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Embedding of dendriform algebras into Rota–Baxter algebras

Research Article

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Abstract: Following a recent work [Bai C., Bellier O., Guo L., Ni X., Splitting of operations, Manin products, and Rota–Baxter operators, Int. Math. Res. Not. IMRN (in press), DOI: 10.1093/imrn/rnr266] we define what is a dendriform di- or trialgebra corresponding to an arbitrary variety Var of binary algebras (associative, commutative, Poisson, etc.). We call such algebras di- or tri-Var-dendriform algebras, respectively. We prove in general that the operad governing the variety of di- or tri-Var-dendriform algebras is Koszul dual to the operad governing di- or trialgebras corresponding to $\mathrm{Var}^!$. We also prove that every di-Var-dendriform algebra can be embedded into a Rota–Baxter algebra of weight zero in the variety Var, and every tri-Var-dendriform algebra can be embedded into a Rota–Baxter algebra of nonzero weight in Var.

MSC: 18D50, 17B69

Keywords: Dendriform algebra • Dialgebra • Trialgebra • Rota–Baxter operator • Operad • Manin product • Conformal algebra

1. Introduction

In [7], Glen Baxter introduced an identity defining what is now called Rota–Baxter operator in developing works of F. Spitzer [36] in fluctuation theory. By definition, a Rota–Baxter operator R of weight λ on an algebra A is a linear map on A such that

$$R(x)R(y) = R\big(xR(y) + R(x)y\big) + \lambda R(xy), \qquad x, y \in A,$$

where λ is a scalar from the base field.

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Later, commutative associative algebras with such an operator were studied by G.-C. Rota and P. Cartier [10, 34]. In 1980s, these operators appeared in the context of Lie algebras independently in works A.A. Belavin and V.G. Drinfeld [8] and M.A. Semenov-Tian-Shansky [35] in research on solutions of classical Young–Baxter equation named in the honour of Chen Ning Yang and Rodney Baxter.

For the present time, numerous connections of Rota–Baxter operators with different areas of mathematics (Young–Baxter equations, operads, Hopf algebras, number theory, etc.) can be found [2, 17, 22, 26]. Also, there is a relation between Rota–Baxter operators and quantum field theory [13, 14].

The notion of a Leibniz algebra introduced by J.-L. Loday [27] originates from cohomology theory of Lie algebras; this is a noncommutative analogue of Lie algebras. Associative dialgebras (now often called diassociative algebras) emerged in the paper by J.-L. Loday and T. Pirashvili [29], they play the role of universal enveloping associative algebras for Leibniz algebras. In [24], a general notion of a dialgebra corresponding to an arbitrary variety Var of "ordinary" algebras (such as associative, alternative, etc.) was introduced (hereinafter, we refer to them as to di-Var-algebras).

Later, J.-L. Loday and M. Ronco [30] introduced a generalization of dialgebras — trialgebras (in the associative case). In this paper, we generalize the definition from [24] to the case of trialgebras. Given a variety Var of Ω-algebras defined by poly-linear identities, we define a corresponding variety called tri-Var-algebras. By a dialgebra (or trialgebra) we mean a di- (or tri-)Var-algebra for some Var.

Dendriform dialgebras were defined by J.-L. Loday [28] in his study of algebraic K-theory. Moreover, they occur to be Koszul-dual to di-As-algebras, where As is the variety of associative algebras. Dendriform trialgebras introduced in [30] are proved to be Koszul-dual to tri-As-algebras. In this paper, we determine what is a di- or tri-Var-dendriform algebra, following [3], for a given variety Var as above. By a dendriform dialgebra (or dendriform trialgebra) we mean a di- (or tri-) Var-dendriform algebra for some Var. The term "dendriform algebra" will stand for either dendriform dialgebra or dendriform trialgebra in contrast to previous works in this topic, where the term "dendriform algebra" means the same as "di-As-dendriform algebra" in this paper. Also, the terms "dendriform trialgebra" or "tridendriform algebra" were used for what we call "tri-As-dendriform algebra".

Dendriform dialgebras (trialgebras) are linear spaces with two (three) operations $\succ, \prec$ (and $\cdot$). For their Koszul duals (dialgebras and trialgebras) their operations usually are denoted by $\vdash, \dashv$, and $\perp$. In this paper, we prefer the latter notations for dendriform structures instead of traditional $\succ, \prec$, and $\cdot$ since our combinatorial approaches to definitions of corresponding varieties are very much similar.

M. Aguiar in [1] was the first who noticed a relation between Rota–Baxter algebras and dendriform algebras. He proved that an associative algebra with a Rota–Baxter operator R of weight zero relative to operations $a \dashv b = aR(b)$, $a \vdash b = R(a)b$ is a di-As-dendriform algebra. Later K. Ebrahimi-Fard [15] generalized this fact to the case of Rota–Baxter algebras of arbitrary weight and obtained as a result both di- and tri-As-dendriform algebras. In the paper by K. Ebrahimi-Fard and L. Guo [18], universal enveloping Rota–Baxter algebras of weight λ for di- and tri-As-dendriform algebras were defined.

The natural question: Whether an arbitrary dendriform algebra can be embedded into its universal enveloping Rota–Baxter algebra of appropriate weight was solved positively in [18] for free dendriform algebras only. Y. Chen and Q. Mo [12] proved that any di-As-dendriform algebra over a field of characteristic zero can be embedded into an appropriate Rota–Baxter algebra of weight zero using the Gröbner–Shirshov bases technique for Rota–Baxter algebras developed in [9].

Also, C. Bai, L. Guo and K. Ni [4] introduced the notion of an $\mathcal{O}$-operator, a generalization of Rota–Baxter operator, and proved that every dendriform algebra can be explicitly obtained from an algebra with an $\mathcal{O}$-operator.

In a recent paper [3], the results of Aguiar and Ebrahimi-Fard were extended to the case of arbitrary operad of Rota–Baxter algebras and dendriform algebras.

In the present work, we completely solve the embedding problem for dendriform algebras: For every di-Var-dendriform algebra A there exists an algebra $B \in$ Var with a Rota–Baxter operator R of weight zero such that A is embedded into B in the sense of [1]. For every tri-Var-dendriform algebra A there exists an algebra $B \in$ Var with a Rota–Baxter operator R of nonzero weight such that A is embedded into B in the sense of [15].

The idea of the construction can be easily illustrated by the following example. Suppose $(A, \dashv, \vdash, \perp)$ is a tri-As-dendriform algebra. Then the direct sum of two isomorphic copies of A, the space $\widehat{A} = A \oplus A'$, endowed with a binary operation

$$a * b = a \dashv b + a \vdash b + a \perp b, \qquad a * b' = (a \vdash b)', \qquad a' * b = (a \dashv b)', \qquad a' * b' = (a \perp b)',$$

$a, b \in A$, is an associative algebra. Moreover, the map $R(a') = a$, $R(a) = -a$ is a Rota–Baxter operator of weight 1 on $\widehat{A}$. The embedding of A into $\widehat{A}$ is given by $a \mapsto a'$, $a \in A$.

In the last section, we introduce and explore a modification of the notion of a trialgebra from [30] which we call a generalized trialgebra (or g-trialgebra, for short). For every variety Var of binary algebras defined by poly-linear identities we define a corresponding variety of g-tri-Var-algebras. This class of systems naturally appears from differential and Γ-conformal algebras, "discrete analogues" of conformal algebras introduced in [20]. The class of g-tri-As-algebras is related with a natural noncommutative analogue of Poisson algebras. The free g-tri-Com-algebra generated by a set X is isomorphic as a linear space to the free Perm-algebra generated by the algebra of polynomials in X. The operad gComTrias governing the variety of g-tri-Com-algebras gives rise to the operads governing the varieties of g-tri-As- and g-tri-Lie-algebras by means of the Manin white product with operads As and Lie, respectively.

Throughout the paper, we identify the notations for a variety of algebras and for the corresponding operad.

2. Operads for di- and trialgebras

Our main object of study is the class of dendriform algebras. In this section, we start with objects from the "dual world" in the sense of Koszul duality.

The notion of an operad once introduced in [32] has had a renaissance since the beginning of 2000s. We address the reader to either of perfect expositions of this notion and its applications in universal algebra, e.g., [19, 25, 31, 37].

Throughout the paper, $\Bbbk$ is an arbitrary base field. All operads are assumed to be families of linear spaces, compositions are linear maps, and the actions of symmetric groups are also linear. By an Ω-algebra we mean a linear space equipped with a family of binary linear operations $\Omega = \{\circ_i : i \in I\}$. Denote by $\mathcal{F}$ the free operad governing the variety of all Ω-algebras. For every natural number $n > 1$, the space $\mathcal{F}(n)$ can be identified with the space spanned by all binary trees with n leaves marked by $x_1, \dots, x_n$, where each vertex (which is not a leaf) has a label from Ω.

Let Var be a variety of Ω-algebras defined by a family S of poly-linear identities of any degree (which is greater than one). An operad governing the variety Var is also denoted by Var. Every algebra from this variety is a functor from Var to Vec, the multi-category of linear spaces with poly-linear maps.

Denote by $\Omega^{(2)}$ and $\Omega^{(3)}$ the sets of binary operations $\{\vdash_i, \dashv_i : i \in I\}$ and $\Omega^{(2)} \cup \{\perp_i : i \in I\}$, respectively. Similarly, let $\mathcal{F}^{(2)}$ and $\mathcal{F}^{(3)}$ stand for the free operads governing the varieties of all $\Omega^{(2)}$- and $\Omega^{(3)}$-algebras, respectively.

We will need the following important operads.

Example 2.1.
Operad Perm introduced in [11] is governing the variety of Perm-algebras [43, p. 17]. Namely, $\mathrm{Perm}(n) = \Bbbk^n$ with a standard basis $e_i^{(n)}$, $i = 1, \dots, n$. Every $e_i^{(n)}$ can be identified with an associative and commutative poly-linear monomial in $x_1, \dots, x_n$ with one emphasized variable x_i.

Example 2.2.
Operad ComTrias introduced in [40] is governing the variety called commutative triassociative algebras in [43, p. 25]. Namely, $\mathrm{ComTrias}(n)$ has a standard basis $e_H^{(n)}$, where $\emptyset \neq H \subseteq \{1, \dots, n\}$. Such an element (corolla) can be identified with a commutative and associative monomial with several emphasized variables x_j, $j \in H$.

2.1. Identities of di- and tri-Var-algebras

Numerous observations made, for example, in [11, 24, 41] lead to the following natural definition.

Definition 2.3.

