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LETTER

Soliton molecules and asymmetric solitons in three fifth order systems via velocity resonance

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Abstract

Soliton molecules can be formed in some possible mechanisms both theoretically and experimentally. In this paper, we introduce a new mechanism, namely the velocity resonant, to find soliton molecules. Under the velocity resonance mechanism, two solitons can form a kink-antikink molecule, an asymmetric soliton, a two-peak soliton and/or a far away bounded molecule depended on the selections of the wave numbers and the distance between two solitons of a molecule. The results are exhibited via three well known fifth order integrable systems which serve as a general fluid model, as well as models many other physical fields.

Soliton molecules and bound states of solitons have been observed experimentally in optics [1–4] and predicted numerically in Bose–Einstein condensates [5]. On the other hand, solitons play a very important role in various modern scientific fields including fluids [6], plasmas [7], fibers [8], optics [9], complex networks [10], quantum field theory [11], gravity [12], Bose–Einstein condensates[13], atmospheric and oceanic dynamics[14] and so on. In addition to optical systems, fluid systems (such as the stratified fluids [15], magnetic fluids [16], quantum fluids [17], fluids of light [18], atomic and molecular gases [19], degenerate electron fluids [20] and superfluid ^{3}He [21]) exhibit abundant soliton structures. Solitary waves and solitons were discovered firstly in fluid. We believe that soliton molecules could be observed in fluids such as oceanic and atmospheric systems.

Resonance is also an important natural phenomena which may lead to terrible disasters (blow up in mathematics). For integrable systems, because of the complexities introduced by nonlinearity, resonances of solitons may lead to various types of new excitations such as the breathers (analytic) [22] or complexitons (singular) [23] (caused by module resonance of wave numbers, say, $|k_2| = |k_1|$, i.e. $k_2 = \pm k_1^*$), soliton fissions and fusions [24], instantons/rogue waves [25] and rational-exponential waves (caused by wave number resonance companied by a vanishing procedure, namely, $k_2 \to \pm k_1 \to 0$) [26], web solitons and lumps (by wave number resonance $\vec{k}_2 = \pm\vec{k}_1$ in high dimensional cases) [27]. In this paper, we try to find a new mechanism to form soliton molecules by introducing velocity resonance $\omega_1/k_1 = \omega_2/k_2$.

One of the important fluid models to describe surface and internal waves can be written as

$$u_t + [\alpha u^2 + \beta u_{xx} + \epsilon(\alpha_1 u^3 + \gamma_1 u u_{xx} + \gamma_2 u_x^2 + \beta_1 u_{xxxx})]_x + o(\epsilon^2) = 0, \tag{1}$$

which has been derived by many authors [28–30]. Some authors (say, Kodama [31] and Fokas and Liu [32]) have proved that the model (1) is asymptotic integrable up to the same order ϵ^2 because it can be asymptotically changed to either the usual Korteweg de-Vries (KdV) equation ($\epsilon = 0$) or the fifth order integrable KdV equation by using suitable transformations. As a matter of fact, similar to the Kodama's idea [31], the transformation

$$u = v + \epsilon\left[\left(\frac{5a}{3\beta^2}\alpha\beta_1 + a_1\right)v^2 + \left(\frac{5b\beta_1}{2\beta} + b_1\right)v_{xx} + \left(\frac{5a\alpha\beta_1}{3\beta^2} - \frac{\gamma_1}{3\beta}\right)v_x\int v\mathrm{d}x\right] \tag{2}$$

with $a_1 \equiv \frac{\gamma_1}{6\beta} - \frac{3\alpha_1}{2\alpha}$, $b_1 \equiv \frac{\gamma_1}{4\alpha} + \frac{\gamma_2}{2\alpha} - \frac{9\beta\alpha_1}{4\alpha^2}$,

$$v_t + \left[\alpha v^2 + \beta v_{xx} + \epsilon\beta_1\left(v_{xxxx} + \frac{5a\alpha}{\beta}vv_{xx} + \frac{5(a-b)\alpha}{\beta}v_x^2 + \frac{5a\alpha^2}{3\beta^2}v^3\right)\right]_x = 0, \tag{3}$$

where a and b are arbitrary constants, solves the original equation (1) approximately up to the same order ϵ. To check the integrability of (3), one can make a Galileo transformation $v \to -\beta^2/(5a\alpha\beta_1\epsilon) + v(x + \beta^2 t/(5a\beta_1\epsilon), t)$ such that the terms αv^2 and βv_{xx} in (3) can be eliminated.

