Domain-Walls and Gauged Supergravities

T.C. de Wit

For my family
In loving memory of my father

The work described in this thesis was performed at the Center for Theoretical Physics in Groningen, with support from the “Stichting voor Fundamenteel Onderzoek der Materie" (FOM).

Printed by Universal Press - Science Publishers / Veenendaal, The Netherlands.
Typeset in Times using LaTeX 2ε.

Rijks*universiteit* Groningen

Domain-Walls and Gauged Supergravities

Proefschrift

ter verkrijging van het doctoraat in de
Wiskunde en Natuurwetenschappen
aan de Rijks*universiteit* Groningen
op gezag van de
Rector Magnificus, dr. F. Zwarts,
in het openbaar te verdedigen op
maandag 13 oktober 2003
om 14.15 uur

door

Tim Cornelis de Wit

geboren op 13 februari 1977
te Waspik

Promotor: Prof. dr. E. A. Bergshoeff

Beoordelingscommissie: Prof. dr. U. Lindström
Prof. dr. K. S. Stelle
Prof. dr. A. Van Proeyen

ISBN-nummer: 90-367-1867-8

Contents

Introduction

Since the birth of particle physics, with the discovery of the electron by Thomson in 1897, much progress has been made in explaining the observable phenomena in nature. In order to explain the properties of particles at the (sub)atomic scale quantum mechanics was developed around the nineteen-twenties. Based on experiments it was realized that all particles in nature have a fundamental property called "spin", the value of which divides them into two classes: bosons and fermions, each with distinct properties. Somewhat earlier, in 1905, Einstein proposed his theory of special relativity, which radically changed our notions of space and time; it showed how both concepts are intricately connected. A combination of special relativity and quantum mechanics finally led to the Standard Model around 1970, which quite successfully describes the interactions between the elementary particles that form the building blocks of all observable matter in the universe. There are three fundamental forces incorporated in the Standard Model: the electromagnetic, the weak, and the strong force. Here the concept of gauge symmetry plays an important role. By making this symmetry local, i.e. introducing coordinate dependent transformation parameters, spin 1 gauge bosons are introduced that mediate the force between two particles. The best known example is the photon that causes an electromagnetic field between two charged particles, causing them to attract or repulse. Similarly, the additional fundamental forces are carried by W/Z bosons and gluons respectively. The Standard Model has been verified to great precision, nevertheless there are some discrepancies. First of all there is the Higgs boson which is responsible for giving masses to the other fundamental particles, but still has not been found.[1] Secondly, the Standard Model contains nineteen fine-tuned parameters – e.g. corresponding to masses of elementary particles – that cannot be theoretically predicted, and is not a fundamental theory.

Another major achievement of 20th century theoretical physics was Einsteins theory of general relativity, dealing with the fourth fundamental force: gravity. The theory was constructed in 1914 in an attempt to implement special relativity into Newtonian gravity and further improved our knowledge about space and time. Some of its successes were the predictions of small deviations of planetary orbits and the deflection of light from heavy objects. More speculative predictions are black holes and gravitational radiation, which both only have been verified indirectly. Furthermore, predictions could be made regarding the evolution of our universe. Although this theory was capable of explaining the interactions between massive objects at relatively large length scales, something goes wrong when trying to describe gravity at small scales where quantum effects become important. Considering that the gravitational force is extremely

[1] There is hope that the new LHC accelerator, due 2006, will provide conclusive experimental proof of its existence.

force	mediating particle	acts on	range	relative strength
strong nuclear force	gluon	quarks	nuclear distances	20
electromagnetism	photon	charged particles	infinite	1
weak nuclear force	W- or Z-boson	quarks and leptons	nuclear distances	10^{-7}
gravity	graviton?	massive particles	infinite	10^{-36}

Table 1: The four fundamental forces. The relative strengths are based on two interacting up-quarks separated by a distance of 10^{-18} m [1].

weak compared to the other three fundamental forces at small scales, see table 1, it is not strange that general relativity theory has only been tested up to approximately 1 mm. An attempt to describe gravity by using similar quantization techniques as used for the Standard Model failed. The theory suffered from infinities since the gravitational coupling constant $\kappa = 8\pi G/c^4$ is not dimensionless and is therefore unsuitable for performing perturbation expansions, which are common in particle physics. The typical length scale where our classical ideas of gravity and space-time lose their validity is given by the Planck length:

$$\ell_P = \sqrt{\frac{hG}{c^3}} \approx 4.1 \cdot 10^{-35}\ \mathrm{m}\,, \tag{1}$$

with h Planck's constant, G Newton's gravitational constant, and c the velocity of light.

Summarizing, at both ends of the scale spectrum two quite successful theories were obtained, that did not seem to be compatible. These arguments show the need for a theory of "quantum gravity", that can handle all four fundamental forces simultaneously. The quest for this unified theory has been the main target for the research done in high energy physics during the last twenty years.

A partial success was reached in 1976 by the discovery of supergravity; an extension of general relativity theory that behaved better at high energies, i.e. the infinities were partially cancelled. The crucial ingredient here was "supersymmetry", a symmetry between bosons and fermions, that predicts that for every boson in nature there exists a corresponding fermionic particle, and vice versa. The gauge theory of supersymmetry is given by supergravity. The spin 2 gauge boson responsible for mediating the gravitational force is called the graviton. Its supersymmetric partner is the so-called gravitino. In order to measure these particles energies would be needed that are way out of the range of our present (and future) accelerators.

The most promising candidate so far for a theory of quantum gravity is superstring theory. String theory assumes that all particles can be represented by different oscillational modes of a string, with a typical length ℓ_S of the order of the Planck length ℓ_P. One of the modes turns out to be a spin 2 particle, behaving like a graviton. Subsequentially it was found that the low energy limit of superstring theory is given by supergravity. There is an intuitive reason why superstring theory is free from infinities. These infinities usually occur at singular points, however a string moving in space-time sweeps out a two dimensional surface, as opposed to a line in the case of a point particle. Exactly this fact causes the interactions not to take place at one single point, but to spread out over a small area. Intuitively that is the reason for string theory to be free from infinities, which usually occur at singular points.

Unfortunately, this theory also has its disadvantages. String theory is only defined perturbatively, i.e. scattering amplitudes are expressed as an infinite expansion in powers of the string coupling constant g_S, associated with the "Feynman-diagrams" of string theory. The main setback however was apparent when there seemed to exist five different superstring theories, whereas we hoped to obtain one unique theory of quantum theory.

This opinion was drastically changed after the discovery of dualities, that enabled us to relate different energy regimes of different theories. An important role was played by the so-called "brane" solutions of string theory. They are solitonic membrane-like objects that can be seen as higher-dimensional generalizations of strings. The five apparently distinct theories and their brane-solutions seemed to be related by a web of dualities, suggesting that they all represented various limits of one single fundamental theory, called "M-theory". Unfortunately there is not much known about this theory. However, by studying the low energy limits of M-theory and the various dualities between them, hopefully we will get closer to a unified theory.

We will now give a brief description of the topics discussed in this thesis. In chapter 1 we will briefly describe the framework of string theory and supergravity, needed to understand the context of the rest of the thesis. Chapter 2 will provide the motivations for the research described in the remainder of the thesis. The main motivation is the concept of "brane-world scenarios", which assumes that our four-dimensional universe can be represented as a four-dimensional brane-solution in five dimensions. With these types of models several problems in cosmology were tried to be solved, e.g. the cosmological constant problem and the hierarchy problem. The branes used in these models separate space-time into two regions and are called "domain-walls". A supersymmetrized version is not easy to construct; the domain-walls have to satisfy several conditions in order to describe the correct vacuum structure of the five-dimensional space-time. The determination of all possible domain-wall candidates requires a knowledge of matter couplings of five-dimensional supergravity. The scalar fields occurring in such theories can be interpreted as coordinates of a manifold. The potential energy of the scalars is given by the scalar potential, which is a function of all the scalars of the scalar manifold. The vacuum-structure of the five-dimensional space-time is determined by the minima of the scalar potential and the geometry of the scalar manifold.

The five-dimensional matter-coupled supergravity theory is a special case of a "gauged supergravity", i.e. a supergravity theory where one or more global symmetries has been made local. One way of constructing these gauged supergravities is by means of dimensional reduction. One starts with a higher-dimensional supergravity theory and "curls up" some extra dimensions to end up effectively with a supergravity in a lower space-time dimension. An extension of this method is called generalized dimensional reduction; here one uses a symmetry of a theory to obtain masses in lower dimensions. In this case, the symmetry used will appear as a gauged symmetry of the reduced theory. When applied to supergravity one can construct gauged supergravities. A general introduction to this topic is given in chapter 3, after which it is applied to eleven- and ten-dimensional supergravity in chapter 4.

The remaining three chapters 5, 6 and 7 provide another method to obtain gauged supergravities: the three-step superconformal program. We used the program in order to obtain a more general matter coupled five-dimensional $\mathcal{N} = 2$ Poincaré supergravity than currently known in the literature. The space-time symmetries in Poincaré supergravity are given by translations and rotations, which are part of the super-Poincaré group. The conformal program extends this group to the largest group of space-time symmetries, namely the superconformal group. By introducing extra symmetries, the corresponding conformal supergravity will contain more structure and will be easier to analyze.

The first step of the program is given in chapter 5 which describes the construction and gauging of the superconformal algebra in five dimensions, resulting in the so-called "Standard Weyl multiplet" which is the minimal representation of the superconformal algebra containing the graviton. The fields in this multiplet are the superconformal background fields.

The second step will be the subject of chapter 6 where we construct various matter multiplet representations of the superconformal algebra, and determine their actions and supersymmetry transformation rules in a background of the Weyl multiplet fields. We will only consider vector-tensor multiplets and hypermultiplets. Both contain scalars that give rise to interesting geometries on the corresponding scalar manifolds.

The last step is given in chapter 7 where the superconformal algebra is "broken down" to the super-Poincaré algebra by making convenient gauge choices for the non-Poincaré symmetries. This "gauge-fixing" process will produce five-dimensional matter coupled Poincaré supergravity, that can be used for many applications. Finally, in appendices A–C, we give our conventions and some in-depth information about the geometrical properties of quaternionic-like manifolds that are generated by the hypermultiplet couplings.

Chapter 1

String theory and supergravity

In this chapter we will briefly review some basic aspects of string theory, supergravity, dualities and (membrane) solutions.

1.1 Free string theory

For obtaining the dynamics of a classical string it is natural to consider the higher dimensional generalization of the relativistic particle. The trajectory of a free relativistic point particle is described by the minimization of the length of its worldline. Equivalently, the action for a free classical string in D dimensions will be proportional to the area of its worldsheet, i.e. the two-dimensional surface it spans in space-time. The worldsheet can be parametrized by the spacelike variable σ ($0 \leq \sigma \leq \ell_s$), the coordinate along the string of length ℓ_s, and timelike variable τ. The embedding of the string worldsheet in Minkowski space-time is given by the functions $X^\mu(\sigma,\tau)$ ($\mu = 0,\ldots,D-1$). The action describing the string dynamics is called the Nambu-Goto action [2, 3],

$$S = -T \int \mathrm{d}\sigma \mathrm{d}\tau \sqrt{|\det(\partial_\alpha X^\mu \partial_\beta X_\mu)|}\,, \qquad (1.1)$$

where T is the string tension given by $\frac{1}{2\pi\alpha'}$ with $\alpha' = \frac{\ell_s^2}{\hbar}$ the so-called Regge-slope. The indices α,β run over σ and τ. Although this form of the action is quite natural, there is a better formulation more suitable for e.g. quantization of the string, without the square root. This action was first discovered by Brink, Di Vecchia, Deser, Howe and Zumino [4, 5] but is better known as the Polyakov action [6]

$$S = -\frac{T}{2} \int \mathrm{d}\sigma \mathrm{d}\tau \sqrt{|\gamma|}\, \gamma^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X_\mu\,, \qquad (1.2)$$

where an auxiliary worldsheet metric $\gamma_{\alpha\beta}$ has been introduced ($\gamma \equiv \det \gamma_{\alpha\beta}$). The two actions are equivalent after eliminating $\gamma_{\alpha\beta}$ by using its equation of motion. The reparametrization invariance of the worldsheet can be used to choose the so-called conformal gauge in which we take the worldsheet metric to be equal to the two-dimensional Minkowskian metric. Then

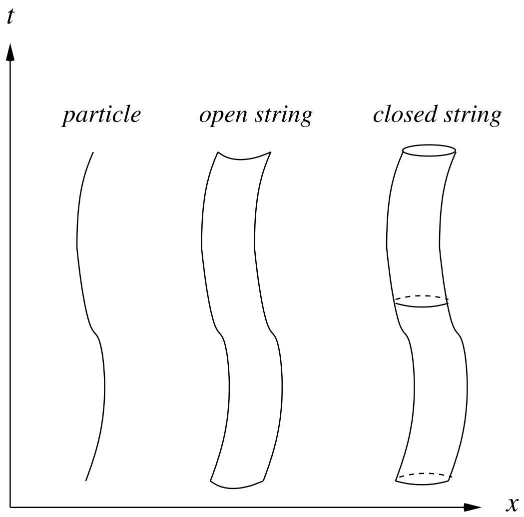

Figure 1.1: Open and closed string worldsheets.

Type of string	Boundary condition	
closed	periodic:	$X^\mu(\sigma + \ell_s) = X^\mu(\sigma)$
open	Dirichlet:	$X^\mu(\sigma) = \text{constant}\,, \quad \sigma = 0, \ell_s$
	Neumann:	$\partial_\sigma X^\mu(\sigma) = 0\,, \quad \sigma = 0, \ell_s$

Table 1.1: Boundary conditions for open and closed strings

using (1.2) the equation of motion for the string can be easily found:

$$\eta^{\alpha\beta}\partial_\alpha\partial_\beta X^\mu = 0\,. \tag{1.3}$$

Note that this wave equation can be solved with two different types of boundary conditions (see table 1.1) describing either open or closed strings (see figure 1.1). The solutions are now fully determined in terms of oscillator expansions, both for left and right moving modes

$$X^\mu(\sigma,\tau) = X^\mu_{\mathrm{L}}(\sigma+\tau) + X^\mu_{\mathrm{R}}(\sigma-\tau)\,. \tag{1.4}$$

At this point, consistently quantizing the string turns out to restrict the space-time dimension to $D = 26$. The oscillation modes of the string behave as particles, having specific mass and energy. Studying the spectrum one finds:

open string:	tachyon (scalar) T_1, massless vector A_μ, $\cdots$
closed string:	tachyon (scalar) T_2, dilaton (scalar) ϕ, graviton $h_{\mu\nu}$ (symmetric, traceless), two-form $B_{\mu\nu}$ (antisymmetric), $\cdots$

The spin-2 graviton particle is believed to be the gauge-particle mediating the gravitational force. So as a surprising result we see that theories with closed strings (or self-interacting open strings) seem to contain gravity! This led people to believe that string theory could form the basis of a theory of quantum gravity. However, the open string spectrum still contains a tachyon as ground state, an unphysical particle with negative mass squared. Furthermore any unified theory of elementary particle physics also should contain fermions. It turns out that including fermions in our theory will provide us with a way to eliminate the tachyon from the spectrum. Also, consistency of the theory will further restrict the number of dimensions to $D = 10$. We can add fermions to (1.2) by again choosing the conformal gauge and adding a kinetic term for a two-component worldsheet Majorana spinor

$$\psi_\mu \equiv \begin{pmatrix} \psi_+^\mu \\ \psi_-^\mu \end{pmatrix} , \tag{1.5}$$

transforming as vectors under the space-time Lorentz group, giving [7]

$$S = -\frac{T}{2} \int \mathrm{d}\sigma \mathrm{d}\tau \left(\partial_\alpha X^\mu \partial^\alpha X_\mu - \mathrm{i}\, \bar{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu \right) , \tag{1.6}$$

where ρ^α is a two dimensional representation of the Clifford algebra. This action turns out to have a worldsheet symmetry called *supersymmetry*, mapping the fermions to bosons and visa versa. Just like in the bosonic case we can have two types of boundary conditions for the open string

$$\begin{aligned} \text{Ramond (R):} \quad & \psi_+^\mu(0,\tau) = \psi_-^\mu(0,\tau) \quad \psi_+^\mu(\ell_s,\tau) = \psi_-^\mu(\ell_s,\tau) , \\ \text{Neveu-Scharz (NS):} \quad & \psi_+^\mu(0,\tau) = \psi_-^\mu(0,\tau) \quad \psi_+^\mu(\ell_s,\tau) = -\psi_-^\mu(\ell_s,\tau) . \end{aligned} \tag{1.7}$$

For the closed string the periodic Ramond or anti-periodic Neveu-Schwarz boundary conditions for left and right moving modes can be chosen independently, resulting in four different sectors: R-R, NS-NS, R-NS and NS-R. Demanding the spectrum of (1.6), apart from worldsheet supersymmetry, also to have space-time supersymmetry, will lead to the so-called Gliozzi-Scherk-Olive(GSO)-projection [8]. Since the fermionic spectrum does not have any negative mass-squared states and the massless sector has to be supersymmetric, this projection will successfully eliminate the tachyonic ground state from the spectrum. This theory, having manifest worldsheet supersymmetry, is called the Neveu-Schwarz-Ramond (NSR) formalism; a GSO-projection is needed to obtain space-time supersymmetry.

There is another formulation of superstring theory, called the Green-Schwarz (GS) formulation. This theory describes the embedding of the bosonic worldsheet in superspace and is therefore manifestly space-time supersymmetric. However, quantization of this theory until recently [9–11] was only possible in the light-cone gauge.

Using either the NSR or the GS formalism, choosing several combinations of the boundary conditions in the open and closed string case turns out to yield five different supersymmetric string theories: Type IIA, Type IIB, Type I, Heterotic $\mathrm{E}_8 \times \mathrm{E}_8$ and Heterotic SO(32). Type IIA and Type IIB are theories of closed strings and contain $\mathcal{N} = 2$ space-time supersymmetry. In Type IIA both supersymmetry parameters have opposite chirality, whereas in Type IIB they are equal. Type I is the only open string theory, and has $\mathcal{N} = 1$ supersymmetry. Both Heterotic theories also have $\mathcal{N} = 1$ and differ by their gauge groups, under which the massless vector fields transform.

1.2 Nonlinear sigma model

Until now we only considered non-interacting strings, moving in a flat Minkowski background. Next consider the closed bosonic string in a more general background consisting of the massless states $(\phi, h_{\mu\nu}, B_{\mu\nu})$, generated by vibrating closed strings in the bulk. The resulting action, invariant under worldsheet reparametrizations, is called the nonlinear sigma model

$$S = -\frac{T}{2}\int \mathrm{d}\sigma\mathrm{d}\tau \left\{\left(\sqrt{|\gamma|}\,\gamma^{\alpha\beta}g_{\mu\nu} - \varepsilon^{\alpha\beta}B_{\mu\nu}\right)\partial_\alpha X^\mu\partial_\beta X^\nu - \alpha'\sqrt{|\gamma|}\,\phi\,\mathcal{R}^{(2)}(\gamma)\right\}, \tag{1.8}$$

where the background metric is given by $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ and $\mathcal{R}^{(2)}$ is the Ricci-scalar of the worldsheet metric $\gamma_{\mu\nu}$. The last term in the action, with ϕ taken equal to 1, is proportional to a topological invariant quantity in two dimensions, called the Euler characteristic χ

$$\chi = \frac{1}{4\pi}\int \mathrm{d}\sigma\mathrm{d}\tau\,\sqrt{|\gamma|}\mathcal{R}^{(2)}(\gamma) = 2(1-g)\,, \tag{1.9}$$

where g denotes the genus of the Riemann surface swept out by the string. A redefinition of the dilaton in terms of its vacuum expectation value: $\phi \to \phi + \langle\phi\rangle$ gives a rescaling of the classical path integral with a factor $e^{\langle\phi\rangle\chi}$. As a consequence, every interaction vertex will have an associated string coupling constant

$$g_s \equiv e^{\langle\phi\rangle}\,. \tag{1.10}$$

Therefore a worldsheet with genus g can be seen as the g-th loop correction for string theory. In contradistinction to the first two terms the topological term is not classically invariant under the worldsheet Weyl symmetry $\gamma_{\alpha\beta} \to \Lambda^2(\sigma,\tau)\gamma_{\alpha\beta}$. It has been included to enable us to get a consistent conformal invariant theory at the quantum level, provided the β-functionals associated to the "coupling constants" ϕ, $h_{\mu\nu}$ and $B_{\mu\nu}$ vanish. In lowest non-trivial approximation in α' one obtains [12]

$$\begin{aligned}
0 &= \beta^{(h)}_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{4}H_{\mu\rho\sigma}H_\nu{}^{\rho\sigma} + 2\nabla_\mu\partial_\nu\phi + \mathcal{O}(\alpha')\,,\\
0 &= \beta^{(B)}_{\mu\nu} = \tfrac{1}{2}\nabla^\rho H_{\rho\mu\nu} - H_{\mu\nu\rho}\nabla^\rho\phi + \mathcal{O}(\alpha')\,,\\
0 &= \beta^{(\phi)} = R + \tfrac{1}{12}H^2 - 4\nabla^\rho\partial_\rho\phi + 4\partial^\rho\phi\,\partial_\rho\phi + \mathcal{O}(\alpha')\,,
\end{aligned} \tag{1.11}$$

where $R_{\mu\nu}(g)$ is the Ricci tensor of space-time, R the corresponding Ricci scalar, and $H_{\mu\nu\rho} = 3\partial_{[\mu}B_{\nu\rho]}$ is the field strength of the two-form. The form of these equations suggests they can be interpreted as equations of motion for the background fields. Indeed they can also be obtained from the following low energy effective action

$$S = \frac{1}{2\kappa_0^2}\int \mathrm{d}^{26}x\,\sqrt{|g|}e^{-2\phi}\left(R(g) + 4(\partial\phi)^2 - \tfrac{1}{2\cdot 3!}H_{\mu\nu\rho}H^{\mu\nu\rho} + \mathcal{O}(\alpha')\right), \tag{1.12}$$

where κ_0 can be related to the gravitational coupling constant, defined in terms of Newton's constant in $D = 26$ as

$$\kappa = \kappa_0 e^{\langle\phi\rangle} = \sqrt{8\pi G_{26}}. \tag{1.13}$$

Observe that the leading order term is not the conventional Einstein-Hilbert kinetic term, due to the dilaton pre-factor. This is because we are currently in the so-called string frame $g = g^{(S)}$. Performing the Weyl-rescaling $g^{(S)}_{\mu\nu} = e^{\phi/2} g^{(E)}_{\mu\nu}$ we get the action in the Einstein frame

$$S = \frac{1}{2\kappa_0^2}\int \mathrm{d}^{26}x\,\sqrt{|g^{(E)}|}\left(R(g^{(E)}) - \tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{2\cdot 3!}e^{-\phi}H_{\mu\nu\rho}H^{\mu\nu\rho} + O(\alpha')\right). \tag{1.14}$$

The analysis above can be repeated for open strings, having an extra massless vector field A_μ in their background, coupling to the string endpoints. This interaction is described by the boundary term

$$S = -\frac{T}{2}\int_{\partial\Sigma} \mathrm{d}\tau\, A_\mu \partial_\tau X^\mu\,, \tag{1.15}$$

which gives rise to the following contribution to the low energy effective action for open strings

$$S = \frac{1}{2\kappa_0}\int \mathrm{d}^{26}x\,\sqrt{|g^{(S)}|}\left(-\tfrac{1}{4}e^{-\phi}F_{\mu\nu}F^{\mu\nu}\right), \tag{1.16}$$

where $F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}$. This analysis, thus far purely bosonic and in $D = 26$, can be extended to the supersymmetric case in $D = 10$, and it then turns out that the low energy effective descriptions of all five superstring theories, except Type I, have one part in common, the so-called "common sector"; namely the NS-NS sector given by the ten dimensional analog of (1.14). The low energy limits of these superstring theories coincide with known supersymmetric extensions of Einstein gravity, called supergravities, which will be described in more detail below.

1.3 Supergravity effective actions

As mentioned in the Introduction, supergravity (sugra), as a gauge theory for supersymmetry, was first introduced in 1976 [13] as an extension of Einstein's theory of general relativity. Although it was not shown to be a finite perturbation theory in all orders, these effective actions are still useful for many applications. Especially because of the remarkable fact that they turned out to describe the low energy effective behavior of string theories. Several different methods can be used to formulate supergravity. One approach is to directly gauge the supersymmetry algebra, comparable to the procedure we will follow in chapters 5, 6 and 7 for constructing off-shell supergravity in five dimensions. In this section we will give some more details about the five different supergravity/superstring theories living in ten dimensions. Also see table 1.2.

1.3.1 Type II

Type II theory gives a description of oriented closed superstrings moving in a background consisting of massless closed string vibration modes. It is called Type II since the theory contains two space-time supersymmetries. Since the left and right moving modes of closed superstrings are decoupled, the states are described by tensorial products of two open string states. The

open	closed	String theory	Low energy limit
	x	IIA	$\mathcal{N}=2$ IIA sugra
	x	IIB	$\mathcal{N}=2$ IIB sugra
x	x	Type I	$\mathcal{N}=1$ sugra coupled to SO(32) YM multiplet
	x	Heterotic SO(32)	$\mathcal{N}=1$ sugra coupled to SO(32) YM multiplet
	x	Heterotic $E_8\times E_8$	$\mathcal{N}=1$ sugra coupled to $E_8\times E_8$ YM multiplet

Table 1.2: superstring theories and their low energy limits.

massless string states transform under the little group SO(8) of the ten dimensional Lorentz group SO(9, 1). These irreducible representations are given by the trivial irrep **1** (dilaton), the fundamental vector $\mathbf{8_v}$, the spinor reps $\mathbf{8_c}, \mathbf{8_s}$ (two gauginos with opposite chirality), **28** (anti-symmetric two-form), $\mathbf{35_v}$ (graviton), $\mathbf{35_s}$ (self-dual four-form), $\mathbf{56_c}, \mathbf{56_s}$ (two gravitinos with opposite chirality). Since the Ramond sector of an open string state transforms under a spinor representation we can distinguish two possibilities for the closed string states. The Ramond sectors of left and right moving string states can either have opposite or equal chirality, leading to two different superstring theories, respectively called IIA and IIB:

$$\begin{aligned} \text{IIA}: &\quad (\mathbf{8_v}\oplus\mathbf{8_c})\otimes(\mathbf{8_v}\oplus\mathbf{8_s})\,, \\ \text{IIB}: &\quad (\mathbf{8_v}\oplus\mathbf{8_c})\otimes(\mathbf{8_v}\oplus\mathbf{8_c})\,. \end{aligned} \tag{1.17}$$

These direct product states can be decomposed into SO(8) irreps to give the full massless spectrum. Both theories have a common NS-NS sector

$$\mathbf{8_v}\otimes\mathbf{8_v} = \mathbf{1}\oplus\mathbf{28}\oplus\mathbf{35_v} = \phi\oplus B_{\mu\nu}\oplus h_{\mu\nu}\,. \tag{1.18}$$

The other bosonic degrees of freedom reside in the R-R sector

$$\begin{aligned} \text{IIA}: &\quad \mathbf{8_c}\otimes\mathbf{8_s} = \mathbf{8_v}\oplus\mathbf{56_v} = C_{(1)}\oplus C_{(3)}\,, \\ \text{IIB}: &\quad \mathbf{8_c}\otimes\mathbf{8_c} = \mathbf{1}\oplus\mathbf{28}\oplus\mathbf{35_c} = C_{(0)}\oplus C_{(2)}\oplus C^+_{(4)}\,, \end{aligned} \tag{1.19}$$

and are therefore called RR gauge fields. The zero-form $C_{(0)}$ is called the axion. The fermionic fields are found in the NS-R and R-NS sectors (chirality denoted by α or $\dot\alpha$)

$$\begin{aligned} \text{IIA/IIB}: &\quad \mathbf{8_v}\otimes\mathbf{8_c} = \mathbf{8_s}\oplus\mathbf{56_c} = \lambda^{\dot\alpha}\oplus\psi^{\alpha}_{\mu}\,, \\ \text{IIA}: &\quad \mathbf{8_v}\otimes\mathbf{8_s} = \mathbf{8_c}\oplus\mathbf{56_s} = \lambda^{\alpha}\oplus\psi^{\dot\alpha}_{\mu}\,. \end{aligned} \tag{1.20}$$

The bosonic truncations of the IIA and IIB actions are given below (in the string frame). For IIA we have

$$\begin{aligned} S_{\text{IIA}} &= \frac{1}{2\kappa^2}\int \mathrm{d}^{10}x\,\sqrt{|g|}\Big\{e^{-2\phi}\Big[R(g)+4(\partial\phi)^2-\tfrac{1}{2\cdot 3!}H^2_{(3)}\Big]-\tfrac{1}{2\cdot 2!}G^2_{(2)} \\ &\quad -\tfrac{1}{2\cdot 4!}G^2_{(4)}\Big\}-\frac{1}{4\kappa^2}\int \mathrm{d}^{10}x\,\mathrm{d}C_{(3)}\wedge \mathrm{d}C_{(3)}\wedge B_{(2)}\,, \end{aligned} \tag{1.21}$$

where the field strengths are defined as follows (see appendix A for our conventions on form notation)

$$H_{(3)} = \mathrm{d}B_{(2)}\,, \quad G_{(2)} = \mathrm{d}C_{(1)}\,, \qquad G_{(4)} = \mathrm{d}C_{(3)} - H_{(3)} \wedge C_{(1)}\,. \tag{1.22}$$

For IIB we have

$$\begin{aligned} S_{\mathrm{IIB}} &= \frac{1}{2\kappa^2}\int \mathrm{d}^{10}x\,\sqrt{|g|}\,\Big\{e^{-2\phi}\Big[R(g) + 4(\partial\phi)^2 - \tfrac{1}{2\cdot 3!}H_{(3)}^2\Big] - \tfrac{1}{2}G_{(1)}^2 - \tfrac{1}{2\cdot 3!}G_{(3)}^2 \\ &\quad - \tfrac{1}{2\cdot 5!}G_{(5)}^2\Big\} - \frac{1}{4\kappa^2}\int \mathrm{d}^{10}x\, C_{(4)} \wedge \mathrm{d}C_{(2)} \wedge H_{(3)}\,, \end{aligned} \tag{1.23}$$

with

$$H_{(3)} = \mathrm{d}B_{(2)}\,, \quad G_{(1)} = \mathrm{d}C_{(0)}\,, \qquad G_{(3)} = \mathrm{d}C_{(2)} - H_{(3)} \wedge C_{(0)}\,. \tag{1.24}$$

The above IIB action is called the non-self-dual action, since the self-duality condition for the four-form gauge field does not follow from the action [14, 15]. The equations of motion have to be supplemented by

$$G_{(5)} = {}^{*}G_{(5)}. \tag{1.25}$$

1.3.2 Type I

Type I string theory is a theory with unoriented open strings and having $\mathcal{N} = 1$ supersymmetry. It also contains a closed string sector due to open string interactions. Since the open string endpoints can interact with a one-form gauge field $A^I_{(1)}$, we can assign charges to them. The only corresponding consistent gauge group turns out to be SO(32). The spectrum can be derived from the IIB spectrum by a specific parity projection Ω on the left and right moving sectors, keeping the left-right symmetric states, i.e. projecting out the NS-NS two-form. The action is given by

$$S_{\mathrm{I}} = \frac{1}{2\kappa^2}\int \mathrm{d}^{10}x\,\sqrt{|g|}\,\Big\{e^{-2\phi}\Big[R(g) + 4(\partial\phi)^2\Big] - \tfrac{1}{2\cdot 3!}H_{(3)}^2 + \tfrac{1}{4}e^{-\phi}F^I_{(2)}F_{(2)I}\Big\}\,, \tag{1.26}$$

where

$$F^I_{(2)} = \mathrm{d}A^I_{(1)} + [A_{(1)}, A_{(1)}]^I\,, \qquad H_{(3)} = \mathrm{d}C_2 + A^I_{(1)} \wedge \mathrm{d}A_{(1)I}\,, \tag{1.27}$$

and the trace runs over all the group generators.

1.3.3 Heterotic

The last two theories are Heterotic superstring $\mathcal{N} = 1$ theories with gauge groups $\mathrm{E}_8 \times \mathrm{E}_8$ and SO(32) respectively. These theories contain oriented closed strings. The action is given by

$$S_{\mathrm{Het}} = \frac{1}{2\kappa^2}\int \mathrm{d}^{10}x\,\sqrt{|g|}e^{-2\phi}\Big\{R(g) + 4(\partial\phi)^2 - \tfrac{1}{2\cdot 3!}H_{(3)}^2 + \tfrac{1}{4}F^I_{(2)}F_{(2)I}\Big\}\,, \tag{1.28}$$

where

$$F^I_{(2)} = \mathrm{d}A^I_{(1)} + [A_{(1)}, A_{(1)}]^I\,, \qquad H_{(3)} = \mathrm{d}B_2 + \tfrac{1}{2}A^I_{(1)} \wedge \mathrm{d}A_{(1)I}\,. \tag{1.29}$$

1.4 Dualities

Although a lot information can be obtained from superstring theories by making use of perturbative techniques, non-perturbative studies of superstring theories turn out to be extremely difficult. Furthermore, we saw in the previous section there are five different consistent theories of quantum gravity at first sight.

The concept of duality could be used to solve this unification problem, by showing that the five superstring theories are connected, suggesting that each of these five theories are merely different vacua of a single theory called M-theory.

Some examples of dualities in physics have been known for a long time. For example, assuming the existence of magnetic monopoles, Maxwell's equations were found to be invariant under the transformation

$$\vec{E} \to \vec{B}\,, \qquad \vec{B} \to -\vec{E}\,, \qquad e \leftrightarrow g\,, \tag{1.30}$$

with e the electric charge and g the magnetic monopole charge. Dirac's theory of monopoles [16] showed that the following quantization condition has to hold

$$e \cdot g = 2\pi n\,, \qquad n \in \mathbb{Z}\,, \tag{1.31}$$

connecting a strongly coupled theory of electrodynamics to a weakly coupled theory of monopoles! Similarly it was found [17] that $e \leftrightarrow g$ is an exact symmetry of $\mathcal{N} = 4$ Yang-Mills theory. In the following sections we will analyze dualities in the context of superstring theory. However, first we will briefly introduce the concept of compactification.

1.4.1 T-duality

This duality, also called target space duality, connects different theories, compactified on inverse radii. If we compactify one dimension, the periodicity along this coordinate y implies that fields can be expanded into their eigenfunctions on the circle

$$\Phi(x^\mu, y) = \sum_n \Phi_n(x^\mu) e^{\mathrm{i}\, p_y y}\,, \tag{1.32}$$

where $p_y \equiv \frac{n}{R}$ is the quantized conjugate momentum of y, and Φ_n are the so-called Kaluza-Klein modes. For more details on compactification and dimensional reduction, see chapter 3. To demonstrate this type of duality, let us consider a theory with coordinate x^9 compactified on a circle of radius R. In the simple case of a theory with only point particles, there are two clearly discernable limits. $R \to \infty$ will lead to a continuous conjugate momentum spectrum in the compact direction, restoring the uncompactified theory. On the other hand, when R shrinks to zero, the momentum will be zero or infinite, effectively decoupling the compact coordinate. However, in the case of string theory, closed strings can wrap w times around the circle, generalizing the periodicity condition to

$$X^\mu(\tau, \sigma + \ell_s) = X^\mu(\tau, \sigma) + 2\pi w R, \tag{1.33}$$

for the string coordinates X^μ, where $w \in \mathbb{N}$ is called the winding number. Inspecting the conjugate momenta of left and right movers along the circle, and the altered mass spectrum, one observes a new 'symmetry' of the theory, called T-duality. It relates a theory compactified on a circle with radius R, winding number w and momentum labelled by n, to another theory compactified on a circle with inverse radius α'/R, and interchanged momentum and winding numbers [18]. Furthermore, the right moving modes are changed by a parity transformation. For the fermionic modes this means that the right moving Ramond ground state alters its chirality; a theory with two opposite chiralities maps to a theory with equal chiralities, i.e. it has been found [19, 20] that the massless sectors of IIA and IIB supergravity are each others T-dual [21]. Also Heterotic SO(32) and Heterotic $E_8 \times E_8$ supergravity turn out to be T-dual. Both dualities are conjectured to hold in the corresponding (non-perturbative) string theory limit.

In the case of open strings it can be shown that the boundary condition changes from Neumann to Dirichlet under T-duality, i.e. their endpoints are localized on the circle. The hyperplane given by $x^9 = c$ turns out to describe a solitonic object in string theory, called a D-brane. This particular class of solutions will be discussed in more detail in section 1.5.2.

Note that T-duality, because of its perturbative nature, does not give us any more insights into the non-perturbative behavior of string theory.

1.4.2 S-duality

Another type of duality is the strong-weak duality. Similarly to the EM-duality it maps between strongly and weakly coupled regimes of different theories, making it particularly useful for obtaining non-perturbative information in one theory, using perturbative methods in the S-dual theory. From (1.10) we see that this duality generally corresponds to changing the sign of the dilaton: $\phi \to -\phi$.

In the supergravity approximation some simple examples are given by the S-duality between Type I and Heterotic SO(32) [22], which can be easily observed after scaling the metric to the Einstein frame. Secondly IIB turns out to be self-dual [23]. To see this, we write the IIB action (1.23) in a manifestly $S\ell(2,\mathbb{R})$ covariant manner. The dilaton and the RR-scalar can be combined into a complex scalar $\tau = C_{(0)} + \mathrm{i}\, e^{-\phi}$, which transforms under the Möbius transformation

$$\tau \to \frac{a\tau + b}{c\tau + d}\,, \tag{1.34}$$

with

$$\Omega = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad ad - bc = 1\,. \tag{1.35}$$

The NS-NS and R-R two-forms transform as a doublet under $S\ell(2,\mathbb{R})$

$$\begin{pmatrix} C_{(2)} \\ B_{(2)} \end{pmatrix} \longrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} C_{(2)} \\ B_{(2)} \end{pmatrix} = \begin{pmatrix} aC_{(2)} + bB_{(2)} \\ cC_{(2)} + dB_{(2)} \end{pmatrix}, \tag{1.36}$$

and the four-form transforms as a singlet. An S-duality transformation can now be seen as a specific $S\ell(2,\mathbb{R})$ transformation with $a = d = 0$ and $b = -c = 1$

$$\phi \to -\phi\,, \qquad C_{(2)} \to B_{(2)}\,, \qquad B_{(2)} \to -C_{(2)}\,, \tag{1.37}$$

mapping IIB onto itself. Since quantum mechanics requires the charge, with respect to the NS-NS two-form of the basic object of string theory, the fundamental string, to be quantized, the symmetry group is broken to the discrete subgroup $S\ell(2,\mathbb{Z})$. Due to the non-perturbative nature of this duality type, $S\ell(2,\mathbb{Z})$ has been conjectured as being the symmetry group of non-perturbative IIB superstring theory.

$D = 11$ **supergravity**

In $D = 11$ dimensions there is only one possible (physical) supergravity theory, having 32 supercharges. This $\mathcal{N} = 1$ supergravity theory was found by Cremmer, Julia and Scherk in 1978 [24], with the bosonic part given by

$$S_{D=11} = \frac{1}{2\kappa_{11}^2} \int \mathrm{d}^{11}x \sqrt{|g|} \left\{ R(g) - \tfrac{1}{2\cdot 4!} G_{(4)}^2 + \tfrac{1}{144^2} \varepsilon^{(4)(4')(3)} G_{(4)} G_{(4')} C_{(3)} \right\} , \tag{1.38}$$

with field strength $G_{(4)} = \mathrm{d}C_{(3)}$. The supersymmetric version of this action will be the starting point of chapter 4.

It was first realized by [25] that compactification of $D = 11$ supergravity onto a circle with radius $R_{11} = (g_s)^{2/3}$ exactly yields $D = 10$ IIA supergravity. It was also found in [26, 27] that compactification onto an interval $R_{11} = S^1/\mathbb{Z}_2$ (called an orbifold) yields Heterotic $E_8 \times E_8$ supergravity. Led by these observations, a unified theory was conjectured, called M-theory, of which the low energy approximation is given by $D = 11$ supergravity. The strong coupling limit (large g_s) of IIA/Heterotic string theory is given by M-theory.

The earlier mentioned $S\ell(2,\mathbb{Z})$ symmetry of IIB supergravity can now be easily explained. Since we know that IIA and IIB are T-dual, a compactification of $D = 11$ supergravity onto two circles $S^1 \times S^1$ with radii R_{11} and R_{10}, should be equal to IIB supergravity compactified on a circle with radius $1/R_{10}$. This is true if $g_{IIB} = R_{11}/R_{10}$. However, since the compact manifold $S^1 \times S^1$ forms a torus, with modular group $S\ell(2,\mathbb{Z})$, the IIB coupling constant g_{IIB} is related to its inverse g_{IIB}^{-1}. It follows that the symmetry group of IIB is given by $S\ell(2,\mathbb{Z})$, and therefore the theory is self-dual.

Some models of M-theory have been proposed, i.e. the matrix model [28], but until now there is still little known about M-theory. However, it is believed that all five superstring theories in ten dimensions somehow should follow from taking some particular low energy limit of M-theory, leading to a web of dualities, as depicted in figure 1.2.

1.5 Solutions

In this section we will discuss several kinds of solutions of the supergravity equations of motion. These solutions have played an essential role in strengthening our belief in dualities in the non-perturbative limit. For reviews on this subject see [29, 30].

1.5.1 p-Branes

The existence of higher rank gauge fields in string theory suggest a further generalization of strings is possible, namely p-branes, $(p + 1)$-dimensional objects in space-time, coupling to a

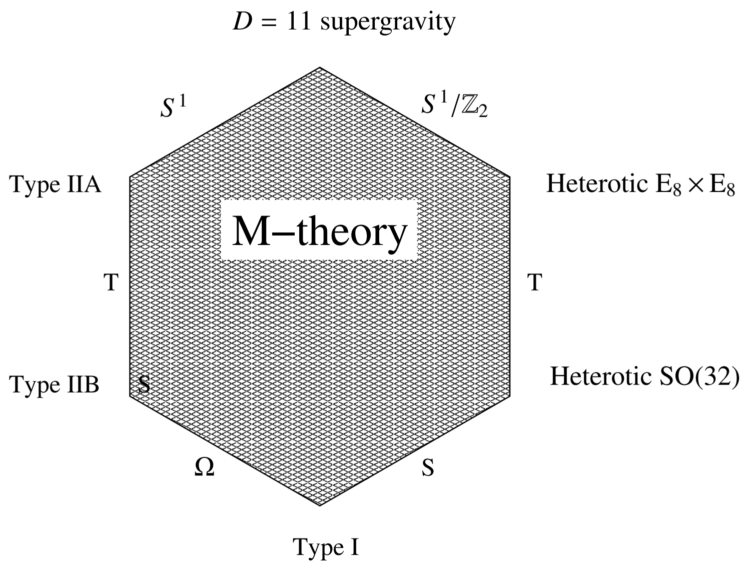

Figure 1.2: M-theory and the web of string theories and their dualities.

$(p+1)$-rank gauge field $A_{\mu_1 \ldots \mu_{p+1}}$ as follows:

$$\int \mathrm{d}^{p+1}\xi\, \partial_{\alpha_1} X^{\mu_1} \ldots \partial_{\alpha_{p+1}} X^{\mu_{p+1}} A_{\mu_1 \ldots \mu_{p+1}} \varepsilon^{\alpha_1 \ldots \alpha_{p+1}} \,, \tag{1.39}$$

in the same way we know a point particle ($p = 0$) couples to a one-form gauge field, and the NS-NS two-form $B_{\mu\nu}$ couples to a string worldsheet. The electrical charge of such an object can be found by a generalization of Gauss' law to be

$$Q_e \sim \int_{S^{D-p-2}} {}^*F_{(p+2)} \,, \tag{1.40}$$

where ${}^*F_{(p+2)}$ is the Hodge-dual (see appendix A) of the $A_{(p+1)}$ field strength, and S^{D-p-2} is a sphere surrounding the p-brane. This charge is conserved due to the equation of motion for the gauge field. Associated with this electrically charged p-brane solution is a dual magnetic $(D-p-4)$-brane, coupling to $\tilde{A}_{(D-p-3)}$, the dualization of the gauge field $A_{(p+1)}$. Its topological magnetic charge is given by

$$Q_m \sim \int_{S^{p+2}} F_{(p+2)} \,, \tag{1.41}$$

which is conserved due to the Bianchi identity. Here we integrated over the transverse space of the p-brane. These charges satisfy

$$Q_e \cdot Q_m = 2\pi n \,, \qquad n \in \mathbb{Z} \,, \tag{1.42}$$

generalizing Dirac's quantization condition for electric and magnetic monopoles. In order to explicitly find solitonic p-brane solutions in a given supergravity theory, let us consider a consistent bosonic truncation, only containing one $(n-1)$-form gauge field

$$S = \frac{1}{2\kappa^2}\int \mathrm{d}^D x\,\sqrt{|g|}\left(R - \tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{2\cdot n!}e^{a\phi}F_{(n)}^2\right). \tag{1.43}$$

In order to solve the equations of motion following from (1.43), a convenient Ansatz is given by

$$\begin{aligned}&\mathrm{d}s^2 = e^{2A(r)}\mathrm{d}x^\mu \mathrm{d}x^\nu \eta_{\mu\nu} + e^{2B(r)}\mathrm{d}y^m \mathrm{d}y^n \delta_{mn}\,, \qquad \phi = \phi(r)\,,\\ &\mu,\nu = 0\ldots p \qquad m,n = p+1,\ldots,D-1\,,\end{aligned} \tag{1.44}$$

with $r \equiv \sqrt{y^m y^n \delta_{mn}}$ the isotropic radial distance in the transverse space. The above Ansatz is consistent with a $P_{(p+1)} \times \mathrm{SO}(D-p-1)$ symmetry of space-time, with Poincaré symmetry along the worldvolume directions. There are two possible solutions of the equations of motion, leading to an electric ($p{=}n{-}2$) or magnetic ($p{=}D{-}p{-}2$) p-brane [31, 32]:

$$\left\{\begin{array}{lcl} \mathrm{d}s^2 &=& H^{\frac{-4(D-p-3)}{\Delta}(D-2)}\mathrm{d}x^2_{(p+1)} + H^{\frac{4(p+1)}{\Delta}(D-2)}\mathrm{d}y^2_{(D-p-1)}\,,\\ e^\phi &=& H^{\frac{2a}{\zeta\Delta}}\,, \qquad \zeta = \left\{\begin{array}{ll} +1 & \text{electric}\\ -1 & \text{magnetic}\end{array}\right.\,,\\ F_{m\mu_1\ldots\mu_{n-1}} &=& \frac{2}{\sqrt{\Delta}}\varepsilon_{\mu_1\ldots\mu_{n-1}}\partial_m(H^{-1}) \qquad \text{electric}\,,\\ F_{m_1\ldots m_n} &=& -\frac{2}{\sqrt{\Delta}}\varepsilon_{m_1\ldots m_n r}\partial_r H \qquad \text{magnetic}\,,\end{array}\right. \tag{1.45}$$

where the harmonic function H satisfies $\nabla^2 H = 0$. For $D-p-1 > 2$, H can be written as $H(r) = 1 + (\frac{r_0}{r})^{D-p-3}$, where r_0 is an integration constant related to the charge in the magnetic case. The constant Δ is given by

$$\Delta = a^2 + \frac{2(p+1)(D-p-3)}{D-2}\,. \tag{1.46}$$

Examples

The simplest example in Type II theories is the electric one-brane, coupling to the NS-NS two-form, called the fundamental string (F1). This solution can be obtained from (1.45) by using $p = 1$, $a = -1$ and $D = 10$. Its magnetic dual is called the Neveu-Schwarz five-brane (NS5). Type II theories also contain RR-gauge fields, allowing for a separate class of solutions as described in the following section.

The $D = 11$ supergravity theory only contains one three-form gauge field, and no dilaton (take $a = 0$). The only sources we can have for a three-form are a two-brane or five-brane solution, so we take $\Delta = 4$ in (1.45). The resulting solutions are called the electric M2-brane [33] and magnetic M5-brane [34]. The compactification of the M2-brane along its spatial direction was found to give exactly the F1 solution of IIA supergravity. The NS5 solution can be obtained by compactifying the M5 brane along a transverse direction.

A special case of p-brane ($p = D-2$) is the so-called domain-wall, a brane with one transverse direction, separating space-time into two regions. As we will see later these objects play an important role in so-called brane-world scenarios.

1.5.2 D-branes

In section 1.4.1 we already encountered D-branes as hyperplanes where open strings can end. They turn out to be a special class of p-brane solutions, coupling to RR potentials, and satisfying Dirichlet boundary conditions along their spacelike coordinates [35], i.e. they are fixed in space. Their dynamics is generated by the open string modes. In the string frame the Dp-brane geometry takes the following simple form

$$\begin{cases} \mathrm{d}s^2 & = & H^{-\frac{1}{2}}\mathrm{d}x^2_{(p+1)} - H^{\frac{1}{2}}\mathrm{d}x^2_{(D-p-1)}\,, \\ e^{-2\phi} & = & H^{\frac{p-3}{2}}\,, \\ F^{RR}_{012\ldots pm} & = & \partial_m H^{-1} \qquad (m = p+1,\ldots,9)\,, \end{cases} \tag{1.47}$$

Since IIA / IIB supergravity only contains odd / even-form gauge potentials, it only contains D$2p$ / D$(2p+1)$-branes. In IIB there are two special cases. There is a D(-1)-brane called the D-instanton, whose position is fixed in space-time, coupling to the axion. There is also a self-dual dyonic D3 brane solution.

