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Abstract: The aim of this paper is to analyze a specific fourth-order matrix spectral problem involving four potentials and two free nonzero parameters and construct an associated integrable hierarchy of bi-Hamiltonian equations within the zero curvature formulation. A hereditary recursion operator is explicitly computed, and the corresponding bi-Hamiltonian formulation is established by the so-called trace identity, showing the Liouville integrability of the obtained hierarchy. Two illustrative examples are novel generalized combined nonlinear Schrödinger equations and modified Korteweg–de Vries equations with four components and two adjustable parameters.

Keywords: matrix spectral problem; Lax pair; integrable hierarchy; nonlinear Schrödinger equations; modified Korteweg–de Vries equations

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1. Introduction

Lax pairs of matrix spectral problems [1] play a central role in the study of mathematical integrability and soliton theory, providing powerful tools for understanding and solving nonlinear partial differential equations arising in physics and mathematics [2,3]. Particularly, one can construct infinitely many symmetries and conserved quantities from associated Lax pairs. Integrable models arise in various areas of physics, including classical mechanics, quantum mechanics, nonlinear optics, fluid dynamics and plasma physics. Examples of integrable models include the Korteweg–de Vries equation, the nonlinear Schrödinger equation, the sine-Gordon equation, and the Toda lattice equation, among others.

Integrable models come in hierarchies and typical examples of integrable hierarchies are the Ablowitz–Kaup–Newell–Segur (AKNS) hierarchy [4] and its various hierarchies of integrable couplings [5]. Matrix Lie algebras are the key to formulate meaningful Lax pairs [6,7], generating integrable models. In mathematics, it has always been intriguing to identify and classify matrix spectral problems that yield integrable hierarchies. There are many examples with one or two potentials but few examples with multiple potentials. In this paper, we would like to present a new matrix spectral problem based on a specific matrix Lie algebra and construct an associated integrable hierarchy with four potentials.

It is known that the zero curvature formulation is a powerful approach for constructing integrable hierarchies, which is briefly stated as follows (see [7,8] for more details). In our discussion, we denote the spectral parameter by λ and a q -dimensional column potential

vector by $u = (u_1, \dots, u_q)^T$. First, take a given loop matrix algebra $\tilde{\mathfrak{g}}$ with the loop parameter λ , and formulate a spatial spectral matrix:

$$\mathcal{M} = \mathcal{M}(u, \lambda) = u_1 h_1(\lambda) + \dots + u_q h_q(\lambda) + h_0(\lambda), \quad (1)$$

where the elements h_1, \dots, h_q are linear independent in $\tilde{\mathfrak{g}}$. We assume that the above element h_0 is pseudo-regular:

$$\text{Im ad}_{h_0} \oplus \text{Ker ad}_{h_0} = \tilde{\mathfrak{g}}, [\text{Ker ad}_{h_0}, \text{Ker ad}_{h_0}] = 0, \quad (2)$$

where ad_{h_0} denotes the adjoint action of h_0 on the Lie algebra $\tilde{\mathfrak{g}}$. This condition is helpful in determining a Laurent series solution $Y = \sum_{n \geq 0} \lambda^{-n} Y^{[n]}$ to a stationary zero curvature equation

$$Y_x = [\mathcal{M}, Y] \quad (3)$$

in the underlying loop algebra $\tilde{\mathfrak{g}}$.

Second, we introduce an infinite sequence of temporal spectral matrices

$$\mathcal{N}^{[m]} = \mathcal{N}^{[m]}(u, \lambda) = (\lambda^m Y)_+ + \Delta_m = \sum_{n=0}^m \lambda^{m-n} Y^{[n]} + \Delta_m, \quad m \geq 0, \quad (4)$$

where $\Delta_m \in \tilde{\mathfrak{g}}$, $m \geq 0$, as the other parts of a sequence of Lax pairs, to generate a hierarchy of integrable models:

$$u_{t_m} = X^{[m]} = X^{[m]}(u), \quad m \geq 0, \quad (5)$$

via the zero curvature equations

$$\mathcal{M}_{t_m} - \mathcal{N}_x^{[m]} + [\mathcal{M}, \mathcal{N}^{[m]}] = 0, \quad m \geq 0. \quad (6)$$

These zero curvature equations represent the solvability conditions of the spatial and temporal matrix spectral problems:

$$\varphi_x = \mathcal{M}(u, \lambda) \varphi, \quad \varphi_{t_m} = \mathcal{N}^{[m]}(u, \lambda) \varphi, \quad m \geq 0. \quad (7)$$

Finally, we furnish Hamiltonian formulations by the so-called trace identity:

$$\frac{\delta}{\delta u} \int \text{tr} \left(Y \frac{\partial \mathcal{M}}{\partial \lambda} \right) dx = \lambda^{-\kappa} \frac{\partial}{\partial \lambda} \lambda^{\kappa} \text{tr} \left(Y \frac{\partial \mathcal{M}}{\partial u} \right), \quad (8)$$

where $\frac{\delta}{\delta u}$ is the variational derivative with respect to u , and κ is a constant, independent of λ , determined by

$$\kappa = -\frac{\lambda}{2} \frac{\partial}{\partial \lambda} \ln |\text{tr}(Y^2)|. \quad (9)$$

for the resulting hierarchy (5). Further, a hereditary recursion operator Φ , which is determined from the recursion relation $X^{[m+1]} = \Phi X^{[m]}$, enables us to establish a bi-Hamiltonian formulation and show the Liouville integrability (see, e.g., [7,9]) for the obtained hierarchy (5).

Many hierarchies of Liouville integrable models have been constructed via the zero curvature formulation (see, e.g., [4–16]). When $q = 2$, namely, in the case of two potentials, we have the AKNS hierarchy [4], the Heisenberg hierarchy [17], the Kaup–Newell hierarchy [18] and the Wadati–Konno–Ichikawa hierarchy [19]. All of the corresponding spectral matrices are 2×2 and contain two potentials, whose spectral problems are of the second order and solvable within the theory of special functions.

In this paper, we would like to construct an integrable hierarchy of combined Liouville integrable models with four potentials via the zero curvature formulation. The key point is to introduce a specific 4×4 matrix spectral problem. The corresponding Hamiltonian formulations are established by an application of the so-called trace identity, and, further, a hereditary recursion operator is computed and used to furnish a bi-Hamiltonian formula-

tion and thus show the Liouville integrability for the resulting hierarchy. Two illustrative examples of novel combined integrable nonlinear Schrödinger and modified Korteweg–de Vries models are presented, together with their uncombined reductions. The final section gives a conclusion and a few concluding remarks. An open question is how to generalize the presented four-component integrable models to six-component or more-component integrable Hamiltonian equation models.

2. A Matrix Spectral Problem and Its Four-Component Integrable Hierarchy

Let δ be an arbitrary constant, r an arbitrary natural number and T a square matrix of order r , whose inverse is given by its negative. Obviously, a set \tilde{g} of block matrices

$$\tilde{g} = \left\{ A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}_{2r \times 2r} \middle| A_4 = TA_1T^{-1}, A_3 = \delta TA_2T^{-1} \right\} \quad (10)$$

forms a matrix Lie algebra, while the matrix commutator $[A, B] = AB - BA$ is taken as its Lie bracket. We will use a special case of this Lie algebra with $r = 2$ and

$$T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad (11)$$

to formulate a specific spectral matrix below.