It is well known that there are three integrable cases, the usual fifth order KdV equation with $\{a, b\}_{KdV} = \left\{\frac{2}{3}, \frac{1}{3}\right\}$, the Sawada-Kotera (SK) model with $\{a, b\}_{SK} = \{1, 1\}$ and the Kaup-Kupershmidt (KK) equation with $\{a, b\}_{KK} = \left\{1, \frac{1}{4}\right\}$. The approximate transformation (2) for the fifth order KdV case is firstly given by Kodama [31], while other two cases have not yet been found elsewhere as far as we know.

For the fifth order KdV and SK cases, the multiple soliton solutions possess the form ($\xi_j = k_j x - \omega_j t + \xi_{j0}$, $\omega_j = \epsilon\beta_1 k_j^5 + \beta k_j^3$, $v \to u$),

$$u = \frac{6\beta}{\alpha}\left\{\ln\left[\sum_{\nu} K_\nu \cosh\left(\sum_{i=1}^{N}\frac{\nu_j\xi_j}{2}\right)\right]\right\}_{xx}, \tag{4}$$

where the summation ν should be done for all possible permutations $\nu_j = 1, -1$ $(j = 1, 2,\ldots,N)$, $K_\nu = \prod_{i<j} a_{ij}$, $a_{ij}^2 = (\nu_i k_i - \nu_j k_j)^2$ for the fifth order KdV case, $a_{ij}^2 = (\nu_i k_i - \nu_j k_j)^2[5\epsilon\beta_1(k_i^2 - \nu_i\nu_j k_i k_j + k_j^2) + 3\beta]$ for the SK case, k_j and ξ_{j0} $(j = 1, 2,\ldots,N)$ are arbitrary constants.

We are interested in finding something new start from (4) except for those from the usual KdV equation ($\epsilon = 0$ in (1)). In additional to the N soliton solutions, the expression (4) includes many kinds of resonant excitations. All known resonant solutions of (4) are singular ones with blow up properties which include the complexiton solutions [23] (or namely singular breathers caused by the resonant condition $|k_i| = |k_j|$, i.e., $k_i = \pm k_j^*$), the rational-exponential function mixed solutions [26] (produced by the resonant conditions $k_i = \pm k_j$) and the rational solutions (generated by further resonant conditions $k_i \to k_j \to 0$).

To find some resonant solutions, we take $N = 2$ as a simple example. For $N = 2$, the expression (4) reduces

$$u = \frac{6\beta}{\alpha}\left\{\ln\left[a_-\cosh\left(\frac{\xi_1+\xi_2}{2}\right) + a_+\cosh\left(\frac{\xi_1-\xi_2}{2}\right)\right]\right\}_{xx} \tag{5}$$

$$= \frac{3\beta}{2\alpha}\frac{\{2a_-a_+[k_2^2\cosh(\xi_1) + k_1^2\cosh(\xi_2)] + a_-^2(k_1+k_2)^2 + a_+^2(k_1-k_2)^2\}}{\left[a_-\cosh\left(\frac{\xi_1+\xi_2}{2}\right) + a_+\cosh\left(\frac{\xi_1-\xi_2}{2}\right)\right]^2}, \tag{6}$$

where $\xi_i = k_i x - \omega_i t + \xi_{i0}$, $i = 1, 2$, $a_\pm^2 = (k_1 \pm k_2)^2$ for the fifth order KdV case, and $a_\pm^2 = (k_1 \pm k_2)^2[5\epsilon\beta_1(k_1^2 \pm k_1k_2 + k_2^2) + 3\beta]$ for the SK case.