A *di-Var-algebra* is a functor from Var ⊗ Perm to Vec, i.e., an $\Omega^{(2)}$-algebra satisfying the following identities:

$$(x_1 \dashv_i x_2) \vdash_j x_3 = (x_1 \vdash_i x_2) \vdash_j x_3, \qquad x_1 \dashv_i (x_2 \vdash_j x_3) = x_1 \dashv_i (x_2 \dashv_j x_3), \tag{1}$$

$$f(x_1, \dots, \dot{x}_k, \dots, x_n), \qquad f \in S, \quad n = \deg f, \quad k = 1, \dots, n, \tag{2}$$

where $i, j \in I$, and $f(x_1, \dots, \dot{x}_k, \dots, x_n)$ stands for the $\Omega^{(2)}$-identity obtained from f by means of replacing all products $\circ_i$ with either $\dashv_i$ or $\vdash_i$ in such a way that all horizontal dashes point to the selected variable x_k.

Example 2.4.

Let $|\Omega| = 1$, and let As be the operad of associative algebras. The variety of di-As-algebras [29] is given by (1) together with

$$x_1 \dashv (x_2 \dashv x_3) = (x_1 \dashv x_2) \dashv x_3, \qquad x_1 \vdash (x_2 \dashv x_3) = (x_1 \vdash x_2) \dashv x_3, \qquad x_1 \vdash (x_2 \vdash x_3) = (x_1 \vdash x_2) \vdash x_3. \tag{3}$$

Example 2.5.

Consider the class of Poisson algebras ($|\Omega| = 2$), where $\circ_1$ is an associative and commutative product (we will denote $x \circ_1 y$ simply by xy) and $\circ_2$ is a Lie product ($x \circ_2 y = [x, y]$) related with $\circ_1$ by means of the following identity:

$$[x_1x_2, x_3] = [x_1, x_3]x_2 + x_1[x_2, x_3].$$

Then a di-Poisson algebra is a linear space equipped with four operations $(\cdot * \cdot)$, $[\cdot * \cdot]$, $* \in \{\vdash, \dashv\}$, satisfying (1) and (2). Commutativity of the first product and anticommutativity of the second one allow to reduce these four operations to only two, since (2) implies

$$(x_1 \dashv x_2) = (x_2 \vdash x_1), \qquad [x_1 \dashv x_2] = -[x_2 \vdash x_1].$$

With respect to the operations

$$xy \stackrel{\text{def}}{=} (x \vdash y), \qquad [x, y] \stackrel{\text{def}}{=} [x \vdash y],$$

the identities (1) and (2) are equivalent to the following system:

$$\begin{gathered} x_1(x_2x_3) = (x_1x_2)x_3, \qquad ([x_1, x_2] + [x_2, x_1])x_3 = 0, \qquad (x_1x_2)x_3 = (x_2x_1)x_3, \\ [x_1, [x_2, x_3]] - [x_2, [x_1, x_3]] = [[x_1, x_2], x_3], \\ [x_1x_2, x_3] = x_1[x_2, x_3] + x_2[x_1, x_3], \qquad [x_1, x_2x_3] = [x_1, x_2]x_3 + x_2[x_1, x_3]. \end{gathered} \tag{4}$$

In [28], a more general class was introduced (without assuming commutativity of the associative product). In [1], the identities (4) defined the operad which is Koszul-dual to the operad of Pre-Poisson algebras.

A similar approach works for trialgebras. There exists a functor $\Psi\colon \mathcal{F}^{(3)} \to \mathcal{F} \otimes \mathrm{ComTrias}$ defined by $\Psi(2)(x_1 \vdash_i x_2) = x_1x_2 \otimes e_2^{(2)}$, $\Psi(2)(x_1 \dashv_i x_2) = x_1x_2 \otimes e_1^{(2)}$, $\Psi(2)(x_1 \perp_i x_2) = x_1x_2 \otimes e_{1,2}^{(2)}$. It is easy to note (see also [41]) that each $\Psi(n)$ is surjective.

We are going to define a canonical family of inverse maps

$$\Phi(n)\colon \mathcal{F}(n) \otimes \mathrm{ComTrias}(n) \to \mathcal{F}^{(3)}(n), \qquad n \geq 1,$$

$\Psi(n)\Phi(n) = \mathrm{id}_{\mathcal{F}(n)\otimes \mathrm{ComTrias}(n)}$. Suppose $u = u(x_1, \dots, x_n) \in \mathcal{F}(n)$ is a non-associative Ω-monomial. Fix l indices $1 \leq k_1 < \dots < k_l \leq n$, and denote the monomial u with l emphasized variables x_{k_j}, $j = 1, \dots, l$, by u^H, $H = \{k_1, \dots, k_l\}$. Now, identify u^H with an element from $\mathcal{F}(n) \otimes \mathrm{ComTrias}(n)$ in the natural way:

$$u^H \equiv u \otimes e^{(n)}_{k_1,\dots,k_l}.$$

It can be considered as a binary tree from $\mathcal{F}(n)$ with l emphasized leaves, see Figure 1.

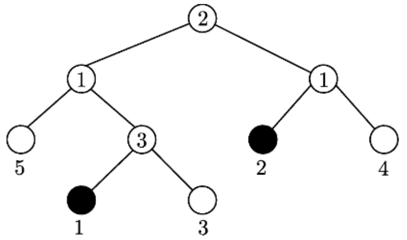

Figure 1. Binary tree representing $u = (x_5 \circ_1 (x_1 \circ_3 x_3)) \circ_2 (x_2 \circ_1 x_4)$ with $H = \{1, 2\}$. Emphasized leaves are colored in black, others — in white.

Now the task is to mark all vertices of u^H with appropriate labels from $\Omega^{(3)}$. For $n = 1$, set $\Phi(1)(x_1 \otimes e_1^1) = x_1$. A monomial $u \in \mathcal{F}(n)$, $n \geq 2$, can be presented as $u = u_1(x_{\sigma(1)}, \dots, x_{\sigma(p)}) \circ_i u_2(x_{\sigma(p+1)}, \dots, x_{\sigma(n)})$, $u_1 \in \mathcal{F}(p)$, $u_2 \in \mathcal{F}(n-p)$, $\sigma \in S_n$. Given a nonempty set of emphasized variables $H = \{k_1, \dots, k_l\} \subseteq \{1, \dots, n\}$, denote

$$H_1 = \sigma^{-1}(H \cap \{\sigma(1), \dots, \sigma(p)\}), \qquad H_2 = \big\{\sigma^{-1}(j) - p \,:\, j \in H \cap \{\sigma(p+1), \dots, \sigma(n)\}\big\}.$$

Then set

$$\Phi(n)(u^H) = \begin{cases} \operatorname{Comp}\big(x_1 \vdash_i x_2,\, u_1^{\vdash},\, \Phi(n-p)(u_2^{H_2})\big)^{\sigma}, & H_1 = \emptyset, \\ \operatorname{Comp}\big(x_1 \dashv_i x_2,\, \Phi(p)(u_1^{H_1}),\, u_2^{\dashv}\big)^{\sigma}, & H_2 = \emptyset, \\ \operatorname{Comp}\big(x_1 \vdash_i x_2,\, \Phi(p)(u_1^{H_1}),\, \Phi(n-p)(u_2^{H_2})\big)^{\sigma}, & H_1, H_2 \neq \emptyset, \end{cases} \tag{5}$$

where Comp is the composition map in the operad $\mathcal{F}^{(3)}$, $v^{\vdash}$ or $v^{\dashv}$ (for $v \in \mathcal{F}(m)$) denote the same polynomial $v(x_1, \dots, x_m)$ with all operations $\circ_j$ replaced with $\vdash_j$ or $\dashv_j$, respectively.

Graphically, in order to compute $\Phi(n)$ one should assign $\perp$ to each vertex which is not a leaf if both left and right branches have emphasized leaves. If only left branch contains an emphasized leaf then assign $\dashv$ to this vertex and to all vertices of the right branch. Symmetrically, if only right branch contains an emphasized leaf then assign $\vdash$ to this vertex and to all vertices of the left branch, see Figure 2.

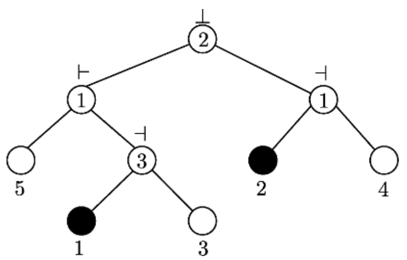

Figure 2. Binary tree with marked vertices representing $\Phi(5)(u^H) = (x_5 \vdash_1 (x_1 \dashv_3 x_3)) \perp_2 (x_2 \dashv_1 x_4)$ for u and H as on Figure 1.

One may extend $\Phi(n)$ by linearity, so, if $f(x_1, \dots, x_n) = \sum_{\xi} \alpha_\xi u_\xi \in \mathcal{F}(n)$, then

$$f(x_1, \dots, \dot{x}_{k_1}, \dots, \dot{x}_{k_l}, \dots, x_n) \overset{\text{def}}{=} \sum_{\xi} \alpha_\xi \Phi(n)(u_\xi^H).$$

It is somewhat similar to the tri-successor procedure from [3].

Definition 2.6.

A *tri*-Var-*algebra* is a functor from Var ⊗ ComTrias to Vec, i.e., an $\Omega^{(3)}$-algebra satisfying the following identities:

$$(x_1 *_i x_2) \vdash_j x_3 = (x_1 \vdash_i x_2) \vdash_j x_3, \qquad x_1 \dashv_i (x_2 *_j x_3) = x_1 \dashv_i (x_2 \dashv_j x_3), \qquad * \in \{\vdash, \dashv, \perp\}, \quad i, j \in I, \tag{6}$$

$$f(x_1, \dots, \dot{x}_{k_1}, \dots, \dot{x}_{k_l}, \dots, x_n), \qquad f \in S, \quad n = \deg f, \quad 1 \le k_1 < \dots < k_l \le n, \quad l = 1, \dots, n. \tag{7}$$

For a variety Var, let us denote by DiVar and TriVar the operads governing di- and tri-Var-algebras, respectively.

Example 2.7.

The only defining identity of the variety As turns into seven identities (7) defining tri-As-algebras. Indeed, each nonempty subset $H \subseteq \{1,2,3\}$ gives rise to an identity of $\Omega^{(3)}$-algebras, $\Omega^{(3)} = \{\vdash, \dashv, \perp\}$. If $|H| = 1$ then these are just the identities of a di-As-algebra (3). For $|H| = 2$, we obtain three identities, e.g., if $H = \{1,3\}$ then the corresponding identity is $x_1 \perp (x_2 \vdash x_3) = (x_1 \dashv x_2) \perp x_3$. If $H = \{1,2,3\}$ then we obtain the relation of associativity for $\perp$. Together with four identities (6), these are exactly the defining identities of what is called triassociative algebras in [43, p. 23].