Let $u = u(x, t) = (u_1, u_2, u_3, u_4)^T$ ($x, t \in \mathbb{R}$) be a four-component potential vector, and α_1, α_2 and δ_1, δ_2 , two pairs of arbitrary constants. Assume that

$$\alpha = \alpha_1 + \alpha_2 \neq 0, \delta_1\delta_2 \neq 0. \quad (12)$$

Motivated by recent studies on matrix spectral problems with four potentials (see, e.g., [20–22] by us and [23,24] by other authors), let us introduce and consider a matrix spectral problem of the form:

$$\varphi_x = \mathcal{M}\varphi = \mathcal{M}(u, \lambda)\varphi, \quad \mathcal{M} = \begin{bmatrix} 0 & \delta_1 u_1 & u_2 & \alpha_1 \lambda \\ \delta_1 u_3 & 0 & \alpha_2 \lambda & u_4 \\ \delta_1 \delta_2 u_4 & -\delta_1 \delta_2 \alpha_2 \lambda & 0 & -\delta_1 u_3 \\ -\delta_1 \delta_2 \alpha_1 \lambda & \delta_1 \delta_2 u_2 & -\delta_1 u_1 & 0 \end{bmatrix}, \quad (13)$$

where λ is again the spectral parameter. This spectral matrix is from the matrix Lie algebra previously defined, with $r = 2$ and T by (11). The spectral problem is not any reduction of the matrix AKNS spectral problem (see, e.g., [25]), but it enables us to generate an integrable hierarchy, each of which is bi-Hamiltonian and possesses a combined structure.

As usual, to construct an associated Liouville integrable hierarchy, we first solve the corresponding stationary zero curvature Equation (3). A solution Y is assumed to be of the form:

$$Y = \begin{bmatrix} \delta_1 a & \delta_1 b & e & f \\ \delta_1 c & -\delta_1 a & f & g \\ \delta_1 \delta_2 g & -\delta_1 \delta_2 f & -\delta_1 a & -\delta_1 c \\ -\delta_1 \delta_2 f & \delta_1 \delta_2 e & -\delta_1 b & \delta_1 a \end{bmatrix} = \sum_{n \geq 0} \lambda^{-n} Y^{[n]}, \quad (14)$$

where all basic objects are taken to be of Laurent series type:

$$\begin{cases} a = \sum_{n \geq 0} \lambda^{-n} a^{[n]}, & b = \sum_{n \geq 0} \lambda^{-n} b^{[n]}, & c = \sum_{n \geq 0} \lambda^{-n} c^{[n]}, \\ e = \sum_{n \geq 0} \lambda^{-n} e^{[n]}, & f = \sum_{n \geq 0} \lambda^{-n} f^{[n]}, & g = \sum_{n \geq 0} \lambda^{-n} g^{[n]}. \end{cases} \quad (15)$$

We take a solution of the above form, because this is the form that the commutator between any matrix in \tilde{g} and the spectral matrix \mathcal{M} takes. Clearly, the corresponding stationary zero curvature Equation (3) leads equivalently to

$$\begin{cases} a_x = \delta_1 c u_1 + \delta_2 g u_2 - \delta_1 b u_3 - \delta_2 e u_4, \\ b_x = \alpha \delta_2 \lambda e - 2\delta_1 a u_1 - 2\delta_2 f u_2, \\ c_x = \alpha \delta_2 \lambda g + 2\delta_1 a u_3 - 2\delta_2 f u_4, \end{cases} \quad (16)$$

$$\begin{cases} e_x = -\alpha \delta_1 \lambda b - 2\delta_1 a u_2 + 2\delta_1 f u_1, \\ g_x = -\alpha \delta_1 \lambda c + 2\delta_1 a u_4 + 2\delta_1 f u_3, \\ f_x = \delta_1 g u_1 - \delta_1 c u_2 + \delta_1 e u_3 - \delta_1 b u_4. \end{cases} \quad (17)$$

These equations exactly generate the initial conditions

$$a_x^{[0]} = 0, \quad b^{[0]} = c^{[0]} = e^{[0]} = g^{[0]} = 0, \quad f_x^{[0]} = 0, \quad (18)$$

and the recursion relations to determine the Laurent series solution Y

$$\begin{cases} b^{[n+1]} = \frac{1}{\alpha \delta_1} (-e_x^{[n]} - 2\delta_1 a^{[n]} u_2 + 2\delta_1 f^{[n]} u_1), \\ c^{[n+1]} = \frac{1}{\alpha \delta_1} (-g_x^{[n]} + 2\delta_1 a^{[n]} u_4 + 2\delta_1 f^{[n]} u_3), \end{cases} \quad (19)$$

$$\begin{cases} e^{[n+1]} = \frac{1}{\alpha \delta_2} (b_x^{[n]} + 2\delta_2 f^{[n]} u_2 + 2\delta_1 a^{[n]} u_1), \\ g^{[n+1]} = \frac{1}{\alpha \delta_2} (c_x^{[n]} + 2\delta_2 f^{[n]} u_4 - 2\delta_1 a^{[n]} u_3), \end{cases} \quad (20)$$

$$\begin{cases} a_x^{[n+1]} = \delta_1 c^{[n+1]} u_1 + \delta_2 g^{[n+1]} u_2 - \delta_1 b^{[n+1]} u_3 - \delta_2 e^{[n+1]} u_4, \\ f_x^{[n+1]} = \delta_1 g^{[n+1]} u_1 - \delta_1 c^{[n+1]} u_2 + \delta_1 e^{[n+1]} u_3 - \delta_1 b^{[n+1]} u_4, \end{cases} \quad (21)$$

where $n \geq 0$. To compute the Laurent series solution concretely, let us take the initial data

$$a^{[0]} = \frac{1}{2}\beta, \quad f^{[0]} = \frac{1}{2}\gamma, \quad (22)$$

where β and γ are a pair of arbitrary constants, and assume the constants of integration to be zero

$$a^{[n]}|_{u=0} = 0, \quad f^{[n]}|_{u=0} = 0, \quad n \geq 1. \quad (23)$$

Under those restrictions, one can work out that

$$\begin{cases} b^{[1]} = \frac{1}{\alpha}(\gamma u_1 - \beta u_2), \quad c^{[1]} = \frac{1}{\alpha}(\gamma u_3 + \beta u_4), \\ e^{[1]} = \frac{1}{\alpha \delta_2}(\delta_1 \beta u_1 + \delta_2 \gamma u_2), \quad g^{[1]} = \frac{1}{\alpha \delta_2}(-\delta_1 \beta u_3 + \delta_2 \gamma u_4), \\ a^{[1]} = f^{[1]} = 0; \end{cases}$$

$$\begin{cases} b^{[2]} = -\frac{1}{\alpha^2 \delta_1 \delta_2}(\delta_1 \beta u_{1,x} + \delta_2 \gamma u_{2,x}), \quad c^{[2]} = \frac{1}{\alpha^2 \delta_1 \delta_2}(\delta_1 \beta u_{3,x} - \delta_2 \gamma u_{4,x}), \\ e^{[2]} = \frac{1}{\alpha^2 \delta_2}(\gamma u_{1,x} - \beta u_{2,x}), \quad g^{[2]} = \frac{1}{\alpha^2 \delta_2}(\gamma u_{3,x} + \beta u_{4,x}), \\ a^{[2]} = \frac{1}{\alpha^2 \delta_2}[(\delta_1 \beta u_3 - \delta_2 \gamma u_4)u_1 + \delta_2(\gamma u_3 + \beta u_4)u_2], \\ f^{[2]} = \frac{1}{\alpha^2 \delta_2}[\delta_1(\gamma u_1 - \beta u_2)u_3 + (\delta_1 \beta u_1 + \delta_2 \gamma u_2)u_4]; \end{cases}$$