The first resonant solution of (6)

$$u = \lim_{k_2\to k_1}\frac{6\beta}{\alpha}\left\{\ln\frac{\mathrm{i}}{k_2-k_1}\left[a_-\cosh\left(\frac{\xi_1+\xi_2}{2}\right) + a_+\cosh\left(\frac{\xi_1-\xi_2}{2}\right)\right]\right\}_{xx}$$
$$= \frac{6\beta}{\alpha}\{\ln[a_1\sinh(k_1x - k_1^3(\beta_1\epsilon k_1^2 + \beta)t + \xi'_{10}) + a_2k_1(x - k_1^2t(5\beta_1\epsilon k_1^2 + 3\beta) + x_0)]\}_{xx}, \tag{7}$$

can be obtained by taking the limit procedure, the wave number resonance conditions, $k_2 \to k_1$, $\xi_{10} \to \xi'_{10} + \mathrm{i}\pi$, $\xi_{20} \to \xi'_{10} + (k_2 - k_1)x_0$, where $\{a_1, a_2\} = \{1, 1\}$ for the KdV case and $\{a_1, a_2\} = \{\sqrt{5\beta_1\epsilon k_1^2 + 3\beta}, \sqrt{15\beta_1\epsilon k_1^2 + 3\beta}\}$ for the SK case.

The second resonant solution of (6) is the simplest rational solution

$$u = -\frac{6\beta}{\alpha(x + x'_0)} \tag{8}$$

of the KdV and SK cases under the resonant condition $k_2 \to k_1 \to 0$.

It is clear that the resonant solutions (7) and (8) are singular at some points expressed by $a_1\sinh(k_1x - k_1^3(\beta_1\epsilon k_1^2 + \beta)t + \xi'_{10}) + a_2k_1(x - k_1^2t(5\beta_1\epsilon k_1^2 + 3\beta) + x_0) = 0$ for (7) and $x = -x'_0$ for (8). These two resonance solutions are not related to soliton molecules.

The third type of resonance solution are also obtained from the wave number resonances but with complex wave numbers, $k_2 \to k_1^* = k - \mathrm{i}\kappa$. If the resonant solution is analytic, then the solution becomes a bound state of two solitons, or a (special type of) soliton molecule [33] and or a breather more popularly. It the resonant solution is singular, the solution is named complexiton [23]. For the solution (5), we have both the breather and complexiton resonant solutions for the SK equation and only the complexiton solutions for the KdV equation.

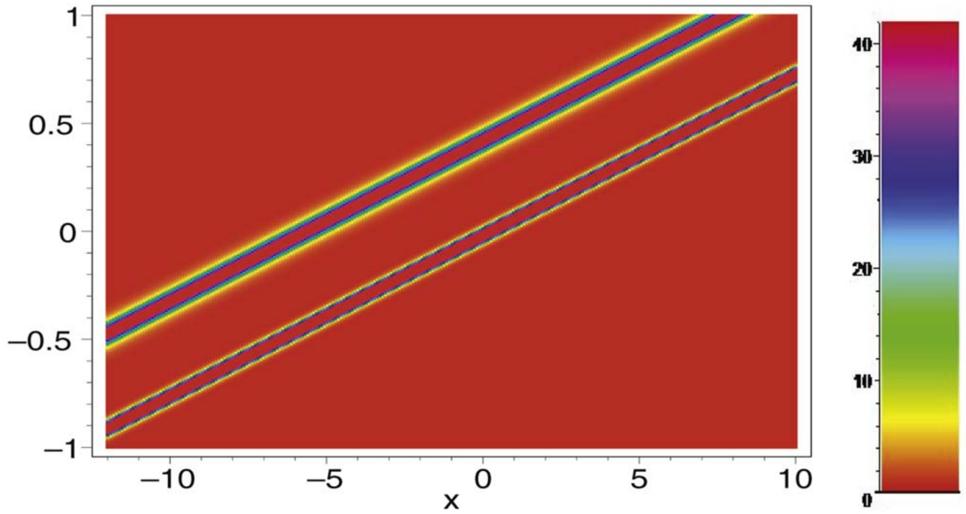

Figure 1. Density plot of the soliton molecule for the SK equation described by equation (6) with (14).

Taking the complex wave number resonant condition,

$$k_2 \to k_1^* = k - \kappa \mathrm{i},\ \xi_{10} = \eta_0 + (\tau_0 + \pi)\mathrm{i},\ \xi_{20} = \eta_0 - \tau_0 \mathrm{i}, \tag{9}$$

we have the complexiton solution,

$$\begin{aligned} u &= \frac{6\beta}{\alpha}\{\ln[\kappa a_1 \sinh(\eta) + k b_1 \sin(\tau)]\}_{xx},\\ \eta &= kx + (\epsilon k^5 - 10\epsilon k^3\kappa^2 + 5\epsilon k\kappa^4 - k^3 + 3k\kappa^2)t + \eta_0,\\ \tau &= \kappa x + (5\epsilon k^4\kappa - 10\epsilon k^2\kappa^3 + \epsilon\kappa^5 - 3k^2\kappa + \kappa^3)t + \tau_0, \end{aligned} \tag{10}$$