Example 2.8.

Let A be an associative algebra. Then the space $A^{\otimes 3}$ with respect to operations

$$a \otimes b \otimes c \vdash a' \otimes b' \otimes c' = abca' \otimes b' \otimes c', \qquad a \otimes b \otimes c \dashv a' \otimes b' \otimes c' = a \otimes b \otimes ca'b'c',$$
$$a \otimes b \otimes c \perp a' \otimes b' \otimes c' = a \otimes bca'b' \otimes c'$$

is a tri-As-algebra.

The following construction invented in [33] for dialgebras also works for trialgebras. Let A be a 0-trialgebra, i.e., an $\Omega^{(3)}$-algebra which satisfies (6). Then $A_0 = \operatorname{Span}\{a \vdash_i b - a \dashv_i b,\ a \vdash_i b - a \perp_i b : a, b \in A,\ i \in I\}$ is an ideal of A. The quotient $\overline{A} = A/A_0$ carries a natural structure of an Ω-algebra. Consider the formal direct sum $\widehat{A} = \overline{A} \oplus A$ with (well-defined) operations

$$\overline{a} \circ_i x = a \vdash_i x, \qquad x \circ_i \overline{a} = x \dashv_i a, \qquad \overline{a} \circ_i \overline{b} = \overline{a \vdash_i b}, \qquad x \circ_i y = x \perp_i y, \tag{8}$$

$\overline{a}, \overline{b} \in \overline{A}$, $x, y \in A$.

Proposition 2.9.

A 0*-trialgebra A is a tri-*Var*-algebra if and only if* $\widehat{A}$ *is an algebra from the variety* Var.

Proof. The claim follows from the following observation. If $f(x_1, \dots, x_n) \in \mathcal{F}(n)$ then the value $f(\overline{a}_1, \dots, \overline{a}_n)$ in $\overline{A} \subset \widehat{A}$ is just the image of $[\Phi(n)(f^H)](a_1, \dots, a_n)$ in $\overline{A}$ for any subset H; moreover, the value of $f(x_1, \dots, \dot{x}_{k_1}, \dots, \dot{x}_{k_l}, \dots, x_n)$ on $a_1, \dots, a_n \in A$ is equal to $f(\overline{a}_1, \dots, a_{k_1}, \dots, a_{k_l}, \dots, \overline{a}_n) \in \widehat{A}$, i.e., one has to add bars to all non-emphasized variables. □

Assuming $x \perp_i y \equiv 0$ for all $x, y \in A$, $i \in I$, we obtain the construction from [33]. This construction turns to be useful in the study of dialgebras, see, e.g., [21, 42].

2.2. Dialgebras and pseudo-algebras

The structure of a di-Var-algebra may be recovered from a structure of a Var-pseudo-algebra over an appropriate bialgebra H. Let us recall this notion from [6]. Suppose H is a cocommutative bialgebra with a coproduct Δ and counit ε. We will use the Swedler notation for Δ, e.g., $\Delta(h) = h_{(1)}\otimes h_{(2)}$, $\Delta^2(h) = (\Delta\otimes \mathrm{id})\Delta(h) = (\mathrm{id}\otimes\Delta)\Delta(h) = h_{(1)}\otimes h_{(2)}\otimes h_{(3)}$, $h\in H$. The operation $F\cdot h = F\Delta^{n-1}(h)$, $F\in H^{\otimes n}$, $h\in H$, turns $H^{\otimes n}$ into a right H-module (the outer product of right regular H-modules).

A unital left H-module C gives rise to an operad (also denoted by C) such that

$$C(n) = \{f\colon C^{\otimes n}\to H^{\otimes n}\otimes_H C \mid f \text{ is } H^{\otimes n}\text{-linear}\}.$$

For example, if $\dim H = 1$ then what we obtain is just a linear space with poly-linear maps. The composition of such maps as well as the action of a symmetric group is defined in [6].

In these terms, if Var is a variety of Ω-algebras defined by a system of poly-linear identities S then a Var-pseudo-algebra structure on an H-module C is a functor from Var to the operad C. Such a functor is determined by a family of $H^{\otimes 2}$-linear maps

$$*_i\colon C\otimes C\to H^{\otimes 2}\otimes_H C$$

satisfying the identities $f^{(*)}(x_1,\dots,x_n) = 0$, $f\in S$, $\deg f = n$, c.f. [23], where $f^{(*)}$ is obtained from f in the following way. Assume a poly-linear Ω-monomial u in the variables $x_1,\dots,x_n$ turns into a word $x_{\sigma(1)}\dots x_{\sigma(n)}$ for some $\sigma=\sigma(u)\in S_n$ after removing all brackets and symbols $\circ_i$, $i\in I$. Denote by $u^{\circledast}$ the expression obtained from the monomial u by means of replacing all $\circ_i$ with $*_i$. Then $u^{\circledast}$ can be considered as a map $C^{\otimes n}\to H^{\otimes n}\otimes_H C$, which may not be $H^{\otimes n}$-linear. However, $u^{(*)} = (\sigma(u)\otimes_H \mathrm{id})\,u^{\circledast}$ is $H^{\otimes n}$-linear. Finally, if $f=\sum\limits_{\xi}\alpha_\xi u_\xi$, $\alpha_\xi\in\Bbbk$, then

$$f^{(*)}(x_1,\dots,x_n) = \sum_{\xi}\alpha_\xi u_\xi^{(*)}.$$

Example 2.10 (c.f. [6]).
Consider an Ω-algebra A, a cocommutative bialgebra H, and define $C = H\otimes A$. Then C is a pseudo-algebra with respect to the operations

$$(f\otimes a) *_i (h\otimes b) = (f\otimes h)\otimes_H (a\circ_i b), \qquad f,h\in H, \quad a,b\in A, \quad i\in I.$$

Such a pseudo-algebra is denoted by $\mathrm{Cur}\,A$ (current pseudo-algebra). If A belongs to Var then, obviously, $\mathrm{Cur}\,A$ is a Var-pseudo-algebra over H.

Given a pseudo-algebra C with operations $*_i$, $i\in I$, one may define operations $\vdash_i,\dashv_i$ on the same space C as follows: if $a*_i b = \sum\limits_{\xi}(h_\xi\otimes f_\xi)\otimes_H d_\xi$ then

$$a\vdash_i b = \sum_{\xi}\varepsilon(h_\xi) f_\xi d_\xi, \qquad a\dashv_i b = \sum_{\xi} h_\xi \varepsilon(f_\xi) d_\xi. \tag{9}$$

Proposition 2.11.
Let C be a Var-pseudo-algebra. Then $C^{(0)}$ is a di-Var-algebra.

Proof. It is enough to verify (1) & (2) on $C^{(0)}$. Indeed, if $a*_i b = \sum\limits_{\xi}(h_\xi\otimes f_\xi)\otimes_H d_\xi$, $d_\xi *_j c = \sum\limits_{\eta}(h'_\eta\otimes f'_\eta)\otimes_H e_\eta$ then

$$(a\vdash_i b)*_j c = \sum_{\eta}\left(\sum_{\xi}\varepsilon(h_\xi) f_\xi d_\xi\right) *_j c = \sum_{\eta,\xi}\big(\varepsilon(h_\xi) f_\xi h'_\eta\otimes f'_\eta\big)\otimes_H e_\eta.$$

Hence,

$$(a \vdash_i b) \vdash_j c = \sum_{\eta,\xi} \varepsilon(h_\xi f_\xi h'_\eta) f'_\eta e_\eta.$$

On the other hand,

$$(a \dashv_i b) *_j c = \sum_\eta \left(\sum_\xi h_\xi \varepsilon(f_\xi) d_\xi \right) *_j c = \sum_{\eta,\xi} \big(h_\xi \varepsilon(f_\xi) h'_\eta \otimes f'_\eta\big) \otimes_H e_\eta,$$

so $(a \vdash_i b) \vdash_j c = (a \dashv_i b) \vdash_j c$ for all $a, b, c \in C$. The second identity in (1) can be proved in the same way. Consider a poly-linear identity $f \in S$. It is straightforward to check, c.f. [24], that if

$$f^{(*)}(a_1, \dots, a_n) = \sum_\xi (h_{1\xi} \otimes \dots \otimes h_{n\xi}) \otimes_H c_\xi$$

then $f(a_1, \dots, \dot{a}_k, \dots, a_n) = \sum_\xi h_{1\xi} \cdots \varepsilon(h_{k\xi}) \cdots h_{n\xi} c_\xi$ in $C^{(0)}$. It is clear that if $f^{(*)}$ vanishes in C then $C^{(0)}$ satisfies (2). □

In particular, if B is a Var-algebra then $(\mathrm{Cur}\, B)^{(0)}$ is a di-Var-algebra.

Proposition 2.12.

If H contains a nonzero element T such that $\varepsilon(T) = 0$ then every di-Var-algebra A embeds into $(\mathrm{Cur}\, \widehat{A})^{(0)}$.

Proof. Recall that $\widehat{A} = \overline{A} \oplus A$, $\mathrm{Cur}\, \widehat{A} = H \otimes \widehat{A}$. Define

$$\iota \colon A \to H \otimes \widehat{A}, \qquad \iota(a) = 1 \otimes \overline{a} + T \otimes a. \tag{10}$$

This map is obviously injective, and

$$\iota(a) *_i \iota(b) = (1 \otimes 1) \otimes_H (1 \otimes \overline{a \vdash_i b}) + (T \otimes 1) \otimes_H (1 \otimes a \dashv_i b) + (1 \otimes T) \otimes_H (1 \otimes a \vdash_i b).$$

Since $\overline{a \vdash_i b} = \overline{a \dashv_i b}$ in $\widehat{A}$, we have

$$\iota(a) \vdash_i \iota(b) = 1 \otimes \overline{a \vdash_i b} + T \otimes a \vdash_i b = \iota(a \vdash_i b), \qquad \iota(a) \dashv_i \iota(b) = 1 \otimes \overline{a \dashv_i b} + T \otimes a \dashv_i b = \iota(a \dashv_i b).$$ □

3. Dendriform di- and trialgebras

Let us first briefly demonstrate relations between dialgebras, dendriform dialgebras, and Manin products in the case when Var = As. The operad Dend in [28] is known to be Koszul dual (see [19] for details on Koszul duality) to the operad DiAs. Since DiAs = As ⊗ Perm and it was noticed in [41, Proposition 15] that for Perm (as well as for ComTrias) the Hadamard product ⊗ coincides with the Manin white product ∘, we have Dend = $(\mathrm{As} \otimes \mathrm{Perm})^! = \mathrm{As} \bullet \mathrm{PreLie}$, where $\mathrm{As}^! = \mathrm{As}$, PreLie is the operad of pre-Lie algebras which is Koszul dual to Perm, • stands for the Manin black product of operads [19].

