Euler's collinear solution to three-body problem in GR

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Abstract

The three-body problem is reexamined in the framework of general relativity. The Newtonian three-body problem admits *Euler's collinear solution*, where three bodies move around the common center of mass with the same orbital period and always line up. The solution is unstable. Hence it is unlikely that such a simple configuration would exist owing to general relativistic forces dependent not only on the masses but also on the velocity of each body. However, we show that the collinear solution remains true with a correction to the spatial separation between masses.

1 Euler's collinear solution in the Newton gravity

The location of each mass M_I $(I = 1, 2, 3)$ is written as $\boldsymbol{X}_I \equiv (x_I, 0)$. Without loss of generality, we assume $x_3 < x_2 < x_1$. Let R_I define the relative position of each mass with respective to the center of mass $\boldsymbol{X}_G \equiv (x_G, 0)$, namely $R_I \equiv x_I - x_G$ ($R_I \neq |\boldsymbol{X}_I|$ unless $x_G = 0$). We choose $x = 0$ between M_1 and M_3. We thus have $R_3 < R_2 < R_1$, $R_3 < 0$ and $R_1 > 0$.

It is convenient to define an important ratio as $R_{23}/R_{12} = z$. Then we have $R_{13} = (1 + z)R_{12}$. The equation of motion in Newton gravity becomes

$$R_1\omega^2 = \frac{M_2}{R_{12}^2} + \frac{M_3}{R_{13}^2}, \tag{1}$$

$$R_2\omega^2 = -\frac{M_1}{R_{12}^2} + \frac{M_3}{R_{23}^2}, \tag{2}$$

$$R_3\omega^2 = -\frac{M_1}{R_{13}^2} - \frac{M_2}{R_{23}^2}, \tag{3}$$

where we define

$$\boldsymbol{R}_{IJ} \equiv \boldsymbol{X}_I - \boldsymbol{X}_J, \tag{4}$$

$$R_{IJ} \equiv |\boldsymbol{R}_{IJ}|. \tag{5}$$

First, we subtract Eq. (2) from Eq. (1) and Eq. (3) from Eq. (2) and use $R_{12} \equiv |\boldsymbol{X}_1 - \boldsymbol{X}_2|$ and $R_{23} \equiv |\boldsymbol{X}_2 - \boldsymbol{X}_3|$. Next, we compute a ratio between them to delete ω^2. Hence we obtain a fifth-order equation as [1]

$$(M_1 + M_2)z^5 + (3M_1 + 2M_2)z^4 + (3M_1 + M_2)z^3 - (M_2 + 3M_3)z^2 - (2M_2 + 3M_3)z - (M_2 + M_3) = 0. \tag{6}$$

Now we have a condition as $z > 0$. Descartes' rule of signs : the number of positive roots either equals to that of sign changes in coefficients of a polynomial or less than it by a multiple of two. According to this rule, Eq. (6) has the only positive root $z > 0$, though such a fifth-order equation cannot be solved in algebraic manners as shown by Galois. After obtaining z, one can substitute it into a difference, for instance between Eqs. (1) and (3). Hence we get ω.

2 What happens in GR ?

2.1 The EIH equation of motion for a many-body system

In order to include the dominant part of general relativistic effects, we take account of the terms at the first post-Newtonian order. Namely, the massive bodies obey the Einstein-Infeld-Hoffman (EIH) equation

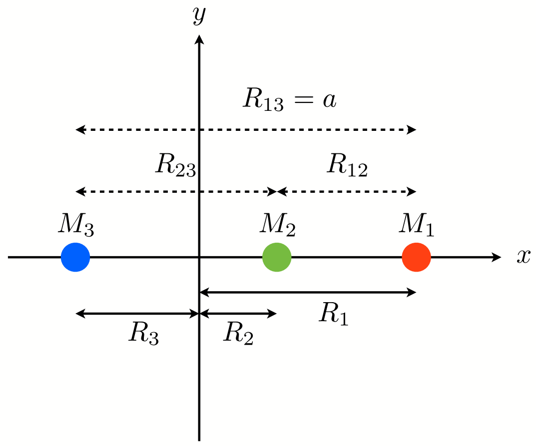

Figure 1: Schematic figure for a classical configuration of three masses denoted by M_1 , M_2 and M_3.