where

$$\{a_1, b_1\} = \{1, 1\}$$

for the KdV case and

$$\{a_1, b_1\} = \{\sqrt{\delta_1}, \sqrt{\delta_2}\} \equiv \{5\beta_1\epsilon k^2 - 15\beta_1\epsilon\kappa^2 + 3\beta,\ 15\beta_1\epsilon k^2 - 5\beta_1\epsilon\kappa^2 + 3\beta\},\ \delta_1\delta_2 > 0$$

for the SK case. It is clearly, the complexiton solution (10) is singular at the line $\kappa a_1 \sinh(\eta) + k b_1 \sin(\tau) = 0$.

It is interesting that for the SK case, by taking

$$k_2 \to k_1^* = k - \kappa \mathrm{i},\ \xi_{10} = \eta_0 + \tau_0 \mathrm{i},\ \xi_{20} = \eta_0 - \tau_0 \mathrm{i}, \tag{11}$$

we can find another complex wave number resonance solution

$$u = \frac{6\beta}{\alpha}\{\ln[\kappa\sqrt{-\delta_1}\cosh(\eta) + k\sqrt{\delta_2}\cos(\tau)]\}_{xx},\ \delta_1\delta_2 < 0, \tag{12}$$

where η and τ are same as in (10). Obviously, the solution (12) is an analytical breather for $|\kappa\sqrt{-\delta_1}| > |k\sqrt{\delta_2}|$ and a singular complexiton for $|\kappa\sqrt{-\delta_1}| < |k\sqrt{\delta_2}|$.

In this paper, we focus on the fourth type of resonance solutions (soliton molecules) from the multiple soliton solutions (4) by introducing a novel type of resonance conditions ($k_i \neq \pm k_j$), the velocity resonances,

$$\frac{k_i}{k_j} = \frac{\epsilon\beta_1 k_i^5 + \beta k_i^3}{\epsilon\beta_1 k_j^5 + \beta k_j^3}. \tag{13}$$

Because of the resonant condition (13), the ith and jth solitons are bounded and form a soliton molecule or an asymmetric soliton under appropriate selections of the solution parameters.

Two-soliton solution (4) exhibits one soliton molecule structure under the resonance condition (13). Figure 1 displays the molecule structure expressed by (6) with the parameter selections ($\alpha = 3,\ \beta = -\beta_1 = 1$ are chosen for the rest of this paper),

$$k_1 = 2\sqrt{21},\ k_2 = 4,\ \epsilon = 0.01,\ \xi_{10} = 0,\ \xi_{20} = 20, \tag{14}$$

for the SK case.

Figure 2 is a three dimensional plot of the soliton molecule (6) of the fifth order KdV case with (14).

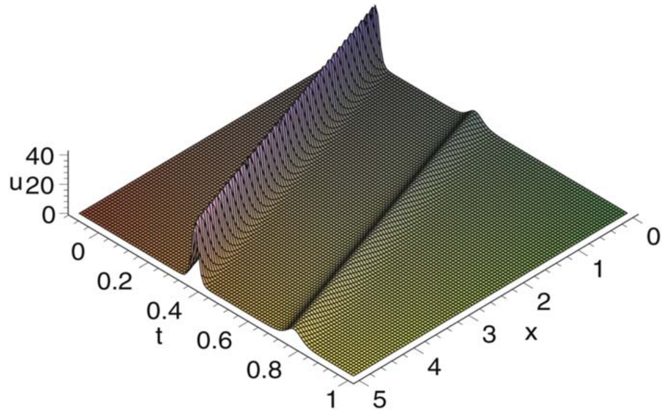

Figure 2. Soliton molecule structure for the fifth order KdV case described by (6) with (14).