$$\begin{cases} b^{[3]} = \frac{1}{\alpha^3 \delta_1 \delta_2} [-\gamma u_{1,xx} + \beta u_{2,xx} + 2\delta_1(\gamma u_3 + \beta u_4)(\delta_1 u_1^2 - \delta_2 u_2^2) \\ \quad - 4\delta_1(\delta_1 \beta u_3 - \delta_2 \gamma u_4)u_1 u_2], \\ c^{[3]} = \frac{1}{\alpha^3 \delta_1 \delta_2} [-\gamma u_{3,xx} - \beta u_{4,xx} + 2\delta_1(\gamma u_1 - \beta u_2)(\delta_1 u_3^2 - \delta_2 u_4^2) \\ \quad + 4\delta_1(\delta_1 \beta u_1 + \delta_2 \gamma u_2)u_3 u_4], \\ e^{[3]} = \frac{1}{\alpha^3 \delta_1 \delta_2} [-\delta_1 \beta u_{1,xx} - \delta_2 \gamma u_{2,xx} + 2\delta_1(\delta_1 \beta u_3 - \delta_2 \gamma u_4)(\delta_1 u_1^2 - \delta_2 u_2^2) \\ \quad + 4\delta_1^2 \delta_2(\gamma u_3 + \beta u_4)u_1 u_2], \\ g^{[3]} = \frac{1}{\alpha^3 \delta_1 \delta_2} [\delta_1 \beta u_{3,xx} - \delta_2 \gamma u_{4,xx} - 2\delta_1(\delta_1 \beta u_1 + \delta_2 \gamma u_2)(\delta_1 u_3^2 - \delta_2 u_4^2) \\ \quad + 4\delta_1^2 \delta_2(\gamma u_1 - \beta u_2)u_3 u_4], \\ a^{[3]} = \frac{1}{\alpha^3 \delta_1 \delta_2} [\delta_1(\gamma u_3 + \beta u_4)u_{1,x} - (\delta_1 \beta u_3 - \delta_2 \gamma u_4)u_{2,x} \\ \quad - \delta_1(\gamma u_1 - \beta u_2)u_{3,x} - (\delta_1 \beta u_1 + \delta_2 \gamma u_2)u_{4,x}], \\ f^{[3]} = \frac{1}{\alpha^3 \delta_2^2} [-(\delta_1 \beta u_3 - \delta_2 \gamma u_4)u_{1,x} - \delta_2(\gamma u_3 + \beta u_4)u_{2,x} \\ \quad + (\delta_1 \beta u_1 + \delta_2 \gamma u_2)u_{3,x} - \delta_2(\gamma u_1 - \beta u_2)u_{4,x}]; \end{cases}$$

and

$$\begin{cases} b^{[4]} = \frac{1}{\alpha^4 \delta_1^2 \delta_2^2} \{ \delta_1 \beta u_{1,xxx} + \delta_2 \gamma u_{2,xxx} - 6\delta_1[\delta_1(\delta_1 \beta u_3 - \delta_2 \gamma u_4)u_1 \\ \quad + \delta_2(\gamma u_3 + \beta u_4)u_2]u_{1,x} - 6\delta_1 \delta_2[\delta_1(\gamma u_3 + \beta u_4)u_1 - (\delta_1 \beta u_3 - \delta_2 \gamma u_4)u_2]u_{2,x} \}, \\ c^{[4]} = \frac{1}{\alpha^4 \delta_1^2 \delta_2^2} \{ -\delta_1 \beta u_{3,xxx} + \delta_2 \gamma u_{4,xxx} + 6\delta_1^2[(\delta_1 \beta u_3 - \delta_2 \gamma u_4)u_1 \\ \quad + \delta_2(\gamma u_3 + \beta u_4)u_2]u_{3,x} - 6\delta_1 \delta_2[\delta_1(\gamma u_3 + \beta u_4)u_1 - (\delta_1 \beta u_3 - \delta_2 \gamma u_4)u_2]u_{4,x} \}, \\ e^{[4]} = \frac{1}{\alpha^4 \delta_1 \delta_2^2} \{ -\gamma u_{1,xxx} + \beta u_{2,xxx} + 6\delta_1[\delta_1(\gamma u_3 + \beta u_4)u_1 - (\delta_1 \beta u_3 - \delta_2 \gamma u_4)u_2]u_{1,x} \\ \quad - 6\delta_1[(\delta_1 \beta u_3 - \delta_2 \gamma u_4)u_1 + \delta_2(\gamma u_3 + \beta u_4)u_2]u_{2,x} \}, \\ g^{[4]} = \frac{1}{\alpha^4 \delta_1 \delta_2^2} \{ -\gamma u_{3,xxx} - \beta u_{4,xxx} + 6\delta_1[\delta_1(\gamma u_3 + \beta u_4)u_1 - (\delta_1 \beta u_3 - \delta_2 \gamma u_4)u_2]u_{3,x} \\ \quad + 6\delta_1[(\delta_1 \beta u_3 - \delta_2 \gamma u_4)u_1 + \delta_2(\gamma u_3 + \beta u_4)u_2]u_{4,x} \}, \\ a^{[4]} = \frac{1}{\alpha^4 \delta_1 \delta_2^2} [-(\delta_1 \beta u_3 - \delta_2 \gamma u_4)u_{1,xx} - \delta_2(\gamma u_3 + \beta u_4)u_{2,xx} - (\delta_1 \beta u_1 + \delta_2 \gamma u_2)u_{3,xx} \\ \quad + \delta_2(\gamma u_1 - \beta u_2)u_{4,xx} + (\delta_1 \beta u_{3,x} - \delta_2 \gamma u_{4,x})u_{1,x} + \delta_2(\gamma u_{3,x} + \beta u_{4,x})u_{2,x} \\ \quad + 3\delta_1^2(\delta_1 \beta u_3^2 - 2\delta_2 \gamma u_3 u_4 - \delta_2 \beta u_4^2)u_1^2 + 6\delta_1 \delta_2(\delta_1 \gamma u_3^2 + 2\delta_1 \beta u_3 u_4 - \delta_2 \gamma u_4^2)u_1 u_2 \\ \quad - 3\delta_1 \delta_2(\delta_1 \beta u_3^2 - 2\delta_2 \gamma u_3 u_4 - \delta_2 \beta u_4^2)u_2^2], \\ f^{[4]} = \frac{1}{\alpha^4 \delta_1 \delta_2^2} [-\delta_1(\gamma u_3 + \beta u_4)u_{1,xx} + (\delta_1 \beta u_3 - \delta_2 \gamma u_4)u_{2,xx} - \delta_1(\gamma u_1 - \beta u_2)u_{3,xx} \\ \quad - (\delta_1 \beta u_1 + \delta_2 \gamma u_2)u_{4,xx} + \delta_1(\gamma u_{3,x} + \beta u_{4,x})u_{1,x} - (\delta_1 \beta u_{3,x} - \delta_2 \gamma u_{4,x})u_{2,x} \\ \quad + 3\delta_1^2(\delta_1 \gamma u_3^2 + 2\delta_1 \beta u_3 u_4 - \delta_2 \gamma u_4^2)u_1^2 - 6\delta_1^2(\delta_1 \beta u_3^2 - 2\delta_2 \gamma u_3 u_4 - \delta_2 \beta u_4^2)u_1 u_2 \\ \quad - 3\delta_1 \delta_2(\delta_1 \gamma u_3^2 - 2\delta_1 \beta u_3 u_4 - \delta_2 \gamma u_4^2)u_2^2]. \end{cases}$$