In general, for a binary operad $\mathcal{P}$ the *successor procedure* described in [3] gives rise to what is natural to call defining identities of di- or tri-$\mathcal{P}$-dendriform algebras. In addition, if $\mathcal{P}$ is quadratic then these $\mathcal{P}$-dendriform algebras are dual to the corresponding di- or tri-$\mathcal{P}^!$-algebras. In this case, obviously, $(\mathcal{P}^! \otimes \mathrm{Perm})^! = \mathcal{P} \bullet \mathrm{PreLie}$ for dialgebras, and $(\mathcal{P}^! \otimes \mathrm{ComTrias})^! = \mathcal{P}^! \bullet \mathrm{PostLie}$ for trialgebras, where $\mathrm{PreLie} = \mathrm{Perm}^!$, $\mathrm{PostLie} = \mathrm{ComTrias}^!$. This observation is closely related with Proposition 3.2 below.

In terms of identities, we do not need $\mathcal{P}$ to be quadratic (in fact, it is easy to generalize the successor procedure even for algebras with n-ary operations, $n \geq 2$).

3.1. Identities of di- and tri-Var-dendriform algebras

Suppose Var is a variety of Ω-algebras defined by a family S of poly-linear identities, as above.

Definition 3.1.
A *tri*-Var-*dendriform algebra* is an $\Omega^{(3)}$-algebra satisfying the identities

$$f^*(x_1,\dots,\dot{x}_{k_1},\dots,\dot{x}_{k_l},\dots,x_n),\qquad f\in S,\quad n=\deg f,\quad 1\le k_1<\dots<k_l\le n,$$

for all $l=1,\dots,n$, where $f^*(x_1,\dots,\dot{x}_{k_1},\dots,\dot{x}_{k_l},\dots,x_n)$ is obtained from f by means of the following procedure (the tri-successor procedure from [3]). Consider a family of maps $\Phi(n)^*\colon \mathcal{F}(n)\otimes \mathrm{ComTrias}(n)\to \mathcal{F}^{(3)}(n)$ defined on monomials in a similar way as in (5), but, instead of $v^{\dashv}$ or $v^{\vdash}$, we have to use v^* which stands for the linear combination of monomials obtained when we replace each operation $\circ_j$ in v with $\vdash_j+\dashv_j+\perp_j$.

Extend $\Phi^*(n)$ by linearity and set

$$f^*(x_1,\dots,\dot{x}_{k_1},\dots,\dot{x}_{k_l},\dots,x_n)\overset{\mathrm{def}}{=}\sum_{\xi}\alpha_\xi\Phi^*(n)(u_\xi^H)$$

for $f(x_1,\dots,x_n)=\sum_\xi\alpha_\xi u_\xi\in\mathcal{F}(n)$, $\alpha_\xi\in\Bbbk$, $H=\{k_1,\dots,k_l\}$. To get the definition of a di-Var-dendriform algebra, it is enough to set $x\perp y=0$ and consider $|H|=1$ only.

Denote by DendDiVar and DendTriVar the operads governing di- and tri-Var-dendriform algebras, respectively.

Proposition 3.2.
If Var *is a quadratic binary operad (and* $|\Omega|<\infty$*) then* $(\mathrm{DiVar})^!=\mathrm{DendDiVar}^!$ *and* $(\mathrm{TriVar})^!=\mathrm{DendTriVar}^!$*, where* $\mathrm{Var}^!$ *stands for the Koszul-dual operad to* Var.

Proof. We consider the trialgebra case in detail since it covers the dialgebra case. Suppose $\mathrm{Var}=\mathcal{P}(E,R)$ is a binary quadratic operad, i.e., a quotient operad of $\mathcal{F}$, $\mathcal{F}(2)=E$, with respect to the operad ideal generated by S_3-submodule $R\subseteq\mathcal{F}(3)$, see [19] for details.

The space E is spanned by $\mu_i\colon x_1\otimes x_2\mapsto x_1\circ_i x_2$ and $\mu_i^{(12)}\colon x_1\otimes x_2\mapsto x_2\circ_i x_1$, $i\in I$. Without loss of generality, we may assume that μ_i, $i\in I$, are linearly independent and

$$\mu_k^{(12)}=\sum_{i\in I}\alpha_{ik}\mu_i+\sum_{j\in I\setminus I'}\beta_{jk}\mu_j^{(12)},\qquad k\in I'\subseteq I,\quad \alpha_{ik}\in\Bbbk,$$

are the only defining identities of Var of degree two, $|I'|=d\ge 0$ (if $\operatorname{char}\Bbbk\ne 2$, these are just commutativity and anti-commutativity). Denote by $N=2|I|-d$ the dimension of E.

The space $\mathcal{F}(3)$ can be naturally identified with the induced S_3-module $\Bbbk S_3\otimes_{\Bbbk S_2}(E\otimes E)$, where $E\otimes E$ is considered as an S_2-module via $(\mu\otimes\nu)^{(12)}=\mu\otimes\nu^{(12)}$, $\mu,\nu\in E$. Namely, the basis of $\mathcal{F}(3)$ consists of expressions

$$\sigma\otimes_{\Bbbk S_2}(\mu\otimes\nu),\qquad \sigma\in\{e,(13),(23)\},$$

μ and ν range over a chosen basis of E. Therefore, $\dim\mathcal{F}(3)=3N^2$. In terms of monomials (or binary trees), for example, $e\otimes_{\Bbbk S_2}(\mu_i\otimes\mu_j)$ corresponds to $(x_1\circ_j x_2)\circ_i x_3$, $e\otimes_{\Bbbk S_2}(\mu_i^{(12)}\otimes\mu_j)$ to $x_3\circ_i(x_1\circ_j x_2)$. A permutation $\sigma\in S_3$ in the first tensor factor permutes variables, e.g., $(13)\otimes_{\Bbbk S_2}(\mu_i^{(12)}\otimes\mu_j^{(12)})$ corresponds to $x_1\circ_i(x_2\circ_j x_3)$.

Recall that $E^\vee$ denotes the dual space to E considered as an S_2-module with respect to sgn-twisted action $\langle\nu^{(12)},\mu\rangle=-\langle\nu,\mu^{(12)}\rangle$, $\nu\in E^\vee$, $\mu\in E$. If $\mathcal{F}^\vee$ is the free binary operad generated by $E^\vee$ then $(\mathcal{F}(3))^\vee\simeq\mathcal{F}^\vee(3)=\Bbbk S_3\otimes_{\Bbbk S_2}(E^\vee\otimes E^\vee)$.

The Koszul-dual operad $\mathrm{Var}^!$ is the quotient of $\mathcal{F}^\vee$ by the operad ideal generated by $R^\perp \subset \mathcal{F}^\vee(3)$, the orthogonal space to R.

By the definition, the operad TriVar is equal to $\mathcal{P}(E^{(3)}, R^{(3)})$, where the initial data $E^{(3)}, R^{(3)}$ are defined as follows. The space $E^{(3)}$ is spanned by $\mu_i^\star$, $(\mu_i^\star)^{(12)}$, $i \in I$, $\star \in \{\vdash, \dashv, \perp\}$, with respect to the relations

$$(\mu_k^\vdash)^{(12)} = \sum_{i\in I} \alpha_{ik}\mu_i^\dashv + \sum_{j\in I\setminus I'} \beta_{jk}(\mu_j^\vdash)^{(12)}, \qquad (\mu_k^\dashv)^{(12)} = \sum_{i\in I} \alpha_{ik}\mu_i^\vdash + \sum_{j\in I\setminus I'} \beta_{jk}(\mu_j^\dashv)^{(12)}, \qquad k \in I'.$$
$$(\mu_k^\perp)^{(12)} = \sum_{i\in I} \alpha_{ik}\mu_i^\perp + \sum_{j\in I\setminus I'} \beta_{jk}(\mu_j^\perp)^{(12)},$$

The S_3-module $R^{(3)}$ is generated by the defining identities of tri-Var-algebras, i.e.,

$$R^{(3)} = \bigl\{\Phi(3)(f^H) : f \in R,\ \emptyset \neq H \subseteq \{1,2,3\}\bigr\} \oplus O^{(3)},$$

and $O^{(3)}$ is the S_3-submodule of $\mathcal{F}^{(3)}$ generated by

$$\begin{matrix} \mu_j^\vdash \otimes \mu_i^\dashv - \mu_j^\vdash \otimes \mu_i^\vdash, & \mu_j^\vdash \otimes \mu_i^\perp - \mu_j^\vdash \otimes \mu_i^\vdash, \\ (\mu_i^\dashv)^{(12)} \otimes \mu_j^\vdash - (\mu_i^\dashv)^{(12)} \otimes \mu_j^\dashv, & (\mu_i^\dashv)^{(12)} \otimes \mu_j^\perp - (\mu_i^\dashv)^{(12)} \otimes \mu_j^\dashv, \end{matrix} \qquad i,j \in I. \tag{11}$$

It is easy to calculate that $\dim E^{(3)} = 3N$, $\dim \mathcal{F}^{(3)}(3) = 27N^2$, $\dim O^{(3)} = 6N^2$, so $\dim R^{(3)} = 6N^2 + 7\dim R$. Denote by $O_+^{(3)}$ the S_3-submodule of $\mathcal{F}^{(3)}$ generated by the first summands of all relations from (11).

Suppose $f \in \mathcal{F}(3)$, $g \in \mathcal{F}^\vee(3)$, and let $H_1, H_2 \subseteq \{1,2,3\}$ be nonempty subsets. It follows from the definition of $\Phi(3)$ that $\bigl\langle \Phi(3)(f^{H_1}), \Phi(3)(g^{H_2})\bigr\rangle = 0$ if $H_1 \neq H_2$. For $H_1 = H_2 = H$, orthogonality of f and g implies $\bigl\langle \Phi(3)(f^H), \Phi(3)(g^H)\bigr\rangle = 0$ as well. Moreover, for every $f \in \mathcal{F}(3)$ we have $\bigl\langle \Phi(3)(f^H), O_+^{(3)}\bigr\rangle = 0$ since neither of terms from $O_+^{(3)}$ appears in images of $\Phi(3)$.