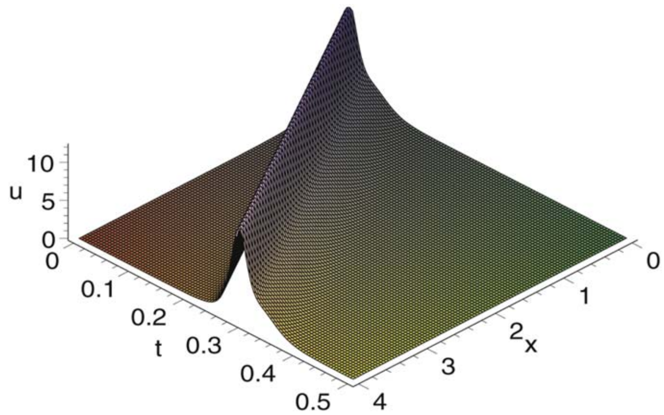

Figure 3. Asymmetric soliton for the fifth order KdV equation described by (6) with (16).

From figures 1, 2 and the expression (6), one can find that two solitons in the molecule are different because $k_1 \neq k_2$ though the velocities of them are the same.

The sizes of the soliton molecule, the distance between two solitons of the molecule, depends on the parameters ξ_{10}, ξ_{20} and k_1 via

$$s^2 = \frac{1}{4\omega_1^3}\beta_1 \epsilon\, k_1^3 (c\xi_{10} - \xi_{20})^2 (k_1^2 + \omega_1^2), \tag{15}$$

where $c = \frac{1}{k_1^2}\sqrt{\frac{-\omega_1}{k_1 \beta_1 \epsilon}}$ and $\omega_1 = k_1^3(\beta + \epsilon\beta_1 k_1^2)$.

It is clear that the soliton molecule will become an asymmetric one soliton solution if two solitons in one molecule are close enough. Figure 3 and figure 4 depict the structures of the asymmetric soliton solutions related to the fifth order KdV and the SK cases, respectively. The parameter selections related to figure 3 are

$$k_1 = 2\sqrt{21}\,,\ k_2 = 4,\ \epsilon = \frac{1}{100},\ \xi_{10} = 0,\ \xi_{20} = 1. \tag{16}$$

The parameters utilized in figure 4 are same as those in figure 3 except for $\xi_{20} = 0.1$.

Figure 5 displays the interaction property for the two-soliton molecule solution (4) of the fifth order KdV equation with $N = 4$, and the parameter selections

$$\begin{aligned} &k_1 = 2\sqrt{21}\,,\ k_2 = 8,\ k_3 = 6,\ k_4 = 4,\ \epsilon = 0.01, \\ &\xi_{10} = \xi_{30} = 0,\ \xi_{20} = 20,\ \xi_{40} = 15. \end{aligned} \tag{17}$$

From the figure 5, one can find that though the interactions among separated solitons elastic, the sizes of the soliton molecules will be changed.

For the KK case (3) with $\{a,\ b\} = \{1,\ 1/4\}$, the multiple soliton solutions do not possess the form (4) [34], however, it is interesting that the resonant cases of (4) with

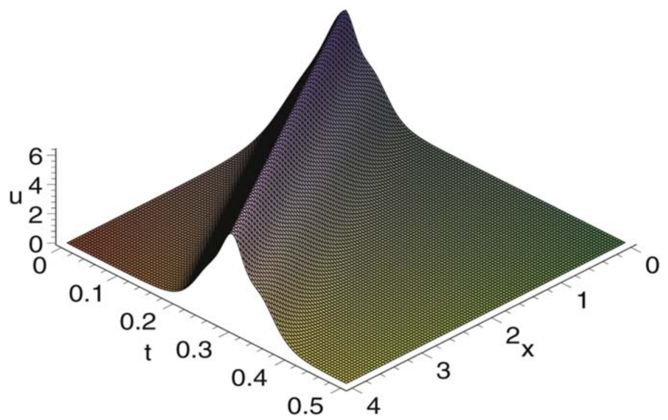

Figure 4. Asymmetric soliton for the SK system described by (6) with the same parameters as in figure 3 except for $\xi_{20} = 0.1$.

