanding this equation we get

$$\begin{aligned} & N(\sigma_0^N - m^N)^2(\delta\sigma\sigma_0 f(m/\sigma_0) + \delta D g(m/\sigma_0)) = \\ &= \widetilde{N}\Lambda^{N-\widetilde{N}}(\sigma_0^{\widetilde{N}} - \mu^{\widetilde{N}})^2(\delta\sigma\sigma_0 f(\mu/\sigma_0) - \delta D g(\mu/\sigma_0)), \end{aligned} \tag{4.6.57}$$

where $f(\beta)$ is defined in (4.6.51) and $g(\beta)$ is [152]

$$g(\beta) = \frac{1}{N}\sum_{k=1}^{N}\frac{1}{|1-\beta e^{2\pi i k/N}|^2} = \frac{1}{|1-\beta^2|}\,. \tag{4.6.58}$$

We are now ready to write down the results for the two Coulomb regions. First, in the $\mathbf{C}\mu$ region, from (4.6.33) we have $\mu > \sigma_{\mu,0} > m$; information that we need to correctly evaluate (4.6.57). Using also the first line of (4.6.56) we obtain for the μ-vacua

$$\mathbf{C}\mu: \quad |\sigma_\mu| = \Lambda\left(\frac{\mu}{\Lambda}\right)^\alpha\left(1 - \frac{u/2}{1-m^2/\sigma_{\mu,0}^2} + \alpha\frac{u/2}{1-\mu^2/\sigma_{\mu,0}^2}\right)\,. \tag{4.6.59}$$

In the $\mathbf{C}m$ we need to find the corrections for both the m-vacua and the Λ-vacua. Using the right values of σ and the second and third line of (4.6.56) we get respectively

$$\mathbf{C}m: \quad \begin{aligned} |\sigma_m| &= \Lambda\left(\tfrac{m}{\Lambda}\right)^{1/\alpha}\left(1 - \tfrac{u}{2\alpha}\tfrac{1}{1-\mu^2/\sigma_{m,0}^2} + \tfrac{u}{2\alpha^2}\tfrac{1}{1-m^2/\sigma_{m,0}^2}\right)\,, \\ |\sigma_\Lambda| &= \Lambda\left(1 - \tfrac{u}{2(1-\alpha)^2}\tfrac{1}{1-m^2/\Lambda^2} - \tfrac{u\alpha}{2(1-\alpha)^2}\tfrac{1}{1-\mu^2/\Lambda^2}\right)\,. \end{aligned} \tag{4.6.60}$$

Let us make a couple of comments. The presented calculation is valid in the large-N limit for all values of masses m and μ in the two Coulomb regions. All the corrections are negative, thus they reduce the VEV of σ. The calculation breaks down at the boundary of the Coulomb regions with the Higgs phases. Notice that the value of D tends to zero if we approach the massless case (we can reach it, for example, through the $\mathbf{C}m$ phase along the $\mu = 0$ axis). This is a strong hint that the theory in the super-conformal regime does not break supersymmetry. Unfortunately, it is not possible to use this small-u expansion to check the same result for the whole super-conformal line, where the corrections calculated above diverge[17] . The factor $1 - \alpha$ in the expression for the Λ-vacua arises naturally if we recall that the theory is conformally invariant for $\alpha = 1$. In this limiting case there are no Λ-vacua. Moreover, the result is consistent with the expectation that a critical value of u appears in the massive case such that the Λ vacua disappear at larger values of u. As in the massless case, the value of u_{crit} should tend to zero as α approaches 1.

For small u, the vacuum energy is simply given by the heterotic deformation in the potential:

$$E = \frac{uN}{4\pi}|\sigma_0|^2 \,. \tag{4.6.61}$$

As expected, vacuum energy is thus larger for larger values of the VEV of σ. While in the $\mathbf{C}\mu$ region all the vacua have the same energy, in the $\mathbf{C}m$ phase the Λ-vacua acquire a much larger energy, as compared to the μ-vacua. As was noticed in the massless case, Λ-vacua become metastable once we turn on the heterotic deformation. In the next section we consider the large u limit and we will assume that σ is always small in the vacuum, which is a consistent assumption once the Λ-vacua have ceased to exist for a sufficiently large u.

Large heterotic deformation

At generic values of the deformation parameter u we can only rely on numerical solutions of the full equations (4.6.4). Unfortunately, this is quite complicated. In this section we will simplify the problem by looking at large values of u. Then we will compare the results with some full numerical calculations done at generic values of u, as a double check of both results.

H*n* phase. As noted in [159] the large u approximation can be exploited by considering $\sigma \ll m$. Finding D from the first line of (4.6.4) and substituting it in the third line, if we ignore σ compared to m , we get (we also ignore terms enhanced by the

[17] We will be able to prove this in the large u limit in the next section.

logarithms)

$$\frac{4\pi}{N}r - \log\left(\frac{4\pi}{N}r + 1\right) = \log\frac{m^2}{u\Lambda^2} - \alpha\log\frac{m^2+\mu^2}{\Lambda^2}\,, \tag{4.6.62}$$

where r is given by

$$r = \frac{N}{4\pi}\log\left(\frac{\sigma m}{\Lambda^2}\right) - \frac{\widetilde{N}}{4\pi}\log\left(\frac{m^2+\mu^2}{\Lambda^2}\right)\,. \tag{4.6.63}$$

We can now find the boundary of this branch with the Coulomb branch of the theory by forcing $r = 0$ in the above equation. It gives us

$$\left(\frac{m}{\Lambda}\right)^2 = u\left(\frac{m^2+\mu^2}{\Lambda^2}\right)^{\alpha}\,. \tag{4.6.64}$$

This boundary is shown in Fig. 4.9. One can see that it gets shifted towards the large values of m as u increases. The value of the phase transition point on the $\mu = 0$ axis is

$$m_* = \Lambda u^{\frac{1}{2-2\alpha}}\,. \tag{4.6.65}$$

H$\boldsymbol{\rho}$ phase. The procedure in this phase is similar. Now we exploit the approximation $\sigma \ll \mu$ valid in the large u limit. The equation for the renormalized coupling is

$$\frac{4\pi}{N}r + \alpha\log\left(\frac{4\pi}{N}r + \alpha\right) = \log\frac{m^2+\mu^2}{\Lambda^2} - \alpha\log\frac{\mu^2}{u\Lambda^2}\,, \tag{4.6.66}$$

where r is given by

$$r = -\frac{\widetilde{N}}{4\pi}\log\left(\frac{\sigma\mu}{\Lambda^2}\right) + \frac{N}{4\pi}\log\left(\frac{m^2+\mu^2}{\Lambda^2}\right) \tag{4.6.67}$$

The boundary between the **H**ρ and Coulomb is parametrized by the following equation

$$\left(\frac{\mu}{\Lambda}\right)^2 = \frac{u}{\alpha}\left(\frac{m^2+\mu^2}{\Lambda^2}\right)^{1/\alpha}\,. \tag{4.6.68}$$

This boundary is shown in Fig. 4.9. The phase transition for $m = 0$ occurs at

$$\mu_* = \Lambda\left(\frac{\alpha}{u}\right)^{\frac{\alpha}{2-2\alpha}}\,. \tag{4.6.69}$$

C phases. In [159] a new phase within the Coulomb regime at large u was found where $\langle\sigma\rangle = 0$ and the residual discrete symmetry was not broken. For the masses which are exponentially small in u, a VEV for σ is restored, and a Coulomb phase with broken symmetry appears. To study how their picture is generalized for the weighted sigma-model, we search for a broken Coulomb phase in the two following regions

$$\begin{aligned} \mathbf{C}\mu : &\quad m \ll \mu, \\ \mathbf{C}m : &\quad \mu \ll m, \end{aligned} \tag{4.6.70}$$

where we shall use the following assumptions[18]

$$\begin{aligned} \mathbf{C}\mu : &\quad \sigma,\, m \ll \mu,\, D \\ \mathbf{C}m : &\quad \sigma,\, \mu \ll m,\, D\,. \end{aligned} \tag{4.6.71}$$

Let us start with the $\mathbf{C}\mu$ phase. By employing the approximations (4.6.71) in (4.6.4) we get the following equations

$$\begin{aligned} D &= \Lambda^2 \left(\left(\frac{\mu}{\Lambda}\right)^2 - \frac{D}{\Lambda^2} \right)^\alpha , \\ 2\alpha\sigma^2 \log\left(\frac{\mu^2 - D}{\Lambda\mu}\right) - u\sigma^2 &= \begin{cases} 2\sigma^2 \log\left(\frac{\sigma}{\Lambda}\right) & m < \sigma \\ 2\sigma^2 \log\left(\frac{m}{\Lambda}\right) & m > \sigma \end{cases} . \end{aligned} \tag{4.6.72}$$

Notice that the equations above admit only the solution $\sigma = 0$ as long as $m > \sigma$. This is the Coulomb symmetric phase. For smaller m, we pick the first line in (4.6.72), wich gives non-trivial values for σ. We can actually determine the boundary $\mathbf{C}\mu$-$\mathbf{C}$s by solving the above equations for $m < \sigma$ and then imposing the condition $\sigma = m$. The $\mathbf{C}\mu$ phase gets extended towards smaller values of μ, when it will eventually meet the $\mathbf{H}\rho$ phase. As a final check, let us further simplify (4.6.72) in the large μ limit

$$\begin{aligned} D &= \Lambda^2 \left(\frac{\mu}{\Lambda}\right)^{2\alpha} , \\ \sigma &= \Lambda \left(\frac{\mu}{\Lambda}\right)^{\alpha} e^{-u/2} , \end{aligned} \tag{4.6.73}$$

which is consistent with the results of [159]. In this region, in fact, our model reduces to the ordinary $\mathbb{CP}^{N-1}$ model, with the new scale $\tilde{\Lambda} = \Lambda(\mu/\Lambda)^\alpha$.