Now, it is easy to see that if $g \in R^\perp \subseteq \mathcal{F}^\vee(3)$ then $\langle f, \Phi^*(3)(g^H)\rangle = 0$ for every $f \in R^{(3)}$. Hence,

$$(R^\perp)^{(3*)} \overset{\text{def}}{=} \bigl\{\Phi^*(3)(g^H) : g \in R^\perp,\ \emptyset \neq H \subseteq \{1,2,3\}\bigr\} \subseteq (R^{(3)})^\perp.$$

On the other hand, $\dim R^\perp = 3N^2 - \dim R$, so $\dim (R^\perp)^{(3*)} = 21N^2 - 7\dim R$. Therefore, $\dim (R^\perp)^{(3*)} + \dim R^{(3)} = 27N^2$ and $(R^\perp)^{(3*)} = (R^{(3)})^\perp$. It remains to recall that, by definition, $\mathrm{DendTriVar} = \mathcal{P}(E^{(3)}, (R^\perp)^{(3*)})$. ☐

Example 3.3.
The defining identities of Perm-algebras are $(x_1x_2)x_3 - (x_2x_1)x_3$ and $x_1(x_2x_3) - (x_1x_2)x_3$ [11]. The corresponding variety of di-Perm-algebras is governed by the operad $\mathrm{DiPerm} = \mathrm{Perm}\otimes\mathrm{Perm} = \mathrm{Perm}\circ\mathrm{Perm}$. Thus, $(\mathrm{DiPerm})^! = \mathrm{Perm}^! \bullet \mathrm{Perm}^! = \mathrm{PreLie}\bullet\mathrm{PreLie}$, where PreLie is the operad governing left-symmetric (pre-Lie) algebras satisfying the identity $(x_1x_2)x_3 - x_1(x_2x_3) = (x_2x_1)x_3 - x_2(x_1x_3)$. By Proposition 3.2, $(\mathrm{DiPerm})^! = \mathrm{DendDiPreLie}$. Defining identities of the variety of di-PreLie-dendriform algebras are easy to construct by Definition 3.1: They coincide with the defining identities of L-dendriform algebras [5]. Hence, the operad governing the class of L-dendriform algebras is equal to $\mathrm{PreLie}\bullet\mathrm{PreLie}$.

3.2. Embedding into Rota–Baxter algebras

Suppose B is an Ω-algebra. A linear map $R\colon B \to B$ is called a Rota–Baxter operator of weight $\lambda \in \Bbbk$ if

$$R(x) \circ_i R(y) = R\bigl(x \circ_i R(y) + R(x) \circ_i y + \lambda x \circ_i y\bigr) \tag{12}$$

for all $x, y \in B$, $i \in I$.

Let A be an $\Omega^{(3)}$-algebra. Consider the isomorphic copy A' of the underlying linear space A (assume $a \in A$ is in the one-to-one correspondence with $a' \in A'$), and define the following Ω-algebra structure on the space $\widehat{A} = A \oplus A'$:

$$a \circ_i b = a \vdash_i b + a \dashv_i b + a \perp_i b, \qquad a \circ_i b' = (a \vdash_i b)', \qquad a' \circ_i b = (a \dashv_i b)', \qquad a' \circ_i b' = (a \perp_i b)', \tag{13}$$

for $a, b \in A$, $i \in I$.

Lemma 3.4.

Given a scalar $\lambda \in \Bbbk$, the linear map $R\colon \widehat{A} \to \widehat{A}$ defined by $R(a') = \lambda a$, $R(a) = -\lambda a$, $a \in A$, is a Rota–Baxter operator of weight λ on the Ω-algebra $\widehat{A}$.

Proof. It is enough to check the relation (12). A straightforward computation shows

$$\begin{aligned} &R(a+b') \circ_i R(x+y') = \lambda^2(-a+b) \circ_i (-x+y) \\ &\quad = \lambda^2\big(a \vdash_i x + a \dashv_i x + a \perp_i x - a \vdash_i y - a \dashv_i y - a \perp_i y - b \vdash_i x - b \dashv_i x - b \perp_i x + b \vdash_i y + b \dashv_i y + b \perp_i y\big). \end{aligned}$$

On the other hand,

$$\begin{aligned} &R\big((a+b') \circ_i R(x+y') + R(a+b') \circ_i (x+y') + \lambda(a+b') \circ (x+y')\big) \\ &\quad = \lambda R\big((a+b') \circ_i (-x+y) + (-a+b) \circ_i (x+y') + (a+b') \circ (x+y')\big) \\ &\quad = \lambda R\big(-a \vdash_i x - a \dashv_i x - a \perp_i x + a \vdash_i y + a \dashv_i y + a \perp_i y \\ &\qquad\qquad - (b \dashv_i x)' + (b \dashv_i y)' - a \vdash_i x - a \dashv_i x - a \perp_i x + b \vdash_i x + b \dashv_i x + b \perp_i x \\ &\qquad\qquad - (a \vdash_i y)' + (b \vdash_i y)' + a \vdash_i x + a \dashv_i x + a \perp_i x + (a \vdash_i y)' + (b \dashv_i x)' + (b \perp_i y)'\big) \\ &\quad = \lambda^2\big(-a \vdash_i y - a \dashv_i y - a \perp_i y + b \dashv_i y + a \vdash_i x + a \dashv_i x + a \perp_i x - b \vdash_i x - b \dashv_i x - b \perp_i x + b \vdash_i y + b \perp_i y\big). \end{aligned}$$ □

Lemma 3.5.

Let A be an $\Omega^{(2)}$-algebra. Then the map $R\colon \widehat{A} \to \widehat{A}$ defined by $R(a') = a$, $R(a) = 0$ is a Rota–Baxter operator of weight $\lambda = 0$ on $\widehat{A}$.

The proof is completely analogous to the previous one. The following statement is well-known in various particular cases, c.f. [1, 15, 16, 38].

Proposition 3.6.

Let B be an Ω-algebra with a Rota–Baxter operator R of weight λ. Assume B belongs to Var*. Then the same linear space B considered as an $\Omega^{(3)}$-algebra with respect to the operations*

$$x \vdash_i y = R(x) \circ_i y, \qquad x \dashv_i y = x \circ_i R(y), \qquad x \perp_i y = \lambda x \circ_i y \tag{14}$$

*is a tri-*Var*-dendriform algebra.*

Proof. Let $u = u(x_1, \dots, x_n) \in \mathcal{F}(n)$ be a poly-linear Ω-monomial. The claim follows from the following relation in B:

$$u^*(x_1, \dots, \dot{x}_{k_1}, \dots, \dot{x}_{k_l}, \dots, x_n) = \lambda^{l-1} u\big(R(x_1), \dots, x_{k_1}, \dots, x_{k_l}, \dots, R(x_n)\big), \tag{15}$$

i.e., in order to get a value of an $\Omega^{(3)}$-monomial in B we have to replace every non-emphasized variable x_i, $i \notin H = \{k_1, \dots, k_l\}$, with $R(x_i)$ and multiply the result by λ^{l-1}.

Relation (15) is clear for $n = 1, 2$. In order to apply induction on n, we have to start with the case when $H = \emptyset$. Recall that $u^*(x_1, \dots, x_n)$ stands for the expression obtained from u by means of replacing each $\circ_i$ with $\vdash_i + \dashv_i + \perp_i$. Then

$$R(u^*(x_1, \dots, x_n)) = u(R(x_1), \dots, R(x_n)), \qquad n \geq 2, \tag{16}$$

in B. Indeed, for $n = 2$ we have exactly the Rota–Baxter relation. If $u = v \circ_i w$, $v = v(x_1, \dots, x_p)$, $w = w(x_{p+1}, \dots, x_n)$, then, by induction,

$$\begin{aligned} R(u^*) &= R(v^* \vdash_i w^* + v^* \dashv_i w^* + v^* \perp_i w^*) = R\big(R(v^*) \circ_i w^* + v^* \circ_i R(w^*) + \lambda v^* \circ_i w^*\big) \\ &= R(v^*) \circ_i R(w^*) = v(R(x_1), \dots, R(x_p)) \circ_i w(R(x_{p+1}), \dots, R(x_n)) = u(R(x_1), \dots, R(x_n)). \end{aligned}$$

Now, let us finish proving (15). If $u = v \circ_i w$, $\deg v = p$, $H = H_1 \dot\cup H_2$ then there are three cases: (a) $H_1, H_2 \neq \emptyset$; (b) $H_1 = \emptyset$; (c) $H_2 = \emptyset$.

In the case (a), $u^*(x_1, \dots, \dot x_{k_1}, \dots, \dot x_{k_l}, \dots, x_n) = \Phi^*(n)(u^H) = \Phi^*(p)(v^{H_1}) \perp_i \Phi^*(n)(w^{H_2})$, and it remains to apply the inductive assumption and the definition of $\perp_i$ from (14). In the case (b), $\Phi^*(n)(u^H) = v^* \vdash_i \Phi^*(n-p)(w^H)$, so for any $a_1, \dots, a_n \in B$ we can apply (16) to get

$$\begin{aligned} \big[\Phi^*(n)(u^H)\big](a_1, \dots, a_n) &= R(v^*(a_1, \dots, a_p)) \circ_i \big[\Phi^*(n-p)(w^H)\big](a_{p+1}, \dots, a_n) \\ &= v(R(a_1), \dots, R(a_p)) \circ_i \lambda^{l-1} w(R(a_{p+1}, \dots, a_{k_1}, \dots, a_{k_l}, \dots, R(a_n)) \\ &= \lambda^{l-1} u(R(a_1, \dots, a_{k_1}, \dots, a_{k_l}, \dots, R(a_n)). \end{aligned}$$

The case (c) is completely analogous. □

Proposition 3.7 (c.f. [1, 38]).

Let B be an Ω-algebra with a Rota–Baxter operator R of weight $\lambda = 0$. Assume B belongs to Var. *Then the same linear space B considered as $\Omega^{(2)}$-algebra with respect to $x \vdash_i y = R(x) \circ_i y$, $x \dashv_i y = x \circ_i R(y)$ is a di-*Var*-dendriform algebra.*

Proof. Note that a di-Var-dendriform algebra is the same as tri-Var-dendriform algebra with $x \perp_i y = 0$ for all x, y, and i. The claim follows from Proposition 3.6. □

Given an Ω-algebra $B \in$ Var with a Rota–Baxter operator $R\colon B \to B$ of weight λ, denote the tri-Var-dendriform algebra obtained by Proposition 3.6 by $B^{(R)}$. If $\lambda = 0$ then $B^{(R)}$ is actually a di-Var-dendriform algebra.

