# The Sixth Superstring

## Bachelor thesis in Physics

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June 10 2013

**Abstract**

The objective of this thesis is to discuss the possible new superstring proposed by Savdeep Sethi in April 2013. In order to do this, we give an introduction to string theory pretty much from the ground up - starting with the 26 dimensional bosonic string and then on to the five different flavours of ten dimensional superstring. Along the way we will discuss some of the dualities and transformations that relate these strings to each other. Savdeep Sethi's superstring will arise from just such a transformation: an orientifold of the type IIB superstring. Eventually we will start to hone in on the modern picture of string theory, which is that the superstrings are all perturbative limits of an 11-dimensional theory called 'M-theory'. We will discuss how Savdeep Sethi's superstring may fit into this web of theories. By construction, the new string looks very similar to the Type I superstring. We will have to find a way to distinguish the two from each other. By comparing the Kaluza-Klein towers that result from their M-theory descriptions, we will find a sharp distinction between Type I and the new superstring. However, we will be left with questions about the consistency of the new string and about its place in the M-theory web.

# Contents

# Chapter 1

# Introduction

## What is string theory?

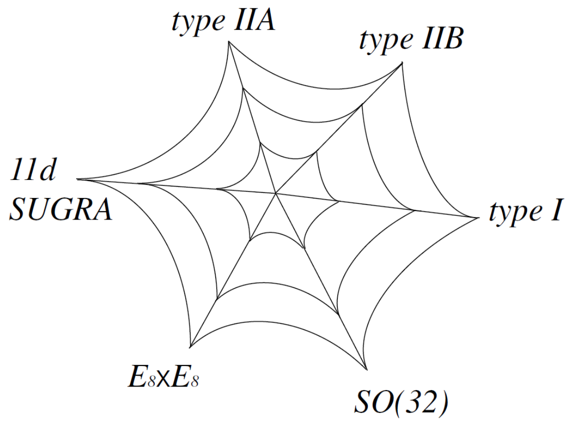
String theory is a theory of nature where the fundamental building blocks are one-dimensional objects called strings. It was first developed in the 1960s as an attempt to describe the strong nuclear force. In this respect, it was eventually superseded by quantum chromodynamics. In the mid-1970s it was realized that string theory could describe a consistent quantum gravity theory. Since then, string theory has been a candidate for a grand unified theory, poised to bring all of the forces of nature together in a single theoretical framework. From this perspective, the big claim of string theory is that the fundamental particles in the standard model are nothing more than the vibrational modes of quantum strings.

Just why exactly is string theory necessary to provide a quantum theory of gravity? This can be seen in several ways. In ordinary quantum field theory, one requires that two fields at space-like separation should (anti)commute [2]. However, when the field in question is the spacetime metric itself, as would be the case in quantum gravity, it is not clear in advance whether two points have space-like separation at all! Secondly, the metric suffers quantum fluctuations, just like any other field. These difficulties make sure that straightforward attempts to build a quantum gravity theory all suffer from uncontrollable infinities. More precisely, they are not renormalizable. String theory gets around these difficulties because it has a built-in length scale, the string length $\ell_s$, which makes it, in some sense, insensitive to the irregularities of spacetime at the smallest scale. This is actually a principle that string theory has in common with all modern quantum gravity theories.

The length scale $\ell_s$ is actually the only dimensionful adjustable parameter in string theory[1], whereas the standard model has 19. [1] This is due to string theory's highly restricted nature. For example, in string theory the dimensionality of spacetime is determined by a *calculation*, instead of a measurement. The bosonic string lives in 26 dimensions, whereas the superstring lives in 10 dimensions. The fact that strings are only consistent in dimensions higher than the 4 we actually observe needs to be accounted for. One solution is that all but 4 of the spacetime dimensions are compactified: they close in on themselves like a circle.

Roughly speaking, there are two different kinds of string theories: bosonic string theory and superstring theory. The bosonic string has mostly been abandoned as a convincing theory of nature, because it does not incorporate fermions. The new string that we intend to examine is a superstring. There are, as-of-yet, five known flavors of superstring. In the 1990s it was realized that these were related by a large number of dualities. The picture that emerged was that the superstrings are all perturbative limit of a more fundamental theory called 'M-theory', which lives in 11 dimensions, and has a low-energy limit called 11-dimensional supergravity. We can summarize with a picture of the so-called web of dualities (taken from Becker, Becker, Schwarz [2]):

[1] This is the only parameter in its *formulation*. There still is a huge landscape of possible solutions.




We will revisit this picture a number of times throughout the rest of the thesis, each time updating it with our new knowledge. The proposed new superstring would represent a seventh spoke on this web.

Let's outline the structure of the rest of the thesis:

## Bosonic strings

We will start by examining the relatively simple bosonic string. This will be useful to gain some familiarity with string theory concepts. Savdeep Sethi's string is not a bosonic string. When we construct the spectrum, we will encounter some familiar particles, and we will immediately see string theory's big claim being actualized to some extent. However, the bosonic string can't provide a full theory of nature by itself, since its spectrum does not contain fermions. To make the bosonic string theory abide by Lorentz invariance, the dimensionality of spacetime will be restricted to D = 26. The bosonic string does not seem to have a place by itself in the M-theory web, but it enters into the description of the heterotic superstrings $E_8 \times E_8$ and *O(32)*. We discuss the bosonic string in Chapter 2.

## Superstrings

The strings that may actually provide us with a convincing theory of nature (they do include fermions, for instance) are all supersymmetric strings, or superstrings. That means that there is a symmetry transformation relating the fermions and the bosons to each other. Every boson is given its own supersymmetric partner. We will construct the superstring in much the same way we did the bosonic string. Only this time we will make things considerably more abstract by introducing anti-commuting Grassman coordinates.

The supersymmetric string comes in (as-of-yet) five different flavours. Savdeep Sethi's proposed string is a potential sixth flavour. We discuss the type IIB, IIa and type I superstrings in Chapter 3. A number of consistency checks restricts the dimension of spacetime in these theories to D = 10, in apparent conflict with the bosonic string.

## Compactification

Superstrings live in ten dimensions. In order to make this consistent with reality, all but 4 of these dimensions have to be compactified. We discuss compactification in Chapter 4.

## Symmetries and dualities, orbifolds and orientifolds

When we discuss the bosonic and supersymmetric strings, we will discuss a number of transformations that relate the different string theories to each other. For example, we will see that the Type IIA string can be obtained from the Type IIB string by an orbifold projection. We will also discuss so-called duality symmetries, which are transformations that identify two theories that at first glance seem to be completely different. T-duality, which relates theories on large compactified dimensions to theories on small compactified dimensions, will feature prominently in Chapter 4. We discuss orbifolds and orientifolds in Chapter 5.

## The new superstring

Once this is done we will be sufficiently equipped to understand Savdeep Sethi's proposition of a new superstring. We will examine whether or not the proposition really leads to a new, inequivalent superstring. In particular, there is doubt over whether the new string is equivalent to the Type I superstring, because they are formulated in a very similar way. The last chapter of the thesis is devoted to discussing the new superstring.

# Chapter 2

# The bosonic string

In this chapter we will mostly follow the development of the bosonic string as described in Becker Becker Schwarz [2], interjected with information from Zwiebach [1] and Tong [3].

## 2.1 Constructing the action

### 2.1.1 The point particle action

We know that free point particles travel along geodesics, which are the paths of minimum proper length. This allows us to easily construct the following action:

$$S_0 = -\alpha \int ds \tag{2.1}$$

In our system of units $\hbar = c = 1$, so we see that the constant $\alpha$ has units of inverse length, which is equivalent to units of mass. We therefore choose to identify this constant with the mass of the point particle. We can easily see that this leads to the correct equations of motion in the non-relativistic limit.

We may parametrize the path of the point particle (its world line) with $X^\mu = X^\mu(\tau)$. Our action becomes:

$$S_0 = -m \int \sqrt{-g_{\mu\nu}(X)\dot{X}^\mu \dot{X}^\nu} d\tau \tag{2.2}$$

Here $g_{\mu\nu}(X)$ is the background metric. An application of the chain rule shows that this action is reparametrization invariant. This will be of great use to us in the future, as it will allow us to choose convenient gauges to work in.

### 2.1.2 The string action

Just as the zero-dimensional point particle traces out a one-dimensional world line on a spacetime diagram, a one-dimensional string traces out a two-dimensional *world sheet*. We parametrized the world line of the point particle using a single variable $\tau$. To work with strings, we have to parametrize their world sheet using two parameters. We call them $\tau$ and $\sigma$ in anticipation of the gauge choice we intend to make, where one parameter, $\tau$, will be akin to a world sheet time variable. We now define a string to be the set of points parametrized by $\sigma$ at a fixed $\tau$.

A string may be either open or closed. If it is open, the string will have to satisfy certain boundary conditions at its end points. If it closed, the embedding functions will have to be periodic in $\sigma$. We will discuss boundary conditions later on in this chapter.

Generalizing the point particle action given above, the string action now becomes:

$$S = -T \int \sqrt{-det(G_{\alpha\beta})} d^2\sigma \tag{2.3}$$

where $G_{\alpha\beta} = g_{\mu\nu}\partial_\alpha X^\mu \partial_\beta X^\nu$ is the induced metric, the $d^2\sigma$ refer to the two parameters of the parametrization and the indices $\alpha, \beta$ run over those same parameters. The constant $T_p$ is called the string tension. Because the action makes explicit reference to the Jacobian $det(G_{\alpha\beta})$, it is manifestly reparametrization invariant. On a flat background, the action becomes:

$$S_{NG} = -T \int \sqrt{(\dot{X} \cdot X')^2 - (\dot{X}^2)(X')^2} d\tau d\sigma \tag{2.4}$$

This is just proportional to the Lorentz-invariant area of the two-dimensional sheet that the string traces out on a spacetime diagram (we call this the *world sheet* of the string). (2.4) is sometimes called the Nambu-Goto action.[2]

Taking a variation in $X^\mu$, the string equations of motion become:

$$\frac{\partial \mathcal{P}^\tau_\mu}{\partial \tau} + \frac{\partial \mathcal{P}^\sigma_\mu}{\partial \sigma} = 0 \tag{2.5}$$

where we have defined $\mathcal{P}^\alpha_\mu \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\alpha X^\mu)}$.

The action (2.4) is not always easy to work with. Its Euler-Lagrange equations are in general very complicated, because the conjugate momenta $\mathcal{P}^\alpha_\mu$ can have very long expressions. They can be made much simpler by choosing a convenient parametrization. We will illustrate how to choose a parametrization in the next subsection. Later on, we propose another action that is equivalent to (2.4) at the classical level: the Polyakov action.

### 2.1.3 Choosing a parametrization

We introduce the following class of gauge choices for $\tau$:

$$n \cdot X = \beta\alpha'(n \cdot p)\tau \tag{2.6}$$

where $\alpha'/2 \equiv \ell_s^2$, $\beta$ is a dimensionless constant and $p^\mu \equiv \int_0^{\sigma_f} \frac{\partial \mathcal{L}}{\partial \dot{X}^\mu} d\sigma$ is the total classical string momentum in the $\mu$ direction, which is conserved in time due to the string equations of motion. We require that $n^\mu$ is either time-like or null. This ensures that the interval between any two points along a string is space-like. Our choice of gauge is *not* Lorentz covariant, because a linear combination of spatial coordinates can never be Lorentz invariant.

We want our $\sigma$ parametrization to satisfy two conditions:

- We want $n \cdot \mathcal{P}^\tau$ to be constant over the world sheet. Substituting this requirement into the equation of motion (2.5) shows that $n \cdot \mathcal{P}^\sigma$ is a world sheet constant as well. In fact, for open strings $n \cdot \mathcal{P}^\sigma = 0$ everywhere, because it is guaranteed to vanish at the string endpoints.

- We want the parametrization range to be $\sigma \in [0, \pi]$ for open strings and $\sigma \in [0, 2\pi]$ for closed strings. This means $\beta$ must be equal to 2 for open strings and 1 for closed strings.

Our conditions may be implemented by requiring the following:

$$(n \cdot p)\sigma = \frac{2\pi}{\beta} \int_0^\sigma d\sigma' n \cdot \mathcal{P}^\tau(\tau, \sigma') \tag{2.7}$$

There still remains an ambiguity in the case of closed strings: how do we choose which point on each string to identify with $\sigma = 0$? We have to select $\sigma = 0$ on one string arbitrarily. We then select $\sigma = 0$ on all other strings by requiring that $n \cdot \mathcal{P}^\sigma$ vanish everywhere, a condition that is automatically satisfied by open strings.

We obtain an expression for $n \cdot \mathcal{P}^\sigma$ from the Nambu-Goto action:

$$n \cdot \mathcal{P}^\sigma = -\frac{1}{2\pi\alpha'} \frac{(X \cdot X')\partial_\tau(n \cdot X)}{\sqrt{(\dot{X} \cdot X')^2 - (\dot{X})^2(X')^2}} \tag{2.8}$$

We must show that $(X \cdot X')$ vanishes at some point on each string, since $\partial_\tau(n \cdot X)$ is a constant. As mentioned, we have to select the point where $\sigma = 0$ on one string arbitrarily. At this point, there is a world sheet tangent vector $v^\mu$ that is orthogonal to $X'^\mu$. We draw the $\sigma = 0$ line along $v^\mu$. This selects $\sigma = 0$ on the neighbouring strings. The full $\sigma = 0$ line is constructed by repeating this process. Because the $\sigma = 0$ line is proportional to $\dot{X}^\mu$, this ensures that $(X \cdot X') = 0$ at one point on each string and, therefore, that $n \cdot \mathcal{P}^\sigma$ vanishes everywhere.

Looking back at equation (2.8), we see that $(X \cdot X')$ actually vanishes everywhere, not just at one point on each string. We can use this fact to simplify the equations of motion. The expression for $\mathcal{P}^\tau$ becomes:

$$\mathcal{P}^{\tau\mu} = \frac{1}{2\pi\alpha'} \frac{X'^2 \dot{X}^\mu}{\sqrt{-\dot{X}^2 X'^2}} \tag{2.9}$$

Taking the $\sigma$ derivative of (2.7), we obtain:

$$n \cdot p = \frac{1}{\beta\alpha'} \frac{X'^2(n \cdot \dot{X})}{\sqrt{-\dot{X}^2 X'^2}} \tag{2.10}$$

Using $n \cdot \dot{X} = \beta\alpha'(n \cdot p)$, we find:

$$\dot{X}^2 + X'^2 = 0 \tag{2.11}$$

The constraints (2.11) along with $(X \cdot X') = 0$ are referred to as the Virasoro constraints. Substituting these into the expressions for the conjugate momenta, we obtain the following:

$$\mathcal{P}^{\sigma\mu} = \frac{1}{2\pi\alpha'} X'^{\mu} \tag{2.12}$$

$$\mathcal{P}^{\tau\mu} = \frac{1}{2\pi\alpha'} \dot{X}^{\mu} \tag{2.13}$$

This means that the equations of motion (2.5) become simple wave equations! [1]

### 2.1.4 The Polyakov action

We now introduce the convenient Polyakov action, which is formulated in terms of an auxiliary metric $h_{\alpha\beta} = (h^{-1})^{\alpha\beta}$. By definition: $h \equiv det(h_{\alpha\beta})$

$$S_\sigma = -\frac{1}{2} T \int d^2\sigma \sqrt{-h} h^{\alpha\beta} \partial_\alpha X \cdot \partial_\beta X \tag{2.14}$$

This action is equivalent to (2.4) at the classical level. To see this, let us take a variation in $h_{\alpha\beta}$ and obtain the equation of motion. We use the formula $\delta h = -h h_{\alpha\beta} \delta h^{\alpha\beta}$, which implies $\delta\sqrt{-h} = -\frac{1}{2}\sqrt{-h} h_{\alpha\beta} \delta h^{\alpha\beta}$. Inserting this into the variation of the action, we obtain the equation of motion for $h^{\alpha\beta}$:

$$\partial_\alpha X \cdot \partial_\beta X - \frac{1}{2} h_{\alpha\beta} h^{\gamma\sigma} \partial_\gamma X \cdot \partial_\sigma X = 0 \tag{2.15}$$

which is equal to the component $T_{\alpha\beta}$ of the energy-momentum tensor. Taking the square root of minus the determinant of both of these terms, we obtain:

$$\sqrt{-det(\partial_\alpha X \cdot \partial_\beta X)} = \frac{1}{2} \sqrt{-h} h^{\gamma\sigma} \partial_\gamma \cdot \partial_\sigma X \tag{2.16}$$

and the equivalence to the Nambu-Goto action is established. [2] The Polyakov action is more convenient for a number of purposes. We will use it as our starting point when we consider the supersymmetric strings in the RNS formalism.

In the next section we classify the symmetries of the Polyakov action. These are reparametrizations, Poincaré transformations, and Weyl transformations. These symmetries will allow us to choose a very convenient gauge to work in: the light-cone gauge.

## 2.2 Symmetries of the Polyakov action and the conformal gauge

### 2.2.1 Poincaré transformations

The action is left unchanged by the following transformations:

$$\delta X^\mu = a^\mu{}_\nu X^\nu + b^\mu \tag{2.17}$$

with

$$\delta h^{\alpha\beta} = 0 \tag{2.18}$$

Where $a^\mu{}_\nu$ is a parameter for infinitesimal Lorentz transformations. The Nambu-Goto action has a Poincaré symmetry as well.

### 2.2.2 Reparametrization

Just like the Nambu-Goto action, the Polyakov action is invariant under reparametrizations. A reparametrizaton has to be accompanied by the following transformation of the auxiliary metric: $h_{\alpha\beta}(\sigma) = \frac{\partial f^\gamma}{\partial \sigma^\alpha} \frac{\partial f^\delta}{\partial \sigma^\beta} h_{\gamma\sigma}(\sigma')$

### 2.2.3 Weyl rescaling

The Polyakov action is invariant under Weyl transformations. A Weyl transformation is a local change of scale that preserves the angles between all lines on the world sheet. They act on the auxiliary metric as $h_{\alpha\beta} \to e^{\phi(\sigma,\tau)} h_{\alpha\beta}$. The symmetry appears because the Polyakov action involves the terms $\sqrt{det(h_{\alpha\beta})}$ and $h^{\alpha\beta}$, which obtain cancelling factors after the Weyl transformation due to the identity $h^{\alpha\beta} = h^{-1}_{\alpha\beta}$. Infinitesimally, we can write $\delta h^{\alpha\beta} = \phi(\sigma,\tau) h^{\alpha\beta}$. Weyl transformations are only symmetries in the two-dimensional case, because the factors that the metric tensor and the Jacobian acquire do not cancel in spaces of any other dimensionality. The requirement of Weyl invariance puts strict limits on what kind of interactions we can add to the theory. [3]

### 2.2.4 Gauge fixing and residual symmetry: the light cone gauge

The auxiliary metric $h_{\alpha\beta}$ has three independent components.

$$h_{\alpha\beta} = \begin{pmatrix} h_{00} & h_{01} \\ h_{10} & h_{11} \end{pmatrix} \tag{2.19}$$

where $h_{01} = h_{10}$. Using the reparametrization invariance, we can gauge away two of the independent components. Applying a Weyl transformation removes the last component. We are therefore free to gauge fix the $h_{\alpha\beta}$ completely. We make the choice $h_{\alpha\beta} = \eta_{\alpha\beta}$, where $\eta_{\alpha\beta}$ is just the Minkowski signature. We call this the conformal gauge. It is equivalent to the class of gauge choices we considered in section (2.1.3). To see this, notice that the Polyakov action becomes:

$$S = \frac{T}{2}\int d^2\sigma(\dot{X}^2 - X'^2) \tag{2.20}$$

which leads to the wave equation

$$(\frac{\partial^2}{\partial\sigma^2} - \frac{\partial^2}{\partial\tau^2})X^\mu = 0 \tag{2.21}$$

This has to be consistent with the equation of motion for $h^{\alpha\beta}$, which has become:

$$T_{\alpha\beta} = \partial_\alpha X \cdot \partial_\beta X - \frac{1}{2}\eta_{\alpha\beta}\eta^{\gamma\sigma}\partial_\gamma X \cdot \partial_\sigma X = 0 \tag{2.22}$$

We have to implement this as a constraint condition. Let's look at the components of $T_{\alpha\beta}$ more closely:

$$T_{01} = \dot{X} \cdot X' = 0 \tag{2.23}$$

$$T_{00} = T_{11} = \frac{1}{2}(\dot{X}^2 + X'^2) = 0 \tag{2.24}$$

These are just the Virasoro constraints we derived in section (2.1.3).

We have not yet used the full range of symmetries possessed by the Polyakov action. There exists a range of additional reparametrizations that can be undone by a Weyl transformation. They are the reparametrizations that act on the metric as:

$$h^{\alpha\beta} \to e^{\phi(\sigma,\tau)}h^{\alpha\beta} \tag{2.25}$$

We can find out what kind of reparametrizations these are by using world-sheet light-cone coordinates:

$$\sigma^\pm \equiv \sigma^0 \pm \sigma^1 \tag{2.26}$$

Spacetime light-cone coordinates are defined slightly differently:

$$X^\pm \equiv \frac{1}{\sqrt{2}}(X^0 \pm X^1) \tag{2.27}$$

The inner product of two vectors in light-cone coordinates takes the form:

$$v \cdot w = -v^+w^- - v^-w^+ + v^iw^i \tag{2.28}$$

In terms of the light-cone coordinates, the metric on the worldsheet becomes:

$$ds^2 = -d\sigma^+d\sigma^- \tag{2.29}$$

So the transformations of the form:

$$\sigma^+ \to \widetilde{\sigma}^+(\sigma^+)\ ,\ \sigma^- \to \widetilde{\sigma}^-(\sigma^-) \tag{2.30}$$

act on the metric as in (2.25). This means that we make the transformation $\tau \to \widetilde{\tau}$ where $\widetilde{\tau}$ can be any solution to the wave equation

$$(\frac{\partial^2}{\partial\sigma^2} - \frac{\partial^2}{\partial\tau^2})\widetilde{\tau} = 0 \tag{2.31}$$

We saw previously that in conformal gauge the spacetime coordinates themselves satisfy the wave equation. We can therefore make the gauge choice:

$$X^+(\sigma,\tau) = x^+ + \ell_s^2 p^+\tau \tag{2.32}$$

where $x^+$ is a constant and $p^+$ is the total string momentum in the $X^+$ direction. This is called the *light-cone gauge.*[3 2] We will make extensive use of it throughout the rest of the thesis.

## 2.3 Boundary conditions: open and closed strings

Taking a variation of the string action of the form

$$X^\mu \to X^\mu + \delta X^\mu \tag{2.33}$$

one obtains the string equations of motion and a boundary term

$$-T\int d\tau [X'_\mu \delta X^\mu|_{\sigma=\pi} - X'_\mu \delta X^\mu|_{\sigma=0}] \tag{2.34}$$

There are several different boundary conditions which make this term vanish

- *Closed string*: A closed string has periodic embedding functions:

$$X^\mu(\sigma,\tau) = X^\mu(\sigma+\pi,\tau) \tag{2.35}$$

- *Neumann boundary conditions*: In this case the $\sigma$ momentum vanishes at the string end points:

$$X'_\mu = 0 \tag{2.36}$$

  at $\sigma = 0, \pi$

- *Dirichlet boundary conditions*: In this case the open string has fixed endpoints:

$$X^\mu(\pi,\tau) = X^\mu_\pi \tag{2.37}$$

$$X^\mu(0,\tau) = X^\mu_0 \tag{2.38}$$

  An open string may satisfy Dirichlet boundary conditions for some of its coordinates and Neumann boundary conditions for others. The coordinates $X^\mu_0$ and $X^\mu_\pi$ represent the locations of D-branes. A D-brane is a hyperplane on which an open string satisfying Dirichlet boundary conditions can end. We return to the subject of D-branes in Chapter 4.[2]

## 2.4 Mode expansion

Before we can quantize the bosonic string, we need to expand the embedding functions $X^\mu$ into oscillator modes, just like we do in ordinary quantum field theory. In terms of the light-cone world sheet coordinates $\sigma^\pm$, the string equation of motion (2.27) takes the form

$$\partial_+\partial_- X^\mu = 0 \tag{2.39}$$

The most general solution is

$$X^\mu(\sigma,\tau) = X^\mu_L(\sigma^+) + X^\mu_R(\sigma^-) \tag{2.40}$$

which has to satisfy Virasoro constraints and the boundary conditions. Any given solution of the form (2.41) has an associated solution in terms of the *dual* coordinate $\widetilde{X}^\mu(\sigma,\tau)$:

$$\widetilde{X}^\mu(\sigma,\tau) = X^\mu_L(\sigma^+) - X^\mu_R(\sigma^-) \tag{2.41}$$

which will come into play when we consider T-duality and D-branes.[2]

### 2.4.1 The closed string

A closed string has periodic embedding functions as indicated in (2.36). A periodic function may be expressed in terms of a Fourier series:

$$X^\mu_R = \frac{1}{2}x^\mu + \frac{1}{2}\ell_s^2\alpha^\mu_0\sigma^- + \frac{i}{2}\ell_s\sum_{n\neq 0}\frac{1}{n}\alpha^\mu_n e^{-2in(\sigma^-)} \tag{2.42}$$

$$X^\mu_L = \frac{1}{2}x^\mu + \frac{1}{2}\ell_s^2\widetilde{\alpha}^\mu_0\sigma^+ + \frac{i}{2}\ell_s\sum_{n\neq 0}\frac{1}{n}\widetilde{\alpha}^\mu_n e^{-2in(\sigma^+)} \tag{2.43}$$

We refer to $\alpha^\mu_n$ and $\widetilde{\alpha}^\mu_n$ as the right- and left-moving oscillators, respectively.