[18] These assumptions are justified because we search for exponentially small values of σ and m (or μ) and we expect D to be large for large u.

The **C**m phase is completely analogous. The correct approximation leads us now to the following equations

$$\begin{aligned} m^2 + D &= \Lambda^2 \left(-\frac{D}{\Lambda^2}\right)^{\alpha} , \\ 2\sigma^2 \log\left(\frac{m^2 - D}{\Lambda m}\right) - u\sigma^2 &= \alpha \begin{cases} 2\sigma^2 \log\left(\frac{\sigma}{\Lambda}\right) & \mu < \sigma \\ 2\sigma^2 \log\left(\frac{\mu}{\Lambda}\right) & \mu > \sigma \end{cases} . \end{aligned} \tag{4.6.74}$$

We proceed as for the **C**μ phase to determine the boundary with the symmetric phase. The result is shown in Fig. 4.9. If we look at very small values of m, we can simplify (4.6.74) a bit more. From the first equation we see that very small m implies $m^2 \gg D$. Finally we get

$$\begin{aligned} -D &= \Lambda^2 \left(\frac{m}{\Lambda}\right)^{2/\alpha} , \\ \sigma &= \Lambda \left(\frac{m}{\Lambda}\right)^{1/\alpha} e^{-u/(2\alpha)} . \end{aligned} \tag{4.6.75}$$

Super-conformal line. Keeping the results of this section in mind, it is now easy to check that supersymmetry is effectively unbroken as we approach the super-conformal line

$$\frac{\mu}{\Lambda} = \left(\frac{m}{\Lambda}\right)^{1/\alpha} . \tag{4.6.76}$$

Since we are looking into the **C**s phase, we put from the beginning $r = 0$ and $\sigma = 0$ in the second line of(4.6.4)

$$\left(m^2 + D\right)^N = \Lambda^{N-\tilde{N}} \left(\mu^2 - D\right)^{\tilde{N}} , \tag{4.6.77}$$

which is clearly solved by $D = 0$ provided that (4.6.76) holds. This condition is enough to show unbroken supersymmetry. One can also directly check that the vacuum energy vanishes. In general, in the **C**s phase D does not vanish, and supersymmetry is generically broken.

4.7 Spectrum of the $\mathbb{WCP}^{N-1}$ Model

As was shown by Witten in the supersymmetric $\mathbb{CP}^{N-1}$ sigma-model photon is massive due to a coupling to fermions and its mass is given by the chiral anomaly [58]. However, the photon remains massless in the bosonic $\mathbb{CP}^{N-1}$ sigma model. It was shown in [159] that once the twisted masses are nonzero and the heterotic deformation is turned on, the photon becomes massless in the symmetric Coulomb phase. The authors also call this phase confining, since existence of long range interactions with massless carrier allows bound states of particles ("kinks"). In $\mathbb{CP}^{N-1}$ sigma-model only $\bar{n}n$ mesons could be formed, our model also admits, in principle, $\bar{\rho}\rho$ and $n\rho$ mesons. Below we calculate the photon mass at different values of twisted masses m and μ as well as the heterotic deformation parameter u, and show that it vanishes in the symmetric Coulomb phase as is prescribed by the unbroken discrete symmetry. Since analogous calculations in supersymmetric sigma-models have been previously performed (see, for instance [158, 159, 38]) here we shall just list our result. Generic expressions for the effective coupling constants can be found in Sec. B.2.

The one-loop effective Lagrangians for the $\mathbb{WCP}^{N_F-1}$ $(0,2)$ sigma-model reads

$$\mathcal{L} = -\frac{1}{4e_\gamma^2}F_{\mu\nu}^2 + \frac{1}{e_{\sigma\,1}^2}(\partial_\mu \mathfrak{Re}\,\sigma)^2 + \frac{1}{e_{\sigma\,2}^2}(\partial_\mu \mathfrak{Im}\,\sigma)^2 + i\mathfrak{Im}(\bar{b}\,\delta\sigma)\epsilon_{\mu\nu}F^{\mu\nu} - V_{\text{eff}}(\sigma) + \text{Fermions}\,. \tag{4.7.1}$$

We shall only consider photon-scalar mixing in this section, that is why we specified only bosonic part of the action. In the above expression we denote $\sigma = \sigma_0 + \delta\sigma$, where σ_0 is the VEV of the field σ in the vacuum where our effective theory lives. In (4.7.1) effective potential $V_{\text{eff}}(\sigma)$ is given by (4.6.3), gauge and scalar couplings can be calculated from the corresponding one-loop Feynman diagrams. Gauge field is coupled to the imaginary part of σ and the mixing can straightforwardly be generalized from [159]. In Fig. 4.11 one-loop diagrams which contribute to the mixing are shown. The result is given by

$$\begin{aligned} b &= \frac{N}{4\pi}\left(\frac{1}{N}\sum_{i=1}^{N-1}\frac{1}{\bar{\sigma}_0 - \bar{m}_i} - \alpha\frac{1}{\tilde{N}}\sum_{i=1}^{\tilde{N}-1}\frac{1}{\bar{\sigma}_0 - \bar{\mu}_i}\right) \\ &= \frac{N}{4\pi}\frac{1}{\bar{\sigma}_0}\left(f\left(\frac{m}{|\sigma_0|}\right) - \alpha f\left(\frac{\mu}{|\sigma_0|}\right)\right)\,, \end{aligned} \tag{4.7.2}$$

where the function $f(\beta)$ was introduced in (4.6.51) and we assumed that $\sigma_0 \neq 0$. If the VEV for σ vanishes at a vacuum (which happens in the symmetric **C**s phase) then the result is different

$$b = \frac{1}{4\pi}\left(-\frac{1}{m} + \frac{\alpha}{\mu}\right)\,. \tag{4.7.3}$$

The photon mass can be obtained by diagonalization of the mass Lagrangian

$$m_\gamma = e_{\sigma\,2} e_\gamma |b| \,. \tag{4.7.4}$$

We can immediately see from (4.7.3), (4.7.4) and the formulae for the couplings (B.2.2), (B.2.3) that in the symmetric **C**s phase photon is massless in the large-N approximation. This result is universal, it is dictated by the unbroken discrete $\mathbb{Z}_{N-\tilde{N}}$ symmetry present in the **C**s phase, and it is independent of the value of the heterotic deformation.

Let us now calculate the photon case for zero and nonzero values of u in the strongly coupled Coulomb phases $\mathbf{C}m$ and $\mathbf{C}\mu$, where discrete symmetries are spontaneously broken by the VEVs of σ.

4.7.1 Undeformed $(2,2)$ Model

If the $(2,2)$ supersymmetry is unbroken the masses of the particles of the same multiplet should be the same

$$m_\gamma = m_\sigma = m_{\text{fermi}} \,. \tag{4.7.5}$$

Using (4.7.4) we can easily find

$$m_\gamma = \frac{A}{\dfrac{|\sigma_0|}{||\sigma_0|^2 - m^2|} + \alpha \dfrac{|\sigma_0|}{||\sigma_0|^2 - \mu^2|}} \,, \tag{4.7.6}$$

where the numerator reads

$$A = \left| f\left(\frac{m}{|\sigma_0|}\right) - \alpha f\left(\frac{\mu}{|\sigma_0|}\right) \right| \,. \tag{4.7.7}$$

Depending on the VEV σ_0 the masses (4.7.5) can have different values, in particular, they can vanish.

Λ-Vacua. The Λ-vacua (4.6.34) appear only in the $\mathbf{C}m$ region. The mass of the $\mathcal{N}=2$ multiplet is given by (4.7.6) with $|\sigma_0| = \Lambda$ and the following numerator

$$A = 2(1-\alpha) \,. \tag{4.7.8}$$

0-Vacua. In the Coulomb phase there are also 0-vacua which are the solutions of the vacua equations in the two regions of the parameter space $\mathbf{C}m$ and $\mathbf{C}\mu$ (see Fig. 4.8). In this case in formulae (4.7.6) and (4.7.7) we should use we have

$$\begin{aligned} A &= 2\alpha\,, \quad |\sigma_0| = \Lambda\left(\frac{m}{\Lambda}\right)^{1/\alpha} \qquad \text{in } \mathbf{C}m \text{ phase} \\ A &= 2\,, \quad |\sigma_0| = \Lambda\left(\frac{\mu}{\Lambda}\right)^{\alpha} \qquad \text{in } \mathbf{C}\mu \text{ phase}\,. \end{aligned} \tag{4.7.9}$$

4.7.2 Deformed $(0, 2)$ Model

As we have observed in the previous sections, Λ vacua become metastable as we increase u and for $u > u_{\text{crit}}$ disappear completely. Keeping this in mind let us focus on 0-vacua, which continue to exist for any value of the deformation, assuming that u is large enough for the approximations we have used in the end of Sec. 4.6 to be valid.