Theorem 3.8.

Let A be an $\Omega^{(3)}$-algebra, and let $\widehat{A}$ be the Ω-algebra defined by (13). Then the following statements are equivalent:

(i) *A is a tri-*Var*-dendriform algebra;*

(ii) *$\widehat{A}$ belongs to* Var.

Proof. (i) ⇒ (ii) Assume A is a tri-Var-dendriform algebra, and let S be the set of defining identities of Var. We have to check that every $f \in S$ holds on $\widehat{A}$.

First, let us compute a monomial in $\widehat{A} = A \oplus A'$ when all its arguments belong to the first summand.

Lemma 3.9.

Suppose $u = u(x_1, \dots, x_n) \in \mathcal{F}(n)$ is a poly-linear Ω-monomial of degree n. Then in the Ω-algebra $\widehat{A}$ we have

$$u(a_1, \dots, a_n) = \sum_H \Phi^*(n)(u^H)(a_1, \dots, a_n), \qquad a_i \in A, \tag{17}$$

where H ranges over all nonempty subsets of $\{1, \dots, n\}$.

Proof. By the definition of multiplication in $\widehat{A}$, $u(a_1,\dots,a_n) = u^*(a_1,\dots,a_n)$, where u^* means the same as in the definition of $\Phi^*(n)$. In particular, for $n = 1,2$ the statement is clear. Proceed by induction on $n = \deg u$. Assume $u = v \circ_i w$, and, without loss of generality, $v = v(x_1,\dots,x_p)$, $w = w(x_{p+1},\dots,x_n)$. Then

$$\begin{aligned} u(a_1,\dots,a_n) &= v^*(a_1,\dots,a_p) \vdash_i \left(\sum_{H_2} \Phi^*(n-p)(w^{H_2})(a_{p+1},\dots,a_n)\right) \\ &+ \left(\sum_{H_1} \Phi^*(p)(v^{H_1})(a_1,\dots,a_p)\right) \perp_i \left(\sum_{H_2} \Phi^*(n-p)(w^{H_2})(a_{p+1},\dots,a_n)\right) \\ &+ \left(\sum_{H_1} \Phi^*(p)(v^{H_1})(a_1,\dots,a_p)\right) \dashv_i w^*(a_{p+1},\dots,a_n), \end{aligned} \tag{18}$$

where H_1 and H_2 range over all nonempty subsets of $\{1,\dots,p\}$ and $\{p+1,\dots,n\}$, respectively. It is easy to see that the overall sum is exactly the right-hand side of (17): The first (second, third) group of summands in (18) corresponds to $H = H_2 \subseteq \{p+1,\dots,n\}$, ($H = H_1 \cup H_2$, $H = H_1 \subseteq \{1,\dots,p\}$, respectively). □

Next, assume that $l > 0$ arguments belong to A'.

Lemma 3.10.

Suppose $u = u(x_1,\dots,x_n) \in \mathcal{F}(n)$ is a poly-linear Ω-monomial of degree n, $H = \{k_1,\dots,k_l\}$ is a nonempty subset of $\{1,\dots,n\}$. Then in the Ω-algebra $\widehat{A}$ we have

$$u(a_1,\dots,a'_{k_1},\dots,a'_{k_l},\dots,a_n) = \left(\Phi^*(n)(u^H)(a_1,\dots,a_n)\right)'.$$

Proof. For $n = 1,2$ the statement is clear. If $u = v \circ_i w$ for some $i \in I$ as above then we have to consider three natural cases: (a) $H \subseteq \{1,\dots,p\}$; (b) $H \subseteq \{p+1,\dots,n\}$; (c) variables with indices from H appear in both v and w. In the case (a), the inductive assumption implies

$$\begin{aligned} u(a_1,\dots,a'_{k_1},\dots,a'_{k_l},\dots,a_n) &= v(a_1,\dots,a'_{k_1},\dots,a'_{k_l},\dots,a_p) \dashv_i w^*(a_{p+1},\dots,a_n) \\ &= \left(\Phi^*(p)(v^H)(a_1,\dots,a_p) \dashv_i w^*(a_{p+1},\dots,a_n)\right)', \end{aligned}$$

and it remains to recall the definition of $\Phi^*(n)$. Case (b) is analogous. In the case (c), $H = H_1 \dot\cup H_2$ as above and

$$u(a_1,\dots,a'_{k_1},\dots,a'_{k_l},\dots,a_n) = \Phi^*(p)(v^{H_1})(a_1,\dots,a_p) \perp_i \Phi^*(n-p)(w^{H_2})(a_{p+1},\dots,a_n)$$

which proves the claim. □

Finally, suppose $f \in S$ is a poly-linear identity of degree n. Then $\Phi^*(n)(f^H)$ is an identity on the $\Omega^{(3)}$-algebra A, so Lemmas 3.9 and 3.10 imply f holds on $\widehat{A}$.

(ii) ⇒ (i) The map $\iota\colon A \to \widehat{A}$, $\iota(a) = a'$, is an embedding of the $\Omega^{(3)}$-algebra A into $\widehat{A}$ equipped with operations (14). Let us choose $\lambda = 1$ and define a Rota–Baxter operator R on $\widehat{A}$ by Lemma 3.4. By Proposition 3.6, $\widehat{A}^{(R)}$ is a tri-Var-dendriform algebra, therefore so is A. □

If $\lambda = 0$ then the simple reduction of Theorem 3.8 by means of Lemma 3.5 leads to

Theorem 3.11.

Suppose A is an $\Omega^{(2)}$-algebra, and let $\widehat{A}$ stands for an Ω-algebra defined by (13) with $x \perp_i y \equiv 0$. Then the following statements are equivalent:

(i) *A is a di-Var-dendriform algebra;*

(ii) *$\widehat{A}$ belongs to Var.*

Remark 3.12.

It is interesting to note that A is a simple di-Var-dendriform algebra if and only if $\widehat{A}$ is a simple Rota–Baxter algebra.

Corollary 3.13.

For every tri- (or di-)Var-dendriform algebra A there exists an algebra $B \in \mathrm{Var}$ with a Rota–Baxter operator R of weight $\lambda \neq 0$ (or $\lambda = 0$, respectively) such that $A \subseteq B^{(R)}$.

Proof. It is enough to consider the case of trialgebras only. Let $\lambda \neq 0$ and let $\widehat{A}^{(\lambda)}$ be an algebra with the same underlying space as $\widehat{A}$ but with new operations $x \circ_i^{(\lambda)} y = (x \circ_i y)/\lambda$. It is clear that $\widehat{A}^{(\lambda)} \in \mathrm{Var}$ and if R is a Rota–Baxter operator on $\widehat{A}$ from Lemma 3.4 then so is R for $\widehat{A}^{(\lambda)}$. Hence, $\widehat{A}^{(\lambda)}$ with respect to the operations (14) is a tri-Var-dendriform algebra by Proposition 3.6. Note that a map $\iota\colon A \to \widehat{A}^{(\lambda)}$ given by $\iota(a) = a' \in A' \subset \widehat{A}^{(\lambda)}$ is an embedding of $\Omega^{(3)}$-algebras. □

Given a tri-Var-dendriform algebra A, its universal enveloping Rota–Baxter algebra $U_\lambda(A)$ of weight λ, c.f. [18], is an algebra in the variety Var with a Rota–Baxter operator R such that

- There is a homomorphism $\varphi_\lambda\colon A \to U_\lambda(A)^{(R)}$ of tri-Var-dendriform algebras;
- For every algebra $B \in \mathrm{Var}$ with a Rota–Baxter operator R' of weight λ and for every homomorphism $\psi\colon A \to B^{(R')}$ of tri-Var-dendriform algebras there exists a unique homomorphism of Rota–Baxter algebras $\chi\colon U_\lambda(A) \to B$ such that $\varphi_\lambda \circ \chi = \psi$.

For a di-Var-dendriform algebra A, its universal enveloping Rota–Baxter algebra of weight zero $U_0(A)$ is defined analogously, see also [12].

It follows from standard universal algebra considerations that for every di- or tri-Var-dendriform algebra A there exists a unique (up to isomorphism) universal enveloping Rota–Baxter algebra $U_\lambda(A)$ ($\lambda = 0$ in the case of dendriform dialgebras). Since there exists $B = \widehat{A}$ (or $\widehat{A}^{(\lambda)}$) such that ψ is injective, the map φ_λ has to be injective.

Corollary 3.14 (c.f. [12]).

Every di-Var-dendriform algebra embeds into its universal enveloping Rota–Baxter algebra of weight $\lambda = 0$ in Var.

Corollary 3.15.

Every tri-Var-dendriform algebra embeds into its universal enveloping Rota–Baxter algebra of weight $\lambda \neq 0$ in Var.

Remark 3.16.

All these results remain valid for dendriform algebras over a commutative ring with a unit provided that λ is invertible.

In [15], another structure of a dendriform dialgebra on an associative Rota–Baxter algebra B of arbitrary weight λ was proposed. In our terms, it corresponds to

$$a \vdash_i b = a \circ_i R(b) + \lambda a \circ_i b, \qquad a \dashv_i b = R(a) \circ_i b, \qquad a, b \in B. \tag{19}$$

Such a construction also admits an embedding of a di-Var-dendriform algebra into an appropriate Rota–Baxter algebra. It is enough to consider the case $\lambda \neq 0$. Indeed, an arbitrary di-Var-dendriform algebra A may be considered as a tri-Var-dendriform algebra with $a \perp_i b = 0$ for all $a, b \in A$, $i \in I$. Theorem 3.8 implies A to be embedded into the Rota–Baxter algebra $\widehat{A}^{(\lambda)} \in \mathrm{Var}$ of weight λ. Since A' is the image of A and $(A')^2 = 0$ in $\widehat{A}$ and hence in $\widehat{A}^{(\lambda)}$, the operations $\dashv_i$ and $\vdash_i$ in (19) coincide with those in (14).

4. Generalized trialgebras

Consider a slightly generalized analogue of trialgebras which we shortly call g-trialgebras.

Definition 4.1.

A generalized *tri*-Var-*algebra* (or *g-tri*-Var-*algebra*) is an $\Omega^{(3)}$-algebra satisfying the identities (1) and (7).