$X_L$ and $X_R$ do not satisfy the periodicity requirement individually, but their sum does. The dual coordinate in fact belongs to an open string with Dirichlet boundary conditions.

The variable $x^\mu$ specifies the location of the center of mass of the string. The zero mode $\alpha^\mu_0$ is equal to $\ell_s p^\mu$. This may be checked by studying the conserved current associated with the spacetime translation symmetry. The same follows for the right-moving zero mode $\widetilde{\alpha}^\mu_0$.

$$\alpha^\mu_0 = \widetilde{\alpha}^\mu_0 = \ell_s p^\mu \tag{2.44}$$

The reality of $X^\mu$ ensures that $\alpha^\mu_n = (\alpha^\mu_{-n})^\star$ and $\widetilde{\alpha}^\mu_n = (\widetilde{\alpha}^\mu_{-n})^\star$.[2]

### 2.4.2 Virasoro constraints

In terms of the light-cone world sheet coordinates, the Virasoro constraints become

$$(\partial_- X)^2 = (\partial_+ X)^2 = 0 \tag{2.45}$$

Let's see what these constraints imply for the oscillator modes. We have:

$$\partial_- X^\mu = \partial_- X^\mu_R = \ell_s \sum_n \alpha^\mu_n e^{-in\sigma^-} \tag{2.46}$$

The constraint becomes:

$$(\partial_- X)^2 = \ell_s^2 \sum_{m,n} \alpha_m \cdot \alpha_{n-m} e^{-in\sigma^-} \equiv 2\ell_s \sum_n L_n e^{-in\sigma^-} \tag{2.47}$$

We have suppressed Lorentz indices for now. The quantity $L_n$ is called the Virasoro mode. We can do the same thing for the second constraint $(\partial_+ X)^2 = 0$ and obtain the right-moving Virasoro mode

$$\widetilde{L}_n \equiv \frac{1}{2} \sum_m \widetilde{\alpha}_{n-m} \cdot \widetilde{\alpha}_m \tag{2.48}$$

Any classical solution must obey the infinite set of constraints

$$L_n = \widetilde{L}_n = 0 \tag{2.49}$$

The case $n = 0$ is special, because the right- and left-moving zero modes are proportional to the total string momentum. The square of the string momentum is equal minus the squared rest mass of the string:

$$p_\mu p^\mu = -M^2 \tag{2.50}$$

This means that the constraints on the Virasoro zero modes tell us the mass of the classical string:

$$M^2 = \frac{4}{\alpha'} \sum_{n>0} \alpha_n \cdot \alpha_{-n} = \frac{4}{\alpha'} \sum_{n>0} \widetilde{\alpha}_n \cdot \widetilde{\alpha}_{-n} \tag{2.51}$$

This relates the number of right- and left-moving oscillators to each other. The constraint is known as *level matching.* We will meet these concepts again, subject to minor adaptations, when we quantize the bosonic string in the next section.[3]

### 2.4.3 The open string

The open string with Neumann boundary conditions has the mode expansion:

$$X^\mu(\tau, \sigma) = x^\mu + \ell_s^2 p^\mu \tau + i\ell_s \sum_{m\neq 0} \frac{1}{m} \alpha^\mu_m e^{-im\tau} cos(m\sigma) \tag{2.52}$$

as may be checked by noting that in this case it is the $\sigma$ derivative of the solution that is periodic. The open string has only one set of oscillator modes, as opposed to the closed string, which has right-movers and left-movers. We will look at the mode expansion of the open string with Dirichlet boundary conditions when we discuss T-duality and D-branes in Chapter 4. The mass formula for the open string becomes[2]:

$$M^2 = \frac{1}{\alpha'} \sum_{n=1}^{\infty} \alpha_n \cdot \alpha_n \tag{2.53}$$

## 2.5 Quantization

There are two methods we can use to quantize the bosonic string. In the first, we apply the standard quantization programme to the oscillator modes and the spacetime coordinates and then impose the Virasoro constraints upon the state space. This is called covariant quantization. In the second, we impose the Virasoro constraints right at the beginning, upon the classical solutions to the equations of motion. Only then do we proceed with the quantization programme. Because we apply this method in the light-cone gauge, this is called light-cone gauge quantization.

Both of these methods have their issues. Covariant quantization leads to negative-norm states, which we will have to decouple from the theory. Light-cone gauge quantization has its own set of problems, which arise because the gauge choice is not Lorentz covariant. The quantum theory is therefore in danger of losing Lorentz invariance, which is unacceptable. This can happen even though the underlying theory of the classical bosonic string is Lorentz invariant. A symmetry of a classical theory that disappears after quantization is called an *anomaly.* We will encounter other anomalies when we discuss the superstrings.

### 2.5.1 Covariant quantization

The classical Poisson brackets for the spacetime coordinates $X^\mu$ and its canonical momentum conjugate $\mathcal{P}^{\tau\mu} = T\dot{X}^\mu$ are given by

$$[\mathcal{P}^\mu(\sigma,\tau), \mathcal{P}^\nu(\sigma',\tau)]_{P.B} = [X^\mu(\sigma,\tau), X^\nu(\sigma',\tau)]_{P.B} = 0 \tag{2.54}$$

$$[\mathcal{P}^\mu(\sigma,\tau), X^\nu(\sigma',\tau)]_{P.B} = \eta^{\mu\nu}\delta(\sigma-\sigma') \tag{2.55}$$

In the rest of this thesis we will omit $\tau$ in the symbol for the momentum conjugate to $X^\mu$ and write: $\mathcal{P}^{\tau\mu} = P^\mu$

Inserting the mode expansions into the Poisson brackets gives the following:

$$[\alpha^\mu_m, \alpha^\nu_n]_{P.B} = [\widetilde{\alpha}^\mu_m, \widetilde{\alpha}^\nu_n]_{P.B} = im\eta^{\mu\nu}\delta_{m+n,0} \tag{2.56}$$

Now we make the standard replacements

$$[...]_{P.B} \to i[...] \tag{2.57}$$

and promote all the physical observables to operators. After defining the lowering and raising operators

$$a^\mu_m = \frac{1}{\sqrt{m}}\alpha^\mu_m \tag{2.58}$$

$$a^{\mu\dagger}_m = \frac{1}{\sqrt{m}}\alpha^\mu_{-m} \tag{2.59}$$

and doing the same for the right-moving oscillators, we find:

$$[a^\mu_m, a^{\nu\dagger}_n] = [\widetilde{a}^\mu_m, a^{\nu\dagger}_n] = \eta^{\mu\nu}\delta_{m_n} \tag{2.60}$$

with $m, n > 0$.

We immediately spot a problem: the commutator of a lowering operator and a raising operator in the time direction is equal to minus one:

$$[a^0_m, a^{0\dagger}_m] = -1 \tag{2.61}$$

This will lead to negative norm states in the spectrum. These states are called ghosts. To see this, let us define the string ground state $|0;k\rangle$, which will be annihilated by the lowering operators:

$$a^\mu_m\,|0;k\rangle = 0 \tag{2.62}$$

The k-index specifies the momentum of the string state:

$$\hat{p}^\mu\,|0;k\rangle = k^\mu\,|0;k\rangle \tag{2.63}$$

We see that a negative norm state is given by:

$$a^{0\dagger}_m\,|0;k\rangle \tag{2.64}$$

which has norm:

$$\langle 0|\, a^0_m a^{0\dagger}_m\,|0\rangle = -1 \tag{2.65}$$

We will comment on how to solve this problem later. Let's start to build the Fock space of the bosonic string. The most general string state has the form:

$$(a^{\mu_1\dagger}_1)^{n_{\mu 1}}(a^{\mu_2\dagger}_2)^{n_{\mu 2}}...(a^{\nu_1\dagger}_1)^{n_{\nu 1}}(a^{\nu_2\dagger}_2)^{n_{\nu 2}}...\,|0;k\rangle \tag{2.66}$$

Each state is interpreted as the one-particle state of a different species of particle in spacetime. The bosonic string therefore carries an infinite number of particles.[2 1] We will discuss exactly what kind of particles are contained within the spectrum in the next section.

### 2.5.2 Dealing with the ghosts

The appearance of negative norm states in the spectrum may remind you of the similar situation that arises when trying to quantize QED in the Gupta-Bleuler formalism. In that case, the problem is solved by imposing the gauge fixing constraint upon the states in the spectrum. Similarly, we will try to fix the spectrum of the bosonic string by imposing the Virasoro constraints.

Recall that we had the classical constraints $L_n = \widetilde{L}_n = 0$. For the open strings the second of these does not apply, since an open string has only a single set of oscillator modes. In the quantum theory, the Virasoro constraints become:

$$L_n\,|phys\rangle = \widetilde{L}_n\,|phys\rangle = 0 \tag{2.67}$$

with $n > 0$. The kets indicated by $|phys\rangle$ are the physical states of the theory.

There is however an ordering ambiguity in the definition of $L_0$ and $\widetilde{L}_0$. This ordering ambiguity may be resolved by choosing a specific ordering and adding an undetermined constant to the constraint upon physical states. In other words, we choose $L_0$ to be:

$$L_0 = \sum_{m=1}^{\infty} \alpha_{-m} \cdot \alpha_m + \frac{1}{2}\alpha_0^2 \tag{2.68}$$

(and choose $\widetilde{L}_0$ in the analogous way) and we change our constraint upon physical states to:

$$(L_0 - a)\,|phys\rangle = (\widetilde{L}_0 - a)\,|phys\rangle = 0 \tag{2.69}$$

for some as-of-yet undetermined constant $a$. In the case of open strings only the first of these applies. For certain critical values of the constant $a$ and the spacetime dimensionality $D$, the Virasoro constraints indeed decouple all negative-norm states from the theory. These values turn out to be $a = 1$ and $D = 26$.[3]

### 2.5.3 The mass operators

The value of the constant $a$ has an effect on the mass operator. For the open string, it changes into:

$$\alpha' M^2 = \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n - a = N - a \tag{2.70}$$

where

$$N \equiv \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n = \sum_{n=1}^{\infty} n a_n^\dagger \cdot a_n \tag{2.71}$$

For the closed string

$$\frac{1}{4}\alpha' M^2 = \sum_{n=1}^{\infty} \alpha_{-n}\alpha_n - a = \sum_{n=1}^{\infty} \widetilde{\alpha}_{-n} \cdot \widetilde{\alpha}_n - a = N - a = \widetilde{N} - a \tag{2.72}$$

This implies $N = \widetilde{N}$, which is the level-matching condition we've already encountered.[3]

### 2.5.4 Light-cone gauge quantization

We will now try to quantize the bosonic string using the second method discussed above. We will implement the Virasoro constraints right at the beginning, before proceeding with the usual quantization programme. In section (2.2.4) we noted that a reparametrization of the form $\sigma^+ \to \widetilde{\sigma}^+(\sigma^+)$ and $\sigma^- \to \widetilde{\sigma}^-(\sigma^-)$ can be undone by a simultaneous Weyl transformation. This allowed us to choose the light-cone gauge

$$X^+ = x^+ + \alpha' p^+ \tau \tag{2.73}$$

The residual reparametrization invariance described above reduces the number of physical degrees of freedom of the theory. To see this, recall that the general solution to the closed-string equations of motion in conformal gauge came in the form:

$$X^\mu = X_L^\mu(\sigma^+) + X_R^\mu(\sigma^-) \tag{2.74}$$

which would seem to imply that there are $2D$ independent solutions. The Virasoro constraints

$$(\partial_+ X)^2 = (\partial_- X)^2 = 0 \tag{2.75}$$

reduce the number of solutions to $2(D-1)$. The residual reparametrization invariance takes away another two solutions, because we can always transform $\sigma^\pm$. The total number of solutions becomes $2(D-2)$. This was the source of our trouble with negative norm states when we did covariant quantization. When we gauge fix the residual reparametrization invariance, which we do when we pick the light-cone gauge, we automatically restrict ourselves to the proper physical degrees of freedom.[3]

Choosing the light-cone gauge has made the oscillator modes of $X^+$ disappear. The dynamics of $X^-$ become fully determined by the center-of-mass momentum $p^+$ and the oscillator modes of the transverse coordinates $X^i$. To see this, note that the Virasoro constraints $(\dot{X} \pm X')^2 = 0$ become:

$$\dot{X}^- \pm X^{-'} = \frac{1}{2p^+\ell_s^2}(\dot{X}^i \pm X^{i'})^2 \tag{2.76}$$

Solving for $X^-$ in terms of $X^i$:

$$\alpha_n^- = \frac{1}{p^+\ell_s}\left(\frac{1}{2}\sum_{i=1}^{D-2}\sum_{m=-\infty}^{+\infty} :\alpha_{n-m}^i\alpha_m^i: -a\delta_{n,0}\right) \tag{2.77}$$

This agrees with our discussion about physical degrees of freedom. The light-cone gauge has eliminated all oscillator modes except those belonging to the $(D-2)$ transverse coordinates. Let's move on to the quantum theory by promoting all the observables to operators. The transverse oscillator modes carry the commutation relations

$$[\alpha_n^i, \alpha_m^j] = [\widetilde{\alpha}_n^i, \widetilde{\alpha}_m^j] = n\eta^{ij}\delta_{n+m}, 0 \tag{2.78}$$

and

$$[x^i, p^j] = i\delta^{ij},\ [x^-, p^+] = -i,\ [x^+, p^-] = -i \tag{2.79}$$

We can obtain the mass operators from (2.73). For the open string:

$$\alpha' M^2 = (N - a) \tag{2.80}$$

where the level operator N:

$$N \equiv \sum_{i=1}^{D-2}\sum_{n=1}^{\infty} \alpha_{-n}^i \alpha_n^i \tag{2.81}$$

now only sums over the transverse oscillators. The constant $a$ arises from the ordering ambiguity of $L_0$, as it did before. For the closed string, we have:

$$\frac{1}{4}\alpha' M^2 = (\widetilde{N} - a) = (N - a) \tag{2.82}$$

which expresses the level matching condition in the light-cone gauge.

Let's construct the state space. We define a ground state $|0;k\rangle$ to be annihilated by all the annihilation operators:

$$\alpha_n^i\,|0;k\rangle = \widetilde{\alpha}_n^i\,|0;k\rangle = 0 \tag{2.83}$$

for $n > 0$. The Fock space is constructed by acting on this ground state with the creation operators $\alpha_{-n}$ and, for the closed string, $\widetilde{\alpha}_{-n}$. We immediately see from the commutation relations defined above that the state space contains no ghosts.

The first excited states of the open string, $\alpha_{-1}^i\,|0;k\rangle$, form a basis for the $(D-2)$-dimensional vector representation of $SO(D-2)$. According to Wigner's classification of representations of the Poincaré group, this means that the first excited states must be *massless*. If they are not, then the theory is not Lorentz invariant. This implies that $a = 1$, just like we saw before.

The dimensionality of spacetime $D$ can be determined by studying the algebra of the Lorentz generators. In order to maintain Lorentz invariance, the following must hold:

$$[J^{i-}, J^{j-}] = 0 \tag{2.84}$$

It can be shown that this is only satisified when $D = 26$.[3 2]

## 2.6 The spectrum

### 2.6.1 The open string

We will now classify the spectrum of the open string at the first few mass levels.

- *N = 0*: At the ground state $|0;k\rangle$ we have a single scalar particle of mass given by $\alpha' M^2 = -1$. This is called the tachyon. The presence of this particle is problematic. For the open string, it implies the instability of the D25-brane. The closed-string tachyon is more mysterious. We will not devote any more time to discussing the tachyon, because it does not appear in the spectrum of the superstrings. A scalar is indicated with • in Young tableaux.

- *N = 1*: These states have the form $\alpha_{-1}^i\,|0;k\rangle$. As we've discussed, they are states of a massless vector boson. In table 1, this is indicated with a single empty box: □

- *N = 2*: The $N = 2$ states are the first with a positive mass squared. They come in two different forms: $\alpha^i_{-2}\,|0;k\rangle$ and $\alpha^i_{-1}\alpha^j_{-1}\,|0;k\rangle$. These have a total of 324 different states. This happens to be equal to the dimensionality of a symmetric second-rank tensor of dimension 25. It turns out that the $N = 2$ states furnish a representation of $SO(25)$. This is to be expected, as massive bosons need to fit into a representation of $SO(D-1)$ in order to maintain Lorentz invariance. Fixing the spacetime dimension at D=26 made sure the Lorentz algebra was realized by the Lorentz generators acting on the state space, so we know the theory is Lorentz invariant. We will therefore find full $SO(25)$ multiplets at each positive mass level. We can identify the $N = 2$ states as belonging to a single spin-2 massive particle. In table 1, an symmetric traceless part is indicated with $\square\square$, an anti-symmetric part with $\begin{smallmatrix}\square\\ \square\end{smallmatrix}$ and so on.[2]

| Level | Excitations | $SO(24)$ | $SO(25)$ |
|---|---|---|---|
| 0 | None (Ground State) | $\bullet$ | $\bullet$ |
| 1 | $\alpha^i_{-1}$ | $\square$ (Massless!) | (Not a rep) |
| 2 | $\alpha^i_{-1}\alpha^j_{-1}$<br>$\alpha^i_{-2}$ | $\square\square \oplus \bullet$<br>$\square$ | $\square\square$ |
| 3 | $\alpha^i_{-1}\alpha^j_{-1}\alpha^k_{-1}$<br>$\alpha^i_{-1}\alpha^j_{-2}$<br>$\alpha^i_{-3}$ | $\square\square\square \oplus \square$<br>$\square\square \oplus \begin{smallmatrix}\square\\ \square\end{smallmatrix} \oplus \bullet$<br>$\square$ | $\square\square\square$<br>$\begin{smallmatrix}\square\\ \square\end{smallmatrix}$ |

*Table 1: An illustration of the first few mass levels of the bosonic open string spectrum using Young tableaux.*

### 2.6.2 The closed string

- *N = 0*: The $N = 0$ state is again a scalar particle of negative mass: a tachyon.
- *N = 1*: Because of the level matching condition, the $N = 1$ states can only come in the form $\alpha^i_{-1}\tilde{\alpha}^j_{-1}\,|0;k\rangle$. This gives $(D-2)^2$ massless particle states, which transform in the $24 \bigotimes 24$ tensor representation of $SO(24)$. Any two-rank tensor may be decomposed into a traceless symmetric part, an anti-symmetric part and a singlet part. Each of these turns out to furnish an irrep of $SO(24)$. The symmetric traceless tensor represents a spin-2 massless particle: a graviton $G_{\mu\nu}$. The anti-symmetric part is associated with the Kalb-Ramond field $B_{\mu\nu}$, which can be seen as a generalized Maxwell field. The singlet is associated with a scalar field called the *dilaton*. The most interesting of these is the graviton. It turns out that any theory of massless spin-2 particles is equivalent to general relativity: we should identify $G_{\mu\nu}$ with the metric of spacetime.[2]

| Level | Excitations | $SO(24)$ | $SO(25)$ |
|---|---|---|---|
| 0 | None (Ground State) | $\bullet$ | $\bullet$ |
| 1 | $\alpha^i_{-1}\tilde{\alpha}^j_{-1}$ | $\square\square \oplus \begin{smallmatrix}\square\\ \square\end{smallmatrix} \oplus \bullet$<br>(Massless!) | (Not a rep) |

*Table 2: The first few mass levels of the bosonic closed string spectrum*

These results are already very promising. The bosonic string is not a valid theory of nature for several reasons - most notably: its spectrum contains no fermions but does contain a problematic tachyon - but we have seen gravity appear out of nowhere! We will solve some of the problems in the next chapter, where we introduce the superstrings.

# Chapter 3

# The supersymmetric string

In this chapter we will discuss the superstrings. These are strings which carry regular bosonic coordinates - the $X^\mu$ we've seen before - as well as fermionic coordinates. They are called superstrings, short for supersymmetric strings, because each superstring theory has a symmetry which mixes the bosonic and bosonic coordinates. Such symmetries are called supersymmetries. In contrast with bosonic strings, superstrings have spacetime fermions in their spectrum.

We will build the theory of the superstrings in much the same way we built the bosonic theory, and we will run into some of the same problems. Indeed, this similarity is the main reason we discussed the bosonic strings in the first place. When we perform consistency checks on the superstring theories, we will find that they live in 10 spacetime dimensions instead of 26. It seems the superstrings and bosonic strings can't live together.

We will encounter several different types of superstring theory. We discuss type IIB, type IIA and type I extensively, and explore the transformations that relate these theories to each other. There are two other types of superstring, the $SO(32)$ and $E_8 \times E_8$ heterotic strings, which we will not discuss in detail.

There are two equivalent ways to build a superstring theory: the Ramond-Neveu-Schwarz (RNS) formalism, which we discuss in the next section, and the Green-Schwarz (GS) formalism. In the RNS formalism, we add the fermionic coordinates $\psi^\mu(\sigma,\tau)$ to the bosonic theory. $\psi^\mu$ are two-component spinors on the world sheet, but transform as a vector under Lorentz transformations. This means we will build a theory in which world-sheet supersymmetry is (almost) manifest, at least at the classical level, but spacetime supersymmetry is rather obscure. We will have to impose supersymmetry upon the spectrum of the quantum theory using the so-called GSO projection. In the GS formalism, we start by adding fermionic coordinates $\theta^{Aa}$, which are spinors on spacetime. As it turns out, these two methods lead to equivalent superstring theories in ten dimensional spacetime.

After we discuss the GS formalism, we will take a look at type II supergravity, the low-energy limit of type II superstring theory. In particular, we will examine an $SL(2,\mathbb{R})$ symmetry of the supergravity action that will be extremely important to us later on.

At the end of this chapter, we will take a look at the modern picture of superstring theory. The superstrings are thought to be connected in a web of dualities, each representing a limit of a theory called M-theory, whose low-energy limit is 11-dimensional supergravity.

In this chapter, we mostly follow the discussion in Becker Becker Schwarz [2], incorporating information from Dabholkar [4] and a few other sources.

## 3.1 The RNS formalism

### 3.1.1 The action and equations of motion

The Polyakov action for the bosonic string (with $\alpha' = \frac{1}{2}$ and $T = \frac{1}{\pi}$) is given by:

$$S = -\frac{1}{2\pi}\int d^2\sigma \partial_\alpha X_\mu \partial^\alpha X^\mu \tag{3.1}$$

which, of course, is in conformal gauge, so it comes with Virasoro constraints. We will incorporate the fermionic coordinates $\psi^\mu$ by adding the standard Dirac action for massless fermions:

$$S = -\frac{1}{2\pi}\int d^2\sigma (\partial_\alpha X_\mu \partial^\alpha X^\mu + \bar{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu) \tag{3.2}$$

where $\rho^\alpha$ are the two-dimensional Dirac matrices. The fermionic coordinates $\psi^\mu$ are required to be Majorana spinors. In the basis that we will use, Majorana spinors are equivalent to real spinors. The above action is in *super*-conformal gauge, which comes with *super*-Virasoro constraints. The precise form of these constraints may be calculated by starting with a more fundamental action (that we will not discuss in detail) with local supersymmetry. What it comes down to is that the energy-momentum tensor must vanish, like before, along with the *supercurrent*. We will come back to this point shortly.