The D-brane low energy effective worldvolume action was found by Leigh [36] by using the same technique used in section 1.2, and is called the Dirac-Born-Infeld action

$$S_{\mathrm{DBI}} = -T_p \int \mathrm{d}^{p+1}\xi\, e^{\phi}\, \sqrt{|g + \mathcal{F}|}\,, \tag{1.48}$$

where T_p is the tension of the Dp-brane, and $\mathcal{F}_{ij} = 2\pi\alpha' F_{ij} - B_{ij}$ ($F = \mathrm{d}A$). When considering D-brane actions in Type II supergravity it turns out one also has to include a Wess-Zumino term

$$S_{\mathrm{WZ}} = T_p \int \left(e^{\mathcal{F}} \wedge C\right)_{(p+1)}\,, \tag{1.49}$$

where $\mathcal{F}$ is given as a formal sum over all odd (IIA) or even (IIB) RR-forms, and the expansion picks out only forms of rank $(p+1)$.

1.5.3 Brane dualities

A lot of evidence for the conjectured dualities has been obtained by inspecting the solutions. The solutions described in the last section are all related by the same dualities, and by dimensional reduction. This is depicted in figure 1.3. Some other solutions that have not been mentioned are Kaluza-Klein ($\mathcal{KK}_D$) monopoles and gravitational waves ($\mathcal{W}_D$).

1.6 BPS states

The presence of p-branes allows the D-dimensional supertranslation algebra to be extended with central charges $Z_{(p)}$ related to the p-brane charges.

$$\left\{Q^i_\alpha, Q^j_\beta\right\} = \delta^{ij}\,(\Gamma^\mu C)_{\alpha\beta}\, P_\mu + \sum_p (\Gamma^{\mu_1\ldots\mu_p} C)_{\alpha\beta}\, Z^{ij}_{\mu_1\ldots\mu_p}\,, \tag{1.50}$$

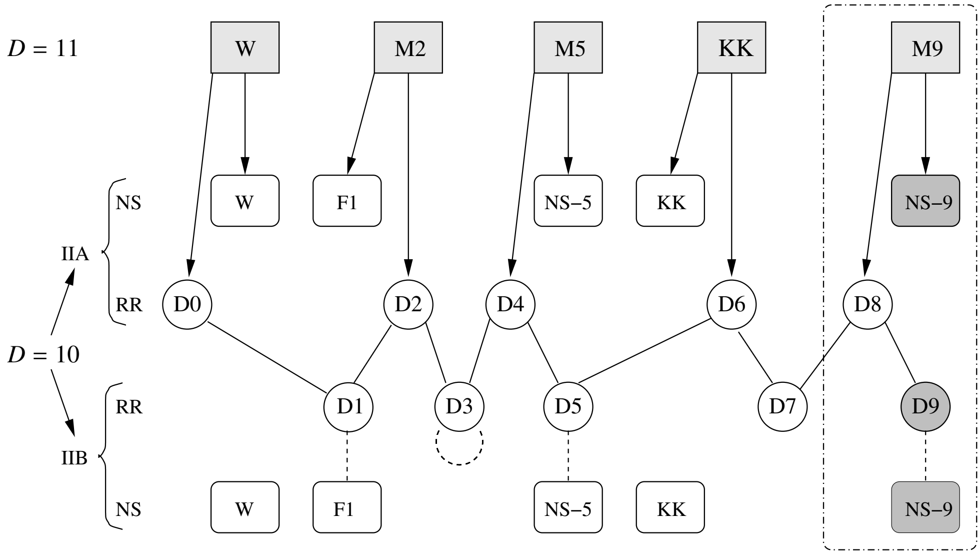

Figure 1.3: Web of dualities between supergravity solutions in $D = 10$ and $D = 11$ [37].

with $i = 1, \dots, \mathcal{N}$. Positivity of the Q^2 operator on the left hand side gives rise to the so-called Bogomol'nyi-Prasad-Sommerfield (BPS) bound [38, 39], symbolically relating the mass and charge by

$$M \geq c|Z| \,. \tag{1.51}$$

States saturating this bound are called BPS states. These states are stable against decay since they minimize the energy for a given charge. Supersymmetry protects these states from renormalization by quantum effects; the mass-charge relation also holds non-perturbatively, therefore these states have played an important role in the study of dualities.

The BPS states we consider are purely bosonic configurations, where the background fermionic fields have been put to zero. Stability and consistency of this solution of the field equations requires the supersymmetry variations of the fermions to vanish, leading to the BPS equations. This provides a convenient way to explicitly derive BPS states. For example, if we consider a Type II background with only one $(p+1)$-form present, the supersymmetry transformation rules can be written as [40]

$$\begin{aligned}
\delta\psi^i_\mu &= (\partial_\mu + \tfrac{1}{4}\omega_\mu{}^{ab}\gamma_{ab})\,\epsilon^i + \frac{(-1)^p}{8(p+2)!}e^\phi\Gamma\cdot F^{(p+2)}\Gamma_\mu P(p)\epsilon^i \equiv D_\mu\epsilon^i = 0\,,\\
\delta\lambda^i &= \Gamma^\mu(\partial_\mu + \tfrac{1}{4}\omega_\mu{}^{ab}\gamma_{ab})\,\phi\epsilon^i + \frac{3-p}{4(p+2)!}e^\phi\Gamma\cdot F^{(p+2)}P(p)\epsilon^i = 0\,,
\end{aligned} \tag{1.52}$$

where $P(p)$ is a p-dependent projection operator and ω_μ^{ab} the spin-connection. The first equation is called the Killing spinor equation. These differential constraints on the background fields are called the BPS equations. Substituting a p-brane Ansatz into these equations allows us to solve for the parameters. One also finds the following algebraic constraint on ϵ

$$\epsilon + \Gamma_{01\ldots p} P(p) \epsilon = 0 \,. \tag{1.53}$$

As a consequence this breaks half of the supersymmetry, which is generally true for objects saturating the BPS-bound. The above procedure is applied in chapter 4 in order to find $\frac{1}{2}$-BPS solutions of $D = 9$ gauged supergravity.

Chapter 2

Scalar potential: Domain-walls and other applications

In this chapter we will give some motivations for the research described in the remainder of this thesis. We will briefly show the relevance of scalar potentials and domain-walls to supergravity, field theories and cosmology. In the next chapters we will give two methods to obtain scalar potentials from supergravity, dimensional reduction and conformal supergravity.

2.1 Gauged and massive supergravities

Gauged supergravities have played an important role during the past 25 years in a broad range of applications. In most of these cases the key factor is the so-called scalar potential. Scalar potentials e.g. in bosonic scalar-gravity models are essentially non-restricted massive deformations; for example ϕ^4 theory coupled to gravity. However, in supersymmetric models, like gauged and massive supergravities, the form of the potential is fully determined. In that case the gauge coupling constant g can be related to the mass parameter m by: $m = \kappa g$. Supersymmetry in general not only restricts the form of the potential; it also imposes a geometrical structure on the collection of scalars in the theory, called the scalar manifold. We will see explicit examples of this in chapters 6 and 7.

Strictly speaking gauged supergravities are defined as supergravity theories where either a subgroup of or the full R-symmetry group is gauged, using one or more vectors present in the theory. In some cases this will involve the coupling of extra matter multiplets, e.g. vector multiplets. In practice the term gauged supergravity is often used to denote a gauging of an arbitrary global symmetry group.

The general procedure of gauging a supergravity theory consists of

- choosing an appropriate gauge group G.
- performing a minimal substitution, i.e. coupling vector fields A^I_μ to matter fields Φ^I by

introducing covariant derivatives

$$D_\mu \Phi^I = \partial_\mu \Phi^I + g\, A^J_\mu \, f_{JK}{}^I \Phi^K \,, \tag{2.1}$$

locally invariant under

$$\delta_G \Phi^I = -g\, \Lambda^J f_{JK}{}^I \Phi^K \,, \qquad \delta_G A^I_\mu = \partial_\mu \Lambda^I - g\, \Lambda^J f_{JK}{}^I A^K_\mu \,, \tag{2.2}$$

where g is the gauge coupling constant, Λ^I is the gauge parameter, and $f_{IJ}{}^K$ are the structure constants encoding the properties of the specific gauge group. Note that non-Abelian gaugings also allow for self-couplings between the vector fields.

- restoring supersymmetry by adding terms to the action and/or transformation rules, making use of the Noether method. This procedure generally gives rise to mass-terms for the fermions and contributes to the scalar potential.

In some specific cases a wide range of possible gauge groups have been classified, e.g. for $\mathcal{N} = 2$, $D = 5$ supergravity in [41]. More details on this subject will be given in chapter 7.

Note that not all potentials necessarily have to come from gauging. A good example of a so-called massive supergravity theory is Romans' [42] deformation of IIA (1.21) in ten dimensions with one mass parameter, consistent with supersymmetry. Although the string theory origin of these theories is somewhat unclear, their importance should not be underestimated. E.g. Romans' theory contains the D8-brane as a natural solution, coupling to the 'zero-form' mass parameter.

2.1.1 Vacua

In this section we will illustrate the way in which the vacua of gauged/massive supergravities are determined by the extrema of the scalar (super)potential. In conventional gravity or supergravity theories the vacua are those solutions of the field equations with maximal symmetry, i.e. the largest number of isometries. Let us first consider the D-dimensional Einstein-Hilbert action, with cosmological constant Λ

$$S = \frac{1}{2\kappa^2} \int \mathrm{d}^D x \sqrt{|g|}\, (R - 2\Lambda) \,. \tag{2.3}$$

The corresponding field equation is given by the vacuum Einstein equation

$$R_{\mu\nu} - \tfrac{1}{2} R\, g_{\mu\nu} + \Lambda\, g_{\mu\nu} = 0 \,, \tag{2.4}$$

which, after taking the trace, gives an expression of the cosmological constant in terms of the Ricci-scalar

$$\Lambda = \tfrac{(D-2)}{2D} R \,. \tag{2.5}$$

Depending on the curvature of space-time, the vacua correspond to de Sitter (dS), anti-de Sitter (AdS) or flat Minkowski space:

anti-de Sitter (AdS)	:	negative curvature
Minkowski	:	zero curvature
de Sitter (dS)	:	positive curvature

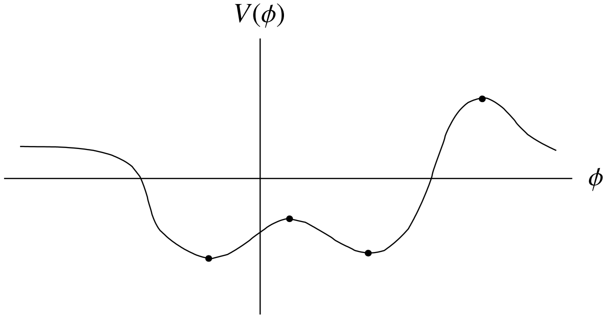

Figure 2.1: Critical points of the scalar potential.

Next consider a slight generalization corresponding to a minimal truncation of a gauged/massive sugra action. The action for a scalar field coupled to gravity is given by

$$S = \frac{1}{2\kappa^2}\int \mathrm{d}^D x\,\sqrt{|g|}\left(R - \tfrac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - V(\phi)\right). \tag{2.6}$$

The equations of motion of (2.6) are given by

$$\begin{aligned}\nabla_\mu\,\partial^\mu\phi &= \frac{\partial V}{\partial\phi}\,,\\ R_{\mu\nu} - \tfrac{1}{2}R\,g_{\mu\nu} &= \tfrac{1}{2}\partial_\mu\phi\,\partial_\nu\phi - \left(\tfrac{1}{4}\partial_\rho\phi\,\partial^\rho\phi + \tfrac{1}{2}V(\phi)\right)g_{\mu\nu}\,.\end{aligned} \tag{2.7}$$

In order to be consistent with maximum symmetry, the vacuum expectation value (v.e.v.) of the scalar field has to be constant, and should correspond to a local extremum of the potential called a critical point, see figure 2.1:

$$\langle\phi\rangle = \phi_c\,, \qquad \left.\frac{\partial V}{\partial\phi}\right|_{\phi=\phi_c} = 0\,. \tag{2.8}$$

At these extrema the equations (2.7) reduce to the field equation describing three different vacuum solutions, depending on the value and sign of the cosmological constant. In the (A)dS cases the cosmological constant is given by

$$\Lambda = \tfrac{1}{2}V(\phi_c)\,. \tag{2.9}$$

In section 2.3 a specific class of vacuum solutions of (2.6) will be discussed, having $(D-1)$-dimensional Poincaré invariance and scalar v.e.v. that are dependent on the D-th coordinate. The geometry of these half-supersymmetric solitons, called domain-walls, interpolates between two conventional vacua with different cosmological constants.

2.1.2 Scalar (super)potential

Supergravity models generically consist of a basic supergravity multiplet coupled to any number of supermultiplet representations of the underlying supersymmetry algebra. Interactions in those models are usually described by three types of potentials for the scalar fields in the theory: the superpotential W and the potential V, derived from W.

The connection between V and W can be made clear by the following toy model. Let us consider the same scalar-gravity model as given in (2.6) and describe scalar fluctuations φ around the AdS vacuum associated with the critical point ϕ_c with $V(\phi_c) < 0$

$$\varphi = \phi - \phi_c \,. \tag{2.10}$$

Expanding the action (2.6) to lowest order around this critical point yields

$$S = \frac{1}{2\kappa^2} \int \mathrm{d}^D x \sqrt{|g|} \left(R - \tfrac{1}{2}\partial_\mu\varphi\, \partial^\mu\varphi - \tfrac{1}{2}M^2\varphi^2 - V(\phi_c) \right) , \tag{2.11}$$

with the following equations of motion

$$\begin{aligned} \left(\nabla_\mu\, \partial^\mu - M^2\right)\varphi &= 0\,, \\ R_{\mu\nu} - \tfrac{1}{2}R\, g_{\mu\nu} + \Lambda g_{\mu\nu} &= \mathcal{O}(\varphi^2)\,, \end{aligned} \tag{2.12}$$

describing a scalar particle of mass M in an AdS background with cosmological constant Λ

$$M^2 \equiv \left.\frac{\partial^2 V}{\partial\phi^2}\right|_{\phi=\phi_c} , \qquad \Lambda \equiv \tfrac{1}{2}V(\phi_c)\,. \tag{2.13}$$

A more general case was considered by Townsend [43]: supergravity coupled to vector multiplets, also called Einstein-Maxwell supergravity. This theory contains multiple scalars ϕ^x that can be interpreted as coordinates on some manifold described by metric g_{xy}

$$S = \frac{1}{2\kappa^2} \int \mathrm{d}^D x \sqrt{|g|} \left(R - \tfrac{1}{2} g_{xy}(\phi)\, \partial_\mu\phi^x\, \partial^\mu\phi^y - V(\phi) \right) . \tag{2.14}$$

In the bosonic case it was shown from stability requirements of the AdS vacua that the potential can be expressed in terms of the superpotential

$$V(\phi) = 4(D-2)^2 \left[2g^{xy}\, \frac{\partial W}{\partial\phi^x}\frac{\partial W}{\partial\phi^y} - \frac{D-1}{D-2}\, W(\phi)^2 \right] . \tag{2.15}$$

A similar result can be obtained by requiring supersymmetry invariance, where the superpotential can be read off from the transformation rules of the fermions

$$\begin{aligned} \delta\psi_\mu &= (\partial_\mu + \tfrac{1}{4}\omega_\mu{}^{ab}\Gamma_{ab})\epsilon + W(\phi)\,\Gamma_\mu\,\epsilon\,, \\ \delta\lambda_x &= g_{xy}\, \partial\!\!\!/\phi^y\epsilon - (D-2)\,\frac{\partial W}{\partial\phi^x}\epsilon\,. \end{aligned} \tag{2.16}$$

These contributions to the scalar potential are called the fermion-shifts. Note that not all potentials in supersymmetric theories can be written in terms of a superpotential. We will see examples of this in chapter 4.

2.1.3 Applications

From the above it should be clear that it is very useful to study the properties of gauged supergravities. In particular the scalar potential can provide crucial information about the vacua, solutions and dynamics of supergravity theories, that can be used in many applications, some of which will be further explained in the following sections.

- The DW/QFT correspondence is a conjectured duality between supergravity on a domain-wall background and a quantum field theory. A special case of this duality is the AdS/CFT duality. Useful properties of field theories can be obtained by studying domain-wall solutions of supergravity, which are fully determined by the form of the scalar potential. By using this duality, the domain-wall geometries describe renormalization group flows in the dual field theory. See sections 2.2 and 2.3 for more details.

- Brane-world scenarios try to describe our four-dimensional world by assuming that we live on the worldvolume of a domain-wall solution in five dimensions. Whether a supersymmetric embedding of these scenarios is possible or not depends on the vacuum solutions of the scalar potential in gauged $\mathcal{N} = 2$ supergravity in five dimensions.

- Inflationary models are used to study several issues in cosmology like the smallness of the cosmological constant, the horizon problem and the isotropy of the universe. These models try to explain the dynamical properties of the universe by studying the scalar-potentials occurring in specific scalar-gravity systems. For certain values of the so-called "slow-roll parameter" cosmic inflation occurs, as the result of the "rolling" of the scalar field towards the minimum of the potential. For a review see [44, 45].

2.2 AdS/CFT

One of the most important developments of the past few years has been the conjecture of Maldacena in 1997, called the AdS/CFT correspondence [46]. This was later generalized to the DW/QFT correspondence [47, 48]. Before giving a brief explanation of this conjecture, let us first give some relevant information about the geometry of anti-de Sitter.

2.2.1 (A)dS geometry

The geometry of AdS_D is given by the SO$(2, D-1)$ invariant hyperboloid in $(D+1)$-dimensional space

$$-X_{-1}^2 - X_0^2 + \sum_{i=1}^{D-1} X_i^2 = -\Re^2 \,, \tag{2.17}$$

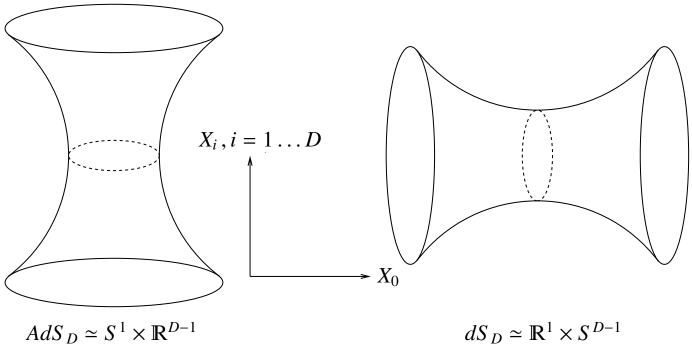

Figure 2.2: AdS_D and dS_D as hyperboloids in $\mathbb{R}^{2,D-1}$.

where $\Re$ denotes the AdS-radius, see figure 2.2. This induces the following line element in terms of so-called horospherical coordinates $\{x_\alpha, U, V\}$:

$$\begin{aligned} U &= X_{-1} + X_{D-1}\,, \\ x_\alpha &= \frac{X_\alpha \Re}{U}\,, \qquad (\alpha = 0, \ldots, D-2)\,, \\ V &= X_{-1} - X_{D-1} = \left(\frac{U}{\Re^2}\right) x^2 + \left(\frac{\Re^2}{U}\right)\,, \\ \mathrm{d}s^2 &= \left(\frac{U}{\Re}\right)^2 \mathrm{d}x^2 + \left(\frac{\Re}{U}\right)^2 \mathrm{d}U^2\,. \end{aligned} \tag{2.18}$$

A more convenient parametrization in the context of brane-world scenarios, that we will encounter further on, are the so-called Poincaré coordinates

$$\mathrm{d}s^2 = e^{-2r/\Re}\mathrm{d}x^2 + \mathrm{d}r^2\,, \qquad e^{-r/\Re} = \frac{U}{\Re}\,. \tag{2.19}$$

2.2.2 Maldacena conjecture

In 1997 a remarkable connection between string theory and gauge theory was conjectured by Maldacena [46], proposing an equivalence between string theory on an $AdS_p \times S^{D-p}$ and a conformal field theory (CFT) in $(p-1)$ dimensions, on the boundary of the AdS space. We will illustrate this statement by briefly describing Maldacenas original motivation.

First imagine we have an open string with both endpoints ending on a single D3-brane. As the lowest mode is given by a vector field with U(1) gauge invariance, this induces a four-dimensional U(1) gauge theory on the brane. Since the D3-brane is a half-BPS solution, breaking half of the total number of supersymmetries, the $D = 4$ U(1) theory has $\mathcal{N} = 4$ supersymmetry.

Now extend this to a system of N parallel D3-branes, separated by a distance r. The open strings stretching between the various branes again induce a U(1) gauge theory on each brane. In the limit of $r \to 0$ we have a stack of coinciding branes and the gauge symmetry on the branes is enhanced from $U(1)^N$ to $U(N)$, which in the low energy limit describes a four-dimensional CFT with gauge group $SU(N)$, known as $D = 4$ $\mathcal{N} = 4$ super-Yang-Mills (SYM) theory. The bosonic symmetry group of this gauge theory is given by the product of the conformal group in four dimensions, $SO(4, 2)$ and the R-symmetry group $SO(6)$.

On the other hand we know that the stack of D3-branes, like any massive object, causes the space-time to curve. Far away from the branes the space-time is given by Minkowski space but in the near-horizon limit, i.e. near the branes, the geometry can be shown to resemble that of the space $AdS_5 \times S^5$. Since the radii of the sphere and of the AdS space are proportional to N, the resemblance gets better for increasing N. The isometry group of this background geometry is given by $SO(4, 2) \times SO(6)$.

Consequently it was conjectured that IIB string theory on a $AdS_5 \times S^5$ background in the large N limit is dual to a CFT on the boundary of AdS_5, given by $\mathcal{N} = 4$ SYM. This statement was later generalized to the so-called Domain-wall/Quantum field theory (DW/QFT) correspondance, that relates supergravity on a near-horizon geometry of a p-brane to a (non-conformal) QFT on the brane. The most striking result of these correspondances is that they relate a gravitational theory, like supergravity or string theory, to a non-gravitational conformal field theory.

In the context of this conjecture, it is useful to study $\mathcal{N} = 8$, $D = 5$ gauged supergravity; the dimensional reduction of IIB on $AdS_5 \times S^5$ gives $SO(6)$ gauged $D = 5$, $\mathcal{N} = 8$ sugra [49, 50]. This reduction is believed to be a consistent nonlinear truncation, meaning that all classical solutions of the five-dimensional theory can be uplifted to IIB solutions. For example, the $SO(6)$-invariant AdS_5 groundstate can be uplifted to an $AdS_5 \times S^5$ vacuum. Therefore the five-dimensional theory should contain all relevant deformations of $\mathcal{N} = 4$ SYM, and all domain-wall solutions in this theory can be uplifted.

During the past few years a considerable amount of evidence has been gathered in many different applications to support the AdS/CFT conjecture. For more details we refer to the reviews [46, 51].

2.3 Domain-walls

Domain-walls are (D-1)-dimensional solutions of the sugra equations of motion, separating two regions of space-time. In the case of a (singular) (D-2)-brane solution, also called 'thin' domain-wall, the brane couples to a volume form which can be dualized to a cosmological constant. The value and/or sign of the cosmological constant usually is different when passing both sides of the domain-wall. The other type of domain-wall is the 'smooth' or 'thick' domain-wall.

2.3.1 Solution

The 'thin' domain-wall solution is a δ-function singularity, given by (1.45) with $p = D - 2$

$$\text{domain-wall} = \begin{cases} \mathrm{d}s^2 &= H(y)^{2\alpha}\mathrm{d}x^2_{(D-1)} + H(y)^{2\beta}\mathrm{d}y^2\,, \\ e^{\varphi} &= H(y)^{-\frac{2a}{\Delta}}\,, \\ F_{(D)} &= \sqrt{\frac{4}{\Delta}}\,\mathrm{d}^d x \wedge \mathrm{d}H^{-1}\,, \\ H(y) &= 1 + Q|y|\,. \end{cases} \tag{2.20}$$

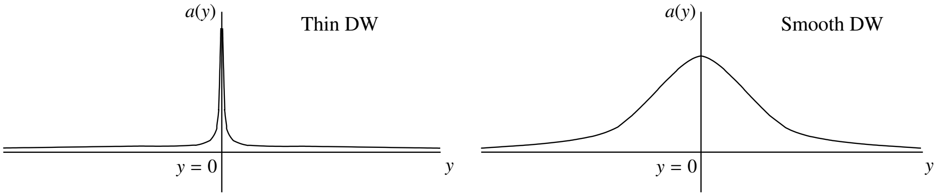

Figure 2.3: Warp-factor for singular and smooth branes.

with

$$\alpha = \frac{2}{\Delta(D-2)}\,, \qquad \beta = \frac{2(D-1)}{\Delta(D-2)}\,, \qquad Q = \sqrt{\Lambda\Delta}\,, \tag{2.21}$$

where the parameter Δ given by

$$\Delta = a^2 - 2\tfrac{D-1}{D-2}\,. \tag{2.22}$$

This expression for Δ is bounded from below by the value Δ_{AdS} corresponding to the AdS vacuum solution of pure supergravity[1]

$$\Delta \geq \Delta_{AdS} \equiv -2\tfrac{D-1}{D-2}\,. \tag{2.23}$$

It was indeed shown [52] that the corresponding domain-wall solution describes two regions of AdS-space. More generally it can be shown that the near-brane limit of solutions of this type are flat Minkowski, and the asymptotic limit far away from the brane describes AdS geometry. Domain-walls therefore are solutions interpolating between two vacua of the theory.

In phenomenological and cosmological models people are usually more interested in non-singular solutions, where there is no δ-function source. A more general domain-wall Ansatz can be written as

$$\mathrm{d}s^2 = a(y)^2 \mathrm{d}x^2_{(D-1)} + \mathrm{d}y^2\,, \tag{2.24}$$

where the function a is called the warp factor; see figure 2.3. For $a(y) = e^{-|y|/L}$ this corresponds to the domain-wall consisting of two slices of AdS in Poincaré coordinates. Depending on the properties of the scalar potential smooth solutions for $a(y)$ can exist corresponding to so-called thick domain-walls.

2.3.2 Toy model: domain-walls as RG-flows

In this section we will show how domain-walls can be associated with renormalization group flows (RG-flows). The application of the AdS/CFT duality shows that supergravity flow equations, connecting critical points of the scalar potential, describe (holographic) RG-flows of quantum field theories, connecting different fixed points.

An exact analysis of the scalar potential is in general not possible, due to the non-trivial geometry of the scalar manifold and the large number of scalars appearing in the potential. Instead of trying to solve the minimization problem at the level of the second order equations of motion, there is a more appealing method.

[1] Corresponding to the case $a = 0$, i.e. constant dilaton; the metric therefore describing AdS-space.

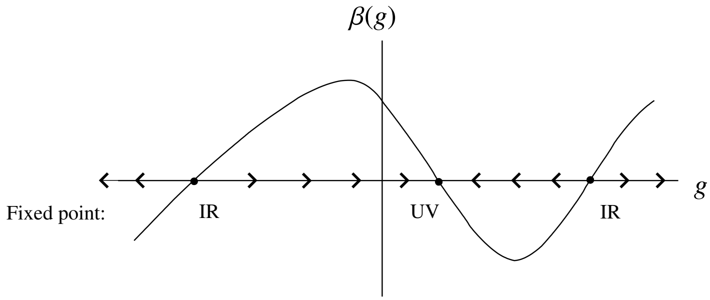

Figure 2.4: Fixed points of the beta-function.

Our starting point is the smooth domain-wall Ansatz in the context of the scalar-gravity toy model (2.6)

$$\mathrm{d}s^2 = e^{2A(y)}\eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + \mathrm{d}y^2\,, \quad \phi = \phi(y)\,. \tag{2.25}$$

As we saw before, at the critical points of $V(\phi)$ the scalar ϕ is constant and the geometry becomes AdS with cosmological constant given by $\Lambda = \frac{1}{2}V(\phi_c)$. However, we also saw that these AdS vacua are dual to a conformal field theory on the boundary of the AdS space-time. Relevant deformations of these CFTs on the field theory side give rise to so-called RG-group flows to other conformal theories. Mapped to its gravity dual this corresponds to scalar fluctuations around AdS space-time. The RG-flow of the coupling constants is described by the U dependence of the scalar fields. These scalars ϕ can be interpreted as coupling constants g and the warp-factor $a(y) = e^{A(y)}$ behaves as a renormalization group scale or energy scale U in the dual field theory side. The expression of the field theory beta-function is conventionally given by

$$\beta(g) \equiv U\frac{\partial g(U)}{\partial U}\,, \tag{2.26}$$

and is depicted in figure 2.4. The arrows denote the flow-direction of the coupling constant g with increasing energy U. The zeroes of the beta-function, called *fixed points*, correspond to scale-independent conformal field theories. These fixed points correspond to critical points of the scalar potential on the supergravity side. There are two types of fixed points: IR points corresponding to low energy scales and UV points at high energy scales, behaving as attractors. A small dictionary mapping between objects in gauge/gravity theory is given in table 2.1.

Returning to equation (2.25), we see that this geometry describes anti-de Sitter space if we take $A(y) = -\frac{y}{L}$. Using the metric-Ansatz (2.25) the equations of motion (2.7) become [53]

$$\begin{aligned} \phi''(y) + (D-1)A'(y)\phi'(y) &= \tfrac{\partial V}{\partial \phi}\,, \\ (D-2)A''(y) + \tfrac{(D-1)(D-2)}{2}A'(y)^2 &= -\tfrac{1}{4}\phi'(y)^2 - \tfrac{1}{2}V(\phi)\,, \\ \tfrac{(D-1)(D-2)}{2}A'(y)^2 &= \tfrac{1}{4}\phi'(y)^2 - \tfrac{1}{2}V(\phi)\,. \end{aligned} \tag{2.27}$$

These equations can be interpreted as Euler-Lagrange equations for the energy functional

$$E = \int_{-\infty}^{\infty} \mathrm{d}y \frac{e^{(D-1)A(y)}}{D-2}\left(-(D-1)(D-2)A'(y)^2 + \frac{1}{2}\phi'(y)^2 + V(\phi)\right)\,. \tag{2.28}$$

Sugra on AdS_D	$(D-1)$-dim. gauge theory
Critical point: AdS space-time	Fixed point: CFT $\beta = 0$
Warp-factor $a(y)$	Energy scale U
Scalar $\phi(y)$	Coupling constant $g(U)$
Domain-wall flow-equations	RG-flow

Table 2.1: A domain-wall/RG-flow dictionary.

Substituting equation (2.15), and using the Bogomol'nyi trick to write E as a sum of squares we obtain

$$\begin{aligned} E &= \int_{-\infty}^{\infty} \mathrm{d}y \frac{e^{(D-1)A(y)}}{D-2} \left(\frac{1}{2} \left[\phi'(y) \mp (D-2) \frac{\partial W}{\partial \phi} \right]^2 - (D-1)(D-2) \left[A'(y) \pm \frac{1}{2} W(\phi) \right]^2 \right) \\ &\quad \pm \left[e^{(D-1)A(y)} W(\phi) \right]_{-\infty}^{\infty} . \end{aligned} \tag{2.29}$$

Written in this form the equations minimizing the energy are easily read off to be

$$\begin{aligned} \phi'(y) &= \mp (D-2) \frac{\partial W}{\partial \phi} , \\ A'(y) &= \pm \frac{1}{2} W(\phi) . \end{aligned} \tag{2.30}$$

These equations describe gradient-flows on the hypersurface given by the functional $W(\phi^i)$ in the scalar manifold. Contrary to the second order equations of motion, the analysis of these first order equations is much simpler. Solutions of the flow-equations are automatically solutions of the equations of motion.

Remarkably, in gauged supergravity theories the flow equations could also have been obtained by plugging the same Ansätze (2.25) and (2.15) into the BPS-equations corresponding to the domain-wall solution

$$\begin{aligned} \delta\psi_\mu &= (\partial_\mu + \tfrac{1}{4} \omega_\mu{}^{ab} \Gamma_{ab}) \epsilon + W(\phi) \Gamma_\mu \epsilon = 0 , \\ \delta\lambda &= \partial\!\!\!/ \phi \epsilon - (D-2) \frac{\partial W}{\partial \phi} \epsilon = 0 . \end{aligned} \tag{2.31}$$

In any theory there are generally different relevant deformations possible, all describing certain RG-flows: IR-UV, UV-UV, IR-IR. For instance, in [54, 55] a flow was constructed from $\mathcal{N} = 4$ SYM to $\mathcal{N} = 1$ SYM by studying the scalar potential of $\mathcal{N} = 8$, $D = 5$ supergravity. Flows of the type IR-IR are of particular interest in the context of supersymmetric brane-world scenarios as we will see in the following section. Figure 2.5 gives an example of a flow between two IR-IR fixed points. The big question however still remains... does there exist a corresponding domain-wall? The answer to this question can be given by studying the scalar potential of the most general matter-coupled gauged supergravity theory.

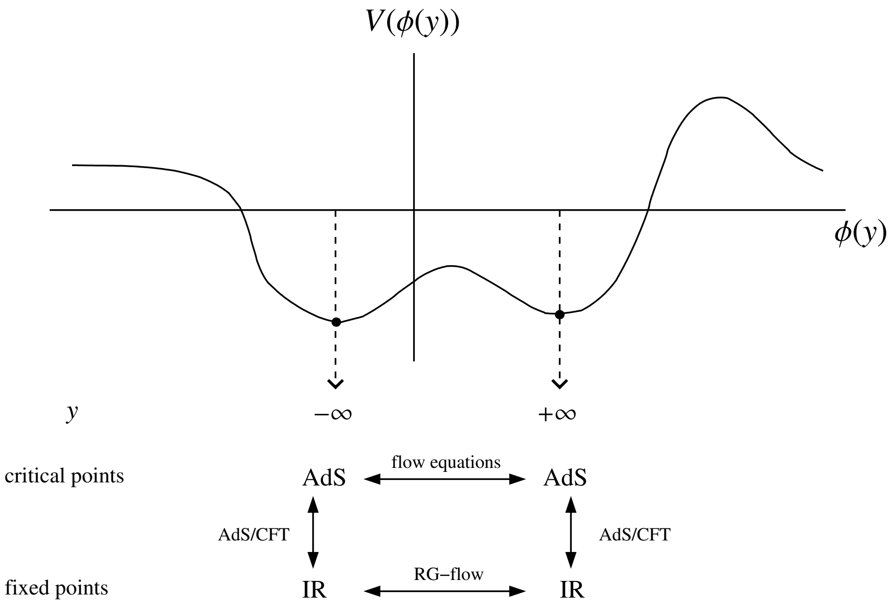

Figure 2.5: Domain-walls as Renormalization Group Flows.

2.4 Brane-world scenarios

The idea of brane-worlds rests on the assumption that our four-dimensional space-time is given by an infinitesimally thin three-brane floating in $(4+n)$ dimensions. Standard model particles are living on the brane but gravity extends in the transverse dimensions. In 1999 Randall and Sundrum proposed two specific brane-world models, motivated to solve a couple of long standing problems in theoretical physics: the hierarchy problem and the cosmological constant problem. The hierarchy problem covers the huge difference of order of magnitude between the Planck scale and the weak scale. Some of the older models tried to explain this using large extra dimensions [56, 57]. Although the idea by Randall and Sundrum is not completely new [58], their approach came with remarkable new insights that stimulated further research until the current moment. In their original two papers they gave two different models, RS I [59] and RS II [60] which will be schematically described below. For more details we refer to the original papers or the reviews [61–63].

2.4.1 Randall-Sundrum I: two branes

The two-brane scenario is a model of five-dimensional gravity on an orbifold $\mathcal{M}_4 \times S^1/\mathbb{Z}_2$ with two three-branes located on both $\mathbb{Z}_2$ fixed points, separated by a distance πr_c. The brane at $y = 0$ is called the "hidden" or "Planck" brane and the one at $y = \pi r_c$ the "visible" or "Standard model" brane (see figure 2.6). The idea is simple:

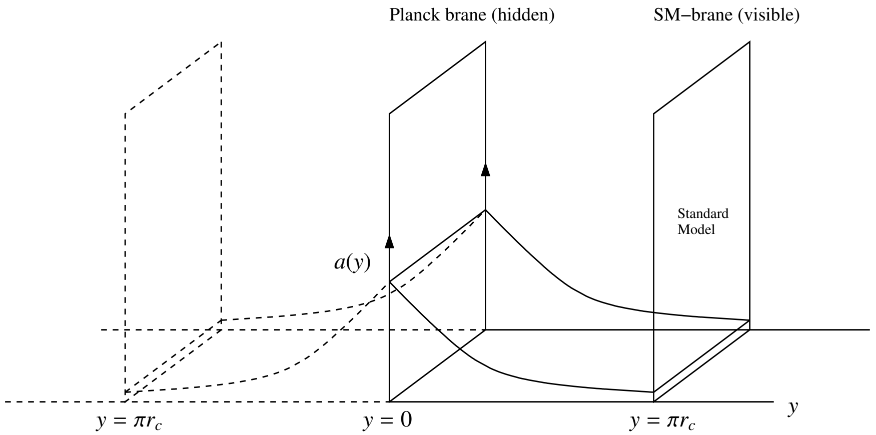

Figure 2.6: The two-brane Randall-Sundrum scenario.

- Symbolically write down an action for the combined system

$$\begin{aligned} S &= S_{\text{gravity}} + S_{\text{vis}} + S_{\text{hid}} , \\ S_{\text{gravity}} &= \int d^4x\, dy\, \sqrt{|G|}\left(2M^3 R - \Lambda\right) , \\ S_{\text{brane}} &= \int d^4x\, \sqrt{|g_{\text{ind}}|}\,(\mathcal{L} - V_{\text{brane}}) , \qquad \text{(for both branes)} \end{aligned} \tag{2.32}$$

  where g_{ind} is the induced metric on the brane, V_{brane} the vacuum energy of the brane, and M the five-dimensional Planck mass.

- Write down an Ansatz for the background metric, possessing four-dimensional Poincaré invariance

$$ds^2 = a(y)^2 \eta_{\mu\nu} dx^\mu dx^\nu + r_c^2 dy^2 , \qquad a(y) = e^{-\sigma(y)} . \tag{2.33}$$

- Deduce the modified Einstein equations from (2.32) and (2.33). These equations are solved by

$$\sigma(y) = r_c|y| \sqrt{\frac{-\Lambda}{24M^3}} , \qquad V_{\text{hid}} = -V_{\text{vis}} = -\Lambda/k , \qquad \Lambda = -24M^3k^2 , \tag{2.34}$$

  where k is some integration constant. We see that the solution of the warp-factor requires the background to consist of two slices of AdS in Poincaré coordinates.

As a result of the above procedure the effective Planck scale on the brane can be calculated to be

$$M_{\text{pl}}^2 = \frac{M^3}{k}\left(1 - e^{-2\pi r_c k}\right) . \tag{2.35}$$

The hierarchy problem can now be solved by taking the five-dimensional Planck scale to be of the order of the weak scale, and considering the effective theory on the visible brane at $y = \pi r_c$. If

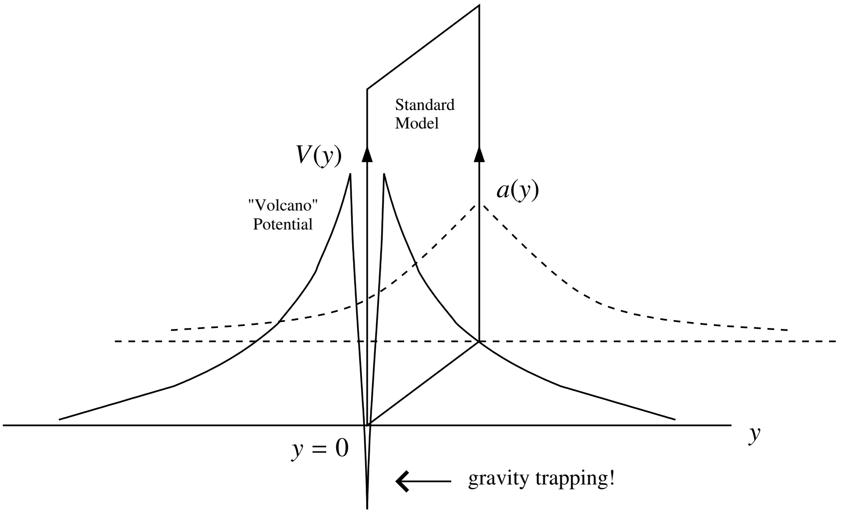

Figure 2.7: The single-brane Randall-Sundrum scenario.

we take $r_c k \approx 50$, this results in a scale hierarchy due to the exponential form of the warp-factor. This was concluded by considering matter fields living on the visible brane. Although solving the hierarchy problem, nevertheless this scenario is still problematic; it lacks the possibility of localization of gravity on the visible brane. Also the presence of a negative tension brane was required. Soon after the RS I model was proposed, another model was suggested, with only one brane, to resolve these problems.

2.4.2 Randall-Sundrum II: one brane

The one-brane scenario initially starts off with the same setup as the one described in the previous section , but the invisible brane is sent to infinity, and is therefore physically removed from the model, see figure 2.7. The brane-tension of the remaining brane is positive and again fine-tuned against the bulk cosmological constant. Instead of solving the hierarchy problem, the warp-factor is now used for the localization of the graviton to the brane. By studying fluctuations of the metric G it was shown that they are effectively described by Newton's equation on the brane, predicting higher order corrections to the Newtonian potential

$$V_N(r) = G_N \frac{m_1 m_2}{r} \left(1 + \frac{1}{r^2 k^2} \right) . \tag{2.36}$$

Although both these models have appealing properties, fermions will have to be included in order to obtain phenomenologically interesting models.

2.4.3 Supersymmetric Randall-Sundrum scenario

The simplest way of including fermions in the theory is by trying to embed the model in a supersymmetric theory. The best candidate for this theory is thought to be $D = 5$, $\mathcal{N} = 2$ supergravity. As we saw in the previous sections some parameter tweaking was necessary for obtaining a consistent model. The main obstruction of a supersymmetric analog however is that the scalar potential is now more restricted, not leaving a lot of room for tweaking the parameters of the solution. Furthermore, the three-brane used in the model should be a valid supergravity solution, namely a domain-wall in five dimensions. Several possible solutions were suggested to resolve this issue.

One solution was given by [64, 65] who considered the insertion of singular brane sources in order to restore supersymmetry in spaces with singularities such as the thin domain-walls. This scenario is conjectured to be the dimensional reduction of the eleven-dimensional Hořava-Witten model [27], on some six-dimensional Calabi-Yau manifold [66, 67].

Another, more appealing, solution would be to find a smooth soliton solution interpolating between two AdS vacua. For such solutions to be compatible with a supersymmetric Randall-Sundrum scenario, the scalar potential should have at least two connected stable IR critical points with the same value of the cosmological constant. Secondly, the flow-equations should be solvable for the smooth domain-wall Ansatz and the corresponding warp-factor should be exponentially decreasing for $y \to \pm\infty$. In order to find such solutions a thorough investigation of the most general gauged $\mathcal{N} = 2$, $D = 5$ sugra is needed. Note that brane-world models can be given a place in string theory, by requiring this five-dimensional theory to follow from a specific Calabi-Yau compactification of M-theory. Alternatively one could try to find an explicit embedding of $\mathcal{N} = 2$ in $\mathcal{N} = 8$ sugra in $D = 5$, which could be related to string theory by the AdS/CFT conjecture.

Let us give a brief description of the field content of ungauged $\mathcal{N} = 2$, $D = 5$ supergravity and its relevant matter multiplets (I labelling the representation of the gauge group):[2]

- (8 + 8) Gravity multiplet: vielbein $e_\mu{}^a$, two gravitinos ψ^i_μ, graviphoton A_μ
- (8 + 8) Vector multiplets: vector A^I_μ, two gauginos ψ^{iI}, scalar σ^I
- (4 + 4) Hyper multiplets: four quaternions q^X, two hyperinos ζ^A

In five dimensions we can also have self-dual tensor multiplets, provided the vectors are in the adjoint representation. Otherwise the tensors can be dualized into vectors. Normally these tensors are self-dual in the sense as explained in [68].

Pure ungauged $\mathcal{N} = 2$, $D = 5$ sugra was constructed by Cremmer in 1980 [69]. A few years later Günaydin, Sierra and Townsend constructed U(1)- and SU(2)-gauged $\mathcal{N} = 2$, $D = 5$ sugra coupled to an arbitrary number of vector multiplets [70–72]. Several years ago Günaydin and Zagermann added tensor-couplings for specific gauge groups [73–75]. Finally, in 2000, Ceresole and Dall'Agata constructed gauged $\mathcal{N} = 2$, $D = 5$ sugra coupled to n_V vectors, n_T tensors and n_H hyper multiplets [76].

The analysis of the scalar potential in these such theories is highly non-trivial. Several simplified cases therefore have been considered in the literature. Many NO-GO theorems have

[2] The bosonic and fermionic degrees of freedom are denoted between parenthesis.

been posed [77–81]. It was found that without hyper-couplings no IR critical points could be found [55, 77, 78, 82]. After including hypermultiplets several solutions were found yielding only one single IR critical point [54, 55, 83]. In [79] multiple critical points were found, having at least one IR direction [3], but they were not connected.

Last year however a possible solution of a smooth domain-wall was found, by Behrndt and Dall'Agata [84], admitting a supersymmetric extension of the Randall-Sundrum scenario. They considered $\mathcal{N} = 2$ sugra coupled to a single hypermultiplet. The crucial ingredient was the restriction to a specific class of non-homogeneous quaternionic manifolds. A generalization to more general non-homogeneous quaternionic manifolds was recently considered by Anguelova and Lazaroiu [85]. Although solutions already have been found, they are not by far the most general solutions possible. First of all because only the coupling of one hypermultiplet was analyzed. Secondly, because a specific type of tensor-couplings was overlooked in the literature, corresponding to non-compact gaugings, which could have surprising implications. We constructed this extension in the context of $\mathcal{N} = 2$, $D = 5$ conformal supergravity [86]; this will be discussed in chapters 5, 6 and 7.

[3] Saddle points of the scalar potential.

Chapter 3

Dimensional reduction

In this chapter we will explain how to obtain massive deformations, i.e. scalar potentials and cosmological constants from dimensional reduction. We start by reviewing some aspects of standard Kaluza-Klein reduction. Then we will introduce a mechanism to generate masses in lower dimensions, called Scherk-Schwarz dimensional reduction, which will be used in chapter 4 to construct gauged and massive supergravities in ten and nine dimensions. For reviews on the subject of dimensional reduction see [87–90].

3.1 Kaluza-Klein dimensional reduction

Even before the advent of string theory, the possibility of extra dimensions was discussed. As early as 1921, a few years after Einstein wrote down his theory of general relativity [91], Kaluza attempted to unify gravitation with electromagnetism by assuming that we live in a five-dimensional universe [92]. By ignoring the extra dimension[1] he managed to obtain the four-dimensional field equations of both gravity and electromagnetism from a five-dimensional theory of pure gravity. Several years later Klein reformulated this theory using the action-principle [93]. He also assumed that the extra coordinate was curled up as a circle with radius of the order of ℓ_p, explaining why this coordinate had never been observed in experiment.

The same mechanism can now also be used for ten-dimensional string theories, in order to try to obtain the four-dimensional world as we observe it and make contact with experiment. For this purpose, one assumes that the ten-dimensional space-time can be written in the form $\mathcal{M}_4 \times K_6$, where $\mathcal{M}_4$ is our four-dimensional space-time, and K_6 a compact sub-manifold of Planckian size. Dimensional reduction also plays an important role in the context of dualities in string theory; many of these results have been obtained by using torus or sphere reductions. In this chapter we will mostly restrict ourselves to compactifications of the type $\mathcal{M}_{D+1} = \mathcal{M}_D \times S^1$, splitting the coordinates $x^{\hat{\mu}}$ into x^{μ} and the compact coordinate z.[2] In section 3.2.2 we will

[1] Kaluza considered our universe to be an isolated four-dimensional subspace of $\mathbb{R}^5$ where all derivatives with respect to the fifth coordinate vanish, the so-called "cylinder condition".

[2] We use hats to denote the dimensionality of a certain object or index. In this section hatted fields are living in $(D+1)$ dimensions and hatted indices run from $0\ldots D$. See appendix B.1 for our conventions.

generalize this to compactifications on higher dimensional compact manifolds.

First of all we observe that all fields in this $(D+1)$-dimensional space have to satisfy the boundary condition

$$\hat{\Phi}(x^\mu, z + 2\pi R_z) = \hat{\Phi}(x^\mu, z)\,, \tag{3.1}$$

and therefore can be Fourier-expanded in terms of the eigenfunctions of the circle

$$\hat{\Phi}(x^\mu, z) = \sum_n \Phi_n(x^\mu) e^{\mathrm{i}\, n\, z/R_z}\,, \tag{3.2}$$

where R_z is the circle-radius. When we insert this Ansatz into the massless Klein-Gordon equation in flat space-time we obtain the equation of motion for a field with mass $M = |\frac{n}{R_z}|$ in the non-compact subspace

$$\hat{\Box}\hat{\Phi}(x^\mu, z) = 0 \quad \rightarrow \quad [\Box + \partial_z\partial^z]\,\Phi_n(x^\mu) \equiv \left[\Box + \left(\tfrac{n}{R_z}\right)^2\right]\Phi_n(x^\mu) = 0\,. \tag{3.3}$$

The infinite set of fields Φ_n is called the tower of massive Kaluza-Klein (KK) modes. Taking the limit where the radius of the circle goes to zero, these KK-modes become infinitely massive and decouple from the massless theory. These modes can be neglected in any effective field theory. In this limit the higher dimensional fields simply become independent of the compact coordinate. This is equivalent to the assumption that the $(D+1)$-dimensional space-time has an isometry along z, with associated Killing vector $K^z = \partial_z$. The process of compactification combined with consistently truncating away the massive modes is called Kaluza-Klein reduction. In the next sections we will give some explicit examples, demonstrating the KK-mechanism.