In other words, we exclude the identities $x_1 \dashv_i (x_2 \perp_j x_3) = x_1 \dashv_i (x_2 \dashv_j x_3)$, $(x_1 \perp_i x_2) \vdash_j x_3 = (x_1 \vdash_i x_2) \vdash_j x_3$ from the definition of a tri-Var-algebra.

For any $\Omega^{(3)}$-algebra A satisfying 0-identities (1) we can also construct (as in the dialgebra case) the Ω-algebra $\widehat{A} = \overline{A} \oplus A$ as follows (similarly as in (8)): $\overline{A} = A/\operatorname{Span}\{a \vdash_i b - a \dashv_i b : a, b \in A,\ i \in I\}$, $\overline{a} \circ_i \overline{b} = \overline{a \vdash_i b}$, $\overline{a} \circ_i b = a \vdash_i b$, $a \circ_i \overline{b} = a \dashv_i b$, $a \circ_i b = a \perp_i b$. An analogue of Proposition 2.9 holds for this construction and provides an equivalent definition of a g-tri-Var-algebra.

Example 4.2.

If Var = Com is the variety of associative and commutative algebras then it is sufficient to consider only two operations $\vdash$ and $\perp$ to define g-tri-Com-algebras. Both these operations are associative, $\perp$ is commutative, and they also satisfy the following identities:

$$x_1 \vdash (x_2 \perp x_3) = (x_1 \vdash x_2) \perp x_3, \qquad (x_1 \vdash x_2) \vdash x_3 = (x_2 \vdash x_1) \vdash x_3.$$

Let us denote the corresponding operad by gComTrias. It is easy to derive from the definition that the free algebra in gComTrias generated by a countable set $X = \{x_1, x_2, \dots\}$ is isomorphic as a linear space to the free algebra in Perm generated by the space of polynomials $\Bbbk[X]$. Its linear basis consists of words

$$u_1 \vdash u_1 \vdash \dots \vdash u_k \vdash u_0, \qquad u_1 \leqslant \dots \leqslant u_k,$$

where u_i are basic monomials of the polynomial algebra $\Bbbk[X]$ with respect to the operation $\perp$ and some linear ordering $\leqslant$.

Proposition 4.3 (c.f. [1]).

(i) *Let A be an Ω-algebra in the variety* Var *with a linear mapping T such that*

$$T(x) \circ_i T(y) = T(x \circ_i T(y)) = T(T(x) \circ_i y), \qquad x, y \in A, \quad i \in I. \tag{20}$$

*Then the space A with respect to operations $x \vdash_i y = T(x) \circ_i y$, $x \dashv_i y = x \circ_i T(y)$, $x \perp_i y = x \circ_i y$ is a g-tri-*Var*-algebra (let us denote it by $A^{(T)}$).*

(ii) *For every di-*Var*-algebra B there exists an Ω-algebra $A \in$ Var and an operator T satisfying (20) such that $B \subseteq A^{(T)}$.*

Proof. (i) Relation (20) implies that (1) hold in $A^{(T)}$. If $f(x_1, \dots, x_n) \in \mathcal{F}(n)$ and $H = \{k_1, \dots, k_l\} \subseteq \{1, \dots, n\}$, $l \geq 1$, then the value of $\Phi(n)(f^H)(a_1, \dots, a_n)$ in $A^{(T)}$ is equal to $f(T(a_1), \dots, a_{k_1}, \dots, a_{k_l}, \dots, T(a_n)) \in A$, i.e., all non-emphasized variables x_i are replaced with $T(x_i)$. Thus, if $A \in$ Var then $A^{(T)}$ is a g-tri-Var-algebra.

(ii) Given a di-Var-algebra B, consider $\widehat{B} = B \oplus \overline{B}$ as in Proposition 2.9 and define a linear mapping $T\colon \widehat{B} \to \widehat{B}$ in such a way that $T(a) = 0$, $\overline{T(\overline{a})} = \overline{a}$, $a \in B$. Then (20) holds trivially, and $B \subseteq \widehat{B}^{(T)}$. □

Example 4.4.

Let $\langle A, \cdot \rangle$ be an algebra in the variety Var with a derivation d such that $d^2 = 0$, see, e.g., [28]. Defining $a \vdash b = d(a) \cdot b$, $a \dashv b = a \cdot d(b)$, $a \perp b = a \cdot b$ we obtain a g-tri-Var-algebra $(A, \vdash, \dashv, \perp)$.

It turns out that g-tri-Var-algebras are closely related with Γ-conformal algebras introduced in [20]. These systems appeared as "discrete analogues" of conformal algebras defined over a group Γ. From the general point of view, these are pseudo-algebras over the group algebra $H=\Bbbk\Gamma$ considered as a Hopf algebra with respect to canonical coproduct $\Delta(\gamma)=\gamma\otimes\gamma$ and counit $\varepsilon(\gamma)=1$, $\gamma\in\Gamma$. Thus, a Γ-conformal algebra of a variety Var is just a Var-pseudo-algebra over $\Bbbk\Gamma$ as defined in subsection 2.2.

Consider a Γ-conformal algebra C with $H^{\otimes 2}$-linear operations $*_i\colon C\otimes C\to H^{\otimes 2}\otimes_H C$, $i\in I$, given by

$$a *_i b=\sum_{\gamma\in\Gamma}(\gamma\otimes 1)\otimes_H c_\gamma^i,\qquad a,b\in C.$$

Then the family of bilinear operations $\vdash_i,\dashv_i,\perp_i$, $i\in I$, on C can be defined as follows, c.f. (9):

$$a\dashv_i b=\sum_{\gamma\in\Gamma}c_\gamma^i,\qquad a\vdash_i b=\sum_{\gamma\in\Gamma}\gamma c_\gamma^i,\qquad a\perp_i b=c_e^i,$$

where e is the unit element of Γ. Denote the $\Omega^{(3)}$-algebra obtained by $C^{(0)}$. The $H^{\otimes 2}$-linearity of $*_i$ implies

$$a\vdash_i b=\sum_{\gamma\in\Gamma}(\gamma a)\perp_i b,\qquad a\dashv_i b=\sum_{\gamma\in\Gamma}a\perp_i(\gamma b),\qquad a,b\in C,\quad i\in I$$

(the sums are finite even if Γ is an infinite group).

Proposition 4.5.

If C is a Γ-conformal algebra of the variety Var *then $C^{(0)}$ is a g-tri-*Var*-algebra.*

Proof. For every $n\geq 1$ and for every $\emptyset\neq K=\{k_1,\dots,k_l\}\subseteq\{1,\dots,n\}$ define a linear map $\Phi_n^K\colon H^{\otimes n}\to H$, $H=\Bbbk\Gamma$, as follows:

$$\Phi_n^K(\gamma_1\otimes\dots\otimes\gamma_n)=\begin{cases}\gamma_{k_1} & \text{if}\quad \gamma_{k_1}=\ldots=\gamma_{k_l},\\ 0 & \text{otherwise.}\end{cases}$$

This is obviously a morphism of right H-modules. Hence, it can be extended to a map $\Phi_n^K\otimes_H \mathrm{id}_C\colon H^{\otimes n}\otimes_H C\to C$ by the rule $F\otimes_H a\mapsto\Phi_n^K(F)a$, $F\in H^{\otimes n}$, $a\in C$. Later we will not distinguish Φ_n^K and $\Phi_n^K\otimes_H\mathrm{id}_C$ since C is fixed.

Lemma 4.6.

For all $f\in\mathcal{F}(n)$, $\emptyset\neq K\subseteq\{1,\dots,n\}$, and $a_1,\dots,a_n\in C$, the following equality holds in $C^{(0)}$:

$$\big(\Phi(n)(f^K)\big)(a_1,\dots,a_n)=\Phi_n^K(f^{(*)}(a_1,\dots,a_n)),\tag{21}$$

where $\Phi(n)$ is the map defined in (5).

Proof. It is enough to prove (21) for all monomials in $\mathcal{F}(n)$. First, let us consider a monomial $v=v(x_1,\dots,x_n)$ such that $v^{(*)}=v^{\circledast}$, see subsection 2.2. Proceed by induction on $n\geq 1$. For $n=1$ the statement is clear. For $n>1$, assume (21) is true for all shorter monomials $w\in\mathcal{F}(m)$, $m<n$, such that $w^{(*)}=w^{\circledast}$. Then $v=v_1(x_1,\dots,x_p)\circ_i v_2(x_{p+1},\dots,x_n)$, $v_j^{(*)}=v_j^{\circledast}$ for $j=1,2$. Suppose

$$v_1^{(*)}(a_1,\dots,a_p)=\sum_\xi F_\xi\otimes_H b_\xi,\qquad F_\xi\in H^{\otimes p},\quad b_\xi\in C;$$

$$v_2^{(*)}(a_{p+1},\dots,a_n)=\sum_\eta G_\eta\otimes_H c_\eta,\qquad G_\eta\in H^{\otimes(n-p)},\quad c_\eta\in C;$$

$$b_\xi *_i c_\eta=\sum_\zeta\big(\alpha_{1,\zeta}^{(\xi,\eta)}\otimes\alpha_{2,\zeta}^{(\xi,\eta)}\big)\otimes_H d_\zeta^{(\xi,\eta)},\qquad \alpha_{j,\zeta}^{(\xi,\eta)}\in\Gamma,\quad d_\zeta^{(\xi,\eta)}\in C.$$

Then

$$v^{(*)}(a_1,\dots,a_n)=\sum_{\xi,\eta,\zeta}\big(F_\xi\alpha_{1,\zeta}^{(\xi,\eta)}\otimes G_\eta\alpha_{2,\zeta}^{(\xi,\eta)}\big)\otimes_H d_\zeta^{(\xi,\eta)}\in H^{\otimes n}\otimes_H C.$$

Without loss of generality we may assume $F_\xi=\gamma_{1,\xi}\otimes\dots\otimes\gamma_{p,\xi}$, $G_\eta=\beta_{1,\eta}\otimes\dots\otimes\beta_{n-p,\eta}$, where $\gamma_{j,\xi},\beta_{j,\xi}\in\Gamma$.

There are three cases: (a) $K_1=K\cap\{1,\dots,p\}=\emptyset$, (b) $K_2=\{j-p:j\in K\cap\{p+1,\dots,n\}\}=\emptyset$, (c) $K_1,K_2\neq\emptyset$.