All these computations allow us to impose $\Delta_r = 0$, $m \geq 0$, to introduce

$$\varphi_{t_m} = \mathcal{N}^{[m]} \varphi = \mathcal{N}^{[m]}(u, \lambda) \varphi, \quad \mathcal{N}^{[m]} = (\lambda^m Y)_+ = \sum_{n=0}^m \lambda^n Y^{[m-n]}, \quad m \geq 0, \quad (24)$$

which are the temporal matrix spectral problems within the zero curvature formulation. The conditions that guarantee the solvability of the spatial and temporal matrix spectral problems in (13) and (24) are the zero curvature equations in (6). They lead to a hierarchy of integrable models with four potentials:

$$u_{t_m} = X^{[m]} = X^{[m]}(u) := (\alpha\delta_2 e^{[m+1]}, -\alpha\delta_1 b^{[m+1]}, \alpha\delta_2 g^{[m+1]}, -\alpha\delta_1 c^{[m+1]})^T, \quad m \geq 0, \quad (25)$$

or more precisely,

$$u_{1,t_m} = \alpha\delta_2 e^{[m+1]}, \quad u_{2,t_m} = -\alpha\delta_1 b^{[m+1]}, \quad u_{3,t_m} = \alpha\delta_2 g^{[m+1]}, \quad u_{4,t_m} = -\alpha\delta_1 c^{[m+1]}, \quad m \geq 0. \quad (26)$$

Taking advantage of the previous derivations, we can present some particular examples. The first nonlinear example is the model of combined integrable nonlinear Schrödinger equations:

$$\left\{ \begin{array}{l} u_{1,t_2} = \frac{1}{\alpha^2\delta_1\delta_2} [-\delta_1\beta u_{1,xx} - \delta_2\gamma u_{2,xx} + 2\delta_1(\delta_1\beta u_3 - \delta_2\gamma u_4)(\delta_1 u_1^2 - \delta_2 u_2^2) \\ \quad + 4\delta_1^2\delta_2(\gamma u_3 + \beta u_4)u_1 u_2], \\ u_{2,t_2} = \frac{1}{\alpha^2\delta_2} [\gamma u_{1,xx} + \beta u_{2,xx} - 2\delta_1(\gamma u_3 + \beta u_4)(\delta_1 u_1^2 - \delta_2 u_2^2) \\ \quad + 4\delta_1(\delta_1\beta u_3 - \delta_2\gamma u_4)u_1 u_2], \\ u_{3,t_2} = \frac{1}{\alpha^2\delta_1\delta_2} [\delta_1\beta u_{3,xx} - \delta_2\gamma u_{4,xx} - 2\delta_1(\delta_1\beta u_1 + \delta_2\gamma u_2)(\delta_1 u_3^2 - \delta_2 u_4^2) \\ \quad + 4\delta_1^2\delta_2(\gamma u_1 - \beta u_2)u_3 u_4], \\ u_{4,t_2} = \frac{1}{\alpha^2\delta_2} [\gamma u_{3,xx} + \beta u_{4,xx} - 2\delta_1(\gamma u_1 - \beta u_2)(\delta_1 u_3^2 - \delta_2 u_4^2) \\ \quad + 4\delta_1(\delta_1\beta u_1 + \delta_2\gamma u_2)u_3 u_4], \end{array} \right. \quad (27)$$

and the second one is the model of combined integrable modified Korteweg–de Vries equations:

$$\left\{ \begin{array}{l} u_{1,t_3} = \frac{1}{\alpha^3\delta_1\delta_2} \{-\gamma u_{1,xxx} + \beta u_{2,xxx} + 6\delta_1[\delta_1(\gamma u_3 + \beta u_4)u_1 - (\delta_1\beta u_3 - \delta_2\gamma u_4)u_2]u_{1,x} \\ \quad - 6\delta_1[(\delta_1\beta u_3 - \delta_2\gamma u_4)u_1 + \delta_2(\gamma u_3 + \beta u_4)u_2]u_{2,x}\}, \\ u_{2,t_3} = \frac{1}{\alpha^3\delta_1\delta_2} \{-\delta_1\beta u_{1,xxx} - \delta_2\gamma u_{2,xxx} + 6\delta_1^2[(\delta_1\beta u_3 - \delta_2\gamma u_4)u_1 + \delta_2(\gamma u_3 + \beta u_4)u_2]u_{1,x} \\ \quad + 6\delta_1\delta_2[\delta_1(\gamma u_3 + \beta u_4)u_1 - (\delta_1\beta u_3 - \delta_2\gamma u_4)u_2]u_{2,x}\}, \\ u_{3,t_3} = -\frac{1}{\alpha^3\delta_1\delta_2} \{-\gamma u_{3,xxx} - \beta u_{4,xxx} + 6\delta_1[\delta_1(\gamma u_3 + \beta u_4)u_1 - (\delta_1\beta u_3 - \delta_2\gamma u_4)u_2]u_{3,x} \\ \quad + 6\delta_1[(\delta_1\beta u_3 - \delta_2\gamma u_4)u_1 + \delta_2(\gamma u_3 + \beta u_4)u_2]u_{4,x}\}, \\ u_{4,t_3} = \frac{1}{\alpha^3\delta_1\delta_2} \{\delta_1\beta u_{3,xxx} - \delta_2\gamma u_{4,xxx} - 6\delta_1^2[(\delta_1\beta u_3 - \delta_2\gamma u_4)u_1 + \delta_2(\gamma u_3 + \beta u_4)u_2]u_{3,x} \\ \quad + 6\delta_1\delta_2[\delta_1(\gamma u_3 + \beta u_4)u_1 - (\delta_1\beta u_3 - \delta_2\gamma u_4)u_2]u_{4,x}\}. \end{array} \right. \quad (28)$$

These provide two typical coupled integrable models, which extend the category of coupled integrable models of nonlinear Schrödinger equations and modified Korteweg–de Vries equations, presented recently (see, e.g., [21,26,27]). One interesting characteristic is that every model contains two linear derivative terms of the highest order, and so, we call them combined models.

Two special cases of $\beta = 1, \gamma = 0$ and $\beta = 0, \gamma = 1$ in the obtained hierarchy are of interest and produce reduced hierarchies of uncombined integrable models.

If we take $\alpha = -\delta_1 = \delta_2 = 1$, $\beta = 1$ and $\gamma = 0$ in the model (27), we obtain a coupled integrable nonlinear Schrödinger-type model:

$$\begin{cases} u_{1,t_2} = -u_{1,xx} + 2u_3(u_1^2 + u_2^2) - 4u_1u_2u_4, \\ u_{2,t_2} = -u_{2,xx} - 2u_4(u_1^2 + u_2^2) + 4u_1u_2u_3, \\ u_{3,t_2} = u_{3,xx} - 2u_1(u_3^2 + u_4^2) + 4u_2u_3u_4, \\ u_{4,t_2} = u_{4,xx} + 2u_2(u_3^2 + u_4^2) - 4u_1u_3u_4. \end{cases} \quad (29)$$

If we take $\alpha = -\delta_1 = \delta_2 = 1$, $\beta = 0$ and $\gamma = 1$ in the model (27), we obtain another coupled integrable nonlinear Schrödinger-type model:

$$\begin{cases} u_{1,t_2} = u_{2,xx} + 2u_4(u_1^2 + u_2^2) - 4u_1u_2u_3, \\ u_{2,t_2} = u_{1,xx} - 2u_3(u_1^2 + u_2^2) + 4u_1u_2u_4, \\ u_{3,t_2} = u_{4,xx} + 2u_2(u_3^2 + u_4^2) - 4u_1u_3u_4, \\ u_{4,t_2} = u_{3,xx} - 2u_1(u_3^2 + u_4^2) + 4u_2u_3u_4. \end{cases} \quad (30)$$