Metric

Let us first consider pure gravity in $(D+1)$ dimensions, given by the Einstein-Hilbert action

$$\hat{S} = \frac{1}{2\kappa_{D+1}^2}\int \mathrm{d}^{D+1}\hat{x}\,\sqrt{|\hat{g}|}\,\hat{R}\,. \tag{3.4}$$

The metric of $(D+1)$-dimensional space-time can be decomposed into the following components: $\hat{g}_{\mu\nu}, \hat{g}_{\mu z}$ and $\hat{g}_{zz}$ naively looking like a metric, vector and scalar in D dimensions respectively. In order to have all components behave correctly under D-dimensional general coordinate transformations (g.c.t.'s) one arrives at the following Ansatz for the metric

$$\hat{g}_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} e^{2\alpha\phi} g_{\mu\nu} + e^{2\beta\phi} A_\mu A_\nu & e^{2\beta\phi} A_\mu \\ e^{2\beta\phi} A_\nu & e^{2\beta\phi} \end{pmatrix}, \tag{3.5}$$

with α and β arbitrary constants. This Ansatz corresponds to the line element

$$\mathrm{d}\hat{s}^2 = e^{2\alpha\phi}\mathrm{d}s^2 + e^{2\beta\phi}\left(\mathrm{d}z + A_\mu \mathrm{d}x^\mu\right)^2\,. \tag{3.6}$$

The scalar ϕ is called the dilaton and A_μ the Kaluza-Klein vector.

Since we will perform reductions of supergravity theories in the next chapter, it is convenient to use the Palatini formalism. Namely in order to covariantly describe spinors we need to define a flat tangent space in each point of space-time. Vielbeins are the orthonormal basis vectors defined over the manifold. The metric can be written in terms of these vielbeins as follows

$$\hat{g}_{\hat{\mu}\hat{\nu}} = \hat{e}_{\hat{\mu}}{}^{\hat{a}} \hat{e}_{\hat{\nu}}{}^{\hat{b}} \hat{\eta}_{\hat{a}\hat{b}} \,. \tag{3.7}$$

We can use the internal Lorentz gauge degrees of freedom to write the vielbein and inverse vielbein in upper triangular form

$$\hat{e}_{\hat{\mu}}{}^{\hat{a}} = \begin{pmatrix} e^{\alpha\phi} e_\mu{}^a & e^{\beta\phi} A_\mu \\ 0 & e^{\beta\phi} \end{pmatrix}, \qquad \hat{e}_{\hat{a}}{}^{\hat{\mu}} = \begin{pmatrix} e^{-\alpha\phi} e_a{}^\mu & -e^{-\alpha\phi} A_a \\ 0 & e^{-\beta\phi} \end{pmatrix}. \tag{3.8}$$

Using the above Ansätze we obtain the following action (for details see appendix B.2)

$$S = \frac{1}{2\kappa_D^2} \int \mathrm{d}^D x\, e \left(R - \tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{4} e^{-2(D-1)\alpha\phi} F(A)^2 \right). \tag{3.9}$$

Note that we had to use the following values for α and β to obtain the canonical Einstein-Hilbert action in the Einstein-frame

$$\alpha^2 = \frac{1}{2(D-1)(D-2)}, \qquad \beta = -(D-2)\alpha \,. \tag{3.10}$$

The D-dimensional gravitational constant is related to the $(D+1)$-dimensional one by the volume of the compact space

$$\kappa_{D+1}^2 = \kappa_D^2 \int \mathrm{d}z = 2\pi R_z \,\kappa_D^2 \,. \tag{3.11}$$

The above mechanism demonstrates how Kaluza and Klein partly succeeded in unifying Maxwell's theory with gravity, by reducing pure gravity from five to four dimensions. The only awkward feature they were not able to explain was the presence of the scalar field. Naively one would put it equal to zero, but this would not be consistent with the field equations; it would imply that the KK-vector should vanish as well.

Forms

Next consider antisymmetric forms in a gravitational background. The generic KK-Ansatz for a $(n-1)$-form gauge field is given by[3]

$$\hat{A}_{\mu_1\ldots\mu_{n-1}} \equiv A_{\mu_1\ldots\mu_{n-1}}\,, \qquad \hat{A}_{\mu_1\ldots\mu_{n-2}\,z} \equiv A_{\mu_1\ldots\mu_{n-2}}\,. \tag{3.12}$$

Using these Ansätze, we obtain the lower-dimensional gauge invariant field-strengths

$$\begin{aligned}
\hat{F}_{\alpha_1\ldots\alpha_{n-1}\underline{z}} &= \hat{e}_{\alpha_1}{}^{\hat{\mu}_1} \ldots \hat{e}_{\alpha_{n-1}}{}^{\hat{\mu}_{n-1}} \hat{e}_{\underline{z}}{}^z \hat{F}_{\mu_1\ldots\mu_{n-1}\,z} \equiv e^{(D-n-1)\phi} F_{\alpha_1\ldots\alpha_{n-1}}\,, \\
\hat{F}_{\alpha_1\ldots\alpha_n} &= \hat{e}_{[\alpha_1}{}^{\mu_1} \ldots \hat{e}_{\alpha_{n-1}}{}^{\mu_{n-1}} \left(\hat{e}_{\alpha_n]}{}^{\mu_n} \hat{F}_{\mu_1\ldots\mu_n} + n\, \hat{e}_{\alpha_n]}{}^z \hat{F}_{\mu_1\ldots\mu_{n-1}\,z} \right) \equiv e^{-n\alpha\phi} F_{\alpha_1\ldots\alpha_n}\,, \\
F_{(n-1)} &= \mathrm{d}A_{(n-2)}\,, \qquad F_{(n)} = \mathrm{d}A_{(n-1)} - n\, F_{(n-1)} \wedge A_{(1)}\,.
\end{aligned} \tag{3.13}$$

Usually the reduction Ansätze will be of a slightly different form. This is because lower-dimensional field redefinitions are already applied on the level of the Ansätze, in order to get a nice expression for the reduced theory.

[3] Flat compact directions are denoted by $\underline{z}$. These differ from the non-flat ones by the vielbein $\hat{e}_z{}^{\underline{z}}$.

3.1.1 Symmetries

Let us now have a look at the symmetries of the lower-dimensional theory, and how they are obtained from the higher-dimensional symmetries. The $(D+1)$-dimensional theory of gravity contains two symmetries:

- General coordinate transformations (g.c.t.)
The g.c.t. can be written infinitesimally as

$$\delta_{g.c.t.} x^{\hat{\mu}} = -\hat{\xi}^{\hat{\mu}} \,, \qquad \delta_{g.c.t.} \hat{g}_{\hat{\mu}\hat{\nu}} = \hat{\xi}^{\hat{\rho}} \partial_{\hat{\rho}} \hat{g}_{\hat{\mu}\hat{\nu}} + 2 \hat{g}_{\hat{\rho}(\hat{\mu}} \partial_{\hat{\nu})} \hat{\xi}^{\hat{\rho}} \,. \tag{3.14}$$

Demanding all fields (e.g. $\hat{g}_{\mu z}$) to stay independent of z after a g.c.t., one derives the following constraints on the g.c.t.-parameter

$$\partial_z \hat{\xi}^{\mu} = 0 \,, \qquad \partial_z \partial_{\hat{\mu}} \hat{\xi}^z = 0 \,, \tag{3.15}$$

which are solved by

$$\hat{\xi}^{\mu} = \xi^{\mu}(x^{\mu}) \,, \qquad \hat{\xi}^z = cz + \lambda(x^{\mu}) \,. \tag{3.16}$$

The g.c.t. corresponding to these parameters also leave invariant the KK-Ansatz (3.6), and give rise to the following transformations after substituting (3.5) and (3.16) into (3.14).

$$\begin{array}{rcllll}
\delta g_{\mu\nu} &=& \underbrace{\xi^{\rho}\partial_{\rho} g_{\mu\nu} + 2 g_{\rho(\mu}\partial_{\nu)}\xi^{\rho}} & - & 2\alpha\beta^{-1} c\, g_{\mu\nu} & \\
\delta A_{\mu} &=& \xi^{\rho}\partial_{\rho} A_{\mu} + A_{\rho}\partial_{\mu}\xi^{\rho} & - & c\, A_{\mu} & + \; \partial_{\mu}\lambda(x) \\
\delta\phi &=& \xi^{\rho}\partial_{\rho}\phi & + & \beta^{-1} c & \\
&& \text{g.c.t.} && \text{scale symmetry} & \text{U(1)}
\end{array} \tag{3.17}$$

The emerging of a U(1) associated with z-independent reparametrizations of the compact coordinate is in fact a generic feature of dimensional reduction. The KK-vector transforms as a true gauge vector. As we will see in section 3.2.2, in general a lower-dimensional gauge group G is generated by the Killing vectors on the internal manifold. The relevance of the scale-symmetry will become clear after discussing the second symmetry.

- Global Weyl symmetry
Actually this is not a symmetry of the action, but of the equations of motion. Under this symmetry the metric scales with a constant factor. Infinitesimally this becomes

$$\delta \hat{g}_{\hat{\mu}\hat{\nu}} = 2a \hat{g}_{\hat{\mu}\hat{\nu}} \,, \tag{3.18}$$

reducing to

$$\delta g_{\mu\nu} = 2a \left[1 - \alpha\beta^{-1} \right] g_{\mu\nu} \,, \qquad \delta A_{\mu} = 0 \,, \qquad \delta\phi = a\beta^{-1} \,. \tag{3.19}$$

By taking two linearly independent combinations of both scale-symmetries we obtain a global dilaton shift symmetry of the lower-dimensional action, and a uniform scaling symmetry[4] that is only valid at the level of the equations of motion

$$a = -\tfrac{c}{D-1} \quad : \quad \delta g_{\mu\nu} = 0 \,, \qquad \delta A_{\mu} = -c A_{\mu} \qquad \delta\phi = -\tfrac{c}{\alpha(D-1)} \,, \tag{3.20}$$

$$a = -c \quad : \quad \delta g_{\mu\nu} = 2a g_{\mu\nu} \,, \qquad \delta A_{\mu} = a A_{\mu} \qquad \delta\phi = 0 \,. \tag{3.21}$$

[4] Also called "trombone" symmetry in the literature [87].

Both types of symmetries will turn out to be of use in chapter 4 when we consider scale symmetries in supergravity.

3.2 Scherk-Schwarz dimensional reduction

Even though the KK-mechanism has some very appealing features, it is clearly not the most general way of dimensional reduction; all massive modes are truncated away and consequently we can never obtain masses for gauge particles in lower dimensions. It also does not provide a natural way of breaking some part of the supersymmetry, which clearly is needed to obtain physically plausible theories. In 1979 Scherk and Schwarz posed an interesting alternative in a series of two papers [94, 95], called generalized dimensional reduction, or also Scherk-Schwarz (SS) reduction.

The general feature of SS-reduction is the usage of symmetries of the higher-dimensional theory to introduce masses in lower dimensions. The generalization consisted of allowing the higher-dimensional fields to depend on the compact coordinate z, in a way prescribed by the symmetries of the action. This assures us that the z-dependence will be completely removed from the equations of motion of the reduced action.

Two types of symmetries can be used:

1. global/internal symmetries: phase, scale and shift symmetries or other global symmetries
2. local/external symmetries: space-time symmetries, such as translations or rotations in the compact manifold

3.2.1 Scherk-Schwarz I

In this section we will restrict ourselves to symmetries of the first kind; the latter type will be briefly explained in section 3.2.2.

In the case of a global U(1) phase-symmetry $\hat{\Phi} \to e^{\mathrm{i}\Lambda}\hat{\Phi}$, we generalize the periodicity condition (3.1) by identifying the two fields up to an extra global phase-transformation, or "twist"

$$\hat{\Phi}(x^\mu, z + 2\pi R_z) = e^{2\pi \mathrm{i} m R_z}\hat{\Phi}(x^\mu, z)\,, \tag{3.22}$$

resulting in the following mode-expansion

$$\hat{\Phi}(x^\mu, z) = e^{\mathrm{i} m z} \sum_n \Phi_n(x^\mu) e^{\mathrm{i} n z / R_z}\,. \tag{3.23}$$

In the limit $R_z \to 0$ the massive modes again decouple, and we are left with the effective Ansatz

$$\hat{\Phi}(x^\mu, z) = e^{\mathrm{i} m z}\Phi_0(x^\mu)\,, \tag{3.24}$$

which can also be obtained by replacing the global symmetry parameter Λ by mz. More generally if there is a global symmetry group G acting on the fields: $\hat{\Phi} \to g(\hat{\Phi})$, we allow for a specific z-dependence in our reduction Ansatz through a symmetry transformation dependent on the compact coordinate(s)

$$\hat{\Phi}(x^\mu, z) = g_z(\hat{\Phi}(x^\mu))\,, \qquad g_z = g(z) \in G\,. \tag{3.25}$$

This particular form of the Ansatz guarantees that the reduced theory will be independent of z, and results in the gauging of the group G, producing a scalar potential or cosmological constant. When a mass term is produced for the gravitino, there is even a spontaneous breaking of supersymmetry. Since the z-dependent transformation in general will not be periodic, as before, going once around the compact coordinate will produce a twist, the so-called *monodromy*

$$\mathcal{M}(g) = g(2\pi R_z)g(0)^{-1}\,, \qquad \mathcal{M} \in G\,. \tag{3.26}$$

Writing the group element in terms of the generators of the Lie algebra

$$g(z) = e^{Mz}\,, \qquad M \in \text{Lie}(G)\,, \tag{3.27}$$

we obtain an expression for the monodromy in terms of M

$$\mathcal{M} = e^{M}\,, \qquad M = g^{-1}\partial_z g\,. \tag{3.28}$$

In practice it turns out that the object M can be interpreted as the mass matrix of the reduced theory, as we will see in chapter 4. The specific function g can now be determined by demanding that M is independent of z.

One might wonder at this point whether the reduced theory we obtain this way is unique. Not every choice for g will necessarily lead to a new reduced theory; provided these functions are in the same conjugacy class, their reduced theories will only differ by a field redefinition. Independent reductions can therefore be classified by the conjugacy classes of the mass matrix M [96–98]. We will see examples of this in the next chapter.

Let us first look at some simple examples of the mechanism described above.

Complex scalar field in a gravitational background

The action for a complex scalar field in a curved background is given by

$$\hat{S} = \frac{1}{2\kappa^2_{D+1}} \int \mathrm{d}^{D+1}\hat{x}\,\hat{e}\left(\hat{R} - \tfrac{1}{2}\,\partial_{\hat{\mu}}\hat{\varphi}(\hat{x})\partial^{\hat{\mu}}\hat{\varphi}^*(\hat{x})\right)\,. \tag{3.29}$$

This action contains two global symmetries: invariance under phase transformations and under shifts.

(1) Phase symmetry: $\hat{\varphi} \to e^{\mathrm{i}\,c}\hat{\varphi}$

Following the above prescription, the corresponding SS-Ansatz becomes:

$$\hat{\varphi}(\hat{x}) = e^{\mathrm{i}\,m_1 y}\varphi(x)\,, \qquad (m_1 \text{ real})\,. \tag{3.30}$$

The reduction of the scalar part of the action then gives:

$$\begin{aligned} \hat{e}\,\partial\hat{\varphi}(\hat{x})\partial\hat{\varphi}^*(\hat{x}) &= \hat{e}\,\partial_{\hat{\mu}}\hat{\varphi}(\hat{x})\partial_{\hat{\nu}}\hat{\varphi}^*(\hat{x})\hat{e}_{\hat{a}}{}^{\hat{\rho}}\hat{e}_{\hat{b}}{}^{\hat{\mu}}\hat{\eta}^{\hat{a}\hat{b}} \\ &= e\left(|\mathcal{D}\varphi(x)|^2 + m_1^2 e^{2(\alpha-\beta)\phi}\,|\varphi|^2\right)\,, \end{aligned} \tag{3.31}$$

where the covariant derivative is defined as: $\mathcal{D}_\mu = \partial_\mu - \mathrm{i}\,m_1 A_\mu$.
This we recognize as the usual expression for the covariant derivative, associated with the

gauging of a U(1) phase transformation of an Abelian gauge theory. So a straightforward interpretation of m_1 is that of a charge for the complex scalar field. Alternatively m_1 can be interpreted as mass parameter. Added up to (3.9), the complete reduced action describes a real scalar ϕ (the dilaton) and a charged complex scalar φ in a curved background. Most importantly, this procedure seems to have produced a scalar potential, describing the interactions between both scalar fields. Finally note that the same result, except for the scalar potential, could have been obtained by a KK-reduction, followed by the gauging of the generated global U(1) symmetry. The scalar potential, however, is gauge-invariant by itself and cannot be constructed by gauging alone. In supersymmetric theories however the scalar potential can always be reconstructed by demanding invariance of the action.

(2) Shift symmetry: $\hat{\varphi} \to \hat{\varphi} + c$.
As SS-Ansatz we now take

$$\hat{\varphi}(\hat{x}) = \hat{\varphi}(x) + m_2 z \qquad (m_2 \text{ complex})\,. \tag{3.32}$$

This time the scalar part of the action reduces as

$$\hat{e}\,\partial\hat{\varphi}(\hat{x})\partial\hat{\varphi}^*(\hat{x}) \quad = \quad e\left(|\mathcal{D}\varphi(x)|^2 + |m_2|^2\, e^{2(\alpha-\beta)\phi}\right)\,, \tag{3.33}$$

with covariant derivative $\mathcal{D}_\mu\varphi = \partial_\mu\varphi - m_2 A_\mu$, invariant under the so-called massive gauge transformations

$$\begin{cases} \delta\varphi &= m_2\,\lambda(x)\,, \\ \delta A_\mu &= \partial_\mu\lambda(x)\,, \end{cases} \tag{3.34}$$

induced by a general coordinate transformation along z in $(D+1)$ dimensions, as we saw earlier in the Kaluza-Klein reduction. The complete reduced action becomes

$$S = \int \mathrm{d}^D x\, e\left[R - \tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{4}\,e^{-2(D-1)\alpha\phi}F^2(A) - \tfrac{1}{2}\,|\mathcal{D}\varphi|^2 - \tfrac{1}{2}\,m_2^2 e^{2(\alpha-\beta)\phi}\right]\,. \tag{3.35}$$

If we now fix the massive gauge transformations, by taking $\Re e(\varphi) = 0$, the action consists of the following parts:

1. a D-dimensional Einstein-Hilbert action + kinetic term for the dilaton
2. $\int \mathrm{d}^D x\, e\left[-\frac{1}{4}e^{-2(D-1)\alpha\phi}F(A)^2 - m_2^2 A^2\right]$,
   which is the well known Proca action for a massive vector field, coupled to gravity, with additional dilaton interaction terms.
3. a scalar potential.

Compared to the KK-reduction, obtained by taking $m_1 = m_2 = 0$, we gained a scalar potential and masses for either the scalar or vector field.

Instead of the two separate global symmetries, we could have used the combination of both symmetries for the Scherk-Schwarz reduction. The Ansatz then becomes

$$\hat{\varphi} = e^{\,\mathrm{i}\,m_1 z}\,(\varphi + m_2 z)\;. \tag{3.36}$$

The reduced action corresponds to the sum of the separate actions corresponding to the phase- and shift symmetry.

$O(N)$ scalar fields

The $O(N)$ N-scalar model in $(D+1)$-dimensional flat space-time, is described by the Lagrangian $\mathcal{L}_{D+1}$

$$\mathcal{L}_{D+1} = -\tfrac{1}{2}\partial_{\hat\mu}\hat{\vec\Phi}^T\partial^{\hat\mu}\hat{\vec\Phi} - \tfrac{1}{2}m^2\hat{\vec\Phi}^T\hat{\vec\Phi}\,, \tag{3.37}$$

invariant under global orthogonal transformations working on the scalar fields

$$\hat{\vec\Phi} \to \hat{\vec\Phi}' = O\hat{\vec\Phi}\,, \qquad O \in O(N)\,, \quad O^T O = O\,O^T = \mathbb{1}\,. \tag{3.38}$$

In order to perform a generalized reduction of this theory, we again use the transformation parameter of the global symmetry (group) to put the z-dependence into, yielding the following Ansatz

$$\hat{\vec\Phi}(x,z) = O(z)\hat{\vec\Phi}(x)\,, \tag{3.39}$$

resulting in the action

$$\mathcal{L}_D = -\tfrac{1}{2}\partial_\mu\vec\Phi^T \underbrace{O^T\,O}_{=\mathbb{1}}\,\partial^\mu\vec\Phi - \tfrac{1}{2}\Big(m^2 + [\underbrace{O^T\,\partial_z O}_{\equiv O^{-1}\partial_z O}]^2\Big)\vec\Phi^T\,\vec\Phi \tag{3.40}$$

The only term in the lower-dimensional action, still containing the matrix $O(z)$, is proportional to $O^{-1}(z)\partial_z O(z)$. Since we do not want this term to contain any z-dependence, the term $O^{-1}(z)\partial_z O(z) \equiv M$ again can be interpreted as a mass matrix, just like in (3.28). The exact form of $O(z)$ is not important since it does not explicitly appear in the reduced theory, but using the properties of $O(z)$, the constraint can in principle be solved.

3.2.2 Non-Abelian reductions and Scherk-Schwarz II

In order to generalize the five-dimensional KK-theory we replace the internal space S^1 with some other (compact) space of higher dimension. The first generalization of the KK-mechanism was first considered by Pauli in 1953 [99]. His starting point was the six-dimensional space-time $\mathcal{M}_4 \times S^2$. The extra dimensions form a two-sphere S^2 with internal symmetry group SO(3) $\simeq$ SU(2)/ U(1). Using an appropriate Ansatz (given below for general case) he constructed a non-Abelian theory with gauge group SU(2), one year before Yang and Mills published their famous paper [100].

In the years after that further generalizations were proposed [101–106], which can be roughly classified in the following possibilities for the compact space E_n:

1. $E_n = T^n$: Torus reduction from $(D+n)$ to D dimensions on T^n, i.e. n successive circle reductions. Each reduction-step will give rise to a KK-vector and a dilaton. Also, p-form gauge fields will reduce to a p-form and $(p-1)$-form gauge field one dimension lower. Further reductions will also create 0-form potentials or *axions* coming from the KK-vector(s) in the compact directions. The reduced theory is ungauged and will finally consist of a plethora of scalars and vectors, in the adjoint of the gauge group U(1)^n.

2. $E_n = G$: Group manifold reduction, where G is the compact Lie-group associated with general coordinate transformations on the compact manifold, which will become the gauge group of the reduced theory.

3. $E_n = G/H$: Coset space reduction, where H is the maximal compact subgroup of G. The most common examples are sphere-reductions: $S^n \simeq \mathrm{SO}(n+1)/\mathrm{SO}(n)$. Note that in most cases coset reductions are preferred above group manifold reductions, since less extra dimensions are needed to obtain a certain gauge group. E.g. in order to obtain the SO(8) gauge group one could use the SO(8) group manifold or the coset SO(8)/ SO(7) corresponding to the seven-sphere. The first case would require $dim(E_n) = 28$ whereas for the coset reduction one only needs $dim(E_n) = 7$.

4. inhomogeneous spaces or spaces without any isometries.

Note that not all these reductions can be performed in a consistent way. The only exceptions where consistency can be understood from group-theoretic arguments are the circle, torus or group-manifold reductions. All other reductions will have to satisfy certain requirements to get a full decoupling of the massless and massive modes in the KK-spectrum [107].

For cases 2 and 3 the generic line-element Ansatz is given by

$$\mathrm{d}\hat{s}^2 = \mathrm{d}s^2 + g_{\alpha\beta}\,[\mathrm{d}z^\alpha + K_I^\alpha(z)A_{I\mu}\mathrm{d}x^\mu][\mathrm{d}z^\beta + K_J^\beta(z)A_{J\nu}\mathrm{d}x^\nu]\,, \tag{3.41}$$

where $g_{\alpha\beta}$ is the internal metric on E_n. The coordinates $x^{\hat{\mu}}$ have been split into x^μ and compact coordinates z^α. The functions $K_I^\alpha(z)$ are the Killing vectors of the internal metric, generating the isometry group G with structure constants $f_{IJ}{}^K$

$$K_I \equiv K_I^\alpha \partial_\alpha \rightarrow \qquad \mathcal{L}_{K_I} K_J = [K_I, K_J] = f_{IJ}{}^K K_K\,. \tag{3.42}$$

These same structure constants also define the lower-dimensional gauge group since they end up in the covariant derivatives of the lower-dimensional vector fields, after reducing the general coordinate transformation of the metric under

$$\delta x^{\hat{\mu}} = -\hat{\xi}^{\hat{\mu}}\,, \qquad \hat{\xi}^\mu = 0\,, \quad \hat{\xi}^\alpha = K_I^\alpha(z)\lambda_I(x)\,. \tag{3.43}$$

The second type of Scherk-Schwarz reductions (SS2) makes use of these z-dependent diffeomorphisms on the group manifold, to assign to some fields a specific dependence on the compact coordinates. For details we refer to [95].

Chapter 4

Scherk-Schwarz reductions and gauged supergravities

In this chapter we will make use of the techniques explained in chapter 3 to construct five different two-parameter massive deformations of the unique nine-dimensional $\mathcal{N} = 2$ supergravity. All of these deformations have a higher-dimensional origin via SS-reduction and correspond to gauged supergravities. Although the ultimate goal is to do a full analysis of the scalar potentials in lower-dimensional gauged supergravities, $D = 4$ and $D = 5$ in specific, in this chapter we will study dimensional reductions from $D = 11$ via $D = 10$ down to $D = 9$; nine-dimensional supergravity shares some of the complexities of the lower-dimensional cases, but is still simple enough to study in full detail. Based on these results we will conclude by making a systematic search for half-supersymmetric domain-walls and non-supersymmetric de Sitter space solutions. Furthermore, we discuss in which sense the supergravities we have constructed can be considered as low-energy limits of compactified superstring theory.

Appendix B.1 contains our conventions and in appendix B.3 we discuss some manipulations with spinors and gamma-matrices in ten and nine dimensions.

This chapter is based on the work published in [108]. In this chapter we will only treat the generalized reduction from $D = 11$ to $D = 10$ in some detail; a detailed description of the reductions to $D = 9$ is given in [108].

4.1 $D = 11$ supergravity

Our starting point is eleven-dimensional supergravity [24], which field content is given by[1]

$$D = 11: \qquad \{\hat{\hat{e}}_{\hat{\hat{\mu}}}{}^{\hat{\hat{a}}}, \hat{\hat{C}}_{\hat{\hat{\mu}}\hat{\hat{\nu}}\hat{\hat{\rho}}}, \hat{\hat{\psi}}_{\hat{\hat{\mu}}}\}\,. \tag{4.1}$$

[1]In order to distinguish between $D = 11, D = 10$ and $D = 9$ we indicate $D = 11$ fields and indices with a double hat, $D = 10$ fields and indices with a single hat and $D = 9$ fields and indices without hat.

$\mathbb{R}^+$	$\hat{\hat{e}}_{\hat{\hat{\mu}}}{}^{\hat{\hat{a}}}$	$\hat{\hat{C}}_{\hat{\hat{\mu}}\hat{\hat{\nu}}\hat{\hat{\rho}}}$	$\hat{\hat{\psi}}_{\hat{\hat{\mu}}}$	$\hat{\hat{\epsilon}}$	$\hat{\hat{\mathcal{L}}}$
$\hat{\hat{\alpha}}$	1	3	$\frac{1}{2}$	$\frac{1}{2}$	9

Table 4.1: The $\mathbb{R}^+$-weights of the $D = 11$ supergravity fields, the supersymmetry parameters $\hat{\hat{\epsilon}}$ and the Lagrangian $\hat{\hat{\mathcal{L}}}$.

The Einstein-frame action and the corresponding supersymmetry transformations, up to quartics, are given by

$$\mathcal{L} = \frac{\hat{\hat{e}}}{2\kappa_{11}^2}\Big[\hat{\hat{R}}(\hat{\hat{\omega}}) - \bar{\hat{\hat{\psi}}}_{\hat{\hat{\mu}}}\hat{\hat{\Gamma}}^{\hat{\hat{\mu}}\hat{\hat{\nu}}\hat{\hat{\rho}}}\hat{\hat{\mathcal{D}}}_{\hat{\hat{\nu}}}(\hat{\hat{\omega}})\hat{\hat{\psi}}_{\hat{\hat{\rho}}} - \tfrac{1}{92}\hat{\hat{G}}_{(4)}\hat{\hat{G}}^{(4)} - \tfrac{1}{92}\Big(\bar{\hat{\hat{\psi}}}_{\hat{\hat{\mu}}}\hat{\hat{\Gamma}}^{\hat{\hat{\mu}}\hat{\hat{\nu}}\hat{\hat{\alpha}}\hat{\hat{\beta}}\hat{\hat{\gamma}}\hat{\hat{\delta}}}\hat{\hat{\psi}}_{\hat{\hat{\nu}}} + 12\bar{\hat{\hat{\psi}}}{}^{\hat{\hat{\alpha}}}\hat{\hat{\Gamma}}^{\hat{\hat{\gamma}}\hat{\hat{\delta}}}\hat{\hat{\psi}}{}^{\hat{\hat{\beta}}}\Big)\hat{\hat{G}}_{\hat{\hat{\alpha}}\hat{\hat{\beta}}\hat{\hat{\gamma}}\hat{\hat{\delta}}}$$
$$+\tfrac{1}{(144)^2}\hat{\hat{\varepsilon}}^{(4)(4')(3)}\hat{\hat{G}}_{(4)}\hat{\hat{G}}_{(4')}\hat{\hat{C}}_{(3)}\Big], \tag{4.2}$$

$$\begin{aligned}
\delta\hat{\hat{e}}_{\hat{\hat{\mu}}}{}^{\hat{\hat{a}}} &= \bar{\hat{\hat{\epsilon}}}\hat{\hat{\Gamma}}^{a}\hat{\hat{\psi}}_{\hat{\hat{\mu}}}\,,\\
\delta\hat{\hat{C}}_{\hat{\hat{\mu}}\hat{\hat{\nu}}\hat{\hat{\rho}}} &= -3\,\bar{\hat{\hat{\epsilon}}}\hat{\hat{\Gamma}}_{[\hat{\hat{\mu}}\hat{\hat{\nu}}}\hat{\hat{\psi}}_{\hat{\hat{\rho}}]}\,,\\
\delta\hat{\hat{\psi}}_{\hat{\hat{\mu}}} &= \hat{\hat{\mathcal{D}}}_{\hat{\hat{\mu}}}(\hat{\hat{\omega}})\hat{\hat{\epsilon}} + \tfrac{1}{192}(\hat{\hat{\Gamma}}^{(4)}\hat{\hat{\Gamma}}_{\hat{\hat{\mu}}} - \tfrac{1}{3}\hat{\hat{\Gamma}}_{\hat{\hat{\mu}}}\hat{\hat{\Gamma}}^{(4)})\hat{\hat{G}}_{(4)}\hat{\hat{\epsilon}}\,,
\end{aligned} \tag{4.3}$$

with the field strength $\hat{\hat{G}}_{(4)} = \mathrm{d}\hat{\hat{C}}_{(3)}$ and $\hat{\hat{\mathcal{D}}}_{\hat{\hat{\mu}}} = \partial_{\hat{\hat{\mu}}} + \frac{1}{4}\hat{\hat{\omega}}_{\hat{\hat{\mu}}}{}^{\hat{\hat{a}}\hat{\hat{b}}}\hat{\hat{\Gamma}}_{\hat{\hat{a}}\hat{\hat{b}}}$. The 11D fermionic field content consists solely of a 32-component gravitino, whose field equation reads

$$X_0(\hat{\hat{\psi}}{}^{\hat{\hat{\mu}}}) \equiv \hat{\hat{\Gamma}}^{\hat{\hat{\mu}}\hat{\hat{\nu}}\hat{\hat{\rho}}}\hat{\hat{\mathcal{D}}}_{\hat{\hat{\nu}}}\hat{\hat{\psi}}_{\hat{\hat{\rho}}} = 0\,, \tag{4.4}$$

where we have set the three-form equal to zero for simplicity.[2] Under supersymmetry this fermionic field equation transforms into

$$\delta_0 X_0(\hat{\hat{\psi}}{}^{\hat{\hat{\mu}}}) = \tfrac{1}{2}\hat{\hat{\Gamma}}^{\hat{\hat{\nu}}}\hat{\hat{\epsilon}}\,[\hat{\hat{R}}^{\hat{\hat{\mu}}}{}_{\hat{\hat{\nu}}} - \tfrac{1}{2}\hat{\hat{R}}\,\hat{\hat{g}}^{\hat{\hat{\mu}}}{}_{\hat{\hat{\nu}}}]\,, \tag{4.5}$$

which implies the bosonic Einstein equation for the metric. The supersymmetry rules and field equations are covariant under an $\mathbb{R}^+$ symmetry with parameter $\hat{\hat{\alpha}}$ [109]. A generic field $\hat{\hat{\Phi}}$ with weight w scales as $\hat{\hat{\Phi}} \to e^{w\hat{\hat{\alpha}}}\hat{\hat{\Phi}}$ under this symmetry. The weights of the $D = 11$ fields under this $\mathbb{R}^+$ are given in table 4.1. Note that the Lagrangian is not invariant but scales with weight $w = 9$. Therefore this $\mathbb{R}^+$ is a symmetry of the equations of motion only.

No massive deformation of the eleven-dimensional supergravity theory is known; in particular, no cosmological constant can be added [110]. One problem with a $D = 11$ supersymmetric cosmological constant is that its reduction gives rise to a $D = 10$ cosmological constant with a dilaton coupling that differs from Romans' massive deformation. A general deformation of $D = 11$ supergravity involving the use of extra Killing vectors has been considered in [111], but we will not pursue this possibility here.

[2] This is because we are only interested in solutions coupling to the metric and the dilaton.

4.2 Massive deformations of $D = 10$ IIA supergravity

As already mentioned in section 1.4.2, the Kaluza-Klein reduction of eleven-dimensional supergravity yields the effective IIA theory in ten dimensions. This gives us the following field content of the $D = 10$ IIA supergravity theory

$$D = 10 \;\; \text{IIA:} \qquad \{\hat{e}_{\hat{\mu}}{}^{\hat{a}}, \hat{B}_{\hat{\mu}\hat{\nu}}, \hat{\phi}, \hat{A}_{\hat{\mu}}, \hat{C}_{\hat{\mu}\hat{\nu}\hat{\rho}}, \hat{\psi}_{\hat{\mu}}, \hat{\lambda}\}\,. \tag{4.6}$$

For this reduction we use the reduction Ansätze[3]

$$\begin{aligned}
&\hat{\hat{e}}_{\hat{\hat{\mu}}}{}^{\hat{\hat{a}}} = \begin{pmatrix} e^{-\hat{\phi}/12}\hat{e}_{\hat{\mu}}{}^{\hat{a}} & -e^{2\hat{\phi}/3}\hat{A}_{\hat{\mu}} \\ 0 & e^{2\hat{\phi}/3} \end{pmatrix}, \qquad \hat{\hat{\psi}}_{\hat{a}} = e^{\hat{\phi}/24}(\hat{\psi}_{\hat{a}} - \tfrac{1}{24}\hat{\Gamma}_{\hat{a}}\hat{\lambda})\,, \qquad \hat{\hat{\epsilon}} = e^{-\hat{\phi}/24}\hat{\epsilon}\,, \\
&\hat{\hat{C}}_{\hat{\mu}\hat{\nu}\hat{\rho}} = \hat{C}_{\hat{\mu}\hat{\nu}\hat{\rho}}\,, \qquad \hat{\hat{C}}_{\hat{\mu}\hat{\nu}x} = -\hat{B}_{\hat{\mu}\hat{\nu}}\,, \qquad \hat{\hat{\psi}}_{\underline{x}} = \tfrac{1}{3}e^{\hat{\phi}/24}\hat{\Gamma}_{\underline{x}}\hat{\lambda}\,.
\end{aligned} \tag{4.7}$$

Applying these to the 11D transformation rules (4.3), we obtain the IIA transformation rules in the Einstein frame and up to quartics[4]:

$$\begin{aligned}
\delta_0\hat{e}_{\hat{\mu}}{}^{\hat{a}} &= \bar{\hat{\epsilon}}\hat{\Gamma}^{\hat{a}}\hat{\psi}_{\hat{\mu}}\,, \\
\delta_0\hat{\psi}_{\hat{\mu}} &= \Big(\hat{\mathcal{D}}_{\hat{\mu}} + \tfrac{1}{48}e^{-\hat{\phi}/2}(\hat{\slashed{H}}\hat{\Gamma}_{\hat{\mu}} + \tfrac{1}{2}\hat{\Gamma}_{\hat{\mu}}\hat{\slashed{H}})\Gamma_{11} \\
&\qquad + \tfrac{1}{16}e^{3\hat{\phi}/4}(\hat{\slashed{F}}\hat{\Gamma}_{\hat{\mu}} - \tfrac{3}{4}\hat{\Gamma}_{\hat{\mu}}\hat{\slashed{F}})\Gamma_{11} + \tfrac{1}{192}e^{\hat{\phi}/4}(\hat{\slashed{G}}\hat{\Gamma}_{\hat{\mu}} - \tfrac{1}{4}\hat{\Gamma}_{\hat{\mu}}\hat{\slashed{G}})\Big)\hat{\epsilon}\,, \\
\delta_0\hat{B}_{\hat{\mu}\hat{\nu}} &= 2e^{\hat{\phi}/2}\bar{\hat{\epsilon}}\Gamma_{11}\hat{\Gamma}_{[\hat{\mu}}(\hat{\psi}_{\hat{\nu}]} + \tfrac{1}{8}\hat{\Gamma}_{\hat{\nu}]}\hat{\lambda})\,, \\
\delta_0\hat{A}_{\hat{\mu}} &= -e^{-3\hat{\phi}/4}\bar{\hat{\epsilon}}\Gamma_{11}(\hat{\psi}_{\hat{\mu}} - \tfrac{3}{8}\hat{\Gamma}_{\hat{\mu}}\hat{\lambda})\,, \\
\delta_0\hat{C}_{\hat{\mu}\hat{\nu}\hat{\rho}} &= -3e^{-\hat{\phi}/4}\bar{\hat{\epsilon}}\hat{\Gamma}_{[\hat{\mu}\hat{\nu}}(\hat{\psi}_{\hat{\rho}]} - \tfrac{1}{24}\hat{\Gamma}_{\hat{\rho}]}\hat{\lambda}) + 3\hat{A}_{[\hat{\mu}}\delta_0\hat{B}_{\hat{\nu}\hat{\rho}]}\,, \\
\delta_0\hat{\lambda} &= \Big(\hat{\slashed{\partial}}\hat{\phi} + \tfrac{1}{12}e^{-\hat{\phi}/2}\hat{\slashed{H}}\Gamma_{11} + \tfrac{3}{8}e^{3\hat{\phi}/4}\hat{\slashed{F}}\Gamma_{11} + \tfrac{1}{96}e^{\hat{\phi}/4}\hat{\slashed{G}}\Big)\hat{\epsilon}\,, \\
\delta_0\hat{\phi} &= \tfrac{1}{2}\bar{\hat{\epsilon}}\hat{\lambda}\,,
\end{aligned} \tag{4.8}$$

with the field strengths:

$$\hat{F} = \mathrm{d}\hat{A}\,, \qquad \hat{H} = \mathrm{d}\hat{B}\,, \qquad \hat{G} = \mathrm{d}\hat{C} + \hat{A} \wedge \hat{H}\,, \tag{4.9}$$

and $\hat{\mathcal{D}}_{\hat{\mu}} = \partial_{\hat{\mu}} + \frac{1}{4}\hat{\omega}_{\hat{\mu}}{}^{\hat{a}\hat{b}}\hat{\Gamma}_{\hat{a}\hat{b}}$. For later purposes we indicate these (undeformed) supersymmetry transformations by δ_0. Upon (massless) reduction with the Ansätze (4.7) the 11D field equation (4.4) splits up into two field equations for the 10D IIA fermionic field content, a gravitino and a dilatino:

$$\begin{aligned}
X_0(\hat{\psi}^{\hat{\mu}}) &\equiv \hat{\Gamma}^{\hat{\mu}\hat{\nu}\hat{\rho}}\hat{\mathcal{D}}_{\hat{\nu}}\hat{\psi}_{\hat{\rho}} - \tfrac{1}{8}(\hat{\slashed{\partial}}\hat{\phi})\hat{\Gamma}^{\hat{\mu}}\hat{\lambda} = 0\,, \\
X_0(\hat{\lambda}) &\equiv \hat{\Gamma}^{\hat{\nu}}\hat{\mathcal{D}}_{\hat{\nu}}\hat{\lambda} - \hat{\Gamma}^{\hat{\nu}}(\hat{\slashed{\partial}}\hat{\phi})\hat{\psi}_{\hat{\nu}} = 0\,,
\end{aligned} \tag{4.10}$$

[3]The flat x^{11}-direction is denoted by $\underline{x}$, and the curved x^{11}-direction by x. The particular dilaton prefactors were conveniently chosen to get the standard form of the IIA transformation rules.

[4]An additional field dependent 10D Lorentz transformation is needed to get the correct transformation rule for e.g. the vielbein: $\delta_Q(\epsilon) = \delta_Q(\hat{\epsilon}) + \delta_M$.

$\mathbb{R}^+$	$\hat{e}_{\hat{\mu}}{}^{\hat{a}}$	$\hat{B}_{\hat{\mu}\hat{\nu}}$	$e^{\hat{\phi}}$	$\hat{A}_{\hat{\mu}}$	$\hat{C}_{\hat{\mu}\hat{\nu}\hat{\rho}}$	$\hat{\psi}_{\hat{\mu}}$	$\hat{\lambda}$	$\hat{\epsilon}$	$\hat{\mathcal{L}}$	Origin
$\hat{\alpha}$	$\frac{9}{8}$	3	$\frac{3}{2}$	0	3	$\frac{9}{16}$	$-\frac{9}{16}$	$\frac{9}{16}$	9	$\hat{\hat{\alpha}}$
$\hat{\beta}$	0	$\frac{1}{2}$	1	$-\frac{3}{4}$	$-\frac{1}{4}$	0	0	0	0	

Table 4.2: The $\mathbb{R}^+$-weights of the $D = 10$ IIA supergravity fields, the supersymmetry parameter $\hat{\epsilon}$ and the Lagrangian $\hat{\mathcal{L}}$.

where we have set the vector, two- and three-form equal to zero. Under supersymmetry these fermionic field equations transform into

$$\begin{aligned} \delta_0 X_0(\hat{\psi}^{\hat{\mu}}) &= \tfrac{1}{2}\hat{\Gamma}^{\hat{\nu}}\hat{\epsilon}\,[\hat{R}^{\hat{\mu}}{}_{\hat{\nu}} - \tfrac{1}{2}\hat{R}\hat{g}^{\hat{\mu}}{}_{\hat{\nu}} - \tfrac{1}{2}(\partial^{\hat{\mu}}\hat{\phi})(\partial_{\hat{\nu}}\hat{\phi}) + \tfrac{1}{4}(\partial\hat{\phi})^2\hat{g}^{\hat{\mu}}{}_{\hat{\nu}}]\,, \\ \delta_0 X_0(\hat{\lambda}) &= \hat{\epsilon}\,[\Box\hat{\phi}]\,, \end{aligned} \tag{4.11}$$

which imply the usual graviton-dilaton field equations. The corresponding action can be deduced from (4.2), using the same reduction Ansätze (4.7), and is e.g. given in [112] (in the string frame). The transformation rules have two $\mathbb{R}^+$-symmetries, one with parameter $\hat{\alpha}$ that scales the Lagrangian and one with parameter $\hat{\beta}$ that leaves the Lagrangian invariant. The first symmetry follows via dimensional reduction from the $D = 11$ $\mathbb{R}^+$-symmetry with parameter $\hat{\hat{\alpha}}$. The weights of these two $\mathbb{R}^+$-symmetries are given in table 4.2. The gauge symmetry associated to the Ramond-Ramond vector, with parameter $\hat{\lambda}$, reads

$$\hat{A} \to \hat{A} - \mathrm{d}\hat{\lambda}\,, \qquad \hat{C} \to \hat{C} - \mathrm{d}\hat{\lambda} \wedge \hat{B}\,. \tag{4.12}$$

The $D = 10$ IIA supergravity theory allows two massive deformations which we discuss one by one below.

4.2.1 Deformation m_{R}: $D = 10$ massive supergravity

The first massive deformation, with mass parameter m_{R}, is due to Romans [42]. In this case (the same is true for all other cases) the supersymmetry transformations receive two types of massive deformations: explicit and implicit ones. The explicit deformations are terms, at most linear in m_{R}, that are added to the original supersymmetry rules. These explicit deformations are denoted by $\delta_{m_{\mathrm{R}}}$ and define the fermion-shifts, used for determining the scalar potential. They are given in terms of a superpotential $W(\hat{\phi})$ and derivatives thereof by

$$m_{\mathrm{R}}: \begin{cases} \delta_{m_{\mathrm{R}}}\hat{\psi}_{\hat{\mu}} &= -\tfrac{1}{8}W\hat{\Gamma}_{\hat{\mu}}\hat{\epsilon}\,, \\ \delta_{m_{\mathrm{R}}}\hat{\lambda} &= 4\frac{\delta W}{\delta\hat{\phi}}\hat{\epsilon}\,, \end{cases} \qquad \text{with } W = \tfrac{1}{4}e^{5\hat{\phi}/4}m_{\mathrm{R}}\,. \tag{4.13}$$

There are further implicit massive deformations to the original supersymmetry rules δ_0, which are given in (4.8), due to the fact that in these rules one must replace all field strengths by corresponding *massive* field strengths which are given by

$$\hat{F} = \mathrm{d}\hat{A} + m_{\mathrm{R}}\hat{B}\,, \qquad \hat{H} = \mathrm{d}\hat{B}\,, \qquad \hat{G} = \mathrm{d}\hat{C} + \hat{A}\wedge\hat{H} + \tfrac{1}{2}m_{\mathrm{R}}\hat{B}\wedge\hat{B}\,. \tag{4.14}$$

The Lagrangian contains terms linear and quadratic in $m_{\rm R}$. Again there are implicit deformations, via the massive field strengths, and explicit deformations. The explicit deformations quadratic in the mass parameter define the scalar potential which can be written in terms of the superpotential $W(\hat{\phi})$ and derivatives thereof by using (2.15) and (2.16).

The linear deformations of the fermionic (gravitino and dilatino) field equations of Romans' theory can be found by requiring closure of the supersymmetry algebra:

$$m_{\rm R} : \begin{cases} X_{m_{\rm R}}(\hat{\psi}^{\hat{\mu}}) & \equiv & m_{\rm R} e^{5\hat{\phi}/4}\hat{\Gamma}^{\hat{\mu}\hat{\nu}}(\frac{1}{4}\hat{\psi}_{\hat{\nu}} + \frac{5}{288}\hat{\Gamma}_{\hat{\nu}}\hat{\lambda})\,, \\ X_{m_{\rm R}}(\hat{\lambda}) & \equiv & m_{\rm R} e^{5\hat{\phi}/4}\hat{\Gamma}^{\hat{\nu}}(-\frac{5}{4}\hat{\psi}_{\hat{\nu}} - \frac{21}{160}\hat{\Gamma}_{\hat{\nu}}\hat{\lambda})\,. \end{cases} \tag{4.15}$$

The undeformed equations, $X_0(\hat{\psi}^{\hat{\mu}})$ and $X_0(\hat{\lambda})$, are given in eqs. (4.10). Under supersymmetry these fermionic field equations, $X_0 + X_{m_{\rm R}}$, transform into the deformed bosonic equations of motion. Since we will only be interested in finding solutions that are carried by the metric and the scalars it is convenient to truncate away all bosonic fields except the metric and the dilaton.[5] After this truncation we find that under supersymmetry the fermionic field equations transform into

$$\begin{aligned} &(\delta_0 + \delta_{m_{\rm R}})(X_0 + X_{m_{\rm R}})(\hat{\psi}^{\hat{\mu}}) = \tfrac{1}{2}\hat{\Gamma}^{\hat{\nu}}\hat{\epsilon}\,[\hat{R}^{\hat{\mu}}{}_{\hat{\nu}} - \tfrac{1}{2}\hat{R}\hat{g}^{\hat{\mu}}{}_{\hat{\nu}} - \tfrac{1}{2}(\partial^{\hat{\mu}}\hat{\phi})(\partial_{\hat{\nu}}\hat{\phi}) + \tfrac{1}{4}(\partial\hat{\phi})^2\hat{g}^{\hat{\mu}}{}_{\hat{\nu}} + \tfrac{1}{4}m_{\rm R}^2 e^{5\hat{\phi}/2}\hat{g}^{\hat{\mu}}{}_{\hat{\nu}}]\,, \\ &(\delta_0 + \delta_{m_{\rm R}})(X_0 + X_{m_{\rm R}})(\hat{\lambda}) = \hat{\epsilon}\,[\Box\hat{\phi} - \tfrac{5}{4}m_{\rm R}^2 e^{5\hat{\phi}/2}]\,. \end{aligned} \tag{4.16}$$

At the right-hand side we find the Romans' bosonic field equations for the metric and the dilaton, one solution of which is the D8-brane. Note that the bosonic field equations contain terms quadratic in the mass parameter.