In the first case, the inductive assumption and (5) imply

$$\begin{aligned}\big(\Phi(n)(v^K)\big)(a_1,\dots,a_n)&=v_1^{\vdash}(a_1,\dots,a_p)\vdash_i\big(\Phi(n-p)(v_2^{K_2})\big)(a_{p+1},\dots,a_n)\\&=\Phi_p^{\{p\}}\big(v_1^{(*)}(a_1,\dots,a_p)\big)\vdash_i\Phi_{n-p}^{K_2}\big(v_2^{(*)}(a_{p+1},\dots,a_n)\big)\\&=\Big(\sum_\xi\gamma_{p,\xi}b_\xi\Big)\vdash_i\Big(\sum_\eta\Phi_{n-p}^{K_2}(G_\eta)c_\eta\Big)=\sum_{\xi,\eta,\zeta}\Phi_{n-p}^{K_2}(G_\eta)\alpha_{2,\zeta}^{(\xi,\eta)}d_\zeta^{(\xi,\eta)}.\end{aligned}$$

On the other hand, since $K\subseteq\{p+1,\dots,n\}$, we may ignore the first p tensor multipliers, so $\Phi_n^K\big(F_\xi\alpha_{1,\zeta}^{(\xi,\eta)}\otimes G_\eta\alpha_{2,\zeta}^{(\xi,\eta)}\big)=\Phi_{n-p}^{K_2}\big(G_\eta\alpha_{2,\zeta}^{(\xi,\eta)}\big)=\Phi_{n-p}^{K_2}(G_\eta)\alpha_{2,\zeta}^{(\xi,\eta)}$, and the claim follows. The case (b) ($K\subseteq\{1,\dots,p\}$) is completely analogous. Consider the third one. If both K_1 and K_2 are nonempty then the inductive assumption and (5) imply

$$\begin{aligned}\big(\Phi(n)(v^K)\big)(a_1,\dots,a_n)&=\big(\Phi(p)(v_1^{K_1})\big)(a_1,\dots,a_p)\perp_i\big(\Phi(n-p)(v_2^{K_2})\big)(a_{p+1},\dots,a_n)\\&=\Phi_p^{K_1}\big(v_1^{(*)}(a_1,\dots,a_p)\big)\perp_i\Phi_{n-p}^{K_2}\big(v_2^{(*)}(a_{p+1},\dots,a_n)\big)\\&=\sum_{\xi,\eta,\zeta}\Phi_2^{\{1,2\}}\Big(\Phi_p^{K_1}(F_\xi)\alpha_{1,\zeta}^{(\xi,\eta)}\otimes\Phi_{n-p}^{K_2}(G_\eta)\alpha_{2,\zeta}^{(\xi,\eta)}\Big)d_\zeta^{(\xi,\eta)}\\&=\sum_{\xi,\eta,\zeta}\Phi_n^K\big(F_\xi\alpha_{1,\zeta}^{(\xi,\eta)}\otimes G_\eta\alpha_{2,\zeta}^{(\xi,\eta)}\big)d_\zeta^{(\xi,\eta)}.\end{aligned}\tag{22}$$

To get the last equation, we used the obvious relation $\Phi_2^{\{1,2\}}\big(\Phi_p^{K_1}(F)\otimes\Phi_{n-p}^{K_2}(G)\big)=\Phi_n^K(F\otimes G)$, $F\in H^{\otimes p}$, $G\in H^{\otimes(n-p)}$. On the other hand, $\Phi(n)^K\big(v^{(*)}(a_1,\dots,a_n)\big)$ by definition is equal to the right-hand side of (22).

To complete the proof, it remains to consider $u=v(x_{\sigma(1)},\dots,x_{\sigma(n)})$, where $v^{(*)}=v^{\circledast}$. In this case, $u^{(*)}(a_1,\dots,a_n)=(\sigma\otimes_H\mathrm{id}_C)\big(v^{(*)}(a_{\sigma(1)},\dots,a_{\sigma(n)})\big)$. By the definition of Φ_n^K, we have

$$\Phi_n^K\big(u^{(*)}(a_1,\dots,a_n)\big)=\Phi_n^{\sigma^{-1}(K)}\big(v^{(*)}(a_{\sigma(1)},\dots,a_{\sigma(n)})\big).$$

On the other hand, $\Phi(n)(u^K)=\big(\Phi(n)(v^{\sigma^{-1}(K)})\big)^\sigma$ by (5). Therefore,

$$\big(\Phi(n)(u^K)\big)(a_1,\dots,a_n)=\Phi(n)\big(v^{\sigma^{-1}(K)}\big)(a_{\sigma(1)},\dots,a_{\sigma(n)}).$$

Since the statement is already proved for v, the relation (21) holds for u as well. □

If for some $f\in\mathcal{F}(n)$ the H-pseudo-algebra C satisfies $f^{(*)}(a_1,\dots,a_n)=0$ for all $a_1,\dots,a_n\in C$, then by Lemma 4.6 the $\Omega^{(3)}$-algebra $C^{(0)}$ satisfies the identities $f(x_1,\dots,\dot{x}_{k_1},\dots,\dot{x}_{k_l},\dots,x_n)=\Phi(n)(f^K)$, $K=\{k_1,\dots,k_l\}$. Hence, this is a g-tri-Var-algebra. □

Remark 4.7.

Relation (21) implies, in particular, that C with respect to $\perp_i$, $i\in I$, is an Ω-algebra from Var. If $|\Gamma|<\infty$ then the operator $T\colon C\to C$, $T(a)=\sum_{\gamma\in\Gamma}\gamma a$, is well-defined, and it satisfies (20). In this case, the structure of a g-tri-Var-algebra on $C^{(0)}$ is given by Proposition 4.3.

There is an interesting question whether a trialgebra or g-trialgebra A can be embedded into $C^{(0)}$ for some pseudo-algebra C. We have a positive answer for tri-Var-algebras, but only for $\operatorname{char}\Bbbk = p > 0$: The mapping ι from (10) realizes such an embedding of A into the Γ-conformal algebra $\operatorname{Cur}\widehat{A}$ when $T = \gamma_1 + \dots + \gamma_p$, where $e \neq \gamma_i$ are pairwise distinct elements of a group Γ such that $|\Gamma| \geq p+1$.

Example 4.8.
A g-tri-As-algebra A with respect to the operations $[x,y] = x \dashv y - x \vdash y$ and $x \cdot y = x \perp y$ turns into a noncommutative dialgebra analogue of a Poisson algebra: The operation $[\cdot,\cdot]$ satisfies the Leibniz identity and $\cdot$ is associative. Moreover, the Poisson identity holds:

$$[xy,z] = x[y,z] + [x,z]y.$$

In [30], the same operations $[\cdot,\cdot]$ and $\cdot$ were considered for tri-As-algebras (in the sense of Definition 2.6). The analogue of a Poisson algebra obtained in this way satisfies one more identity $[x, yz - zy] = [x,[y,z]]$ which does not appear in the case of generalized trialgebras.

It is natural to conjecture that, as in the case of tri-Var-algebras, the operad governing the variety of g-tri-Var-algebras can be obtained by the white product procedure in the case when Var is quadratic. Let us recall the definition of a white product of quadratic binary operads [19]. For an S_2-module E, denote $\Bbbk S_3 \otimes_{\Bbbk S_2} (E\otimes E)$ by $\mathcal{F}(E)(3)$. In $\mathcal{F}(E)(3)$, the transposition $(12) \in S_2$ acts on the tensor square $E\otimes E$ as $\mathrm{id}\otimes(12)$. If $\mathcal{P}_1 = \mathcal{P}(E_1,R_1)$ and $\mathcal{P}_2 = \mathcal{P}(E_2,R_2)$ are two quadratic binary operads then the Manin white product $\mathcal{P}_1 \circ \mathcal{P}_2$ is the sub-operad in $\mathcal{P}_1 \otimes \mathcal{P}_2$ generated by $E_1 \otimes E_2$ (here $(12) \in S_2$ acts on $E_1\otimes E_2$ as $(12)\otimes(12)$). Consider the S_3-linear injection

$$\Sigma\colon (\mathcal{F}(E_1\otimes E_2))(3) \to \mathcal{F}(E_1)(3)\otimes \mathcal{F}(E_2)(3)$$

given by $\Sigma\colon \sigma \otimes_{\Bbbk S_2} \big((e_1\otimes\mu_1)\otimes(e_2\otimes\mu_2)\big) \mapsto (\sigma\otimes_{\Bbbk S_2}(e_1\otimes e_2))\otimes(\sigma\otimes_{\Bbbk S_2}(\mu_1\otimes\mu_2))$.

Denote the image of Σ by $\mathcal{D}(E_1,E_2)$. The images of defining identities of an algebra over $\mathcal{P}_1\otimes\mathcal{P}_2$ under Σ have to fall into $R = R_1\otimes\mathcal{F}(E_2)(3) + \mathcal{F}(E_1)(3)\otimes R_2$, so to compute the white product one has to find the intersection of $\mathcal{D}(E_1,E_2)$ and R. This is a routine problem of linear algebra, but the amount of computations is usually very large.

In our case, the operad $\mathcal{P}_1 = \mathrm{gComTrias}$ is defined by 3-dimensional $E_1 = \Bbbk e \oplus \Bbbk e^{(12)} \oplus \Bbbk f$, $f^{(12)} = f$, and 17-dimensional subspace $R_1 \subset \mathcal{F}(E_1)(3)$. The operad of associative algebras has 2-dimensional $E_2 = \Bbbk\mu\oplus\Bbbk\mu^{(12)}$ and 6-dimensional R_2. In $\mathcal{F}(E_1\otimes E_2)$, one has to interpret $e\otimes\mu$ as $x_1 \vdash x_2$, $e^{(12)}\otimes\mu$ as $x_1 \dashv x_2$, $e\otimes\mu^{(12)}$ as $x_2 \dashv x_1$, $e^{(12)}\otimes\mu^{(12)}$ as $x_2\vdash x_1$, $f\otimes\mu$ as $x_1 \perp x_2$, and $f\otimes\mu^{(12)}$ as $x_2\perp x_1$.

A simple computer program allowed us to make sure that gComTrias $\circ$ As and gComTrias $\circ$ Lie define the varieties of g-tri-As- and g-tri-Lie-algebras, respectively. In particular, the class of g-tri-Lie-algebras consists of linear spaces L with two operations $[x,y] = x\vdash y$ and $(x,y) = x\perp y$ such that L is a Leibniz algebra with respect to $[\cdot,\cdot]$ and Lie algebra with respect to $(\cdot,\cdot)$. These operations are related by one binary-quadratic relation $([x,y],z) = [x,(y,z)] + ([x,z],y)$. Such a relation has recently appeared in [39]. We conjecture that a similar relation holds for every quadratic binary operad.

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