Similarly, if we take $\alpha = -\delta_1 = \delta_2 = 1$, $\beta = 1$ and $\gamma = 0$ in the model (28), we obtain a coupled integrable modified Korteweg–de Vries-type model:

$$\begin{cases} u_{1,t_3} = -u_{2,xxx} - 6(u_1u_4 - u_2u_3)u_{1,x} + 6(u_1u_3 - u_2u_4)u_{2,x}, \\ u_{2,t_3} = -u_{1,xxx} + 6(u_1u_3 - u_2u_4)u_{1,x} - 6(u_1u_4 - u_2u_3)u_{2,x}, \\ u_{3,t_3} = u_{4,xxx} - 6(u_1u_4 - u_2u_3)u_{3,x} - 6(u_1u_3 - u_2u_4)u_{4,x}, \\ u_{4,t_3} = u_{3,xxx} - 6(u_1u_3 - u_2u_4)u_{3,x} - 6(u_1u_4 - u_2u_3)u_{4,x}. \end{cases} \quad (31)$$

If we take $\alpha = -\delta_1 = \delta_2 = 1$, $\beta = 0$ and $\gamma = 1$ in the model (28), we obtain another coupled integrable modified Korteweg–de Vries-type model:

$$\begin{cases} u_{1,t_3} = u_{1,xxx} - 6(u_1u_3 - u_2u_4)u_{1,x} + 6(u_1u_4 - u_2u_3)u_{2,x}, \\ u_{2,t_3} = u_{2,xxx} + 6(u_1u_4 - u_2u_3)u_{1,x} - 6(u_1u_3 - u_2u_4)u_{2,x}, \\ u_{3,t_3} = u_{3,xxx} - 6(u_1u_3 - u_2u_4)u_{3,x} - 6(u_1u_4 - u_2u_3)u_{4,x}, \\ u_{4,t_3} = u_{4,xxx} - 6(u_1u_4 - u_2u_3)u_{3,x} - 6(u_1u_3 - u_2u_4)u_{4,x}. \end{cases} \quad (32)$$

These models are different from the vector AKNS integrable models [25]. In each pair, the two models just exchange the first component with the second component, carrying two sign changes, and the third component with the fourth component, carrying no sign change, in the vector fields on the right hand sides. Moreover, all those four models still commute with each other and so they are symmetries to each other.

3. Recursion Operator and Bi-Hamiltonian Formulation

To establish a bi-Hamiltonian formulation [28,29] and show the Liouville integrability for the resulting hierarchy (26), one can make use of the so-called trace identity (8) associated with the spatial matrix spectral problem (13). Substituting the spectral matrix \mathcal{M} by (13) and the Laurent series solution Y determined by (14) into the trace identity engenders

$$-\frac{\delta}{\delta u} \int \lambda^{-(n+1)} \alpha \delta_2 f^{[n+1]} dx = \lambda^{-\kappa} \frac{\partial}{\partial \lambda} \lambda^{\kappa-n} (\delta_1 c^{[n]}, \delta_2 g^{[n]}, \delta_1 b^{[n]}, \delta_2 e^{[n]})^T, \quad n \geq 0, \quad (33)$$

since we have

$$\text{tr}(Y \frac{\partial \mathcal{M}}{\partial \lambda}) = -2\alpha \delta_1 \delta_2 f, \quad \text{tr}(Y \frac{\partial \mathcal{M}}{\partial u}) = (2\delta_1^2 c, 2\delta_1 \delta_2 g, 2\delta_1^2 b, 2\delta_1 \delta_2 e)^T. \quad (34)$$

Checking with $n = 2$ determines $\kappa = 0$, and consequently, one arrives at

$$\frac{\delta}{\delta u} \mathcal{H}^{[n]} = (\delta_1 c^{[n+1]}, \delta_2 g^{[n+1]}, \delta_1 b^{[n+1]}, \delta_2 e^{[n+1]})^T, \quad n \geq 0, \quad (35)$$

where the Hamiltonian functionals are computed as follows:

$$\mathcal{H}^{[n]} = \int \frac{\alpha \delta_2}{n+1} f^{[n+2]} dx, \quad n \geq 0. \quad (36)$$

This enables us to furnish the following Hamiltonian formulations for the resulting hierarchy (26):

$$u_{t_m} = X^{[m]} = J_1 \frac{\delta \mathcal{H}^{[m]}}{\delta u}, \quad m \geq 0, \quad (37)$$

where J_1 is the Hamiltonian operator:

$$J_1 = \left[\begin{array}{cc|cc} 0 & & 0 & \alpha \\ & & -\alpha & 0 \\ \hline 0 & \alpha & & \\ -\alpha & 0 & & 0 \end{array} \right], \quad (38)$$

and $\mathcal{H}^{[m]}$ are the functionals given by (36). It follows from the Hamiltonian theory that there exists an interrelation $S = J_1 \frac{\delta \mathcal{H}}{\delta u}$ between a symmetry S and a conserved functional \mathcal{H} of the same model.

It is a common characteristic property that the vector fields $X^{[n]}$ constitutes an abelian algebra:

$$[[X^{[n_1]}, X^{[n_2]}]] := X^{[n_1]'}(u)[X^{[n_2]}] - X^{[n_2]'}(u)[X^{[n_1]}] = 0, \quad n_1, n_2 \geq 0, \quad (39)$$

which can be derived from an abelian algebra of Lax operators:

$$[[\mathcal{N}^{[n_1]}, \mathcal{N}^{[n_2]}]] := \mathcal{N}^{[n_1]'}(u)[X^{[n_2]}] - \mathcal{N}^{[n_2]'}(u)[X^{[n_1]}] + [\mathcal{N}^{[n_1]}, \mathcal{N}^{[n_2]}] = 0, \quad n_1, n_2 \geq 0. \quad (40)$$

Such a commutative property of vector fields still holds true under reciprocal transformations [30], and more discussions about the isospectral zero curvature equations is given in [31].

Furthermore, based on the recursion relations in (19)–(21), directly from the recursion relation $X^{[m+1]} = \Phi X^{[m]}$, where $X^{[m]}$, $m \geq 0$, are defined by (25), we can derive a hereditary recursion operator $\Phi = (\Phi_{jk})_{4 \times 4}$ [29] for the hierarchy (26) as follows:

$$\left\{ \begin{array}{l} \Phi_{11} = \frac{1}{\alpha}(-2\delta_1 u_1 \partial^{-1} u_4 + 2\delta_1 u_2 \partial^{-1} u_3), \quad \Phi_{12} = \frac{1}{\alpha}(-\frac{1}{\delta_1} \partial_x + 2\delta_1 u_1 \partial^{-1} u_3 + 2\delta_2 u_2 \partial^{-1} u_4, \\ \Phi_{13} = \frac{1}{\alpha}(2\delta_1 u_1 \partial^{-1} u_2 + 2\delta_1 u_2 \partial^{-1} u_1), \quad \Phi_{14} = \frac{1}{\alpha}(-2\delta_1 u_1 \partial^{-1} u_1 + 2\delta_2 u_2 \partial^{-1} u_2); \end{array} \right. \quad (41)$$

$$\left\{ \begin{array}{l} \Phi_{21} = \frac{1}{\alpha}(\frac{1}{\delta_2} \partial_x - \frac{2\delta_1^2}{\delta_2} u_1 \partial^{-1} u_3 - 2\delta_1 u_2 \partial^{-1} u_4), \quad \Phi_{22} = \frac{1}{\alpha}(-2\delta_1 u_1 \partial^{-1} u_4 + 2\delta_1 u_2 \partial^{-1} u_3), \\ \Phi_{23} = \frac{1}{\alpha}(-\frac{2\delta_1^2}{\delta_2} u_1 \partial^{-1} u_1 + 2\delta_1 u_2 \partial^{-1} u_2), \quad \Phi_{24} = \frac{1}{\alpha}(-2\delta_1 u_1 \partial^{-1} u_2 - 2\delta_1 u_2 \partial^{-1} u_1); \end{array} \right. \quad (42)$$