Let's discuss some of the specifics regarding the new mathematical concepts we've introduced. Firstly, we choose the basis in which the Dirac matrices take the following form:

$$\rho^0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{3.3}$$

$$\rho^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{3.4}$$

As mentioned, in this basis Majorana spinors become equivalent to real spinors. Secondly, the fermionic coordinates $\psi^\mu$ have two components, which we will label $\psi^\mu_+$ and $\psi^\mu_-$. In the classical theory, $\psi^\mu$ is made of Grassman numbers, which means that:

$$\{\psi^\mu, \psi^\nu\} = 0 \tag{3.5}$$

In the quantum theory, we will of course promote these to operators and endow them with other anti-commutation relations. Lastly, the conjugate of a spinor is given by:

$$\bar{\psi} = i\psi^\dagger \rho^0 \tag{3.6}$$

We can now return to the action and express the fermionic part a bit more conveniently:

$$S_f = \frac{i}{\pi} \int d^2\sigma (\psi_- \partial_+ \psi_- + \psi_+ \partial_- \psi_+) \tag{3.7}$$

which leads to the simple equations of motion:

$$\partial_+ \psi_- = 0, \quad \partial_- \psi_+ = 0 \tag{3.8}$$

to be supplemented by the super-Virasoro constraints. We see that these equations describe left- and right-moving waves respectively.[2]

### 3.1.2 World-sheet supersymmetry

The superstring action in superconformal gauge is invariant under the following transformations:

$$\delta X^\mu = \bar{\epsilon}\psi^\mu \tag{3.9}$$

$$\delta \psi^\mu = \rho^\alpha \partial_\alpha X^\mu \epsilon \tag{3.10}$$

where $\epsilon$ is a constant infinitesimal real spinor, consisting of anti-commuting Grassmann numbers. These are called the supersymmetry transformations. They may be seen as generalized translations on the world-sheet, as can be checked by calculating the action of the commutator upon the coordinates $X^\mu$ and $\psi^\mu$. The result is:

$$[\delta_{\epsilon 1}, \delta_{\epsilon 2}] X^\mu = a^\alpha \partial_\alpha X^\mu, \quad [\delta_{\epsilon 1}, \delta_{\epsilon 2}] \psi^\mu = a^\alpha \partial_\alpha \psi^\mu \tag{3.11}$$

where $\delta_{\epsilon 1}$ represent infinitesimal supersymmetry transformations and $a^\alpha$ are constants.

The supersymmetry described here is *global*, because the parameter $\epsilon$ does not depend on the world-sheet coordinates $\tau$ and $\sigma$. In the more fundamental theory described above, the supersymmetry is *local*, but it becomes global in superconformal gauge.[2]

### 3.1.3 The super-Virasoro constraints

As mentioned above, the solutions to our equations of motion have to satisfy the super-Virasoro constraints. This implies the vanishing of the energy-momentum tensor, which now takes the form:

$$T_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X_\mu + \frac{1}{4}\bar{\psi}^\mu \rho_\alpha \partial_\beta \psi_\mu + \frac{1}{4}\bar{\psi}\rho_\beta \partial_\alpha \psi_\mu \tag{3.12}$$

up to a trace part that can be seen to vanish automatically due to the local Weyl invariance of the fundamental theory. The energy-momentum tensor represents the conserved current associated with infinitesimal translations.

The super-Virasoro constraints also demand that the supercurrent $J^\alpha_A$ vanish. The supercurrent is the conserved current associated with supersymmetry transformations. In this case, the local *super*-Weyl invariance of the fundamental theory makes sure the supercurrent only has two independent components, which we label $J_+$ and $J_-$:

$$J_+ = \psi^\mu_+ \partial_+ X_\mu = 0, \quad J_- = \psi^\mu_- \partial_- X_\mu \tag{3.13}$$

In summary, the super-Virasoro constraints take the form:

$$J_+ = J_- = T_{++} = T_{--} \tag{3.14}$$

and the other components of these two tensors vanish due to (super)-Weyl invariance.

When we try to quantize the theory, we will run into negative-norm states again. We will solve this problem in essentially the same way as before. Firstly, we could impose the super-Virasoro constraints upon the states of the spectrum after quantizing covariantly. Secondly, we could use the residual symmetry of the fundamental theory to choose the light-cone gauge and obtain a spectrum manifestly free of negative-norm states.[2]

### 3.1.4 Mode expansion

We still need to provide boundary conditions for $\psi^\mu$. The boundary conditions for the bosonic coordinates $X^\mu$ work out exactly as before. Consider the fermionic part of the superstring action in superconformal gauge:

$$S_f = \frac{i}{\pi} \int d^2\sigma (\psi_- \partial_+ \psi_- + \psi_+ \partial_- \psi_+) \tag{3.15}$$

Taking a variation in $\psi_\pm$, we find the equations of motion and the following boundary term:

$$\delta S = \frac{i}{\pi} \int d\tau (\psi_+ \delta\psi_+ - \psi_- \delta\psi_-)|_{\sigma=\pi} - (\psi_+ \delta\psi_+ - \psi_- \delta\psi_-)|_{\sigma=0} \tag{3.16}$$

which we must make vanish by introducing boundary conditions. For open strings, the two terms must vanish separately. This is satisfied when:

$$\psi^\mu_+(\sigma, \tau) = \pm \psi^\mu_-(\sigma, \tau) \tag{3.17}$$

for $\tau = 0, \pi$. We can choose, by manner of convention, that

$$\psi^\mu_+(0, \tau) = \psi^\mu_-(0, \tau) \tag{3.18}$$

The other sign choice is not physically different. We still have to make a sign choice at the other end of the string. This leads to two physically different sets of boundary conditions:

- *Ramond boundary conditions*: We make the choice:

$$\psi^\mu_+(\pi, \tau) = \psi^\mu_-(\pi, \tau) \tag{3.19}$$

  The state space of strings carrying Ramond boundary conditions is called the R sector

- *Neveu-Schwarz boundary conditions*: We make the choice:

$$\psi^\mu_+(\pi, \tau) = -\psi^\mu_-(\pi, \tau) \tag{3.20}$$

  The state space of strings carrying Neveu-Schwarz boundary conditions is called the NS sector

As we've seen, $\psi^\mu_\pm$ describe left- and right-moving waves:

$$\psi^\mu_+(\tau, \sigma) = \psi^\mu_+(\tau + \sigma), \quad \psi^\mu_-(\tau, \sigma) = \psi^\mu_-(\tau - \sigma) \tag{3.21}$$

To see how these boundary conditions affect the mode expansions, let's bring the $\psi^\mu_\pm$ together in a single fermion field $\Psi^\mu$ defined over $\sigma \in [-\pi, \pi]$.

$$\Psi^\mu(\tau, \sigma) = \begin{cases} \psi^\mu_+(\tau, \sigma) & \sigma \in [0, \pi] \\ \psi^\mu_-(\tau, -\sigma) & \sigma \in [-\pi, 0] \end{cases} \tag{3.22}$$

Using the boundary conditions, we see that:

$$\Psi^\mu(\tau, \pi) = +\Psi^\mu(\tau, -\pi) \tag{3.23}$$

for Ramond boundary conditions and

$$\Psi^\mu(\tau, \pi) = -\Psi^\mu(\tau, -\pi) \tag{3.24}$$

for Neveu-Schwarz boundary conditions. We see that $\Psi^\mu$ is anti-periodic for Ramond boundary conditions and periodic for Neveu-Schwarz boundary conditions. An anti-periodic function can be expanded with fractionally moded exponentials, whereas a periodic functions can be expanded with integrally moded exponentials. We thus obtain the following mode expansions[1]:

- In the R sector:

$$\psi_-^\mu(\sigma,\tau) = \frac{1}{\sqrt{2}}\sum_{n\in\mathbb{Z}} d_n^\mu e^{-in(\tau-\sigma)} \tag{3.25}$$

$$\psi_+^\mu(\sigma,\tau) = \frac{1}{\sqrt{2}}\sum_{n\in\mathbb{Z}} d_n^\mu e^{-in(\tau+\sigma)} \tag{3.26}$$

- In the NS sector:

$$\psi_-^\mu(\sigma,\tau) = \frac{1}{\sqrt{2}}\sum_{r\in\mathbb{Z}+1/2} b_r^\mu e^{-ir(\tau-\sigma)} \tag{3.27}$$

$$\psi_+^\mu(\sigma,\tau) = \frac{1}{\sqrt{2}}\sum_{r\in\mathbb{Z}+1/2} b_r^\mu e^{-ir(\tau+\sigma)} \tag{3.28}$$

From now on, we will use the letters m and n for integers and the letters r and s for half-integers.

Now let's examine the closed string mode expansion of the fermionic coordinates. For the closed string, the possible boundary conditions are:

$$\psi_\pm(\sigma,\tau) = \pm\psi_\pm(\sigma+\pi,\tau) \tag{3.29}$$

The periodic choice corresponds to the R sector and the anti-periodic choice corresponds to the NS sector. We can choose a sector for the left- and right-movers separately. Again, the mode expansion for NS coordinates must be fractionally moded whereas the expansion of the R coordinates must be integrally moded. We obtain:

$$\psi_-^\mu(\sigma,\tau) = \sum_{n\in\mathbb{Z}} d_n^\mu e^{-2in(\tau-\sigma)} \quad \text{or} \quad \psi_-^\mu(\sigma,\tau) = \sum_{r\in\mathbb{Z}+1/2} b_r^\mu e^{-2ir(\tau-\sigma)} \tag{3.30}$$

for the right-movers and:

$$\psi_+^\mu(\sigma,\tau) = \sum_{n\in\mathbb{Z}} \tilde{d}_n^\mu e^{-2in(\tau+\sigma)} \quad \text{or} \quad \psi_+^\mu(\sigma,\tau) = \sum_{r\in\mathbb{Z}+1/2} \tilde{b}_r^\mu e^{-2ir(\tau+\sigma)} \tag{3.31}$$

for the left-movers.

We can pair these up in four different ways: NS-NS, R-R, NS-R, and R-NS. In the next section, where we consider covariant quantization, we will see that the R-NS and NS-R sector carry spacetime fermions![2]

### 3.1.5 Covariant quantization

We endow the bosonic oscillator modes with the same commutation relations as before:

$$[\alpha_m^\mu, \alpha_n^\nu] = m\delta_{m+n}\eta^{\mu\nu} \tag{3.32}$$

and similarly for the right-movers of the closed string. We give $\psi_\pm^\mu$ the following anticommutation relations:

$$\{\psi_A^\mu(\sigma,\tau), \psi_B^\nu(\sigma',\tau)\} = \pi\eta^{\mu\nu}\delta_{AB}\delta(\sigma-\sigma') \tag{3.33}$$

This gives the fermionic oscillators the following anticommutation relations:

$$\{b_r^\mu, b_s^\nu\} = \eta^{\mu\nu}\delta_{r+s,0} \tag{3.34}$$

$$\{d_m^\mu, d_n^\nu\} = \eta^{\mu\nu}\delta_{m+n,0} \tag{3.35}$$

We can immediately see that these relations lead to negative-norm states, unless a certain set of physical state conditions can make them decouple. As we've mentioned, the super-Virasoro constraints will do the job. Let's restrict our discussion to the open strings for now.

To proceed, we have to define ground states for the NS and R sectors:

$$\alpha_m^\mu\,|0\rangle_R = d_m^\mu\,|0\rangle_R = 0 \quad \text{for m} > 0 \tag{3.36}$$

$$\alpha_m^\mu\,|0\rangle_{NS} = b_r^\mu\,|0\rangle_{NS} = 0 \quad \text{for m,r} > 0 \tag{3.37}$$

The ground state of the NS sector is unique, because each of the operators that can act upon it changes its mass. It therefore represents a scalar particle in spacetime. All the oscillators that can act upon the NS ground state are spacetime vectors, so any state coming from the NS sector is a spacetime boson.

Conversely, the R-sector ground state is degenerate, because the $d_0^\mu$ operator does not change its mass. To see this, we need to examine the super-Virasoro operators of the quantum theory.

For the bosonic string, we had the Virasoro operators $L_m$ that were the modes of the energy-momentum tensor:

$$L_m = \frac{1}{\pi}\int_{-\pi}^{\pi} d\sigma e^{im\sigma} T_{++} \tag{3.38}$$

which leads to the familiar relation:

$$L_m = \frac{1}{2}\sum_{n\in\mathbb{Z}} : \alpha_{-n}\cdot\alpha_{m+n} : \tag{3.39}$$

The $L_0$ operator factored into the definition of the mass squared operator and the physical state condition $(L_0 - a)\,|0;k\rangle = 0$. This was the implementation of the Virasoro constraints in the quantum theory. Let's repeat this process for the superstring. We now have super-Virasoro constraint, which means we must also construct operators corresponding to the modes of the supercurrent. In the NS sector, we have:

$$G_r = \frac{\sqrt{2}}{\pi}\int_{-\pi}^{\pi} d\sigma e^{ir\sigma} J_+ = \sum_{n\in\mathbb{Z}} \alpha_{-n}\cdot b_{r+n} \tag{3.40}$$

and in the R sector, we have:

$$F_m = \frac{\sqrt{2}}{\pi}\int_{-\pi}^{\pi} d\sigma e^{im\sigma} J_+ = \sum_{n\in\mathbb{Z}} \alpha_{-n}\cdot\alpha_{m+n} \tag{3.41}$$

We must also add a fermionic part to the $L_m$ operators. In the NS sector, we find:

$$L_m^{(f)} = \frac{1}{2}\sum_{r\in\mathbb{Z}+1/2} (r+\frac{m}{2}) : b_{-r}\cdot b_{m+r} \tag{3.42}$$

and in the R sector:

$$L_m^{(f)} = \frac{1}{2}\sum_{n\in\mathbb{Z}} : d_{-n}\cdot d_{m+n} \tag{3.43}$$

We see from the anticommutation properties of $d_r^\mu$ that $d_0^\mu$ commutes with $L_0^{(f)}$ and therefore with the mass squared operator. This means that the R sector ground state cannot be a spacetime scalar. To see what kind of particle it is, we need to look at the anticommutation relations for $d_0^\mu$:

$$\{d_0^\mu, d_0^\nu\} = \eta^{\mu\nu} \tag{3.44}$$

This is essentially the spacetime d-dimensional Clifford algebra (where d has yet to be determined):

$$\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu} \tag{3.45}$$

so we identify the zero modes with the d-dimensional Γ-matrices, that is:

$$d_0^\mu\,|a\rangle = \frac{1}{\sqrt{2}}\Gamma^\mu_{ba}\,|b\rangle \tag{3.46}$$

where we indicate the different ground states with $|a\rangle$. This means that the ground state of the R sector furnishes a representation of the Clifford algebra. We say that $|0\rangle_R$ is a spinor in d dimensions. In general, a spinor in d dimensions has $2^{[d/2]}$ complex components. We only use Majorana spinors, so the ground state has $2^{[d/2]}$ real components (See the Appendix for a review of spinors). We will find further reductions in the next section, where we introduce chirality. Because all of the oscillators that can act on $|0\rangle_R$ transform as vectors, we see that all R-sector states are spacetime fermions.

Let's take a detailed look at the physical-state conditions for the superstring. In Chapter 2, we found that the physical-state conditions for the bosonic string were:

$$L_n\,|phys\rangle = \widetilde{L}_n\,|phys\rangle = 0 \tag{3.47}$$

for n > 0, and:

$$(L_0 - a)\,|phys\rangle = (\widetilde{L}_0 - a)\,|phys\rangle = 0 \tag{3.48}$$

As we've mentioned, this is the implementation of the Virasoro constraints in the quantum theory. In the quantum theory of the superstrings, we have super-Virasoro constraints, so the physical state conditions are going to include the supercurrent modes $G_r$ and $F_n$ as well. We also find a different normal ordering constant ($a_R$ and $a_{NS}$) for each sector. The full set of physical state conditions is:

$$G_r \left|phys\right\rangle = 0, \quad \mathrm{r} > 0 \tag{3.49}$$

$$L_m \left|phys\right\rangle = 0, \quad \mathrm{m} > 0 \tag{3.50}$$

$$(L_0 - a_{NS}) \left|phys\right\rangle = 0 \tag{3.51}$$

in the NS sector, and:

$$F_n \left|phys\right\rangle = 0, \quad \mathrm{n} \geq 0 \tag{3.52}$$

$$L_m \left|phys\right\rangle = 0, \quad \mathrm{m} > 0 \tag{3.53}$$

$$(L_0 - a_R) \left|phys\right\rangle = 0 \tag{3.54}$$

in the R sector.

Let's take an explicit look at the $F_0$ equation:

$$\left( p \cdot \Gamma + \frac{2\sqrt{2}}{\ell_s} \sum_{n=1}^{\infty} (\alpha_{-n} \cdot d_n + d_{-n} \cdot \alpha_n) \right) \left|phys\right\rangle = 0 \tag{3.55}$$

This is called the Dirac-Ramond equation. In the case of the ground state, it reduces to the massless Dirac equation. This means that the $F_0$ equation takes away half of the independent components of $\left|0\right\rangle_R$, which, as we will see shortly, is necessary in order to maintain supersymmetry in the quantum theory.

These physical-state conditions turn out to be just right to decouple the negative-norm states from the theory, as long as $a_{NS} = 1/2$, $a_R = 0$ and the dimensionality of spacetime $D = 10$! This can be checked by repeating the analysis of spurious states as described in Chapter 2, this time incorporating the algebra of the supercurrent Virasoro operators $G_r$ and $F_m$.[2]

When we quantized the bosonic string, we found we could decouple the negative-norm states by applying the Virasoro constraints to the spectrum or by going back to the start and working in the light-cone gauge. It turns out we can do the same thing for the superstring. We will do so in the next section.

### 3.1.6 Light-cone gauge quantization

In Chapter 2, we saw that the Polyakov action still had some residual symmetry left after we chose the conformal gauge. We could still perform reparametrizations that could be undone by a Weyl transformation. This meant that were able to choose a gauge in which the $X^+$ coordinate had only a single independent Fourier component: the light-cone gauge. It turns out that we can do the same thing in superstring theory. The formalism with local supersymmetry that we've described above has a similar residual symmetry after gauge fixing. This time we can perform reparametrizations to be undone by a *super*-Weyl transformation. We can use this to make the choice:

$$X^+(\sigma, \tau) = x^+ + p^+\tau \tag{3.56}$$

and

$$\psi^+(\sigma, \tau) = 0 \quad \text{(NS sector)} \tag{3.57}$$

This is the light-cone gauge in superstring theory. In the R sector we have to keep the zero mode of the $psi^+$ coordinate, which is a Dirac matrix. Just as in the bosonic theory, the light-cone gauge removes the independent degrees of freedom of the $X^-$ coordinate, except for the zero mode. It now removes the independent degrees of freedom of the $psi^-$ coordinate as well. This means that we build the Fock space using only transverse creation operators. We obtain a spectrum manifestly free of negative-norm states.

Let's take a moment to analyze the spectrum of the open superstring at the first few mass levels. The mass formula in the NS sector is:

$$\alpha' M^2 = \sum_{n=1}^{\infty} \alpha^i_{-n}\alpha^i_n + \sum_{r=1/2}^{\infty} r b^i_{-r} b^i_r - a_{NS} \tag{3.58}$$

The analysis of spurious states in covariant quantization required that $a_{NS} = \frac{1}{2}$. As always, we define a ground state to be annihilated by the positive modes $\alpha^i_n$ and $b^i_r$.

$$\alpha^i_n \left|0; k\right\rangle_{NS} = 0 \tag{3.59}$$

$$b^i_r \left|0; k\right\rangle_{NS} = 0 \tag{3.60}$$

for (n, r >0). The index k indicated the momentum of the ground state, as usual. We can see from the mass formula that every oscillator mode changes the mass of the state it acts upon. This means that the ground state is a spacetime scalar. Because $a_{NS} = \frac{1}{2}$, the ground state of the NS sector is a tachyon. In the next section, we will decouple the tachyon ground state from the theory with a so-called GSO projection (named after Gliozzi, Scherk, and Olive).

The first excited state of the NS sector is given by:

$$b^i_{-1/2} \left|0;k\right\rangle_{NS} \tag{3.61}$$

This is a spacetime vector of $Spin(8)$, so according to Wigner's classification it must be massless. This verifies that $a_{NS} = \frac{1}{2}$.

In the Ramond sector the mass formula becomes:

$$\alpha' M^2 = \sum_{n=1}^{\infty} \alpha^i_{-n}\alpha^i_n + \sum_{n=1}^{\infty} n d^i_{-n} d^i_n \tag{3.62}$$

The ground state of the R sector is annihilated by the positive modes:

$$\alpha^i_n \left|0;k\right\rangle_R = d^i_n \left|0;k\right\rangle_R = 0 \tag{3.63}$$

for n >0. The R sector ground state must also satisfy the physical state condition $F_0 \left|phys\right\rangle = 0$, which in this case reduces to the massless Dirac equation. As we've discussed, the R sector ground state is a Majorana spinor. A Majorana spinor in ten dimensions has $2^{[d/2]} = 32$ real components. The massless Dirac equation takes away one half of the independent components, reducing their number to 16. In order to have unbroken supersymmetry, we need to have the same number of fermionic and bosonic physical degrees of freedom at each mass level. This means we have to take away another 8 degrees of freedom.

In order to find out how to do this, we need to know how the ground state transforms. Note that the zero modes in light-cone gauge $d^i_0$ are all vectors of $Spin(8)$, the covering group of $SO(8)$. $Spin(8)$ has three inequivalent irreps that will be relevant to our purposes: the vector representation **8v**, the spinor representation **8s**, and the conjugate spinor representation **8c**. These three representations are related by a so-called triality symmetry.[4]

We now make the following definition:

$$\sqrt{2} g_m = d^{2m-1}_0 + i d^{2m}_0 \tag{3.64}$$

with (m = 1,...,4). These oscillators satisfy the following anticommutation relations:

$$\{g_m, g^\dagger_n\} = \delta_{mn}, \quad \{g_m, g_n\} = \{g^\dagger_m, g^\dagger_n\} = 0 \tag{3.65}$$

This amounts to choosing an embedding of $SO(8)$ into the direct product of $SU(4)$ and $U(1)$, so that $g_m$ transform as the fundamental representation **4** of $SU(4)$ with $\frac{1}{2}$ units of $U(1)$ charge. In this embedding, the representations of $Spin(8)$ transform as:

$$\mathbf{8v} = \mathbf{4}(\mathbf{\tfrac{1}{2}}) + \bar{\mathbf{4}}(-\mathbf{\tfrac{1}{2}}) \tag{3.66}$$

$$\mathbf{8s} = \mathbf{4}(-\mathbf{\tfrac{1}{2}}) + \bar{\mathbf{4}}(\mathbf{\tfrac{1}{2}}) \tag{3.67}$$

$$\mathbf{8c} = \mathbf{6}(\mathbf{0}) + \mathbf{1}(\mathbf{1}) + \mathbf{1}(-\mathbf{1}) \tag{3.68}$$

where we denote U(1) charge between parentheses. We can see that the embedding is possible by noting that $SO(6) \sim SU(4)$ and $SO(2) \sim U(1)$. The embedding into $SU(4) \times U(1)$ now corresponds to another embedding into $SO(6)$ and $SO(2)$: $SO(8) \supset SO(6) \times SO(2)$. Let's examine the transformation properties of the ground state by acting upon the vacuum (denoted $\left|0\right\rangle$) with $g^\dagger_m$:

$$\begin{array}{ll}
\left|0\right\rangle & \mathbf{1}(\mathbf{1}) \\
g^\dagger_m \left|0\right\rangle & \bar{\mathbf{4}}(\tfrac{1}{2}) \\
g^\dagger_m g^\dagger_n \left|0\right\rangle & \mathbf{6}(\mathbf{0}) \\
g^\dagger_m g^\dagger_n g^\dagger_p \left|0\right\rangle & \mathbf{4}(-\mathbf{\tfrac{1}{2}}) \\
g^\dagger_m g^\dagger_n g^\dagger_p g^\dagger_q \left|0\right\rangle & \mathbf{1}(-\mathbf{1})
\end{array}$$

These relations are simple enough to determine. For example, the product of two 4-dimensional vectors is a 4-by-4 tensor, which may be decomposed into a traceless symmetric part, an anti-symmetric part and a trace part. From the group theory of $SU(N)$, we know that these are all irreps. In terms of Young tableaux, $\square \otimes \square = \square\square \oplus \begin{array}{c}\square\\[-4pt]\square\end{array} \oplus \bullet$. Because of the anticommutation relations of $g_m$, only the anti-symmetric part of the tensor $g^\dagger_m g^\dagger_n$ remains, which has 6 independent components, so the $SU(4)$ representation is **6**.

We see from examining the $SU(4)$ and $U(1)$ numbers that the R sector ground state must transform as $\mathbf{8s}+\mathbf{8c}$. What we set out to do was find a way to remove 8 degrees of freedom from the R sector ground state in order to save supersymmetry. A natural proposition would be to restrict our attention to spinors of only **8s** or **8c**. This may be implemented by using the chirality matrix $\Gamma_{11}$:

$$\Gamma_{11} \equiv \Gamma_1\Gamma_2...\Gamma_9 \tag{3.69}$$

Any spinor in even dimensions may be decomposed into two chirality eigenstates with eigenvalues $+1$ or $-1$. Each of these corresponds with one of the two irreps **8s** and **8c**. We therefore propose to restrict our attention to spinors of definite chirality. This is known as the Weyl condition.