Romans' theory is not known to have a higher-dimensional supergravity origin; neither is it a gauged supergravity. A candidate symmetry of the Lagrangian to be gauged is the $\hat{\beta}$ symmetry of table 4.2. However, the candidate gauge field $\hat{A}_{\hat{\mu}}$ has a nontrivial weight under $\hat{\beta}$. This means that the curl $\mathrm{d}\hat{A}$ transforms with a non-covariant term proportional to $\hat{A}\wedge\mathrm{d}\hat{\lambda}$. Such a term cannot be cancelled by adding an extra term, such as $\hat{B}$, to the definition of the $\hat{A}$ curvature. In short, the $\hat{\beta}$-symmetry cannot be gauged [113]. The same table shows that on the other hand $\hat{A}_{\hat{\mu}}$ has weight zero under the $\hat{\alpha}$-symmetry which is a symmetry of the equations of motion only. This $\hat{\alpha}$-symmetry can indeed be gauged at the level of the equations of motion. This gauging leads to the $D = 10$ gauged supergravity discussed below.

4.2.2 Deformation m_{11}: $D = 10$ gauged supergravity

The second massive deformation, with mass parameter m_{11}, has been considered in [114, 115] and is a gauged supergravity. It can be obtained by generalized Scherk-Schwarz reduction of $D = 11$ supergravity using the $\mathbb{R}^+$ symmetry $\hat{\hat{\alpha}}$ of table 4.1 [115]. The corresponding reduction Ansätze can be obtained by adding the appropriate factors $e^{w m_{11} x}$ to the Ansätze in (4.7), using the corresponding weights in table 4.1. This reduction leads to the following explicit massive deformations of the $D = 10$ IIA supersymmetry rules

$$m_{11} : \begin{cases} \delta_{m_{11}}\hat{\psi}_{\hat{\mu}} & = & \frac{9}{16}m_{11}e^{-3\hat{\phi}/4}\hat{\Gamma}_{\hat{\mu}}\Gamma_{11}\hat{\epsilon}\,, \\ \delta_{m_{11}}\hat{\lambda} & = & \frac{3}{2}m_{11}e^{-3\hat{\phi}/4}\Gamma_{11}\hat{\epsilon}\,. \end{cases} \tag{4.17}$$

[5]Note that a further truncation to $\phi = c$ is inconsistent.

The implicit massive deformations of the original supersymmetry rules δ_0 are given by the massive bosonic field strengths

$$\mathrm{D}\hat{\phi} = \mathrm{d}\hat{\phi} + \tfrac{3}{2}m_{11}\hat{A}\,, \quad \hat{F} = \mathrm{d}\hat{A}\,, \quad \hat{H} = \mathrm{d}\hat{B} + 3m_{11}\hat{C}\,, \quad \hat{G} = \mathrm{d}\hat{C} + \hat{A}\wedge\hat{H}\,, \tag{4.18}$$

while the covariant derivative of the supersymmetry parameter $\hat{\mathcal{D}}_{\hat{\mu}}\hat{\epsilon}$ is replaced by

$$\hat{D}_{\hat{\mu}}\hat{\epsilon} = (\partial_{\hat{\mu}} + \tfrac{1}{4}\hat{\omega}_{\hat{\mu}}{}^{\hat{a}\hat{b}}\hat{\Gamma}_{\hat{a}\hat{b}} + \tfrac{9}{16}m_{11}\hat{\Gamma}_{\hat{\mu}}\hat{\not{A}})\hat{\epsilon}\,. \tag{4.19}$$

The gauge vector in the definition of the covariant derivative is required to make the derivative of the supersymmetry parameter *and* the spin connection $\mathbb{R}^+$-covariant.

The linear deformations of the fermionic field equations read in this case

$$m_{11}: \begin{cases} X_{m_{11}}(\hat{\psi}^{\hat{\mu}}) & \equiv & m_{11}e^{-3\hat{\phi}/4}\Gamma_{11}\hat{\Gamma}^{\hat{\mu}\hat{\nu}}(-\tfrac{9}{2}\hat{\psi}_{\hat{\nu}} + \tfrac{17}{48}\hat{\Gamma}_{\hat{\nu}}\hat{\lambda})\,, \\ X_{m_{11}}(\hat{\lambda}) & \equiv & m_{11}e^{-3\hat{\phi}/4}\Gamma_{11}\hat{\Gamma}^{\hat{\nu}}(\tfrac{3}{2}\hat{\psi}_{\hat{\nu}} - \tfrac{9}{16}\hat{\Gamma}_{\hat{\nu}}\hat{\lambda})\,. \end{cases} \tag{4.20}$$

We first consider the truncation where all bosonic fields except the metric and the dilaton are set equal to zero. Under supersymmetry the fermionic field equations transform into

$$\begin{aligned}(\delta_0 + \delta_{m_{11}})(X_0 + X_{m_{11}})(\hat{\psi}^{\hat{\mu}}) &= \tfrac{1}{2}\hat{\Gamma}^{\hat{\nu}}\hat{\epsilon}\,[\hat{R}^{\hat{\mu}}{}_{\hat{\nu}} - \tfrac{1}{2}\hat{R}\hat{g}^{\hat{\mu}}{}_{\hat{\nu}} - \tfrac{1}{2}(\partial^{\hat{\mu}}\hat{\phi})(\partial_{\hat{\nu}}\hat{\phi}) + \tfrac{1}{4}(\partial\hat{\phi})^2\hat{g}^{\hat{\mu}}{}_{\hat{\nu}} \\ &\quad + 36m_{11}^2e^{-3\hat{\phi}/2}\hat{g}^{\hat{\mu}}{}_{\hat{\nu}}] + \Gamma_{11}\hat{\epsilon}[3m_{11}e^{-3\hat{\phi}/4}\partial^{\hat{\mu}}\hat{\phi}]\,, \\ (\delta_0 + \delta_{m_{11}})(X_0 + X_{m_{11}})(\hat{\lambda}) &= \hat{\epsilon}\,[\Box\hat{\phi}] + \hat{\Gamma}^{\hat{\nu}}\Gamma_{11}\hat{\epsilon}[9m_{11}e^{-3\hat{\phi}/4}\partial_{\hat{\nu}}\hat{\phi}]\,. \end{aligned} \tag{4.21}$$

The terms involving Γ_{11} are part of the vector field equation. Therefore, to obtain a consistent truncation, we must further truncate the dilaton to zero. One is then left with only the metric satisfying the Einstein equation with a positive cosmological constant, a solution of which is 10D de Sitter space [115].

The reduced theory is a gauged supergravity where the $\mathbb{R}^+$ symmetry $\hat{\alpha}$ of table 4.2 has been gauged. In particular, the gauge parameter and transformation of the Ramond-Ramond potentials read as follows[6]

$$\hat{\alpha}: \qquad \Lambda = e^{w_{\hat{\alpha}}m_{11}\hat{\lambda}} \quad \text{with} \quad \hat{A}\to\hat{A} - \mathrm{d}\hat{\lambda}\,, \quad \hat{C}\to e^{3m_{11}\hat{\lambda}}(\hat{C} - \mathrm{d}\hat{\lambda}\wedge\hat{B})\,, \tag{4.22}$$

where $w_{\hat{\alpha}}$ are the weights under $\hat{\alpha}$. We note that one can take two different limits of the $\hat{\alpha}$ gauge transformations. First, the limit $m_{11} \to 0$ leads to the massless gauge transformations (4.12). Note that $\hat{C}$ transforms trivially under this gauge symmetry in the sense that $\hat{C}$ can be made gauge-invariant after a simple field-redefinition. Secondly, one can take the limit that $\hat{\alpha}$ is constant. This leads to the ungauged $\mathbb{R}^+$ $\hat{\alpha}$-symmetry of table 4.2.

A noteworthy feature of the $D = 10$ gauged supergravity is that no Lagrangian can be defined for it. In the search for supersymmetric domain-wall solutions in five dimensions other examples of gauged supergravity theories without a Lagrangian have been found [86]; we will encounter these in chapter 6. Note that one can write down a Lagrangian for the ungauged theory. The reason that one cannot write down a Lagrangian after gauging is that the symmetry that is gauged is not a symmetry of the Lagrangian but only of the equations of motion. It would be instructive to construct the $D = 10$ gauged supergravity from the ungauged theory by gauging the $\hat{\alpha}$-symmetry. Apparently, it shows that one can gauge symmetries that leave a Lagrangian invariant up to a scale factor.

[6] It is understood that each field with $w_{\hat{\alpha}} \neq 0$ is multiplied by Λ.

4.3 $D = 10$ IIB supergravity

The other ten-dimensional supergravity theory is chiral IIB, which field content is given by

$$D = 10 \text{ IIB:} \qquad \{\hat{e}_{\hat{\mu}}{}^{\hat{a}}, \hat{\phi}, \hat{\chi}, \hat{B}^{(1)}_{\hat{\mu}\hat{\nu}}, \hat{B}^{(2)}_{\hat{\mu}\hat{\nu}}, \hat{D}_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}}, \hat{\psi}_{\hat{\mu}}, \hat{\lambda}\}\,. \tag{4.23}$$

The supersymmetry transformation rules of ten-dimensional IIB supergravity read (in complex notation)

$$\begin{aligned}
\delta\hat{e}_{\hat{\mu}}{}^{\hat{a}} &= \tfrac{1}{2}\bar{\hat{\epsilon}}\,\hat{\Gamma}^{\hat{a}}\hat{\psi}_{\hat{\mu}} + \text{h.c.}\,,\\
\delta\hat{\psi}_{\hat{\mu}} &= \hat{D}_{\hat{\mu}}\hat{\epsilon} - \tfrac{1}{16\cdot 5!}\,\mathrm{i}\,\hat{\not{G}}^{(5)}\hat{\Gamma}_{\hat{\mu}}\hat{\epsilon}\\
&\quad + \tfrac{1}{16\cdot 3!}\,\mathrm{i}\,e^{\hat{\phi}/2}\left(\hat{\Gamma}_{\hat{\mu}}\hat{\Gamma}^{(3)} + 2\hat{\Gamma}^{(3)}\hat{\Gamma}_{\hat{\mu}}\right)\left(\hat{H}^{(1)} - \hat{\tau}\hat{H}^{(2)}\right)_{(3)}\hat{\epsilon}^*\,,\\
\delta\hat{\lambda} &= -e^{\hat{\phi}}\hat{\not{\partial}}\hat{\tau}\hat{\epsilon}^* - \tfrac{1}{2\cdot 3!}e^{\hat{\phi}/2}\hat{\Gamma}^{(3)}\left(\hat{H}^{(1)} - \hat{\tau}\hat{H}^{(2)}\right)_{(3)}\hat{\epsilon}\,,\\
\delta\hat{B}^{(1)}_{\hat{\mu}\hat{\nu}} &= -e^{\hat{\phi}/2}\hat{\tau}^*\left(\bar{\hat{\epsilon}}^*\hat{\Gamma}_{[\hat{\mu}}\hat{\psi}_{\hat{\nu}]} - \tfrac{1}{8}\,\mathrm{i}\,\bar{\hat{\epsilon}}\,\hat{\Gamma}_{\hat{\mu}\hat{\nu}}\hat{\lambda}\right) + \text{h.c.}\,,\\
\delta\hat{B}^{(2)}_{\hat{\mu}\hat{\nu}} &= -e^{\hat{\phi}/2}\left(\bar{\hat{\epsilon}}^*\,\hat{\Gamma}_{[\hat{\mu}}\hat{\psi}_{\hat{\nu}]} - \tfrac{1}{8}\,\mathrm{i}\,\bar{\hat{\epsilon}}\,\hat{\Gamma}_{\hat{\mu}\hat{\nu}}\hat{\lambda}\right) + \text{h.c.}\,,\\
\delta\hat{D}_{\hat{\mu}\hat{\nu}\hat{\lambda}\hat{\rho}} &= 2\,\mathrm{i}\,\bar{\hat{\epsilon}}\,\hat{\Gamma}_{[\hat{\mu}\hat{\nu}\hat{\lambda}}\hat{\psi}_{\hat{\rho}]} - \tfrac{3}{2}\,\varepsilon_{ij}\hat{B}^{(i)}_{[\hat{\mu}\hat{\nu}}\delta\hat{B}^{(j)}_{\hat{\lambda}\hat{\rho}]} + \text{h.c.}\,,\\
\delta\hat{\chi} &= -\tfrac{1}{4}e^{-\hat{\phi}}\bar{\hat{\epsilon}}\hat{\lambda}^* + \text{h.c.}\,,\\
\delta\hat{\phi} &= \tfrac{1}{4}\,\mathrm{i}\,\bar{\hat{\epsilon}}\hat{\lambda}^* + \text{h.c.}\,,
\end{aligned} \tag{4.24}$$

with the complex scalar $\hat{\tau} = \hat{\chi} + ie^{-\hat{\phi}}$ and the field strengths

$$\vec{\hat{H}} = \mathrm{d}\vec{\hat{B}}\,, \qquad \hat{G} = \mathrm{d}\hat{D} + \tfrac{1}{2}\vec{\hat{B}}^T\eta\vec{\hat{H}}\,, \qquad \eta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{4.25}$$

The field strength $\hat{G}$ is subject to a self-duality constraint:

$$\hat{G}_{\hat{\mu}_1\cdots\hat{\mu}_5} = -\tfrac{1}{5!}\hat{\varepsilon}_{\hat{\mu}_1\cdots\hat{\mu}_{10}}\hat{G}^{\hat{\mu}_{10}\cdots\hat{\mu}_6}\,, \tag{4.26}$$

which can be used to eliminate the four form potential $C_{(4)}$, after a dimensional reduction to $D = 9$.

The covariant derivative of the IIB Killing spinor reads

$$\hat{D}_{\hat{\mu}}\hat{\epsilon} = (\partial_{\hat{\mu}} + \tfrac{1}{4}\hat{\omega}_{\hat{\mu}}^{\hat{a}\hat{b}}\Gamma_{\hat{a}\hat{b}} + \tfrac{1}{4}\,\mathrm{i}\,e^{\hat{\phi}}\partial_{\hat{\mu}}\hat{\chi})\hat{\epsilon}\,. \tag{4.27}$$

The corresponding action can be found in [112]. The IIB supersymmetry rules transform covariant under the $S\ell(2,\mathbb{R})$ transformations (omitting indices)

$$\begin{aligned}
&\hat{\tau} \to \frac{a\hat{\tau} + b}{c\hat{\tau} + d}\,, \qquad \vec{\hat{B}} \to \Omega\vec{\hat{B}}\,, \qquad \hat{D} \to \hat{D}\,, \qquad \text{with } \Omega = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in S\ell(2,\mathbb{R})\,,\\
&\hat{\psi}_{\hat{\mu}} \to \left(\frac{c\,\hat{\tau}^* + d}{c\,\hat{\tau} + d}\right)^{1/4}\hat{\psi}_{\hat{\mu}}\,, \qquad \hat{\lambda} \to \left(\frac{c\,\hat{\tau}^* + d}{c\,\hat{\tau} + d}\right)^{3/4}\hat{\lambda}\,, \qquad \hat{\epsilon} \to \left(\frac{c\,\hat{\tau}^* + d}{c\,\hat{\tau} + d}\right)^{1/4}\hat{\epsilon}\,.
\end{aligned} \tag{4.28}$$

Here we have used the vector notation $\vec{\hat{B}} = (\hat{B}^{(1)}, \hat{B}^{(2)})^T$. The group $S\ell(2,\mathbb{R})$ contains a set of three one-parameter conjugacy classes defining one compact and two non-compact subgroups. We will describe them shortly. Each of the subgroups is generated by a $S\ell(2,\mathbb{R})$ group element Ω with $\det\Omega = 1$. As a global symmetry group of IIB supergravity, $S\ell(2,\mathbb{R})$ is suitable for performing a Scherk-Schwarz reduction to $D = 9$. There are three different cases to consider, corresponding to the three different subgroups listed below.

1. One non-compact subgroup $\mathbb{R}$ is generated by

$$\Omega_p = e^{\frac{1}{2}\hat{\zeta}(\sigma_1 + \mathrm{i}\sigma_2)} = \begin{pmatrix} 1 & \hat{\zeta} \\ 0 & 1 \end{pmatrix}. \tag{4.29}$$

   Each element defines a parabolic conjugacy class with $\mathrm{Tr}\,\Omega = 2$. These parabolic transformations leave the combination $(\hat{B}^{(2)})^2$ invariant. Therefore the invariant metric is given by diag (0,1). The action of the $\mathbb{R}$ $\hat{\zeta}$-symmetry on the fields can not be expressed by assigning weights to the standard basis of fields given in (4.23).

2. An $\mathrm{SO}(1,1)^+$ subgroup which is generated by elements

$$\Omega_h = e^{\hat{\gamma}\sigma_3} = \begin{pmatrix} e^{\hat{\gamma}} & 0 \\ 0 & e^{-\hat{\gamma}} \end{pmatrix}. \tag{4.30}$$

   Each element defines a hyperbolic conjugacy class with $\mathrm{Tr}\,\Omega > 2$. These hyperbolic transformations leave the combination $\hat{B}^{(1)}\hat{B}^{(2)}$ invariant. After diagonalization this leads to an invariant metric given by diag (1,-1). The weights corresponding to the $\mathrm{SO}(1,1)^+$ $\hat{\gamma}$-symmetry are given in table 4.3.

3. There is a SO(2) subgroup which is generated by elements Ω of $S\ell(2,\mathbb{R})$ with

$$\Omega_e = e^{\mathrm{i}\hat{\theta}\sigma_2} = \begin{pmatrix} \cos\hat{\theta} & \sin\hat{\theta} \\ -\sin\hat{\theta} & \cos\hat{\theta} \end{pmatrix}. \tag{4.31}$$

   Each element defines an elliptic conjugacy class with $\mathrm{Tr}\,\Omega < 2$. The elliptic transformations leave $(\hat{B}^{(1)})^2 + (\hat{B}^{(2)})^2$ invariant. After diagonalization this leads to an invariant metric given by diag (1,1). The action of the SO(2) $\hat{\theta}$-symmetry on the fields can not be expressed by assigning weights to the standard real basis of fields given in (4.23).

Table 4.3 contains the weights of the $\hat{\gamma}$-symmetry defined above[7] and of a new $\mathbb{R}^+$ symmetry $\hat{\delta}$ which is *not* a subgroup of $S\ell(2,\mathbb{R})$ and that does not leave the Lagrangian invariant. One could combine $S\ell(2,\mathbb{R})$ with this new $\mathbb{R}^+$ into a $G\ell(2,\mathbb{R})$ symmetry that leaves the IIB equations of motion invariant. Its action is the product of the two separate transformations: $\tilde{\Omega} = \Omega\Lambda_{\hat{\delta}}$. This exhausts all the symmetries of $D = 10$ IIB supergravity.

The IIB supergravity theory is not known to have massive deformations. One of the reasons for this is that there is no candidate vector field like in the IIA case.

[7] The other two symmetries defined above cannot be defined in terms of weights of real fields only.

$\mathbb{R}^+$	$\hat{e}_{\hat{\mu}}{}^{\hat{a}}$	$e^{\hat{\phi}}$	$\hat{\chi}$	$\hat{B}^{(1)}_{\hat{\mu}\hat{\nu}}$	$\hat{B}^{(2)}_{\hat{\mu}\hat{\nu}}$	$\hat{D}_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}}$	$\hat{\psi}_{\hat{\mu}}$	$\hat{\lambda}$	$\hat{\epsilon}$	$\hat{\mathcal{L}}$	symmetry
$\hat{\gamma}$	0	-2	2	1	-1	0	0	0	0	0	$SO(1,1)^+$
$\hat{\delta}$	1	0	0	2	2	4	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	8	$\mathbb{R}^+$

Table 4.3: The scaling weights of the $D = 10$ IIB supergravity fields, the supersymmetry parameter $\hat{\epsilon}$ and the Lagrangian $\hat{\mathcal{L}}$.

$\mathbb{R}^+$	$e_\mu{}^a$	e^ϕ	e^φ	χ	A_μ	$A^{(1)}_\mu$	$A^{(2)}_\mu$	$B^{(1)}_{\mu\nu}$	$B^{(2)}_{\mu\nu}$	$C_{\mu\nu\rho}$	ψ_μ	λ	$\tilde{\lambda}$	ϵ	$\mathcal{L}$	Origin
α	$\frac{9}{7}$	0	$\frac{6}{\sqrt{7}}$	0	3	0	0	3	3	3	$\frac{9}{14}$	$-\frac{9}{14}$	$-\frac{9}{14}$	$\frac{9}{14}$	9	11D
β	0	$\frac{3}{4}$	$\frac{\sqrt{7}}{4}$	$-\frac{3}{4}$	$\frac{1}{2}$	$-\frac{3}{4}$	0	$-\frac{1}{4}$	$\frac{1}{2}$	$-\frac{1}{4}$	0	0	0	0	0	IIA
γ	0	-2	0	2	0	1	-1	1	-1	0	0	0	0	0	0	IIB
δ	$\frac{8}{7}$	0	$-\frac{4}{\sqrt{7}}$	0	0	2	2	2	2	4	$\frac{4}{7}$	$-\frac{4}{7}$	$-\frac{4}{7}$	$\frac{4}{7}$	8	IIB

Table 4.4: The scaling weights of the nine-dimensional supergravity fields, the supersymmetry parameter ϵ and the Lagrangian $\mathcal{L}$.

4.4 Massive deformations of $D = 9$, $\mathcal{N} = 2$ supergravity

The Kaluza-Klein reduction of either (massless) IIA or IIB supergravity gives the unique $D = 9$, $\mathcal{N} = 2$ massless supergravity theory. Its field content is given by

$$D = 9 : \qquad \{e_\mu{}^a, \phi, \varphi, \chi, A_\mu, A^{(1)}_\mu, A^{(2)}_\mu, B^{(1)}_{\mu\nu}, B^{(2)}_{\mu\nu}, C_{\mu\nu\rho}, \psi_\mu, \lambda, \tilde{\lambda}\}\,. \tag{4.32}$$

The supersymmetry rules are given in [108]. The massless nine-dimensional theory inherits several global symmetries from its parents: two $\mathbb{R}^+$ symmetries α,β from IIA supergravity and one $\mathbb{R}^+$ symmetry δ plus a full $S\ell(2,\mathbb{R})$ symmetry from IIB supergravity. The latter leads in particular to an SO(2) symmetry θ, an $SO(1,1)^+$ symmetry γ and an $\mathbb{R}$-symmetry ζ. The weights of all these symmetries, except for the SO(2) θ-symmetry and $\mathbb{R}$ ζ-symmetry, and their higher-dimensional origin are given in table 4.4 (see also [109]).

It turns out that only three out of the four scalings given in table 4.4 are linearly independent, due to the relation

$$\tfrac{4}{9}\alpha - \tfrac{8}{3}\beta = \gamma + \tfrac{1}{2}\delta\,. \tag{4.33}$$

We observe the following pattern. Using (4.33) to eliminate one of the scaling-symmetries we are left with three independent scaling-symmetries. Each of the three gauge fields $A_\mu, A^{(1)}_\mu, A^{(2)}_\mu$ has weight zero under the linear combination of *two* out of these three symmetries: one is a symmetry of the action, the other is a symmetry of the equations of motion only.

mass parameters	$S\ell(2,\mathbb{R})$
(m_1, m_2, m_3)	triplet
$(m_4, \tilde{m}_4)$	doublet
(m_{11}, m_{IIA})	doublet
m_{IIB}	singlet

Table 4.5: This table indicates the different multiplets that the $D = 9$ mass parameters form under $S\ell(2,\mathbb{R})$.

The $D = 9$ $S\ell(2,\mathbb{R})$ symmetry acts in the following way:

$$\begin{aligned}
&\tau \to \frac{a\tau + b}{c\tau + d}\,, \qquad \vec{A} \to \Omega\vec{A}\,, \qquad \vec{B} \to \Omega\vec{B}\,, \qquad \text{with } \Omega = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in S\ell(2,\mathbb{R})\,,\\
&\psi_\mu \to \left(\frac{c\,\tau^* + d}{c\,\tau + d}\right)^{1/4} \psi_\mu\,, \qquad \lambda \to \left(\frac{c\,\tau^* + d}{c\,\tau + d}\right)^{3/4} \lambda\,, \\
&\tilde{\lambda} \to \left(\frac{c\,\tau^* + d}{c\,\tau + d}\right)^{-1/4} \tilde{\lambda}\,, \qquad \epsilon \to \left(\frac{c\,\tau^* + d}{c\,\tau + d}\right)^{1/4} \epsilon\,,
\end{aligned} \tag{4.34}$$

while φ and C are invariant. We have used a vector notation for the two vectors and two anti-symmetric tensors, like in $D = 10$. Again one can combine $S\ell(2,\mathbb{R})$ with an $\mathbb{R}^+$ symmetry to form $G\ell(2,\mathbb{R})$ with parameter $\tilde{\Omega} = \Omega\Lambda_{\mathbb{R}^+}$.

In addition to the global symmetries there is a number of local symmetries. In particular, the gauge transformations of the vectors read

$$\begin{aligned}
A^{(1)} &\to A^{(1)} - \mathrm{d}\lambda^{(1)}\,, \qquad & A^{(2)} &\to A^{(2)} - \mathrm{d}\lambda^{(2)}\,,\\
A &\to A - \mathrm{d}\lambda\,, & \vec{B} &\to \vec{B} - \vec{A} \wedge \mathrm{d}\lambda\,.
\end{aligned} \tag{4.35}$$

We now turn to massive deformations of the 9D theory. Applying a SS dimensional reduction of the higher-dimensional supergravities we obtain a number of massive deformations in nine dimensions, as illustrated in figure 4.1. By employing the different global symmetries of 11D, IIA and IIB supergravity we obtain seven deformations of the unique $D = 9$ supergravity. Since the procedure is quite straightforward – though tedious – we will not give any details here; these can be found in [108].

Note that the different massive deformations can be related. Symmetries of the massless theory become field redefinitions in the massive theory that only act on the massive deformations. This means that the mass parameters transform under such transformations: they have a scaling weight under the different scaling symmetries and fall in multiplets of $S\ell(2,\mathbb{R})$. In table 4.5 the multiplet structure of the massive deformations under $S\ell(2,\mathbb{R})$ is given. The mass parameter $\tilde{m}_4$ is defined as the S-dual partner of m_4 and can not be obtained by a SS reduction of IIA supergravity.

All these deformations correspond to a gauging of a 9D global symmetry. In particular, it is always the symmetry that is employed in the SS reduction Ansatz that becomes gauged upon

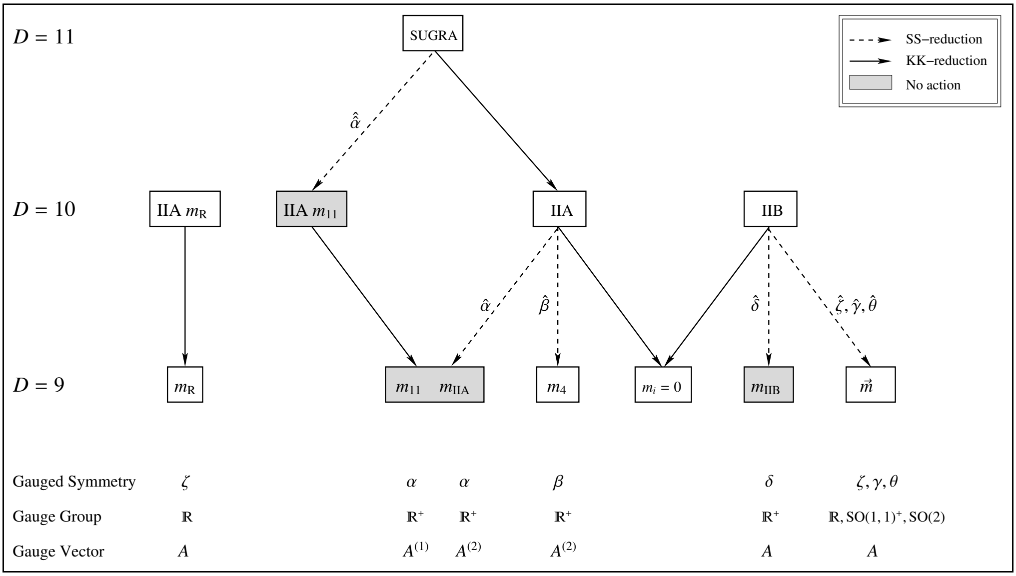

Figure 4.1: Overview of all reductions performed in [108]. These cases can all be interpreted as gauged supergravities, with gauged symmetry and corresponding gauge field as given in the figure. Mass parameters in the same box, such as $m_{11}, m_{\rm IIA}$ or m_1, m_2, m_3, form a multiplet under $S\ell(2,\mathbb{R})$. Further details of these cases will be given below. Note that the two ways of obtaining the $\mathbb{R}$-gauging give rise to the massive T-duality of [116], provided that $m_1 = m_2 = m_R$ and $m_3 = 0$.

reduction. The corresponding gauge vector is always provided by the metric, i.e. it is the Kaluza-Klein vector of the dimensional reduction. In all but one case this is the complete story and one finds an Abelian gauged supergravity. It turns out that there is one exception where we find a *non-Abelian* gauge symmetry. This can be understood from the following general rule.[8] As we noted, the Kaluza-Klein vector gauges the symmetry employed in the SS reduction Ansatz. The fate of either of the remaining two gauge vectors is restricted to three possibilities:

- The vector is a singlet under the gauge symmetry and its field strength acquires no modification, e.g. $A^{(1)}$ in the $m_{\rm IIA}$ deformation.
- The vector transforms under the gauge symmetry and its field strength acquires a massive deformation proportional to a two-form. The degrees of freedom of the vector are eaten up by the two-form via the Stückelberg mechanism, e.g. A in the $m_{\rm IIA}$ deformation.
- The vector transforms under the gauge symmetry and its field strength acquires no massive deformation proportional to a two-form. In this case we must have gauge enhancement to preserve covariance, e.g. $A^{(1)}$ in the m_4 deformation.

All cases we find in $D = 9$ are consistent with this rule of thumb. Details can be found in [108].

[8] We thank Sergio Ferrara for clarifying discussions on this issue.

4.5 Combining massive deformations

In this section we would like to consider combining the massive deformations discussed in the previous section. The resulting theories will have more mass parameters characterizing the different deformations. However, not all combinations will turn out to be consistent with supersymmetry. This inconsistency only appears when turning to the bosonic field equations: the supersymmetry algebra with a combination of massive deformations always closes, as can be seen from the following argument.

Suppose one has a supergravity with one massive deformation m and supersymmetry transformations $\delta_0 + \delta_m$. In all cases discussed in this chapter the massive deformation of the supersymmetry rules satisfies the following property: $\delta_m(\text{boson}) = 0$. In other words, only the supersymmetry variations of the fermions receive massive corrections. This implies that the issue of the closure of the supersymmetry algebra is a calculation with m-independent parts and parts linear in m but no parts of higher order in m.[9] On the one hand $[\delta(\epsilon_1), \delta(\epsilon_2)]$ has no terms quadratic in m since one of the two δ's acts on a boson. On the other hand the supersymmetry algebra closes modulo fermionic field equations which also have only terms independent of and linear in m. Therefore, given the closure of the massless algebra, the closure of the massive supersymmetry algebra only requires the cancellation of terms linear in m.

In the previous sections we have not checked the closure of the massive supersymmetry algebras since this was guaranteed by the higher-dimensional origin, i.e. Scherk-Schwarz reduction of supergravity leads to a gauged supergravity. However, the argument of linearity allows us to combine different massive deformations. Suppose one has two massive supersymmetry algebras with transformations $\delta_0 + \delta_{m_a}$ and $\delta_0 + \delta_{m_b}$. Both supersymmetry algebras close modulo fermionic field equations with (different) massive deformations. Then the combined massive algebra with transformation $\delta_0 + \delta_{m_a} + \delta_{m_b}$ also closes modulo fermionic field equations whose massive deformations are given by the sum of the separate massive deformations linear in m_a and m_b. The closure of the combined algebra is guaranteed by the closure of the two massive algebras since it requires a cancellation at the linear level.

Under supersymmetry variation of the fermionic field equations, one in general finds linear *and* quadratic deformations of the bosonic equations of motion. In addition to these corrections, we find that there are also 'non-dynamical' equations posing constraints on the mass parameters. Solving these equations generically excludes the possibility of combining massive deformations by requiring mass parameters to vanish. At first sight, one might seem surprised that the supersymmetry variation of the fermionic equations of motion leads to constraints other than the bosonic field equations. However, one should keep in mind that the multiplets involved cannot be linearized around a Minkowski vacuum solution. Therefore, the usual rules for linearized (Minkowski) multiplets do not apply here.

We find that generically adding massive deformations is possible whenever the $D = 10$ symmetries, giving rise to the separate massive deformations, can be combined in $D = 10$ as symmetries of IIA or IIB supergravity only. The combined $D = 9$ supergravity is then a gauged supergravity which just follows by performing a SS reduction on the combined $D = 10$ symmetry.

[9]That is, up to cubic order in fermions. We have not checked the higher-order fermionic terms but, based upon dimensional arguments, we do not expect that these rule out the possibility of combining massive deformations.

In the first subsection we will discuss the situation in $D = 10$ and in the next subsection we will review the $D = 9$ situation; see [108] for details.

4.5.1 Combining massive deformations in 10D

The 10D IIA supergravity theory has two massive deformations parameterized by m_{R} and m_{11}. Can we combine these two massive deformations? Based on the linearity argument presented above one would expect a closed supersymmetry algebra. The bosonic field equations (with up to quadratic deformations) can be derived by applying the supersymmetry transformations (with only linear deformations) to the fermionic field equations (containing only linear deformations). For simplicity, we truncate all bosonic fields to zero except the metric and the dilaton. We thus find

$$
\begin{aligned}
&(\delta_0 + \delta_{m_{\mathrm{R}}} + \delta_{m_{11}})(X_0 + X_{m_{\mathrm{R}}} + X_{m_{11}})(\hat{\psi}^{\hat{\mu}}) \\
&= \tfrac{1}{2}\hat{\Gamma}^{\hat{\nu}}\hat{\epsilon}\,[\hat{R}^{\hat{\mu}}{}_{\hat{\nu}} - \tfrac{1}{2}\hat{R}\hat{g}^{\hat{\mu}}{}_{\hat{\nu}} - \tfrac{1}{2}(\partial^{\hat{\mu}}\hat{\phi})(\partial_{\hat{\nu}}\hat{\phi}) + \tfrac{1}{4}(\partial\hat{\phi})^2\hat{g}^{\hat{\mu}}{}_{\hat{\nu}} + \tfrac{1}{4}m_{\mathrm{R}}^2 e^{5\hat{\phi}/2}\hat{g}^{\hat{\mu}}{}_{\hat{\nu}} + 36 m_{11}^2 e^{-3\hat{\phi}/2}\hat{g}^{\hat{\mu}}{}_{\hat{\nu}}] \\
&\quad + \Gamma_{11}\hat{\epsilon}[3m_{11}e^{-3\hat{\phi}/4}\partial^{\hat{\mu}}\hat{\phi}] + \Gamma_{11}\hat{\Gamma}^{\hat{\mu}}\hat{\epsilon}\,[\tfrac{15}{4}m_{\mathrm{R}}m_{11}e^{\hat{\phi}/2}]\,, \qquad (4.36) \\
&(\delta_0 + \delta_{m_{\mathrm{R}}} + \delta_{m_{11}})(X_0 + X_{m_{\mathrm{R}}} + X_{m_{11}})(\hat{\lambda}) \\
&= \hat{\epsilon}\,[\Box\hat{\phi} - \tfrac{5}{4}m_{\mathrm{R}}^2 e^{5\hat{\phi}/2}] + \hat{\Gamma}^{\hat{\nu}}\Gamma_{11}\hat{\epsilon}[9m_{11}e^{-3\hat{\phi}/4}\partial_{\hat{\nu}}\hat{\phi}] + \Gamma_{11}\hat{\epsilon}\,[\tfrac{33}{2}m_{\mathrm{R}}m_{11}e^{\hat{\phi}/2}]\,.
\end{aligned}
$$

At the right-hand side we find four different structures. Three of them correspond to the field equations of the metric, dilaton and RR vector. The vector field equation corresponds to the terms linear in m_{11} and containing Γ_{11}. They show us that truncating the RR vector to zero forces us to further truncate the dilaton to $\phi = c$. More interesting is the fourth structure which is bilinear in $m_{\mathrm{R}}m_{11}$. It leads to the constraint $m_{\mathrm{R}}m_{11} = 0$. This constraint cannot be a remnant of a higher-rank form field equation due to its lack of Lorentz indices. It could only fit in the scalar field equation but the Γ_{11} factor prevents this. It is an extra constraint which does not restrict degrees of freedom but rather restricts mass parameters.

We conclude that, even though the closure of the algebra is a linear calculation and therefore always works for combinations, the bosonic field equations exclude the possibility of the combination of massive deformations in $D = 10$ dimensions.

4.5.2 Combining massive deformations in 9D

Repeating the above analysis – i.e. requiring that the fermionic field equations transform under supersymmetry to a complete set of bosonic field equations – restricts us to five cases, each containing two non-zero mass parameters:

- **Case 1** with $\{m_{\mathrm{IIA}}, m_4\}$: this combination can also be obtained by Scherk-Schwarz reduction of IIA employing a linear combination of the symmetries $\hat{\alpha}$ and $\hat{\beta}$, guaranteeing its consistency. It is also a gauging of both this symmetry and (for $m_4 \neq 0$) the parabolic subgroup of $S\ell(2,\mathbb{R})$ in 9D, giving the non-Abelian gauge group A(1).
- **Case 2,3,4** with $\{\vec{m}, m_{\mathrm{IIB}}\}$: as in the case with $m_{\mathrm{IIB}} = 0$ and only $\vec{m}$ this combination contains three different, inequivalent cases depending on $\vec{m}^2$ (depending crucially on the fact that m_{IIB} is a singlet under $S\ell(2,\mathbb{R})$):

  – **Case 2** with $\{\vec{m}, m_{\rm IIB}\}$ and $\vec{m}^2 = 0$.
  – **Case 3** with $\{\vec{m}, m_{\rm IIB}\}$ and $\vec{m}^2 > 0$.
  – **Case 4** with $\{\vec{m}, m_{\rm IIB}\}$ and $\vec{m}^2 < 0$.

  All these combinations can also be obtained by Scherk-Schwarz reduction of IIB employing a linear combination of the symmetries $\hat{\delta}$ and (one of the subgroups of) $S\ell(2,\mathbb{R})$, guaranteeing its consistency. All cases (assuming that $m_{\rm IIB} \neq 0$) correspond to the gauging of an Abelian non-compact symmetry in 9D. Only the special case $\{\vec{m}^2 < 0, m_{\rm IIB} = 0\}$ corresponds to a SO(2)-gauging.

- **Case 5** with $\{m_4 = -\frac{12}{5} m_{\rm IIA}, m_2 = m_3\}$: this case can be understood as the generalized dimensional reduction of Romans' massive IIA theory, employing the $\mathbb{R}^+$ symmetry that is not broken by the $m_{\rm R}$ deformations: $\hat{\beta} - \frac{5}{12}\hat{\alpha}$. It gauges both this linear combination of $\mathbb{R}^+$'s and the parabolic subgroup of $S\ell(2,\mathbb{R})$ in 9D, giving the non-Abelian gauge group A(1).

Another solution to the quadratic constraints has parameters $\{m_{\rm IIA}, m_{11}\}$, but this combination does not represent a new case. It can be obtained from only $m_{\rm IIA}$ (and thus a truncation of case 1) via an $S\ell(2,\mathbb{R})$ field redefinition (since they form a doublet). Thus the most general deformations are the five cases given above, all containing two mass parameters. All five of these are gauged theories and have a higher-dimensional origin. Both case 1 and case 5 have a non-Abelian gauge group provided $m_4 \neq 0$.

4.6 Solutions

In the first part of this chapter we constructed a gauged supergravity with 32 supersymmetries in $D = 10$; after that we illustrated how to obtain a variety of gauged supergravities in $D = 9$, using the same methods. They all have in common that there is a scalar potential. Our next goal is to make a systematic search for solutions that are based on this scalar potential. In the next subsections we will search for two types of solutions: (i) 1/2 BPS domain-wall (DW) solutions and (ii) maximally symmetric solutions with constant scalars, i.e. de Sitter (dS), Minkowski (Mink) or anti-de Sitter (AdS) solutions.

4.6.1 1/2 BPS domain-wall solutions

The authors of [117] already made a systematic search for half-supersymmetric DW solutions of the gauged supergravities corresponding to the cases 3, 4 and 5. Due to a one-to-one relationship with seven-branes in $D = 10$ dimensions [111] they could even make a systematic investigation of the quantization of the mass parameters by using the results of [118, 119].

The goal of this subsection is to investigate whether the five massively deformed supergravities we found in subsection 4.5.2 allow new half-supersymmetric DW solutions. In other words, we will derive all 1/2 BPS seven-brane solutions to the nine-dimensional supergravities described in the previous sections. This analysis should lead, as a check of our calculations, to at least all the solutions of [117]. Since we are looking for 1/2 BPS solutions it is convenient to solve the Killing spinor equations, which are obtained by setting the supersymmetry variation of the gravitino and dilatinos to zero. In this way we solve first order equations instead of second order equations which we would encounter if we were to solve the field equations directly. For static configurations a solution to the Killing spinor equation is also a solution to the field equations, so we do not have to check explicitly that the field equations are satisfied. The projector[10] for a DW is given by $\frac{1}{2}(1 \pm \gamma_y)$, where y denotes the transverse direction. We find that, in order to make a projection operator in the Killing spinor equations, we are forced to set all mass parameters to zero except for $\vec{m}$, which corresponds to cases 3, 4 and 5 of section 4.5. This is a consistent combination of masses and we obtain three classes of domain-wall solution which were discussed in detail in [117]. As it turns out, there are no more half-supersymmetric DW solutions.

To summarize, we find that there are no new codimension-one 1/2 BPS solutions to the $D = 9$ supergravity theories we obtained in the previous sections, as compared to the three classes of domain-wall solutions given in [117].

4.6.2 Solutions with constant scalars

In this subsection we will consider solutions with all three scalars constant. This is a consistent truncation in two cases, both of which have two mass parameters. In this truncation one is left with the metric only satisfying the Einstein equation with a cosmological term

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = -\Lambda g_{\mu\nu} \,, \tag{4.37}$$

with Λ quadratic in the two mass parameters. Depending on the sign of this term one then has anti-de Sitter, Minkowski or de Sitter geometry.

We find that solutions with constant scalars are possible in the following massive supergravities:

- $D = 10$ with $\{m_{11}\}$ has $\Lambda = 36 {m_{11}}^2 e^{-3\hat{\phi}/2}$, which gives rise to de Sitter_{10} [115], breaking all supersymmetry. The $D = 11$ origin of this solution is Mink_{11} written in a basis where the x-dependence is of the required form [115]

$$\text{Mink}_{11}: \qquad \mathrm{d}s^2 = e^{2m_{11}x}(-\mathrm{d}t^2 + e^{2m_{11}t}\mathrm{d}x_9^2 + \mathrm{d}x^2)\,. \tag{4.38}$$

- $D = 9$, **Case 1** with $\{m_{\text{IIA}} = -\frac{2}{3} m_4\}$ has $\Lambda = \frac{63}{4} {m_4}^2 e^{\phi - 3\varphi/\sqrt{7}}$, which gives rise to de Sitter_9, breaking all supersymmetry. This case follows from the reduction of Mink_{10} by using a combination of IIA scale symmetries that leave the dilaton invariant (since Minkowski has vanishing dilaton) so that, after reduction, one is left with a non-trivial geometry only.

- $D = 9$, **Case 4** with $\{m_{\text{IIB}}, m_3\}$ has $\Lambda = 28 {m_{\text{IIB}}}^2 e^{4\varphi/\sqrt{7}}$, which gives rise to de Sitter_9 for non-vanishing m_{IIB}. This case follows from the reduction of Mink_{10} by using a combination of IIB scale symmetries that leave the dilaton invariant. Note that for vanishing m_{IIB} this reduces to Mink_9, despite the presence of m_3 [120]. For either m_{IIB} or m_3 non-zero this solution breaks all supersymmetry.

[10] From a general analysis of the possible projectors in nine dimensions, i.e. demanding that the projector squares to itself and that its trace is half of the spinor dimension, in order to yield a 1/2 BPS state, we find that there is a second projector given by $\frac{1}{2}(1 \pm \mathrm{i}\,\gamma_t)$. This projector would give a euclidean DW, i.e. a DW having time as a transverse direction. Note that such a Euclidean DW can never be 1/2 BPS since if there existed a Killing spinor it would square to a Killing vector in the *transverse* direction, i.e. time, which is not an isometry of the euclidean DW.

4.7 Conclusions

In this chapter we have illustrated how to construct five different $D = 9$ massive deformations with 32 supersymmetries, each containing two mass parameters. We found in [108] that all these five theories have a higher-dimensional origin via SS reduction from $D = 10$ dimensions. Furthermore, the massive deformations gauge a global symmetry of the massless theory. The gauge groups we obtained are the Abelian groups SO(2), $SO(1,1)^+$, $\mathbb{R}$, $\mathbb{R}^+$ and the unique two-dimensional non-Abelian Lie group A(1) of scalings and translations on the real line.

We have analyzed the possibility of combining massive deformations to obtain more general massive supergravities that are not gauged or do not have a higher-dimensional origin. Our analysis shows that the only possible combinations are the five two-parameter deformations, which are all gauged and can be uplifted. We have not made a systematic search for massive $D = 9$ supergravities that are not the combination of gaugings and we cannot exclude that there are more possibilities; this requires a separate calculation. In this context, it is of interest to point out that examples of massive supergravities like Romans have been found in lower dimensions, e.g. [121, 122]. In these cases the compactification manifolds are such that the candidate gauge fields are truncated away.

It is intriguing that some of the gauged supergravities we have constructed result from gauging an $\mathbb{R}^+$ scale symmetry that does not leave the Lagrangian invariant but scales it with a factor. Apparently, it is possible to gauge such symmetries at the level of the equations of motion.

Finally we would like to address the question of whether the gauged supergravities we constructed can be interpreted as the leading terms in a low-energy approximation to (compactified) superstring theory. The nine-dimensional massive deformations split up in two categories: those where only the theory to lowest order in α' has a higher-dimensional origin and those where also the higher-derivative corrections can be obtained from 10D. The latter category can be derived using symmetries that extend to all orders in α'. We have two such symmetries:

- The $S\ell(2,\mathbb{R})$ (or rather its $S\ell(2,\mathbb{Z})$ subgroup) symmetry of IIB. Thus the $\vec{m} = (m_1, m_2, m_3)$ deformations correspond to the low-energy limits of three different sectors of compactified IIB string theory (depending on $\vec{m}^2 = \frac{1}{4}({m_1}^2 + {m_2}^2 - {m_3}^2)$). In [117] DW solutions were constructed for all three sectors. Of these only the D7-brane has a well-understood role in IIB string theory.

- The linear combination $\frac{1}{12}\hat{\alpha} + \hat{\beta}$ of $\mathbb{R}^+$-symmetries of IIA. Thus one can define a massive deformation m_s within case 1 with $\{m_{\text{IIA}} = \frac{1}{12}m_s, m_4 = m_s\}$ which corresponds to the low-energy limit of a sector of compactified IIA string theory. No vacuum solution has been constructed for this sector. It would be very interesting to try to find a vacuum solution and understand which role it plays in IIA string theory.

In fact, one can have a better understanding of the m_s massive deformation and the $\frac{1}{12}\hat{\alpha} + \hat{\beta}$ symmetry of IIA from the following point of view. The combination $\frac{1}{12}\hat{\alpha} + \hat{\beta}$ of IIA can be understood from its 11D origin as the general coordinate transformation $x^{11} \to \lambda x^{11}$; one can easily check that this is indeed the case by comparing with (3.17). This explains why all α' corrections transform covariantly under this specific $\mathbb{R}^+$: the higher-order corrections in 11D are invariant under general coordinate transformations and upon reduction they must transform covariantly under the reduced g.c.t.'s, among which is the $\frac{1}{12}\hat{\alpha} + \hat{\beta}$ scaling-symmetry.

Chapter 5

Conformal supergravity

As we saw in the previous chapters, Scherk-Schwarz dimensional reduction can be used as a tool to obtain scalar potentials in lower dimensional gauged supergravity theories. There are unfortunately many different ways in which the compactification process can be performed and therefore it is a priori not clear how to obtain the most general vacuum solutions.

Another approach to this problem is the construction of matter coupled Poincaré supergravity in lower dimensions. One possibility is the explicit coupling of matter multiplets of the super Poincaré algebra to the supergravity multiplet. The method we will use in the following chapters however, is that of conformal supergravity for reasons explained below.

Conformal supergravities have been constructed in various dimensions (for a review, see [123]) but not yet in five dimensions. By using conformal tensor calculus, conformal supergravities form an elegant way to construct general couplings of Poincaré-supergravities to matter [124]. It also provides a method to find the auxiliary fields of off-shell Poincaré supergravities, like e.g. for $\mathcal{N} = 1, D = 4$ supergravity [125]. The reason for using a theory based on the superconformal group instead of the Poincaré group is the presence of more symmetries, generally resulting in more structure and therefore simplifying the calculations. Furthermore, the conformal group is the largest possible group of space-time symmetries and turns out to be gauge equivalent to the Poincaré group. In the five-dimensional case these matter coupled supergravities have recently attracted renewed attention for reasons motivated in chapter 2.

Although quite some progress has been made in these areas, it is clear that it is important to have an independent derivation of the most general matter couplings derived in [76] where most of the current results are based on. Especially since past experience has shown that superconformal constructions lead to new insights in the structure of matter couplings. A recent example is the insight in relations between hyper-Kähler cones and quaternionic manifolds, based on the study of superconformal invariant matter couplings with hypermultiplets [126]. For these reasons a superconformal construction of general matter couplings in $\mathcal{N} = 2, D = 5$ is desirable.

The superconformal program

The procedure will be as follows.