$$\left\{ \begin{array}{l} \Phi_{31} = \frac{1}{\alpha}(2\delta_1 u_3 \partial^{-1} u_4 + 2\delta_1 u_4 \partial^{-1} u_3), \quad \Phi_{32} = \frac{1}{\alpha}(-2\delta_1 u_3 \partial^{-1} u_3 + 2\delta_2 u_4 \partial^{-1} u_4), \\ \Phi_{33} = \frac{1}{\alpha}(-2\delta_1 u_3 \partial^{-1} u_2 + 2\delta_1 u_4 \partial^{-1} u_1), \quad \Phi_{34} = \frac{1}{\alpha}(-\frac{1}{\delta_1} \partial_x + 2\delta_1 u_3 \partial^{-1} u_1 + 2\delta_2 u_4 \partial^{-1} u_2); \end{array} \right. \quad (43)$$

$$\left\{ \begin{array}{l} \Phi_{41} = \frac{1}{\alpha}(-\frac{2\delta_1^2}{\delta_2} u_3 \partial^{-1} u_3 + 2\delta_1 u_4 \partial^{-1} u_4), \quad \Phi_{42} = \frac{1}{\alpha}(-2\delta_1 u_3 \partial^{-1} u_4 - 2\delta_1 u_4 \partial^{-1} u_3), \\ \Phi_{43} = \frac{1}{\alpha}(\frac{1}{\delta_2} \partial_x - \frac{2\delta_1^2}{\delta_2} u_3 \partial^{-1} u_1 - 2\delta_1 u_4 \partial^{-1} u_2), \quad \Phi_{44} = \frac{1}{\alpha}(-2\delta_1 u_3 \partial^{-1} u_2 + 2\delta_1 u_4 \partial^{-1} u_1). \end{array} \right. \quad (44)$$

The hereditariness of the operator Φ [32] means that Φ satisfies

$$L_{\Phi X}\Phi = \Phi L_X\Phi, \quad (45)$$

where the Lie derivative $L_X\Phi$ is defined by

$$(L_X\Phi)Z = \Phi[X, Z] - [X, \Phi Z], \quad (46)$$

in which X and Z are arbitrary vector fields. Observe that an operator $\Psi = \Psi(x, t, u, u_x, \dots)$ is a recursion operator of an evolution equation $u_t = X(u)$ [33] if and only if the operator Ψ needs to satisfy

$$\frac{\partial \Psi}{\partial t} + L_X\Psi = 0. \quad (47)$$

In the above example, we can easily verify that the autonomous operator Φ is a recursion operator of the first model $u_{t_0} = X^{[0]}$, i.e., we have $L_{X^{[0]}}\Phi = 0$. Then, based on these two facts, we can have

$$L_{X^{[m]}}\Phi = L_{\Phi X^{[m-1]}}\Phi = \Phi L_{X^{[m-1]}}\Phi = \dots = \Phi^m L_{X^{[0]}}\Phi = 0, \quad m \geq 1. \quad (48)$$

It then follows that Φ provides a common recursion operator for all models in the obtained hierarchy (26).

With some additional analysis, we can see that J_1 and $J_2 = \Phi J_1$ constitute a Hamiltonian pair. This means that an arbitrary linear combination of J_1 and J_2 is again Hamiltonian, i.e., it satisfies

$$\int (Z^{[1]})^T J'(u) [JZ^{[2]}] Z^{[3]} dx + \text{cycle}(Z^{[1]}, Z^{[2]}, Z^{[3]}) = 0, \quad (49)$$

where $Z^{[1]}, Z^{[2]}$ and $Z^{[3]}$ are arbitrary vector fields, and thus the hierarchy (26) possesses a bi-Hamiltonian formulation [28]:

$$u_{t_m} = X^{[m]} = J_1 \frac{\delta \mathcal{H}^{[m]}}{\delta u} = J_2 \frac{\delta \mathcal{H}^{[m-1]}}{\delta u}, \quad m \geq 1. \quad (50)$$

Moreover, we can observe that the associated Hamiltonian functionals also commute with each other under the corresponding two Poisson brackets [7]:

$$\{\mathcal{H}^{[n_1]}, \mathcal{H}^{[n_2]}\}_{J_1} = \int \left(\frac{\delta \mathcal{H}^{[n_1]}}{\delta u} \right)^T J_1 \frac{\delta \mathcal{H}^{[n_2]}}{\delta u} dx = 0, \quad n_1, n_2 \geq 0, \quad (51)$$

and

$$\{\mathcal{H}^{[n_1]}, \mathcal{H}^{[n_2]}\}_{J_2} = \int \left(\frac{\delta \mathcal{H}^{[n_1]}}{\delta p} \right)^T J_2 \frac{\delta \mathcal{H}^{[n_2]}}{\delta u} dx = 0, \quad n_1, n_2 \geq 0. \quad (52)$$

In summary, each model in the obtained hierarchy (26) is bi-Hamiltonian and Liouville integrable, possessing infinitely many commuting symmetries $\{X^{[n]}\}_{n=0}^{\infty}$ and conserved functionals $\{\mathcal{H}^{[n]}\}_{n=0}^{\infty}$. Two specific examples of such novel nonlinear combined Liouville integrable Hamiltonian models are the two models in (27) and (28), which involve two pairs of arbitrary constants.

4. Concluding Remarks

A Liouville integrable hierarchy with four potentials has been derived from a specific 4×4 matrix spectral problem, along with its hereditary recursion operator and bi-Hamiltonian formulation. The success comes from a particular Laurent series solution of the corresponding stationary zero curvature equation. The resulting integrable models involve two arbitrary constants and contain diverse specific four-component examples of integrable models, both combined and uncombined. However, it is still open to us how

to generalize the presented 4×4 matrix spectral problem so that integrable models with more potentials can be generated.

Studying algebraic or geometric structures of soliton solutions is a fascinating area of research with wide-ranging implications in mathematics and physics. The resulting integrable models should possess diverse soliton solutions, due to their nice integrable properties. One can try to apply different powerful and effective approaches, such as the Riemann–Hilbert technique [34], the Zakharov–Shabat dressing method [35], the Darboux transformation [36–40], the algebro–geometric method [41–45], the decomposition method [46–53] and the determinant approach [54]. In addition to solitons, other kinds of interesting nonlinear wave solutions such as lump, kink, breather and rogue wave solutions, including their interaction solutions (see, e.g., [55–64]), are also of great interest, and one can often compute those nonlinear wave solutions from solitons by taking special wave number reductions. Moreover, conducting nonlocal group reductions or equivalently similarity transformations, for matrix spectral problems, one can derive nonlocal reduced integrable models as well as study their soliton solutions (see, e.g., [65–68]).

Integrable models and Lax pairs are closely related. There is a huge diversity of multi-component integrable models, which have close connections to various areas of mathematics, including algebraic geometry, Lie groups, Lie algebras and Riemann surfaces. Identifying and classifying multi-component integrable models from Lax pairs is crucial for advancing our understanding of complex nonlinear mathematical and physical problems. It enables us to uncover dynamical behaviors of nonlinear waves and gain insights into a wide range of nonlinear phenomena across different branches of science and mathematics.