All in all, the R sector ground state has been reduced to an irreducible spinor of $Spin(8)$. An irreducible spinor has 8 independent components, so spacetime supersymmetry is saved at the massless level. Still, because we have no tachyonic states in the R sector, the supersymmetry appears to be broken anyway. As mentioned, we will decouple the tachyon from the spectrum with a GSO projection. It turns out that this will suffice to maintain spacetime supersymmetry.[2]

### 3.1.7 GSO projection

We will now try to solve some of the problems with the superstring spectrum. Most importantly, we want to decouple the tachyon from the spectrum. Secondly, we want to make sure we have the same number of bosonic and fermionic degrees of freedom at each mass level. We will do this by truncating the superstring spectrum in a specific way. We now the define the G-parity operator, which counts the number of fermion excitations:

$$G \equiv (-1)^{F+1} = (-1)^{\sum_{r=1/2}^{\infty} b^i_{-r} b^i_r + 1} \quad \text{(NS sector)} \tag{3.70}$$

$$G = \Gamma_{11}(-1)^{\sum_{r=1/2}^{\infty} d^i_{-n} d^i_n} \quad \text{(R sector)} \tag{3.71}$$

As mentioned, spinors that are eigenstates of the chirality matrix $\Gamma_{11}$ are said to have positive or negative chirality, depending on whether the eigenvalue is $+1$ or $-1$. Spinors with a definite chirality are also called Weyl spinors. We can define a chirality projection operator as follows:

$$P_\pm \equiv \frac{1}{2}(1 \pm \Gamma_{11}) \tag{3.72}$$

In the NS sector, we only keep the states with positive G-parity. In the R sector, we can either keep states with positive or negative G-parity, depending on whether we want to keep states with positive or negative chirality. This is content of the GSO projection. We immediately see that the GSO projection removes the NS sector tachyon from the spectrum. The first excited state in the NS sector, $b^i_{-1/2}\left|0\right\rangle_{NS}$, has positive G-parity and survives GSO projection. This is the new NS sector ground state. As discussed above, it happens to be massless. The GSO projection in the R sector ensures that we only keep states of definite chirality. This means that the number of fermionic and bosonic degrees of freedom are now equal at the massless level. It turns out that the GSO projection leaves an equal number of fermionic and bosonic degrees of freedom at *every* mass level. This suggests, but doesn't prove, that spacetime supersymmetry has been saved. Later on in this chapter, we will briefly discuss a formalism (equivalent to the one presented here) in which spacetime supersymmetry is manifest.[2]

### 3.1.8 Closed string spectrum

Let's reiterate some of our previous discussion on closed superstrings. We build the closed-string spectrum by tensoring left- and right-moving states. This results in four different sectors (R-R, R-NS, NS-R, and NS-NS), each of them corresponding to a choice of boundary conditions on the left- and right-moving spinors $\psi^\pm$. In the NS sector, we keep only states with positive G-parity. In the R sector, we keep states with either positive or negative G-parity depending on the chirality of the ground state. We can build two different closed-string theories (called Type IIA and Type IIB) by keeping states of either equal or opposite G-parity in the left- and right-moving R sectors. In the Type IIB theory the left- and right-moving R sector states have equal chirality, chosen to be positive by convention. We denote them $\left|+\right\rangle_R$. The massless spectrum of Type IIB becomes:

$$\left|+\right\rangle_R \otimes \left|+\right\rangle_R \quad \text{(R-R sector)} \tag{3.73}$$

$$\tilde{b}^i_{-1/2}\left|0\right\rangle_{NS} \otimes b^j_{-1/2}\left|0\right\rangle_{NS} \quad \text{(NS-NS sector)} \tag{3.74}$$

$$\tilde{b}^i_{-1/2}\left|0\right\rangle_{NS} \otimes \left|+\right\rangle_R \quad \text{(NS-R sector)} \tag{3.75}$$

$$|+\rangle_R \otimes b^i_{-1/2}\,|0\rangle_{NS} \quad \text{(R-NS sector)} \tag{3.76}$$

In the Type IIA theory the left- and right-moving R sector states have opposite chirality. We denote them $|+\rangle_R$ and $|-\rangle_R$. The Type IIA massless spectrum becomes:

$$|-\rangle_R \otimes |+\rangle_R \quad \text{(R-R sector)} \tag{3.77}$$

$$\tilde{b}^i_{-1/2}\,|0\rangle_{NS} \otimes b^j_{-1/2}\,|0\rangle_{NS} \quad \text{(NS-NS sector)} \tag{3.78}$$

$$\tilde{b}^i_{-1/2}\,|0\rangle_{NS} \otimes |+\rangle_R \quad \text{(NS-R sector)} \tag{3.79}$$

$$|-\rangle_R \otimes b^i_{-1/2}\,|0\rangle_N S \quad \text{(R-NS sector)} \tag{3.80}$$

We can summarize this in the language of group theory:

Type IIA: $(\mathbf{8v} \oplus \mathbf{8s}) \otimes (\mathbf{8v} \oplus \mathbf{8c})$

Type IIB: $(\mathbf{8v} \oplus \mathbf{8c}) \otimes (\mathbf{8v} \oplus \mathbf{8c})$

With the **8v** coming from the NS sector and the **8c** or **8s** coming from the R sector. Let's take an explicit look at the particle content of the massless Type II spectra. The NS-NS (or $\mathbf{8v} \otimes \mathbf{8v}$) spectrum is the same for Type IIA and IIB. The tensor product decomposes into a traceless symmetric part, an anti-symmetric part and a trace part. These are the graviton $g_{ij}$, the Kalb-Ramond 2-form field $B_2$ and the dilaton $\phi$. As we mentioned in Chapter 2, these three particles are common to all string theories. The R-NS and NS-R sectors of Type IIA and Type IIB theory give rise to spin 3/2 particles called gravitini (56 states) and a spin 1/2 particle called the dilatino (8 states). In Type IIB theory the R-NS and NS-R fermions have the same chirality, whereas in Type IIA theory they have opposite chirality. The R-R sector contains bosons. In Type IIB theory, the $\mathbf{8c} \otimes \mathbf{8c}$ coming from the R-R sector may be decomposed as:

$$\lambda_1^{\dot{a}} \lambda_2^{\dot{b}} \sim \lambda_1^T \lambda_2 \oplus \lambda_1^T \Gamma_{ij} \lambda_2 \oplus \lambda_1^T \Gamma_{ijkl} \lambda_2 \tag{3.81}$$

where we denote the spinor indices of conjugate spinors with $\dot{a}$. The matrices $\Gamma_{ij}$ and $\Gamma_{ijkl}$ are the totally antisymmetrized products of Dirac matrices. The first term corresponds to a scalar particle $\chi$, which is called the axion. It is sometimes denoted $C_0$. The second term corresponds to a 2-form field $B_2'$. The last term corresponds to a self-dual 4-form field $D_4$. The above decomposition is possible because the anti-symmetrized products of $\Gamma$ matrices in d dimensions form a basis in the space $Mat(2^{[d/2]}, \mathbb{C})$. For more information, see the Appendix. In the Type IIA theory, the decomposition of the R-R sector changes into:

$$\lambda_i^a \lambda_2^{\dot{b}} \sim \lambda_1^T \Gamma_i \lambda_2 \oplus \lambda_1^T \Gamma_{ijk} \lambda_2 \tag{3.82}$$

where we denote spinor indices of **8s** spinors with an undotted $a$. The first term corresponds to a vector $A_i$ and the second to a 3-form $C_3$.

In summary, the massless spectrum of Type IIB is:

| | |
|---|---|
| NS-NS sector: | graviton $g_{ij}$, 2-form $B_ij$, dilaton $\phi$ |
| R-R sector: | axion $\chi$ or $C_0$, 2-form $B_2'$, self-dual 4-form $D_4$ |
| NS-R sector: | gravitino $\psi_{i\dot{a}}$, dilatino |
| R-NS sector: | gravitino $\psi_{j\dot{b}}$, dilatino |

and the massless spectrum of Type IIA is[42]:

| | |
|---|---|
| NS-NS sector: | graviton $g_{ij}$, 2-form $B_2$, dilaton $\phi$ |
| R-R sector: | vector $A_i$, 3-form $C_3$ |
| NS-R sector: | gravitino $\psi_{i\dot{b}}$, dilatino |
| R-NS sector: | gravitino $\psi_{ja}$, dilatino |

## 3.2 The GS formalism

We've now built superstring theory in the RNS formalism, which is built from an action with world-sheet supersymmetry. As mentioned, there is a second formalism which has manifest spacetime supersymmetry, called the Green-Schwarz (GS) formalism. In the RNS formalism, spacetime supersymmetry was rather obscure, appearing only after we'd performed the clunky GSO projection. We chose to build superstring theory in the RNS formalism anyway, because it is mathematically simpler than the GS formalism. We can still learn something from examining the GS formalism. Much of our discussion in later chapters will be more conveniently expressed in GS language. We therefore devote this section to it, skipping some of the (not particularly illuminating) mathematical detail. In this section, we will encounter the Type I superstring theory for the first time. Such $\mathcal{N} = 1$ supersymmetric theories are highly restricted by anomaly cancellation, which implies that the gauge group of these theories must be either $SO(32)$ or $E_8 \times E_8$. We discuss anomaly cancellation very briefly at the end of this section.

The basic world-sheet fields in the GS formalism are the bosonic field $X^\mu(\sigma,\tau)$ and the spacetime Majorana spinor field $\Theta^{Aa}(\sigma,\tau)$. The index $a$ labels the 32 components of a spinor in $D=10$. The index $A=1,2,...,\mathcal{N}$ labels the $\mathcal{N}$ supersymmetries of the theory. For the Type II strings, the number of supersymmetries is $\mathcal{N}=2$. All other superstrings have $\mathcal{N}=1$. This will become clear in a moment.

### 3.2.1 Point particle action

Let's start out by constructing a spacetime supersymmetric action for a point particle. The ordinary relativistic point particle action is:

$$S=-m\int\sqrt{-\dot{X}^\mu\dot{X}_\mu}d\tau \tag{3.83}$$

The supersymmetry transformations now take the form:

$$\delta\Theta^{Aa}=\epsilon^{Aa} \tag{3.84}$$

$$\delta X^\mu=\bar{\epsilon}^A\Gamma^\mu\Theta^A \tag{3.85}$$

The important difference between these supersymmery transformations and the RNS versions (3.9)-(3.10) is that these do not contain any reference to a world-sheet coordinate. We say that they are spacetime supersymmetries. The commutator of two spacetime supersymmetry transformations acts upon $X^\mu$ as an infinitesimal spacetime translation:

$$[\delta_1,\delta_2]X^\mu=-2\bar{\epsilon_1}^A\Gamma^\mu\epsilon_2^A \tag{3.86}$$

By comparison with the analogous relation in the RNS formalism, we see again that the transformations deserve to be called *spacetime* supersymmetries.

We have to adapt the point particle action slightly to accommodate these supersymmetry transformations. A simple way to do this is to introduce the manifestly supersymmetric combination $\Pi_0^\mu$:

$$\Pi_0^\mu=\dot{X}^\mu-\bar{\Theta}^A\Gamma^\mu\dot{\Theta}^A \tag{3.87}$$

We make the replacement $\dot{X}^\mu\to\Pi_0^\mu$. The action becomes:

$$S_1=-m\int\sqrt{-\Pi_0\cdot\Pi_0}d\tau \tag{3.88}$$

This is actually not quite the correct action yet. We can add a second term to it that enhances the supersymmetry by ensuring the saturation of a BPS bound (we will examine BPS bounds in the next chapter). This new term introduces a local fermionic symmetry called $\kappa$ symmetry, which decouples half the degrees of freedom of $\Theta$. This turns out to be necessary in order to build a supersymmetric quantum theory. [2]

The action we've constructed describes massive supersymmetric point particles (or D0 branes). These actually appear in the Type IIA superstring theory. The Type II superstrings have $\mathcal{N}=2$ supersymmetry, so there are two (Majorana-Weyl) spinor coordinates $\Theta^{1a}$ and $\Theta^{2a}$.

### 3.2.2 Superstring action

Let's move on to the superstring action. In Chapter 2, we generalized the point particle action to the Nambu-Goto action:

$$S_{NG}=-\frac{1}{\pi}\int d^2\sigma\sqrt{-det(\partial_\alpha X^\mu\partial_\beta X^\mu)} \tag{3.89}$$

We now want to make it supersymmetric by making the same replacement as before. The action becomes:

$$S_1=-\frac{1}{\pi}\int d^2\sigma\sqrt{-det(\Pi_\alpha\cdot\Pi_\beta)} \tag{3.90}$$

with

$$\Pi_\alpha^\mu=\partial_\alpha X^\mu-\bar{\Theta}^A\Gamma^\mu\partial_\alpha\Theta^A \tag{3.91}$$

As before, this is not the complete action. It still needs a term that enhances the supersymmetry and introduces a $\kappa$ symmetry to gauge away one half of the independent degrees of freedom of $\Theta$. The GS action constructed in this way is very difficult to quantize covariantly. This is because the equations of motion for $X^\mu$ and $\Theta^A$ are non-linear, and because of the constraint conditions required by $\kappa$ symmetry. These problems have actually not been completely resolved yet. We have no choice but to do most of our analysis in the light-cone gauge, which, as always, is possible due to residual conformal symmetry after gauge fixing. The light-cone gauge now takes the form:

$$X^+ = x^+ + p^+\tau \tag{3.92}$$

$$\Gamma^+\Theta^A = 0 \tag{3.93}$$

with

$$\Gamma^\pm \equiv \frac{1}{\sqrt{2}}(\Gamma^0 \pm \Gamma^9) \tag{3.94}$$

Only the transverse coordinates $X^i$ remain as independent degrees of freedom. This reduction has to be reflected in the fermionic sector, because we want to find the same number of fermionic and bosonic degrees of freedom at each mass level. A general spinor in $D = 10$ has $2^{[5]} = 32$ components. Majorana-Weyl conditions reduce the spinor to 16 real components. Because we are working with $\mathcal{N} = 2$ theories, the two fermionic coordinates now have a total of 32 degrees of freedom. The $\kappa$ symmetry takes away one half and the equations of motion take away another half. We are left with 8 fermionic degrees of freedom, which matches the bosonic degrees of freedom. In the light-cone gauge, the $\Theta$ equations of motion become:

$$(\frac{\partial}{\partial_\tau} + \frac{\partial}{\partial_\sigma})\Theta^1 = 0 \tag{3.95}$$

$$(\frac{\partial}{\partial_\tau} - \frac{\partial}{\partial_\sigma})\Theta^2 = 0 \tag{3.96}$$

which means $\Theta^1$ and $\Theta^2$ describe right- and left-moving waves on the string respectively. Let's change our notation slightly to indicate the various reductions we've found on $\Theta^A$. Now that each $\Theta^A$ has 8 independent components, they form an eight-dimensional spinor representation of $Spin(8)$. As we mentioned in the last section, $Spin(8)$ has two inequivalent spinor representations **8s** and **8c**, which are related to the two possible ten-dimensional chiralities. We now denote the 8 surviving components of $\Theta^A$ by $S^a_A$ if it is an **8s** spinor, and $S^{\dot{a}}_A$ if it is an **8c** spinor. In Type IIA, the right- and left-moving spinors need to have opposite chirality:

$$\mathbf{8s} + \mathbf{8c} = (S^a_1, S^{\dot{a}}_2) \tag{3.97}$$

In Type IIB, the right- and left-movers need to have the same chirality:

$$\mathbf{8s} + \mathbf{8s} = (S^a_1, S^a_2) \tag{3.98}$$

The equations of motion now become:

$$\partial_+\partial_- X^i = 0 \tag{3.99}$$

$$\partial_+ S^a_1 = 0 \tag{3.100}$$

$$\partial_- S_2^{\mathrm{a\ or\ }\dot{a}} \tag{3.101}$$

These equations are the same as the ones in the RNS formalism, except that the fermions are now $Spin(8)$ spinors instead of vectors. We can construct a light-cone gauge action that gives these equations of motion. For the Type IIB string:

$$S = -\frac{1}{2\pi}\int d^2\sigma \partial_\alpha X_i \partial^\alpha X^i + \frac{1}{\pi}\int d^2\sigma (S^a_1 \partial_+ S^a_1 + S^a_2 \partial_- S^a_2) \tag{3.102}$$

Replacing $S^a_2$ by $S^{\dot{a}}_2$ gives the Type IIA string action. We will use this expression to examine some perturbative symmetries of Type IIB superstring theory later on in this chapter.[2]

### 3.2.3 Quantization

Let's proceed to quantization. The treatment of the bosonic coordinates is exactly the same as before. The fermionic coordinates are endowed with the following anticommutation relations:

$$\{S^{Aa}(\sigma,\tau), S^{Bb}(\sigma',\tau)\} = \pi\delta^{ab}\delta^{AB}\delta(\sigma - \sigma') \tag{3.103}$$

where $A, B$ indicate the $\mathcal{N}$ supercharges and $a, b$ indicate the spacetime spinor indices. We still have to perform a mode expansion of the spinor coordinates. The structure of the mode expansion of each coordinate is determined by the boundary conditions. For open strings, the bosonic coordinates satisfy Neumann boundary coordinate. In the presence of D-branes, Dirichlet boundary conditions are possible as well. Only one set of open-string fermionic boundary conditions is consistent with supersymmetry:

$$S^{1a}(\tau, 0) = S^{2a}(\tau, 0) \;\text{ and }\; S^{1a}(\tau, \pi) = S^{2a}(\tau, \pi) \tag{3.104}$$

The supersymmetry transformation on $\Theta^A$ is $\delta\Theta^A = \epsilon^A$. We see that the boundary conditions are only consistent with supersymmetry when $\epsilon^1 = \epsilon^2$. This means that all open strings have $\mathcal{N} = 1$ supersymmetry. Such strings appear in the Type I superstring theory. The mode expansions for the fermionic fields of open strings become:

$$S^{1a} = \frac{1}{\sqrt{2}} \sum_{-\infty}^{\infty} S_n^a e^{-in(\tau-\sigma)} \tag{3.105}$$

$$S^{2a} = \frac{1}{\sqrt{2}} \sum_{-\infty}^{\infty} S_n^a e^{-in(\tau+\sigma)} \tag{3.106}$$

The Fourier modes will be endowed with the following anticommutation relations:

$$\{S_m^a, S_n^b\} = \delta_{m+n,0}\delta^{ab} \tag{3.107}$$

For closed strings, the fermionic fields are periodic:

$$S^{Aa}(\sigma,\tau) = S^{Aa}(\sigma+\pi,\tau) \tag{3.108}$$

The mode expansions become:

$$S^{1a} = \sum -\infty^{\infty} S_n^a e^{-2in(\tau-\sigma)} \tag{3.109}$$

$$S^{2a} = \sum -\infty^{\infty} \tilde{S}_n^a e^{-2in(\tau+\sigma)} \tag{3.110}$$

The modes satisfy the same anticommutation relations as before, except now there is of course a second set of right-moving oscillators:

$$\{S_n^a, S_m^b\} = \delta^{ab}\delta_{m+n,0} \tag{3.111}$$

$$\{\tilde{S}_n^a, \tilde{S}_m^b\} = \delta^{ab}\delta_{m+n,0} \tag{3.112}$$

In the Type IIB theory, the two spinors have the same chirality, whereas in the Type IIA they have opposite chirality. The closed string sector of Type I superstring theory is described by a left-right symmetrization of Type IIB theory. This is called an orientifold projection. We will examine orientifolds in detail in Chapter 5. They are the key to obtaining the new superstring theory.[2]

### 3.2.4 The massless spectrum

We can now analyze the superstring spectrum. We'll start with the open strings of Type I theory. We find the following mass formula:

$$\alpha' M^2 = \sum_{n=1}^{\infty} (\alpha_{-n}^i \alpha_n + n S_{-n}^a S_n^a) \tag{3.113}$$

This indicates that the ground state is massless, and there is no tachyon in the spectrum. We won't need to perform a GSO projection on the GS formalism spectrum. The ground state is degenerate, because the mass formula commutes with the zero modes $S_0^a$. We can repeat the analysis we did on the R sector ground state to find the transformation properties of the GS formalism ground state. Let's start by rewriting in the same way as before:

$$\sqrt{2} b_m = (S_0^{2m} + i S_0^{2m}),\ m = 1, ..., 4 \tag{3.114}$$

The oscillators $b_m$ satisfy the following anticommutation relations:

$$\{b_m, b_n^\dagger\} = \delta_{mn}, \quad \{b_m, b_n\} = 0 \quad \{b_m^\dagger, b_n^\dagger\} \tag{3.115}$$

As before, we have implicitly chosen an embedding $SO(8) \sim SU(4) \times SU(1)$. The oscillators $b_m$ transform in the fundamental representation $\mathbf{4}$ with one half unit of $U(1)$ charge. The irreps of $Spin(8)$ now decompose as:

$$\mathbf{8v} = \mathbf{6}(0) + \mathbf{1}(1) + \mathbf{1}(-1) \tag{3.116}$$

$$\mathbf{8s} = \mathbf{4}(\frac{1}{2})\bar{\mathbf{4}}(-\frac{1}{2}) \tag{3.117}$$

$$\mathbf{8c} = \mathbf{4}(-\frac{1}{2}) + \bar{\mathbf{4}}(\frac{1}{2}) \tag{3.118}$$

Let's examine the action of $b_m^\dagger$ on the ground state:

$$\begin{array}{ll} |0\rangle & \mathbf{1}(1) \\ b_m^\dagger\,|0\rangle & \bar{\mathbf{4}}(\tfrac{1}{2}) \\ b_m^\dagger b_n^\dagger\,|0\rangle & \mathbf{6}(0) \\ b_m^\dagger b_n^\dagger b_p^\dagger\,|0\rangle & \mathbf{4}(-\tfrac{1}{2}) \\ b_m^\dagger b_n^\dagger b_p^\dagger b_q^\dagger\,|0\rangle & \mathbf{1}(-1) \end{array}$$

Looking at the $SU(4)$ and $U(1)$ numbers, we see that the GS ground state must transform as $\mathbf{8v}$ + $\mathbf{8c}$. Other than that the spinor $\mathbf{8s}$ has been replaced by the vector $\mathbf{8v}$, this transforms in the same way as the R sector ground state (before GSO projection). The similarity between the RNS and GS constructions has its roots in the triality symmetry of $Spin(8)$.[4] We see that the massless open string spectrum is exactly the same as in the RNS formalism after GSO projection. We can build the rest of the Fock space by acting on the ground state with the negative modes $S^a_{-n}$ and $\alpha^i_{-n}$. The spectrum is guaranteed to be supersymmetric, because the supersymmetry generators can be expressed in terms of these oscillators[2].

We can find the massless spectrum of the Type II strings by tensoring right- and left-moving ground states, which have almost the same structure as the open string ground state. We now have right- and left- moving zero modes $S^a_0$ and $\tilde{S}^a_0$. If the right- and left-moving zero modes have different chirality, the previous analysis has to be modified slightly, and we find that the right-moving ground state must transform as $\mathbf{8v}$ + $\mathbf{8s}$ instead of $\mathbf{8v}$ + $\mathbf{8c}$. This clearly corresponds to Type IIB string theory. If the right- and left-moving zero modes have the same chirality, we just obtain $\mathbf{8v}$ + $\mathbf{8c}$ again. This corresponds to Type IIA string theory. Performing the tensor products, we find exactly the same massless Type II spectrum as before:

$$\begin{array}{ll} \text{Type IIA:} & (\mathbf{8v}\oplus\mathbf{8s})\otimes(\mathbf{8v}\oplus\mathbf{8c}) \\ \text{Type IIB:} & (\mathbf{8v}\oplus\mathbf{8c})\otimes(\mathbf{8v}\oplus\mathbf{8c}) \end{array}$$

## 3.3 A look at supergravity

String theory in the low-energy limit may be well described by a theory that only incorporates the interactions of the massless particles, since the massive ones are too heavy to be observed. Such theories are called *supergravities*. They suffer from the usual problem of quantum gravity theories, which is that they are not renormalizable. Non-renormalizability is not necessarily a big problem for theories describing the low-energy limit of a more fundamental theory. Even though supergravity theories are not fundamental, they do contain some of the features of the theories whose low-energy limit they describe. For example, we will see that the Type IIB supergravity action has an $SL(2,\mathbb{R})$ symmetry, which does carry over into the Type IIB string theory in the form of a smaller $SL(2,\mathbb{Z})$ symmetry. Supergravity theories are described with rather convoluted algebra, and involve some concepts from general relativity, so we will not have time to examine them in very great detail. We will try to sketch their general structure and then examine more closely some of the features that will be relevant to our purposes. Specifically, we are interested in the field content of 11-dimensional supergravity, which is the low-energy limit of M-theory, and the $SL(2,\mathbb{R})$ symmetry of Type IIB supergravity.[2]

### 3.3.1 11-dimensional supergravity

We start with 11-dimensional supergravity. This was the earliest supergravity theory to be constructed, in the late 1970s. Eventually, it fell out of favor due to the problems of non-renormalizability and its lack of chirality. In the modern picture, which is that 11-dimensional supergravity is the low-energy limit of M-theory, these concerns are no longer very important.