- In this chapter we take the first step in this investigation by constructing the $\mathcal{N} = 2, D = 5$

conformal supergravity theory. In our construction we use the methods developed first for $\mathcal{N}=1$, $D=4$ [127, 128], which were inspired by the geometrical methods of [129]. They are based on gauging the conformal superalgebra [130] which in our case is $F^2(4)$. The superconformal multiplet one obtains this way contains all the (independent) gauge fields of the superconformal algebra and is called the Weyl multiplet. In general one needs to include matter fields to have an equal number of bosons and fermions. Although there are two sets of auxiliary fields one can use, in this chapter we will restrict to the one leading to the so-called Standard Weyl multiplet.

- The second step will be performed in chapter 6, where we construct the actions for matter multiplets in the background of the Standard Weyl multiplet. This step already produces a nice geometrical framework on the scalar manifolds resulting from this construction.
- Finally, in chapter 7 we will gaugefix the symmetries not present in the Poincaré algebra and construct $\mathcal{N}=2$, $D=5$ matter coupled Poincaré supergravity.

This chapter is based on the work published in [131]. Note that many details have been left out for reasons of brevity and clarity; we refer the reader to [131, 132] for more details. Note that shortly after our publication interesting results have been obtained on conformal supergravity in five dimensions [133] that have some overlap with our work.

For more information on the conformal supergravity approach, see [134–136].

5.1 Definition of rigid conformal (super-)symmetry

We start this chapter by giving a short review of rigid conformal supersymmetry; for a more extended discussion, see e.g. [137]. We first introduce conformal symmetry and in a second step extend this to conformal supersymmetry. Given a space-time with a metric tensor $g_{\mu\nu}(x)$, the conformal transformations are defined as the general coordinate transformations that leave "angles" invariant. The parameters of these special coordinate transformations define a conformal Killing vector $k^\mu(x)$. The defining equation for this conformal Killing vector is given by

$$\delta_{\text{g.c.t.}}(k)g_{\mu\nu}(x) \equiv \nabla_\mu k_\nu(x) + \nabla_\nu k_\mu(x) = \omega(x)g_{\mu\nu}(x)\,, \tag{5.1}$$

where $\omega(x)$ is an arbitrary function, $k_\mu = g_{\mu\nu}k^\nu$ and the covariant derivative is given by $\nabla_\mu k_\nu = \partial_\mu k_\nu - \Gamma^\rho_{\mu\nu}k_\rho$. In flat D-dimensional Minkowski space-time, (5.1) implies

$$\partial_{(\mu}k_{\nu)}(x) - \tfrac{1}{D}\eta_{\mu\nu}\partial_\rho k^\rho(x) = 0\,. \tag{5.2}$$

In dimensions $D > 2$, the conformal algebra is finite-dimensional. The solutions of (5.2) are given by

$$k^\mu(x) = \xi^\mu + \lambda_M^{\mu\nu}x_\nu + \lambda_D x^\mu + \left(x^2\Lambda_K^\mu - 2x^\mu x\cdot\Lambda_K\right). \tag{5.3}$$

Corresponding to the parameters ξ^μ are the translations P_μ, the parameters $\lambda_M^{\mu\nu}$ correspond to Lorentz rotations $M_{\mu\nu}$, to λ_D are associated the dilatations D, and Λ_K^μ are the parameters of 'special conformal transformations' K_μ. Thus, the full set of conformal transformations δ_C can be expressed as follows:

$$\delta_C = \xi^\mu P_\mu + \lambda_M^{\mu\nu}M_{\mu\nu} + \lambda_D D + \Lambda_K^\mu K_\mu\,. \tag{5.4}$$

The commutators between different generators define the conformal algebra which is isomorphic to the algebra of $\mathrm{SO}(D,2)$.

We wish to consider representations of the conformal algebra on fields $\phi^\alpha(x)$ where α stands for a collection of internal indices referring to the stability subalgebra of $x^\mu = 0$. From the expression (5.3) for the conformal Killing vector, we deduce that this algebra is isomorphic to the algebra generated by $M_{\mu\nu}$, D and K_μ. We denote the generators of this stability subalgebra by $\Sigma_{\mu\nu}$, Δ and κ_μ. Applying the theory of induced representations, it follows that any representation (Σ, Δ, κ) of the stability subalgebra induces a representation of the full conformal algebra with the following transformation rules [135] (we suppress any internal indices):

$$\begin{aligned}
\delta_P\phi(x) &= \xi^\mu\partial_\mu\phi(x)\,, \\
\delta_M\phi(x) &= \tfrac{1}{2}\lambda_M^{\mu\nu}(x_\nu\partial_\mu - x_\mu\partial_\nu)\phi(x) + \delta_\Sigma(\lambda_M)\phi(x)\,, \\
\delta_D\phi(x) &= \lambda_D x^\lambda\partial_\lambda\phi(x) + \delta_\Delta(\lambda_D)\phi(x)\,, \\
\delta_K\phi(x) &= \lambda_K^\mu(x^2\partial_\mu - 2x_\mu x^\lambda\partial_\lambda)\phi(x) \\
&\quad + \Big(\delta_\Delta(-2x\cdot\Lambda_K) + \delta_\Sigma(-4x_{[\mu}\lambda_{K\nu]}) + \delta_\kappa(\lambda_K)\Big)\phi(x)\,.
\end{aligned} \tag{5.5}$$

We now look at the non-trivial representation (Σ, Δ, κ) that we use in this chapter.

- Firstly, concerning the Lorentz representations, in this chapter we will encounter anti-symmetric tensors $\phi_{a_1\cdots a_n}(x)$ $(n = 0, 1, 2, \ldots)$ and spinors $\psi_\alpha(x)$:

$$\begin{aligned}
\delta_\Sigma(\lambda_M)\phi_{a_1\cdots a_n}(x) &= -n(\lambda_M)_{[a_1}{}^b\phi_{|b|a_2\cdots a_n]}(x)\,, \\
\delta_\Sigma(\lambda_M)\psi(x) &= -\tfrac{1}{4}\lambda_M^{ab}\gamma_{ab}\psi(x)\,.
\end{aligned} \tag{5.6}$$

- Secondly, we consider the dilatations. For most fields, the Δ transformation is just determined by a number w, which is called the Weyl weight of ϕ^α:

$$\delta_\Delta(\lambda_D)\phi^\alpha(x) = w\lambda_D\phi^\alpha(x)\,. \tag{5.7}$$

  An exception is given in the next chapter for the scalars of the hypermultiplet, on which dilatation transformations are realized nonlinearly. Namely, for scalar fields it is often convenient to consider the set of fields ϕ^α as the coordinates of a scalar manifold with affine connection $\Gamma_{\alpha\beta}{}^\gamma$. With this understanding, the transformation of ϕ^α under dilatations can be characterized by

$$\delta_\Delta(\lambda_D)\phi^\alpha = \lambda_D k^\alpha(\phi)\,. \tag{5.8}$$

  Requiring dilatational invariance of kinetic terms determined by a metric $g_{\alpha\beta}$, leads to the interpretation of the vector k^α as a homothetic Killing vector, i.e. it should satisfy the conformal Killing equation (5.1) for *constant* $\omega(x)$:

$$\mathfrak{D}_\alpha k_\beta + \mathfrak{D}_\beta k_\alpha = (D-2)g_{\alpha\beta}\,, \tag{5.9}$$

  where D denotes the space-time dimension and $\mathfrak{D}_\alpha k_\beta = \partial_\alpha k_\beta - \Gamma_{\alpha\beta}{}^\gamma k_\gamma$. However, (5.5) shows that the Δ-transformation also enters in the special conformal transformation. It turns out that invariance of the kinetic terms under these special conformal transformations restricts $k^\alpha(\phi)$ further to a so-called *exact* homothetic Killing vector, i.e.

$$k_\alpha = \partial_\alpha \mathcal{K}\,, \tag{5.10}$$

for some function $\mathcal{K}(\phi)$. One can show that the restrictions (5.9) and (5.10) are equivalent to

$$\mathfrak{D}_\alpha k^\beta \equiv \partial_\alpha k^\beta + \Gamma_{\alpha\gamma}{}^\beta k^\gamma = w\delta_\alpha{}^\beta\,. \tag{5.11}$$

The constant w is identified with the Weyl weight of ϕ^α and is given by $w = (D-2)/2$, i.e. 3/2 in five dimensions. The proof of the necessity of (5.11) can be extracted from [138], see also [139, 140]. In these papers the conditions for conformal invariance of a sigma model with either gravity or supersymmetry are investigated. Note that the condition (5.11) can be formulated *independent* of a metric. Only an affine connection is necessary.

For the special case of a zero affine connection, the homothetic Killing vector is given by $k^\alpha = w\phi^\alpha$ and the transformation rule (5.8) reduces to $\delta_\Delta(\lambda_D)\phi^\alpha = w\lambda_D\phi^\alpha$. Note that the homothetic Killing vector $k^\alpha = w\phi^\alpha$ is indeed exact with $\mathcal{K}$ given by

$$\mathcal{K} = \tfrac{1}{(D-2)} k^\alpha g_{\alpha\beta} k^\beta\,. \tag{5.12}$$

- Finally, all (non-gauge)fields that we will discuss in this thesis are invariant under the internal special conformal transformations, i.e. $\delta_\kappa\phi^\alpha = 0$.

We next consider the extension to conformal supersymmetry. The parameters of these supersymmetries define a conformal Killing spinor $\epsilon^i(x)$ whose defining equation is given by

$$\nabla_\mu\epsilon^i(x) - \tfrac{1}{D}\gamma_\mu\gamma^\nu\nabla_\nu\epsilon^i(x) = 0\,. \tag{5.13}$$

In D-dimensional Minkowski space-time this equation implies

$$\partial_\mu\epsilon^i(x) - \tfrac{1}{D}\gamma_\mu\partial\!\!\!/\epsilon^i(x) = 0\,. \tag{5.14}$$

The solution to this equation is given by

$$\epsilon^i(x) = \epsilon^i + \mathrm{i}\,x^\mu\gamma_\mu\eta^i\,, \tag{5.15}$$

where the (constant) parameters ϵ^i correspond to "ordinary" supersymmetry transformations Q^i_α and the parameters η^i define special conformal supersymmetries generated by $S_\alpha^{\,i}$. The conformal transformation (5.3) and the supersymmetries (5.15) do not form a closed algebra. To obtain closure, one must introduce additional R-symmetry generators. In particular, in the case of 8 supercharges Q^i_α in $D = 5$, there is an additional SU(2) R-symmetry with generators $U_{ij} = U_{ji}$ $(i = 1, 2)$. Thus, the full set of superconformal transformations δ_C is given by:

$$\delta_C = \xi^\mu P_\mu + \lambda_M^{\mu\nu} M_{\mu\nu} + \lambda_D D + \Lambda_K^\mu K_\mu + \Lambda^{ij} U_{ij} + \mathrm{i}\,\bar\epsilon Q + \mathrm{i}\,\bar\eta S\,. \tag{5.16}$$

The factors of i in the last two terms appear due to the reality properties, as explained in appendix A. The full superconformal algebra $F^2(4)$ formed by (anti-)commutators between the (bosonic and fermionic) generators will be given in section 5.2.1.

To construct field representations of the superconformal algebra, one can again apply the method of induced representations. In this case one must use superfields $\Phi^a(x^\mu, \theta^i_\alpha)$, where a stands for a collection of internal indices referring to the stability subalgebra of $x^\mu = \theta^i_\alpha = 0$. This algebra is isomorphic to the algebra generated by $M_{\mu\nu}, D, K_\mu, U_{ij}$ and $S^{\ i}_{\alpha}$.

An additional complication, not encountered in the bosonic case, is that the representation one obtains is reducible. To obtain an irreducible representation, one must impose constraints on the superfield. It is at this point that the transformation rules become nonlinear in the fields. In this chapter we will follow a different approach; instead of working with superfields we will work with the component "ordinary" fields. The different nonlinear transformation rules are obtained by imposing the superconformal algebra.

In the supersymmetric case, we must specify the SU(2)-properties of the different fields as well as the behavior under S-supersymmetry. Concerning the SU(2), we will only encounter scalars ϕ, doublets ψ^i and triplets $\phi^{(ij)}$ whose transformations are given by

$$\begin{aligned} \delta_{\mathrm{SU(2)}}(\Lambda^{ij})\phi &= 0\,, \\ \delta_{\mathrm{SU(2)}}(\Lambda^{ij})\psi^i(x) &= -\Lambda^i{}_j\psi^j(x)\,, \\ \delta_{\mathrm{SU(2)}}(\Lambda^{ij})\phi^{ij}(x) &= -2\Lambda^{(i}{}_k\phi^{j)k}(x)\,. \end{aligned} \tag{5.17}$$

Note that the scalars of the hypermultiplet will also have an SU(2) transformation despite the absence of an i index, as we will see in the following chapter in section 6.3.2.

This leaves us with specifying how a given field transforms under the special supersymmetries generated by $S^{\ i}_{\alpha}$. In superfield language the full S-transformation is given by a combination of an x-dependent translation in superspace, with parameter $\epsilon^i(x) = \mathrm{i}\, x^\mu\gamma_\mu\eta^i$, and an internal S-transformation. This is a perfect analogy to the bosonic case. In terms of component fields, the same is true. The x-dependent contribution is obtained by making the substitution

$$\epsilon^i \to \mathrm{i}\, \not{x}\eta^i \tag{5.18}$$

in the Q-supersymmetry rules. The internal S-transformations can be deduced by imposing the superconformal algebra.

5.2 Gauging the Superconformal Algebra

In this section we will construct the Standard Weyl multiplet by using the methods developed first for $\mathcal{N} = 1$ in four dimensions [128]. They are based on gauging the conformal superalgebra [130] which, in our case, is $F^2(4)$. We start by giving the commutation relations defining the $F^2(4)$ algebra. Next we discuss the general method, and then apply this to construct the full nonlinear Standard Weyl multiplet. For clarity, we have collected the final results in section 5.3.

5.2.1 The $D = 5$ superconformal algebra $F^2(4)$

Our starting point is the five dimensional superconformal algebra. There exist many varieties of superconformal algebras when one allows for central charges [141, 142]. However, so far a suitable superconformal Weyl multiplet has only been constructed from those superconformal algebras[1] that appear in the Nahm's classification [144]. In that classification there appears one exceptional algebra, which is $F(4)$. The particular real form that we need here is denoted by $F^2(4)$, see tables 5 and 6 in [137].

As we saw in section 5.1, the algebra consists of the bosonic generators M_{ab}, P_a, K_a, D and the fermionic generators $Q_{i\alpha}$ and $S_{i\alpha}$, where $a, b, \ldots$ are Lorentz indices, α is a spinor index and $i = 1, 2$ is an SU(2) index. M_{ab} is the Lorentz generator, P_a are the conformal transformations, K_a is the special conformal transformation, D the dilatation, $Q_{i\alpha}$ and $S_{i\alpha}$ are the supersymmetry and the special supersymmetry generators, respectively, which are symplectic Majorana spinors, 8 real components in total. Finally, $U^{ij} = U^{ji}$ are the generators of the SU(2) R-symmetry group. For more details on the $F^2(4)$ algebra and the rigid superconformal transformations, see [137]. The non-trivial (anti)commutation relations of the generators defining the $F^2(4)$ algebra are given by

$$\begin{array}{rclrcl}
[P_a, M_{bc}] &=& \eta_{a[b}P_{c]}\,, & [K_a, M_{bc}] &=& \eta_{a[b}K_{c]}\,,\\
{}[D, P_a] &=& P_a\,, & [D, K_a] &=& -K_a\,,\\
\left[M_{ab}, M^{cd}\right] &=& -2\delta_{[a}{}^{[c}M_{b]}{}^{d]}\,, & [P_a, K_b] &=& 2(\eta_{ab}D + 2M_{ab})\,,\\[2ex]
[M_{ab}, Q_{i\alpha}] &=& -\tfrac{1}{4}(\gamma_{ab}Q_i)_\alpha\,, & [M_{ab}, S_{i\alpha}] &=& -\tfrac{1}{4}(\gamma_{ab}S_i)_\alpha\,,\\
{}[D, Q_{i\alpha}] &=& \tfrac{1}{2}Q_{i\alpha}\,, & [D, S_{i\alpha}] &=& -\tfrac{1}{2}S_{i\alpha}\,,\\
{}[K_a, Q_{i\alpha}] &=& \mathrm{i}\,(\gamma_a S_i)_\alpha\,, & [P_a, S_{i\alpha}] &=& -\mathrm{i}\,(\gamma_a Q_i)_\alpha\,,\\[2ex]
\left\{Q_{i\alpha}, Q_{j\beta}\right\} &=& -\tfrac{1}{2}\varepsilon_{ij}(\gamma^a)_{\alpha\beta}P_a\,, & \left\{S_{i\alpha}, S_{j\beta}\right\} &=& -\tfrac{1}{2}\varepsilon_{ij}(\gamma^a)_{\alpha\beta}K_a\,,\\
\left\{Q_{i\alpha}, S_{j\beta}\right\} &=& \multicolumn{4}{l}{-\tfrac{1}{2}\,\mathrm{i}\left(\varepsilon_{ij}C_{\alpha\beta}D + \varepsilon_{ij}(\gamma^{ab})_{\alpha\beta}M_{ab} + 3C_{\alpha\beta}U_{ij}\right)\,,}\\[2ex]
[Q_{i\alpha}, U_{kl}] &=& \varepsilon_{i(k}Q_{l)\alpha}\,, & [S_{i\alpha}, U_{kl}] &=& \varepsilon_{i(k}S_{l)\alpha}\,,\\
\left[U_{ij}, U^{kl}\right] &=& 2\delta_{(i}{}^{(k}U_{j)}{}^{l)}\,,
\end{array} \tag{5.19}$$

where $C_{\alpha\beta}$ is the charge conjungation matrix, see appendix A. The first six commutation relations define the bosonic conformal subgroup SO(5, 2).

We give below some of the commutators of the (rigid) superconformal algebra expressed in terms of commutators of variations of the fields. The commutators between Q- and S-supersymmetry are given by

$$[\delta_Q(\epsilon_1), \delta_Q(\epsilon_2)] = \delta_P\left(\tfrac{1}{2}\bar{\epsilon}_2\gamma_\mu\epsilon_1\right), \tag{5.20}$$

$$[\delta_S(\eta), \delta_Q(\epsilon)] = \delta_D\left(\tfrac{1}{2}\,\mathrm{i}\,\bar{\epsilon}\eta\right) + \delta_M\left(\tfrac{1}{2}\,\mathrm{i}\,\bar{\epsilon}\gamma^{ab}\eta\right) + \delta_U\left(-\tfrac{3}{2}\,\mathrm{i}\,\bar{\epsilon}^{(i}\eta^{j)}\right), \tag{5.21}$$

$$[\delta_S(\eta_1), \delta_S(\eta_2)] = \delta_K\left(\tfrac{1}{2}\bar{\eta}_2\gamma^a\eta_1\right). \tag{5.22}$$

Note that to verify these commutators one should use not only the internal but the *full* superconformal transformation rules including the x-dependent translations (5.5) and Q-supersymmetries (5.18).

[1]One exception is the ten dimensional Weyl multiplet [143], which is not based on a known algebra.

Generators	P_a	M_{ab}	D	K_a	U_{ij}	$Q_{\alpha i}$	$S_{\alpha i}$
Fields	$e_\mu{}^a$	ω_μ^{ab}	b_μ	$f_\mu{}^a$	V_μ^{ij}	ψ_μ^i	ϕ_μ^i
Parameters	ξ^a	λ^{ab}	Λ_D	Λ_K^a	Λ^{ij}	ϵ^i	η^i

Table 5.1: The gauge fields and parameters of the superconformal algebra $F^2(4)$.

5.2.2 The gauge fields and their curvatures

The $D = 5$ conformal supergravity theory is based on the superconformal algebra $F^2(4)$ whose generators are those in table 5.1. As a first step we assign to every generator of the superconformal algebra a gauge field. These gauge fields and the names of the corresponding gauge parameters are given in table 5.1.

The transformations are generated by operators according to

$$\delta = \xi^a P_a + \lambda^{ab} M_{ab} + \Lambda_D D + \Lambda_K^a K^a + \Lambda^{ij} U_{ij} + \mathrm{i}\,\bar{\epsilon} Q + \mathrm{i}\,\bar{\eta} S\,. \tag{5.23}$$

Gauge fields h_μ^A in general transform as

$$\delta_B(\epsilon^B) h_\mu^A = \partial_\mu \epsilon^A + \epsilon^C h_\mu^B f_{BC}{}^A\,, \tag{5.24}$$

where the structure constants $f_{BC}{}^A$ can be read off from the algebra (5.19). We find

$$\begin{aligned}
\delta e_\mu{}^a &= \mathcal{D}_\mu \xi^a - \lambda^{ab} e_{\mu b} - \Lambda_D e_\mu{}^a + \tfrac{1}{2}\bar{\epsilon}\gamma^a \psi_\mu\,,\\
\delta \omega_\mu{}^{ab} &= \mathcal{D}_\mu \lambda^{ab} - 4\xi^{[a} f_\mu{}^{b]} - 4\Lambda_K^{[a} e_\mu{}^{b]} + \tfrac{1}{2}\,\mathrm{i}\,\bar{\epsilon}\gamma^{ab}\phi_\mu - \tfrac{1}{2}\,\mathrm{i}\,\bar{\eta}\gamma^{ab}\psi_\mu\,,\\
\delta b_\mu &= \partial_\mu \Lambda_D - 2\xi^a f_{\mu a} + 2\Lambda_K^a e_{\mu a} + \tfrac{1}{2}\,\mathrm{i}\,\bar{\epsilon}\phi_\mu + \tfrac{1}{2}\,\mathrm{i}\,\bar{\eta}\psi_\mu\,,\\
\delta f_\mu{}^a &= \mathcal{D}_\mu \Lambda_K^a - \lambda^{ab} f_{\mu b} + \Lambda_D f_\mu{}^a + \tfrac{1}{2}\bar{\eta}\gamma^a \phi_\mu\,,\\
\delta V_\mu^{ij} &= \partial_\mu \Lambda^{ij} - 2\Lambda^{(i}{}_\ell V_\mu^{j)\ell} - \tfrac{3}{2}\,\mathrm{i}\,\bar{\epsilon}^{(i}\phi_\mu^{j)} + \tfrac{3}{2}\,\mathrm{i}\,\bar{\eta}^{(i}\psi_\mu^{j)}\,,\\
\delta \psi_\mu^i &= \mathcal{D}_\mu \epsilon^i + \mathrm{i}\,\xi^a \gamma_a \phi_\mu^i - \tfrac{1}{4}\lambda^{ab}\gamma_{ab}\psi_\mu^i - \tfrac{1}{2}\Lambda_D \psi_\mu^i - \Lambda^i{}_j \psi_\mu^j - \mathrm{i}\, e_\mu^a \gamma_a \eta^i\,,\\
\delta \phi_\mu^i &= \mathcal{D}_\mu \eta^i - \tfrac{1}{4}\lambda^{ab}\gamma_{ab}\phi_\mu^i + \tfrac{1}{2}\Lambda_D \phi_\mu^i - \Lambda^i{}_j \phi_\mu^j - \mathrm{i}\,\Lambda_K^a \gamma_a \psi_\mu^i + \mathrm{i}\, f_\mu^a \gamma_a \epsilon^i\,,
\end{aligned} \tag{5.25}$$

where $\mathcal{D}_\mu$ is the covariant derivative with respect to dilatations, Lorentz rotations and SU(2) transformations:

$$\begin{aligned}
\mathcal{D}_\mu \xi^a &= \partial_\mu \xi^a + b_\mu \xi^a + \omega_\mu{}^{ab}\xi_b\,,\\
\mathcal{D}_\mu \lambda^{ab} &= \partial_\mu \lambda^{ab} + 2\omega_{\mu c}{}^{[a}\lambda^{b]c}\,,\\
\mathcal{D}_\mu \Lambda_K^a &= \partial_\mu \Lambda_K^a - b_\mu \Lambda_K^a + \omega_\mu{}^{ab}\Lambda_{Kb}\,,\\
\mathcal{D}_\mu \epsilon^i &= \partial_\mu \epsilon^i + \tfrac{1}{2} b_\mu \epsilon^i + \tfrac{1}{4}\omega_\mu^{ab}\gamma_{ab}\epsilon^i - V_\mu^{ij}\epsilon_j\,,\\
\mathcal{D}_\mu \eta^i &= \partial_\mu \eta^i - \tfrac{1}{2} b_\mu \eta^i + \tfrac{1}{4}\omega_\mu^{ab}\gamma_{ab}\eta^i - V_\mu^{ij}\eta_j\,.
\end{aligned} \tag{5.26}$$

The corresponding curvatures can be calculated by using the general rule

$$R_{\mu\nu}{}^A = 2\partial_{[\mu} h_{\nu]} + h_\nu^C h_\mu^B f_{BC}{}^A\,. \tag{5.27}$$

The structure constants can again be read off from the (anti)commutator expressions (5.19) and we obtain the following curvatures (terms proportional to vielbeins are underlined for later use):

$$
\begin{aligned}
R_{\mu\nu}{}^{a}(P) &= 2\partial_{[\mu}e_{\nu]}{}^{a} + \underline{2\omega_{[\mu}{}^{ab}e_{\nu]b}} + \underline{2b_{[\mu}e_{\nu]}{}^{a}} - \tfrac{1}{2}\bar{\psi}_{[\mu}\gamma^{a}\psi_{\nu]}\,,\\
R_{\mu\nu}{}^{ab}(M) &= 2\partial_{[\mu}\omega_{\nu]}{}^{ab} + 2\omega_{[\mu}{}^{ac}\omega_{\nu]c}{}^{b} + \underline{8f_{[\mu}{}^{[a}e_{\nu]}{}^{b]}} + \mathrm{i}\,\bar{\phi}_{[\mu}\gamma^{ab}\psi_{\nu]}\,,\\
R_{\mu\nu}(D) &= 2\partial_{[\mu}b_{\nu]} - \underline{4f_{[\mu}{}^{a}e_{\nu]a}} - \mathrm{i}\,\bar{\phi}_{[\mu}\psi_{\nu]}\,,\\
R_{\mu\nu}{}^{a}(K) &= 2\partial_{[\mu}f_{\nu]}{}^{a} + 2\omega_{[\mu}{}^{ab}f_{\nu]b} - 2b_{[\mu}f_{\nu]}{}^{a} - \tfrac{1}{2}\bar{\phi}_{[\mu}\gamma^{a}\phi_{\nu]}\,,\\
R_{\mu\nu}{}^{ij}(V) &= 2\partial_{[\mu}V_{\nu]}{}^{ij} - 2V_{[\mu}{}^{k(i}V_{\nu]k}{}^{j)} - 3\,\mathrm{i}\,\bar{\phi}^{(i}_{[\mu}\psi^{j)}_{\nu]}\,,\\
R_{\mu\nu}{}^{i}(Q) &= 2\partial_{[\mu}\psi^{i}_{\nu]} + \tfrac{1}{2}\omega_{[\mu}{}^{ab}\gamma_{ab}\psi^{i}_{\nu]} + b_{[\mu}\psi^{i}_{\nu]} - 2V_{[\mu}{}^{ij}\psi_{\nu]j} + \underline{2\,\mathrm{i}\,\gamma_{a}\phi^{i}_{[\mu}e_{\nu]}{}^{a}}\,,\\
R_{\mu\nu}{}^{i}(S) &= 2\partial_{[\mu}\phi^{i}_{\nu]} + \tfrac{1}{2}\omega_{[\mu}{}^{ab}\gamma_{ab}\phi^{i}_{\nu]} - b_{[\mu}\phi^{i}_{\nu]} - 2V_{[\mu}{}^{ij}\phi_{\nu]j} - 2\,\mathrm{i}\,\gamma_{a}\psi^{i}_{[\mu}f_{\nu]}{}^{a}\,.
\end{aligned}
\tag{5.28}
$$

Since the transformation laws given above satisfy the $F^2(4)$ superalgebra, we have constructed a gauge theory of $F^2(4)$. However, this is not a gauge theory of diffeomorphisms of space-time yet; this can only be realized if we take the spin connection as a composite field that depends on the vielbein. So far we have it as an independent field.[2]

Furthermore, we see that the number of bosonic and fermionic degrees of freedom do not match. The gauge fields together have 96 + 64 degrees of freedom. Therefore, we can not have a supersymmetric theory with invertible general coordinate transformations generated by the square of supersymmetry operations.

5.2.3 Curvature constraints

The solution to the problems described above is well known. In order to convert the P-gauge transformations into general coordinate transformations and to obtain irreducibility we need to impose curvature constraints. This will define some gauge fields to be dependent fields.

We will consider the fünfbein as an invertible field. Then some of the curvatures in (5.28) are linear in some gauge fields. This is shown by the underlined terms in (5.28). Therefore, we can impose constraints on these curvatures that are solvable for these gauge fields. Such constraints are called conventional constraints, and imposing them reduces the Weyl multiplet, such that we get closer to an irreducible multiplet. The conventional constraints are

$$
\begin{aligned}
R_{\mu\nu}{}^{a}(P) &= 0 \quad (50)\,,\\
e^{\nu}{}_{b}R_{\mu\nu}{}^{ab}(M) &= 0 \quad (25)\,,\\
\gamma^{\mu}R_{\mu\nu}{}^{i}(Q) &= 0 \quad (40)\,.
\end{aligned}
\tag{5.29}
$$

In brackets we denoted the number of restrictions each constraint imposes. These constraints are similar to those for other Weyl multiplets in four dimensions with $\mathcal{N}=1$ [127,130], $\mathcal{N}=2$ [145] or $\mathcal{N}=4$ [146], or in six dimensions for the $(1,0)$ [147] or $(2,0)$ [148] Weyl multiplets.

[2] One might think that the field equations can determine the spin connection as a dependent gauge field. This can indeed be done for the spin connection, but it is not known how to generalize this for the gauge fields of special (super)conformal symmetries, which we also want to be dependent gauge fields.

Field	#	Gauge	SU(2)	w	Field	#	Gauge	SU(2)	w
		Elementary gauge fields					Dependent gauge fields		
$e_\mu{}^a$	9	P^a	1	-1	$\omega_\mu^{[ab]}$	–	$M^{[ab]}$	1	0
b_μ	0	D	1	0	$f_\mu{}^a$	–	K^a	1	1
$V_\mu^{(ij)}$	12	SU(2)	3	0					
ψ_μ^i	24	Q_α^i	2	$-\frac{1}{2}$	ϕ_μ^i	–	S_α^i	2	$\frac{1}{2}$
		Dilaton Weyl multiplet					Standard Weyl multiplet		
$B_{[\mu\nu]}$	6	$\delta B_{\mu\nu} = 2\partial_{[\mu}\Lambda_{\nu]}$	1	0	$T_{[ab]}$	10		1	1
A_μ	4	$\delta A_\mu = \partial_\mu \Lambda$	1	0					
σ	1		1	1	D	1		1	2
ψ^i	8		2	$\frac{3}{2}$	χ^i	8		2	$\frac{3}{2}$

Table 5.2: Fields of the Weyl multiplets, and their roles. The upper half contains the fields that are present in all versions. They are the gauge fields of the superconformal algebra (see section 5.2). The fields at the right-hand side of the upper half are dependent fields. The symbol # indicates the off-shell degrees of freedom. The lower half contains the extra matter fields that appear in the two versions of the Weyl multiplet. In the left half we have those of the Dilaton Weyl multiplet, and at the right those of the Standard Weyl multiplet. We also indicated the (generalized) gauge-symmetries of the fields A_μ and $B_{\mu\nu}$.

5.2.4 Adding matter fields

After imposing the constraints we are left with 21 bosonic and 24 fermionic degrees of freedom. The independent fields are those in the upper left part of table 5.2. In order to get matching bosonic and fermionic degrees of freedom, we have to introduce extra matter fields in the multiplet, to obtain the full Weyl multiplet. There turns out to be two possibilities for a $D = 5$ Weyl multiplet, each with 32 + 32 degrees of freedom. The auxiliary fields $(A_\mu, B_{\mu\nu}, \sigma, \psi^i)$ lead to the Dilaton Weyl multiplet, whereas the set (T_{ab}, D, χ^i) leads to the Standard Weyl multiplet. The latter type is the Weyl multiplet one would expect when comparing to four and six dimensional theories with eight supercharges. Furthermore, since both Weyl multiplets are related by field redefinitions [131] we will restrict ourselves to the Standard Weyl multiplet from now on.

Modified constraints. The extra matter fields will change the transformations of the gauge fields. In fact, for the transformation of a general gauge field h_μ^I we will have (apart from the general coordinate transformations):

$$\delta_J(\epsilon^J)h_\mu^I = \partial_\mu\epsilon^I + \epsilon^J h_\mu{}^A f_{AJ}{}^I + \epsilon^J M_{\mu J}{}^I \,, \tag{5.30}$$

where we use the index I to denote all gauge transformations apart from general coordinate transformations, and an index A includes the translations.

The last term depends on the matter fields, and its explicit form has to be determined below. But also the second term has contributions from matter fields. This is due to the fact that the structure 'functions' of the final algebra $f_{IJ}{}^K$ are modified from those of the $F^2(4)$ algebra which was used for (5.25). These extra terms also lead to modified curvatures

$$\widehat{R}_{\mu\nu}{}^I = 2\partial_{[\mu}h_{\nu]}{}^I + h_\nu{}^B h_\mu{}^A f_{AB}{}^I - 2h_{[\mu}{}^J M_{\nu]J}{}^I \,. \tag{5.31}$$

The commutator of two supersymmetry-transformations will also change. In particular we will find transformations with field-dependent parameters. They can be conveniently written as so-called covariant general coordinate transformations, which are defined as

$$\delta_{cgct}(\xi) = \delta_{gct}(\xi) - \delta_I(\xi^\mu h_\mu{}^I) \,, \tag{5.32}$$

namely a combination of general coordinate transformations and all the other transformations whose parameter ϵ^I is replaced by $\xi^\mu h_\mu{}^I$.

Note that the curvature modifications also lead to modified curvature constraints:[3]

$$R_{\mu\nu}{}^a(P) = 0 \,, \qquad e^\nu{}_b \widehat{R}_{\mu\nu}{}^{ab}(M) = 0 \,, \qquad \gamma^\mu \widehat{R}_{\mu\nu}{}^i(Q) = 0 \,. \tag{5.33}$$

In general one can add extra terms to the constraints (5.33), which just amount to redefinitions of the composite fields. By choosing suitable terms simplifications were obtained in four and six dimensions. In this case one could e.g. add a term $T_{\mu b}T^{ba}$ to the second constraint rendering all the constraints invariant under S-supersymmetry, but in five dimensions this turns out to be impossible. Therefore we keep the constraints as written above.

Due to these constraints the fields $\omega_\mu{}^{ab}$, $f_\mu{}^a$ and ϕ_μ^i are no longer independent, but can be expressed in terms of the other fields. In order to write down the explicit solutions of these constraints, it is useful to extract the terms which have been underlined in (5.28). We define $\widehat{R}'$ as the curvatures without these terms. Formally,

$$\widehat{R}'_{\mu\nu}{}^I = \widehat{R}_{\mu\nu}{}^I + 2h^J_{[\mu} e_{\nu]}{}^a f_{aJ}{}^I \,, \tag{5.34}$$

where $f_{aJ}{}^I$ are the structure constants in the $F^2(4)$ algebra that define commutators of translations with other gauge transformations. Then the solutions to the constraints are

$$\begin{aligned}
\omega_\mu^{ab} &= 2e^{\nu[a}\partial_{[\mu}e_{\nu]}{}^{b]} - e^{\nu[a}e^{b]\sigma}e_{\mu c}\partial_\nu e_\sigma{}^c + 2e_\mu{}^{[a}b^{b]} - \tfrac{1}{2}\bar\psi^{[b}\gamma^{a]}\psi_\mu - \tfrac{1}{4}\bar\psi^b\gamma_\mu\psi^a \,, \\
\phi_\mu^i &= -\tfrac{1}{12}\,\mathrm{i}\,(\gamma^{ab}\gamma_\mu - \tfrac{1}{2}\gamma_\mu\gamma^{ab})\widehat{R}'_{ab}{}^i(Q) \,, \\
f_\mu^a &= \tfrac{1}{6}\mathcal{R}_\mu{}^a - \tfrac{1}{48}e_\mu{}^a\mathcal{R} \,, \qquad \mathcal{R}_{\mu\nu} \equiv \widehat{R}'_{\mu\rho}{}^{ab}(M)e_a{}^\rho e_{\nu b} \,, \qquad \mathcal{R} \equiv \mathcal{R}_\mu{}^\mu \,.
\end{aligned} \tag{5.35}$$

The constraints imply further relations between the curvatures through Bianchi identities.

[3]Note that the third constraint implies that $\gamma_{[\mu\nu}\widehat{R}_{\rho\sigma]}{}^i(Q) = 0$.

Modified transformation rules. To obtain all the extra transformations one imposes the superconformal algebra, but at the same time allowing modifications of the algebra by field-dependent quantities. The techniques are the same as already used in four and six dimensions in [145, 146], and were described in detail in [147].

For the fields in the upper left corner of table 5.2, we now have to specify the extra parts M in (5.30). This will in fact only apply to Q-supersymmetry. The other transformations are as in (5.25). The full supersymmetry transformations of these fields are

$$\begin{aligned}\delta_Q e_\mu{}^a &= \tfrac{1}{2}\bar{\epsilon}\gamma^a\psi_\mu\,,\\ \delta_Q\psi_\mu^i &= \mathcal{D}_\mu\epsilon^i + \mathrm{i}\,\gamma\cdot T\gamma_\mu\epsilon^i\,,\\ \delta_Q V_\mu{}^{ij} &= -\tfrac{3}{2}\,\mathrm{i}\,\bar{\epsilon}^{(i}\phi_\mu^{j)} + \mathrm{i}\,\bar{\epsilon}^{(i}\gamma\cdot T\psi_\mu^{j)} + 4\bar{\epsilon}^{(i}\gamma_\mu\chi^{j)}\,,\\ \delta_Q b_\mu &= \tfrac{1}{2}\,\mathrm{i}\,\bar{\epsilon}\phi_\mu - 2\bar{\epsilon}\gamma_\mu\chi\,,\end{aligned} \tag{5.36}$$

where $\mathcal{D}_\mu\epsilon$ is given in (5.27).

The modification M in (5.30) is the last term of the transformations of ψ_μ^i, V_μ^{ij} and b_μ. The second term in the transformation of V_μ^{ij} on the other hand is due to the fact that the structure constants have become structure functions, and in particular there appears a new T-dependent SU(2) transformation in the anti-commutator of two supersymmetries. We will give the full new algebra in section 5.3.

The transformation rules for the matter fields of the Standard Weyl multiplet are as follows (Q and S supersymmetry)

$$\begin{aligned}\delta T_{ab} &= \tfrac{1}{2}\,\mathrm{i}\,\bar{\epsilon}\gamma_{ab}\chi - \tfrac{3}{32}\,\mathrm{i}\,\bar{\epsilon}\widehat{R}_{ab}(Q)\,,\\ \delta\chi^i &= \tfrac{1}{4}\epsilon^i D - \tfrac{1}{64}\gamma\cdot\widehat{R}^{ij}(V)\epsilon_j + \tfrac{1}{8}\,\mathrm{i}\,\gamma^{ab}\not{D}T_{ab}\epsilon^i - \tfrac{1}{8}\,\mathrm{i}\,\gamma^a D^b T_{ab}\epsilon^i\\ &\quad -\tfrac{1}{4}\gamma^{abcd}T_{ab}T_{cd}\epsilon^i + \tfrac{1}{6}T^2\epsilon^i + \tfrac{1}{4}\gamma\cdot T\eta^i\,,\\ \delta D &= \bar{\epsilon}\not{D}\chi - \tfrac{5}{3}\,\mathrm{i}\,\bar{\epsilon}\gamma\cdot T\chi - \mathrm{i}\,\bar{\eta}\chi\,.\end{aligned} \tag{5.37}$$

There are no explicit gauge fields here, as should be the case for ‘matter’, i.e. non-gauge fields. These are all hidden in the covariant derivatives and covariant curvatures. The covariant derivative for any matter field Φ is given by the rule

$$D_a\Phi = e_a^\mu\left(\partial_\mu - \delta_I(h_\mu^I)\right)\Phi\,. \tag{5.38}$$

The last term represents a sum over all transformations except general coordinate transformations, with parameters replaced by the corresponding gauge fields. In practice, the Lorentz transformations and SU(2) transformations follow directly from the index structure and lead to additions similar to those in (5.27). For the Weyl transformations there is a term $-w\,b_\mu\Phi$, where w is the Weyl weight of the field that can be found in table 5.2, and then there remain the terms for Q and S supersymmetry. There are no K transformations for any matter field in five dimensions.

The covariant curvatures in (5.37) are given by the general rule (5.31), e.g.

$$\begin{aligned}\widehat{R}_{\mu\nu}{}^i(Q) &= R_{\mu\nu}{}^i(Q) + 2\,\mathrm{i}\,\gamma\cdot T\gamma_{[\mu}\psi_{\nu]}^i\,,\\ \widehat{R}_{\mu\nu}{}^{ij}(V) &= R_{\mu\nu}{}^{ij}(V) - 8\bar{\psi}_{[\mu}^{(i}\gamma_{\nu]}\chi^{j)} - \mathrm{i}\,\bar{\psi}_{[\mu}^{(i}\gamma\cdot T\psi_{\nu]}^{j)}\,,\end{aligned} \tag{5.39}$$

where $R_{\mu\nu}{}^i(Q)$ and $R_{\mu\nu}{}^{ij}(V)$ are those given in (5.28). Given the transformation rules in (5.36) and (5.37), we could calculate the transformations of the dependent fields. Their transformation rules are now determined by their definition due to the constraints. An equivalent way of expressing this is that their transformation rules are modified w.r.t. (5.25) due to the non-invariance of the constraints under these transformations. We have chosen the constraints to be invariant under all bosonic symmetries without modifications. Therefore, only the Q- and S-supersymmetries of the dependent fields are modified to get invariant constraints.

This finishes our discussion of the Standard Weyl multiplet. The final results for this multiplet have been collected in section 5.3.

5.3 Results for the Weyl multiplet

In this section we collect the essential results of the previous sections, and give the supersymmetry algebra, which is modified by field-dependent terms. The transformation under dilatation is for each field $\delta_D\Phi = w\Lambda_D\Phi$, where the Weyl weight w can be found in table 5.2. The Lorentz, and SU(2) transformations are evident from the index structure, and our normalizations can be found in (5.25).

5.3.1 The Standard Weyl multiplet

The Q- and S-supersymmetry and K-transformation rules for the independent fields of the Standard Weyl multiplet are

$$\begin{aligned}
\delta e_\mu{}^a &= \tfrac{1}{2}\bar\epsilon\gamma^a\psi_\mu\,,\\
\delta\psi_\mu^i &= \mathcal{D}_\mu\epsilon^i + \mathrm{i}\,\gamma\cdot T\gamma_\mu\epsilon^i - \mathrm{i}\,\gamma_\mu\eta^i\,,\\
\delta V_\mu{}^{ij} &= -\tfrac{3}{2}\,\mathrm{i}\,\bar\epsilon^{(i}\phi_\mu^{j)} + 4\bar\epsilon^{(i}\gamma_\mu\chi^{j)} + \mathrm{i}\,\bar\epsilon^{(i}\gamma\cdot T\psi_\mu^{j)} + \tfrac{3}{2}\,\mathrm{i}\,\bar\eta^{(i}\psi_\mu^{j)}\,,\\
\delta T_{ab} &= \tfrac{1}{2}\,\mathrm{i}\,\bar\epsilon\gamma_{ab}\chi - \tfrac{3}{32}\,\mathrm{i}\,\bar\epsilon\widehat{R}_{ab}(Q)\,,\\
\delta\chi^i &= \tfrac{1}{4}\epsilon^i D - \tfrac{1}{64}\gamma\cdot\widehat{R}^{ij}(V)\epsilon_j + \tfrac{1}{8}\,\mathrm{i}\,\gamma^{ab}\not{D}T_{ab}\epsilon^i - \tfrac{1}{8}\,\mathrm{i}\,\gamma^a D^b T_{ab}\epsilon^i\\
&\quad -\tfrac{1}{4}\gamma^{abcd}T_{ab}T_{cd}\epsilon^i + \tfrac{1}{6}T^2\epsilon^i + \tfrac{1}{4}\gamma\cdot T\eta^i\,,\\
\delta D &= \bar\epsilon\not{D}\chi - \tfrac{5}{3}\,\mathrm{i}\,\bar\epsilon\gamma\cdot T\chi - \mathrm{i}\,\bar\eta\chi\,,\\
\delta b_\mu &= \tfrac{1}{2}\,\mathrm{i}\,\bar\epsilon\phi_\mu - 2\bar\epsilon\gamma_\mu\chi + \tfrac{1}{2}\,\mathrm{i}\,\bar\eta\psi_\mu + 2\Lambda_{K\mu}\,.
\end{aligned}\tag{5.40}$$

The covariant derivative $\mathcal{D}_\mu\epsilon$ is given in (5.27). For other covariant derivatives, see the general rule (5.38), with more explanation below that equation. The covariant curvatures $\widehat{R}(Q)$ and $\widehat{R}(V)$ are given explicitly in (5.39). The expressions for the dependent fields are given in (5.35), where the prime indicates the omission of the underlined terms in (5.28).

5.3.2 Modified superconformal algebra

The original algebra given in (5.19) is no longer satisfied on the Weyl multiplet; the algebra closes up to matter field modifications. These modifications can be written as superconformal transformations, with field dependent parameters. The algebra realized on the Weyl multiplet therefore is also called a 'soft' algebra. This is also the algebra that all matter multiplets will have to satisfy, apart from possible additional transformations under which the fields of the Weyl multiplet do not transform, and possibly field equations if these matter multiplets are on-shell.

The full commutator of two supersymmetry transformations is given by

$$[\delta_Q(\epsilon_1), \delta_Q(\epsilon_2)] = \delta_{cgct}(\xi_3^\mu) + \delta_M(\lambda_3^{ab}) + \delta_S(\eta_3) + \delta_U(\lambda_3^{ij}) + \delta_K(\Lambda_{K3}^a) . \tag{5.41}$$

Note that the Dilaton Weyl multiplet also gives rise to field dependent gauge transformations, which have been omitted here. The covariant general coordinate transformations have been defined in (5.32). The parameters appearing in (5.41) are

$$\begin{aligned}
\xi_3^\mu &= \tfrac{1}{2}\bar{\epsilon}_2\gamma_\mu\epsilon_1 , \\
\lambda_3^{ab} &= -\,\mathrm{i}\,\bar{\epsilon}_2\gamma^{[a}\gamma\cdot T\gamma^{b]}\epsilon_1 , \\
\lambda_3^{ij} &= \mathrm{i}\,\bar{\epsilon}_2^{(i}\gamma\cdot T\epsilon_1^{j)} , \\
\eta_3^i &= -\tfrac{9}{4}\,\mathrm{i}\;\bar{\epsilon}_2\epsilon_1\chi^i + \tfrac{7}{4}\,\mathrm{i}\;\bar{\epsilon}_2\gamma_c\epsilon_1\gamma^c\chi^i \\
&\quad +\tfrac{1}{4}\,\mathrm{i}\;\bar{\epsilon}_2^{(i}\gamma_{cd}\epsilon_1^{j)}\left(\gamma^{cd}\chi_j + \tfrac{1}{4}\widehat{R}^{cd}{}_j(Q)\right) , \\
\Lambda_{K3}^a &= -\tfrac{1}{2}\bar{\epsilon}_2\gamma^a\epsilon_1 D + \tfrac{1}{96}\bar{\epsilon}_2^i\gamma^{abc}\epsilon_1^j\widehat{R}_{bcij}(V) \\
&\quad +\tfrac{1}{12}\,\mathrm{i}\,\bar{\epsilon}_2\left(-5\gamma^{abcd}D_bT_{cd} + 9D_bT^{ba}\right)\epsilon_1 \\
&\quad +\bar{\epsilon}_2\left(\gamma^{abcde}T_{bc}T_{de} - 4\gamma^cT_{cd}T^{ad} + \tfrac{2}{3}\gamma^aT^2\right)\epsilon_1 .
\end{aligned} \tag{5.42}$$

For the Q, S commutators we find the following algebra:

$$\begin{aligned}
[\delta_S(\eta), \delta_Q(\epsilon)] &= \delta_D(\tfrac{1}{2}\,\mathrm{i}\,\bar{\epsilon}\eta) + \delta_M(\tfrac{1}{2}\,\mathrm{i}\,\bar{\epsilon}\gamma^{ab}\eta) + \delta_U(-\tfrac{3}{2}\,\mathrm{i}\,\bar{\epsilon}^{(i}\eta^{j)}) + \delta_K(\Lambda_{3K}^a) , \\
[\delta_S(\eta_1), \delta_S(\eta_2)] &= \delta_K(\tfrac{1}{2}\bar{\eta}_2\gamma^a\eta_1) .
\end{aligned} \tag{5.43}$$

with

$$\Lambda_{3K}^a = \tfrac{1}{6}\bar{\epsilon}\left(\gamma\cdot T\gamma_a - \tfrac{1}{2}\gamma_a\gamma\cdot T\right)\eta . \tag{5.44}$$

This concludes our description of the Standard Weyl multiplet.

In this chapter we have taken the first step in the superconformal tensor calculus by constructing the Standard Weyl multiplet for $\mathcal{N} = 2$ conformal supergravity theory in five dimensions. We explained how the Weyl multiplet could be obtained by gauging the superconformal algebra $F^2(4)$. The results of this chapter will be our starting point for the construction of general supergravity/matter couplings in five dimensions.