of motion as [2, 3]

$$
\begin{aligned}
\frac{d^2\boldsymbol{r}_K}{dt^2} &= \sum_{A\neq K} \boldsymbol{r}_{AK}\frac{1}{r_{AK}^3}\Bigg[1-4\sum_{B\neq K}\frac{1}{r_{BK}} - \sum_{C\neq A}\frac{1}{r_{CA}}\left(1-\frac{\boldsymbol{r}_{AK}\cdot\boldsymbol{r}_{CA}}{2r_{CA}^2}\right) \\
&\qquad +\boldsymbol{v}_K^2 + 2\boldsymbol{v}_A^2 - 4\boldsymbol{v}_A\cdot\boldsymbol{v}_K - \frac{3}{2}\left(\frac{\boldsymbol{v}_A\cdot\boldsymbol{r}_{AK}}{r_{AK}}\right)^2\Bigg] \\
&\quad - \sum_{A\neq K}(\boldsymbol{v}_A-\boldsymbol{v}_K)\frac{\boldsymbol{r}_{AK}\cdot(3\boldsymbol{v}_A-4\boldsymbol{v}_K)}{r_{AK}^3} \\
&\quad +\frac{7}{2}\sum_{A\neq K}\sum_{C\neq A}\boldsymbol{r}_{CA}\frac{1}{r_{AK}r_{CA}^3}.
\end{aligned}
$$

2.2 The seventh-order equation

Similarly to the above Newtonian case, we obtain a seventh-order equation as [4]

$$
F(z) \equiv \sum_{k=0}^{7} A_k z^k = 0, \tag{7}
$$

where we define the mass ratio as $\nu_I \equiv M_I/M$ for $M \equiv \sum_I M_I$ and

$$
\begin{aligned}
A_7 &= \frac{M}{a}\left[-4-2(\nu_1-4\nu_3)+2(\nu_1^2+2\nu_1\nu_3-2\nu_3^2)-2\nu_1\nu_3(\nu_1+\nu_3)\right], \\
A_6 &= 1-\nu_3+\frac{M}{a}\Big[-13-(10\nu_1-17\nu_3)+2(2\nu_1^2+8\nu_1\nu_3-\nu_3^2) \\
&\qquad +2(\nu_1^3-2\nu_1^2\nu_3-3\nu_1\nu_3^2-\nu_3^3)\Big],
\end{aligned}
$$

$$\begin{aligned} A_5 &= 2+\nu_1-2\nu_3+\frac{M}{a}\left[-15-(18\nu_1-5\nu_3)+4(5\nu_1\nu_3+4\nu_3^2)\right. \\ &\qquad \left.+6(\nu_1^3-\nu_1\nu_3^2-\nu_3^3)\right], \\ A_4 &= 1+2\nu_1-\nu_3+\frac{M}{a}\left[-6-2(5\nu_1+2\nu_3)-4(2\nu_1^2-\nu_1\nu_3-4\nu_3^2)\right. \\ &\qquad \left.+2(3\nu_1^3+\nu_1^2\nu_3-2\nu_1\nu_3^2-3\nu_3^3)\right], \end{aligned}$$

$$\begin{aligned} A_3 &= -(1-\nu_1+2\nu_3)+\frac{M}{a}\left[6+2(2\nu_1+5\nu_3)+4(-4\nu_1^2-\nu_1\nu_3+2\nu_3^2)\right. \\ &\qquad \left.-2(-3\nu_1^3-2\nu_1^2\nu_3+\nu_1\nu_3^2+3\nu_3^3)\right], \\ A_2 &= -(2-2\nu_1+\nu_3)+\frac{M}{a}\left[15+(-5\nu_1+18\nu_3)-4(4\nu_1^2+5\nu_1\nu_3)\right. \\ &\qquad \left.-6(-\nu_1^3-\nu_1^2\nu_3+\nu_3^3)\right], \end{aligned}$$

$$\begin{aligned} A_1 &= -(1-\nu_1)+\frac{M}{a}\left[13+(-17\nu_1+10\nu_3)-2(-\nu_1^2+8\nu_1\nu_3+2\nu_3^2)\right. \\ &\qquad \left.-2(-\nu_1^3-3\nu_1^2\nu_3-2\nu_1\nu_3^2+\nu_3^3)\right], \\ A_0 &= \frac{M}{a}\left[4+2(-4\nu_1+\nu_3)-2(-2\nu_1^2+2\nu_1\nu_3+\nu_3^2)+2\nu_1\nu_3(\nu_1+\nu_3)\right]. \end{aligned}$$

This seventh-order equation is symmetric for exchanges between ν_1 and ν_3, only if one makes a change $z \to 1/z$. This symmetry seems to validate the complicated form of each coefficient.