Let's take a look at the field content of 11-dimensional supergravity. Firstly, the theory needs to have a graviton in order to contain gravity. The graviton is a traceless symmetric tensor of $SO(D-2)$. In 11 dimensions, it has 44 degrees of freedom. Because we are building a supersymmetric theory, we need to incorporate the gauge field of local supersymmetry, which is the gravitino $\Psi_M$, where $M$ represents a spatial index. The gravitino field carries an implicit spinor index as well. For each value of its spatial index, the gravitino field is a 32-component Majorana spinor. Because spinors are included, we are now interested in transformation properties under $Spin(9)$ instead of $SO(9)$. $Spin(9)$ has a real spinor representation of dimension 16. The tensor product of a vector and a spinor in $Spin(9)$ decomposes as follows:

$$\mathbf{9}\times\mathbf{16}=\mathbf{128}+\mathbf{16} \tag{3.119}$$

The kinetic term for the gravitino field takes the following form:

$$S_\Psi \sim \int \bar{\Psi}_M \Gamma^{MNP}\partial_N \Psi_P d^D x \tag{3.120}$$

This term is invariant up to a total derivative under the variation

$$\delta\Psi_M = \partial_M \epsilon \tag{3.121}$$

where $\epsilon$ is an infinitesimal Majorana spinor. This local symmetry implies that the physical degrees of freedom of the gravitino field are carried by the **128** alone. We now have 128 fermionic degrees of freedom, but only 44 bosonic degrees of freedom. The missing 84 bosonic degrees of freedom are exactly the right number to be filled by an anti-symmetric rank-3 tensor $A_{MNP}$, or, equivalently, a three-form field $A_3$. To see this, note that the three-form field has the the following gauge invariance:

$$A_3 \to A_3 + d\lambda_2 \tag{3.122}$$

where $\lambda_2$ is a two-form. This is the case in general because the $d^2$ operator is zero. The gauge invariance ensures that the independent physical polarizations are transverse. This means that the three-form has $9 \cdot 8 \cdot 7/3! = 84$ independent degrees of freedom. This is the full field content of 11-dimensional supergravity. It is is considerably simpler than the massless spectrum of Type IIA and Type IIB superstring theory. The requirement of the $A_3$ gauge invariance places very strong constraints on the form of the 11-dimensional supergravity action. It is actually unique up to normalization conventions. It will not be very illuminating for us to examine the action explicitly. We only note that Type IIA supergravity can be obtained by dimensional reduction of 11-dimensional supergravity.

To see how this is done, let's examine the fermionic fields. The analysis of the bosonic fields is too lengthy to include here. The R-NS and NS-R sector of Type IIA superstring theory consists, at the massless level, of two Majorana-Weyl gravitinos of opposite chirality. In 11-dimensional supergravity, we have the gravitino $\Psi_M$, which is a 32-component Majorana spinor for each value of the spatial index $M$. These can be decomposed into two 16-component Majorana-Weyl spinors of opposite chirality. We see that the first 10 spatial components of $\Psi_M$ have the structure of the 10-dimensional gravitinos. The last component, $\Psi_{11}$, contains the two dilatinos, each of which has 8 polarization states due to the massless Dirac equation. The action of Type IIA supergravity may be obtained by integrating over the compactified spatial coordinate.[2]

This is the first hint of a relation between an as-of-yet undetermined 11-dimensional theory (whose low-energy limit is 11-dimensional supergravity) and superstring theory. We will come back to this point when we consider compactification in Chapter 4.

### 3.3.2 Type IIB supergravity

Type IIB supergravity cannot be obtained from 11-dimensional supergravity by dimensional reduction. We immediately see this from the chirality properties of the Type IIB gravitinos. As we saw in the last section, the type IIB massless spectrum has a four-form field $C_4$, which has a self-dual field strength $\widetilde{F}_5$. This becomes problematic when we try to construct an action, because a term of the form

$$\int |F_5|^2 d^{10}x \tag{3.123}$$

which is the form that field strength terms in supergravity actions normally take, does not implement the self-duality condition. This problem can be solved by writing down an action that gives the correct field equations when supplemented by the self-duality condition. As mentioned, what we are interested in is the $SL(2,\mathbb{R})$ symmetry of the type IIB action. It isn't very illuminating to work out the algebra that establishes the symmetry explicitly, so we not included it here. The details are given in Becker, Becker, Schwarz[2]. Let's see what, specifically, the $SL(2,\mathbb{R})$ symmetry entails. An element $\Lambda$ of $SL(2,\mathbb{R})$ is a 2-by-2 real matrix with determinant 1:

$$\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{3.124}$$

with $ad - bc = 1$. To see how this acts on the field content of Type IIB supergravity, let's define the complex scalar $\tau$ as follows:

$$\tau = C_0 + ie^{-\phi} \tag{3.125}$$

where $C_0$ is the R-R sector axion (sometimes denoted $\chi$) and $\phi$ is the NS-NS sector dilaton. The action of $SL(2,\mathbb{R})$ becomes:

$$\tau \to \frac{a\tau + b}{c\tau + d} \tag{3.126}$$

This relation will be extremely important to us later on. It is the key to how the new superstring emerges out of the Type IIB superstring. To examine the action of the symmetry on the other fields, let's put the two-form fields $B_2$ and $B_2'$ together in a vector:

$$\begin{pmatrix} B \\ B' \end{pmatrix} \to \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \begin{pmatrix} B \\ B' \end{pmatrix} \tag{3.127}$$

where the action of $SL(2,\mathbb{R})$ is carried out by the matrix multiplication. The symmetry leaves the four-form field $D_4$ and the metric invariant.

In the full type IIB string theory, the full $SL(2, \mathbb{R})$ symmetry can't survive. The theory has states which carry charges with respect to the two-form fields $B_2$ and $B_2'$. The charges must satisfy certain Dirac quantization conditons, which are analogous to the quantization conditions of the electric and magnetic charges in electrodynamics with magnetic monopoles. After a general $SL(2, \mathbb{R})$ transformation, the charges would no longer satisfy the quantization conditions. We explore the subject of string charge and brane charge in the next chapter. The symmetry must be reduced to $SL(2, \mathbb{Z})$, which is the group of 2-by-2 matrices with determinant 1 carrying integer components. The $SL(2, \mathbb{Z})$ symmetry is generated by the following three elements, denoted T, S, and R respectively:

$$\tau \to \tau + 1, \quad \Lambda = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tag{3.128}$$

$$\tau \to -1/\tau, \quad \Lambda = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{3.129}$$

$$\tau \to \tau, \quad \Lambda = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.130}$$

The second of these, denoted S, is of special interest. When $C_0$ is set to 0, it relates a theory at strong coupling to a theory at weak coupling. This is known as S-duality, which also relates the type I string to the heterotic strings.

Let's take a step back and explain why exactly we think the $SL(2, \mathbb{Z})$ symmetry should extend to the full Type IIB superstring theory. The $SL(2, \mathbb{Z})$ symmetry is a so-called nonperturbative symmetry, which means that it is not evident order by order in perturbation theory. If we want to prove the existence of a nonperturbative symmetry, we would need to have full knowledge of the quantum theory, including its strong coupling behaviour. However, we know that supersymmetry places very strict constraints upon the behaviour of the quantum theory. With enough supersymmetry, there are semiclassical quantities that are expected to receive no quantum corrections at all. The massless spectrum and the associated equations of motions are examples of such semiclassical quantities. If we can establish a duality at this level, it constitutes powerful evidence that the duality exists in the full quantum theory. A second semiclassical quantity that is not expected to receive any quantum corrections is the mass spectrum of the BPS-saturated states. BPS states are protected from quantum corrections by supersymmetry.[2] We will come back to this point later.

## 3.4 Perturbative symmetries of Type IIB

In the last section we established a nonperturbative $SL(2, \mathbb{Z})$ symmetry of Type IIB string theory. We now want to examine a number of perturbative Type IIB symmetries which will be relevant to us in the following chapters. The difference between nonperturbative and perturbative symmetries is that the perturbative symmetry is evident order-by-order in perturbation theory.[4]

The first symmetry we want to discuss is worldsheet parity $\Omega$, which takes the worldsheet coordinate $\sigma$ to $2\pi - \sigma$. It therefore transforms right-movers into left-movers and vice versa. Because Type IIB right- and left-movers have the same chirality, this is a symmetry.

Let's look at the action of $\Omega$ on the Type IIB massless spectrum. In the NS-NS sector, we had the symmetric traceless tensor $g_{ij}$, the scalar $\phi$ and the anti-symmetric two-form $B_2$. From the symmetry properties, we see that the metric $g_{ij}$ and the dilaton $\phi$ are even under $\Omega$, and $B_2$ is odd. In the R-R sector, the anticommutation properties of the ground state spinors make sure that $C_0$ and $D_4$ are odd and $B_2'$ is even under $\Omega$. In Chapter 5, we will obtain the closed-string spectrum of Type I theory by projecting out the states of Type IIB which are not invariant under $\Omega$. This is called an orientifold projection.

The second symmetry we want to discuss is $(-1)^{F_L}$, where $F_L$ is the spacetime fermion number of the left-movers. $(-1)^{F_L}$ takes $S^a$ into $-S^a$, which, as we can see from the light-cone gauge action in the GS formalism, is a symmetry of Type IIB theory. The NS-NS and NS-R states of the massless spectrum are even under $(-1)^{F_L}$. The R-NS and R-R states are odd. We will obtain Type IIA by projecting out the states of Type IIB which are odd under $(-1)^{F_L}$. This is called an orbifold projection, which we also discuss in Chapter 5. The difference between an orbifold and an orientifold is that the orbifold only deals with spacetime symmetries (such as $(-1)^{F_L}$), whereas the orientifold contains worldsheet symmetries (such as $\Omega$) as well.

The two symmetries discussed here actually fit into a group isomorphic to $D_4$. It has the following elements:

$$G = \{1, \Omega(-1^{F_L}), (-1^F), \Omega(-1^{F_R}), (-1^{F_L}), \Omega(-1^F), (-1^{F_R})\} \tag{3.131}$$

where $(-1^{F_R}) = \Omega(-1^{F_L})\Omega$ and $(-1^F) = (-1^{F_L+F_R})$.[4]

## 3.5 Anomaly cancellation

Early on in superstring theory history, it was thought that the quantum theories of superstrings could not be consistent due to anomalies. Anomalies occur when a symmetry of a classical theory disappears after quantization. Certain anomalies, such as gauge anomalies (the breaking of local gauge symmetry), are especially problematic. They indicate that the quantum theory is inconsistent. The first "superstring revolution" occurred when it was shown that $\mathcal{N} = 1$ superstrings are free of gauge anomalies, but only when the gauge group is $SO(32)$ or $E_8 \times E_8$.[2] The $\mathcal{N} = 2$ superstrings (Type IIA or Type IIB) are free of anomalies as well, even though they have quite different gauge groups (empty for Type IIB and $U(1)$ for IIA). The restriction of $\mathcal{N} = 1$ strings to these two gauge groups strongly limits what kind of new theories are possible. This is one reason the proposal of Savdeep Sethi's new string is interesting.[6]

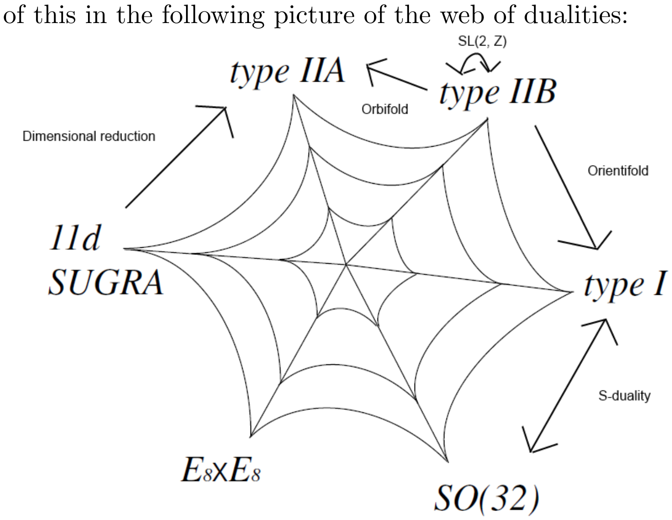
## 3.6 The heterotic strings

Aside from the Type IIA, Type IIB and Type I superstrings, there are also two so-called heterotic strings, which have $\mathcal{N} = 1$ supersymmetry. They are named after their gauge groups: $E_8 \times E_8$ and $SO(32)$. They are constructed by combining a left-moving superstring with a right-moving bosonic string in a certain way. We will not examine how this works out in detail. For now, let's just note that the $SO(32)$ and Type I superstrings are related by S-duality.[2]

## 3.7 The web of dualities

Let's take stock of what we've learned so far. We examined $\mathcal{N} = 2$ Type IIB and Type IIA superstring theory, which are connected by an orbifold projection of a nonperturbative spacetime symmetry. We saw that Type IIA supergravity could be obtained from 11-dimensional supergravity by dimensional reduction. This suggested the existence of an 11-dimensional theory (M-theory) which relates to the full Type IIA superstring theory in the same way. We found that the $\mathcal{N} = 1$ Type I string could be obtained from Type IIB by projecting out a nonperturbative worldsheet symmetry. We also learned that Type IIB is self-dual with duality group $SL(2, \mathbb{Z})$. One of the elements of $SL(2, \mathbb{Z})$ related the theory at weak coupling to the theory at strong coupling. This was called S-duality. S-duality also relates Type I theory to the $SO(32)$ heterotic string.

We can summarize all of this in the following picture of the web of dualities:



# Chapter 4

# Compactification

In the previous chapters we built the bosonic and supersymmetric string theories. We found that the bosonic string lives in $D = 26$ and the superstring in $D = 10$. In order to make this consistent with reality, either all but 4 of these dimensions have to be compactified. Compactification changes the predictions of the theories considerably. In this chapter we will discuss simple circular and toroidal compactifications, which entail that either one or two spatial coordinates satisfy

$$X^\mu \sim X^\mu + 2\pi R \tag{4.1}$$

where R is the radius of the compactified dimension. The symbol $\sim$ indicates that the two points are identified. Compactifications were first introduced in Kaluza-Klein theory, which was an attempt to unify general relativity with electrodynamics in 5-dimensional spacetime. The most simple effect of compactification we want to discuss is the emergence of so-called Kaluza-Klein towers. These are towers of massive states coming from a field which is normally massless. We will examine Kaluza-Klein towers in the context of 11-dimensional supergravity at the end of this chapter. Our first topic will be T-duality, which is a duality between a theory compactified on $R$ and another theory compactified on $\frac{\alpha'}{R}$. Objects called D-branes are required to appear in any theory with T-duality. D-branes are hyperplanes on which open strings may end. Just as a point particle acquires electric charge when it couples to the one-form Maxwell gauge field, a string acquires charge when it couples to a two-form Kalb-Ramond gauge field. In general, a p-brane may couple electrically to a (p + 1)-form gauge field. This fact will be crucial in our analysis of orientifold planes in Chapter 5. This chapter contains information from Becker Becker Schwarz [2], Zwiebach [1], and Dabholkar [4].

## 4.1 T-duality and D-branes in the bosonic theory

### 4.1.1 Closed strings

We will start our exploration of these new concepts in the most simple way possible, focusing on closed bosonic strings. A closed bosonic string compactified on a circle satisfies the following boundary conditions:

$$X^{25}(\sigma + \pi, \tau) = X^{25}(\sigma, \tau) + 2\pi RW \tag{4.2}$$

where the index 25 denotes the single compactified dimension, $R$ is the radius of the compactification, and $W$ is an integer known as the string's winding number. The winding number indicates how many times the closed string is wrapped around the circle. We may indicate the geometry of the spacetime under consideration by $(\mathbb{R}^{24,1} \times S^1)$, where $S^1$ stands for the circularly compactified dimension and $\mathbb{R}^{24,1}$ indicate the 24 noncompact spatial dimensions and the time dimension. The new boundary conditions change the mode expansions for the $X^25$ coordinates:

$$X^{25}(\sigma, \tau) = x^{25} + 2\alpha' p^{25}\tau + 2RW\sigma + \text{oscillators} \tag{4.3}$$

We can rewrite the third term slightly using:

$$w \equiv \frac{mR}{\alpha'} \tag{4.4}$$

We then obtain:

$$X^{25}(\sigma, \tau) = x^{25} + 2\alpha' p^{25}\tau + 2\alpha' w\sigma + \text{oscillators} \tag{4.5}$$

This seems to suggest that the winding modes and the momentum are on equal footing. Let's explore this idea. We can split the expansion of $X^25$ into left- and right-movers:

$$X_R^2 5 = \frac{1}{2}(x^{25} - \tilde{x}^{25}) + (\alpha' \frac{K}{R} - WR)(\tau - \sigma) + \text{oscillators} \tag{4.6}$$

$$X_L^25 = \frac{1}{2}(x^{25} + \tilde{x}^{25}) + (\alpha' \frac{K}{R} - WR)(\tau + \sigma) + \text{oscillators} \tag{4.7}$$

We can incorporate the first factors in the second term into the zero modes $\alpha_0^{25}$ and $\tilde{\alpha}_0^{25}$:

$$\sqrt{2\alpha'}\alpha_0^{25} = \alpha' \frac{K}{R} - WR \tag{4.8}$$

$$\sqrt{2\alpha'}\tilde{\alpha}_0^{25} = \alpha' \frac{K}{R} + WR \tag{4.9}$$

In terms of these zero modes, find:

$$p^{25} = \frac{1}{\sqrt{2\alpha'}}(\alpha_0^{25} + \tilde{\alpha}_0^{25}) \tag{4.10}$$

$$w = \frac{1}{\sqrt{2\alpha'}}(-\alpha_0^{25} + \tilde{\alpha}_0^{25}) \tag{4.11}$$

which confirms our suspicion that $w$ is a form of momentum. Because the $x^{25}$ dimension is compactified, the values of $p$ become quantized, just like the momentum states of a particle in a box:

$$p^{25} = \frac{K}{R}, \quad K \in \mathbb{Z} \tag{4.12}$$

We see that we have a momentum inversely proportional to $R$ and a momentum proportional to $R$. We could make the guess that transforming $R \to \alpha'/R$ and exchanging $W \leftrightarrow N$ leaves the momentum spectrum invariant. Let's quantize the string and see if this suspicion is confirmed. The mass squared of the string is given by:

$$M^2 = -\sum_{\mu=0}^{24} p_\mu p^\mu \tag{4.13}$$

which receives no contributions from the momentum in the compactified dimension. The physical state conditions $(L_0 - 1)|phys\rangle = 0$ and $(\tilde{L}_0 - 1)|phys\rangle = 0$ appear essentially unchanged. They receive contributions from the oscillators of the compactified dimension as well. The mass-shell condition now becomes:

$$\alpha' M^2 = \alpha' \left[ (\frac{K}{R})^2 + (\frac{WR}{\alpha'})^2 \right] + 2N_L + 2N_R - 4 \tag{4.14}$$

subject to the level-matching condition

$$N_R - N_L = WK \tag{4.15}$$

We see that the mass spectrum is invariant under the transformation $R \to \alpha'/R$ and the interchange $W \leftrightarrow K$. This is called T-duality. It is actually a symmetry of the full interacting theory of the bosonic string. In a slightly modified form, it relates several superstring theories to each other as well. Because T-duality maps a closed string theory to another closed string theory, the closed string is self-dual. After compactification, the massless 26-dimensional tensor states of the closed string in $\mathbb{R}^{25,1}$ rearrange themselves into 25-dimensional tensors. Among these are two one-form Maxwell gauge fields, a Kalb-Ramond field and a graviton. This is analogous to the situation in Kaluza-Klein theory, where compactification of a fifth dimension rearranges the five-dimensional metric $g_{\mu\nu}$ into a four-dimensional $g_{mn}$, a vector $g_{m5}$ (which is a Maxwell field), and a four-dimensional dilaton [1].

We can express the transformations of T-duality in a different way:

$$X_R \to -X_R, \quad \text{and} \quad X_L \to X_L \tag{4.16}$$

along with $R \to \alpha'/R$. This is how T-duality will act on open strings as well (open strings can always be continuously contracted to a point, so they have no notion of winding number). We will examine open bosonic strings in a moment. First, we want to comment on why we think T-duality is an exact symmetry of bosonic string theory rather than merely a coincidental overlapping of spectra. The T-duality transformation introduces the dual coordinate $\tilde{X}$:

$$X = X_L + X_R \to \tilde{X} = X_L - X_R \tag{4.17}$$

We can define a momentum conjugate to the dual coordinate:

$$\tilde{\mathcal{P}}^\tau = \frac{1}{2\pi\alpha'}(\dot{X}_L - \dot{X}_R) \tag{4.18}$$

And we can postulate the following commutation relation:

$$[\tilde{X}(\tau,\sigma),\mathcal{P}^{\tau}(\tau,\sigma')]=i\delta(\sigma-\sigma') \tag{4.19}$$

which leads to the commutation relations:

$$[\tilde{x}_0,w]=i \tag{4.20}$$

$$[\tilde{x}_0,p]=[\tilde{x}_0,\alpha_n]=[\tilde{x}_0,\tilde{\alpha}_n]=0,\quad n\neq 0 \tag{4.21}$$

$$[x_0,\tilde{x}_0]=0 \tag{4.22}$$

We want to find a mapping from the operators of the theory at radius $R$ to the operators of the theory at radius $\alpha'/R$. This mapping must respect all commutation relations and map one Hamiltonian into the other. This is satisfied by the following mapping:

$$\begin{pmatrix} x_0\to\tilde{x}_0 \\ \tilde{x}_0\to x_0\end{pmatrix},\quad \begin{pmatrix} p\to w \\ w\to p\end{pmatrix},\quad \begin{pmatrix} \alpha_n\to-\alpha_n \\ \tilde{\alpha}_n\to\tilde{\alpha}_n\end{pmatrix} \tag{4.23}$$

This constitutes a proof that the two theories are equivalent, at least before interactions are included. T-duality is actually an exact symmetry of the full interacting quantum theory[1], but we will not attempt to prove this.

### 4.1.2 Open strings

The action of T-duality on open strings is the mapping of $X$ onto its dual coordinate $\tilde{X}$, along with, of course, the inversion of the compactification radius $R\to\alpha'/R$. There is no interpretation in terms of winding numbers here, since Neumann open strings are topologically equivalent to a point. The expansion of a coordinate satisfying Neumann boundary conditions is:

$$X(\tau,\sigma)=x+p\tau+i\sum_{n\neq0}\frac{1}{n}\alpha_n e^{-in\tau}cos(n\sigma) \tag{4.24}$$

(with $\ell_s=1$). Splitting this into the left- and right-movers:

$$X_R(\tau-\sigma)=\frac{x-\tilde{x}}{2}+\frac{1}{2}p(\tau-\sigma)+\frac{i}{2}\sum_{n\neq0}\frac{1}{n}\alpha_n e^{-in(\tau-\sigma)} \tag{4.25}$$

$$X_L(\tau+\sigma)=\frac{x+\tilde{x}}{2}+\frac{1}{2}p(\tau+\sigma)+\frac{i}{2}\sum_{n\neq0}\frac{1}{n}\alpha_n e^{-in\tau+\sigma} \tag{4.26}$$

Performing the T-duality transformation yields:

$$\tilde{X}(\tau,\sigma)=\tilde{x}+p\sigma+\sum_{n\neq0}\frac{1}{n}\alpha_n e^{-in\tau}sin(n\sigma) \tag{4.27}$$

This is just the expansion of a coordinate satisfying Dirichlet boundary conditions! Specifically, the boundary conditions become:

$$\tilde{X}(\tau,0)=\tilde{x},\quad\text{and}\quad \tilde{X}(\tau,\pi)=\tilde{x}+\frac{\pi K}{R}=\tilde{x}+2\pi K\tilde{R} \tag{4.28}$$

The mode expansion for the dual string carries no momentum, but there now appears a winding number in the boundary conditions. The Dirichlet string is not topologically equivalent to a point, since the end points of the string are fixed.

Dirichlet boundary conditions are problematic in two ways: Firstly, they break Poincaré invariance. Secondly, the string momentum is not conserved. There needs to be a physical object to which the string ends are attached in order to account for the momentum non-conservation. These objects are p-dimensional hyperplanes called Dp-branes, short for Dirichlet branes. We can simultaneously compactify and T-dualize other directions as well. The corresponding coordinates interchange Neumann and Dirichlet boundary conditions.