Chapter 6

Matter-couplings of conformal supergravity

6.1 Introduction

In the previous chapter the first step in the conformal program has been performed by constructing the Standard Weyl multiplet of $\mathcal{N} = 2$ conformal supergravity in five dimensions. We will now take the next step in the program and introduce the different $D = 5$ matter multiplets with eight conformal supersymmetries together with the corresponding actions (when they exist). Apart from reasons given before, there is a rather different, more general, motivation of why the $D = 5$ matter-coupled supergravities are interesting to study. The reason is that they belong to the class of theories with eight supersymmetries [149]. Such theories are especially interesting since the geometry, determined by the kinetic terms of the scalars, contains undetermined functions. Theories with thirty-two supersymmetries have no matter multiplets while the geometry of those with sixteen supersymmetries is completely determined by the number of matter multiplets. Of course, theories with four supersymmetries allow for more general geometries. The restricted class of geometries, in the case of eight supersymmetries, makes these theories especially interesting and manageable. For instance, the work of Seiberg and Witten [150, 151] heavily relies on the presence of eight supersymmetries. Theories with eight supersymmetries are thus the maximally supersymmetric theories that, on the one hand, are not completely determined by the number of matter multiplets in the model and, on the other hand, allow arbitrary functions in their definition, i.e. continuous deformations of the metric of the manifolds.

The geometry related to supersymmetric theories with eight real supercharges is called 'special geometry'. Special geometry was first found in [152, 153] for local supersymmetry and in [154, 155] for rigid supersymmetry. It occurs in Calabi-Yau compactifications of type II superstrings as the moduli space of these manifolds [156–161].

In the following sections we will introduce the relevant basic superconformal matter multiplets: the vector-tensor multiplet and the hypermultiplet. We will start by discussing them in a rigid superconformal context, at which level we already find all the interesting geometry. A local superconformal extension will be given in the last section.

Field	SU(2)	w	# d.o.f.
A_μ^I	1	0	$4n$
Y^{ijI}	3	2	$3n$
σ^I	1	1	$1n$
ψ^{iI}	2	$3/2$	$8n$

Table 6.1: The off-shell non-Abelian vector multiplet, where n labels the number of vector multiplets.

6.2 The vector-tensor multiplet

In this section, we will discuss superconformal vector multiplets that transform in arbitrary representations of the gauge group. From work on $\mathcal{N} = 2$, $D = 5$ Poincaré matter couplings [73] it is known that vector multiplets transforming in representations other than the adjoint have to be dualized to tensor fields. We define a vector-tensor multiplet to be a vector multiplet transforming in a reducible representation that contains the adjoint representation as well as another, arbitrary representation.

We will show that the analysis of [73] can be extended to superconformal vector multiplets. In doing this we will generalize the gauge transformations for the tensor fields [73] by allowing them to transform into the field-strengths for the adjoint gauge fields. These more general gauge transformations are consistent with supersymmetry, even after breaking the conformal symmetry.

The vector-tensor multiplet contains *a priori* an arbitrary number of tensor fields. The restriction to an even number of tensor fields is not imposed by the closure of the algebra. If one demands that the field equations do not contain tachyonic modes, an even number is required [68]. Closely related to this is the fact that one can only construct an action for an even number of tensor multiplets. But supersymmetry without an action allows the more general possibility. Note that these main results are independent of the use of superconformal or super-Poincaré algebras.

6.2.1 Adjoint representation

We will start with giving the transformation rules for a vector multiplet in the adjoint representation [133]. An off-shell vector multiplet has 8 + 8 real degrees of freedom whose SU(2) labels and Weyl weights we have indicated in table 6.1.

The gauge transformations that we consider satisfy the commutation relations ($I = 1, \ldots, n$)

$$\left[\delta_G(\Lambda_1^I), \delta_G(\Lambda_2^J)\right] = \delta_G(\Lambda_3^K)\,, \qquad \Lambda_3^K = g\Lambda_1^I\Lambda_2^J f_{IJ}{}^K\,. \tag{6.1}$$

The gauge fields A_μ^I ($\mu = 0, 1, \ldots, 4$) and general matter fields of the vector multiplet as e.g. X^I

transform under gauge transformations with parameters Λ^I according to

$$\delta_G(\Lambda^J)A^I_\mu = \partial_\mu\Lambda^I + gA^J_\mu f_{JK}{}^I\Lambda^K\,, \qquad \delta_G(\Lambda^J)X^I = -g\Lambda^J f_{JK}{}^I X^K\,, \tag{6.2}$$

where g is the coupling constant of the gauge group G. The expression for the gauge-covariant derivative of X^I and the field-strengths are given by

$$\mathcal{D}_\mu X^I = \partial_\mu X^I + gA^J_\mu f_{JK}{}^I X^K\,, \qquad F^I_{\mu\nu} = 2\partial_{[\mu}A^I_{\nu]} + gf_{JK}{}^I A^J_\mu A^K_\nu\,. \tag{6.3}$$

The field-strength satisfies the Bianchi identity

$$\mathcal{D}_{[\mu}F^I_{\nu\lambda]} = 0\,. \tag{6.4}$$

The rigid Q- and S-supersymmetry transformation rules for the off-shell Yang-Mills multiplet are given by [133]

$$\begin{aligned}
\delta A^I_\mu &= \tfrac{1}{2}\bar\epsilon\gamma_\mu\psi^I\,,\\
\delta Y^{ijI} &= -\tfrac{1}{2}\bar\epsilon^{(i}\not{\mathcal{D}}\psi^{j)I} - \tfrac{1}{2}\,\mathrm{i}\,g\bar\epsilon^{(i}f_{JK}{}^I\sigma^J\psi^{j)K} + \tfrac{1}{2}\,\mathrm{i}\,\bar\eta^{(i}\psi^{j)I}\,,\\
\delta\psi^{iI} &= -\tfrac{1}{4}\gamma\cdot F^I\epsilon^i - \tfrac{1}{2}\,\mathrm{i}\,\not{\mathcal{D}}\sigma^I\epsilon^i - Y^{ijI}\epsilon_j + \sigma^I\eta^i\,,\\
\delta\sigma^I &= \tfrac{1}{2}\,\mathrm{i}\,\bar\epsilon\psi^I\,.
\end{aligned} \tag{6.5}$$

The commutator of two Q-supersymmetry transformations yields a translation with an extra G-transformation

$$[\delta(\epsilon_1), \delta(\epsilon_2)] = \delta_P\left(\tfrac{1}{2}\bar\epsilon_2\gamma_\mu\epsilon_1\right) + \delta_G\left(-\tfrac{1}{2}\,\mathrm{i}\,\sigma\bar\epsilon_2\epsilon_1\right). \tag{6.6}$$

Note that even though we are considering rigid superconformal symmetry, the algebra (6.6) contains a field-dependent term on the right-hand side. Such soft terms are commonplace in local superconformal symmetry but here they already appear at the rigid level. In Hamiltonian language, it means that the algebra is satisfied modulo constraints.

6.2.2 Reducible representation

Starting from n vector multiplets we now wish to consider a more general set of fields $\mathcal{H}^{\widetilde{I}}_{\mu\nu}$ ($\widetilde{I} = 1,\ldots,n+m$). We write $\mathcal{H}^{\widetilde{I}}_{\mu\nu} = \{F^I_{\mu\nu}, B^M_{\mu\nu}\}$ with $\widetilde{I} = (I, M)$ ($I = 1,\ldots,n$; $M = n+1,\ldots n+m$). The first part of these fields corresponds to the generators in the adjoint representation. These are the fields that we used in subsection 6.2.1. The other fields form a tensor multiplet which may transform in an arbitrary, possibly reducible, representation. Properties of the tensor multiplet fields are given in table 6.2. The representation matrix can be written as

$$(t_I)_{\widetilde{J}}{}^{\widetilde{K}} = \begin{pmatrix} (t_I)_J{}^K & (t_I)_J{}^N \\ (t_I)_M{}^K & (t_I)_M{}^N \end{pmatrix}, \qquad \begin{cases} I,J,K = 1,\ldots,n \\ M,N = n+1,\ldots,n+m\,. \end{cases} \tag{6.7}$$

It is understood that the $(t_I)_J{}^K$ are in the adjoint representation, i.e.

$$(t_I)_J{}^K = f_{IJ}{}^K\,. \tag{6.8}$$

Field	SU(2)	w	# d.o.f.
$B^M_{\mu\nu}$	1	0	$3m$
ϕ^M	1	1	$1m$
λ^{iM}	2	3/2	$4m$

Table 6.2: The on-shell tensor multiplet, where m labels the number of tensor multiplets.

If $m \neq 0$, then the representation $(t_I)_{\widetilde{J}}{}^{\widetilde{K}}$ is reducible. We will see that this representation can be more general than assumed so far in treatments of vector-tensor multiplet couplings. The requirement that m is even will only appear when we demand the existence of an action in section 6.4.2, or if we require absence of tachyonic modes. The matrices t_I satisfy commutation relations

$$[t_I, t_J] = -f_{IJ}{}^K t_K\,, \qquad \text{or} \qquad t_{I\widetilde{N}}{}^{\widetilde{M}} t_{J\widetilde{M}}{}^{\widetilde{L}} - t_{J\widetilde{N}}{}^{\widetilde{M}} t_{I\widetilde{M}}{}^{\widetilde{L}} = -f_{IJ}{}^K t_{K\widetilde{N}}{}^{\widetilde{L}}\,. \tag{6.9}$$

If the index $\tilde{L}$ is a vector index, then this relation is satisfied using the matrices as in (6.8).

Requiring the closure of the superconformal algebra, we find Q- and S-supersymmetry transformation rules for the vector-tensor multiplet and a set of constraints. The transformations are

$$\begin{aligned}
\delta\mathcal{H}^{\widetilde{I}}_{\mu\nu} &= -\bar{\epsilon}\gamma_{[\mu}\mathcal{D}_{\nu]}\psi^{\widetilde{I}} + \mathrm{i}\, g\bar{\epsilon}\gamma_{\mu\nu} t_{(\widetilde{J}\widetilde{K})}{}^{\widetilde{I}}\sigma^{\widetilde{J}}\psi^{\widetilde{K}} + \mathrm{i}\,\bar{\eta}\gamma_{\mu\nu}\psi^{\widetilde{I}}\,,\\
\delta Y^{ij\widetilde{I}} &= -\tfrac{1}{2}\bar{\epsilon}^{(i}\not{\mathcal{D}}\psi^{j)\widetilde{I}} - \tfrac{1}{2}\,\mathrm{i}\, g\bar{\epsilon}^{(i}\Big(t_{[\widetilde{J}\widetilde{K}]}{}^{\widetilde{I}} - 3t_{(\widetilde{J}\widetilde{K})}{}^{\widetilde{I}}\Big)\sigma^{\widetilde{J}}\psi^{j)\widetilde{K}} + \tfrac{1}{2}\,\mathrm{i}\,\bar{\eta}^{(i}\psi^{j)\widetilde{I}}\,,\\
\delta\psi^{i\widetilde{I}} &= -\tfrac{1}{4}\gamma\cdot\mathcal{H}^{\widetilde{I}}\epsilon^i - \tfrac{1}{2}\,\mathrm{i}\,\not{\mathcal{D}}\sigma^{\widetilde{I}}\epsilon^i - Y^{ij\widetilde{I}}\epsilon_j + \tfrac{1}{2}g t_{(\widetilde{J}\widetilde{K})}{}^{\widetilde{I}}\sigma^{\widetilde{J}}\sigma^{\widetilde{K}}\epsilon^i + \sigma^{\widetilde{I}}\eta^i\,,\\
\delta\sigma^{\widetilde{I}} &= \tfrac{1}{2}\,\mathrm{i}\,\bar{\epsilon}\psi^{\widetilde{I}}\,.
\end{aligned} \tag{6.10}$$

The curly derivatives denote gauge-covariant derivatives as in (6.3) with the replacement of structure constants by general matrices t_I according to (6.8). We have extended the range of the generators from I to $\widetilde{I}$ in order to simplify the transformation rules with the understanding that

$$(t_M)_{\widetilde{J}}{}^{\widetilde{K}} = 0\,. \tag{6.11}$$

We find that the supersymmetry algebra (6.6) is satisfied provided the representation matrices are restricted to

$$t_{(\widetilde{J}\widetilde{K})}{}^{I} = 0\,, \tag{6.12}$$

and provided the following two constraints on the fields are imposed:

$$\begin{aligned}
L^{ij\widetilde{I}} &\equiv t_{(\widetilde{J}\widetilde{K})}{}^{\widetilde{I}}\Big(2\sigma^{\widetilde{J}}Y^{ij\widetilde{K}} - \tfrac{1}{2}\,\mathrm{i}\,\bar{\psi}^{i\widetilde{J}}\psi^{j\widetilde{K}}\Big) = 0\,, && (6.13)\\
E^{\widetilde{I}}_{\mu\nu\lambda} &\equiv \frac{3}{g}\mathcal{D}_{[\mu}\mathcal{H}_{\nu\lambda]}{}^{\widetilde{I}} - \varepsilon_{\mu\nu\lambda\rho\sigma} t_{(\widetilde{J}\widetilde{K})}{}^{\widetilde{I}}\Big(\sigma^{\widetilde{J}}\mathcal{H}^{\rho\sigma\widetilde{K}} + \tfrac{1}{4}\,\mathrm{i}\,\bar{\psi}^{\widetilde{J}}\gamma^{\rho\sigma}\psi^{\widetilde{K}}\Big) = 0\,. && (6.14)
\end{aligned}$$

For $\widetilde{I} = I$, the constraint (6.14) reduces to the Bianchi identity (6.4). The tensor $F^I_{\mu\nu}$ can therefore be seen as the curl of a gauge vector A^I_μ. Moreover, the constraint (6.13) is trivially satisfied for

$\widetilde{I} = I$. We conclude that the fields with indices $\widetilde{I} = I$ form an off-shell vector multiplet in the adjoint representation of the gauge group.

On the other hand, when $\widetilde{I} = M$, the constraint (6.14) does not permit the fields $B^M_{\mu\nu}$ to be written as the curl of a gauge field and they should be seen as independent tensor fields. Instead, the constraint (6.14) is a massive self-duality condition that puts the tensors $B^M_{\mu\nu}$ *on-shell.* The constraint (6.13) implicitly allows us to eliminate the fields Y^{ijM} altogether. The general vector-tensor multiplet can then be interpreted as a set of m on-shell tensor multiplets in the background of n off-shell vector multiplets.

Using (6.12) we have reduced the representation matrices t_I to the following block-upper-triangular form:

$$(t_I)_{\widetilde{J}}{}^{\widetilde{K}} = \begin{pmatrix} f_{IJ}{}^K & (t_I)_J{}^N \\ 0 & (t_I)_M{}^N \end{pmatrix}. \tag{6.15}$$

In [73] it is mentioned that, "since terms of the form $B^M \wedge F^I \wedge A^J$ appear to be impossible to supersymmetrize in a gauge invariant way (except possibly in very special cases) we shall also assume that $C_{MIJ} = 0$". This corresponds, as we will see in section 6.4.2, to the assumption that the representation is completely reducible, i.e. $t_{IJ}{}^N = 0$, meaning that gauge transformations do not mix the pure Yang-Mills field-strengths and the tensor fields. However, we find that off-diagonal generators *are* allowed, both when requiring closure of the superconformal algebra and when writing down an action. We thus allow reducible, but not necessarily completely reducible representations.

The constraints (6.13) and (6.14), with $\widetilde{I} = M$, do not yet form a supersymmetric set; successive variations under S-supersymmetry and Q-supersymmetry lead to the equations of motion for the spinors $\psi^{i\widetilde{I}}$ and scalars $\sigma^{\widetilde{I}}$ [86]. Although this procedure generates a set of constraints, transforming to each other under Q- and S-supersymmetry, they do not seem to form a multiplet by themselves. That is, the algebra is *not* realized on this set of transformation rules.

6.3 The hypermultiplet

In this subsection, we discuss hypermultiplets in five dimensions. As for the tensor multiplets, there is in general no known off-shell formulation with a finite number of auxiliary fields.[1] Therefore, the supersymmetry algebra already leads to the equations of motion.

A single hypermultiplet contains four real scalars and two spinors subject to the symplectic Majorana reality condition. For r hypermultiplets, we introduce real scalars $q^X(x)$, with $X = 1, \ldots, 4r$, and spinors $\zeta^A(x)$ with $A = 1, \ldots, 2r$. The properties of the hypermultiplet fields are given in table 6.3. To formulate the symplectic Majorana condition, we introduce two matrices $\rho_A{}^B$ and $E_i{}^j$, with

$$\rho\rho^* = -\mathbb{1}_{2r}, \qquad EE^* = -\mathbb{1}_2\,. \tag{6.16}$$

This defines symplectic Majorana conditions for the fermions and supersymmetry transformation parameters [163]:

$$\alpha C\gamma_0 \zeta^B \rho_B{}^A = \left(\zeta^A\right)^*, \qquad \alpha C\gamma_0 \epsilon^j E_j{}^i = \left(\epsilon^i\right)^*, \tag{6.17}$$

[1] An off-shell tensor-formulation *can* be constructed by extending the algebra with central charges [162]. A similar procedure could also possibly be used to obtain an off-shell hypermultiplet.

Field	SU(2)	w	# d.o.f.
q^X	2	3/2	$4r$
ζ^A	1	2	$4r$

Table 6.3: The on-shell hypermultiplet, where r labels the number of hypermultiplets.

where C is the charge conjugation matrix, and α is an irrelevant number of modulus 1. We can always adopt the basis where $E_i{}^j = \varepsilon_{ij}$, and will further restrict to that.

The scalar fields are interpreted as coordinates of some target space, and requiring the on-shell closure of the superconformal algebra imposes certain conditions on the target space, which we derive below. Superconformal hypermultiplets in four space-time dimensions were discussed in [164]; our discussion is somehow similar, but we extend it to the case where an action is not needed, in the spirit explained in [149].

6.3.1 Rigid supersymmetry

We will show how the closure of the supersymmetry transformation laws leads to a 'hypercomplex manifold'. The closure of the algebra on the bosons leads to the defining equations for this geometry, whereas the closure of the algebra on the fermions and its further consistency leads to equations of motion in this geometry, independent of an action.

The rigid supersymmetry transformations for the hypermultiplet are given by

$$\begin{aligned} \delta(\epsilon)q^X &= -\mathrm{i}\,\bar\epsilon^i\zeta^A f^X_{iA}\,, \\ \delta(\epsilon)\zeta^A &= \tfrac{1}{2}\,\mathrm{i}\,\partial\!\!\!/ q^X f^{iA}_X\epsilon_i - \zeta^B\omega_{XB}{}^A(\delta(\epsilon)q^X)\,, \end{aligned} \tag{6.18}$$

where the functions $f^X_{iA}(q)$, $f^{iA}_X(q)$ and $\omega_{XB}{}^A(q)$ satisfy reality properties consistent with reality of q^X and the symplectic Majorana conditions, e.g.

$$\left(f^{iA}_X\right)^* = f^{jB}_X E_j{}^i\rho_B{}^A\,, \qquad \left(\omega_{XA}{}^B\right)^* = \left(\rho^{-1}\omega_X\rho\right)_A{}^B\,. \tag{6.19}$$

A priori the functions f^X_{iA} and f^{iA}_X are independent, but the commutator of two supersymmetries on the scalars only gives a translation if one imposes

$$\begin{aligned} f^{iA}_Y f^X_{iA} &= \delta^X_Y\,, \qquad f^{iA}_X f^X_{jB} = \delta^i_j\delta^A_B\,, \\ \mathfrak{D}_Y f^X_{iB} &\equiv \partial_Y f^X_{iB} - \omega_{YB}{}^A f^X_{iA} + \Gamma_{ZY}{}^X f^Z_{iB} = 0\,, \end{aligned} \tag{6.20}$$

where $\Gamma_{XY}{}^Z$ is some object, symmetric in the lower indices. This means that f^X_{iA} can be interpreted as vielbeins on the hyperscalar manifold, i.e. f^X_{iA} and f^{iA}_X are each others inverse and are covariantly constant with connections Γ and ω. It also implies that ρ is covariantly constant. The conditions (6.20) encode all the constraints on the target space that follow from imposing the supersymmetry algebra. Below, we show that there are no further geometrical constraints coming from the fermion commutator; instead this commutator defines the equations of motion for the on-shell hypermultiplet.

Reparametrizations

The supersymmetry transformation rules (6.18) are covariant with respect to two kinds of reparametrizations. The first ones are the target space diffeomorphisms, $q^X \to \widetilde{q}^X(q)$, under which f_{iA}^X transforms as a vector, $\omega_{XA}{}^B$ as a one-form, and $\Gamma_{XY}{}^Z$ as a connection. The second set are the reparametrizations which act on the tangent space indices $A, B, \ldots$ On the fermions, they act as

$$\zeta^A \to \widetilde{\zeta}^A(q) = \zeta^B U_B{}^A(q)\,, \tag{6.21}$$

where $U(q)_A{}^B$ is any invertible matrix. In general, such a transformation brings us into a basis where the fermions depend on the scalars q^X. In this sense, the hypermultiplet is written in a special basis where q^X and ζ^A are independent fields. The supersymmetry transformation rules (6.18) are covariant under (6.21) if we transform $f_X^{iA}(q)$ as a vector and $\omega_{XA}{}^B$ as a connection,

$$\omega_{XA}{}^B \to \widetilde{\omega}_{XA}{}^B = [(\partial_X U^{-1})U + U^{-1}\omega_X U]_A{}^B\,. \tag{6.22}$$

These considerations lead us to define the covariant variation of the fermions:

$$\widehat{\delta}\zeta^A \equiv \delta\zeta^A + \zeta^B\omega_{XB}{}^A\delta q^X\,, \tag{6.23}$$

for any transformation δ (supersymmetry, conformal transformations,...). Two models related by either target space diffeomorphisms or fermion reparametrizations of the form (6.21) are equivalent; they are different coordinate descriptions of the same system. Thus, in a covariant formalism, the fermions can be functions of the scalars. However, the expression $\partial_X\zeta^A$ only makes sense if one compares different bases. But in the same way also, the expression $\zeta^B\omega_{XB}{}^A$ only makes sense if one compares different bases, as the connection has no absolute value. The only covariant object is the covariant derivative

$$\mathfrak{D}_X\zeta^A \equiv \partial_X\zeta^A + \zeta^B\omega_{XB}{}^A\,. \tag{6.24}$$

We will frequently use the covariant transformations (6.23). It can similarly be used on target-space vectors or tensors. E.g. for a quantity Δ^X:

$$\widehat{\delta}\Delta^X = \delta\Delta^X + \Delta^Y\Gamma_{ZY}{}^X\,\delta q^Z\,. \tag{6.25}$$

Geometry

The geometry of the target space is that of a *hypercomplex* manifold. It is a weakened version of hyperkähler geometry where no hermitian covariantly constant metric is defined. We refer the reader to appendix C for an introduction to these manifolds, references and the mathematical context in which they can be situated.

The crucial ingredient is a triplet of complex structures, the hypercomplex structure, defined as

$$J^\alpha{}_X{}^Y \equiv -\mathrm{i}\, f_X^{iA}(\sigma^\alpha)_i{}^j f_{jA}^Y\,. \tag{6.26}$$

Using (6.20), they are covariantly constant and satisfy the quaternion algebra

$$J^\alpha J^\beta = -\mathbb{1}_{4r}\delta^{\alpha\beta} + \varepsilon^{\alpha\beta\gamma}J^\gamma\,. \tag{6.27}$$

At some places we also use a doublet notation, for which

$$J_X{}^Y{}_i{}^j \equiv \mathrm{i}\, J^\alpha{}_X{}^Y (\sigma^\alpha)_i{}^j = 2 f_X^{jA} f_{iA}^Y - \delta_i^j \delta_X^Y \,. \tag{6.28}$$

The same transition between doublet and triplet notation is also used for other SU(2)-valued quantities.

The holonomy group of such a space is contained in $\mathrm{G}\ell(r,\mathbb{H}) = \mathrm{SU}^*(2r)\times\mathrm{U}(1)$, the group of transformations acting on the A, B-indices. This follows from the integrability conditions on the covariantly constant vielbeins f_X^{iA}, which relates the curvatures of the $\omega_{XA}{}^B$ and $\Gamma_{XY}{}^Z$ connections (see appendix C.2 for conventions on the curvatures),

$$R_{XYZ}{}^W = f_{iA}^W f_Z^{iB} \mathcal{R}_{XYB}{}^A \,, \qquad \delta_j^i\, \mathcal{R}_{XYB}{}^A = f_W^{iA} f_{jB}^Z R_{XYZ}{}^W \,, \tag{6.29}$$

such that the Riemann curvature only lies in $\mathrm{G}\ell(r,\mathbb{H})$. Moreover, from the cyclicity properties of the Riemann tensor, it follows that

$$\begin{aligned} f_{Ci}^X f_{jD}^Y \mathcal{R}_{XYB}{}^A &= -\tfrac{1}{2}\varepsilon_{ij} W_{CDB}{}^A \,, \\ W_{CDB}{}^A \equiv f_C^{iX} f_{iD}^Y \mathcal{R}_{XYB}{}^A &= \tfrac{1}{2} f_C^{iX} f_{iD}^Y f_{jB}^Z f_W^{Aj} R_{XYZ}{}^W \,, \end{aligned} \tag{6.30}$$

where W is symmetric in all its three lower indices. For a more detailed discussion on hypercomplex manifolds and their curvature relations, we refer to appendix C. There we show that, in contrast with hyperkähler manifolds, hypercomplex manifolds are not necessarily Ricci flat; instead, the Ricci tensor is antisymmetric and defines a closed two-form.

So far we have only used the commutator of supersymmetry on the hyperscalars, and this led us to the geometry of hypercomplex manifolds. Before continuing, we want to see what the independent objects are that determine the theory, and what the independent constraints are. We start in the supersymmetric theory from the vielbeins f_X^{iA}. They have to be real in the sense of (6.19) and invertible. With these vielbeins, we can construct the complex structures as in (6.26). In the developments above, the only remaining independent equation is the covariant constancy of the vielbein in (6.20). This equation contains the affine connection $\Gamma_{XY}{}^Z$ and the $\mathrm{G}\ell(r,\mathbb{H})$-connection $\omega_{XA}{}^B$. These two objects can be determined from the vielbeins if and only if the ('diagonal') Nijenhuis tensor (C.24) vanishes. Indeed, for vanishing Nijenhuis tensor, the 'Obata'-connection [165]

$$\Gamma_{XY}{}^Z = -\tfrac{1}{6}\left(2\partial_{(X} J^\alpha{}_{Y)}{}^W + \varepsilon^{\alpha\beta\gamma} J^\beta{}_{(X}{}^U \partial_{|U|} J^\gamma{}_{Y)}{}^W\right) J^\alpha{}_W{}^Z \,, \tag{6.31}$$

leads to covariantly constant complex structures. Moreover, one can show that any torsionless connection that leaves the complex structures invariant is equal to this Obata connection (similar to the fact that a connection that leaves a metric invariant is the Levi-Civita connection). With this connection one can then construct the $\mathrm{G}\ell(r,\mathbb{H})$-connection

$$\omega_{XA}{}^B = \tfrac{1}{2} f_Y^{iB}\left(\partial_X f_{iA}^Y + \Gamma_{XZ}^Y f_{iA}^Z\right) , \tag{6.32}$$

such that the vielbeins are covariantly constant.

Dynamics

Now we consider the commutator of supersymmetry on the fermions, which will determine the equations of motion for the hypermultiplets. Using (6.20), (6.29) and (6.30), we compute the supersymmetry commutator on the fermions, and find

$$[\delta(\epsilon_1),\delta(\epsilon_2)]\zeta^A = \tfrac{1}{2}\partial_a\zeta^A\bar{\epsilon}_2\gamma^a\epsilon_1 + \tfrac{1}{4}\Gamma^A\bar{\epsilon}_2\epsilon_1 - \tfrac{1}{4}\gamma_a\Gamma^A\bar{\epsilon}_2\gamma^a\epsilon_1\,. \tag{6.33}$$

On-shell closure of the algebra on the fermion requires the last two terms to vanish. The Γ^A are therefore called non-closure functions, and define the equations of motion for the fermions,

$$\Gamma^A = \not{\mathfrak{D}}\zeta^A + \tfrac{1}{2}W_{CDB}{}^A\zeta^B\bar{\zeta}^D\zeta^C = 0\,, \tag{6.34}$$

where we have introduced the covariant derivative with respect to the transformations (6.23)

$$\mathfrak{D}_\mu\zeta^A \equiv \partial_\mu\zeta^A + (\partial_\mu q^X)\zeta^B\omega_{XB}{}^A\,. \tag{6.35}$$

By varying the fermion equation of motion under supersymmetry, we derive the corresponding equations of motion for the scalar fields:

$$\widehat{\delta}(\epsilon)\Gamma^A = \tfrac{1}{2}\,\mathrm{i}\,f^{iA}_X\epsilon_i\Delta^X\,, \tag{6.36}$$

where

$$\Delta^X = \Box q^X - \tfrac{1}{2}\bar{\zeta}^B\gamma_a\zeta^D\partial^a q^Y f^{iC}_Y f^X_{iA}W_{BCD}{}^A - \tfrac{1}{4}\mathfrak{D}_Y W_{BCD}{}^A\bar{\zeta}^E\zeta^D\bar{\zeta}^C\zeta^B f^{iY}_E f^X_{iA}\,, \tag{6.37}$$

and the covariant Laplacian is given by

$$\Box q^X = \partial_a\partial^a q^X + \left(\partial_a q^Y\right)\left(\partial^a q^Z\right)\Gamma_{YZ}{}^X\,. \tag{6.38}$$

In conclusion, the supersymmetry algebra imposes the hypercomplex constraints (6.20) and the equations of motion (6.34) and (6.37). These form a multiplet, as (6.36) has the counterpart

$$\widehat{\delta}(\epsilon)\Delta^X = -\,\mathrm{i}\,\bar{\epsilon}^i\not{\mathfrak{D}}\Gamma^A f^X_{iA} + 2\,\mathrm{i}\,\bar{\epsilon}^i\Gamma^B\bar{\zeta}^C\zeta^D f^Y_{Bi}\mathcal{R}^X{}_{YCD}\,, \tag{6.39}$$

where the covariant derivative of Γ^A is defined similar to (6.35). In the following, we will derive further constraints on the target space geometry from requiring the presence of conformal symmetry.

6.3.2 Superconformal symmetry

Now we define transformation rules for the hypermultiplet under the full (rigid) superconformal group. The scalars do not transform under special conformal transformations and special supersymmetry, but under dilatations and SU(2) transformations, we parametrize

$$\begin{aligned}\delta_D(\Lambda_D)q^X &= \Lambda_D k^X(q)\,,\\ \delta_{\mathrm{SU}(2)}(\Lambda^{ij})q^X &= \Lambda^{ij}k^X_{ij}(q)\,,\end{aligned} \tag{6.40}$$

for some unknown functions $k^X(q)$ and $k^X_{ij}(q)$.

To derive the appropriate transformation rules for the fermions, we first note that the hyperinos should be invariant under special conformal symmetry. This is due to the fact that this symmetry changes the Weyl weight with one. The special supersymmetry transformation of ζ^A can be read off from the $[K, Q]$-commutator, giving rise to

$$\delta_S(\eta^i)\zeta^A = -k^X f_X^{iA}\eta_i\,. \tag{6.41}$$

Realizing the commutator of regular and special supersymmetry (5.21) on the scalars, we determine the expression for the generator of SU(2) transformations in terms of the dilatations and complex structures,

$$k_{ij}^X = \tfrac{1}{3}k^Y J_Y{}^X{}_{ij} \qquad \text{or} \qquad k^{\alpha X} = \tfrac{1}{3}k^Y J^\alpha{}_Y{}^X\,. \tag{6.42}$$

Realizing (5.21) on the hyperinos, we determine the covariant variations

$$\widehat{\delta}_D\zeta^A = 2\Lambda_D\zeta^A\,, \qquad \widehat{\delta}_{\mathrm{SU}(2)}\zeta^A = 0\,, \tag{6.43}$$

and furthermore the commutator (5.21) only closes if we impose

$$\mathfrak{D}_Y k^X = \tfrac{3}{2}\delta_Y{}^X\,, \tag{6.44}$$

which also implies

$$\mathfrak{D}_Y k^{\alpha X} = \tfrac{1}{2}J^\alpha{}_Y{}^X\,. \tag{6.45}$$

Note that (6.44) is imposed by supersymmetry. In a more usual derivation, where one considers symmetries of the Lagrangian, we would find this constraint by imposing dilatation invariance of the action, see (5.11). Our result, though, does not require the existence of an action. The relations (6.44) and (6.42) further restrict the geometry of the target space, and it is easy to derive that the Riemann tensor has four zero eigenvectors,

$$k^X R_{XYZ}{}^W = 0\,, \qquad k^{\alpha X} R_{XYZ}{}^W = 0\,. \tag{6.46}$$

Also, under dilatations and SU(2) transformations, the hypercomplex structure is scale invariant and rotated into itself,

$$\begin{aligned}\Lambda_D\left(k^Z\partial_Z J^\alpha{}_X{}^Y - \partial_Z k^Y J^\alpha{}_X{}^Z + \partial_X k^Z J^\alpha{}_Z{}^Y\right) &= 0\,,\\ \Lambda^\beta\left(k^{\beta Z}\partial_Z J^\alpha{}_X{}^Y - \partial_Z k^{\beta Y} J^\alpha{}_X{}^Z + \partial_X k^{\beta Z} J^\alpha{}_Z{}^Y\right) &= -\epsilon^{\alpha\beta\gamma}\Lambda^\beta J^\gamma{}_X{}^Y\,.\end{aligned} \tag{6.47}$$

All properties derived above are similar to those derived from superconformal hypermultiplets in four space-time dimensions [164, 166]. There, the $\mathrm{Sp}(1)\times \mathrm{G}\ell(r,\mathbb{H})$ sections, or simply, hypercomplex sections, were introduced

$$A^{iB}(q) \equiv k^X f_X^{iB}\,, \qquad (A^{iB})^* = A^{jC}E_j{}^i\rho_C{}^B\,, \tag{6.48}$$

which allow for a coordinate independent description of the target space. This means that all equations and transformation rules for the sections can be written without the occurrence of the q^X fields.

6.3.3 Symmetries

We now assume the action of a symmetry group on the hypermultiplet. We have no action, but the 'symmetry' operation should leave invariant the set of equations of motion. The symmetry algebra must commute with the supersymmetry algebra (and later with the full superconformal algebra). This leads to hypermultiplet couplings to a non-Abelian gauge group G. The symmetries are parametrized by

$$\begin{aligned} \delta_G q^X &= -g\Lambda_G^I k_I^X(q)\,, \\ \widehat{\delta}_G \zeta^A &= -g\Lambda_G^I t_{IB}{}^A(q)\zeta^B\,. \end{aligned} \tag{6.49}$$

The vectors k_I^X depend on the scalars and generate the algebra of G with structure constants $f_{IJ}{}^K$,

$$k_{[I|}^Y \partial_Y k_{|J]}^X = -\tfrac{1}{2} f_{IJ}{}^K k_K^X\,. \tag{6.50}$$

The commutator of two gauge transformations (6.1) on the fermions requires the following constraint on the field-dependent matrices $t_I(q)$,

$$[t_I, t_J]_B{}^A = -f_{IJ}{}^K t_{KB}{}^A - 2k_{[I|}^X \mathfrak{D}_X t_{|J]B}{}^A + k_I^X k_J^Y \mathcal{R}_{XYB}{}^A\,. \tag{6.51}$$

Requiring the gauge transformations to commute with supersymmetry leads to further relations between the quantities k_I^X and $t_{IB}{}^A$, allowing us to determine $t_I(q)$ in terms of the vielbeins f_X^{iA} and the vectors k_I^X

$$t_{IA}{}^B = \tfrac{1}{2} f_{iA}^Y \mathfrak{D}_Y k_I^X f_X^{iB}\,, \tag{6.52}$$

if the vectors k_I^X satisfy the constraint

$$f_A^{Y(i} f_X^{j)B} \mathfrak{D}_Y k_I^X = 0\,. \tag{6.53}$$

Equation (6.53) can be expressed as the vanishing of the commutator of $\mathfrak{D}_Y k_I^X$ with the complex structures:[2]

$$\left(\mathfrak{D}_X k_I^Y\right) J^\alpha{}_Y{}^Z = J^\alpha{}_X{}^Y \left(\mathfrak{D}_Y k_I^Z\right)\,, \tag{6.54}$$

which is equivalent to the vanishing of the Lie derivative of the complex structure in the direction of the vector k_I

$$(\mathcal{L}_{k_I} J^\alpha)_X{}^Y \equiv k_I^Z \partial_Z J^\alpha{}_X{}^Y - \partial_Z k_I^Y J^\alpha{}_X{}^Z + \partial_X k_I^Z J^\alpha{}_Z{}^Y = 0\,. \tag{6.55}$$

According to part C.5 of the appendix, this means that (6.55) is a special case of the statement that the vector k_I normalizes the hypercomplex structures. The vanishing of this Lie derivative, or (6.53), is expressed by saying that the gauge transformations act *triholomorphic*. Thus, it says that all the symmetries are embedded in $\mathrm{G}\ell(r,\mathbb{H})$.

Vanishing of the gauge-supersymmetry commutator on the fermions requires

$$\mathfrak{D}_Y t_{IA}{}^B = k_I^X \mathcal{R}_{YXA}{}^B\,. \tag{6.56}$$

Using (6.52) this implies a new constraint,

$$\mathfrak{D}_X \mathfrak{D}_Y k_I^Z = R_{XWY}{}^Z k_I^W\,. \tag{6.57}$$

[2]This can be seen directly from lemma C.2.2 in appendix C.

Note that this equation is in general true for any Killing vector of a metric. As we have no metric here, we could not rely on this fact, but here the algebra imposes this equation. It turns out that (6.53) and (6.57) are sufficient for the full commutator algebra to hold.

A further identity can be derived: substituting (6.56) into (6.51) one gets

$$[t_I, t_J]_B{}^A = -f_{IJ}{}^K t_{KB}{}^A - k_I^X k_J^Y \mathcal{R}_{XYB}{}^A\,. \tag{6.58}$$

The group of gauge symmetries should also commute with the superconformal algebra, in particular with dilatations and SU(2) transformations. This leads to

$$k^Y \mathfrak{D}_Y k_I^X = \tfrac{3}{2} k_I^X\,, \qquad k^{\alpha Y}\mathfrak{D}_Y k_I^X = \tfrac{1}{2} k_I^Y J^\alpha{}_Y{}^X\,, \tag{6.59}$$

coming from the scalars, and there are no new constraints from the fermions or from other commutators. Since $\mathfrak{D}_Y k_I^X$ commutes with $J^\alpha{}_Y{}^X$, the second equation in (6.59) is a consequence of the first one.

In the above analysis, we have taken the parameters Λ^I to be constants. In the following, we also allow for local gauge transformations. The gauge coupling is done by introducing vector multiplets and defining the covariant derivatives

$$\begin{aligned}
\mathfrak{D}_\mu q^X &\equiv \partial_\mu q^X + g A_\mu^I k_I^X\,,\\
\mathfrak{D}_\mu \zeta^A &\equiv \partial_\mu \zeta^A + \partial_\mu q^X \omega_{XB}{}^A \zeta^B + g A_\mu^I t_{IB}{}^A \zeta^B\,.
\end{aligned} \tag{6.60}$$

The commutator of two supersymmetries should now also contain a local gauge transformation, in the same way as for the multiplets of the previous sections, see (6.6). This requires an extra term in the supersymmetry transformation law of the fermion,

$$\widehat{\delta}(\epsilon)\zeta^A = \tfrac{1}{2}\,\mathrm{i}\,\not{\mathfrak{D}} q^X f_X^{iA}\epsilon_i + \tfrac{1}{2} g\sigma^I k_I^X f_{iX}^A \epsilon^i\,. \tag{6.61}$$

With this additional term, the commutator on the scalars closes, whereas on the fermions, it determines the equations of motion

$$\Gamma^A \equiv \not{\mathfrak{D}}\zeta^A + \tfrac{1}{2} W_{BCD}{}^A \bar\zeta^C \zeta^D \zeta^B - g(\,\mathrm{i}\, k_I^X f_{iX}^A \psi^{iI} + \,\mathrm{i}\,\zeta^B \sigma^I t_{IB}{}^A) = 0\,, \tag{6.62}$$

with the same conventions as in (6.33).

Acting on Γ^A with supersymmetry determines the equation of motion for the scalars

$$\begin{aligned}
\Delta^X = {} & \square q^X - \tfrac{1}{2}\bar\zeta^B \gamma_a \zeta^D \mathfrak{D}^a q^Y f_Y^{iC} f_{iA}^X W_{BCD}{}^A - \tfrac{1}{4}\mathfrak{D}_Y W_{BCD}{}^A \bar\zeta^E \zeta^D \bar\zeta^C \zeta^B f_E^{iY} f_{iA}^X\\
& - g\left(2\,\mathrm{i}\,\bar\psi^{iI}\zeta^B t_{IB}{}^A f_{iA}^X - k_I^Y J_Y{}^X{}_{ij} Y^{ijI}\right) + g^2 \sigma^I \sigma^J \mathfrak{D}_Y k_I^X k_J^Y\,.
\end{aligned} \tag{6.63}$$

The first line is the same as in (6.37), the second line contains the corrections due to the gauging. The gauge-covariant Laplacian is here given by

$$\square q^X = \partial_a \mathfrak{D}^a q^X + g\mathfrak{D}_a q^Y \partial_Y k_I^X A^{aI} + \mathfrak{D}_a q^Y \mathfrak{D}^a q^Z \Gamma_{YZ}^X\,. \tag{6.64}$$

The equations of motions Γ^A and Δ^X still satisfy the same algebra with (6.36) and (6.39).

6.4 Rigid superconformal actions

In this section, we will present rigid superconformal actions for the multiplets discussed in the previous section. We will see that demanding the existence of an action is more restrictive than only considering equations of motion. For the different multiplets, we find that new geometric objects have to be introduced.

6.4.1 Vector multiplet action

The rigid superconformal invariant action describing n vector multiplets was obtained from tensor calculus using an intermediate linear multiplet in [167]. The Abelian part can be obtained by just taking the cubic action of the improved vector multiplet as given in [131], adding indices I, J, K on the fields and multiplying with the symmetric tensor C_{IJK}. For the non-Abelian case, we need conditions expressing the gauge invariance of this tensor:

$$f_{I(J}{}^{H} C_{KL)H} = 0\,. \tag{6.65}$$

Moreover one has to add a few more terms, e.g. to complete the Chern–Simons term to its non-Abelian form. This leads to the action

$$\begin{aligned}\mathcal{L}_{\text{vector}} \;=\;& \Big[\Big(-\tfrac{1}{4}F^I_{\mu\nu}F^{\mu\nu J} - \tfrac{1}{2}\bar{\psi}^I \not{\mathcal{D}}\psi^J - \tfrac{1}{2}\mathcal{D}_a\sigma^I\mathcal{D}^a\sigma^J + Y^I_{ij}Y^{ijJ}\Big)\sigma^K \\ & - \tfrac{1}{24}\varepsilon^{\mu\nu\lambda\rho\sigma}A^I_\mu\Big(F^J_{\nu\lambda}F^K_{\rho\sigma} + \tfrac{1}{2}g[A_\nu, A_\lambda]^J F^K_{\rho\sigma} + \tfrac{1}{10}g^2[A_\nu, A_\lambda]^J[A_\rho, A_\sigma]^K\Big) \\ & - \tfrac{1}{8}\,\mathrm{i}\,\bar{\psi}^I\gamma\cdot F^J\psi^K - \tfrac{1}{2}\,\mathrm{i}\,\bar{\psi}^{iI}\psi^{jJ}Y^K_{ij} + \tfrac{1}{4}\,\mathrm{i}\,g\bar{\psi}^L\psi^H\sigma^I\sigma^J f_{LH}{}^K\Big]C_{IJK}\,.\end{aligned} \tag{6.66}$$

The equations of motion for the fields of the vector multiplet following from the action (6.66) are

$$0 = L^{ij}_I = \varphi^i_I = E^a_I = N_I\,, \tag{6.67}$$

where we have defined

$$\begin{aligned} L^{ij}_I \;\equiv\;& C_{IJK}\Big(2\sigma^J Y^{ijK} - \tfrac{1}{2}\,\mathrm{i}\,\bar{\psi}^{iJ}\psi^{jK}\Big)\,, \\ \varphi^i_I \;\equiv\;& C_{IJK}\Big(\mathrm{i}\,\sigma^J\not{\mathcal{D}}\psi^{iK} + \tfrac{1}{2}\,\mathrm{i}\,(\not{\mathcal{D}}\sigma^J)\psi^{iK} + Y^{ikJ}\psi^K_k - \tfrac{1}{4}\gamma\cdot F^J\psi^{iK}\Big) \\ & - gC_{IJK}f_{LH}{}^K\sigma^J\sigma^L\psi^{iH}\,, \\ E_{aI} \;\equiv\;& C_{IJK}\Big[\mathcal{D}^b\Big(\sigma^J F_{ba}{}^K + \tfrac{1}{4}\,\mathrm{i}\,\bar{\psi}^J\gamma_{ba}\psi^K\Big) - \tfrac{1}{8}\varepsilon_{abcde}F^{bcJ}F^{deK}\Big] \\ & - \tfrac{1}{2}gC_{JKL}f_{IH}{}^J\sigma^K\bar{\psi}^L\gamma_a\psi^H - gC_{JKH}f_{IL}{}^J\sigma^K\sigma^L\mathcal{D}_a\sigma^H\,, \\ N_I \;\equiv\;& C_{IJK}\Big(\sigma^J\Box\sigma^K + \tfrac{1}{2}\mathcal{D}^a\sigma^J\mathcal{D}_a\sigma^K - \tfrac{1}{4}F^J_{ab}F^{abK} - \tfrac{1}{2}\bar{\psi}^J\not{\mathcal{D}}\psi^K + Y^{ijJ}Y_{ij}{}^K\Big) \\ & + \tfrac{1}{2}\,\mathrm{i}\,gC_{IJK}f_{LH}{}^K\sigma^J\bar{\psi}^L\psi^H\,.\end{aligned} \tag{6.68}$$

These equations themselves transform as a linear multiplet in the adjoint representation of the gauge group for which the transformation rules have been given in appendix A of [86].

6.4.2 The vector-tensor multiplet action

We will now generalize the vector action (6.66) to an action for the vector-tensor multiplets (with n vector multiplets and m tensor multiplets) discussed in section 6.2.2.

The supersymmetry transformation rules for the vector-tensor multiplet (6.10) were obtained from those for the vector multiplet (6.5) by replacing all contracted indices by the extended range of tilde indices. In addition, extra terms of $O(g)$ had to be added to the transformation rules. Similar considerations apply to the generalization of the action, as we will see below.

To obtain the generalization of the Chern-Simons (CS) term, it is convenient to rewrite this CS-term as an integral in six dimensions which has a boundary given by the five-dimensional Minkowski space-time. The six-form appearing in the integral is given by

$$I_{\text{vector}} = C_{IJK} F^I \wedge F^J \wedge F^K \,, \tag{6.69}$$

where we have used form notation. This six-form is both gauge-invariant and closed, by virtue of (6.65) and the Bianchi identities (6.4). It can therefore be written as the exterior derivative of a five-form which is gauge-invariant up to a total derivative. The space-time integral over this five-form is the CS-term given in the second line of (6.66).

We now wish to generalize (6.69) to the case of vector-tensor multiplets. It turns out that the generalization of (6.69) is somewhat surprising. We find

$$I_{\text{vec-tensor}} = C_{\widetilde{I}\widetilde{J}\widetilde{K}} \mathcal{H}^{\widetilde{I}} \wedge \mathcal{H}^{\widetilde{J}} \wedge \mathcal{H}^{\widetilde{K}} - \tfrac{3}{g} \Omega_{MN} \mathcal{D}B^M \wedge \mathcal{D}B^N \,. \tag{6.70}$$

The tensor Ω_{MN} is antisymmetric and invertible, and it restricts the number of tensor multiplets to be *even*

$$\Omega_{MN} = -\Omega_{NM} \,, \qquad \Omega_{MP}\Omega^{MR} = \delta_P{}^R \,. \tag{6.71}$$

The covariant derivative of the tensor field is given by

$$\begin{aligned} \mathcal{D}_\lambda B^N_{\rho\sigma} &= \partial_\lambda B^N_{\rho\sigma} + g\, A^I_\lambda t_{I\widetilde{J}}{}^N \mathcal{H}^{\widetilde{J}}_{\rho\sigma} \\ &= \partial_\lambda B^N_{\rho\sigma} + g\, A^I_\lambda t_{IJ}{}^N F^J_{\rho\sigma} + g\, A^I_\lambda t_{IP}{}^N B^P_{\rho\sigma} \,. \end{aligned} \tag{6.72}$$

To see why (6.70) is a closed six-form, we write out the first term of (6.70)

$$C_{\widetilde{I}\widetilde{J}\widetilde{K}} \mathcal{H}^{\widetilde{I}} \wedge \mathcal{H}^{\widetilde{J}} \wedge \mathcal{H}^{\widetilde{K}} = C_{IJK} F^I \wedge F^J \wedge F^K + 3 C_{IJM} F^I \wedge F^J \wedge B^M + 3 C_{IMN} F^I \wedge B^M \wedge B^N \,. \tag{6.73}$$

Since the B^M tensors in (6.73) do not satisfy a Bianchi identity, we also need the second term in (6.70) to render it a closed six-form. This requirement of closure leads to the following relations between the C and Ω tensors:

$$C_{IJM} = t_{(IJ)}{}^N \Omega_{NM} \,, \qquad C_{IMN} = \tfrac{1}{2} t_{IM}{}^P \Omega_{PN} \,. \tag{6.74}$$

We stress that the tensor $C_{\widetilde{I}\widetilde{J}\widetilde{K}}$ is not a fundamental object: the essential data for the vector-tensor multiplet are the representation matrices $t_{I\widetilde{J}}{}^{\widetilde{K}}$, the Yang-Mills components C_{IJK}, and the symplectic matrix Ω_{MN}. The tensor components of the C tensor are derived quantities, and we can summarize (6.74) as

$$C_{M\widetilde{J}\widetilde{K}} = t_{(\widetilde{J}\widetilde{K})}{}^P \Omega_{PM} \,. \tag{6.75}$$

From (6.74), we deduce that the second term of (6.73) only depends on the off-diagonal (between vector and tensor multiplets) transformations. The first term of (6.73) will induce the usual five-dimensional CS-term. The generalized CS-term induced by the third term of (6.73) was given in [73]. With our extension to also allow for the off-diagonal term in (6.15), we also get CS-terms induced by the C_{IJM} components, which were not present in [73].