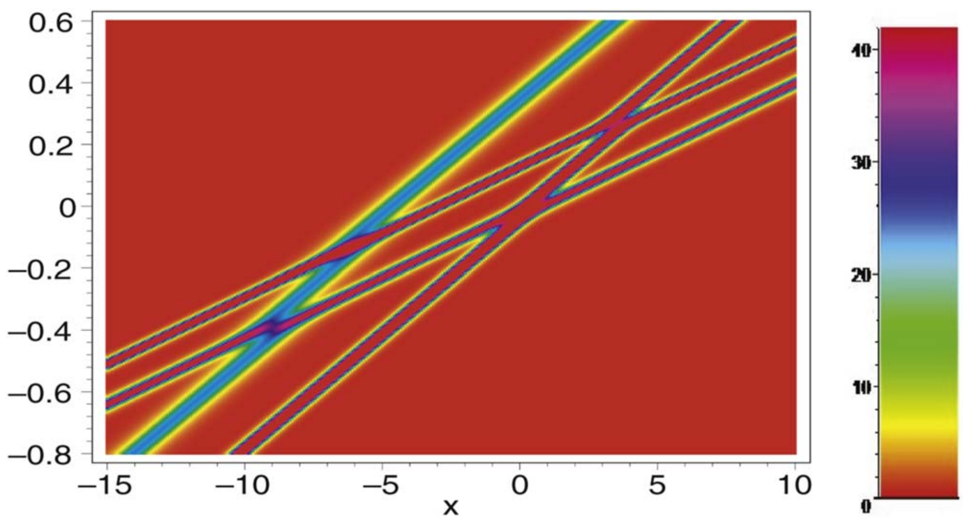

Figure 5. Plot of the interaction property between two soliton molecules for the fifth order KdV equation described by (4) along with (17) and $N = 4, \alpha = 3, \beta = \beta_1 = 1$.

$$N = 2n,\ \frac{k_i}{k_{n+i}} = \frac{\omega_i}{\omega_{n+i}} = \pm 1,\ i = 1,\ldots,n \tag{18}$$

do exist for all n. Here, we just list the details for $n = 1$ and $n = 2$.

The resonant solution (4) with (18) and $n = 1$ for the KK equation (3) with $\{a, b\} = \{1, 1/4\}$ is of the form $(v \to u,\ \xi = kx - \omega t + \xi_0)$,

$$\begin{aligned} u &= \frac{a_0\beta}{\alpha} + \frac{a_1\beta}{\alpha}\{\ln[c + \cosh(\xi)]\}_{xx}, \\ \omega &= 5k\beta_1\epsilon a_0^2 + k(5\beta_1\epsilon k^2 + 2\beta)a_0 + k^3(\beta_1\epsilon k^2 + \beta), \end{aligned} \tag{19}$$

where the parameters a_0, a_1 and c can be taken in three nonequivalent ways

$$a_0 = -\frac{(c^2 - 4)k^2}{4(c^2 - 1)} - \frac{\beta}{5\beta_1\epsilon},\ a_1 = 3, \tag{20}$$

$$a_0 = -\frac{k^2}{4} - \frac{\beta}{5\beta_1\epsilon},\ a_1 = \frac{3}{2},\ c^2 = 1, \tag{21}$$

and

$$a_0 = -2k^2 - \frac{\beta}{5\beta_1\epsilon},\ a_1 = 12,\ c^2 = 1, \tag{22}$$

while k, ξ_0 and c in the first case (20) remain free.

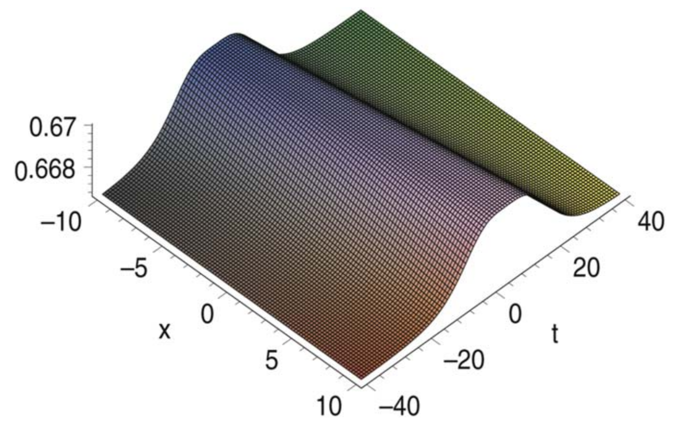

Figure 6. Kink-antikink molecule for the KK equation described by (19) with the parameter selections (23).