In the $\mathbf{C}m$ phase we get

$$\begin{aligned} \frac{4\pi}{Ne_\gamma^2} &= \frac{1}{m^2} + \frac{\alpha}{3}\frac{1}{\Lambda^2}\left(\frac{\Lambda}{m}\right)^{2/\alpha} + \frac{2\alpha}{3}\frac{1}{\Lambda^2\left(\frac{m}{\Lambda}\right)^{2/\alpha}e^{-\frac{u}{\alpha}} - \mu^2}\,, \\ \frac{4\pi}{Ne_{\sigma\,2}^2} &= \frac{1}{m^2} + \frac{\alpha}{\Lambda^2\left(\frac{m}{\Lambda}\right)^{2/\alpha}e^{-\frac{u}{\alpha}} - \mu^2}\,, \end{aligned} \tag{4.7.10}$$

where we have neglected all the terms as in the calculation of VEV D and σ_0. The photon mass by means of (4.7.4) is then given by

$$m_\gamma = \sqrt{6}\,\Lambda\left(\frac{\Lambda}{m}\right)^{1/\alpha}\left(\left(\frac{m}{\Lambda}\right)^{2/\alpha} - \left(\frac{\mu}{\Lambda}\right)^2 e^{u/\alpha}\right)\mathrm{e}^{-\frac{u}{2\alpha}}\,, \tag{4.7.11}$$

where we have used (4.7.7) which implies that $A = \alpha$ in the $\mathbf{C}m$ phase. The above expression may seem to diverge at large u, but we do not need to forget that the expression in the parentheses above should be bigger that zero for all u. The bigger u is the smaller is μ and the whole expression becomes suppressed.

Analogously, the photon mass in the $\mathbf{C}\mu$ phase reads

$$m_\gamma = \sqrt{6}\Lambda\left(\frac{\Lambda}{\mu}\right)^{\alpha}\left(\left(\frac{\mu}{\Lambda}\right)^{2\alpha} - \left(\frac{\mu}{\Lambda}\right)^2 e^{u}\right)e^{-u/2}\,, \tag{4.7.12}$$

where we used that $A = 2$.

4.8 Conclusions

We have started this Chapter with constructing of the heterotic $\mathcal{N} = (0,1)$ supersymmetric O(N) sigma model. Similarly to the $\mathbb{CP}^N$ sigma model it can be solved at large N. As a result we found the effective potential of the theory, which allowed us to get the spectrum. It appeared to be very much reminiscent of the spectrum of the $(0,2)$ $\mathbb{CP}^N$ sigma model in spite of the fact that the latter possesses a Kähler structure, and therefore a larger supersymmetry, whereas the former does not. For all values of the deformation parameter there is a massless fermion. For $\gamma = 0$ it is the extra left-handed sterile fermion, while for $\gamma \neq 0$ it is a Goldstino (mixture of the additional left-handed fermion and the initial right-handed one) corresponding to the supersymmetry breaking. The existence of the Goldstino is the evidence of the fact that the supersymmetry becomes broken at any nonzero value of the deformation parameter. Another proof of the supersymmetry breaking is the presence of the nonzero vacuum energy density. Also fields from the same multiplet acquire different masses when the deformation is turned on.

The low energy one-loop effective potential has two vacua. From the naive point of view there should be one kink (and one antikink) interpolating between those vacua. However, such an argument leads to a wrong conclusion about the number of kinks for the $\mathbb{CP}^N$ sigma model. For the latter kink dynamics has been studied to a very high extent (see [35, 40, 174] and [160]). It was found that the different kinks correspond to the different integration contours in the σ plane and that the number of kinks interpolating between two different vacua depends not only on N but also on the number of vacua separating those two. Therefore, the question about the number of kinks in the O(N) theory should be worked out more carefully.

In the current Chapter we used Majorana $\mathcal{N} = 1$ superfield formalism while constructing the action of the O(N) sigma-model. The heterotic deformation was rendered by the coupling of the extra chiral superfield to the auxiliary superfield we have used to build up the undeformed theory. We then generalize this construction to the heterotic $(0,2)$ $\mathbb{CP}^N$ and weighted $\mathbb{CP}^N$ sigma models by adding an additional auxiliary superfield and modifying the interaction between the chiral superfield and the auxiliary superfields. One may now try to use our methods for investigating sigma models with twisted masses and their chiral deformations.

Perhaps at this point the most intriguing question to be answered is from which four-dimensional bulk theory does the O(N) sigma model originate from (if any). The analogy with U(N) Yang-Mills theory and $\mathbb{CP}^{N-1}$ sigma models discussed in [147] and references therein is not clear – what number of supersymmetries does the bulk theory have to have? It is interesting to understand by means of what (extended object or mechanism) the bulk supersymmetry gets broken.

Further on in this Chapter we solved, in the large-N approximation, a particular kind of two-dimensional $\mathcal{N} = (0,2)$ non-linear sigma-model which we referred to as "heterotic" $\mathbb{WCP}^{N_F}$. As it was already noticed, we didn't study the most general kind of heterotic deformations, rather, we focused on the particular case relevant to the study of non-Abelian vortices. The main result of the Chapter is the determination of the phase diagram of the theory summarized in Fig. 4.9, which generalize the well-known $\mathcal{N} = (2,2)$ case (see Fig. 4.8), once a heterotic deformation is turned on. In addition to the two Coulomb phases $\mathbf{C}m$, $\mathbf{C}\mu$ and the two Higgs phases $\mathbf{H}n$ and $\mathbf{H}\rho$, already present at zero values of the deformation parameter, a new $\mathbf{C}s$ [159] phase emerges around what we called the super-conformal line: $\mu/\Lambda = (m/\Lambda)^{1/\alpha}$. On this line some excitations become massless, and the theory is described by a super-conformal theory of the minimal A_{N-1} type [181, 182, 183, 184]. A discrete $\mathbb{Z}_{N-\widetilde{N}}$ symmetry is broken in all phases but it is preserved in the $\mathbf{C}s$ phase. Supersymmetry is also generically broken. The vacuum energy and the expectation value of the auxiliary field D vanish as we approach the super-conformal line suggesting that supersymmetry is unbroken on the line. The $\mathbf{C}\mu$ phase contains two well-defined sets of vacua which we called μ-vacua and Λ-vacua (in this region $\mu < \Lambda$). Once the heterotic deformation is turned on, the Λ vacua become metastable. For sufficiently large values of the deformation ($u > u_{crit}$), the Λ vacua do not exist at all. All the phase transitions look like being of the second order [159], but it is important to stress that this is an effect of the leading order large-N approximation: at finite N, they should rather look like sharp crossovers.

The vacuum diagram in Fig. 4.9 also gives us the spectrum of non-Abelian vortices in the associated $\mathcal{N} = 1$ four-dimensional gauge theory. In particular, supersymmetry breaking means that vortices are not BPS saturated at the quantum level. Furthermore, the vacuum energy (see (4.6.61)) is translated into a correction to the classical formula

$T = 2\pi\xi$ for the tension of vortices[19] . In the $\mathbf{C}m$ regime, for example, $N - \widetilde{N}$ vortices become metastable and eventually disappear from the spectrum. Moreover, as was shown in [58] and later discussed in [59, 63, 152, 159], the fundamental fields n's and ρ's, together with their fermionic superpartners, can be interpreted as kinks interpolating between two vacua, or vortices. As already mentioned, kinks correspond to monopoles in the four-dimensional gauge theory. The study of spectrum of the model we considered, thus, gives informations about the monopole spectrum in $\mathcal{N} = 1$ theories.

The discovery that non-Abelian vortices are the precise link between two and four-dimensions is a recent exciting result in the study of supersymmetric gauge theories. Given the tighten relationship between two-dimensional $\mathcal{N} = (2, 2)$ and four-dimensional $\mathcal{N} = 2$ theories, it is tempting to explore systems with less supersymmetry, to find if and how the physics of $\mathcal{N} = 1$ theories is "seen" on the two-dimensional $\mathcal{N} = (0, 2)$ theory, and vice versa. Interesting results have already been found when no additional hypermultiplets have been considered [164, 165] where a qualitative matching of the supersymmetry braking pattern and of the meson spectrum was observed. Our results are the first important step to extend this line of research when an additional number of flavors $\widetilde{N}$ is included. In this case, physics of $\mathcal{N} = 1$ SQCD varies dramatically [185, 186]. The most remarkable feature is the existence of an electric-magnetic duality (Seiberg duality). More recently it was found that dynamical supersymmetry breaking is a quite common feature [187]. It would be interesting to search for signs of these phenomena on the two-dimensional side. With the better understanding of the quantum physics of vortices when additional flavors are included, it should be possible to extend, for example, the analysis made in [149, 188, 189], where the role of vortices in the context of Seiberg duality was investigated. In these works, the dual quarks of the "electric" theory were interpreted as monopoles of the "magnetic" theory.

The investigations on "heterotic" $\mathbb{WCP}^{N_F-1}$ are by no means finished here. First of all we think it may be interesting to further study the model on the super-conformal line. This line as a direct counterpart in the four-dimensional gauge theory, where the coalescence of multiple vacua give rise to the appearance of super-conformal vacua called Argyres-Douglas points, where the relevant degrees of freedom are mutually non-local [112, 180] . An analysis of this kind has been initiated for example in [181]. A

[19] ξ is the four-dimensional Fayet-Iliopoulos

comprehensive study of kinks is also in order. The spectrum of kinks is in fact related to the monopole spectrum in four dimensions. This study started in [160]. A careful study of kinks interpolating between different kind of vacua is still to be done.