Gauge invariance of the first term of (6.70) requires that the tensor C satisfies a modified version of (6.65)

$$f_{I(J}{}^{H}C_{KL)H} = t_{I(J}{}^{M}t_{KL)}{}^{N}\Omega_{MN}\,. \tag{6.76}$$

In addition to this, the second term of (6.70) is only gauge invariant if the tensor Ω satisfies

$$t_{I[M}{}^{P}\Omega_{N]P} = 0\,, \tag{6.77}$$

such that the second relation in (6.74) is consistent with the symmetry (MN). The two conditions (6.76) and (6.77) combined with the definition (6.75) imply the following generalization of (6.65)

$$t_{I(\widetilde{J}}{}^{\widetilde{M}}C_{\widetilde{K}\widetilde{L})\widetilde{M}} = 0\,. \tag{6.78}$$

The superconformal action for the combined system of $m = 2k$ tensor multiplets and n vector multiplets contains the CS-term induced by (6.70) and the generalization of the vector action (6.66) to the extended range of indices. Some extra terms are necessary to complete it to an invariant action: we need mass terms and/or Yukawa coupling for the fermions at $\mathcal{O}(g)$, and a scalar potential at $\mathcal{O}(g^2)$. We thus find the following action:

$$\begin{aligned}
\mathcal{L}_{\text{vec-tensor}} \;=\;& \Big(-\tfrac{1}{4}\mathcal{H}^{\widetilde{I}}_{\mu\nu}\mathcal{H}^{\mu\nu\widetilde{J}} - \tfrac{1}{2}\bar{\psi}^{\widetilde{I}}\not{\mathcal{D}}\psi^{\widetilde{J}} - \tfrac{1}{2}\mathcal{D}_a\sigma^{\widetilde{I}}\mathcal{D}^a\sigma^{\widetilde{J}} + Y^{\widetilde{I}}_{ij}Y^{ij\widetilde{J}}\Big)\sigma^{\widetilde{K}}C_{\widetilde{I}\widetilde{J}\widetilde{K}} \\
& + \frac{1}{16g}\varepsilon^{\mu\nu\lambda\rho\sigma}\Omega_{MN}B^{M}_{\mu\nu}\left(\partial_\lambda B^{N}_{\rho\sigma} + 2g t_{IJ}{}^{N}A^{I}_{\lambda}F^{J}_{\rho\sigma} + g\, t_{IP}{}^{N}A^{I}_{\lambda}B^{P}_{\rho\sigma}\right) \\
& -\tfrac{1}{24}\varepsilon^{\mu\nu\lambda\rho\sigma}C_{IJK}A^{I}_{\mu}\Big(F^{J}_{\nu\lambda}F^{K}_{\rho\sigma} + f_{FG}{}^{J}A^{F}_{\nu}A^{G}_{\lambda}\Big(-\tfrac{1}{2}g\,F^{K}_{\rho\sigma} + \tfrac{1}{10}g^2 f_{HL}{}^{K}A^{H}_{\rho}A^{L}_{\sigma}\Big)\Big) \\
& - \tfrac{1}{8}\varepsilon^{\mu\nu\lambda\rho\sigma}\Omega_{MN}t_{IK}{}^{M}t_{FG}{}^{N}A^{I}_{\mu}A^{F}_{\nu}A^{G}_{\lambda}\left(-\tfrac{1}{2}g\,F^{K}_{\rho\sigma} + \tfrac{1}{10}g^2 f_{HL}{}^{K}A^{H}_{\rho}A^{L}_{\sigma}\right) \\
& + \Big(-\tfrac{1}{8}\,\mathrm{i}\,\bar{\psi}^{\widetilde{I}}\gamma\cdot\mathcal{H}^{\widetilde{J}}\psi^{\widetilde{K}} - \tfrac{1}{2}\,\mathrm{i}\,\bar{\psi}^{i\widetilde{I}}\psi^{j\widetilde{J}}Y^{\widetilde{K}}_{ij}\Big)C_{\widetilde{I}\widetilde{J}\widetilde{K}} + \\
& +\tfrac{1}{4}\,\mathrm{i}\,g\bar{\psi}^{\widetilde{I}}\psi^{\widetilde{J}}\sigma^{\widetilde{K}}\sigma^{\widetilde{L}}\Big(t_{[\widetilde{I}\widetilde{J}]}{}^{\widetilde{M}}C_{\widetilde{M}\widetilde{K}\widetilde{L}} - 4t_{(\widetilde{I}\widetilde{K})}{}^{\widetilde{M}}C_{\widetilde{M}\widetilde{J}\widetilde{L}}\Big) \\
& -\tfrac{1}{2}g^2\sigma^{K}C_{KMN}t_{I\widetilde{L}}{}^{M}\sigma^{I}\sigma^{\widetilde{L}}t_{J\widetilde{P}}{}^{N}\sigma^{J}\sigma^{\widetilde{P}}\,.
\end{aligned} \tag{6.79}$$

To check the supersymmetry of this action, one needs all the relations between the various tensors given above. Another useful identity implied by the previous definitions is

$$t_{(\widetilde{I}\widetilde{J})}{}^{M}C_{\widetilde{K}\widetilde{L}M} = -t_{(\widetilde{K}\widetilde{L})}{}^{M}C_{\widetilde{I}\widetilde{J}M}\,. \tag{6.80}$$

The terms in the action containing the fields of the tensor multiplets can also be obtained from the field equations following from the on-shell closure of the algebra in section 6.2.2. Note however that the equations of motion for the vector multiplet fields, obtained from this action, are similar to those given in (6.68), but with the contracted indices running over the extended range of vector and tensor components. Furthermore, the A^I_μ equation of motion gets corrected by a term proportional to the self-duality equation for $B^M_{\mu\nu}$:

$$\frac{\delta S_{\text{vec-tensor}}}{\delta A^I_a} = E^a_I + \tfrac{1}{12} g \varepsilon^{abcde} A^J_b E_{cde}{}^M t_{JI}{}^N \Omega_{MN} \,. \tag{6.81}$$

Finally, we remark that the action (6.79) is invariant under supersymmetry for the general form (6.15) of the representation matrices $(t_I)_{\widetilde{J}}{}^{\widetilde{K}}$.

We thus conclude that in order to write down a superconformal action for the vector-tensor multiplet, we need to introduce another geometrical object, namely a gauge-invariant anti-symmetric invertible tensor Ω_{MN}. This symplectic matrix will restrict the number of tensor multiplets to be even. We can still allow the transformations to have off-diagonal terms between vector and tensor multiplets, if we adapt (6.65) to (6.78). In this way, we have constructed more general matter couplings than were known so far. Terms of the form $A \wedge F \wedge B$ did not appear in previous papers. We see that such terms appear generically in our Lagrangian by allowing off-diagonal gauge transformations for the tensor fields.

6.4.3 The hypermultiplet

Until this point, the equations of motion we derived found their origin in the fact that we had an open superconformal algebra; the non-closure functions Γ^A, together with their supersymmetric partners Δ^X yielded these equations of motion. We discovered a hypercomplex scalar manifold $\mathcal{M}$, whose properties are described in appendix C.

Now, we will introduce an action to derive the field equations of the hypermultiplet. An important point to note is that the necessary data for the scalar manifold we had in the previous section are not sufficient anymore. This is not specific to our setting, but is a general property of nonlinear sigma models. In such models, the kinetic term for the scalars is multiplied by a scalar-dependent symmetric tensor $g_{\alpha\beta}(\phi)$,

$$S = -\tfrac{1}{2} \int \mathrm{d}^D x \; g_{\alpha\beta}(\phi) \partial_\mu \phi^\alpha \partial^\mu \phi^\beta \,, \tag{6.82}$$

in which α and β run over the dimensions of the scalar manifold. The tensor g is interpreted as the metric on the target space $\mathcal{M}$. As the field equations for the scalars should now also be covariant with respect to coordinate transformations on the target manifold, the connection on the tangent bundle $T\mathcal{M}$ should be the Levi-Civita connection. Only in that particular case, the field equations for the scalars will be covariant. In other words, in $\Box\phi^\alpha + \cdots = 0$ the Levi-Civita connection on $T\mathcal{M}$ will be used in the covariant box.

Therefore, in order to be able to write down an action, we will need to introduce a metric on the scalar manifold. However, this metric will also restrict the possible target spaces for the theory.

Observe that most steps in this section do not depend on the use of superconformal symmetry.[3] Only at the end of this section we make explicit use of this symmetry.

[3] Of course, the form of the field equations does reflect the superconformal symmetry.

Without gauged isometries

To start with, we take the non-closure functions Γ^A to be proportional to the field equations for the fermions ζ^A. In other words, we ask

$$\frac{\delta S_{\text{hyper}}}{\delta \bar{\zeta}^A} = 2C_{AB}\Gamma^B\,. \tag{6.83}$$

In general, the tensor C_{AB} could be a function of the scalars and bilinears of the fermions. If we try to construct an action with the above Ansatz, it turns out that the tensor has to be anti-symmetric in AB and

$$\frac{\delta C_{AB}}{\delta \zeta^C} = 0\,, \tag{6.84}$$

$$\mathfrak{D}_X C_{AB} = 0\,. \tag{6.85}$$

This means that the tensor does not depend on the fermions and is covariantly constant.[4]

This tensor C_{AB} will be used to raise and lower indices according to the NW–SE convention similar to ε_{ij}:

$$A_A = A^B C_{BA}\,, \qquad A^A = C^{AB} A_B\,, \tag{6.86}$$

where ε^{ij} and C^{AB} for consistency should satisfy

$$\varepsilon_{ik}\varepsilon^{jk} = \delta_i{}^j\,, \qquad C_{AC}C^{BC} = \delta_A{}^B\,. \tag{6.87}$$

We may choose C_{AB} to be constant. For this choice, the connection ω_{XAB} is symmetric, so the structure group $G\ell(r,\mathbb{H})$ breaks to $USp(2r-2p,2p)$. The signature is the signature of d_{CB}, which is defined as $C_{AB} = \rho_A{}^C d_{CB}$ where $\rho_A{}^C$ was given in (6.16). However, we will allow C_{AB} also to be non-constant, but covariantly constant.

We now construct the metric on the scalar manifold as

$$g_{XY} = f_X^{iA} C_{AB}\varepsilon_{ij} f_Y^{jB}\,. \tag{6.88}$$

The above-mentioned requirement that the Levi-Civita connection should be used (as $\Gamma_{XY}{}^Z$) is satisfied due to (6.85). Indeed, this guarantees that the metric is covariantly constant, such that the affine connection is the Levi-Civita one. On the other hand we have seen already that for covariantly constant complex structures we have to use the Obata connection. Hence, the Levi-Civita and Obata connection should coincide, and this is obtained from demanding (6.85) using the Obata connection. This makes us conclude that we can only write down an action for a hyperkähler scalar manifold.

We can now write down the action for the rigid hypermultiplets. It has the following form:

$$S_{\text{hyper}} = \int \mathrm{d}^5 x \left(-\tfrac{1}{2} g_{XY}\partial_a q^X \partial^a q^Y + \bar{\zeta}_A \not{\mathfrak{D}}\zeta^A - \tfrac{1}{4} W_{ABCD}\bar{\zeta}^A\zeta^B\bar{\zeta}^C\zeta^D \right), \tag{6.89}$$

[4] This can also easily be seen by using the Batalin-Vilkovisky formalism.

where the tensor W_{ABCD} can be proven to be completely symmetric in all of its indices (see appendix C). The field equations derived from this action are

$$\begin{aligned} \frac{\delta S_{\text{hyper}}}{\delta\bar{\zeta}^A} &= 2C_{AB}\Gamma^B\,, \\ \frac{\delta S_{\text{hyper}}}{\delta q^X} &= g_{XY}\Delta^Y - 2\bar{\zeta}_A\Gamma^B\omega_{XB}{}^A\,. \end{aligned} \tag{6.90}$$

Also remark that due to the introduction of the metric, the expression of Δ^X simplifies to

$$\Delta^X = \Box q^X - \bar{\zeta}^A \not{\mathfrak{D}} q^Y \zeta^B \mathcal{R}^X{}_{YAB} - \tfrac{1}{4}\mathfrak{D}^X W_{ABCD}\bar{\zeta}^A\zeta^B\zeta^C\zeta^D\,. \tag{6.91}$$

Conformal invariance

Due to the presence of the metric, the condition for the homothetic Killing vector (6.44) implies that k_X is the derivative of a scalar function as in (5.10). This scalar function $\mathcal{K}(q)$ is called the hyperkähler potential [139, 164, 168]. It determines the conformal structure, but should be restricted to

$$\mathfrak{D}_X\mathfrak{D}_Y\mathcal{K} = \tfrac{3}{2}g_{XY}\,. \tag{6.92}$$

The relation with the homothetic Killing vector is

$$k_X = \partial_X\mathcal{K}\,, \qquad \mathcal{K} = \tfrac{1}{3}k_X k^X\,. \tag{6.93}$$

Note that this implies that, when $\mathcal{K}$ and the complex structures are known, one can compute the metric with (6.92), using the formula for the Obata connection (6.31).

With gauged isometries

With a metric, the symmetries of section 6.3.3 should be isometries, i.e.

$$\mathfrak{D}_X k_{YI} + \mathfrak{D}_Y k_{XI} = 0\,. \tag{6.94}$$

This makes the requirement (6.57) superfluous, but we still have to impose the triholomorphicity expressed by either (6.53) or (6.54) or (6.55).

In order to integrate the equations of motion to an action we have to define (locally) triples of 'moment maps', according to

$$\partial_X P_I^\alpha = -\tfrac{1}{2}J^\alpha{}_{XY}k_I^Y\,. \tag{6.95}$$

The integrability condition that makes this possible is the triholomorphic condition.

In the kinetic terms of the action, the derivatives should now be covariantized with respect to the new transformations. Supersymmetry invariance of the action also forces us to include some new terms proportional to g and g^2

$$\begin{aligned} S^g_{\text{hyper}} = \int d^5x\Big(&-\tfrac{1}{2}g_{XY}\mathfrak{D}_a q^X\mathfrak{D}^a q^Y + \bar{\zeta}_A\not{\mathfrak{D}}\zeta^A - \tfrac{1}{4}W_{ABCD}\bar{\zeta}^A\zeta^B\bar{\zeta}^C\zeta^D \\ &-g\left(2\,\mathrm{i}\,k_I^X f_{iX}^A\bar{\zeta}_A\psi^{iI} + \mathrm{i}\,\sigma^I t_{IB}{}^A\bar{\zeta}_A\zeta^B - 2P_{Iij}Y^{Iij}\right) - \tfrac{1}{2}g^2\sigma^I\sigma^J k_I^X k_{JX}\Big)\,, \end{aligned} \tag{6.96}$$

[where the covariant derivatives $\mathfrak{D}$ now also include gauge-covariantization proportional to g as in (6.60)], while the field equations have the same form as in (6.90).

Alternatively one can use the method explained in [169] to construct the action. Since the field equations are linear in the non-closure functions and the Lagrangian should vanish on-shell, we expect that the action itself can in fact be written as a linear combination of non-closure functions, in the form of Σ[field]$\times$[non-closure]:

$$\mathcal{L} = \tfrac{1}{3}k_X\Delta^X + \bar{\zeta}_A\Gamma^A\,. \tag{6.97}$$

The two coefficients can be fixed by looking at the normalization of the kinetic terms. Substituting the non-closure functions into (6.97) and partial integrating the covariant box, we indeed find the correct action (6.96). This method is believed to be correct for any on-shell multiplet. Note however that supersymmetry is a necessary ingredient. The invariance of the hypermultiplet action under supersymmetry can easily be checked by using the following transformation rules for the non-closure functions:

$$\begin{aligned} \delta\Delta^X &= -\mathrm{i}\,\bar{\epsilon}^i \not{\mathcal{D}}\Gamma^A f_{iA}^X + 2\,\mathrm{i}\,\bar{\epsilon}^i\Gamma^B\zeta^C\zeta^D f_{iB}^Y\mathcal{R}^X{}_{YCD} + 2\bar{\eta}^i\Gamma^A f_{iA}^X + \Delta^Y\Gamma_{ZY}{}^X\delta q^Z\,,\\ \delta\Gamma^A &= \tfrac{1}{2}\,\mathrm{i}\,f_X^{iA}\epsilon_i\Delta^X - \delta q^X\omega_{XB}{}^A\Gamma^B\,, \end{aligned} \tag{6.98}$$

where the covariant derivative is given by

$$\mathcal{D}_\mu\Gamma^A = (\partial_\mu - 3b_\mu + \tfrac{1}{4}\omega_\mu{}^{ab}\gamma_{ab})\Gamma^A + \partial_\mu q^X\omega_{XB}{}^A\Gamma^B + gA_\mu^I t_{IB}{}^A\Gamma^B\,. \tag{6.99}$$

Supersymmetry of the action imposes

$$k_I^X J^\alpha{}_{XY}k_J^Y = 2f_{IJ}{}^K P_K^\alpha\,. \tag{6.100}$$

As only the derivative of P appears in the defining equation (6.95), one may add an arbitrary constant to P. But that changes the right-hand side of (6.100). One should then consider whether there is a choice of these coefficients such that (6.100) is satisfied. This is the question about the center of the algebra, which is discussed in [170, 171]. For simple groups there is always a solution.[5] For Abelian theories the constant remains undetermined. This free constant is the so-called Fayet–Iliopoulos term.

In a conformal invariant theory, the Fayet–Iliopoulos term is not possible, since dilatation invariance of the action requires

$$3P_I^\alpha = k^X\partial_X P_I^\alpha\,. \tag{6.101}$$

Thus, P_I^α is completely determined [using (6.95) or (6.59)] as (see also [172])

$$-6P_I^\alpha = k^X J^\alpha{}_{XY}k_I^Y = -\tfrac{2}{3}k^X k^Z J^\alpha{}_Z{}^Y\mathfrak{D}_Y k_{IX}\,. \tag{6.102}$$

[5] We thank Gary Gibbons for a discussion on this subject.

The proof of the invariance of the action under the complete superconformal group, uses the equation obtained from (6.59) and (6.95):

$$k^{X\alpha}\mathfrak{D}_X k_I^Y = \partial^Y P_I^\alpha\,. \tag{6.103}$$

If the moment map P_I^α has the value that it takes in the conformal theory, then (6.100) is satisfied due to (6.50). Indeed, one can multiply that equation with $k_X k^Z J^\alpha{}_Z{}^W \mathfrak{D}_W$ and use (6.54), (6.57) and (6.46). Thus, in the superconformal theory, the moment maps are determined and there is no further relation to be obeyed, i.e. the Fayet–Iliopoulos terms of the rigid theories are absent in this case.

To conclude, isometries of the scalar manifold that commute with dilatations, see (6.59), can be gauged. The resulting theory has an extra symmetry group G, whose algebra is generated by the corresponding Killing vectors.

Gathering together our results (6.79) and (6.96) the total Lagrangian describing the most general couplings of vector/tensor multiplets to hypermultiplets with rigid superconformal symmetry is given by

$$\mathcal{L}_{\rm total} = \mathcal{L}_{\rm vec-tensor} + \mathcal{L}^g_{\rm hyper}\,. \tag{6.104}$$

Summarizing, in this section the actions of rigid superconformal vector/tensor-hypermultiplet couplings have been constructed. The full answer is (6.104). We found that the existence of an action requires the presence of additional tensorial objects. A review of all the independent objects needed to determine the transformation laws, or to determine the action, are given in table 6.4. Note that these objects could already be introduced at the level of rigid supersymmetry. In the next section these results will be generalized to the local case, by coupling the matter multiplets to the Weyl multiplet, but this will not introduce any new constraints.

	ALGEBRA (no action)		ACTION	
multiplets	objects	Def/restriction	objects	Def/restriction
Vect.	${f_{[IJ]}}^K$	Jacobi identities	$C_{(IJK)}$	${f_{I(J}}^H C_{KL)H} = 0$ ▲
Vect./Tensor	$(t_I)_{\widetilde{J}}{}^{\widetilde{K}}$ $\widetilde{I} = (I, M)$	$[t_I, t_J] = -{f_{IJ}}^K t_K$ ${t_{IJ}}^K = {f_{IJ}}^K, \qquad {t_{IM}}^J = 0$	$\Omega_{[MN]}$	invertible ${f_{I(J}}^H C_{KL)H} = {t_{I(J}}^M {t_{KL)}}^N \Omega_{MN}$ ${t_{I[M}}^P \Omega_{N]P} = 0$
Hyper	${f_X}^{iA}$	invertible and real using ρ Nijenhuis condition: ${N_{XY}}^Z = 0$	$C_{[AB]}$	$\mathfrak{D}_X C_{AB} = 0$
Hyper + gauging	k_I^X	$\mathfrak{D}_X \mathfrak{D}_Y k_I^Z = {R_{XWY}}^Z k_I^W$ ◀ $k_{[I|}^Y \partial_Y k_{\|J]}^X = -\frac{1}{2} {f_{IJ}}^K k_K^X$ $\mathcal{L}_{k_I} J^\alpha = 0$ ◀	P_I^α ▲	$\mathfrak{D}_X k_{YI} + \mathfrak{D}_Y k_{XI} = 0$ $\partial_X P_I^\alpha = -\frac{1}{2} {J^\alpha}_{XY} k_I^Y$ ▲ $k_I^X J_{XY}^\alpha k_J^Y = 2 {f_{IJ}}^K P_K^\alpha$ ▲
Hyper + conformal	k^X ◀	$\mathfrak{D}_Y k^X = \frac{3}{2} {\delta_Y}^X$ ◀	$\mathcal{K}$	$\mathfrak{D}_X \mathfrak{D}_Y \mathcal{K} = \frac{3}{2} g_{XY}$
Hyper + conformal + gauged		$k^Y \mathfrak{D}_Y k_I^X = \frac{3}{2} k_I^X$		

Table 6.4: Various matter couplings with or without action. We indicate which are the geometrical objects that determine the theory and what are the independent constraints. The symmetries of the objects are already indicated when they appear first. In general, the equations should also be valid for the theories in the rows below (apart from the fact that 'hyper+gauging' and 'hyper+conformal' are independent, but both are used in the lowest row). However, the symbol ▲ indicates that these equations are not to be taken over below. E.g. the moment map P_I^α itself is completely determined in the conformal theory, and it should therefore no longer be given as an independent quantity. For the rigid theory without conformal invariance, only constant pieces can be undetermined by the given equations, and they are the Fayet–Iliopoulos terms. On the other hand, the equations indicated by ◀ have not to be taken over for the theories with an action, as they are then satisfied due to the Killing equation or are defined by $\mathcal{K}$.

6.5 Local superconformal multiplets

In this section we will extend the supersymmetry to a local conformal supersymmetry, by making use of the off-shell 32 + 32 Standard Weyl multiplet constructed in chapter refch:weyl. We restrict ourselves here to the Standard Weyl multiplet. One may wonder whether the use of the dilaton Weyl multiplet could lead to other matter couplings. Though we can not exclude this, we do not expect a physically different result. Whether the conformal gauge-fixing program will also be insensitive to the choice of Weyl multiplet, remains to be seen.

The procedure for extending the rigid superconformal transformation rules for the various matter multiplets is to introduce covariant derivatives with respect to the superconformal symmetries. These derivatives contain the superconformal gauge fields which, in turn, will also transform to additional matter fields as explained in chapter 5.

Since in the previous sections we have explained most of the subtleties concerning the possible geometrical structures, we can be brief here. We will obtain our results in two steps. First, we require that the local superconformal commutator algebra, as it is realized on the standard Weyl multiplet (5.41)–(5.44) is also realized on the matter multiplets (with possible additional transformations under which the fields of the standard Weyl multiplet do not transform, and possibly field equations if the matter multiplet is on-shell). Note that this local superconformal algebra is a modification of the rigid superconformal algebra (5.22), (5.20) where all modifications involve the fields of the standard Weyl multiplet.

Now we will apply a standard Noether procedure to extend the rigid supersymmetric actions to a locally supersymmetric one. This will introduce the full complications of coupling the matter multiplets to conformal supergravity.

6.5.1 Vector-tensor multiplet

The local supersymmetry rules are given by

$$
\begin{aligned}
\delta A_\mu^I &= \tfrac12\bar\epsilon\gamma_\mu\psi^I - \tfrac12\,\mathrm{i}\,\sigma^I\bar\epsilon\psi_\mu\,,\\
\delta B_{ab}^M &= -\bar\epsilon\gamma_{[a}D_{b]}\psi^M - \tfrac12\,\mathrm{i}\,\sigma^M\bar\epsilon\widehat{R}_{ab}(Q) + \mathrm{i}\,\bar\epsilon\gamma_{[a}\gamma\cdot T\gamma_{b]}\psi^M\\
&\quad + \mathrm{i}\,g\bar\epsilon\gamma_{ab}t_{(\widetilde{J}\widetilde{K})}{}^M\sigma^{\widetilde{J}}\psi^{\widetilde{K}} + \mathrm{i}\,\bar\eta\gamma_{ab}\psi^M\,,\\
\delta Y^{ij\widetilde{I}} &= -\tfrac12\bar\epsilon^{(i}\not{D}\psi^{j)\widetilde{I}} + \tfrac12\,\mathrm{i}\,\bar\epsilon^{(i}\gamma\cdot T\psi^{j)\widetilde{I}} - 4\,\mathrm{i}\,\sigma^{\widetilde{I}}\bar\epsilon^{(i}\chi^{j)}\\
&\quad -\tfrac12\,\mathrm{i}\,g\bar\epsilon^{(i}\left(t_{[\widetilde{J}\widetilde{K}]}{}^{\widetilde{I}} - 3t_{(\widetilde{J}\widetilde{K})}{}^{\widetilde{I}}\right)\sigma^{\widetilde{J}}\psi^{j)\widetilde{K}} + \tfrac12\,\mathrm{i}\,\bar\eta^{(i}\psi^{j)\widetilde{I}}\,,\\
\delta\psi^{i\widetilde{I}} &= -\tfrac14\gamma\cdot\widehat{\mathcal{H}}^{\widetilde{I}}\epsilon^i - \tfrac12\,\mathrm{i}\,\not{D}\sigma^{\widetilde{I}}\epsilon^i - Y^{ij\widetilde{I}}\epsilon_j + \sigma^{\widetilde{I}}\gamma\cdot T\epsilon^i + \tfrac12 g t_{(\widetilde{J}\widetilde{K})}{}^{\widetilde{I}}\sigma^{\widetilde{J}}\sigma^{\widetilde{K}}\epsilon^i + \sigma^{\widetilde{I}}\eta^i\,,\\
\delta\sigma^{\widetilde{I}} &= \tfrac12\,\mathrm{i}\,\bar\epsilon\psi^{\widetilde{I}}\,.
\end{aligned}
\tag{6.105}
$$

The covariant derivatives are given by

$$
\begin{aligned}
D_\mu\sigma^{\widetilde{I}} &= \mathcal{D}_\mu\sigma^{\widetilde{I}} - \tfrac12\,\mathrm{i}\,\bar\psi_\mu\psi^{\widetilde{I}}\,,\\
\mathcal{D}_\mu\sigma^{\widetilde{I}} &= (\partial_\mu - b_\mu)\sigma^{\widetilde{I}} + g t_{J\widetilde{K}}{}^{\widetilde{I}}A_\mu^J\sigma^{\widetilde{K}}\,,\\
D_\mu\psi^{i\widetilde{I}} &= \mathcal{D}_\mu\psi^{i\widetilde{I}} + \tfrac14\gamma\cdot\widehat{\mathcal{H}}^{\widetilde{I}}\psi_\mu^i + \tfrac12\,\mathrm{i}\,\not{D}\sigma^{\widetilde{I}}\psi_\mu^i + Y^{ij\widetilde{I}}\psi_{\mu j} - \sigma^{\widetilde{I}}\gamma\cdot T\psi_\mu^i
\end{aligned}
\tag{6.106}
$$

$$
\begin{aligned}
& -\tfrac{1}{2} g t_{(\widetilde{J}\widetilde{K})}{}^{\widetilde{I}} \sigma^{\widetilde{J}} \sigma^{\widetilde{K}} \psi^i_\mu - \sigma^{\widetilde{I}} \phi^i_\mu \,, \\
\mathcal{D}_\mu \psi^{i\widetilde{I}} \;\; = \;\; & (\partial_\mu - \tfrac{3}{2} b_\mu + \tfrac{1}{4} \gamma_{ab}\, \omega_\mu{}^{ab}) \psi^{i\widetilde{I}} - V_\mu^{ij} \psi_j^{\widetilde{I}} + g t_{J\widetilde{K}}{}^{\widetilde{I}} A_\mu^J \psi^{i\widetilde{K}} \,.
\end{aligned}
$$

The covariant curvature $\widehat{\mathcal{H}}^{\widetilde{I}}_{\mu\nu}$ should be understood as having components $(\widehat{F}^I_{\mu\nu}, B_{\mu\nu})$ and

$$
\widehat{F}^I_{\mu\nu} = 2\partial_{[\mu} A^I_{\nu]} + g f_{JK}{}^I A^J_\mu A^K_\nu - \bar{\psi}_{[\mu} \gamma_{\nu]} \psi^I + \tfrac{1}{2}\,\mathrm{i}\, \sigma^I \bar{\psi}_{[\mu} \psi_{\nu]} \,. \tag{6.107}
$$

The locally superconformal constraints needed to close the algebra are given by the following extensions of (6.13) and (6.14) (which are non-zero only for $\widetilde{I}$ in the tensor multiplet range)

$$
\begin{aligned}
L^{ijM} \;\; \equiv \;\; & t_{(\widetilde{J}\widetilde{K})}{}^M \left(2\sigma^{\widetilde{J}} Y^{ij\widetilde{K}} - \tfrac{1}{2}\,\mathrm{i}\, \bar{\psi}^{i\widetilde{J}} \psi^{j\widetilde{K}} \right) = 0 \,, \\
E^M_{\mu\nu\lambda} \;\; \equiv \;\; & \frac{3}{g} D_{[\mu} B_{\nu\lambda]}{}^M - \varepsilon_{\mu\nu\lambda\rho\sigma} t_{(\widetilde{J}\widetilde{K})}{}^M \left(\sigma^{\widetilde{J}} \widehat{\mathcal{H}}^{\rho\sigma\widetilde{K}} - 8\sigma^{\widetilde{J}} \sigma^{\widetilde{K}} T^{\rho\sigma} + \tfrac{1}{4}\,\mathrm{i}\, \bar{\psi}^{\widetilde{J}} \gamma^{\rho\sigma} \psi^{\widetilde{K}} \right) \\
& - \frac{3}{2g} \bar{\psi}^M \gamma_{[\mu} \widehat{R}_{\nu\lambda]}(Q) \\
= \;\; & 0 \,.
\end{aligned} \tag{6.108}
$$

Here, the supercovariant derivative on the tensor is defined as

$$
\begin{aligned}
D_{[\mu} B^M_{\nu\rho]} \;\; = \;\; & \partial_{[\mu} B^M_{\nu\rho]} - 2b_{[\mu} B^M_{\nu\rho]} + \bar{\psi}_{[\mu} \gamma_\nu D_{\rho]} \psi^M + \tfrac{1}{2}\,\mathrm{i}\, \sigma^M \bar{\psi}_{[\mu} \widehat{R}_{\nu\rho]}(Q) \\
& - \mathrm{i}\, \bar{\psi}_{[\mu} \gamma_\nu \gamma \cdot T \gamma_{\rho]} \psi^M - \mathrm{i}\, \bar{\phi}_{[\mu} \gamma_{\nu\rho]} \psi^M \\
& - \mathrm{i}\, g \bar{\psi}_{[\mu} \gamma_{\nu\rho]} \psi^{\widetilde{K}} \sigma^{\widetilde{J}} t_{(\widetilde{J}\widetilde{K})}{}^M + g t_{J\widetilde{K}}{}^M A^J_{[\mu} \widehat{\mathcal{H}}^{\widetilde{K}}_{\nu\rho]} \,.
\end{aligned} \tag{6.109}
$$

Analogously to subsection 6.2.2, the full set of constraints could be obtained by varying these constraints under supersymmetry.

The action, invariant under local superconformal symmetry, can be obtained by replacing the rigid covariant derivatives in (6.79) by the local covariant derivatives (6.106) and adding extra terms proportional to gravitinos or matter fields of the Weyl multiplet, determined by supersymmetry. It is convenient at this point to introduce a new S-invariant tensorfield $\widetilde{B}^M_{\mu\nu}$ which is defined as

$$
B^M_{\mu\nu} = \widetilde{B}^M_{\mu\nu} - \bar{\psi}_{[\mu} \gamma_{\nu]} \psi^M + \tfrac{1}{2}\,\mathrm{i}\, \sigma^M \bar{\psi}_{[\mu} \psi_{\nu]} \;,
$$

such that the symbol $\widehat{\mathcal{H}}^{\widetilde{I}}_{\mu\nu}$ can be written as

$$
\widehat{\mathcal{H}}^{\widetilde{I}}_{\mu\nu} = H^{\widetilde{I}}_{\mu\nu} - \bar{\psi}_{[\mu} \gamma_{\nu]} \psi^{\widetilde{I}} + \tfrac{1}{2}\,\mathrm{i}\, \sigma^{\widetilde{I}} \bar{\psi}_{[\mu} \psi_{\nu]} \,, \qquad H^{\widetilde{I}}_{\mu\nu} \equiv (F^I_{\mu\nu}, \widetilde{B}^M_{\mu\nu}) \,.
$$

The action then reads

$$
\begin{aligned}
e^{-1} \mathcal{L}^{\rm conf}_{\rm vec-ten} \;\; = \;\; & \Big[\Big(-\tfrac{1}{4} \widehat{\mathcal{H}}^{\widetilde{I}}_{\mu\nu} \widehat{\mathcal{H}}^{\mu\nu\widetilde{J}} - \tfrac{1}{2} \bar{\psi}^{\widetilde{I}} \not{D} \psi^{\widetilde{J}} + \tfrac{1}{3} \sigma^{\widetilde{I}} \Box^c \sigma^{\widetilde{J}} + \tfrac{1}{6} D_a \sigma^{\widetilde{I}} D^a \sigma^{\widetilde{J}} + Y^{\widetilde{I}}_{ij} Y^{ij\widetilde{J}} \Big) \sigma^{\widetilde{K}} \\
& - \tfrac{4}{3} \sigma^{\widetilde{I}} \sigma^{\widetilde{J}} \sigma^{\widetilde{K}} \left(D + \tfrac{26}{3} T_{ab} T^{ab} \right) + 4 \sigma^{\widetilde{I}} \sigma^{\widetilde{J}} \widehat{\mathcal{H}}^{\widetilde{K}}_{ab} T^{ab} \\
& - \tfrac{1}{8}\,\mathrm{i}\, \bar{\psi}^{\widetilde{I}} \gamma \cdot \widehat{\mathcal{H}}^{\widetilde{J}} \psi^{\widetilde{K}} - \tfrac{1}{2}\,\mathrm{i}\, \bar{\psi}^{i\widetilde{I}} \psi^{j\widetilde{J}} Y^{\widetilde{K}}_{ij} + \mathrm{i}\, \sigma^{\widetilde{I}} \bar{\psi}^{\widetilde{J}} \gamma \cdot T \psi^{\widetilde{K}} - 8\,\mathrm{i}\, \sigma^{\widetilde{I}} \sigma^{\widetilde{J}} \bar{\psi}^{\widetilde{K}} \chi
\end{aligned}
$$

$$
\begin{aligned}
&+\tfrac{1}{6}\sigma^{\widetilde{I}}\bar{\psi}_\mu\gamma^\mu\Big(\mathrm{i}\,\sigma^{\widetilde{J}}\not{D}\psi^{\widetilde{K}}+\tfrac{1}{2}\,\mathrm{i}\,\not{D}\sigma^{\widetilde{J}}\psi^{\widetilde{K}}-\tfrac{1}{4}\gamma\cdot\widehat{\mathcal{H}}^{\widetilde{J}}\psi^{\widetilde{K}}+2\sigma^{\widetilde{J}}\gamma\cdot T\psi^{\widetilde{K}}-8\sigma^{\widetilde{J}}\sigma^{\widetilde{K}}\chi\Big)\\
&-\tfrac{1}{6}\bar{\psi}_a\gamma_b\psi^{\widetilde{I}}\left(\sigma^{\widetilde{J}}\widehat{\mathcal{H}}^{ab\widetilde{K}}-8\sigma^{\widetilde{J}}\sigma^{\widetilde{K}}T^{ab}\right)-\tfrac{1}{12}\sigma^{\widetilde{I}}\bar{\psi}_\lambda\gamma^{\mu\nu\lambda}\psi^{\widetilde{J}}\widehat{\mathcal{H}}^{\widetilde{K}}_{\mu\nu}\\
&+\tfrac{1}{12}\,\mathrm{i}\,\sigma^{\widetilde{I}}\bar{\psi}_a\psi_b\left(\sigma^{\widetilde{J}}\widehat{\mathcal{H}}^{ab\widetilde{K}}-8\sigma^{\widetilde{J}}\sigma^{\widetilde{K}}T^{ab}\right)+\tfrac{1}{48}\,\mathrm{i}\,\sigma^{\widetilde{I}}\sigma^{\widetilde{J}}\bar{\psi}_\lambda\gamma^{\mu\nu\lambda\rho}\psi_\rho\widehat{\mathcal{H}}^{\widetilde{K}}_{\mu\nu}\\
&-\tfrac{1}{2}\sigma^{\widetilde{I}}\bar{\psi}^i_\mu\gamma^\mu\psi^{j\widetilde{J}}Y^{\widetilde{K}}_{ij}+\tfrac{1}{6}\,\mathrm{i}\,\sigma^{\widetilde{I}}\sigma^{\widetilde{J}}\bar{\psi}^i_\mu\gamma^{\mu\nu}\psi^j_\nu Y^{\widetilde{K}}_{ij}-\tfrac{1}{24}\,\mathrm{i}\,\bar{\psi}_\mu\gamma_\nu\psi^{\widetilde{I}}\bar{\psi}^{\widetilde{J}}\gamma^{\mu\nu}\psi^{\widetilde{K}}\\
&+\tfrac{1}{12}\,\mathrm{i}\,\bar{\psi}^i_\mu\gamma^\mu\psi^{j\widetilde{I}}\bar{\psi}^{\widetilde{J}}_i\psi^{\widetilde{K}}_j-\tfrac{1}{48}\sigma^{\widetilde{I}}\bar{\psi}_\mu\psi_\nu\bar{\psi}^{\widetilde{J}}\gamma^{\mu\nu}\psi^{\widetilde{K}}+\tfrac{1}{24}\sigma^{\widetilde{I}}\bar{\psi}^i_\mu\gamma^{\mu\nu}\psi^j_\nu\bar{\psi}^{\widetilde{J}}_i\psi^{\widetilde{K}}_j\\
&-\tfrac{1}{12}\sigma^{\widetilde{I}}\bar{\psi}_\lambda\gamma^{\mu\nu\lambda}\psi^{\widetilde{J}}\bar{\psi}_\mu\gamma_\nu\psi^{\widetilde{K}}+\tfrac{1}{24}\,\mathrm{i}\,\sigma^{\widetilde{I}}\sigma^{\widetilde{J}}\bar{\psi}_\lambda\gamma^{\mu\nu\lambda}\psi^{\widetilde{K}}\bar{\psi}_\mu\psi_\nu\\
&+\tfrac{1}{48}\,\mathrm{i}\,\sigma^{\widetilde{I}}\sigma^{\widetilde{J}}\bar{\psi}_\lambda\gamma^{\mu\nu\lambda\rho}\psi_\rho\bar{\psi}_\mu\gamma_\nu\psi^{\widetilde{K}}+\tfrac{1}{96}\sigma^{\widetilde{I}}\sigma^{\widetilde{J}}\sigma^{\widetilde{K}}\bar{\psi}_\lambda\gamma^{\mu\nu\lambda\rho}\psi_\rho\bar{\psi}_\mu\psi_\nu\Big]C_{\widetilde{I}\widetilde{J}\widetilde{K}}\\
&+\frac{1}{16g}e^{-1}\varepsilon^{\mu\nu\rho\sigma\tau}\Omega_{MN}\widetilde{B}^M_{\mu\nu}\Big(\partial_\rho\widetilde{B}^N_{\sigma\tau}+2g\,t_{IJ}{}^NA^I_\rho F^J_{\sigma\tau}+g\,t_{IP}{}^NA^I_\rho\widetilde{B}^P_{\sigma\tau}\Big)\\
&-\tfrac{1}{24}e^{-1}\varepsilon^{\mu\nu\lambda\rho\sigma}C_{IJK}A^I_\mu\Big(F^J_{\nu\lambda}F^K_{\rho\sigma}+f_{FG}{}^JA^F_\nu A^G_\lambda\Big(-\tfrac{1}{2}g\,F^K_{\rho\sigma}+\tfrac{1}{10}g^2f_{HL}{}^KA^H_\rho A^L_\sigma\Big)\Big)\\
&-\tfrac{1}{8}e^{-1}\varepsilon^{\mu\nu\lambda\rho\sigma}\Omega_{MN}t_{IK}{}^Mt_{FG}{}^NA^I_\mu A^F_\nu A^G_\lambda\left(-\tfrac{1}{2}g\,F^K_{\rho\sigma}+\tfrac{1}{10}g^2f_{HL}{}^KA^H_\rho A^L_\sigma\right)\\
&+\tfrac{1}{4}\,\mathrm{i}\,g\bar{\psi}^{\widetilde{I}}\psi^{\widetilde{J}}\sigma^{\widetilde{K}}\sigma^{\widetilde{L}}\left(t_{[\widetilde{I}\widetilde{J}]}{}^{\widetilde{M}}C_{\widetilde{M}\widetilde{K}\widetilde{L}}-4t_{(\widetilde{I}\widetilde{K})}{}^{\widetilde{M}}C_{\widetilde{M}\widetilde{J}\widetilde{L}}\right)\\
&-\tfrac{1}{4}g\bar{\psi}_\mu\gamma^\mu\psi^{\widetilde{I}}\sigma^{\widetilde{J}}\sigma^{\widetilde{K}}\sigma^{\widetilde{L}}t_{(\widetilde{J}\widetilde{K})}{}^{\widetilde{M}}C_{\widetilde{M}\widetilde{I}\widetilde{L}}-\tfrac{1}{2}g^2\sigma^I\sigma^J\sigma^K\sigma^{\widetilde{M}}\sigma^{\widetilde{N}}t_{J\widetilde{M}}{}^Pt_{K\widetilde{N}}{}^QC_{IPQ}\,,
\end{aligned}
\tag{6.110}
$$

where the superconformal d'Alembertian is defined as

$$
\begin{aligned}
\Box^c\sigma^{\widetilde{I}} &= D^aD_a\sigma^{\widetilde{I}}\\
&= \left(\partial^a-2b^a+\omega_b{}^{ba}\right)D_a\sigma^{\widetilde{I}}+gt_{J\widetilde{K}}{}^{\widetilde{I}}A^J_aD^a\sigma^{\widetilde{K}}-\tfrac{1}{2}\,\mathrm{i}\,\bar{\psi}_\mu D^\mu\psi^{\widetilde{I}}-2\sigma^{\widetilde{I}}\bar{\psi}_\mu\gamma^\mu\chi\\
&\quad+\tfrac{1}{2}\bar{\psi}_\mu\gamma^\mu\gamma\cdot T\psi^{\widetilde{I}}+\tfrac{1}{2}\bar{\phi}_\mu\gamma^\mu\psi^{\widetilde{I}}+2f_\mu{}^\mu\sigma^{\widetilde{I}}-\tfrac{1}{2}g\bar{\psi}_\mu\gamma^\mu t_{\widetilde{J}\widetilde{K}}{}^{\widetilde{I}}\psi^{\widetilde{J}}\sigma^{\widetilde{K}}\,.
\end{aligned}
\tag{6.111}
$$

6.5.2 Hypermultiplet

Imposing the local superconformal algebra we find the following supersymmetry rules:

$$
\begin{aligned}
\delta q^X &= -\,\mathrm{i}\,\bar{\epsilon}^i\zeta^Af^X_{iA}\,,\\
\widehat{\delta}\zeta^A &= \tfrac{1}{2}\,\mathrm{i}\,\not{D}q^Xf^{iA}_X\epsilon_i-\tfrac{1}{3}\gamma\cdot Tk^Xf^A_{iX}\epsilon^i+\tfrac{1}{2}g\sigma^Ik^X_If^A_{iX}\epsilon^i+k^Xf^A_{iX}\eta^i\,.
\end{aligned}
\tag{6.112}
$$

The covariant derivatives are given by

$$
\begin{aligned}
D_\mu q^X &= \mathcal{D}_\mu q^X+\,\mathrm{i}\,\bar{\psi}^i_\mu\zeta^Af^X_{iA}\,,\\
\mathcal{D}_\mu q^X &= \partial_\mu q^X-b_\mu k^X-V^{jk}_\mu k^X_{jk}+gA^I_\mu k^X_I\,,\\
D_\mu\zeta^A &= \mathcal{D}_\mu\zeta^A-k^Xf^A_{iX}\phi^i_\mu+\tfrac{1}{2}\,\mathrm{i}\,\not{D}q^Xf^A_{iX}\psi^i_\mu+\tfrac{1}{3}\gamma\cdot Tk^Xf^A_{iX}\psi^i_\mu-g\tfrac{1}{2}\sigma^Ik^X_If^A_{iX}\psi^i_\mu\\
\mathcal{D}_\mu\zeta^A &= \partial_\mu\zeta^A+\partial_\mu q^X\omega_{XB}{}^A\zeta^B+\tfrac{1}{4}\omega_\mu{}^{bc}\gamma_{bc}\zeta^A-2b_\mu\zeta^A+gA^I_\mu t_{IB}{}^A\zeta^B\,.
\end{aligned}
\tag{6.113}
$$

Similar to section 6.3, requiring closure of the commutator algebra on these transformation rules yields the equation of motion for the fermions

$$\begin{aligned}\Gamma^A_{\text{conf}} &= \not{D}\zeta^A + \tfrac{1}{2}W_{CDB}{}^A\zeta^B\bar{\zeta}^D\zeta^C - \tfrac{8}{3}\,\mathrm{i}\,k^X f^A_{iX}\chi^i + 2\,\mathrm{i}\,\gamma\cdot T\zeta^A \\ &\quad -g\left(\mathrm{i}\,k^X_I f^A_{iX}\psi^{iI} + \mathrm{i}\,\sigma^I t_{IB}{}^A\zeta^B\right). \end{aligned} \tag{6.114}$$

The scalar equation of motion can be obtained from varying (6.114):

$$\widehat{\delta_Q}\Gamma^A = \tfrac{1}{2}\,\mathrm{i}\,f^{iA}_X\Delta^X\epsilon_i + \tfrac{1}{4}\gamma^\mu\Gamma^A\bar{\epsilon}\psi_\mu - \tfrac{1}{4}\gamma^\mu\gamma^\nu\Gamma^A\bar{\epsilon}\gamma_\nu\psi_\mu\,, \tag{6.115}$$

where

$$\begin{aligned}\Delta^X_{\text{conf}} &= \Box^c q^X - \bar{\zeta}^B\gamma^a\zeta^C D_a q^Y \mathcal{R}^X{}_{YBC} + \tfrac{8}{9}T^2k^X \\ &\quad +\tfrac{4}{3}Dk^X + 8\,\mathrm{i}\,\bar{\chi}^i\zeta^A f^X_{iA} - \tfrac{1}{4}\mathcal{D}^X W_{ABCD}\bar{\zeta}^A\zeta^B\bar{\zeta}^C\zeta^D \\ &\quad -g(2\,\mathrm{i}\,\bar{\psi}^{iI}\zeta^B t_{IB}{}^A f^X_{iA} - k^Y_I J_Y{}^X{}_{ij}Y^{Iij}) \\ &\quad +g^2\sigma^I\sigma^J\mathfrak{D}_Y k^X_I k^Y_J\,, \end{aligned} \tag{6.116}$$

and the superconformal d'Alembertian is given by

$$\begin{aligned}\Box^c q^X &\equiv D_a D^a q^X \\ &= \partial_a D^a q^X - \tfrac{5}{2}b_a D^a q^X - \tfrac{1}{2}V^{jk}_a J_Y{}^X{}_{jk}D^a q^Y + \mathrm{i}\,\bar{\psi}^i_a D^a\zeta^A f^X_{iA} \\ &\quad +2f_a{}^a k^X - 2\bar{\psi}_a\gamma^a\chi k^X + 4\bar{\psi}^{(j}_a\gamma^a\chi^{k)}k^X_{jk} - \bar{\psi}^i_a\gamma^a\gamma\cdot T\zeta^A f^X_{iA} \\ &\quad -\bar{\phi}^i_a\gamma^a\zeta^A f^X_{iA} + \omega_a{}^{ab}D_b q^X - \tfrac{1}{2}g\bar{\psi}^a\gamma_a\psi^I k^X_I - D_a q^Y\partial_Y k^X_I A^{aI} \\ &\quad +D_a q^Y D^a q^Z\Gamma^X_{YZ}\,. \end{aligned} \tag{6.117}$$

Note that so far we did not require the presence of an action. Introducing a metric, the locally conformal supersymmetric action is given by

$$\begin{aligned} e^{-1}\mathcal{L}^{\text{conf}}_{\text{hyper}} &= -\tfrac{1}{2}g_{XY}\mathcal{D}_a q^X\mathcal{D}^a q^Y + \bar{\zeta}_A\not{D}\zeta^A + \tfrac{4}{9}Dk^2 + \tfrac{8}{27}T^2k^2 \\ &\quad -\tfrac{16}{3}\,\mathrm{i}\,\bar{\zeta}_A\chi^i k^X f^A_{iX} + 2\,\mathrm{i}\,\bar{\zeta}_A\gamma\cdot T\zeta^A - \tfrac{1}{4}W_{ABCD}\bar{\zeta}^A\zeta^B\bar{\zeta}^C\zeta^D \\ &\quad -\tfrac{2}{9}\bar{\psi}_a\gamma^a\chi k^2 + \tfrac{1}{3}\bar{\zeta}_A\gamma^a\gamma\cdot T\psi^i_a k^X f^A_{iX} + \tfrac{1}{2}\,\mathrm{i}\,\bar{\zeta}_A\gamma^a\gamma^b\psi^i_a\mathcal{D}_b q^X f^A_{iX} \\ &\quad +\tfrac{2}{3}f_a{}^a k^2 - \tfrac{1}{6}\,\mathrm{i}\,\bar{\psi}_a\gamma^{ab}\phi_b k^2 - \bar{\zeta}_A\gamma^a\phi^i_a k^X f^A_{iX} \\ &\quad +\tfrac{1}{12}\bar{\psi}^i_a\gamma^{abc}\psi^j_b\mathcal{D}_c q^Y J_Y{}^X{}_{ij}k_X - \tfrac{1}{9}\,\mathrm{i}\,\bar{\psi}^a\psi^b T_{ab}k^2 + \tfrac{1}{18}\,\mathrm{i}\,\bar{\psi}_a\gamma^{abcd}\psi_b T_{cd}k^2 \\ &\quad -g\Big(\mathrm{i}\,\sigma^I t_{IB}{}^A\bar{\zeta}_A\zeta^B + 2\,\mathrm{i}\,k^X_I f^A_{iX}\bar{\zeta}_A\psi^{iI} + \tfrac{1}{2}\sigma^I k^X_I f^A_{iX}\bar{\zeta}_A\gamma^a\psi^i_a \\ &\qquad + \bar{\psi}^i_a\gamma^a\psi^{jI}P_{Iij} - \tfrac{1}{2}\,\mathrm{i}\,\bar{\psi}^i_a\gamma^{ab}\psi^j_b\sigma^I P_{Iij}\Big) \\ &\quad + 2gY^I_{ij}P^{ij}_I - \tfrac{1}{2}g^2\sigma^I\sigma^J k^X_I k_{JX}\,. \end{aligned} \tag{6.118}$$

Indeed, no further constraints other than those given in section 6.3 were necessary in this local case. In particular, the target space is still hypercomplex or, when an action exists, hyperkähler.