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References

1. Lax, P.D. Integrals of nonlinear equations of evolution and solitary waves. *Comm. Pure Appl. Math.* **1968**, *21*, 467–490. [\[CrossRef\]](#)
2. Ablowitz, M.J.; Segur, H. *Solitons and the Inverse Scattering Transform*; SIAM: Philadelphia, PA, USA, 1981.
3. Das, A. *Integrable Models*; World Scientific: Teaneck, NJ, USA, 1989.
4. Ablowitz, M.J.; Kaup, D.J.; Newell, A.C.; Segur, H. The inverse scattering transform-Fourier analysis for nonlinear problems. *Stud. Appl. Math.* **1974**, *53*, 249–315. [\[CrossRef\]](#)
5. Ma, W.X. Integrable couplings and matrix loop algebras. In *Nonlinear and Modern Mathematical Physics*; Ma, W.X., Kaup, D., Eds.; AIP Conference Proceedings; American Institute of Physics: Melville, NY, USA, 2013; Volume 1562, pp. 105–122.
6. Drinfel’d, V.; Sokolov, V.V. Lie algebras and equations of Korteweg–de Vries type. *Sov. J. Math.* **1985**, *30*, 1975–2036. [\[CrossRef\]](#)
7. Tu, G.Z. On Liouville integrability of zero-curvature equations and the Yang hierarchy. *J. Phys. A Math. Gen.* **1989**, *22*, 2375–2392.
8. Ma, W.X. A new hierarchy of Liouville integrable generalized Hamiltonian equations and its reduction. *Chin. Ann. Math. Ser. A* **1992**, *13*, 115–123.
9. Liu, C.S. How many first integrals imply integrability in infinite-dimensional Hamilton system. *Rep. Mat. Phys.* **2011**, *67*, 109–123. [\[CrossRef\]](#)
10. Antonowicz, M.; Fordy, A.P. Coupled KdV equations with multi-Hamiltonian structures. *Phys. D* **1987**, *28*, 345–357. [\[CrossRef\]](#)
11. Xia, T.C.; Yu, F.J.; Zhang, Y. The multi-component coupled Burgers hierarchy of soliton equations and its multi-component integrable couplings system with two arbitrary functions. *Physica A* **2004**, *343*, 238–246. [\[CrossRef\]](#)
12. Manukure, S. Finite-dimensional Liouville integrable Hamiltonian systems generated from Lax pairs of a bi-Hamiltonian soliton hierarchy by symmetry constraints. *Commun. Nonlinear Sci. Numer. Simul.* **2018**, *57*, 125–135. [\[CrossRef\]](#)
13. Liu, T.S.; Xia, T.C. Multi-component generalized Gerdjikov-Ivanov integrable hierarchy and its Riemann–Hilbert problem. *Nonlinear Anal. Real World Appl.* **2022**, *68*, 103667. [\[CrossRef\]](#)
14. Wang, H.F.; Zhang, Y.F. Application of Riemann–Hilbert method to an extended coupled nonlinear Schrödinger equations. *J. Comput. Appl. Math.* **2023**, *420*, 114812. [\[CrossRef\]](#)
15. Gerdjikov, V.S. Nonlinear evolution equations related to Kac-Moody algebras $A_r^{(1)}$: spectral aspects. *Turkish J. Math.* **2022**, *46*, 1828–1844. [\[CrossRef\]](#)
16. Ma, W.X. AKNS type reduced integrable hierarchies with Hamiltonian formulations. *Rom. J. Phys.* **2023**, *68*, 116. [\[CrossRef\]](#)

17. Takhtajan, L.A. Integration of the continuous Heisenberg spin chain through the inverse scattering method. *Phys. Lett. A* **1977**, *64*, 235–237. [\[CrossRef\]](#)
18. Kaup, D.J.; Newell, A.C. An exact solution for a derivative nonlinear Schrödinger equation. *J. Math. Phys.* **1978**, *19*, 798–801. [\[CrossRef\]](#)
19. Wadati, M.; Konno, K.; Ichikawa, Y.H. New integrable nonlinear evolution equations. *J. Phys. Soc. Jpn.* **1979**, *47*, 1698–1700. [\[CrossRef\]](#)
20. Ma, W.X. A Liouville integrable hierarchy with four potentials and its bi-Hamiltonian structure. *Rom. Rep. Phys.* **2023**, *75*, 115.
21. Ma, W.X. Four-component combined integrable equations possessing bi-Hamiltonian formulations. *Mod. Phys. Lett. B* **2024**, *38*, to appear.
22. Ma, W.X. Four-component integrable hierarchies and their Hamiltonian structures. *Commun. Nonlinear Sci. Numer. Simul.* **2023**, *126*, 107460. [\[CrossRef\]](#)
23. Zhang, Y.F. A few expanding integrable models, Hamiltonian structures and constrained flows. *Commun. Theor. Phys.* **2011**, *55*, 273–290. [\[CrossRef\]](#)
24. Zhaqilao. A generalized AKNS hierarchy, bi-Hamiltonian structure, and Darboux transformation. *Commun. Nonlinear Sci. Numer. Simul.* **2012**, *17*, 2319–2332. [\[CrossRef\]](#)
25. Ma, W.X. Reduced AKNS spectral problems and associated complex matrix integrable models. *Acta Appl. Math.* **2023**, *187*, 17. [\[CrossRef\]](#)
26. Ma, W.X. Four-component integrable hierarchies of Hamiltonian equations with $(m + n + 2)$ th-order Lax pairs. *Theoret. Math. Phys.* **2023**, *216*, 1180–1188. [\[CrossRef\]](#)
27. Ma, W.X. A four-component hierarchy of combined integrable equations with bi-Hamiltonian formulations. *Appl. Math. Lett.* **2024**, *153*, 109025. [\[CrossRef\]](#)
28. Magri, F. A simple model of the integrable Hamiltonian equation. *J. Math. Phys.* **1978**, *19*, 1156–1162. [\[CrossRef\]](#)
29. Fuchssteiner, B.; Fokas, A.S. Symplectic structures, their Bäcklund transformations and hereditary symmetries. *Phys. D* **1981**, *4*, 47–66. [\[CrossRef\]](#)
30. Ma, W.X. The commutative property of reciprocal transformations and dimensional deformations. *Qual. Theory Dyn. Syst.* **2024**, *23*, 2. [\[CrossRef\]](#)
31. Ma, W.X. The algebraic structure of zero curvature representations and application to coupled KdV systems. *J. Phys. A Math. Gen.* **1993**, *26*, 2573–2582. [\[CrossRef\]](#)
32. Fuchssteiner, B. Application of hereditary symmetries to nonlinear evolution equations. *Nonlinear Anal. Theory Methods Appl.* **1979**, *3*, 849–862. [\[CrossRef\]](#)
33. Olver, P.J. Evolution equations possessing infinitely many symmetries. *J. Math. Phys.* **1977**, *18*, 1212–1215. [\[CrossRef\]](#)
34. Novikov, S.P.; Manakov, S.V.; Pitaevskii, L.P.; Zakharov, V.E. *Theory of Solitons: The Inverse Scattering Method*; Consultant Bureau: New York, NY, USA, 1984.
35. Doktorov, E.V.; Leble, S.B. *A Dressing Method in Mathematical Physics*; Springer: Dordrecht, The Netherlands, 2007.
36. Matveev, V.B.; Salle, M.A. *Darboux Transformations and Solitons*; Springer: Berlin, Germany, 1991.
37. Xia, B.Q.; Zhou, R.G.; Qiao, Z.J. Darboux transformation and multi-soliton solutions of the Camassa-Holm equation and modified Camassa-Holm equation. *J. Math. Phys.* **2016**, *57*, 103502. [\[CrossRef\]](#)
38. Geng, X.G.; Li, R.M.; Xue, B. A vector general nonlinear Schrödinger equation with $(m + n)$ components. *J. Nonlinear Sci.* **2020**, *30*, 991–1013. [\[CrossRef\]](#)
39. Ye, R.S.; Zhang, Y. A vectorial Darboux transformation for the Fokas–Lenells system. *Chaos Solitons Fractals* **2023**, *169*, 113233. [\[CrossRef\]](#)
40. Ma, W.X. Binary Darboux transformation of vector nonlocal reverse-time integrable NLS equations. *Chaos Solitons Fractals* **2024**, *180*, 114539. [\[CrossRef\]](#)
41. Belokolon, E.D.; Bobenko, A.I.; Enol'skii, V.Z.; Its, A.R.; Matveev, V.B. *Algebro-Geometric Approach to Nonlinear Integrable Equations*; Springer: Berlin, Germany, 1994.
42. Geng, X.G. Algebraic-geometrical solutions of some multidimensional nonlinear evolution equations. *J. Phys. A Math. Gen.* **2003**, *36*, 2289–2303. [\[CrossRef\]](#)
43. Geng, X.G.; Liu, W.; Xue, B. Finite genus solutions to the coupled Burgers hierarchy. *Results Math.* **2019**, *74*, 11. [\[CrossRef\]](#)
44. Gao, H.; Wang, D.S. Optical undular bores in Riemann problem of photon fluid with quintic nonlinearity. *Phys. Rev. E* **2023**, *108*, 024222. [\[CrossRef\]](#) [\[PubMed\]](#)
45. Gao, H.; Wang, D.S.; Zhao, P. Algebro-geometric solutions of the modified Jaulent-Miodek hierarchy. *Int. J. Geom. Methods Mod. Phys.* **2024**, *21*, 2350239. [\[CrossRef\]](#)
46. Cao, C.W.; Geng, X.G. Classical integrable systems generated through nonlinearization of eigenvalue problems. In *Nonlinear Physics*; Gu, C.H., Li, Y.S., Tu, G.Z., Eds.; Research Reports in Physics; Springer: Berlin/Heidelberg, Germany, 1990; pp. 68–78.
47. Ma, W.X.; Strampp, W. An explicit symmetry constraint for the Lax pairs and the adjoint Lax pairs of AKNS systems. *Phys. Lett. A* **1994**, *185*, 277–286. [\[CrossRef\]](#)
48. Xu, X.X. A generalized Wadati-Konno-Ichikawa hierarchy and its binary nonlinearization by symmetry constraints. *Chaos Solitons Fractals* **2003**, *15*, 475–486. [\[CrossRef\]](#)