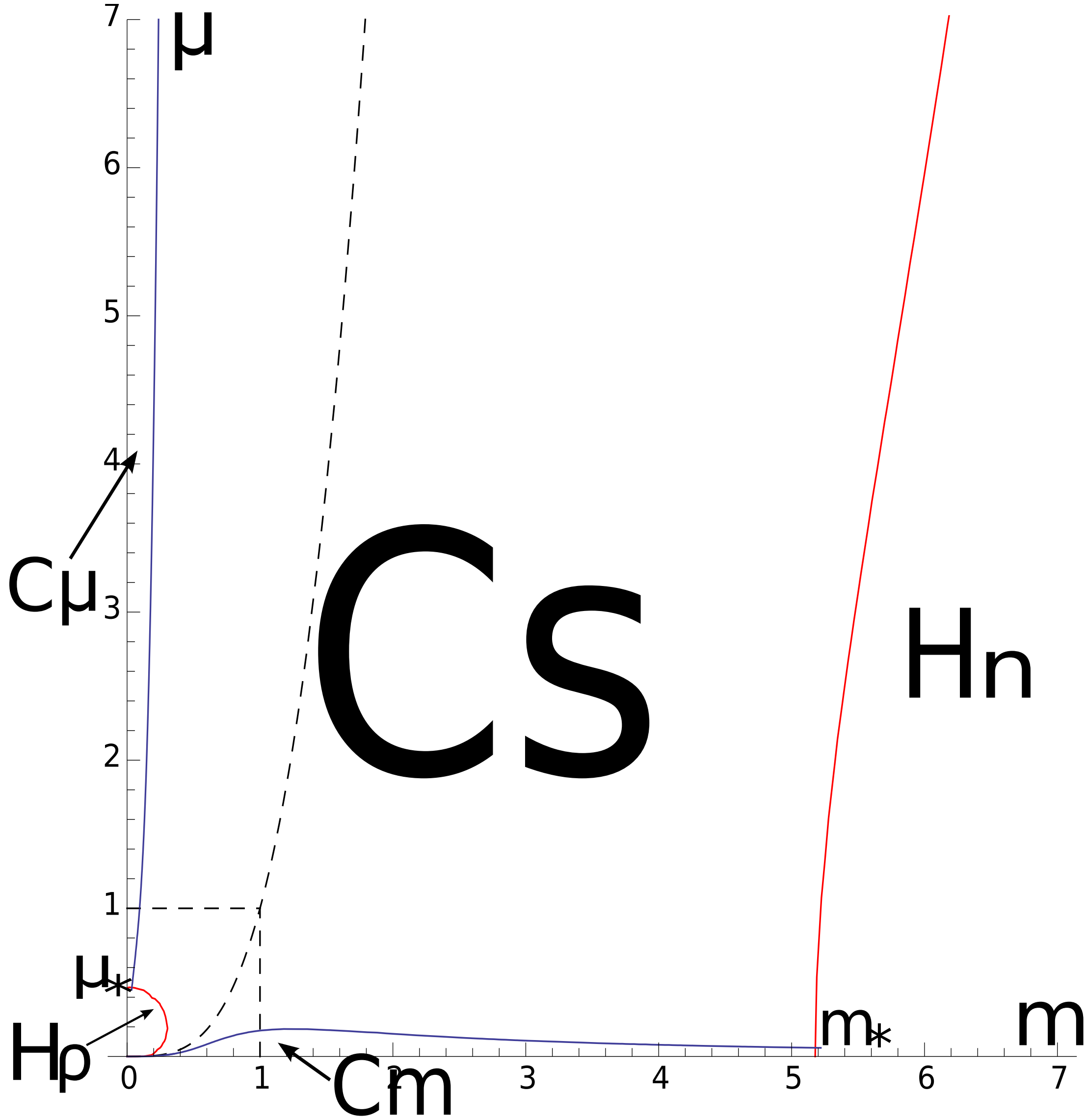

Figure 4.9: Phase Diagram for $u = 10$ and $\alpha = .3$. At $u = 0$ the **C** phase of the theory had non-zero VEV for σ everywhere but on the graph $\mu/\Lambda = (m/\Lambda)^{1/\alpha}$. As we increase u the domain with unbroken $\mathbb{Z}_{\tilde{N}}$ symmetry (**Cs** phase) gets widened pushing **C** phases with broken symmetry towards the axes. At very large values of u the latter phase occupies only two small domains as is shown in the figure. The Higgs phases **H**n and **H**ρ are also pushed apart by blowing **Cs** phase, one can see it from (4.6.65) and (4.6.69). In the limit $u \to \infty$ the theory has only **Cs** phase. To show all the phases, we magnified **C**m region by a factor of 10^7 and the **C**μ by a factor of 20.

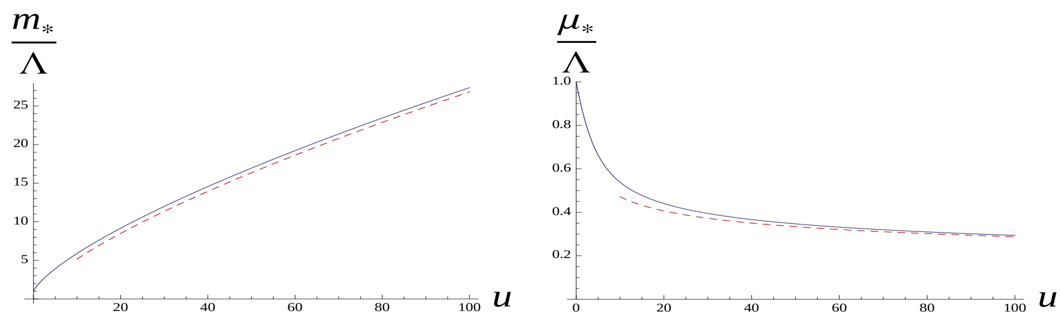

Figure 4.10: Dependence of m_* (left plot) and μ_* (right plot) as a function of u. Here we compare numerical solutions (solid lines) and the analytical values (4.6.65) and (4.6.69) (dashed lines) for $\alpha = 0.3$.

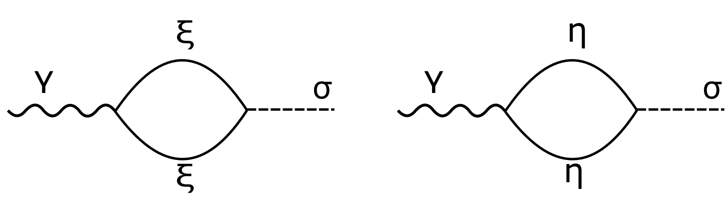

Figure 4.11: One-loop diagrams which contribute to the the photon-scalar anomalous mixing.

Chapter 5

Conclusions and Open Questions

In this concluding Chapter we would like to highlight some directions of possible future work on $\mathcal{N} = 2$ supersymmetry and outline the connection with other developments which were not represented in the bulk of the current thesis.

In Chapter 3 we only briefly discussed wall crossing phenomena in two dimensional models, however, there is a vast literature existent nowadays on this topic. In four dimensions WCF seems to be more intricate than in 2d, however, as recent developments suggest [190, 191, 192], they may not me as complicated as they seem at least for certain class of theories (so-called compete $\mathcal{N} = 2$ theories). According to the authors whose papers we have just cited, a 4d BPS quiver (a picture which uniquely describes BPS spectrum of the theory in a chamber of the moduli space of the theory with certain identifications) can be viewed as a scheme of 2d vacua (which minimize twisted superpotential) and arrows between the nodes represent simple kinks interpolating between these vacua! One can travel to another chamber in the moduli space, and when the corresponding wall between the two chambers is crossed, a certain transformation of the quiver has to be made, which is called mutation (physically it roughly corresponds to a Seiberg duality [185] in the four dimensional language). Such a correspondence exists for all complete theories in 4d. It would be interesting to find a relationship between this fact and the 4d/2d duality we have discussed in the current thesis. Indeed, for our story we need a BPS vortex and the 2d theory can be identified as its worldvolume low energy effective theory. A typical example of such theory, as we have discussed it in the text, would be a SQCD. However, the class of theories considered in [190, 191, 192] is wider, for instance it includes theories which do not have any flavors, e.g. pure Yang-Mills theory. Yet, the relationship with a 2d theory can be found. One can pose the following question: What is the symmetry (both on the 2d side and in the 4d side of the duality) which relates two pairs of dual theories?

In Chapter 4 we have mostly discussed the 2d (0,2) theory. As we have reported in the text, a relationship with the 4d bulk theory was established in [151], where the (0,2) theory was derived as a wolrdvolume theory of the heterotic vortex string. It would be nice to have an independent confirmation of this observation, which to the best of our knowledge is not present in the literature. We have presented in the text a family of superconformal theories in two dimensions and a concise way to related them to the 4d counterparts (4.6.45). For $\mathcal{N}=1$ theories in 4d and for $(0,2)$ theories in 2d the proposed answer cannot be made as simple as in the $\mathcal{N}=2$ case as there is no Seiberg-Witten curve any longer. Yet, some methods of investigating such SCFTs are present in the literature and we should take advantage of them in order to answer the question posed above.

The list can be completed with many other interesting problems related to non-abelian mirror symmetry, Langlands duality in connection with the 4d/2d correspondence.

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Appendix A

Notations

A.1 4d Notations

In this work we are intended to use Euclidean signature. We benefit from this while studying static configurations, where in the gauge $A_4 = 0$ the Lagrangian is nothing but the energy density. Below we list some definitions and conventions. Raising and lowering of spinor indices is performed by means of Levi-Civita symbol

$$\epsilon^{12} = -\epsilon^{21} = -\epsilon_{12} = \epsilon_{21} = 1\,, \tag{A.1.1}$$

the same for the Levi-Civita symbol with dotted indices. The definition is consistent with

$$\epsilon^{\alpha\beta}\epsilon_{\beta\gamma} = \delta^{\alpha}_{\gamma}\,. \tag{A.1.2}$$

Scalars can then be obtained by contracting spinor indices, for example $\eta_\alpha\chi^\alpha = -\eta^\alpha\chi_\alpha$. Vector indices are contracted with Euclidean metric δ_{mn}.