### 4.1.3 D-branes and gauge fields

Let's consider the quantization of a string attached to a Dp-brane. The mode expansion for Dirichlet coordinates considered above leads to the usual commutation relations for the oscillators. The mass shell condition becomes:

$$M^2 = \frac{1}{\alpha'}(-1 + \sum_{n=1}^{\infty}\sum_{i=2}^{p} n a_n^{i\dagger} a_n^i) + \sum_{m=1}^{\infty}\sum_{a=p+1}^{d} m a_m^{a\dagger} a_m^a \tag{4.29}$$

where $p$ indicates the transverse coordinates which still satisfy Neumann boundary conditions. The second term sums over the Dirichlet coordinates. Note that we are working in light-cone gauge and that we have not assumed any compactification. The ground state is now labeled by the momentum in the transverse Neumann directions (denoted $\vec{p}$), and the light-cone momentum $p^+$. There is no total momentum in the Dirichlet directions. To summarize, the ground state is labeled:

$$\left|p^+, \vec{p}\right\rangle \tag{4.30}$$

We build the Fock space by acting with oscillators normal to the brane or oscillators tangent to the brane. The various fields may be assigned a Schrödinger wavefunction $\Psi_{i_1,\ldots,i_p,a_1,\ldots,a_q}(\tau, p^+, \vec{p})$. This indicates that the fields live on the Dp-brane world volume. The wavefunction has no dependence on momentum in directions normal to the brane. The wavefunction may therefore be Fourier transformed from momentum space into the Dp-brane world volume[1]. This is the most natural interpretation, but of course the argument presented here does not constitute a proof.

We can see from the mass formula that the first excited states are massless. We can act with an oscillator tangent to the brane to obtain:

$$a_1^{i\dagger}\left|p^+, \vec{p}\right\rangle \tag{4.31}$$

This is an $SO((p+1)-2)$ massless vector boson. It constitutes a $U(1)$ gauge field on the world volume of the D-brane. Acting with the normal oscillators $a_1^{a\dagger}$ yields a massless scalar for each Dirichlet direction. These states represent displacements of the D-brane.

Let's consider the case where we have $N$ parallel Dp-branes. Each end of the open string may end on any one of the D-branes, so the string obtains an additional $N^2$ degrees of freedom. These degrees of freedom are sometimes called Chan-Paton charges. They were introduced into string theory before the discovery of D-branes, as an algebraic tool without a clear physical interpretation. We may include the Chan-Paton degrees of freedom with the indices $[ij]$ in the ground states:

$$\left|p^+, \vec{p}; [ij]\right\rangle \tag{4.32}$$

Let's focus on the case $N = 2$. For strings starting on one D-brane and ending on the other, the mass shell condition becomes:

$$M^2 = \frac{1}{\alpha'}(N-1) + (\frac{\bar{x}_2^a - \bar{x}_1^a}{2\pi\alpha'})^2 \tag{4.33}$$

where $\bar{x}_i^a$ denote the positions of the D-branes. Let's take a look at the first excited states. Acting with oscillators normal to the brane, we get:

$$a_1^{a\dagger}\left|p^+, \vec{p}; [12]\right\rangle \tag{4.34}$$

at first glance, these appear to be $(d-p)$ massive scalar fields. For the tangent oscillators, we get:

$$a_i^{i\dagger}\left|p^+, \vec{p}; [12]\right\rangle \tag{4.35}$$

these are $(p-1)$ massive states. In order to fit these into an $SO(p)$ representation, as is required for vector bosons by Wigner's classification, we require an extra degree of freedom. This extra state can come from the scalar fields. A certain linear combination of scalar states becomes part of the vector, in what has been called a "stringy Higgs mechanism". When the separation between the D-branes is zero, we get the same particle content as in the case where the string starts and ends on the same D-brane. We therefore find $N^2 = 4$ massless $U(1)$ gauge fields. These actually interact with each other to form a $U(2)$ massless gauge field. This is a general result: $N$ coincident D-branes carry a $U(N)$ massless gauge field on their world volume! If only $N_0 \leq N$ of the D-branes coincide, we obtain a $U(N_0) \times U(1)^{N-N_0}$ gauge symmetry.[2] These facts will be crucial when we consider orientifolds.

### 4.1.4 Wilson lines

In a compactified space, a potential with a vanishing field strength can have physical effects. These effects are characterized by a so-called Wilson line, which is a matrix $U$ that breaks a $U(N)$ gauge symmetry to its subgroup of elements commuting with $U$. We will have to understand Wilson lines to know how T-duality works on configurations of non-coincident D-branes.

Let's first review what exactly we mean by a $U(N)$ gauge symmetry. The easiest way to do this is to examine the gauge transformation of Maxwell's equations, which have $U(1)$ gauge symmetry. Consider the classical Hamiltonian for a charged particle with mass $m$ and charge $q$:

$$H = \frac{1}{2m}(\vec{p} - q\vec{A})^2 + q\Phi \tag{4.36}$$

The Schrödinger equation becomes:

$$i\frac{\partial\psi}{\partial t} = \frac{1}{2m}(\frac{\nabla}{i} - q\vec{A})^2\psi + q\Phi\psi \tag{4.37}$$

The Schrödinger equation is invariant under the following simultaneous transformations:

$$\vec{A} \to \vec{A}' = \vec{A} + \nabla\chi \tag{4.38}$$

$$\Phi \to \Phi' = \Phi - \frac{\partial\chi}{\partial t} \tag{4.39}$$

$$\Phi \to \Phi' = e^{iq\chi}\Psi \tag{4.40}$$

where $\chi(x)$ is a function of spacetime. Let's denote the phase factor with $U(x)$:

$$U(x) \equiv e^{iq\chi(x)} \tag{4.41}$$

Now, remember that Maxwell gauge transformations are often expressed in the following way:

$$A_\mu \to A'_\mu = A_\mu + \partial_\mu\chi \tag{4.42}$$

where we interpret $\chi(x)$ as the gauge parameter. We will have to shift our viewpoint on this in order to work with gauge fields on compactified spaces. More fundamentally, it is $U(x)$ that should be considered the gauge parameter. We can rewrite the gauge transformation as:

$$A_\mu \to A'_\mu = A_\mu - \frac{i}{q}(\partial_\mu U)U^{-1} \tag{4.43}$$

It is easy to see that $U(x)$ is an element of $U(1)$. In general, $U(N)$ is the group of N-by-N unitary matrices, so an element $u$ of $U(1)$ simply satisfies $u^*u = 1$. The argument is completed by noting that Maxwell gauge transformations performed in sequence satisfy the rules of group multiplication. In other words:

$$\psi(x) \to U_2(x)\psi(x) \to U_1(x)(U_2(x)\psi(x)) = (U_1U_2)(x)\psi(x) \tag{4.44}$$

Let's return to the subject of Wilson lines. Consider a space $R^n \times S^1$ where the $x$ dimension has been compactified into a circle. A vector potential $A_x = c$ where $c$ is constant now satisfies all the field equations. A field with such a vector potential has vanishing field strength, but may still have physical effects. A gauge transformation of $A_x$ takes the form:

$$A_x \to A_x + \frac{\partial\chi}{\partial x} \tag{4.45}$$

Under the assumption that $chi(x)$ is the gauge parameter, it would be natural to require:

$$\chi(x + 2\pi R) \stackrel{?}{=} \chi(x) \tag{4.46}$$

We can now define the holonomy or Wilson line of the gauge field:

$$W \equiv \exp(iw) = \exp(iq\oint dxA_x) \tag{4.47}$$

The gauge transformation acts upon $w$ as:

$$w \to w' = w + q(\chi(x_0 + 2\pi R) - \chi(x_0) \tag{4.48}$$

where the integral has been taken around the circle once starting from an arbitrary point $x_0$. If (4.46) is correct, then $w$ does not change under the gauge transformation. If, however, we interpret $U(x)$ as the gauge parameter, then it is $U(x)$ that should be periodic on the circle:

$$U(x + 2\pi R) = U(x) \tag{4.49}$$

This implies that $w' = w + 2\pi m$, with $m \in \mathbb{Z}$, under gauge transformations. In other words, we should make the identification:

$$w \sim w + 2\pi m, \quad m \in \mathbb{Z} \tag{4.50}$$

The natural interpretation is that $w$ is really an angle of some kind. To emphasize this, let's rewrite:

$$\theta \equiv w = q \oint dx A_x \tag{4.51}$$

There are many choices of $\chi$ that are not single-valued on the circle, but do lead to a single-valued $U(x)$. The simplest one is the following:

$$q\chi = (2\phi m)\frac{x}{2\pi R} = \frac{mx}{R}, \quad m \in \mathbb{Z} \tag{4.52}$$

A gauge transformation changes this field by $\frac{m}{R}$. So we can make the identification:

$$qA_x \sim qA_x + \frac{m}{R} \tag{4.53}$$

When $A_x$ is constant, we can write:

$$qA_x = \frac{\theta}{2\pi R} \tag{4.54}$$

The presence of the Wilson line has significant physical effects. In particular, the energy levels of a charged particle are shifted according to:

$$E_n = \frac{1}{2m}\left(\frac{n}{R} - \frac{\theta}{2\pi R}\right)^2 \tag{4.55}$$

as may be checked by solving the Schrödinger equation.

Now let's apply these ideas to D-branes. T-duality will provide a very simple physical interpretation to the angle variable that specifies the Wilson line, relating it to the locations of D-branes along compactified circles.

Consider a Dp-brane that wraps around the circular dimension $x$. After T-dualizing $x$, it is transformed into a D$(p-1)$-brane located at some point on the dual circle. We can see this from the fact that T-duality turns Neumann boundary conditions into Dirichlet boundary conditions. Now let's say the Dp-brane carried a gauge field that satisified:

$$q \oint A_x dx = \theta \tag{4.56}$$

It turns out that $\theta$ specifies the position of the D$(p-1)$-brane along the dual circle! We can see this by adding another Dp-brane to the configuration. Let's say the Dp-branes now have gauge fields such that the angular variables become $\theta_1$ and $\theta_2$. The momentum in the circular direction of a string stretched between the two D-branes is shifted according to:

$$\frac{n}{R} \to \frac{n}{R} - \frac{\theta_2}{2\pi R} + \frac{\theta_1}{2\pi R} \tag{4.57}$$

with $n \in \mathbb{Z}$ because the momentum is quantized. This shift occurs due to the coupling of the string endpoints to the Maxwell field on the Dp-brane world volume. The mass-squared formula becomes[1]:

$$M^2 = \left(\frac{2\pi n - (\theta_2 - \theta_1)}{2\pi R}\right)^2 + \frac{1}{\alpha'}(N-1), \quad n \in \mathbb{Z} \tag{4.58}$$

This in fact precisely coincides with the mass-squared formula for a string stretched between two $D(p-1)$-branes in the T-dual description, as may be seen by comparing with (4.33). We have seen that the relative positions of D-branes determine the structure of the gauge fields that live on their world volume. For example, $N$ coincident D-branes produce a $U(N)$ gauge field, whereas only a $U(N_0) \times U(1)^{N-N_0}$ gauge symmetry is produced when $N_0 \leq N$ D-branes are coincident. Wilson lines do in fact break gauge symmetry in the same way. For a general $U(N)$ gauge field, we can define the holonomy matrix, or Wilson line:

$$U \equiv \exp i \int_0^{2\pi R} A dx \tag{4.59}$$

where $A$ is a hermitian matrix specifying the gauge potential, which may be transformed into diagonal form by a constant gauge transformation:

$$A = -\frac{1}{2\pi R}\mathrm{diag}(\theta_1, \theta_2, ...\theta_N) \tag{4.60}$$

In the presence of this Wilson line, the $U(N)$ gauge symmetry is broken to the subgroup of its elements that commute with $U$.[2] When all the eigenvalues of $U$ are distinct, the symmetry is broken to $U(1)^N$. Just like before, the $\theta_n$ variables specify the positions of D$(p-1)$-branes in the T-dual world.

We will encounter Wilson lines again when we discuss Type I' superstring theory in the next chapter.

### 4.1.5 String charge

Our last topic before we move on to the superstring theory will be string charge. Just like a point particle may couple electrically to the Maxwell field and acquire a charge [1], a string may couple "electrically" to the Kalb-Ramond field. To see how this works out, let's look at the point particle action in the presence of a Maxwell field:

$$S = -m \int ds + e \int A_\mu(x) dx^\mu - \frac{1}{4\kappa_0^2} \int d^D x F_{\mu\nu} F^{\mu\nu} \tag{4.61}$$

where $\kappa_0^2$ is a constant with dimension $M^{D-4}$, which is necessary in order to make the units work out. The second term takes this form because the world-line is parametrized with a single variable $\tau$, which defines a tangent vector $dx^\mu/d\tau$ that forms a Lorentz scalar when multiplied by the gauge field $A_\mu$. Strings can couple to Kalb-Ramond fields in an analogous way. This time we have two tangent vectors $\frac{\partial X^\mu}{\partial \tau}$ and $\frac{\partial X^\nu}{\partial \sigma}$, which may form a Lorentz scalar when multiplied by $B_{\mu\nu}$:

$$-\int d\tau d\sigma \frac{\partial X^\mu}{\partial \tau} \frac{\partial X^\nu}{\partial \sigma} B_{\mu\nu}(X(\tau,\sigma)) \tag{4.62}$$

This term in the action leads to a conserved vector charge that is tangent to the string, pointing in the direction of increasing $\sigma$.[1]

The previous discussion suggests that Dp-branes may also couple electrically to some tensor field. In the bosonic theory, this doesn't work out for $p > 1$, because there is no three-form (or higher) field in the massless spectrum. As we already know, the superstrings do contain massless higher-form fields, which means the D-branes may acquire their own "electric" charge. This implies that D-branes can be stable in superstring theories, an issue that is more subtle in the bosonic theory due to the existence of tachyons. In the next section, we will generalize this discussion using the language of differential forms. The existence of D-brane charge will be crucial in our discussion of orientifolds.

## 4.2 T-duality and D-branes in superstring theory

We will now extend the previous considerations into superstring theory. When we add D-branes, open superstrings become a possibility. We mentioned in Chapter 2 that only $\mathcal{N} = 1$ supersymmetry is consistent with open string boundary conditions. Adding D-branes to Type II theory breaks at least half of the supersymmetries, along with the $SO(1,9)$ Lorentz symmetry, which is reduced to $SO(1,p) \times SO(9-p)$. The world volume of a single D-brane carries a $U(1)$ gauge field coming from the NS sector states $b^\mu_{-1/2} |NS\rangle$, where the index $\mu$ indicates the directions tangent to the D-brane. This can be seen from quantizing the superstring under the appropriate boundary conditions. It works out exactly the same way as in the bosonic theory. A number $N$ of coincident D-branes carry $N$ interacting gauge fields which together form a $U(N)$ gauge symmetry.

In Type II theory, D-branes can be stable. This is possible due to the presence of massless higher-form gauge fields to which the D-branes may couple elecrically (or, as we will see, magnetically).

### 4.2.1 D-brane charges

To see this, we need to revisit our previous discussion of string charge, this time using the language of differential forms. In general, an n-form gauge field in the is given by:

$$A_n = \frac{1}{n!} A_{\mu_1 \mu_2 \ldots \mu_n} dx\mu_1 \wedge dx^{\mu_2} \wedge \ldots \wedge dx^{\mu_n} \tag{4.63}$$

The field strength is given by $F_{n+1} = dA_n$. This field strength has a gauge invariance $\delta A_n = d\Sigma_{n-1}$ where $\Sigma_{n-1}$ is an (n - 1)-form. This follows from the fact that the square of an exterior derivative equals zero.

Let's take a look at what Maxwell theory looks like expressed in differential forms. The free field equations become:

$$dF = 0 \tag{4.64}$$

$$d \star F = 0 \tag{4.65}$$

where $\star F$ represents the Hodge dual of $F$. The hodge dual of an $n$ form in $D$ dimensions is a $(D-n)$-form.[2] When we incorporate sources, the field equations become:

$$dF = \star J_m \tag{4.66}$$

[1]We have already seen that string endpoints do in fact couple to Maxwell fields. However, the two endpoints carry opposite charge, so a string as a whole is neutral with respect to a Maxwell field

$$d \star F = \star J_e \tag{4.67}$$

where $J_e$ and $J_m$ are one-forms proportional to the current density $J_\mu = (\rho, \vec{j})$. These equations presume magnetic monopoles exist, even though they have not been discovered yet. In $D = 4$, we can calculate electric and magnetic charges in the following way:

$$e = \int_{S^2} \star F \tag{4.68}$$

$$g = \int_{S^2} F \tag{4.69}$$

where $e$ is the electric charge and $g$ is the magnetic charge. $S^2$ indicates that the integral is to be carried out over a two-sphere surrounding the sources. Dirac discovered that electric and magnetic charge have to be connected by a quantization condition:

$$e \cdot g \in 2\pi\mathbb{Z} \tag{4.70}$$

We can generalize the above discussion to higher dimensions to investigate D-brane charges. Remember that point particles may couple electrically to Maxwell fields because of the presence of the following term in the action:

$$S_{int} = e \int A_\mu dx^\mu = e \int d\tau A_\mu \frac{dx^\mu}{d\tau} \tag{4.71}$$

That this leads to an electric charge $e$ may be checked by taking a variation of $A_\mu$, which leads to a continuity equation. The result is consistent with the integration of the field strength hodge dual over the two-sphere. In the general case, the integration must be carried out over a $(D-2)$-sphere, because of hodge dual of the field strength is itself a $(D-2)$-form. The magnetic charge is calculated by the integration over the field strength, which in the case of the Maxwell field is a two-form. In $D$-dimensions, a two-sphere $S^2$ can surround a $(D-4)$-dimensional object. For example, $\mathbb{R}^{9,1}$ the D6-branes carry magnetic charge with respect to the Maxwell field.

Let's generalize further to incorporate higher-form fields. The electric coupling of an n-form field now occurs to the world volume of an $(n-1)$-brane. This is reflected in the following term in the action:

$$S_{int} = \mu_p \int A_{p+1} \tag{4.72}$$

where $\mu_p$ represents the p-brane charge. This integral can be rewritten:

$$\int A_{p+1} = \frac{1}{(p+1)!} \int A_{\mu_1 \ldots \mu_{p+1}} \frac{\partial x^{\mu_1}}{\partial \sigma^0} \cdots \frac{\partial x^{\mu_{p+1}}}{\partial \sigma^P} d^{p+1}\sigma \tag{4.73}$$

That this leads to an electric charge $\mu_p$ can be verified by taking a variation to obtain a continuity equation. The result is consistent with the integration $\mu_p = \int \star F_{p+2}$, which must now be carried out over a $(D-p-2)$ sphere. Likewise, the n-form gauge field may couple magnetically to the $(D-p-4)$-brane. The higher-form gauge field charges have to satisfy a Dirac quantization condition as well. In ten dimensions, this becomes[2]:

$$\mu_p \mu_{6-p} \in 2\pi\mathbb{Z} \tag{4.74}$$

### 4.2.2 Stability of D-branes

Let's take a look at which D-branes, specifically, couple to which fields in the Type II massless spectra. As we've discussed before, a charged D-brane must be stable. In the Type IIA massless spectrum, we find the one-form $A$ and the three-form $C_3$. These couple electically to D0-branes (which are point particles), and D2-branes, respectively. They couple magnetically to D6- and D4-branes. Altogether, this implies the stability of the Dp-branes with $p = 0, 2, 4, 6$. This regularity suggests that D8-branes might be stable as well. A D8-brane would couple electrically to a nine-form gauge field. Such a field is nondynamical, because its field equation $dF_{10} = 0$ is automatically satisfied due to the anti-symmetry of wedge products. Similarly, in Type IIB theory, we find that Dp-branes with odd p are guaranteed to be stable, although the analysis is rather more subtle in this case.[2]

The presence of stable D-branes reduces the supersymmetry to $\mathcal{N} = 1$. For this reason, they are sometimes referred to as half-BPS D-branes. The requirement of supersymmetry decouples the tachyon from the open string spectrum. D-branes which do not have the right dimensionality to couple to a gauge field are not protected by a conserved charge and are therefore unstable. They break the full supersymmetry of the theory. This leaves open strings attached to them free to carry a tachyon. An unstable D-brane can decay into closed-string radiation.

The inclusion of D-branes in Type II theory is actually necessary in order to keep the $SL(2, \mathbb{Z})$ symmetry of Type IIB.[4] As we mentioned in the last chapter, nonperturbative dualities can be tested by investigating the spectrum of BPS states. It can be shown that a string wound around a compactified direction is charged with respect to $B_2$. Such a string is in a BPS state if it carries no momentum in the compactified direction. Under the element S of $SL(2,\mathbb{Z})$, the fields $B_2$ and $B_2'$ are transformed into each other, so in order for $SL(2,\mathbb{Z})$ to hold, we need to find states which are charged with respect to $B_2'$. There are no such string states in the BPS spectrum due to a general rule that perturbative states couple to the field strength of R-R fields and not to the potential. However, the $B_2'$ may also couple to D1-branes. If we include these into the theory, the $SL(2,\mathbb{Z})$ symmetry is saved. The D1-brane is sometimes referred to as the D-string.

### 4.2.3 Type II T-duality

T-duality mapped the theory of closed bosonic strings on $R$ to the same theory on $\alpha'/R$. We therefore said that closed bosonic string theory is self-dual under T-duality. The closed strings of Type II superstring theory, however, are not self-dual. T-duality actually maps a IIA string on $R$ to a IIB string on $\alpha'/R$. We will examine how this works out in this subsection.

Let's take type II theory with one direction, $X^9$, say, compactified on a circle of radius $R$. The action of T-duality on the bosonic coordinates is exactly the same as before:

$$X_L^9 \to X_L^9, \quad \text{and} \quad X_R^9 \to -X_R^9 \tag{4.75}$$

In the RNS formalism, the same thing happens to the fermion $\psi^9$:

$$\psi_L^9 \to \psi_L^9 \quad \text{and} \quad \psi_R^9 \to -\psi_R^9 \tag{4.76}$$

This flips the chirality of the right-moving R-sector fermion. To see this, note that the transformation flips the sign of the zero modes:

$$d_0^9 \to -d_0^9 \tag{4.77}$$

In the previous chapter we saw that the R-sector zero modes furnish a representation of the Dirac algebra, which may be expressed as:

$$\Gamma^\mu = \sqrt{2} d_0^\mu \tag{4.78}$$

So the transformation takes $\Gamma^9$ into $-\Gamma^9$, which also flips the sign of the chirality operator $\Gamma^{11}$. The difference between Type IIB and Type IIA theory was in the relative chirality of the left- and right-moving R sector fermions. We therefore see that T-duality maps IIA into IIB and vice versa. This fact will be of use to us when we examine the M-theory description of Savdeep Sethi's new string.

Just like in the bosonic theory, the transformations of T-duality interchange Neumann and Dirichlet boundary conditions of open strings. This implies that T-duality transforms Dp-branes into D$(p-1)$-branes. This means that the stable half-BPS D-branes of Type IIB get transformed into the stable half-BPS D-branes of Type IIA (and vice versa), which is what we would like to see.

In the GS formalism in light-cone gauge, the T-duality transformations for Type IIB take the following form:

$$S_1^a \to \Gamma\Gamma^9 S^a \tag{4.79}$$

$$S_2^a \to S_2^a \tag{4.80}$$

where $\Gamma = \Gamma^1\Gamma^2 ... \Gamma^8$. This flips the chirality of the right-moving fermions. We get a left-moving conjugate spinor **8c** and a right-moving spinor **8s**, so T-duality maps IIB into IIA as expected.[2]

Lastly, let's see how T-duality transforms the perturbative symmetries of Type IIB theory. We obtain the following[4]:

$$\Omega \text{ in IIB } \to I_9\Omega \text{ in IIA} \tag{4.81}$$

where $I_9$ is given by:

$$I_9 : \ (X_L^9, X_R^9) \to (-X_L^9, -X_R^9) \tag{4.82}$$

The action of $I_9$ on the fermions is to flip the chirality of both of them. $\Omega$ by itself is of course not a symmetry of Type IIA theory, because it takes a theory with a left-moving spinor and right-moving conjugate spinor into a theory with right-moving spinor and a left-moving conjugate spinor. $I_9$ flips them back and we obtain a proper symmetry of Type IIA.

## 4.3 Kaluza-Klein towers and 11-dimensional supergravity

When we compactify a $D$-dimensional theory, a field which is ordinarily massless may appear to be massive in the dimensionally reduced theory. For example, let's take a massless scalar $\phi$ in a $D$-dimensional theory. When we compactify down to $(D-1)$ dimensions, the field may be Fourier expanded on the circular coordinate:

$$\phi(\vec{x},0) = \sum_{n=-\infty}^{\infty} \phi^{(n)}(\vec{x})e^{in\theta/R} \tag{4.83}$$

where $\theta$ represents the circular coordinate, $R$ is the radius of compactification, and $\vec{x}$ indicate the $D-1$ noncompact dimensions. The field satisifies the $D$-dimensional Klein-Gordon equation on the full spacetime:

$$\partial^\mu \partial_\mu \phi = 0 \tag{4.84}$$

More explicitly:

$$(\partial^A \partial_A + \partial^\theta \partial_\theta)\phi(\vec{x},\theta) = 0 \tag{4.85}$$

where the index $A$ indicates the noncompact dimensions. Carrying out the derivatives with respect to $\theta$ and taking the perspective of an observer in the lower-dimensional spacetime, the equation becomes:

$$(\partial^\mu \partial_\mu - \frac{N^2}{R})\phi(\vec{x}), \quad N \in \mathbb{Z} \tag{4.86}$$

which is just the Klein-Gordon equation for a massive scalar. We therefore see that a whole tower of massive states emerges from the compactification. This is called a Kaluza-Klein tower. We will apply the previous analysis to the dimensional reduction of 11-dimensional supergravity. The field under consideration will be the full massless supergraviton. We will find that there is a duality between M-theory compactified on a torus $(S^1 \times S^1)$ and Type IIB superstring theory compactified on a circle $S^1$.