Figure 2 shows a numerical example for $M_1 : M_2 : M_3 = 1 : 2 : 3$, $R_{12} = 1$ and $a/M = 100$, where the post-Newtonian correction is of the order of one percent. In this figure, we employ the inertial frame $(\bar{x}, \bar{y})$ but not the corotating frame (x, y). We assume $x_3 < x_2 < x_1$ throughout this paper. This figure suggests that as an alternative initial condition we can assume $x_1 < x_2 < x_3$, which is realized at $t = T/2$ (T=orbital period) in this figure. It is natural that this is a consequence of the parity symmetry in our formulation. It should be noted also that the location of each mass at $t = T/2$ is advanced compared with that at $t = T_N/2$ (a half of the *Newtonian* orbital period). This may correspond to the periastron advance (in circular orbits).

Finally, we focus on the restricted three-body problem so that we can put $z = z_N(1+\varepsilon)$ for the Newtonian root z_N. Substitution of this into Eq. (7) gives the post-Newtonian correction as

$$\varepsilon = -\frac{\sum_k A_{PNk} z_N^k}{\sum_k k A_{Nk} z_N^k}, \tag{8}$$

where A_{Nk} and A_{PNk} denote the Newtonian and post-Newtonian parts of A_k, respectively. For a binary system of comparable mass stars, the correction ε is $O(M/a)$. This implies that a corrected length is of the order of the Schwarzschild radius.

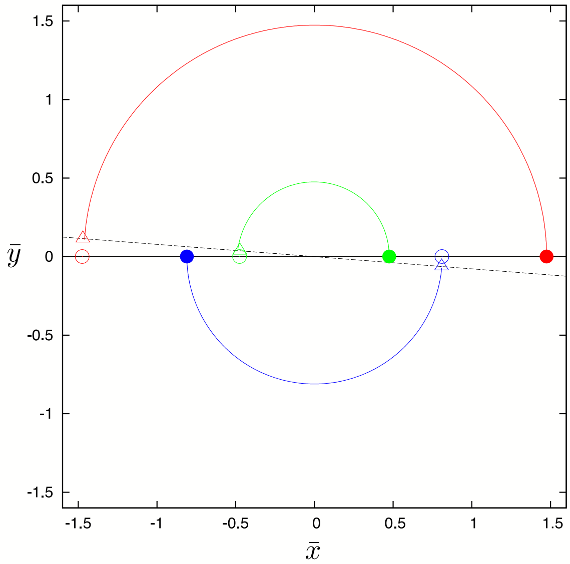

Fig 2: $M_1 : M_2 : M_3 = 1 : 2 : 3$, $a/M = 100$

For the Sun-Jupiter system, general relativistic corrections to L_1, L_2 and L_3 become +30, −38, +1 [m], respectively, where the positive sign is chosen along the direction from the Sun to the Jupiter. Such corrections suggest a potential role of the general relativistic three (or more) body dynamics for high precision astrometry in our solar system and perhaps also for gravitational waves astronomy.

3 Conclusion

We obtained a general relativistic version of Euler's collinear solution for the three-body problem at the post-Newtonian order [4]. Studying global properties of the seventh-order equation that we have derived is left as future work.

It is interesting also to include higher post-Newtonian corrections, especially 2.5PN effects in order to elucidate the secular evolution of the orbit due to the gravitational radiation reaction at the 2.5PN order. One might see probably a shrinking collinear orbit as a consequence of a decrease in the total energy and angular momentum, if such a radiation reaction effect is included. This is a testable prediction.

It may be important also to search other solutions, notably a relativistic counterpart of the Lagrange's triangle solution (so-called L_4 and L_5 in the restricted three-body problem). Clearly it seems much more complicated to obtain relativistic corrections to the Lagrange orbit.

References

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