Sigma matrices

$$\begin{aligned}(\sigma^m)_{\alpha\dot{\alpha}} &= (-i\tau^1,\, -i\tau^2,\, -i\tau^3,\, 1)\,,\\ (\bar{\sigma}^m)_{\dot{\alpha}\alpha} &= (i\tau^1,\, i\tau^2,\, i\tau^3,\, 1) = ((\sigma^m)_{\alpha\dot{\alpha}})^\dagger\,,\end{aligned} \tag{A.1.3}$$

where $\tau^{1,2,3}$ are standard Pauli matrices. Thus for σ^m the undotted index goes first, whereas for $\bar{\sigma}^m$ it is the last.

Lorentz projectors

$$(\sigma^{mn})_{\alpha\beta} = \tfrac{1}{4}\left((\sigma^m)_{\alpha\dot\alpha}(\bar\sigma^n)^{\dot\alpha}{}_{\beta} - (\sigma^n)_{\alpha\dot\alpha}(\bar\sigma^m)^{\dot\alpha}{}_{\beta}\right) ,$$
$$(\bar\sigma^{mn})_{\dot\alpha\dot\beta} = \tfrac{1}{4}\left((\bar\sigma^m)_{\dot\alpha\alpha}(\sigma^n)^{\alpha}{}_{\dot\beta} - (\bar\sigma^n)_{\dot\alpha\alpha}(\sigma^m)^{\alpha}{}_{\dot\beta}\right) . \qquad \text{(A.1.4)}$$

Chiral and antichiral electromagnetic field strength

$$F_{\alpha\beta} = -\tfrac{1}{2}F_{mn}(\sigma^m)_{\alpha\dot\alpha}(\sigma^n)_{\beta}{}^{\dot\alpha} = -(\vec E + \vec B)\cdot\vec\sigma\,\sigma^2 ,$$
$$\bar F^{\dot\alpha\dot\beta} = \tfrac{1}{2}F_{mn}(\bar\sigma^m)^{\dot\alpha\alpha}(\bar\sigma^n)^{\dot\beta}{}_{\alpha} = (-\vec E + \vec B)\cdot\vec\sigma\,\sigma^2 . \qquad \text{(A.1.5)}$$

A.2 2d Notations

Gamma matrices

$$\gamma^0 = \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \quad \gamma^1 = i\sigma_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} , \quad \gamma^5 = \gamma^0\gamma^1 = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \qquad \text{(A.2.1)}$$

Antisymmetric symbol

$$\varepsilon_{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} . \qquad \text{(A.2.2)}$$

Left and right coordinates

$$x_L = x_0 + x_1, \qquad \partial_0 = \partial_L + \partial_R, \qquad \partial_L = \tfrac{1}{2}\left(\partial_0 + \partial_1\right) ,$$
$$x_R = x_0 - x_1, \qquad \partial_1 = \partial_L - \partial_R, \qquad \partial_R = \tfrac{1}{2}\left(\partial_0 - \partial_1\right) . \qquad \text{(A.2.3)}$$

Left and right fermions

$$\psi = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix} \qquad \text{(A.2.4)}$$

are eigenstates of γ^5

$$\gamma^5\psi_{R,L} = \pm\psi_{R,L} . \qquad \text{(A.2.5)}$$

Derivatives and integrals

$$\int d^2\theta\,\bar\theta\theta = \int d\theta_1\,d\theta_2\bar\theta\theta = \int d\theta_1\,d\theta_2\,2i\theta_2\theta_1 = 2i ,$$
$$\frac{\partial}{\partial\bar\theta_\alpha}\theta_\beta = \gamma^0_{\alpha\beta} . \qquad \text{(A.2.6)}$$

Contraction of indices for Majorana fermions

$$\bar{\psi}\theta = \psi^\dagger\gamma^0\psi = \psi^T\gamma^0\psi = i\theta_2\psi_1 - i\theta_1\psi_2 = \bar{\theta}\psi\,, \tag{A.2.7}$$

$$\bar{\theta}\gamma^{0,1}\theta = (\theta_1)^2 = (\theta_2)^2 = 0\,, \tag{A.2.8}$$

$$\begin{aligned} \bar{\theta}\theta &= 2i\theta_2\theta_1 = -2i\theta_1\theta_2\,, \\ \theta_\alpha\theta_\beta &= \frac{i}{2}\epsilon_{\alpha\beta}\,\bar{\theta}\theta = -\tfrac{1}{2}\gamma^0_{\alpha\beta}\bar{\theta}\theta\,, \\ \bar{\theta}_\alpha\theta_\beta &= \tfrac{1}{2}\delta_{\alpha\beta}\,\bar{\theta}\theta\,. \end{aligned} \tag{A.2.9}$$

Some relations for gamma matrices

$$\begin{aligned} \gamma^{\mu T} &= -\gamma^0\gamma^\mu\gamma^0\,, \\ \gamma^{\mu\dagger} &= \gamma^0\gamma^\mu\gamma^0\,. \end{aligned} \tag{A.2.10}$$

Supersymmetry transformations. Coordinate transformations

$$\begin{aligned} x_\mu &\to x_\mu + i\bar{\varepsilon}\gamma_\mu\theta, \\ \theta_\alpha &\to \theta_\alpha + \varepsilon_\alpha, \\ \bar{\theta}_\alpha &\to \bar{\theta}_\alpha + \bar{\varepsilon}_\alpha. \end{aligned} \tag{A.2.11}$$

Chiral superfield

$$\Phi = \phi + \bar{\theta}\psi + \tfrac{1}{2}\bar{\theta}\theta F\,, \tag{A.2.12}$$

obeys the following supertransformations

$$\begin{aligned} \delta n &= \bar{\varepsilon}\psi, \\ \delta\psi &= -i\partial_\mu n\gamma^\mu\varepsilon + \varepsilon F, \\ \delta F &= -i\bar{\varepsilon}\gamma^\mu\partial_\mu\psi\,. \end{aligned} \tag{A.2.13}$$

A natural generalization of the chiral superfield is the isovector superfield

$$\mathcal{N}^i = n^i + \bar{\theta}\psi^i + \tfrac{1}{2}\bar{\theta}\theta F^i\,, \quad i = 1, \ldots, N\,. \tag{A.2.14}$$

Supertransformations act as

$$\delta\Phi = \bar{\epsilon}\mathfrak{Q}\Phi \tag{A.2.15}$$

Supersymmetry generators

$$\mathfrak{Q}_\alpha = \frac{\partial}{\partial \bar{\theta}_\alpha} - i\,(\gamma^\mu \theta)_\alpha\,\partial_\mu\,, \tag{A.2.16}$$

Covariant derivative

$$\mathcal{D}_\alpha = \frac{\partial}{\partial \bar{\theta}_\alpha} + i\,(\gamma^\mu \theta)_\alpha\,\partial_\mu\,, \tag{A.2.17}$$

anticommutes with the supercharge

$$\{\mathfrak{Q}_\alpha, \mathcal{D}_\beta\} = 0\,. \tag{A.2.18}$$

Chiral Notation. One can use the following identification

$$x^\mu = \gamma^\mu_{\alpha\beta} x^{\alpha\beta}\,, \quad \mu = 1,2\,, \quad \alpha,\beta = 1,2\,. \tag{A.2.19}$$

Having done so we can write

$$\mathfrak{Q}_\alpha = \epsilon_{\alpha\beta}\frac{\partial}{\partial\theta_\beta} + \theta_\beta \partial_{\alpha\beta}\,. \tag{A.2.20}$$

Accordingly we have

$$\begin{aligned}
\{\mathfrak{Q}_1, \mathfrak{Q}_1\} &= 2\partial_{12} = 2\left(i\frac{\partial}{\partial t} - i\frac{\partial}{\partial x}\right) = 2(\mathcal{H} + \mathcal{P})\,,\\
\{\mathfrak{Q}_2, \mathfrak{Q}_2\} &= 2\partial_{21} = 2\left(i\frac{\partial}{\partial t} + i\frac{\partial}{\partial x}\right) = 2(\mathcal{H} - \mathcal{P})\,,\\
\{\mathfrak{Q}_1, \mathfrak{Q}_2\} &= 0\,,
\end{aligned} \tag{A.2.21}$$

where $\mathcal{H}$ and $\mathcal{P}$ are energy and momentum charges respectively. Covariant derivative reads

$$\mathcal{D}_\alpha = i\epsilon_{\alpha\beta}\frac{\partial}{\partial\theta_\beta} - i\theta_\beta \partial_{\alpha\beta}\,. \tag{A.2.22}$$