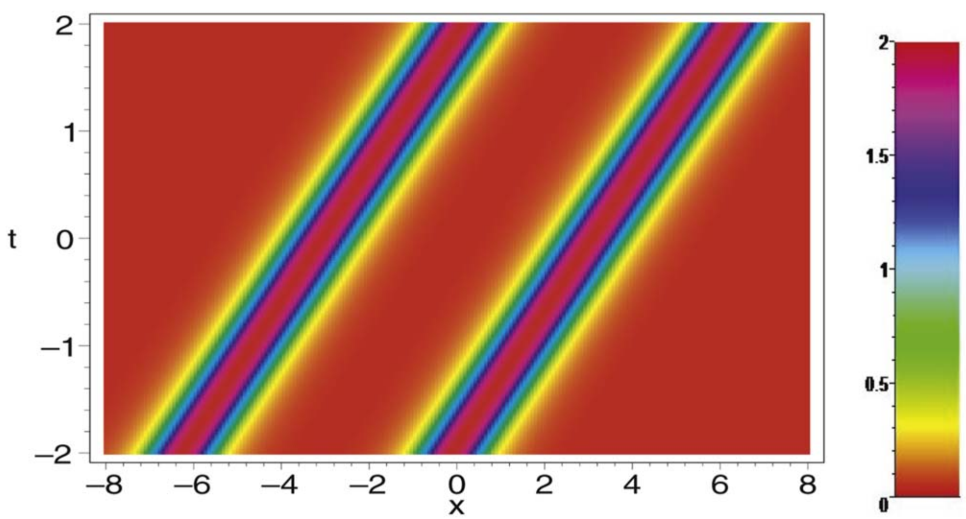

Figure 7. Soliton molecule structure for the KK equation described by (19) where the parameters are fixed by (25).

The resonant soliton solution (19) with (20) shows abundant structures because of the existence of the free parameter c which does not exist for the KdV and SK cases (6). For $-1 < c < 1$ and $1 < c < 2$, the solution (19) with (20) can be viewed as an analytic single soliton with one peak. For $c = 2$, the resonant solution (19) with (20) can be considered as a bounded kink-antikink soliton (a kink-antikink molecule), or a fat soliton, and/or a plateau soliton as shown in figure 6 with the parameter selections

$$c = 2,\ \epsilon = 0.1,\ k = 0.1,\ \xi_0 = 0. \tag{23}$$

This critical solution, $c = 2$ case, may be called a plateau soliton because of the property

$$u_x|_{\xi=0} = u_{xx}|_{\xi=0} = u_{xxx}|_{\xi=0} = 0 \tag{24}$$

at the center, $\xi = 0$, of the soliton.

For $c > 2$, the solution (19) with (20) is appropriately a two-soliton molecule, of which the two peaks will separate as c increasing. Figure 7 displays the structure of the soliton molecule (19) with the parameter selections

$$c = 3000,\ \epsilon = 0.1,\ k = 2.828428,\ \xi_0 = 0. \tag{25}$$

Different from the SK and the fifth order KdV cases, two solitons in one molecule of the KK equation are completely identical.

The second and third types of the soliton solutions (19) with (21) or (22) are only single analytical peaked solitons for $c = 1$.

The resonant solution (4) with (18) and $n = 2$ for the KK equation (3) with $\{a, b\} = \{1, 1/4\}$ possesses the form ($v \to u$),

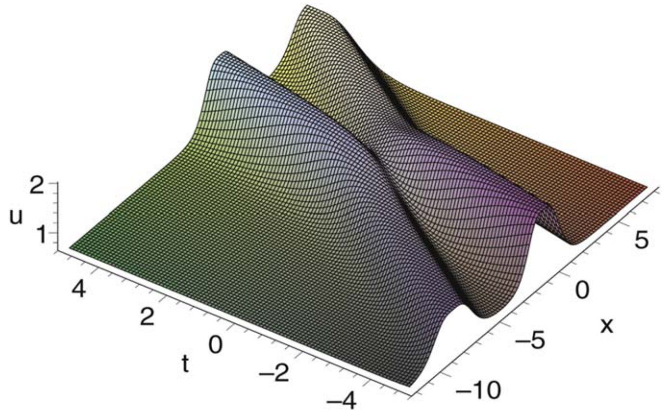

Figure 8. Interactions between two kink-antikink molecules of the KK equation described by (26) with (27).