49. Zhao, Q.L.; Li, Y.X. The binary nonlinearization of generalized Toda hierarchy by a special choice of parameters. *Commun. Nonlinear Sci. Numer. Simul.* **2011**, *16*, 3257–3268. [\[CrossRef\]](#)
50. Dong, H.H.; Su, J.; Yi, F.J.; Zhang, T.Q. New Lax pairs of the Toda lattice and the nonlinearization under a higher-order Bargmann constraint. *J. Math. Phys.* **2012**, *53*, 033708. [\[CrossRef\]](#)
51. Yu, J.; He, J.S.; Cheng, Y. Binary nonlinearization of the nonlinear Schrödinger equation under an implicit symmetry constraint. *Acta Math. Appl. Sin. Engl. Ser.* **2014**, *30*, 379–388. [\[CrossRef\]](#)
52. Li, X.Y.; Zhao, Q.L. A new integrable symplectic map by the binary nonlinearization to the super AKNS system. *J. Geom. Phys.* **2017**, *121*, 123–137. [\[CrossRef\]](#)
53. Hu, B.B.; Xia, T.C. The binary nonlinearization of the super integrable system and its self-consistent sources. *Int. J. Nonlinear Sci. Numer. Simul.* **2017**, *18*, 285–292. [\[CrossRef\]](#)
54. Aktosun, T.; Busse, T.; Demontis, F.; van der Mee, C. Symmetries for exact solutions to the nonlinear Schrödinger equation. *J. Phys. A Math. Theor.* **2010**, *43*, 025202. [\[CrossRef\]](#)
55. Zhang, Y.; Dong, H.H.; Zhang, X.E.; Yang, H.W. Rational solutions and lump solutions to the generalized $(3 + 1)$ -dimensional shallow water-like equation. *Comput. Math. Appl.* **2017**, *73*, 246–252. [\[CrossRef\]](#)
56. Cheng, L.; Zhang, Y.; Lin, M.J. Lax pair and lump solutions for the $(2 + 1)$ -dimensional DJKM equation associated with bilinear Bäcklund transformations. *Anal. Math. Phys.* **2019**, *9*, 1741–1752. [\[CrossRef\]](#)
57. Sulaiman, T.A.; Yusuf, A.; Abdeljabbar, A.; Alquran, M. Dynamics of lump collision phenomena to the $(3 + 1)$ -dimensional nonlinear evolution equation. *J. Geom. Phys.* **2021**, *169*, 104347. [\[CrossRef\]](#)
58. Yusuf, A.; Sulaiman, T.A.; Abdeljabbar, A.; Alquran, M. Breather waves, analytical solutions and conservation law using Lie–Bäcklund symmetries to the $(2 + 1)$ -dimensional Chaffee–Infante equation. *J. Ocean Eng. Sci.* **2023**, *8*, 145–151. [\[CrossRef\]](#)
59. Manukure, S.; Chowdhury, A.; Zhou, Y. Complexiton solutions to the asymmetric Nizhnik–Novikov–Veselov equation. *Int. J. Mod. Phys. B* **2019**, *33*, 1950098. [\[CrossRef\]](#)
60. Zhou, Y.; Manukure, S.; McAnally, M. Lump and rogue wave solutions to a $(2 + 1)$ -dimensional Boussinesq type equation. *J. Geom. Phys.* **2021**, *167*, 104275. [\[CrossRef\]](#)
61. Ma, W.X. Lump waves in a spatial symmetric nonlinear dispersive wave model in $(2 + 1)$ -dimensions. *Mathematics* **2023**, *11*, 4664. [\[CrossRef\]](#)
62. Manukure, S.; Zhou, Y. A study of lump and line rogue wave solutions to a $(2 + 1)$ -dimensional nonlinear equation. *J. Geom. Phys.* **2021**, *167*, 104274. [\[CrossRef\]](#)
63. Yang, S.X.; Wang, Y.F.; Zhang, X. Conservation laws, Darboux transformation and localized waves for the N -coupled nonautonomous Gross–Pitaevskii equations in the Bose–Einstein condensates. *Chaos Solitons Fractals* **2023**, *169*, 113272. [\[CrossRef\]](#)
64. Zhang, X.Q.; Ren, B. Resonance soliton, breather and interaction solutions of the modified Kadomtsev–Petviashvili-II equation. *J. Nonlinear Math. Phys.* **2023**, *30*, 1272–1281. [\[CrossRef\]](#)
65. She, Y.L.; Yao, R.X. Novel particular solutions, breathers, and rogue waves for an integrable nonlocal derivative nonlinear Schrödinger equation. *Adv. Math. Phys.* **2022**, *2022*, 7670773.
66. Li, Y.; Tian, S.F. Inverse scattering transform and soliton solutions of an integrable nonlocal Hirota equation. *Commun. Pure Appl. Anal.* **2022**, *21*, 293–313. [\[CrossRef\]](#)
67. Wu, J.P. A novel Riemann–Hilbert approach via t -part spectral analysis for a physically significant nonlocal integrable nonlinear Schrödinger equation. *Nonlinearity* **2023**, *36*, 2021–2037. [\[CrossRef\]](#)
68. Ma, W.X. Integrable non-local nonlinear Schrödinger hierarchies of type $(-\lambda^*, \lambda)$ and soliton solutions. *Rep. Math. Phys.* **2023**, *92*, 19. [\[CrossRef\]](#)

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