Appendix B

SUSY Gauge Theories

B.1 Superfield Formalism

In this section we present the superfield derivation of the Lagrangian (4.5.1) of the weighted sigma model without twisted masses. Inclusion of twisted masses (4.5.7) can naturally be realized in the brane picture. Here we briefly review the derivation of the Lagrangian given in Ref. [38]

$$\mathcal{L}^{het}_{W\mathbb{CP}^N} = \int d^2\theta \Bigg[\tfrac{1}{4}\varepsilon_{\beta\alpha}(\mathcal{D}_\alpha + i\mathcal{A}_\alpha)\mathcal{N}_i^\dagger(\mathcal{D}_\beta - i\mathcal{A}_\beta)\mathcal{N}_i + \tfrac{1}{4}\varepsilon_{\beta\alpha}(\mathcal{D}_\alpha - i\mathcal{A}_\alpha)\mathcal{R}_j^\dagger(\mathcal{D}_\beta + i\mathcal{A}_\beta)\mathcal{R}_j + \\ + i\mathcal{S}\left(\sum_{i=1}^{N}\mathcal{N}_i^\dagger\mathcal{N}_i - \sum_{j=1}^{\widetilde{N}}\mathcal{R}_j^\dagger\mathcal{R}_j - r_0\right) \\ + \tfrac{1}{4}\varepsilon_{\beta\alpha}\mathcal{D}_\alpha\mathcal{B}^\dagger\,\mathcal{D}_\beta\mathcal{B} + \Big(i\,\omega\,\mathcal{B}(\mathcal{S} - \tfrac{i}{2}\bar{\mathcal{D}}\gamma^5\mathcal{A}) + \text{H.c.}\Big)\Bigg], \tag{B.1.1}$$

where the covariant derivative is given in (4.5.2) and in the third line of the above Lagrangian it is implied that

$$\overline{\mathcal{D}}\gamma^5\mathcal{A} = \mathcal{D}_\alpha\left(\gamma^0\gamma^5\right)_{\alpha\beta}\mathcal{A}_\beta\,, \tag{B.1.2}$$

isovector superfields are represented by

$$\mathcal{N}^i = n^i + \bar\theta\xi^i + \tfrac{1}{2}\bar\theta\theta F^i, \quad i = 1,\ldots,N, \\ \mathcal{R}^j = \rho^j + \bar\theta\eta^j + \tfrac{1}{2}\bar\theta\theta G^j, \quad j = 1,\ldots,\widetilde{N}\,, \tag{B.1.3}$$

constraint superfield

$$\mathcal{S} = \sigma_1 + \bar{\theta}u + \tfrac{1}{2}\bar{\theta}\theta D \tag{B.1.4}$$

gets multiplied by the D-term constraint in the second line of (B.4.4), and spinor superfield under the proper gauge[1]

$$\mathcal{A}_\alpha = -i(\gamma^\mu\theta)_\alpha A_\mu + (\gamma^5\theta)_\alpha\sigma_2 + \bar{\theta}\theta\, v_\alpha\,.$$

The heterotic deformation is conducted by the chiral field

$$\mathcal{B} = -\bar{\theta}\zeta + \tfrac{1}{2}\bar{\theta}\theta\bar{\mathcal{F}}\mathcal{F}. \tag{B.1.5}$$

which by definition contains only the right-handed fermion:

$$\zeta = \begin{pmatrix} \zeta_R \\ 0 \end{pmatrix} \tag{B.1.6}$$

In the formulae (B.4.4)-(B.1.5) θ is a Majorana spinor, σ_1, σ_2, A_μ, u_α, v_α and D are real fields, while ζ and $\mathcal{F}$ are complex fields. The complex-valued parameter ω stands for the heterotic deformation. The complex-valued fields σ and λ from (**??**) can be assembled using the components of $\mathcal{S}$ and $\mathcal{A}_\alpha$ as follows

$$\sigma = \sigma_1 + i\sigma_2\,, \quad \lambda_\alpha = u_\alpha + iv_\alpha\,. \tag{B.1.7}$$

B.2 One-loop Effective Action

Below we list generic expressions for the effective couplings of (4.7.1) in terms of D, σ, u and twisted masses. Let us first for completeness specify the full action including the fermionic part

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4e_\gamma^2}F_{\mu\nu}^2 + \frac{1}{e_{\sigma\,1}^2}(\partial_\mu \mathfrak{Re}\,\sigma)^2 + \frac{1}{e_{\sigma\,2}^2}(\partial_\mu \mathfrak{Im}\,\sigma)^2 + i\frac{1}{e_\lambda^2}\bar{\lambda}\gamma^\mu\nabla_\mu\lambda + \frac{i}{2}\bar{\zeta}_R\partial_L\zeta_R \\ &+ i\mathfrak{Im}(\bar{b}\,\sigma)\epsilon_{\mu\nu}F^{\mu\nu} - V_{\text{eff}}(\sigma) - (i\Gamma\bar{\sigma}\bar{\lambda}\lambda + i\omega\lambda_L\zeta_R + \text{H.c.})\,.\end{aligned} \tag{B.2.1}$$

The gauge coupling can be calculated using the wavefunction renormalization for the photon Fig. B.1 and reads[2]

[1] see Ref. [38] for further details

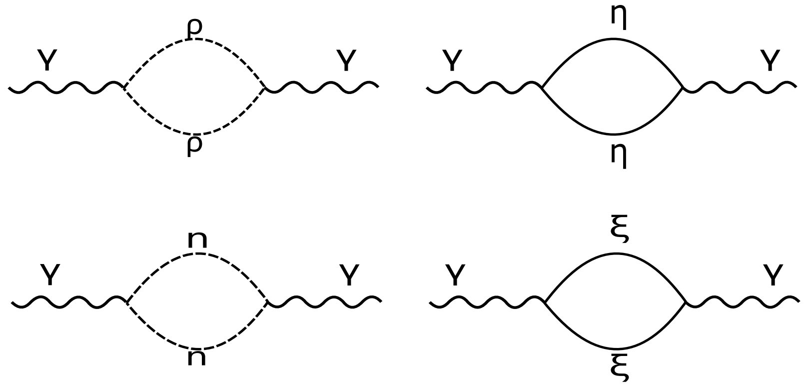

Figure B.1: Four series of one-loop diagrams which give photon wavefunction renormalization.

$$\frac{1}{e_\gamma^2} = \frac{1}{4\pi}\sum_{i=1}^{N-1}\left[\frac{1}{3}\frac{1}{|\sigma_0 - m_i|^2 + D} + \frac{2}{3}\frac{1}{|\sigma_0 - m_i|^2}\right] + \frac{1}{4\pi}\sum_{i=1}^{\tilde{N}-1}\left[\frac{1}{3}\frac{1}{|\sigma_0 - \mu_i|^2 - D} + \frac{2}{3}\frac{1}{|\sigma_0 - \mu_i|^2}\right]. \tag{B.2.2}$$

Feynman diagrams corresponding to the scalar couplings renormalization can be found in Fig. B.2. Performing the integrals we obtain

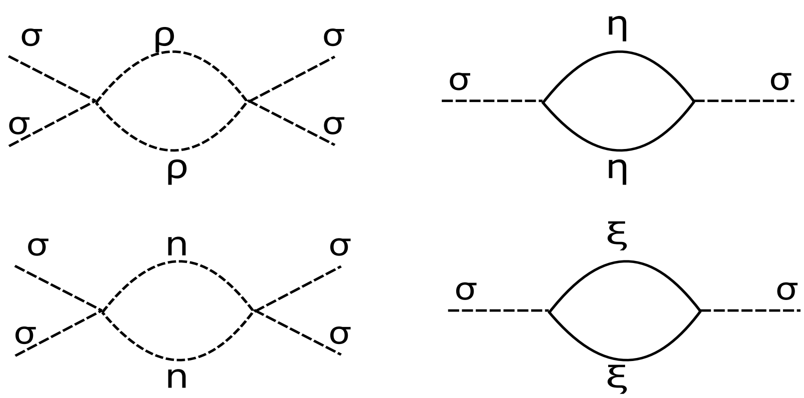

Figure B.2: Four series of one-loop diagrams which give scalar wavefunction renormalization.

[2] We assume for simplicity that for the vacuum $\mathfrak{Im}\,\sigma_0 = 0$.

$$\frac{1}{e^2_{\sigma\,1}} = \frac{1}{4\pi}\sum_{i=1}^{N-1}\frac{1}{|\sigma_0 - m_i|^2}\left[\frac{1}{3} + \frac{2}{3}\frac{|\sigma_0 - m_i|^4}{(|\sigma_0 - m_i|^2 + D)^2}\right]$$
$$+ \frac{1}{4\pi}\sum_{i=1}^{\tilde{N}-1}\frac{1}{|\sigma_0 - \mu_i|^2}\left[\frac{1}{3} + \frac{2}{3}\frac{|\sigma_0 - \mu_i|^4}{(|\sigma_0 - \mu_i|^2 - D)^2}\right],$$
$$\frac{1}{e^2_{\sigma\,2}} = \frac{1}{4\pi}\sum_{i=1}^{N-1}\frac{1}{|\sigma_0 - m_i|^2} + \frac{1}{4\pi}\sum_{i=1}^{\tilde{N}-1}\frac{1}{|\sigma_0 - \mu_i|^2}\,. \tag{B.2.3}$$

We see that real and imaginary components of the σ_0 field acquire different renormalizations, in particular, if the SUSY is broken, their couplings are different. The fermion coupling renormalization given by the diagrams in Fig. B.3 reads

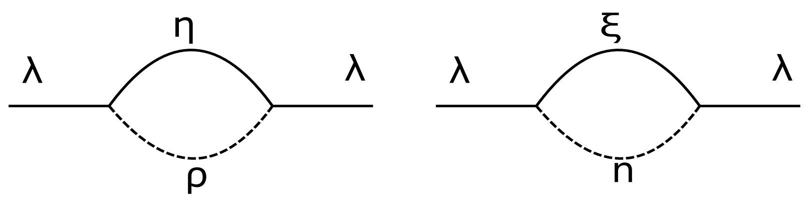

Figure B.3: Two series of one-loop diagrams which give scalar wavefunction renormalization.

$$\frac{1}{e^2_\lambda} = \frac{N - \tilde{N}}{4\pi}\frac{2}{D} - \frac{1}{2\pi}\sum_{i=1}^{N-1}\frac{|\sigma_0 - m_i|^2}{D^2}\log\frac{|\sigma_0 - m_i|^2 + D}{|\sigma_0 - m_i|^2}$$
$$- \frac{1}{2\pi}\sum_{i=1}^{\tilde{N}-1}\frac{|\sigma_0 - \mu_i|^2}{D^2}\log\frac{|\sigma_0 - \mu_i|^2 - D}{|\sigma_0 - \mu_i|^2}\,. \tag{B.2.4}$$