### 4.3.1 A note on BPS states

Before we can proceed, we want to clarify some of our discussion about BPS states. These are states that are part of a so-called shortened supersymmetry multiplet. Their mass is tied to a conserved charge of a massless gauge field. More explicitly, in supersymmetry theories (not specific to string theory) there is a so-called central-charge matrix, whose elements $\pm Z_i$ (with $|Z_1| \geq |Z_2| \geq ... \geq 0$) are electric and magnetic charges belonging to the massless gauge fields of the theory. Crucially, these charges commute with all the generators of the supersymmetry algebra. These generators realize the supersymmetry transformations given in the previous chapter, and as such they square to spacetime translations. The matrix

$$\begin{pmatrix} M & Z \\ Z^\dagger & M \end{pmatrix} \tag{4.87}$$

is required by the supersymmetry algebra to be positive semidefinite. This implies that the eigenvalues of this matrix $M \pm |Z_i|$ are nonnegative. This implies that the mass of the states is bounded below[2]:

$$M \geq |Z_1| \tag{4.88}$$

All of this is still rather abstract at this point. The only way to clarify the discussion is to delve deeper into the mathematics of supersymmetry theories, which we do not have time for. The important part is the following: the states which have $M = |Z_1|$ are the BPS states. They are protected by supersymmetry from quantum corrections. In other words, we can reliably extrapolate their weak-coupling mass formulas to strong coupling. This is what makes them so useful for analyzing proposed dualities. If we can establish that a duality holds among the BPS states, it provides evidence that the duality holds in the full theory, including at strong coupling.

### 4.3.2 M-theory - Type II duality

We've seen that 11-dimensional supergravity compactified on a circle gives Type IIA supergravity, which suggested that the same link exists between M-theory and the full Type IIA superstring theory. We've also seen that Type IIA compactified on a circle $R$ is equivalent under T-duality to Type IIB compactified on the dual circle $\alpha'/R$. This seems to suggest that there exists a duality between M-theory compactified on a torus and Type IIB on a circle. We will now try to verify this.

As we noted earlier in this chapter, the $SL(2,\mathbb{Z})$ symmetry may transform the fundamental string of Type IIB theory (the F-string, which couples electrically to $B_2$) into the D-string (which couples to $B_2'$), or vice versa. The transformation is more general than that, and we obtain a whole tower of $(p, q)$ strings, where $p$ indicates their charge with respect to $B_2$ and $q$ indicates their charge with respect to $B'_2$. The $(p, q)$ strings are all supersymmetric, which uniquely determines their tensions. The result is[2]:

$$T_{(p,q)} = |p - q\tau_b|T_F = T_F\sqrt{(p - q\frac{\theta_0}{2\pi})^2 + \frac{q^2}{g_s^2}} \tag{4.89}$$

where $\tau_b$ is equal to the vacuum expectation value of the axion-dilaton field $\tau$:

$$\tau_b =< \tau >= \frac{\theta_0}{2\pi} + \frac{i}{g_s} \tag{4.90}$$

In general, a string's tension is related to $\ell_s$ by:

$$T = \frac{1}{2\pi\ell_s^2} \tag{4.91}$$

We have rewritten the axion $C_0$ into an angular variable $\theta_0$ to emphasize its periodicity. Remember that the $SL(2, \mathbb{Z})$ acts upon $\tau$ as:

$$\tau \to \frac{a\tau + b}{c\tau + d}, \quad ad - bc = 1 \tag{4.92}$$

which implies that $C_0 \sim C_0 + 1$. This fact will be crucial to us later on, when we introduce Savdeep Sethi's new superstring. As can be seen from the above formula, $SL(2, \mathbb{Z})$ takes a theory at weak coupling to a theory at strong coupling, so at most one of the $(p, q)$ strings may be weakly coupled at the same time. Let's write down the weak-coupling mass spectrum of Type IIB on a circle:

$$M_B^2 = (\frac{K}{R_B})^2 + (2\pi R_B W T_{(p,q)})^2 + 4\pi T_{(p,q)}(N_L + N_R) \tag{4.93}$$

subject to the level-matching condition:

$$N_R - N_L = KW \tag{4.94}$$

We want to consider the spectrum of all the $(p, q)$ strings at once and compare it to the Kaluza-Klein spectrum of 11-dimensional supergravity on a torus. However, the mass formula is meaningless at strong coupling due to quantum corrections, and at most one of the $(p, q)$ strings is weakly coupled. The solution is to focus on just the BPS states. Their mass formulas are protected by supersymmetry and don't receive quantum corrections. The BPS states are those with $N_L = 0$ or $N_R = 0$. We can map the three integers $W$,$p$, and $q$ into two integers $n_1$ and $n_2$ in the following way:

$$(n_1, n_2) = (Wp, Wq) \tag{4.95}$$

where W is the greatest common divisor of $n_1$ and $n_2$. We can therefore specify all the BPS using $K$, $n_1$, $n_2$, and the left-moving oscillator structure satisfying $N_L = |WK|$. We can rewrite the second term in the mass formula as:

$$(2\pi R_B W T_{(p,q)})^2 = (|n_1 - n_2\tau_B|T_F)^2 \tag{4.96}$$

Now let's look at the M-theory Kaluza-Klein spectrum that results from the torus compactification. A torus may be specified by two complex periods $w_1$ and $w_2$, which define the identifications

$$z \sim z + w_1, \quad z \sim z + w_2 \tag{4.97}$$

This defines a parallelogram on the complex plane, whose opposite edges are identified. The real and imaginary axes represent the two directions which will become the torus after compactification. We obtain the same parallelogram when we switch to the alternative periods $w'_1$ and $w'_2$:

$$w'_1 = aw_1 + bw_2, \quad w'_2 = cw_1 + dw_2 \tag{4.98}$$

with $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$. In other words, the constants fit into a 2-by-2 matrix with determinant 1. We see that the alternative torus periods fit into a group isomorphic to $SL(2, \mathbb{Z})$. This is called the modular group of the torus.

Let's take a torus with periods $w_1 = 2\pi R_{11}$ and $w_2 = 2\pi R_{11}\tau_M$, where $\tau_M$ is called the complex structure[6]. Under the torus identifications specified by these periods, we see that a single-valued wave function takes the form:

$$\psi_{n_1,n_2} \sim \exp\left[\frac{i}{R_{11}}(n_2 x - \frac{n_2\mathrm{Re}\tau_M - n_1}{\mathrm{Im}\tau_M} y)\right] \tag{4.99}$$

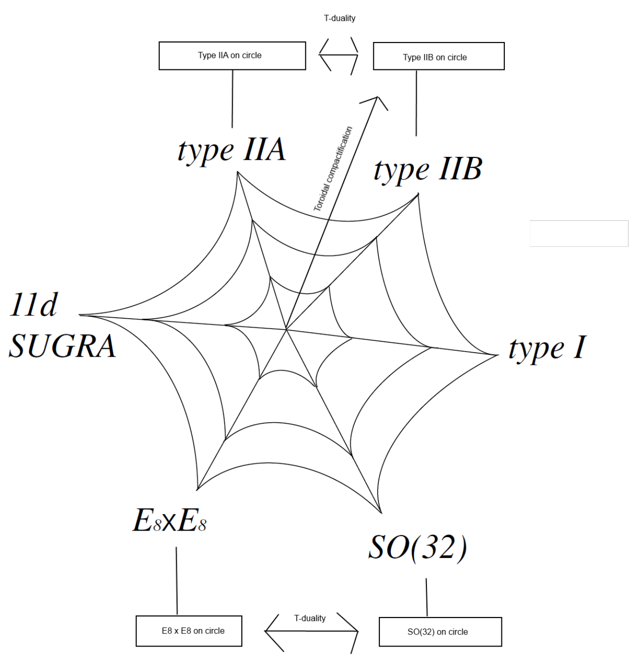
In the same way we obtained the Kaluza-Klein tower for the massless scalar, the tower can now be obtained by taking the derivatives in the compactified directions. The result is:

$$M_{KK}^2 = \frac{1}{R_{11}^2}\left[n_2^2 + \frac{(n_2 \mathrm{Re}\tau_M - n_1)^2}{(\mathrm{Im}\tau_M)^2}\right] = \frac{|n_1 - n_2\tau_M|^2}{(R_{11}\mathrm{Im}\tau_M)^2} \tag{4.100}$$

If we make the identification $\tau_M = \tau_B$, this is identical to the winding-mode term in the Type IIA mass formula, up to a normalization constant. The discrepancy in the normalization is due to the fact that these spectra are measured in different metrics. This result confirms our suspicion that there exists a duality between M-theory on a torus $(S^1 \times S^1) \times R^{8,1}$ and Type IIB on $(S^1) \times R^{8,1}$. It also implies that the $SL(2,\mathbb{Z})$ symmetry of compactified Type IIB has an elegant M-theory interpretation as the modular group of a toroidal compactification [2].

## 4.4 The web of dualities: part II

Let's summarize what we've learned about the web of dualities in this chapter. We now know that Type IIA and Type IIB compactified on a circle are equivalent to each other under T-duality. We've also seen that there is a duality between Type IIB on a circle and M-theory on a torus. It turns out that T-duality links the two heterotic strings $SO(32)$ and $E_8 \times E_8$ as well (although we haven't discussed this). To summarize:




The link coming from the center of this web represents a connection to M-theory.

# Chapter 5

# Orbifolds and Orientifolds

A number of times throughout the previous chapter, we've mentioned orbifold and orientifold projections. These projections truncate a theory's spectrum down to the states which are invariant under a certain symmetry. By doing so, we can often obtain the spectrum of a different consistent string theory. For example, we've suggested that Type I superstring theory may be obtained from Type IIB by keeping only the states which are invariant under world-sheet parity. This is not the whole story. For orbifolds, we need to add a sector of strings which satisfy boundary conditions that are twisted by an element of the orbifold group. For orientifolds, we need to deal with orientifold planes. An orientifold plane is the set of points that are invariant under the subgroup of spacetime symmetries in the orientifold group. Like D-branes, they can couple to gauge fields to acquire a charge. This charge must be cancelled by adding the right number of D-branes carrying opposite charge. Just like Type I, Savdeep Sethi's superstring is obtained from Type IIB by an orientifold projection. We will see how this works out in Chapter 6.

## 5.1 General features of orbifolds

As a warmup to the subject of orientifolds, we first discuss orbifolds, which are a bit simpler. In mathematics, an orbifold is obtained from a smooth manifold with discrete isometry group $G$ by identifying all points which are transformed into each other by the action of an element of $G$. For example, we can obtain the orbifold $S^1/\mathbb{Z}_2$ by identifying the points $x \sim -x$ on a circle $S^1$. The result is a line interval.[2] We will not go any further into the geometrical interpretation of orbifolds. For our purposes, an orbifold $A'$ is a theory which is obtained from a theory $A$ with discrete spacetime symmetry group $G$ by projecting onto states which are invariant under $G$. Symbolically:

$$A' = A/G \tag{5.1}$$

In closed string theory, we need to add a so-called twisted sector. In the twisted sector, we find strings which are closed up to an element $\hat{\alpha}$ of the orbifold group $G$:

$$\Phi(\sigma + 2\pi, \tau) = \hat{\alpha}\Phi(\sigma, \tau) \tag{5.2}$$

We must perform the projection onto $G$-invariant states in both the twisted and the untwisted sector[4].

## 5.2 Type IIA as an orbifold

Let's look at an example of an orbifold projection. As we discussed in Chapter 3, Type IIA superstring theory may be obtained from Type IIB by projecting onto states invariant under the perturbative symmetry $(-1)^{F_L}$. This operator clearly squares to 1, so the orbifold group will be isomorphic to $Z_2$.

In the untwisted sector, the projection removes all R-NS and R-R states, which have odd left-moving fermion numbers. The NS-NS and NS-R states all survive. In the language of group theory, we now obtain a massless spectrum given by:

$$\mathbf{8v} \oplus (\mathbf{8v} \oplus \mathbf{8c}) \tag{5.3}$$

In the twisted sector, the boundary conditions (in the GS formalism) become the following:

$$S^{1a}(\sigma + 2\pi) = (-1)^{F_L} S^{1a}(\sigma) = -S^{1a}(\sigma) \tag{5.4}$$

$$S^{2a}(\sigma + 2\pi) = S^{2a}(\sigma) \tag{5.5}$$

The situation is unchanged for the right-movers. We obtain a ground state which transforms as $\mathbf{8v}\oplus\mathbf{8s}$. For the left-movers, the ground state actually becomes tachyonic due to the twisted boundary conditions. This isn't a problem, because the ground state is projected out by the orbifold projection. The first left-moving excited state is given by:

$$S^a_{-1/2}\left|0\right\rangle \tag{5.6}$$

where the operator now carries a half-integer index due to the anti-symmetric boundary conditions. These states are massless. They transform as:

$$\mathbf{8s}\otimes(\mathbf{8v}\oplus\mathbf{8c}) \tag{5.7}$$

Altogether, we obtain the following massless spectrum[4]:

$$(\mathbf{8v}\oplus\mathbf{8s})\oplus(\mathbf{8v}\oplus\mathbf{8c}) \tag{5.8}$$

which is just the Type IIA massless spectrum derived in Chapter 3.

## 5.3 General features of orientifolds

The difference between orientifolds and orbifold is that an orientifold group can contain world sheet orientation reversal $\Omega$ as well as spacetime symmetries. We can write this as:

$$G = G_1 \cup \Omega G_2 \tag{5.9}$$

where $G_1$ and $G_2$ are spacetime symmetry groups. When $G_2$ is non-empty, $\Omega$ becomes a local gauge symmetry, so after the orientifold projection the theory only includes unoriented surfaces.

The set of points in spacetime which are left invariant under $G_2$ form the orientifold plane[2]. Like a Dp-brane, a p-dimensional orientifold plane (an Op-plane) may couple to an R-R $(p+1)$-form. This means that the orientifold plane may be charged with respect to those fields. If it is charged, then it acts as a source term for the field equations of the $(p+1)$-form $A_{p+1}$:

$$dH_{p+2} = \star J_{7-p} \tag{5.10}$$

$$d\star H_{p+2} = \star J_{p+1} \tag{5.11}$$

where $H_{p+2}$ is the field strength of $A_{p+1}$ and $J_n$ are the source terms. Consistency now requires that:

$$\int_{\Sigma_k} \star J_{10-k} = 0 \tag{5.12}$$

for all closed surfaces $\Sigma_k$. In a compact space, this is analogous to the fact that electric field lines coming from a charge must end on an opposite charge.[4] The only way to cancel the contribution of the Op-plane charge to the above integral is to add the appropriate number of Dp-branes carrying opposite charge.

## 5.4 Type I as an orientifold

Let's see how the Type I superstring emerges from an orientifold projection of Type IIB theory. The relevant symmetry in this case is world-sheet parity:

$$\Omega : \sigma \to -\sigma \tag{5.13}$$

The orientifold group is now isomorphic to $Z_2$, because we are not using any spacetime symmetries. This amounts to choosing the trivial group for $G_1$ and $G_2$. We've already examined the action of $\Omega$ upon the particle content of the Type IIB massless spectrum. In the NS-NS sector we keep the dilaton and the graviton. From the NS-R and R-NS sectors, we keep only the symmetric combination of gravitino fields $(\Psi^\mu_1 + \Psi^\mu_2)$ and one of the dilatinos. In the R-R sector we keep only the two-form $C_2$ and discard $C_0$ and $D_4$. We can now see this by counting degrees of freedom. The Type I theory is required to be supersymmetric, so we need to have an equal number of fermionic and bosonic degrees of freedom. The gravitino and dilatino, which are fermions, have $56+8=64$ combined degrees of freedom. This is equal to the number of combined degrees of freedom of the graviton, dilaton, and the two-form $C_2$, so we see that the above choice is correct. This is the content of the untwisted sector.

Because $G_2$ is the trivial group (consisting of only the identity element) in this case, the entire spacetime is invariant under its action. The orientifold plane becomes a 9-plane. It can couple to a 10-form R-R sector field (these do not appear after quantizing the spectrum because they are nondynamical). It can be shown that the orientifold plane carries -32 units of charge with respect to this field. We can cancel this by adding 32 D9-branes, each of which has unit charge. These 32 D9-branes carry a $U(32)$ massless gauge field on their world volume. After carrying out the orientifold projection, only an $SO(32)$ gauge symmetry remains. This is just what is necessary in order to make gauge anomalies disappear! [4]

Because the only massless R-R field in the Type I spectrum is $C_2$, the only stable Dp-branes (apart from the D9-branes) are the D1-brane (which couples to $C_2$ electrically) and the D5-brane (which couples magnetically). The Type I fundamental string itself does not couple to any field (the $B_2$ Kalb-Ramond field is discarded by the orientifold projection), so it isn't stable. Fundamental Type I strings are only long-lived at weak coupling. [2]

### 5.4.1 Type I'

We know that Type IIA on $\mathbb{R}^{8,1} \times S^1$ is T-dual to Type IIB on $\mathbb{R}^{8,1} \times \tilde{S}^1$, where $\tilde{S}^1$ indicates the dual circle. This suggests that taking an orientifold of Type IIA will yield a theory that is T-dual to Type I on $\mathbb{R}^{8,1} \times S^1$. This T-dual counterpart of Type I is known as Type I' or Type IA superstring theory.

Let's take a look at the T-dual description of Type I. The T-duality transformation in Type II theory may be expressed as:

$$X_R^9 \to -X_R^9, \quad \text{and} \quad \psi_R^9 \to -\psi_R^9 \tag{5.14}$$

$$X^9 = X_L^9 + X_R^9 \to \tilde{X}^9 = X_L^9 - X_R^9 \tag{5.15}$$

where $X^9$ lies along the circle $S^1$. In the Type I theory, world-sheet parity $\Omega$ is gauged. $\Omega$ exchanges left- and right-movers, so after the T-duality transformation it corresponds to

$$\tilde{X}^9 \to -\tilde{X}^9 \tag{5.16}$$

We have previously denoted this coordinate inversion as $I_9$. The gauging of $\Omega$ actually requires us to identify $\tilde{X}^9 \sim -\tilde{X}^9$, which describes an orbifold of the dual circle. As discussed in Chapter 3, we can combine $\Omega$ and $I_9$ into a symmetry of Type IIA. We can therefore form an orientifold group out of these two operations. T-duality now entails that the compactified Type IIB orientifold $(\mathbb{R}^{8,1} \times S^1)/\Omega$ is equivalent to the Type IIA orientifold $(R^{8,1} \times \tilde{S}^1)/\Omega \cdot I_9$. The set of points that are invariant under $I_9$ form two 8-dimensional hyperplanes located at $\tilde{X}^9 = 0$ and $\tilde{X}^9 = \pi\tilde{R}$, where $\tilde{R}$ is the radius of the dual circle. Each of these planes carries -8 units of R-R charge. This charge needs to be cancelled by adding 16 D8-branes along the circle. In the rest of the thesis, we will place 8 D8-branes at each of the two orientifold planes. This configuration is dual to Type I with a Wilson line breaking the gauge symmetry to $SO(16) \times SO(16)$. [5]

## 5.5 Heterotic-Type I S-duality

The field contents of Type I and $SO(32)$ heterotic supergravity are exactly the same. Moreover, their actions are mapped into each other by an inversion of the dilaton:

$$\phi \to -\phi \tag{5.17}$$

combined with a Weyl rescaling of the metric $g_{\mu\nu} \to e^{-\phi} g_{\mu\nu}$. Since the string coupling is given by $g_s = e^{-\phi}$, this suggests that the $SO(32)$ and Type I superstrings are related by S-duality. This is verified by a number of nonperturbative tests. [2] The heterotic-Type I S-duality has a very nice interpretation as a classical symmetry of M-theory compactified on a torus. This was discovered by Horava and Witten in 1995 [5]. Let's briefly outline their argument.

The low-energy limit of M-theory on the familiar $Z_2$ orbifold $\mathbb{R}^{9,1} \times S^1/Z_2$ is a ten-dimensional supergravity theory with $\mathcal{N} = 1$ chiral supersymmetry. There are three supergravities with this structure: Type I, $E_8 \times E_8$, and $SO(32)$ heterotic. The low-energy limit of M-theory on the orbifold must converge to one of these nodes. By examining the strong coupling limit, the $SO(32)$ and Type I supergravities may be ruled out. In the strong coupling limit, the compactification radius becomes very large. Far from the fixed hyperplanes of the orbifold, the strongly-coupled theory is indistinguishable from strongly-coupled Type IIA, which is just M-theory on $\mathbb{R}^{9,1} \times S^1$. We know that the strongly-coupled limits of Type I and $SO(32)$ heterotic are related to each other's weakly-coupled limit by S-duality. This rules out the possibility that M-theory on the orbifold reduces to $SO(32)$ heterotic or Type I. $E_8 \times E_8$ has (at the time Witten and Horava published their article) previously unknown strong coupling behaviour, so it can't be ruled out. This suggests that M-theory on $\mathbb{R}^{9,1} \times S^1/Z_2$ is $E_8 \times E_8$ superstring theory.

As discussed in the previous section, the T-dual of Type I superstring theory on $\mathbb{R}^{8,1} \times S^1$ is related to the Type IIA orientifold $R^{8,1} \times S^1/Z_2$, which is called Type I'. Because Type IIA is just M-theory on $\mathbb{R}^{9,1} \times S^1$, we hope that M-theory on $R^{8,1} \times S^1/Z_2 \times S^1$ is equivalent to Type I' on a $R^{8,1} \times S^1$. On the other hand, we have argued that M-theory on $\mathbb{R}^{8,1} \times S^1 \times S^1/Z_2$ is $E_8 \times E_8$ on $\mathbb{R}^{8,1} \times S^1$. From the M-theory perspective, it is quite transparent that these two theories should be dual to each other. We don't presently know what form this duality should take in the superstring theory. For starters, the two theories have completely different gauge groups. Witten and Horava argue that the natural solution would be to turn on a Wilson line in the $E_8 \times E_8$ theory, to break the gauge group to $SO(16) \times SO(16)$.[12] It is possible that the two theories are equivalent after turning on the Wilson line. This is not like any of the dualities we've seen before. We can relate it to a more familiar duality by T-dualizing both theories. The Type I' theory will turn into Type I on $R^{8,1} \times S^1$. The $E_8 \times E_8$ theory will turn into $SO(32)$ heterotic (with broken gauge symmetry $SO(16) \times SO(16)$ in each case). Now, we know that $SO(32)$ heterotic and Type I are S-dual to each other. The suggestion is that the M-theory symmetry of exchanging the circles manifests itself as the S-duality in the superstring theory. A simple calculation (carried out in the paper by Horava and Witten) confirms this.

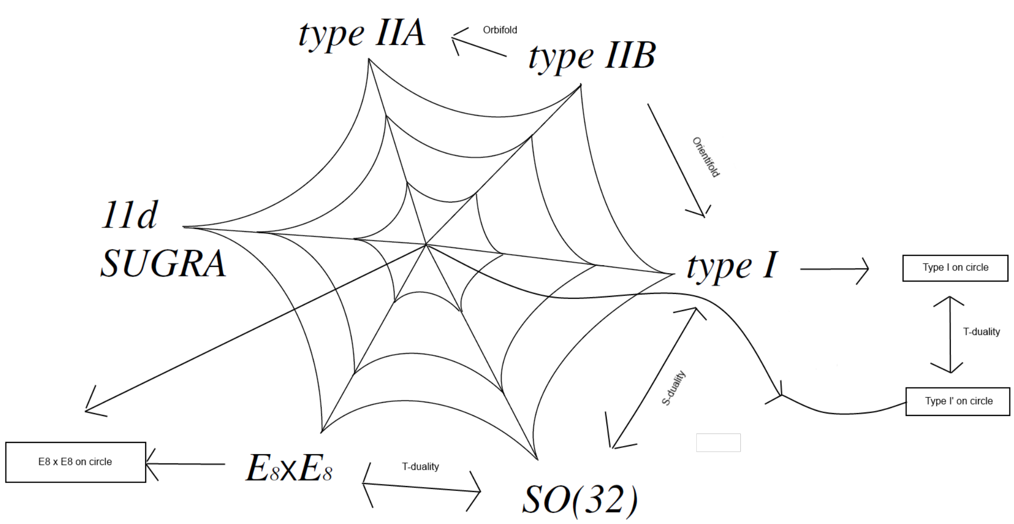
The exchange of $S^1$ and $S^1/Z_2$ may actually be implemented by an element of $SL(2,\mathbb{Z})$ acting upon the torus. Specifically, the element that takes

$$\tau_M \to -\frac{1}{\tau} \tag{5.18}$$

exchanges $S^1$ and $S^1/Z_2$. It inverts the string coupling, and exchanges $SO(32)$ heterotic string theory and Type I[6].