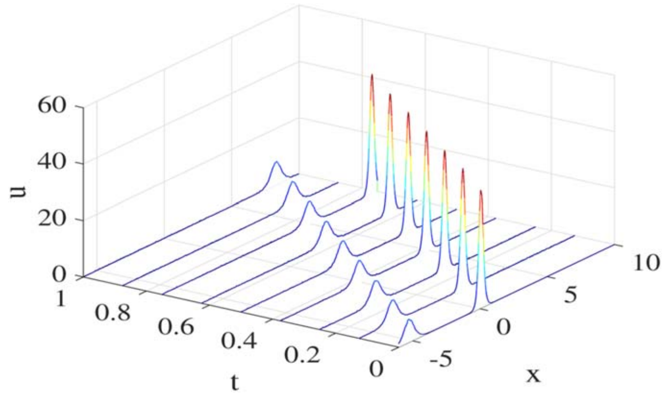

Figure 9. The numerical evolution of the molecule shown in figure 2 under the 10% random perturbation.

$$u = -\frac{\beta^2}{5\alpha\beta_1\epsilon} + \frac{3\beta}{\alpha}(\ln\psi)_{xx} \tag{26}$$

with $\psi = 8(k_1^2 - k_2^2)^2 + 12k_1^2k_2^2 + 4a_+a_-[\cosh(\xi_1) + \cosh(\xi_2)] + a_+^2\cosh(\xi_1 - \xi_2) + a_-^2\cosh(\xi_1 + \xi_2)$, where $a_\pm^2 = (k_1 \pm k_2)^2(k_1^2 \pm k_1k_2 + k_2^2)$, $\xi_i = k_i x - \left(\beta_1\epsilon k_i^5 - \frac{k_i\beta^2}{5\epsilon\beta_1}\right)t + \xi_{i0}$, $(i = 1, 2)$ and k_1, k_2, ξ_{10} and ξ_{20} are arbitrary constants.

Figure 8 displays the interaction between two kink-antikink molecules described by (26) with the parameters

$$k_1 = 2,\ k_2 = 1.6,\ \epsilon = 0.1,\ \xi_{10} = \xi_{20} = 0. \tag{27}$$

Finally, we numerically check the stability of the soliton molecule (shown in figure 2) for the KdV equation by using the initial condition

$$u(x, 0) = \frac{[1088\cosh(2\sqrt{21}x) + 5712\cosh(4x + 20) + 4624](1 + 0.1\text{rand(x)})}{[(2\sqrt{21} - 4)\cosh(\sqrt{21}x + 2x + 10) + (2\sqrt{21} + 4)\cosh(2x - \sqrt{21}x + 10)]^2} \tag{28}$$

where the random function $0 < \text{rand}(x) < 1$ are taken as the random perturbation to the original exact solution at $t = 0$. The figure 9 shows us that the soliton molecule (6) is very stable under the 10% random perturbation.

In summary, soliton molecules can be found not only in the optical systems [2, 4] but also in fluid systems, which may be applied for the oceanic surface and internal waves [29, 30], plasma waves [35], electromagnetic waves in discrete transmission lines [36] and so on. The fluid model (1) is integrable only in three special cases involving the fifth order KdV equation, the SK equation and the KK equation. However, everyone of three integrable models can be used to solve the original nonintegrable fluid system (1) approximately up to the same order of ϵ. Though the integrable systems are well known, the soliton molecules and asymmetric solitons of these models have not yet been reported previously. The soliton molecules for the fifth order KdV and SK equations are quite similar, however, the soliton molecules of the KK equation are very different to those of the fifth order KdV and SK cases. For the SK and fifth order KdV systems, two solitons in the molecule possess different amplitudes and widths while two solitons in the soliton molecule of the KK system possess completely the same properties except for the positions. The kink-antikink molecules with arbitrary wave numbers exist only for the KK equation. Soliton molecules can be considered as some types of special soliton resonance solutions. Soliton molecules are stable in the sense that the soliton-molecule is stable under small perturbations. When two solitons of the molecule is close enough, the molecule looks like one peak soliton that is asymmetric for the SK and fifth order KdV cases, and symmetric for the KK equation. Both the soliton molecules and the asymmetric solitons obtained in this letter may be observed in the fluid systems such as the atmospheric and oceanic dynamics, the stratified fluids, the magnetic fluids, the quantum fluids, the fluids of light, the atomic and molecular gases, the degenerate electron fluids, the superfluid ^{3}He [14–21], and other physical systems where the model equation (1) is valid approximately.

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