The Yukawa coupling can be found as the mass renormalization using Fig. B.3 and is given by (equivalently one could compute the corresponding triangular graph)

$$\Gamma = \frac{1}{4\pi}\frac{2}{D}\left[\sum_{i=1}^{N-1}\log\frac{|\sigma_0 - m_i|^2 + D}{|\sigma_0 - m_i|^2} + \sum_{i=1}^{\tilde{N}-1}\log\frac{|\sigma_0 - \mu_i|^2 - D}{|\sigma_0 - \mu_i|^2}\right]. \tag{B.2.5}$$

The sums in the above formulae can be done explicitly. Below we list those formulae we used in Sec. 4.7. Using

$$\frac{1}{N}\sum_{k=1}^{N-1}\frac{1}{|1 - \gamma e^{\frac{2\pi i k}{N}}|^2} = \frac{1}{|1 - \gamma^2|}\,, \tag{B.2.6}$$

and, for nonzero D

$$\frac{1}{N}\sum_{k=1}^{N-1}\frac{1}{1-\gamma\cos\frac{2\pi ik}{N}}=\frac{1}{\sqrt{1-\gamma^2}}\,, \tag{B.2.7}$$

the value of the gauge coupling (B.2.2) can be evaluated and reads

$$\begin{aligned}\frac{1}{e_\gamma^2}=\frac{N}{4\pi}&\left[\frac{1}{3}\frac{1}{\sqrt{\left(|\sigma_0|^2+m^2+D\right)^2-4|\sigma_0|^2m^2}}+\frac{2}{3}\frac{1}{||\sigma_0|^2-m^2|}\right]\\ &+\frac{\tilde N}{4\pi}\left[\frac{1}{3}\frac{1}{\sqrt{\left(|\sigma_0|^2+\mu^2-D\right)^2-4|\sigma_0|^2\mu^2}}+\frac{2}{3}\frac{1}{||\sigma_0|^2-\mu^2|}\right]\,,\end{aligned} \tag{B.2.8}$$

whereas the coupling for the imaginary part of σ is given by

$$\frac{1}{e_{\sigma\,2}^2}=\frac{N}{4\pi}\frac{1}{||\sigma_0|^2-m^2|}+\frac{\tilde N}{4\pi}\frac{1}{||\sigma_0|^2-\mu^2|}\,. \tag{B.2.9}$$

Thus if the SUSY is unbroken we can get from the above two formulae and (B.2.4) the following

$$\frac{1}{e_\gamma^2}=\frac{1}{e_{\sigma\,1}^2}=\frac{1}{e_{\sigma\,2}^2}=\frac{1}{e_\lambda^2}=\frac{N}{4\pi}\left(\frac{1}{||\sigma_0|^2-m^2|}+\alpha\frac{1}{||\sigma_0|^2-\mu^2|}\right)\,. \tag{B.2.10}$$

B.3 Supersymmetry Algebra and Central Charges

$\mathcal{N}=2$ supersymmetry algebra in four dimensions has the following form

$$\begin{aligned}\{Q_\alpha^I,\bar Q_{J\,\dot\alpha}\}&=2P_{\alpha\dot\alpha}\delta_J^I+2Z_{\alpha\dot\alpha}\delta_J^I\,,\\ \{Q_\alpha^I,Q_\beta^J\}&=\epsilon_{\alpha\beta}\epsilon^{IJ}Z_{\rm mon}+(Z_{\rm d.w.})_{\alpha\beta}^{IJ}\,.\end{aligned} \tag{B.3.1}$$

There are three types on central charges: string, monopole and domain wall types. The full global symmetry of the theory is $SU(2)_L\times SU(2)_R\times SU(2)_{\mathcal{R}}\times SU(2)_c$. It is broken by the Omega background in the NS limit to $SU(2)_{R+\mathcal{R}}\times SU(2)_c$. Twisted supercharges

$$\bar Q=\delta_I^{\dot\alpha}\bar Q_{\dot\alpha}^I\,,\quad Q_m=(\bar\sigma_m)^{I\alpha}Q_{I\alpha}\,,\quad \bar Q_{mn}=(\bar\sigma_{mn})_I^{\dot\alpha}\bar Q_{\dot\alpha}^I\,. \tag{B.3.2}$$

The former operator above is also known as BRST operator. The transformations can be inverted as

$$Q_\alpha^I=\tfrac{1}{2}(\sigma^m)_\alpha^I Q_m\,,\quad \bar Q_{\dot\alpha J}=\tfrac{1}{2}\epsilon_{\dot\alpha J}\bar Q+\tfrac{1}{2}(\bar\sigma_{mn})_{\dot\alpha J}\bar Q^{mn}\,. \tag{B.3.3}$$

Plugging these formulae into (B.3.1) we get the twisted version of the supersymmetry algebra **check on the domain wall charge**

$$\{\bar{Q}, Q_m\} = 8P_m + 8Z_m\,, \quad \{Q_m, \bar{Q}_{nk}\} = 4(\delta_{mk}\delta_{ln} - \delta_{ml}\delta_{kn} - \varepsilon_{mlnk})(P^l + Z^l)\,,$$
$$\{Q_m, Q_n\} = 2\delta_{mn}(Z_{mon} - Z_{d.w.})\,, \quad \{\bar{Q}, \bar{Q}\} = 4(\bar{Z}_{mon} - \bar{Z}_{d.w.})\,,$$
$$\{\bar{Q}, \bar{Q}_{mn}\} = 0\,, \quad \{\bar{Q}_{mn}, \bar{Q}_{pq}\} = 2i(\delta_{mp}\delta_{nq} - \delta_{mq}\delta_{np} - \epsilon_{mnpq})(\bar{Z}_{mon} - \bar{Z}_{d.w.})\,. \quad \text{(B.3.4)}$$

B.4 Alternative Superfield Formulation of the Heterotic $\mathcal{N} = (0, 2)$ $\mathbb{CP}^{N-1}$ sigma model

Distler and Kachru [171] constructed a very wide class of heterotic $(0, 2)$ Calabi-Yau sigma models and provided gauged formulation for them. Their construction employs fiber bundles over complex projective spaces and includes heterotic $(0, 2)$ $\mathbb{CP}^{N-1}$ sigma model in it. In this section we give a simple alternative derivation of the $(0, 2)$ heterotic (weighted) $\mathbb{CP}^N$ sigma model from the Majorana formalism we used above.

Recall that for the O(N) sigma model we used the following trick – the constraint on the isovector superfield was replaced by a different one, but it did not affect the lowest component constraint; thereby the geometry of the theory was not deformed. In order to apply this trick for the $\mathbb{CP}^{N-1}$ model we write its action in the form found in [193, 194] and deform it by adding the coupling of chiral field

$$\mathcal{B} = -\bar{\theta}\zeta + \tfrac{1}{2}\bar{\theta}\theta\bar{\mathcal{F}}\mathcal{F}, \quad \text{(B.4.1)}$$

constraint superfield

$$\mathcal{S} = \sqrt{2}\sigma_1 + \sqrt{2}\bar{\theta}u + \tfrac{1}{2}\bar{\theta}\theta D \quad \text{(B.4.2)}$$

and spinor superfield

$$\mathcal{A}_\alpha = -i(\gamma^\mu\theta)_\alpha A_\mu + \sqrt{2}(\gamma^5\theta)_\alpha\sigma_2 + \sqrt{2}\bar{\theta}\theta\, v_\alpha,$$

where θ is a Majorana spinor, σ_1, σ_2, A_μ, u_α, v_α and D are real fields, while ζ and $\mathcal{F}$ are complex ones [3] . Introducing now the complex isovector superfield

$$\mathcal{N}^i = n^i + \bar{\theta}\xi^i + \tfrac{1}{2}\bar{\theta}\theta F^i, \quad \text{(B.4.3)}$$

[3] Note that for the present section we changed the notations of the fields in order for the reader to see the equivalence with the Lagrangian from [158] more easily. Also for convenience we consider ζ to be a right-handed fermion and use the definition $\partial_{L,R} = \partial_0 \pm \partial_1$ instead of one mentioned in Appendix.

we can write the Lagrangian of the model in the following form

$$\mathcal{L}_{\mathbb{CP}^N} = \int d^2\theta \Big[\tfrac{1}{2}\varepsilon_{\beta\alpha}(\mathcal{D}_\alpha + i\mathcal{A}_\alpha)\mathcal{N}_i^\dagger(\mathcal{D}_\beta - i\mathcal{A}_\beta)\mathcal{N}_i + i\mathcal{S}(\mathcal{N}_i^\dagger\mathcal{N}_i - r_0) \\ +\tfrac{1}{4}\varepsilon_{\beta\alpha}\mathcal{D}_\alpha\mathcal{B}^\dagger\,\mathcal{D}_\beta\mathcal{B} + \big(i\,\omega\,\mathcal{B}(\mathcal{S} - \tfrac{i}{2}\overline{\mathcal{D}}\gamma^5\mathcal{A}) + \text{H.c.}\big)\Big]\,, \tag{B.4.4}$$

where ω is the complex-valued deformation parameter and $\overline{\mathcal{D}}\gamma^5\mathcal{A} = \mathcal{D}_\alpha\left(\gamma^0\gamma^5\right)_{\alpha\beta}\mathcal{A}_\beta$. Some comments about the Lagrangian are due. The advantage of the superfield formulation is that the supersymmetry is manifest without an explicit check. Although the Lagrangian for the undeformed theory is written using the $\mathcal{N} = (1,1)$ formulation it possesses $\mathcal{N} = (2,2)$ symmetry due to the Kähler structure of the target space [176]. The field $\mathcal{B}$ is a superfield only with respect to the half of supertransformations $\theta_R \to \theta_R + \varepsilon_R$, therefore the symmetry of the deformed Lagrangian is $\mathcal{N} = (0,2)$ one. Also it should be noted that the field $\mathcal{A}_\alpha$ has the form (B.4.3) only if one considers a particular gauge. Starting from the most general expression for the real spinor field

$$\mathcal{A}_\alpha = a_\alpha + A\theta_\alpha - i(\gamma^\mu\theta)_\alpha A_\mu + \sqrt{2}(\gamma^5\theta)_\alpha\sigma_2 + \bar\theta\theta\left(v_\alpha + \tfrac{i}{2}(\gamma^\mu\partial_\mu a)_\alpha\right), \tag{B.4.5}$$

one can use the following gauge transformations

$$\mathcal{A}_\alpha \to \mathcal{A}_\alpha - \mathcal{D}_\alpha\Phi, \tag{B.4.6}$$

with Φ being the scalar real superfield, to eliminate a_α and A. The undeformed Lagrangian is obviously gauge invariant, while the invariance of the deformation is more subtle. The transformation of the term $\mathcal{D}_\alpha\left(\gamma^0\gamma^5\right)_{\alpha\beta}\mathcal{A}_\beta$ is proportional to $\left(\gamma^0\gamma^5\right)_{\alpha\beta}\mathcal{D}_\alpha\mathcal{D}_\beta\Phi$, which is identically zero since the operators $\mathcal{D}_1$ and $\mathcal{D}_2$ anticommute.