## 5.6 The web of dualities: part III

Let's update the web of dualities again. We have learned that Type IIA follows from Type IIB by a $Z_2$ orbifold of $(-1)^{F_L}$, which we already mentioned in Chapter 2. Likewise, Type I superstring theory is obtained from Type IIB by an orientifold of world-sheet parity. The T-dual theory of Type I, called Type I', is obtained by orientifolding with the combined operation of world-sheet parity and spacetime orientation reversal. We also know that Type I' on a circle is obtained from M-theory on $\mathbb{R}^{8,1} \times S^1 \times S^1/Z_2$. Similarly, $E_8 \times E_8$ is obtained from M-theory on $\mathbb{R}^{8,1} \times S^1/Z_2 \times S^1$. A T-duality transformation on both of these yields $SO(32)$ heterotic and Type I superstring theory, which are related by S-duality. Let's summarize with a picture:




[1]Remember that we are working with Type I' with 8 D8-branes located at each orientifold plane. This is the T-dual of Type I with a Wilson line breaking the gauge symmetry to $SO(16) \times SO(16)$. This has to be done because T-duality actually links $E_8 \times E_8$ with broken $SO(16) \times SO(16)$ symmetry to the $SO(32)$ string with broken $SO(16) \times SO(16)$ symmetry. The heterotic strings with full unbroken gauge symmetry are actually self-dual under T-duality.

[2]It should be clear that the difference implied by writing $\mathbb{R}^{8,1} \times S^1 \times S^1/Z_2$ instead of $\mathbb{R}^{8,1} \times S^1/Z_2 \times S^1$ lies in the ordering of applying the orbifold and the circular compactification.

# Chapter 6

# Savdeep Sethi's Superstring

## 6.1 A different orientifold

We are finally ready to introduce Savdeep Sethi's superstring. After all this introduction, it will seem like an almost disappointingly simple proposal. Let's reiterate some of our previous discussion. The $SL(2,\mathbb{Z})$ symmetry of Type IIB acts on the axion-dilaton field $\tau \equiv C_0 + \frac{i}{g_s} = C_0 + ie^{-\phi}$ as:

$$\tau \to \frac{a\tau + b}{c\tau + d} \tag{6.1}$$

The choice $(a, b, c, d) = (1, 1, 0, 1)$ satisfies $ad - bc = 1$ and requires us to identify:

$$C_0 \sim C_0 + 1 \tag{6.2}$$

Type I superstring theory is obtained by taking an orientifold with world-sheet parity $\Omega$ of Type IIB. The axion $C_0$ is odd under $\Omega$:

$$\Omega C_0 = -C_0 \tag{6.3}$$

so ordinarily the orientifold projection would require $C_0 = 0$. However, $C_0 = -\frac{1}{2}$ is identified with $C_0 = +\frac{1}{2}$ under $SL(2,\mathbb{Z})$. There is therefore a second possibility of orientifolding around $C_0 = \frac{1}{2}$! This is the new superstring.[6] There are several things that can go wrong at this point. Firstly, the proposed theory might end up being equivalent to Type I superstring theory. Secondly, the theory might be inconsistent. The last possibility is that the theory is consistent, but doesn't fit into the web of M-theory backgrounds. An example of such a theory is massive Type IIA supergravity. It will not be possible for us pick out one these possibilities with certainty. We will try to rule out the first possibility, by comparing the BPS spectra of Type I and the new string from the perspective of M-theory. Secondly, we will take a look at the strong-coupling limit of the new superstring, which appears quite different from Type I's strong-coupling limit (which is described by $SO(32)$'s weak-coupling limit).

## 6.2 M-theory BPS spectrum

Let's look at the M-theory description of the Type I superstring. The simplest way to do this is to first compactify on a circle and performing a T-duality transformation to Type I'. As we saw in the last chapter, this is described by M-theory on $\mathbb{R}^{8,1} \times S^1 \times S^1/Z_2$. The orientifold uses the combined action of world-sheet and spacetime parity, $\Omega$ and $I$. We can describe the two circles by using angular coordinates $(\theta_1, \theta_2)$. As we noted in Chapter 3, M-theory has a massless three-form $A_3$. This field is odd with respect to world-sheet parity. The coordinate $\theta_2$ is of course odd with respect to $I$. The orientifold produces two O8-planes corresponding to the set of points left invariant by its spacetime part. Their charge has to be cancelled by 16 D8-branes distributed across the circle.

Let's reiterate our discussion about the Type IIB/M-theory duality. A torus is specified using two periods $w_1$ and $w_2$. We can normalize one of these to unity by rescaling the torus metric. We now have the periods $w_1 = 1$ and $w_2 = \tau_M$. To obtain Type IIB theory, we have to identify $\tau_M$ with the axion-dilaton field expectation value $\tau_B$. We will just call both of these $\tau$ from now on. Moreover, we define:

$$\tau_1 \equiv \mathrm{Re}\tau \tag{6.4}$$

$$\tau_2 \equiv \mathrm{Im}\tau \tag{6.5}$$

The torus identifications have become:

$$z = \theta_1 + i\theta_2 \sim z + (\hat{n} + \hat{m}\tau) \tag{6.6}$$

with integer $(\hat{n}, \hat{m})$, and we are working with the torus metric:

$$ds^2 = \frac{A}{\tau_2} dz d\bar{z} \tag{6.7}$$

where $A$ is the area of the torus, which is given by:

$$R_B = \frac{\ell_p^3}{A} \tag{6.8}$$

where $\ell_p$ is the 11-dimensional Planck scale.

We can express the Kaluza-Klein spectrum coming from the supergravity multiplet as follows:

$$M^2_{(n,m)} = \frac{1}{A\tau_2}[n^2\tau_2^2 + (m - \tau_1 n)^2] \tag{6.9}$$

which is just the same formula we obtained in Chapter 4, but expressed in a different way. Each of these modes, of course, is BPS. From the Type IIB perspective, the interpretation of the $(n, m)$ mode is an $(n, m)$ string wrapped on a circle with radius $R_B$. This is just what we saw in Chapter 4.

Let's examine the action on the M-theory torus (with $\tau_1 = C_0 = 0$) of the orientifold that produces Type I' superstring theory:

$$\mathcal{I} : (\theta_1, \theta_2, A_3) \to (\theta_1, -\theta_2, -A_3) \tag{6.10}$$

Only states which are invariant under this quotient action survive. This forces states with Kaluza-Klein momenta $(0, m)$ to appear in combination with $(0, -m)$. From the Type I perspective, this is just the statement that there are no stable fundamental string winding modes, due to the absence of an NS-NS two-form.

The question is now how the new string appears in M-theory. What really needs to be done is to study the perturbative orientifold with $C_0 = \frac{1}{2}$ and deduce its behaviour under T-duality. Unfortunately, we are unable to do this. The best we can do is guess that the quotient action should be unchanged. The T-duality between Type I and Type I' is straightforward, so it is unlikely to be affected by the expectation value of $C_0$. This implies that $\mathcal{I}$ takes $\theta_2 \to \theta_2$, as before. The precise statement of the T-duality under consideration here is that Type I with broken $SO(16) \times SO(16)$ gauge symmetry is dual to Type IIA on an orientifold with 8 D8-branes coincident with each orientifold plane. Savdeep Sethi argues that the situation at the orientifold planes may be different for the new strings. We have to keep this in mind in what follows.

The torus identifications (6.6) are invariant under the quotient action for both $\tau_1 = 0$ and $\tau_1 = \frac{1}{2}$. The natural conclusion is that the second possibility corresponds to the new string. However, the quotient action on this new torus produces only a single boundary for an orientifold plane to be located.At the boundary we find only a single set of $SO(16)$ gauge bosons, whereas in the conventional Type I' case there are two sets located at the two boundaries. The implications of this are unclear. It could mean that the quotient action we have guessed is wrong. According to Sethi, it could mean that there is an alternate mechanism for anomaly cancelation.

The BPS spectrum changes considerably due to the introduction of $C_0 = \frac{1}{2}$. The Kaluza-Klein state combinations that can survive the quotient action are now given by:

$$|(n, m)\rangle \pm |(n, n - m)\rangle \tag{6.11}$$

We can see this from examining the requirement on real-valued wavefunctions (4.74). The quotient action acts as $y \to -y$ so a the second factor in the exponential is mapped according to:

$$n_2 \mathrm{Re}\tau_M - n_1 = \frac{1}{2} n_2 - n_1 \to -n_2 + n_1 \tag{6.12}$$

whereas the first factor is unchanged. This may be implemented by the mapping $(n_1, n_2) \to (n_1, n_1 - n_2)$. This mapping squares to 1, or in other words: $(n_1, n_1 - n_2) \to (n_1, n_2)$, so we see that (6.11) is correct. This implies that the surviving states have momenta given by $(2m, m)$. The spectrum is limited to:

$$M^2_{(2m,m)} = \frac{4m^2\tau_2}{A} \tag{6.13}$$

All of these states have mass larger than the D-string (the $(1, 0)$ string) wrapping modes in Type I.

The spectrum we have obtained is quite different from the Type I' case, but Savdeep Sethi argues that the collection of stable excitations is the same. In particular, the new string has the same low-energy limit supergravity theory as Type I, so its usefulness as a calculation tool is limited.

## 6.3 The strong coupling limit

Let's take a look at the strong coupling limit of the new string theory. The strong coupling limit of regular Type I theory is described by the $SO(32)$ heterotic string, by way of S-duality. In the previous chapter, we saw how this duality could be related to the classical symmetries of M-theory on a torus. In particular, the element of the modular group $SL(2,\mathbb{Z})$ that takes

$$\tau \to -\frac{1}{\tau} \tag{6.14}$$

exchanges $S^1$ and $S^1/Z_2$. From the 10-dimensional perspective, it inverts the string coupling and exchanges Type I string theory with $SO(32)$ heterotic string theory:

$$g_{s_I} \to g_{s_{\text{het}}} = \frac{1}{g_{s_I}} \tag{6.15}$$

When $\tau = \frac{1}{2}$, things are quite different. The $SL(2,\mathbb{Z})$ element that exchanges the circles no longer simply inverts the string coupling. Let's see if another element of the modular group can invert the string coupling. This is only possible with:

$$c\tau_1 + d = 0, \quad \tau_2 \to \frac{1}{c^2\tau_2} \tag{6.16}$$

An element of $SL(2,\mathbb{Z})$ that implements this is given by:

$$TST^2S = \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix} \tag{6.17}$$

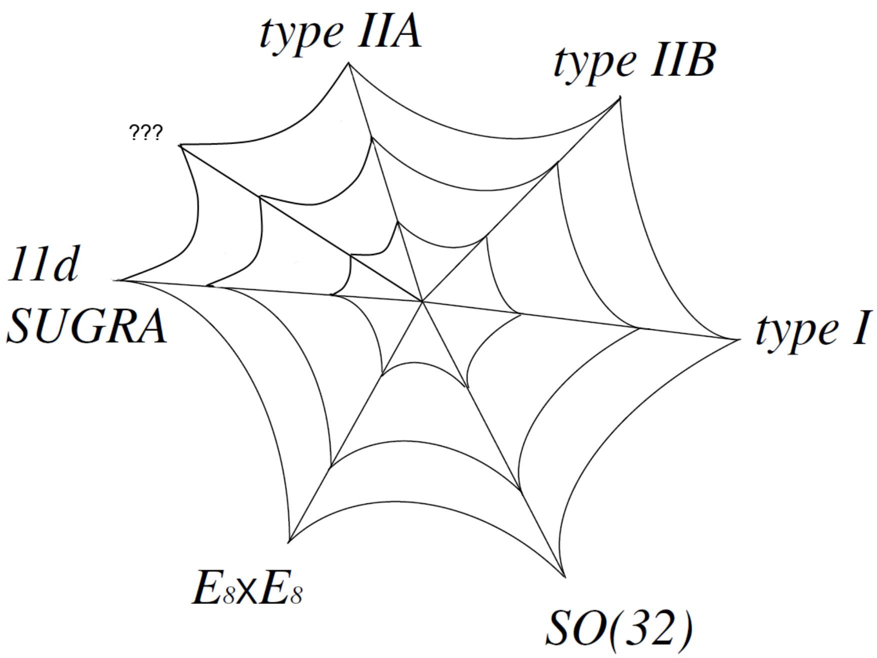
where the $SL(2,\mathbb{Z})$ generators are as we saw them before in Chapter 3:

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{6.18}$$

As required, this leaves $\tau_1$ invariant and maps a small value of the coupling $\tau_2$ to a large value of $\tau_2$: $\tau_2 \to \frac{1}{4\tau_2}$. The change of basis we've considered here is not a symmetry of the M-theory description due to the orbifold. It is a mapping between two possibly distinct theories, one with large $\tau_2$ and one with small $\tau_2$. The factor of 4 in the $\tau$ transformation fits nicely with the BPS spectrum. Savdeep Sethi argues that this suggests a light closed string provides the strong coupling limit description.

# Conclusion

We have seen how the new superstring emerges from an orientifold of Type IIB theory. The new superstring is by construction very similar to the Type I superstring, so we looked for ways in which to distinguish the two from each other. We found quite distinct Kaluza-Klein spectra, and the strong-coupling limit appeared to be different as well. Sethi actually presents another argument, from K-theory, that the theories are distinct, but we have not developed the necessary background to discuss this. All in all, we have gathered some strong evidence that the new superstring is in fact distinct, but we haven't proved this. There was, for instance, some uncertainty about whether we'd guessed the correct quotient action to obtain the BPS spectrum. Moreover, when we outlined Sethi's proposal, we mentioned that three things could go wrong: the theory might be equivalent to Type I (which we've addressed), the theory might be inconsistent, or the theory might not fit into the M-theory web (even though it is consistent). We have not addressed the latter two points at all. Let's still be cautiously optimist and update the web of dualities with a new spoke labeled with a question mark, representing our uncertainty as well as the fact that Sethi never proposes a name for the new string:




## Acknowledgements

I would like to thank my supervisor Diederik Roest and Andrea Borghese for their guidance and for proposing the subject.

# Appendix A

# Review of spinors

We review some facts about spinors, focusing on even dimensions $d = 2 + 2k$. This information is adapted from Green, Schwarz, Witten[8] and Wipf[7].

## A.1 Gamma matrices in d dimensions

The Clifford algebra (or Dirac algebra) is the algebra generated by the elements $\Gamma^0, ..., \Gamma^{d-1}$, satisfying the anticommutation relations:

$$\Gamma^\mu \Gamma^\nu + \Gamma^\nu \Gamma^\mu \equiv \{\Gamma^\mu, \Gamma^\nu\} = 2\eta_{\mu\nu} \tag{A.1}$$

where $\eta_{\mu\nu}$ is the Minkowski signature. We can give explicit representations of this algebra in terms of the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{A.2}$$

Let's construct the representation. For even (and $n = 0$) matrices $Gamma^n$ we find:

$$\Gamma^0 = \sigma_1 \otimes \sigma_0 \otimes \sigma_0 \otimes ... \tag{A.3}$$

$$\Gamma^2 = i\sigma_3 \otimes \sigma_1 \otimes \sigma_0 \otimes ... \tag{A.4}$$

$$\Gamma^4 = i\sigma_3 \otimes \sigma_3 \otimes \sigma_1 \otimes ... \tag{A.5}$$

and for odd matrices we find:

$$\Gamma^1 = i\Gamma^2 \otimes \Gamma^0 \otimes \Gamma^0 \otimes ... \tag{A.6}$$

$$\Gamma^3 = i\sigma_3 \otimes \sigma_2 \otimes \sigma_0 \otimes ... \tag{A.7}$$

For even dimensions, this representation is $2^{[d/2]}$-dimensional. It is unique up to a unitary transformation by $U$[1]:

$$\Gamma^\mu \to U\Gamma^\mu U^{-1} \tag{A.8}$$

In even dimensions, the anti-symmetrized products of the gamma-matrices provide a complete set of $2^{[d/2]} \times 2^{[d/2]}$ matrices:

$$\Gamma^{\mu_1 \cdots \mu_n} = \Gamma^{[\mu_1}\Gamma^{\mu_2}...\Gamma^{\mu_n]} \tag{A.9}$$

## A.2 Spinors of $SO(1, d-1)$

Define a set of raising and lowering operators as:

$$\Gamma^{0\pm} = \frac{1}{2}(\pm\Gamma^0 + \Gamma^1) \tag{A.10}$$

$$\Gamma^{a\pm} = \frac{1}{2}(\Gamma^{2a} \pm i\Gamma^{2a+1}), \quad a = 1, 2, ..., k \tag{A.11}$$

These satisfy the following anticommutation relations:

$$\{\Gamma^{a+}, \Gamma^{b-}\} = \delta^{ab} \tag{A.12}$$

[1] For odd dimensions, there is a second representation that is not unitary equivalent

$$\{\Gamma^{a-},\Gamma^{b-}\} = \{\Gamma^{a+},\Gamma^{b+}\} = 0 \tag{A.13}$$

Since $(\Gamma^{a\pm})^2 = 0$, there is a state $\xi$ such that $\Gamma a{-}\xi = 0$ for all $a$. Given such a state, we can construct $2^{k+1}$ states by acting with $\Gamma^{a+}$ 0 or one times for all values of $a$. These $2^{k+1}$ states form a Dirac spinor $|s\rangle$:

$$|s\rangle = |s_0, s_1, ..., s_k\rangle = (\Gamma^{k+})^{s_k+1/2} \cdot ... \cdot (\Gamma^{0+})^{s_0+1/2}\dot{\xi} \tag{A.14}$$

where $s_j = \pm\frac{1}{2}$ for all $j$.

The Dirac spinor furnishes a representation of $SO(1,d-1)$. To see this, define the combinations:

$$\Sigma^{\mu\nu} \equiv \frac{1}{2i}\Gamma^{\mu\nu} \tag{A.15}$$

$$S_a \equiv i^{\delta_{a,0}}\Sigma^{2a,2a+1} = \Gamma^{a+}\Gamma^{a-} - \frac{1}{2} \tag{A.16}$$

The Dirac spinors furnish a representation of the $SO(1,d-1)$ Lorentz algebra, given by:

$$S_a\,|s\rangle = s_a\,|s\rangle \tag{A.17}$$

In even dimensions, a Dirac spinor is not, in general, irreducible. We can decompose it into two Weyl spinors of opposite chirality. Let's define the chirality matrix:

$$\Gamma = i^{-k}\Gamma^0\Gamma^1...\Gamma^{d-1} \tag{A.18}$$

If the Dirac spinor has an even number of $s_a$ equal to $\frac{1}{2}$, the chirality is positive:

$$\Gamma\,|s\rangle = 1, \quad \text{even number of } s_a = \frac{1}{2} \tag{A.19}$$

If it has an odd number of $s_a$, the chirality is negative:

$$\Gamma\,|s\rangle = -1, \quad \text{odd number of } s_a = \frac{1}{2} \tag{A.20}$$

A Dirac spinor of definite chirality is also called a Weyl spinor. In dimensions $d$ satisfying $d = 2\text{mod}8$, a Majorana condition can be applied at the same time. For our purposes, a Majorana spinor is a spinor with real components.

## A.3 Spin transformations

Let's write down the commutation relations of the $\Sigma$:

$$[\Sigma_{\mu\nu},\Sigma_{\rho\sigma}] = i(\eta_{\mu\rho}\Sigma_{\nu\sigma} + \eta_{\nu\sigma}\Sigma_{\mu\rho} - \eta_{\mu\sigma}\Sigma_{\nu\rho} - \eta_{\nu\rho}\Sigma_{\mu\sigma}) \tag{A.21}$$

We see again that $\Sigma$ furnish a representation of the Lorentz algebra. It is called the spin representation. Let's label a family of transformations by $S(s)$:

$$S(s) = e^{i\frac{s}{2}(\omega,\Sigma)}, \quad (\omega,\Sigma) = \omega_{\mu\nu}\Sigma^{\mu\nu} \tag{A.22}$$

with initial value $S(0) = \not{1}$. It can be shown that:

$$S^{-1}(s)\Gamma^\rho S(s) = (e^{s\omega})^\rho_\sigma\Gamma^\Sigma \tag{A.23}$$

With antisymmetry $(\omega_{\mu\nu})$, the matrix $(e^\omega)^\rho_\sigma$ represents a Lorentz transformation. So we can rewrite:

$$S^{-1}\Gamma^\rho S = \Lambda^\rho{}_\sigma\Gamma^\sigma \tag{A.24}$$

From the Lorentz covariance of the Dirac equation, it can be shown that a Dirac spinor transforms as:

$$\psi(x) \to S\psi(\Lambda^{-1}x) \tag{A.25}$$

We say that the spinor transforms with the spin-transformation $S$. We can compare this to the Lorentz transformations on vector fields $A^\mu(x)$ and scalar fields $\phi(x)$:

$$\phi(x) \to \phi(\Lambda^{-1}x), \quad A^\mu(x) \to \Lambda^\mu{}_\nu A^\nu(\Lambda^{-1}x) \tag{A.26}$$

## A.4 Fierz identity

The matrix $\Gamma^0 = (\Gamma^0)^{-1}$ conjugates the $\Gamma$ and $\Sigma$ matrices into their adjoints:

$$\Gamma^0\Gamma^\mu\Gamma^0 = \Gamma^{\mu\dagger}, \quad \Gamma^0\Sigma_{\mu\nu}\Gamma^0 = \Gamma^\dagger_{\mu\nu} \tag{A.27}$$

This implies that $\Gamma^0$ conjugates the adjoint of $S$ into the inverse of $S$:

$$\Gamma^0 S^\dagger \Gamma^0 = \Gamma^0 e^{-\frac{i}{2}(\omega,\Sigma)^\dagger}\Gamma^0 = S^{-1} \tag{A.28}$$

We now define conjugate spinors as:

$$\bar{\psi} \equiv \psi^\dagger\Gamma^0 \tag{A.29}$$

It transforms with the inverse spin rotation under Lorentz transformations:

$$\bar{\psi} \to (S\psi)^\dagger\Gamma^0 = \bar{\psi}S^{-1} \tag{A.30}$$

We can see that the following objects $A^{\mu_1\cdots\mu_n}$ are antisymmetric tensors:

$$A^{\mu_1\cdots\mu_n} \equiv \bar{\psi}\Gamma^{\mu_1\cdots\mu_n}\psi \tag{A.31}$$

This follows from the transformation properties of $\psi$ and $\bar{\psi}$:

$$A^{\mu_1\cdots\mu_n} \to \bar{\psi}S^{-1}\Gamma^{\mu_1\cdots\mu_n}S\psi \tag{A.32}$$

Noting that $S^{-1}\Gamma^\rho S = \Lambda^\rho{}_\sigma\Gamma^\sigma$, we can shuffle the $S^{-1}$ to the right through all the terms in the antisymmetrized product to find:

$$\bar{\psi}S^{-1}\Gamma^{\mu_1\cdots\mu_n}\psi = \Lambda^{\mu_1}_{\nu_1}...\Lambda^{\mu_n}_{\nu_n}A^{\nu_1\cdots\nu_n} \tag{A.33}$$

which establishes that the objects $A^{\mu_1\cdots\mu_n}$ are antisymmetric tensors.

Any bilinear object $\bar{\psi}\chi$ may actually be decomposed in terms of such antisymmetric tensors. This is because the gamma-matrices $\Gamma^\mu$ in $d$ dimensions provide an orthonormal basis for the linear space $\mathrm{Mat}(\Delta, \mathbb{Z})$, which is the space of all $\Delta = 2^{[d/2]}$ matrices with complex components. This means that the matrix $M$ given by $M_\alpha{}^\beta = \psi_\alpha\bar{\chi}^\beta$ may be decomposed according to:

$$\psi\bar{\chi} = -\frac{1}{\Delta}\sum_n \frac{1}{n!}(-1)^{n(n-1)/2}\Gamma_{\mu_1\ldots\mu_n}(\bar{\chi}\Gamma^{\mu_1\cdots\mu_n}\psi) \tag{A.34}$$

This is called the Fierz identity. We used it fact in Chapter 3 to decompose direct products such as $\mathbf{8c}\otimes\mathbf{8s}$ into antisymmetric tensor irreps of $SO(8)$, which was crucial in determining the field content of the Type II massless spectrum.

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