Carrying out the calculations and integrating out the auxiliary fields F and $\mathcal{F}$ we recover the Lagrangian of the $(0,2)$ $\mathbb{CP}^{N-1}$ sigma model considered in [158]

$$\begin{aligned}\mathcal{L}_{\mathbb{CP}^N} &= |\nabla_\mu n_i|^2 + i\bar\xi_L^i\nabla_R\xi_L^i + i\bar\xi_R^i\nabla_L\xi_R^i - 2|\sigma|^2|n_i|^2 - D\left(|n_i|^2 - r_0\right) \\ &+ \left[i\sqrt{2}\bar n_i\left(\lambda_L\xi_R^i - \lambda_R\xi_L^i\right) - i\sqrt{2}\sigma\bar\xi_R^i\xi_L^i + \text{H.c.}\right] - 4|\omega|^2|\sigma|^2 \\ &+ \tfrac{i}{2}\bar\zeta_R\partial_L\zeta_R - \left[i\sqrt{2}\omega\lambda_L\zeta_R + \text{H.c.}\right],\end{aligned} \tag{B.4.7}$$

where the following complex fields have been introduced

$$\sigma = \sigma_1 + i\sigma_2\,, \quad \lambda_\alpha = u_\alpha + iv_\alpha, \tag{B.4.8}$$

and $\nabla_\mu = \partial_\mu - iA_\mu$ is a usual notation for the covariant derivative.

Analogously, the weighted $\mathbb{CP}^N$ sigma-model considered in [160] which emerges from the reduction of $\mathcal{N}=1$ supersymmetric QCD with gauge group $\mathrm{U}(N_c)$ and N_f flavors can be easily deformed to the chiral version by (B.4.4), where the constraint $\mathcal{N}_i^\dagger \mathcal{N}_i = r_0$ is replaced by

$$\sum_{i=1}^{N_c} \mathcal{N}_i^\dagger \mathcal{N}_i - \sum_{i=N_c+1}^{N_f} \mathcal{N}_i^\dagger \mathcal{N}_i = r_0 \ . \tag{B.4.9}$$

In components the Lagrangian reads

$$\begin{aligned} \mathcal{L}^{\mathrm{w}}_{\mathbb{CP}^N} &= |\nabla_\mu n_i|^2 + |\nabla_\mu \rho_i|^2 + i\bar{\xi}^i_L \nabla_R \xi^i_L + i\bar{\xi}^i_R \nabla_L \xi^i_R + i\bar{\eta}^i_L \nabla_R \eta^i_L + i\bar{\eta}^i_R \nabla_L \eta^i_R \\ &- 2|\sigma|^2 |n_i|^2 - 2|\sigma|^2 |\rho_i|^2 - D\left(|n_i|^2 - |\rho_i|^2 - r_0\right) - 4|\omega|^2|\sigma|^2 \\ &+ \left[i\sqrt{2}\bar{n}_i \left(\lambda_L \xi^i_R - \lambda_R \xi^i_L\right) - i\sqrt{2}\sigma \bar{\xi}^i_R \xi^i_L + \text{H.c.}\right] \\ &+ \left[-i\sqrt{2}\bar{\rho}_i \left(\bar{\lambda}_L \eta^i_R - \bar{\lambda}_R \eta^i_L\right) + i\sqrt{2}\bar{\sigma} \bar{\eta}^i_R \eta^i_L + \text{H.c.}\right] \\ &+ \tfrac{i}{2}\bar{\zeta}_R \partial_L \zeta_R - \left[i\sqrt{2}\omega \lambda_L \zeta_R + \text{H.c.}\right], \end{aligned} \tag{B.4.10}$$

with

$$\begin{aligned} \mathcal{N}^i &= n^i + \bar{\theta}\xi^i + \tfrac{1}{2}\bar{\theta}\theta F^i, \quad i = 1, \ldots, N_c, \\ \mathcal{N}^{N_c+i} &= \rho^i + \bar{\theta}\eta^i + \tfrac{1}{2}\bar{\theta}\theta F^i, \quad i = 1, \ldots, N_f - N_c. \end{aligned} \tag{B.4.11}$$

Appendix C

Liouville and Toda Theories

C.1 Gaudin model from Liouville CFT

Here we discuss the Gaudin model – the key tool in our AGT construction, its relations with the XXX spin chain and how it appears in conformal field theories.

Gaudin model from XXX chain. The Gaudin model is the simplest example of the Hitchin system on a sphere with marked points [195]. It is also known to be a large impurity limit of an anisotropic twisted XXX spin chain. This fact can be realized both in the transfer matrix at the classical limit and in the Bethe ansatz equations in the quantum case. We shall be interested in the quantum case and upon the proper limit Bethe ansatz equations for the Gaudin model can be obtained. Let us start with Bethe equations for anisotropic $XXX_{\frac{S}{2}}$ spin chain[1] with twist $q = e^{2\pi i\hat{\tau}}$

$$\prod_{a=1}^{N} \frac{\lambda_i - \nu_a + \frac{\epsilon}{2} S_a}{\lambda_i - \nu_a - \frac{\epsilon}{2} S_a} = q \prod_{\substack{j=1 \\ j \neq i}}^{K} \frac{\lambda_i - \lambda_j - \epsilon}{\lambda_i - \lambda_j + \epsilon} \,. \tag{C.1.1}$$

By taking logarithms of both parts of the above equations, then rescaling

$$\lambda_i \mapsto x\lambda_i, \quad \nu_a \mapsto x\nu_a, \quad \hat{\tau} \mapsto \frac{\hat{\tau}}{x} \,, \tag{C.1.2}$$

[1] We measure spectral parameters λ_i in units of $i\epsilon$ here.

and sending $x \to \infty$ we arrive at the following set of equations

$$\frac{\log q}{\epsilon} - \sum_{a=1}^{N} \frac{S_a}{\lambda_i - \nu_a} = \sum_{\substack{j=1 \\ j\neq i}}^{K} \frac{2}{\lambda_i - \lambda_j}\,, \tag{C.1.3}$$

which are nothing but Bethe equations for the Gaudin model. The anisotropies ν_a at each site still play the role of the inhomogenities in the model, while the twist q in the XXX chain play the role of the external field in the Gaudin system. As we can see the latter vanishes as $\epsilon \to \infty$.

Bethe ansatz equations for the Gaudin model. Let us now recall how the Bethe ansatz equations for the rational Gaudin model with the Lie algebra symmetry $\mathfrak{g}$ are derived. For our purposes we need merely $\mathfrak{g} = \mathfrak{sl}(2)$ and and L points on the sphere. At each point we fix a representation $V(\nu_1), \ldots, V(\nu_L)$ of $\mathfrak{sl}(2)$ algebra with some dominant weights ν_a, $a = 1, \ldots, L$. According to the Bethe ansatz prescription [107] we construct the following operator

$$S(u) = \sum_{a=1}^{4} \frac{\mathcal{H}_a}{u - z_a} + \sum_{a=1}^{4} \frac{\Delta(\nu_a)}{(u - z_a)^2}\,, \tag{C.1.4}$$

where $\mathcal{H}_a$ are Gaudin Hamiltonians at each site of the lattice

$$\mathcal{H}_a = \sum_{b\neq a} \frac{\mathfrak{J}^{(a)}_{\alpha} \mathfrak{J}^{\alpha\,(b)}}{z_a - z_b}\,, \tag{C.1.5}$$

where $\alpha = 1, \ldots, \dim(\mathfrak{g})$, $\mathfrak{J}^{(b)}_{\alpha}$ of the acts with $\mathfrak{J}_{\alpha} \in \mathfrak{sl}(2)$ on the b-th site of the spin chain and with identity on the others. $\Delta(\nu_a)$ are eigenvalues of the $\mathrm{U}(\mathfrak{sl}(2))$ quadratic Casimir acting on $V(\nu_a)$. For such a system Bethe ansatz equations for the sector with κ_a Bethe roots read as follows

$$\sum_{b=1}^{L} \frac{\nu_b \epsilon}{t_i - z_b} - \sum_{\substack{j=1 \\ j\neq i}}^{\kappa_a} \frac{2\epsilon}{t_i - t_j} = 0, \qquad i = 1, \ldots, \kappa_a\,. \tag{C.1.6}$$