This action leads to the following dynamical equations

$$\begin{aligned}
\frac{\delta \mathcal{S}^{\text{conf}}_{\text{hyper}}}{\delta \bar\zeta^A} &= 2\, C_{AB}\Gamma^B_{\text{conf}}\,, \\
\frac{\delta \mathcal{S}^{\text{conf}}_{\text{hyper}}}{\delta q^X} &= g_{XY}\left(\Delta^Y_{\text{conf}} - 2\bar\zeta_A\Gamma^B_{\text{conf}}\omega^Y{}_B{}^A - \mathrm{i}\,\bar\psi^i_a\gamma^a\Gamma^A_{\text{conf}} f^Y_{iA}\right). \qquad (6.119)
\end{aligned}$$

Again, this action can also be obtained by using the [field]×[non-closure] method. The transformation rules for the non-closure functions now get gravitino-corrections:

$$\begin{aligned}
\delta\Delta^X &= -\mathrm{i}\,\bar\epsilon^i \not{D}\Gamma^A f^X_{iA} + 2\,\mathrm{i}\,\bar\epsilon^i\Gamma^B\zeta^C\zeta^D f^Y_{iB}\mathcal{R}^X{}_{YCD} + 2\bar\eta^i\Gamma^A f^X_{iA} + \Delta^Y\Gamma_{ZY}{}^X\delta q^Z \\
&\quad + \tfrac14\,\mathrm{i}\, f^X_{iA}\bar\psi^i_\mu\Gamma^A\bar\epsilon\psi^\mu - \tfrac14\,\mathrm{i}\, f^X_{iA}\bar\psi^i_\mu\gamma_a\Gamma^A\gamma^a\bar\epsilon\gamma_a\psi^\mu\,, \\
\delta\Gamma^A &= \tfrac12\,\mathrm{i}\, f^{iA}_X\epsilon_i\Delta^X + \tfrac14\gamma^\mu\Gamma^A\bar\epsilon\psi_\mu - \tfrac14\gamma^\mu\gamma^a\Gamma^A\bar\epsilon\gamma_a\psi_\mu - \delta q^X\omega_{XB}{}^A\Gamma^B\,, \qquad (6.120)
\end{aligned}$$

where the covariant derivative is given by

$$\begin{aligned}
D_\mu\Gamma^A &= \mathcal{D}_\mu\Gamma^A + \tfrac13\,\mathrm{i}\,\Delta^A_i\psi^i_\mu - \tfrac18\gamma^\nu\Gamma^A\bar\psi_\mu\psi_\nu + \tfrac18\gamma^\nu\gamma^a\Gamma^A\bar\psi_\nu\gamma_a\psi_\nu\,, \\
\mathcal{D}_\mu\Gamma^A &= (\partial_\mu - 3b_\mu + \tfrac14\omega_\mu{}^{ab}\gamma_{ab})\Gamma^A + \partial_\mu q^X\omega_{XB}{}^A\Gamma^B + gA^I_\mu t_{IB}{}^A\Gamma^B\,. \qquad (6.121)
\end{aligned}$$

There are several ways to determine the coefficients of the gravitino terms; for example by trying to close the $[Q, S]$ commutator on Γ^A and Δ^X, or by using the non-closure functions in the $[Q, Q]$ of ζ^A and $D_\mu A^{iA}$, like explained in [147, p.19-21]. The extra term in the Ansatz can be obtained by requiring S-invariance of the action:

$$e^{-1}\mathcal{L} = \tfrac13 k_X\Delta^X + (\bar\zeta_A - \tfrac13\,\mathrm{i}\,\bar\psi^i_\mu\gamma^\mu k_X f^X_{iA})\Gamma^A\,. \qquad (6.122)$$

Substituting the non-closure functions into (6.122) and partial integrating the covariant box, we again find (6.118).

The Lagrangians (6.110) and (6.118) are the starting point for obtaining matter couplings to Poincaré supergravity. This involves a gauge fixing of the local scale and SU(2) symmetries, which will be studied in the next chapter.

Chapter 7

Gauge fixing

The general idea of this chapter can be illustrated by using a scalar-gravity toy model in four dimensions. We start with a conformally invariant action for a scalar field φ

$$\mathcal{L} = \sqrt{|g|}\left[\tfrac{1}{2}(\partial\varphi)^2 + \tfrac{1}{12}R\varphi^2\right], \tag{7.1}$$

which is invariant under the following local dilatations

$$\delta\varphi = \Lambda_D\varphi\,, \qquad \delta g_{\mu\nu} = -2\Lambda_D g_{\mu\nu}\,. \tag{7.2}$$

This dilatation symmetry can be gauge fixed by choosing the gauge $\varphi = \sqrt{6}/\kappa$; this leads to the Poincaré action

$$\mathcal{L} = \tfrac{1}{2\kappa^2}\sqrt{|g|}\,R\,. \tag{7.3}$$

Therefore the actions (7.1) and (7.3) are *gauge equivalent*. Alternatively, we could have chosen new coordinates ($g'_{\mu\nu} = g_{\mu\nu}\varphi^2$), such that the resulting action is manifestly invariant under the dilatation symmetry. Although φ still transforms under dilatations, the field does not appear in the action anymore. The scalar φ has no physical degrees of freedom, and is called a "compensating scalar". Note that the scalar kinetic term has the wrong sign; this is a generic feature of compensating scalars which we will also encounter in the more complicated case of conformal supergravity.

The same mechanism will be used in this chapter to obtain five-dimensional matter coupled Poincaré supergravity. In chapter 5 the Poincaré algebra was extended to the local superconformal algebra $F^2(4)$. We constructed the minimal representation of the superconformal algebra containing the graviton, called the Standard Weyl multiplet. The fields of the Standard Weyl multiplet and their properties were given in table 5.2. Next, in chapter 6, we constructed vector-tensor multiplets and hypermultiplets in the background of this Weyl multiplet, see tables 6.1, 6.2 and 6.3 for the contents and properties of these multiplets. The corresponding actions, equations of motion and transformation rules were given in (6.110) and (6.118). As a third and final step we now want to gaugefix the extra symmetries, not belonging to the super-Poincaré algebra, and obtain Poincaré supergravity coupled to vector-tensor multiplets and hypermultiplets. As compensators we will need one hypermultiplet and one vector-tensor multiplet. Therefore, the starting point of this chapter will be local $D = 5, \mathcal{N} = 2$ conformal supergravity coupled to $(n_V + n_T + 1)$ vector-tensor multiplets and $(n_H + 1)$ hypermultiplets[1]:

$$\mathcal{L}_{total} = \mathcal{L}_{Vector-Tensor} + \mathcal{L}_{Hyper} . \tag{7.4}$$

This chapter is based on work to be published in [173, 174].

7.1 Conformal geometry

The superconformal tensor calculus performed in the last chapter resulted in the construction of a hypercomplex manifold spanned by $(4n_H + 4)$ hyperscalars $q^{\hat{X}}$ ($\hat{X} = 1, \ldots, 4n_H + 4$). In this chapter we will for simplicity assume the presence of a covariantly constant hermitian metric, which promotes the hypercomplex manifold to a hyperkähler manifold. This manifold includes the four scalars of the compensating hypermultiplet. In the end these compensating scalars will be removed from the manifold; therefore it is convenient to split these coordinates off by making a specific coordinate choice on the hyperkähler manifold. Details about this procedure can be found in [174].[2] Here we will skip the technical details and only give some of the relevant results. We use the hat-notation for objects that are defined on the "higher dimensional" hyperkähler manifold, spanned by the $q^{\hat{X}}$.

As we saw in chapter 6, the manifold contains three generic isometries, generated by the SU(2) Killing vectors $\hat{\vec{k}}^{\hat{X}}$. These isometries were gauged using the vectors of the vector-tensor multiplets. Using Frobenius' theorem it can be shown that the three-dimensional subspace spanned by the direction of the three SU(2) transformations can be parametrized by coordinates z^α ($\alpha = 1, 2, 3$). Furthermore, using the homothetic Killing equation (6.44), one more coordinate z^0 can be singled out, which is associated with the dilatation transformation. The other directions are indicated by q^X ($X = 1, \ldots, 4n_H$). Thus, we split the coordinates on the hyperkähler manifold as $\{q^{\hat{X}}\} = \{z^0, z^\alpha, q^X\}$. Similarly we can split the tangent space index as $\{\hat{A}\} = \{i, A\}$ ($i = 1, 2; A = 1, \ldots, 2n_H$), where i is an SU(2) index. Note that throughout this chapter we will work in this coordinate basis. In this basis the Killing vectors take on the following form

$$\hat{k}^{\hat{X}}(z^0, z^\alpha, q) = \{3z^0, 0, 0\} , \qquad \hat{\vec{k}}^{\hat{X}}(z^0, z^\alpha, q) = \{0, \vec{k}^\alpha(z^0, z^\alpha), 0\} . \tag{7.5}$$

We will choose the following metric parametrization:

$$\begin{aligned} d\hat{s}^2 &\equiv \hat{g}_{\hat{X}\hat{Y}} dq^{\hat{X}} dq^{\hat{Y}} . \\ &= -\frac{(dz^0)^2}{z^0} + z^0 h_{XY}(z^\alpha, q) dq^X dq^Y \\ &\quad -\hat{g}_{\alpha\beta}(z^0, z^\alpha, q)[dz^\alpha + A^\alpha_X(z^0, z^\alpha, q)dq^X][dz^\beta + A^\beta_Y(z^0, z^\alpha, q)dq^Y] , \end{aligned} \tag{7.6}$$

where we have chosen the signs and factors for later convenience. The object h_{XY} denotes the metric on the subspace spanned by the coordinates q^X, and $A^\alpha_X(z, q) \equiv \hat{f}^\alpha_{ij} \hat{f}^{ij}_X$.

[1] In comparison with chapter 6 we have: $n = n_V + 1$, $m = n_T$ and $r = n_H + 1$.

[2] In this reference we also discuss the case without a hyperscalar-metric.

Note the resemblance of (7.6) with the generic form of the Kaluza-Klein Ansatz (3.41). This is not that surprising, since we are in fact performing a "dimensional reduction" of the scalar manifold. In the above coordinate basis we find the following expressions for the vielbeins[3]:

$$\begin{array}{lll} \hat{f}_{ij}^{0} = -\,\mathrm{i}\,\varepsilon_{ij}\sqrt{\tfrac{1}{2}z^0}\,, & \hat{f}_{ij}^{\alpha} = \sqrt{\tfrac{1}{2z^0}}\vec{k}^{\alpha}\cdot\vec{\sigma}_{ij}\,, & \hat{f}_{ij}^{X} = 0\,, \\ \hat{f}_{iA}^{0} = 0\,, & \hat{f}_{iA}^{\alpha} = -f_{iA}^{X}A_X^{\alpha}\,, & \hat{f}_{iA}^{X} = f_{iA}^{X}\,, \\ \hat{f}_{0}^{ij} = \mathrm{i}\,\varepsilon^{ij}\sqrt{\tfrac{1}{2z^0}}\,, & \hat{f}_{\alpha}^{ij} = \sqrt{\tfrac{1}{2z^0}}\vec{k}_{\alpha}\cdot\vec{\sigma}^{ij}\,, & \hat{f}_{X}^{ij} = \sqrt{\tfrac{1}{2z^0}}\vec{k}_{\alpha}\cdot\vec{\sigma}^{ij}A_X^{\alpha}\,, \\ \hat{f}_{0}^{iA} = 0\,, & \hat{f}_{\alpha}^{iA} = 0\,, & \hat{f}_{X}^{iA} = f_{X}^{iA}\,. \end{array} \tag{7.7}$$

Using the above expressions for the vielbeins, we find the following complex structures:

$$\begin{array}{lll} \widehat{\vec{J}_0}{}^0 = 0\,, & \widehat{\vec{J}_\alpha}{}^0 = \vec{k}_\alpha\,, & \widehat{\vec{J}_X}{}^0 = A_X^{\alpha}\vec{k}_\alpha\,, \\ \widehat{\vec{J}_0}{}^\beta = \tfrac{1}{z^0}\vec{k}^\beta\,, & \widehat{\vec{J}_\alpha}{}^\beta = \tfrac{1}{z^0}\vec{k}_\alpha\times\vec{k}^\beta\,, & \widehat{\vec{J}_X}{}^\beta = \tfrac{1}{z^0}A_X^{\gamma}\vec{k}_\gamma\times\vec{k}^\beta - \vec{J}_X{}^Z A_Z^\beta\,, \\ \widehat{\vec{J}_0}{}^Y = 0\,, & \widehat{\vec{J}_\alpha}{}^Y = 0\,, & \widehat{\vec{J}_X}{}^Y = \vec{J}_X{}^Y\,. \end{array} \tag{7.8}$$

Covariant constancy of the vielbeins furthermore leads to the expressions for the $G\ell(n_H+1,\mathbb{H})$ connections. The non-zero components are given by:

$$\begin{array}{ll} \hat{\omega}_{\alpha i}{}^{j} = -\,\mathrm{i}\,\tfrac{1}{2z^0}\vec{k}_\alpha\cdot\vec{\sigma}_i{}^j\,, & \hat{\omega}_{\alpha A}{}^{B} = \tfrac{1}{2}f_Y^{iB}\partial_\alpha f_{iA}^Y\,, \\ \hat{\omega}_{X i}{}^{j} = A_X^\alpha\hat{\omega}_{\alpha i}{}^{j}\,, & \hat{\omega}_{0A}{}^{B} = \tfrac{1}{2}f_Y^{iB}\partial_0 f_{iA}^Y + \tfrac{1}{2z^0}\delta_A^B\,, \\ \hat{\omega}_{X i}{}^{A} = \mathrm{i}\,\sqrt{\tfrac{1}{2z^0}}\varepsilon_{ik}\hat{f}_X^{kA}\,, & \hat{\omega}_{XA}{}^{i} = -\,\mathrm{i}\,\sqrt{\tfrac{z^0}{2}}\varepsilon^{ij}f_{jA}^Y h_{YX}\,. \end{array} \tag{7.9}$$

Using these results, some other relevant expressions can be derived

$$\hat{C}_{AB} = z^0 C_{AB}\,, \qquad \hat{C}_{ij} = \varepsilon_{ij}\,, \qquad \hat{C}_{iA} = 0\,, \tag{7.10}$$

$$\hat{W}_{ABC}{}^{D} = \mathcal{W}_{ABC}{}^{D}\,, \tag{7.11}$$

$$\widehat{\vec{P}_I} = \vec{P}_I\,, \tag{7.12}$$

$$\hat{k}_I^{\hat{X}} = \{0, -2\vec{k}^\alpha(\vec{\omega}_X k_I^X - \tfrac{1}{z^0}\vec{P}_I), k_I^X\}\,, \tag{7.13}$$

where $\mathcal{W}_{ABC}{}^{D}$ is the 'quaternionic Weyl tensor' defined in (C.54).

We found that for each point in the subspace $\{z^\alpha\}$, corresponding to a specific gauge fixing, the $\{q^X\}$ space describes a quaternionic-Kähler manifold. These manifolds are all related to each other by coordinate redefinitions.

We point out in appendix C.3 that the connections on a quaternionic manifold are not uniquely defined; a certain ξ-transformation can be performed to choose a convenient gauge for the connections. The gauge chosen in [174] leads to the following expressions for the $G\ell(n_H,\mathbb{H})$ and SU(2) connections:

$$\hat{\omega}_{XA}{}^{B} = \omega_{XA}{}^{B}\,, \qquad \vec{\omega}_X = -\tfrac{1}{2z^0}A_X^\alpha\vec{k}_\alpha\,. \tag{7.14}$$

Note that before gauge fixing the unhatted objects are dependent on the z-coordinates.

[3] These expressions do not represent reduction Ansätze, because the fields on the right hand side still depend on $\{z^0, z^\alpha\}$.

7.2 Gauge fixing

The actions given in (6.110) and (6.118) are invariant under the full supercovariant group. In order to break the symmetries that are not present in the Poincaré algebra, we will impose the necessary gauge conditions in the following subsections.

7.2.1 Preliminaries

The first step in the gaugefixing process will be the elimination of the dependent gauge fields ϕ_μ^i and $f_\mu{}^a$, associated to S- and K-symmetry respectively. Using the relations (5.35) together with the definitions of the supercovariant curvatures, we find the following expressions for the dependent gauge fields

$$\begin{aligned} f_a{}^a &= \tfrac{1}{16}\Big(-R(\hat\omega) - \tfrac{1}{3}\bar\psi_\rho\gamma^{\rho\mu\nu}\mathcal{D}_\mu\psi_\nu \\ &\quad + \tfrac{1}{3}\bar\psi^i_a\gamma^{abc}\psi^j_b V_{cij} + 16\bar\psi_a\gamma^a\chi - 4\,\mathrm{i}\,\bar\psi^a\psi^b T_{ab} + \tfrac{4}{3}\,\mathrm{i}\,\bar\psi^b\gamma_{abcd}\psi^a T^{cd}\Big)\,, \\ \hat\omega_\mu{}^{ab} &= \omega_\mu{}^{ab}(e) - \tfrac{1}{2}\bar\psi^{[b}\gamma^{a]}\psi_\mu - \tfrac{1}{4}\bar\psi^b\gamma_\mu\psi^a + 2e_\mu{}^{[a}b^{b]}\,, \\ \phi_\mu^i &= \tfrac{1}{2}\,\mathrm{i}\,\gamma^\nu\mathcal{D}_{[\mu}\psi^i_{\nu]} - \tfrac{1}{12}\,\mathrm{i}\,\gamma_\mu{}^{\nu\rho}\mathcal{D}_\nu\psi^i_\rho - \tfrac{1}{2}\,\mathrm{i}\,V_{[\mu}{}^{ij}\gamma^a\psi_{a]\,j} + \tfrac{1}{12}\,\mathrm{i}\,V_a{}^{ij}\gamma_\mu{}^{ab}\psi_{b\,j} \\ &\quad - T^a{}_\mu\psi^i_a - \tfrac{1}{3}T^{ab}\gamma_{b\mu}\psi^i_a - \tfrac{2}{3}T_{b\mu}\gamma^{ab}\psi^i_a - \tfrac{1}{3}T_{bc}\gamma^{abc}{}_\mu\psi_a \\ &\quad - \tfrac{1}{12}\,\mathrm{i}\,(\gamma^{ab}\gamma_\mu - \tfrac{1}{2}\gamma_\mu\gamma^{ab})b_a\psi^i_b\,, \end{aligned} \tag{7.15}$$

with

$$\mathcal{D}_\mu = \partial_\mu + \tfrac{1}{4}\hat\omega_\mu{}^{ab}\gamma_{ab}\,. \tag{7.16}$$

We only need the contracted version of $f_\mu{}^a$ since the other components do not appear in the action or transformation rules. Also, in order to simplify notation we will choose not to eliminate $\hat\omega_\mu{}^{ab}$ in most places.

First of all we observe that, after writing out all covariant derivatives, the gauge field b_μ does not appear in the action. This can be argued from K-invariance of the action. Although this prohibits us from determining its equation of motion, we will choose the conventional gauge choice for K-symmetry, namely $b_\mu = 0$.

This still leaves us with one more gauge field corresponding to a non-Poincaré symmetry: the SU(2) gauge field V_μ^{ij}. Solving for its equation of motion, corresponding to the action (7.4), gives us the following expression

$$\begin{aligned} V_\mu^{ij} &= \tfrac{9}{2k^2}\Big(\hat g_{\hat X\hat Y}(\partial_\mu q^{\hat X} + gA^I_\mu k^{\hat X}_I)k^{ij\hat Y} + \tfrac{1}{2}\,\mathrm{i}\,k^{\hat X}\hat f^{i\hat A}_{\hat X}\bar\zeta_{\hat A}\gamma_{\mu\nu}\psi^{\nu\,j} - \mathrm{i}\,k^{ij\hat X}\hat f^{\hat A}_{k\hat X}\bar\zeta_{\hat A}\gamma_\nu\gamma_\mu\psi^{k\nu} \\ &\quad - \tfrac{1}{2}C_{\widetilde I\widetilde J\widetilde K}\sigma^{\widetilde K}\bar\psi^{i\widetilde I}\gamma_\mu\psi^{j\widetilde J} + \tfrac{1}{4}\,\mathrm{i}\,C_{\widetilde I\widetilde J\widetilde K}\sigma^{\widetilde K}\sigma^{\widetilde I}\bar\psi^{i\widetilde J}\gamma_{\mu\nu}\psi^{j\nu}\Big)\,. \end{aligned} \tag{7.17}$$

The action contains four auxiliary matter fields: D, T_{ab} and χ^i from the Weyl multiplet, and $Y^{\widetilde I}_{ij}$ from the vector-tensor multiplet. Both D and χ^i appear as Lagrange multipliers in the action, leading to the following constraints, respectively

$$D \ : \qquad C - \tfrac{1}{3}k^2 = 0\,, \qquad \text{with} \quad C \equiv C_{\widetilde I\widetilde J\widetilde K}\sigma^{\widetilde I}\sigma^{\widetilde J}\sigma^{\widetilde K}\,, \tag{7.18}$$

$$\chi^i \ : \qquad -8\,\mathrm{i}\,C_{\widetilde I\widetilde J\widetilde K}\sigma^{\widetilde I}\sigma^{\widetilde J}\psi^{\widetilde K}_i - \tfrac{4}{3}\Big(C - \tfrac{1}{3}k^2\Big)\gamma^\mu\psi_{\mu i} + \tfrac{16}{3}\,\mathrm{i}\,A^{\hat A}_i\zeta_{\hat A} = 0\,. \tag{7.19}$$

The equations of motion for $Y^{\widetilde{I}}_{ij}$ and T_{ab} are given by

$$Y^{ij\widetilde{J}}C_{\widetilde{I}\widetilde{J}\widetilde{K}}\sigma^{\widetilde{K}} = -g\,\delta^{L}_{\widetilde{I}}\,\hat{P}^{ij}_{L} + \tfrac{1}{4}\,\mathrm{i}\,C_{\widetilde{I}\widetilde{J}\widetilde{K}}\bar{\psi}^{i\widetilde{J}}\psi^{j\widetilde{K}}\,, \tag{7.20}$$

$$\begin{aligned}T_{ab} = \tfrac{9}{64k^2}\Big(&4\sigma^{\widetilde{I}}\sigma^{\widetilde{J}}\widehat{\mathcal{H}}^{\widetilde{K}}_{ab}C_{\widetilde{I}\widetilde{J}\widetilde{K}} + \sigma^{\widetilde{I}}\sigma^{\widetilde{J}}\bar{\psi}^{\widetilde{K}}\gamma_{[a}\psi_{b]}C_{\widetilde{I}\widetilde{J}\widetilde{K}} + \sigma^{\widetilde{I}}\sigma^{\widetilde{J}}\bar{\psi}^{\widetilde{K}}\gamma_{abc}\psi^{c}C_{\widetilde{I}\widetilde{J}\widetilde{K}}\\ &+\mathrm{i}\,\sigma^{\widetilde{I}}\bar{\psi}^{\widetilde{J}}\gamma_{ab}\psi^{\widetilde{K}}C_{\widetilde{I}\widetilde{J}\widetilde{K}} + \tfrac{2}{3}k^{\hat{X}}f^{\hat{A}}_{i\hat{X}}\bar{\zeta}_{\hat{A}}\gamma_{[a}\psi^{i}_{b]} + \tfrac{2}{3}k^{\hat{X}}f^{\hat{A}}_{i\hat{X}}\bar{\zeta}_{\hat{A}}\gamma_{abc}\psi^{ic} + 2\,\mathrm{i}\,\bar{\zeta}_{\hat{A}}\gamma_{ab}\zeta^{\hat{A}}\Big)\,,\end{aligned} \tag{7.21}$$

which have been simplified by making use of (7.18).

7.2.2 Gauge choices and decomposition rules

Apart from the K-gauge we already introduced to fix the special conformal (K-)symmetry, we will have to choose gauges for the other non-Poincaré (super)symmetries as well.

D-gauge

Having written out all dependent gaugefields in the action, the kinetic terms for the graviton and the gravitino become

$$e^{-1}\mathcal{L}_{\mathrm{EH+RS}} = \tfrac{1}{24}\left(C + k^2\right)\left(R(\hat{\omega}) + \bar{\psi}_{\mu}\gamma^{\mu\nu\rho}\mathcal{D}_{\nu}\psi_{\rho}\right)\,. \tag{7.22}$$

Similarly to the example given in (7.1)–(7.3) we can demand canonical factors for the Einstein-Hilbert and Rarita-Schwinger kinetic terms means by imposing the following D-gauge:

$$\tfrac{1}{24}\left(C + k^2\right) = -\frac{1}{2\kappa^2}\,. \tag{7.23}$$

Note that in order to get the conventional mass-dimensions for the Rarita-Schwinger term, we identify the superconformal gravitino ψ^{C}_{μ} in terms of the gravitino ψ^{P}_{μ} from the super-Poincaré multiplet as follows:

$$\psi^{C}_{\mu} \equiv \kappa\psi^{P}_{\mu}\,. \tag{7.24}$$

The index P will be suppressed in the rest of this chapter. If we combine the D-gauge (7.23) and the equation of motion for D (7.18) we obtain

$$k^2 = -\frac{9}{\kappa^2}\,, \qquad C = -\frac{3}{\kappa^2}\,. \tag{7.25}$$

The first constraint implies that $z^0 = \frac{1}{\kappa^2}$, whereas the second constraint effectively eliminates one of the vector-tensor scalars.

S-gauge

Off-diagonal kinetic terms like e.g. $\bar{\psi}_{\mu}\mathcal{D}\psi$ or $\bar{\psi}^{i}_{\mu}\mathcal{D}\zeta^{A}$ appear in the action with one overall coefficient. A canonical form of the action requires the vanishing of these terms, which can be accomplished by demanding the overall coefficient to vanish. This leads to the following constraint, called the S-gauge:

$$C_{\widetilde{I}\widetilde{J}\widetilde{K}}\sigma^{\widetilde{I}}\sigma^{\widetilde{J}}\psi^{\widetilde{K}}_{i} = 0\,. \tag{7.26}$$

This constraint effectively eliminates one of the gauginos.

SU(**2**)-gauge

The gauge for dilatations was chosen such that $z^0 = \frac{1}{\kappa^2}$. Similarly we may also choose a gauge for SU(2). Such a gauge would be a specific point in the three-dimensional space of the z^α. Any fixed value of these coordinates would fix a gauge, however we will leave this arbitrary. The dependence of objects on the hyperkähler manifold on the coordinates z^α thus describes the gauge dependence. By fixing the SU(2) gauge, i.e choosing z^α to be constant, all fields become particular functions of the quaternionic-Kähler coordinates q^X only. These functions may be different for different gauge choices, but once we make a choice, which is not relevant for further considerations, they are fixed.

Using both the S-gauge (7.26) and the equation of motion for D (7.18) in equation (7.19) we also get a constraint on the hyperino

$$A_i^{\hat{A}} \zeta_{\hat{A}} = 0\,. \tag{7.27}$$

In our coordinate basis, we obtain the following expression for the sections $A^i_{\hat{A}}$

$$A^i_{\hat{A}} \equiv \varepsilon^{ij} k_{\hat{X}} \hat{f}^{\hat{X}}_{j\hat{A}} = -3\varepsilon^{ij} \hat{f}^0_{j\hat{A}} = -3\,\mathrm{i}\,\sqrt{\tfrac{z^0}{2}}\delta^i_{\hat{A}}\,. \tag{7.28}$$

After applying the D-gauge, i.e. fixing one degree of freedom of $q^{\hat{X}}$ by (7.25), equation (7.28) plays the role of SU(2) gauge since it fixes three of the degrees of freedom contained in $A^i_{\hat{A}}$. Moreover, combining it with (7.27) one discovers that our choice of coordinates on the hyperkähler manifold is consistent with the hyperinos of the compensating multiplet carrying no physical information:

$$\zeta_i \equiv \zeta^j \varepsilon_{ji} = 0\,. \tag{7.29}$$

Decomposition rules

As a consequence of the gauge choices, the corresponding transformation parameters can be expressed in terms of the others by so-called decomposition rules. These rules will enable us to eliminate the parameters $\Lambda_D, \Lambda_K^a, \Lambda^{ij}_{\mathrm{SU}(2)}, \eta^i$ and determine the transformation rules for the remaining symmetries in section 7.3. For example, the requirement that the K-gauge should be invariant under the most general superconformal transformation, i.e. $\delta b_\mu = 0$, leads to the decomposition rule for Λ_K^a:

$$\Lambda_K^a = -\tfrac{1}{2} e^{\mu a}\left(\partial_\mu \Lambda_D + \tfrac{1}{2}\,\mathrm{i}\,\bar{\epsilon}\phi_\mu - 2\bar{\epsilon}\gamma_\mu\chi + \tfrac{\kappa}{2}\,\mathrm{i}\,\bar{\eta}\psi_\mu\right). \tag{7.30}$$

Similarly, demanding $\delta z^0 = 0$ yields

$$\Lambda_D = 0\,. \tag{7.31}$$

The decomposition rule for η^i can be found by varying the S-gauge and demanding that

$$\delta\left(C_{\widetilde{I}\widetilde{J}\widetilde{K}}\sigma^{\widetilde{I}}\sigma^{\widetilde{J}}\psi^{i\widetilde{K}}\right) = 0\,. \tag{7.32}$$

We find

$$\begin{aligned}\eta^i \;=\; & -\tfrac{\kappa^2}{12} C_{\widetilde{I}\widetilde{J}\widetilde{K}}\sigma^{\widetilde{I}}\sigma^{\widetilde{J}}\gamma\cdot\widehat{\mathcal{H}}^{\widetilde{K}}\epsilon^i + \tfrac{1}{3} g\sigma^I P_I^{ij}\epsilon_j + \tfrac{1}{32\kappa^2}\,\mathrm{i}\,\gamma^{ab}\epsilon^i\bar{\zeta}_A\gamma_{ab}\zeta^A \\ & + \tfrac{\kappa^2}{16}\,\mathrm{i}\,C_{\widetilde{I}\widetilde{J}\widetilde{K}}\sigma^{\widetilde{I}}\left(\gamma^a\epsilon_j\bar{\psi}^{i\widetilde{J}}\gamma_a\psi^{j\widetilde{K}} - \tfrac{1}{16}\gamma^{ab}\epsilon^i\bar{\psi}^{\widetilde{J}}\gamma_{ab}\psi^{\widetilde{K}}\right).\end{aligned} \tag{7.33}$$

The SU(2) decomposition rule can be found by requiring that $\delta z^\alpha = 0$:

$$\Lambda^{ij}_{\mathrm{SU}(2)} = \omega^{ij}_X(\delta_Q + \delta_G)q^X + \kappa^2 g \Lambda^I_G P^{ij}_I\,. \tag{7.34}$$

7.2.3 Hypersurfaces

The gauge condition for the vector/tensor scalars (7.25), defines a $(n_V + n_T)$-dimensional hypersurface of scalars φ^x called a "very special real" manifold, embedded into a $(n_V + n_T + 1)$-dimensional space spanned by the scalars $h^{\widetilde{I}}(\varphi)$. In order to find the kinetic term for the scalars φ^x we need to identify the embedding metric g_{xy}. At this point it is convenient to rescale the $C_{\widetilde{I}\widetilde{J}\widetilde{K}}$ symbol and to redefine our scalars, in order to get a convenient normalization:

$$\begin{aligned}\sigma^{\widetilde{I}} &\equiv \alpha h^{\widetilde{I}}\,, \qquad \alpha = \sqrt{\tfrac{3}{2\kappa^2}}\,,\\ C_{\widetilde{I}\widetilde{J}\widetilde{K}} &\equiv -\tfrac{2}{\alpha} N_{\widetilde{I}\widetilde{J}\widetilde{K}}\,,\\ \mathcal{N} &\equiv N_{\widetilde{I}\widetilde{J}\widetilde{K}} h^{\widetilde{I}} h^{\widetilde{J}} h^{\widetilde{K}} = 1\,.\end{aligned} \tag{7.35}$$

The metric on the $h^{\widetilde{I}}$-manifold can be determined by substituting the equation of motion for T_{ab} (7.21) back into the action, and defining the kinetic term for the vectors/tensors as

$$\mathcal{L}_{\mathrm{kin,vec-ten}} = -\tfrac{1}{4} a_{\widetilde{I}\widetilde{J}} \widehat{\mathcal{H}}^{\widetilde{I}}_{\mu\nu} \widehat{\mathcal{H}}^{\mu\nu\widetilde{J}}\,. \tag{7.36}$$

We then find

$$a_{\widetilde{I}\widetilde{J}} = -2N_{\widetilde{I}\widetilde{J}\widetilde{K}} h^{\widetilde{K}} + 3 h_{\widetilde{I}} h_{\widetilde{J}}\,, \tag{7.37}$$

where

$$h_{\widetilde{I}} \equiv a_{\widetilde{I}\widetilde{J}} h^{\widetilde{J}} = N_{\widetilde{I}\widetilde{J}\widetilde{K}} h^{\widetilde{J}} h^{\widetilde{K}} \qquad \Rightarrow \qquad h_{\widetilde{I}} h^{\widetilde{I}} = 1\,. \tag{7.38}$$

In the following we will assume that $a_{\widetilde{I}\widetilde{J}}$ is invertible; this enables us to solve (7.20) for $Y^{ij\widetilde{I}}$.

$$Y^{ij\widetilde{I}} = -\left(a^{\widetilde{I}\widetilde{J}} - \tfrac{3}{2} h^{\widetilde{I}} h^{\widetilde{J}}\right)\left(g\,\delta^L_{\widetilde{J}} \hat{P}^{ij}_L - \frac{\kappa}{\sqrt{6}}\,\mathrm{i}\, N_{\widetilde{J}\widetilde{K}\widetilde{L}} \bar{\psi}^{i\widetilde{K}} \psi^{j\widetilde{L}}\right)\,. \tag{7.39}$$

This expression is needed to eliminate $Y^{ij\widetilde{I}}$ from the action and transformation rules. For convenience we introduce the following notation:

$$h^{\widetilde{I}}_x \equiv -\sqrt{\tfrac{3}{2}} h^{\widetilde{I}}_{,x}(\varphi)\,, \quad \rightarrow \quad h_{\widetilde{I}x} \equiv a_{\widetilde{I}\widetilde{J}} h^{\widetilde{J}}_x(\varphi) = \sqrt{\tfrac{3}{2}} h_{\widetilde{I},x}(\varphi)\,. \tag{7.40}$$

It follows from (7.38) that:

$$h_{\widetilde{I}} h^{\widetilde{I}}_x = h^x_{\widetilde{I}} h^{\widetilde{I}} = 0\,. \tag{7.41}$$

Let us now focus on the embedding manifold, spanned by the scalars φ^x. We define the embedding metric on this surface as

$$g_{xy} = h^{\widetilde{I}}_x h^{\widetilde{J}}_y a_{\widetilde{I}\widetilde{J}}\,. \tag{7.42}$$

This metric indeed gives the required kinetic term for φ. Apart from a metric, we can also introduce vielbeins f_x^a that are covariantly constant with respect to the spin-connection $\omega_x^{\tilde{a}\tilde{b}}$ and Levi-Civita connection Γ^z_{xy}, defined on this manifold:

$$\begin{aligned}
\Gamma^z_{xy} &= \tfrac{1}{2} g^{zw}\left(-g_{xy,w} + g_{wx,y} + g_{yw,x}\right), \qquad\qquad g_{xy;z} = 0\,,\\
g_{xy} &= \eta_{\tilde{a}\tilde{b}} f_x^{\tilde{a}} f_y^{\tilde{b}}\,, \\
f^{\tilde{a}}_{y;x} &= f^{\tilde{a}}_{y,x} + \omega_x^{\tilde{a}\tilde{b}} f_{y\tilde{b}} - \Gamma^z_{xy} f_z^{\tilde{a}} = 0\,, \qquad f_{[x,y]} = \omega_{[x}^{\tilde{a}\tilde{b}} f_{y]\tilde{b}}\,,\\
h^{\tilde{a}}_{\tilde{I}} &\equiv f_x^{\tilde{a}} h^x_{\tilde{I}}\,.
\end{aligned} \tag{7.43}$$

For future usage we also give the following useful relations, that follow from the above:

$$\begin{aligned}
h_{\tilde{I}x;y} &= h_{\tilde{I}x,y} - \Gamma^z_{xy} h_{\tilde{I}z} = \sqrt{\tfrac{2}{3}}(h_{\tilde{I}} g_{xy} + T_{xyz} h^z_{\tilde{I}})\\
T_{xyz} &\equiv \sqrt{\tfrac{3}{2}} h_{\tilde{J}x;y} h_z^{\tilde{J}} = -\sqrt{\tfrac{3}{2}} h_{\tilde{J}x} h^{\tilde{J}}_{z;y} = h_x^{\tilde{I}} h_y^{\tilde{J}} h_z^{\tilde{K}} N_{\tilde{I}\tilde{J}\tilde{K}}\\
\Gamma^w_{xy} &= h^w_{\tilde{I}} h^{\tilde{I}}_{x,y} + \sqrt{\tfrac{2}{3}} T_{xyz} g^{zw}\,,
\end{aligned} \tag{7.44}$$

The $(n_V + n_T + 1)$ gauginos $\psi^{\tilde{I}}$ are also still constrained fields, due to the S-gauge. In order to translate these to $(n_V + n_T)$ unconstrained gauginos on the embedding space, we introduce $\lambda^{i\tilde{a}}$, which transforms as a vector in the tangent space. As we will see later, a convenient choice is given by (for agreement with the literature [76]):

$$\lambda^{i\tilde{a}} \equiv -h^a_{\tilde{I}} \psi^{i\tilde{I}}\,, \qquad \psi^{i\tilde{I}} = -h^{\tilde{I}}_{\tilde{a}} \lambda^{i\tilde{a}}\,. \tag{7.45}$$

Note that this choice for $\psi^{i\tilde{I}}$ indeed solves the S-gauge (7.26).

7.3 Results

The scalar potential. We will now determine the scalar potential like it appears in the gauge-fixed action. We will have to take into account all terms in (6.110) and (6.118) of order g^2. As the solution for Y^{ijI} (7.39) contains a term linear in g, the Y^2 and gYP terms in the actions will both contribute to the scalar potential:

$$V_{\text{scalar}} = C_{\tilde{I}\tilde{J}\tilde{K}} Y^{\tilde{I}}_{ij} Y^{ij\tilde{J}} \sigma^{\tilde{K}} - \tfrac{1}{2} g^2 \sigma^I \sigma^J \sigma^K \sigma^{\tilde{M}} \sigma^{\tilde{N}} t_{J\tilde{M}}{}^P t_{K\tilde{N}}{}^Q C_{IPQ} + 2g Y^I_{ij} \hat{P}^{ij}_I - \tfrac{1}{2} g^2 \sigma^{\tilde{I}} \sigma^{\tilde{J}} \hat{k}^{\hat{X}}_{\tilde{I}} \hat{k}_{\tilde{J}\hat{X}}\,. \tag{7.46}$$

After performing the rescaling of C and σ into N and h, substituting the expression for Y and applying the specific coordinate basis, the potential can be simplified to:

$$V_{\text{scalar}} = \frac{g^2}{\kappa^4}\left[2W^x W_x - 4\vec{P}\cdot\vec{P} + 2\vec{P}^x\cdot\vec{P}_x + 2\mathcal{N}_{iA}\mathcal{N}^{iA}\right], \tag{7.47}$$

where we defined the following quantities

$$\begin{aligned}
W^x &\equiv \tfrac{\sqrt{6}}{4} h^I K_I^x = -\tfrac{3}{4} t_{J\tilde{M}}{}^P h^J h^{\tilde{M}} h^x_P\,, \qquad K^x_I \equiv -\sqrt{\tfrac{3}{2}} t_{I\tilde{M}}{}^{\tilde{P}} h^{\tilde{M}} h^x_{\tilde{P}}\,,\\
\vec{P} &\equiv \kappa^2 h^I \vec{P}_I\,, \qquad \vec{P}_x \equiv \kappa^2 h^I_x \vec{P}_I\,, \qquad \mathcal{N}^{iA} \equiv \tfrac{\sqrt{6}}{4} h^I k^X_I\,.
\end{aligned} \tag{7.48}$$

The composite object $\vec{P}$, containing the moment map, also occurs with its derivative and is indeed the superpotential for this scalar potential.

The action. After applying the special coordinate basis, substituting the expressions for the dependent gauge fields and matter fields, and "reducing" the objects on the hyperkähler manifold to the quaternionic-Kähler manifold, we obtain the following action:

$$
\begin{aligned}
e^{-1}\mathcal{L} \;=\;& \tfrac{1}{2\kappa^2}R(\omega) - \tfrac{1}{4}a_{\widetilde{I}\widetilde{J}}\widetilde{\mathcal{H}}^{\widetilde{I}}_{\mu\nu}\widetilde{\mathcal{H}}^{\widetilde{J}\mu\nu} - \tfrac{1}{2}g_{xy}\mathcal{D}_a\varphi^x\mathcal{D}^a\varphi^y - \tfrac{1}{2\kappa^2}h_{XY}\mathcal{D}_a q^X\mathcal{D}^a q^Y \\
&+\frac{1}{16g}e^{-1}\varepsilon^{\mu\nu\rho\sigma\tau}\Omega_{MN}\widetilde{B}^M_{\mu\nu}\Big(\partial_\rho\widetilde{B}^N_{\sigma\tau} + 2g t_{IJ}{}^N A^I_\rho F^J_{\sigma\tau} + g\, t_{IP}{}^N A^I_\rho \widetilde{B}^P_{\sigma\tau}\Big) \\
&-\tfrac{1}{2}\bar{\psi}_\rho\gamma^{\rho\mu\nu}\mathcal{D}_\mu\psi_\nu - \tfrac{1}{2}\bar{\lambda}_x \not{\mathcal{D}}\lambda^x + \tfrac{1}{\kappa^2}\bar{\zeta}_A\not{\mathcal{D}}\zeta^A \\
&+\tfrac{g^2}{\kappa^4}\Big(-2W_xW^x + 4\vec{P}\cdot\vec{P} - 2\vec{P}^x\cdot\vec{P}_x - 2\mathcal{N}_{iA}\mathcal{N}^{iA}\Big) \\
&+\tfrac{\kappa}{12}\sqrt{\tfrac{2}{3}}e^{-1}\varepsilon^{\mu\nu\lambda\rho\sigma}N_{IJK}A^I_\mu\Big[F^J_{\nu\lambda}F^K_{\rho\sigma} + f_{FG}{}^J A^F_\nu A^G_\lambda\Big(-\tfrac{1}{2}g\,F^K_{\rho\sigma} + \tfrac{1}{10}g^2 f_{HL}{}^K A^H_\rho A^L_\sigma\Big)\Big] \\
&-\tfrac{1}{8}e^{-1}\varepsilon^{\mu\nu\lambda\rho\sigma}\Omega_{MN}t_{IK}{}^M t_{FG}{}^N A^I_\mu A^F_\nu A^G_\lambda\Big(-\tfrac{1}{2}g\,F^K_{\rho\sigma} + \tfrac{1}{10}g^2 f_{HL}{}^K A^H_\rho A^L_\sigma\Big) \\
&-\tfrac{1}{4}\kappa h_{\widetilde{I}x}H^{\widetilde{I}}_{bc}\bar{\psi}_a\gamma^{abc}\lambda^x - \tfrac{3}{8\sqrt{6}}\kappa\,\mathrm{i}\,h_{\widetilde{I}}H^{cd\widetilde{I}}\bar{\psi}^a\gamma_{abcd}\psi^b + \tfrac{1}{4}\sqrt{\tfrac{2}{3}}\kappa\,\mathrm{i}\,T_{xyz}h^z_{\widetilde{I}}\bar{\lambda}^x\gamma\cdot H^{\widetilde{I}}\lambda^y \\
&+\tfrac{1}{8\sqrt{6}}\kappa\,\mathrm{i}\,h_{\widetilde{I}}\bar{\lambda}^x\gamma\cdot H^{\widetilde{I}}\lambda_x + \tfrac{1}{4}\sqrt{\tfrac{3}{2\kappa^2}}\,\mathrm{i}\,h_{\widetilde{I}}\bar{\zeta}_A\gamma\cdot H^{\widetilde{I}}\zeta^A + \tfrac{1}{2}\,\mathrm{i}\,\bar{\psi}_a\not{\mathcal{D}}\varphi^x\gamma^a\lambda_x + \mathrm{i}\,\tfrac{1}{\kappa}\bar{\zeta}_A\gamma^a\not{\mathcal{D}}q^X\psi^i_a f^A_{iX} \\
&-g\Big(\sqrt{\tfrac{3}{2}}\tfrac{1}{\kappa^3}\,\mathrm{i}\,h^I t_{IB}{}^A\bar{\zeta}_A\zeta^B - 2\,\mathrm{i}\,\tfrac{1}{\kappa^2}k^X_I f^A_{iX}h^I_x\bar{\zeta}_A\lambda^{ix} + \sqrt{\tfrac{3}{2}}\tfrac{1}{\kappa^2}h^I k^X_I f^A_{iX}\bar{\zeta}_A\gamma^a\psi^i_a - \kappa\bar{\psi}^i_a\gamma^a\lambda^{jx}h^I_x P_{Iij} \\
&-\tfrac{1}{2}\sqrt{\tfrac{3}{2}}\kappa\,\mathrm{i}\,h^I P_{Iij}\bar{\psi}^i_a\gamma^{ab}\psi^j_b + \sqrt{\tfrac{2}{3}}\kappa\,\mathrm{i}\,T_{xyz}h^{Iz}P_{Iij}\bar{\lambda}^{ix}\lambda^{jy} + \tfrac{\kappa}{2\sqrt{6}}\,\mathrm{i}\,h^I P_{Iij}\bar{\lambda}^{ix}\lambda^j_x \\
&+\sqrt{\tfrac{3}{2}}\tfrac{1}{\kappa}\,\mathrm{i}\,h^{\widetilde{I}}_x h^{\widetilde{J}}_y\bar{\lambda}^x\lambda^y h^{\widetilde{K}}h^{\widetilde{L}}(t_{\widetilde{I}\widetilde{J}}{}^{\widetilde{M}}N_{\widetilde{M}\widetilde{K}\widetilde{L}} + t_{\widetilde{K}\widetilde{I}}{}^{\widetilde{M}}N_{\widetilde{M}\widetilde{J}\widetilde{L}}) - \tfrac{3}{4}\tfrac{1}{\kappa}\bar{\psi}_a\gamma^a\lambda^x h^{\widetilde{I}}_x h^{\widetilde{J}}h_{\widetilde{K}}t_{\widetilde{J}\widetilde{I}}{}^{\widetilde{K}}\Big) \\
&-\tfrac{\kappa^2}{16}\bar{\psi}^i_a\psi^{ja}\bar{\lambda}^x_i\lambda_{jx} - \tfrac{\kappa^2}{16}\bar{\psi}^i_a\gamma_b\psi^{ja}\bar{\lambda}^d_i\gamma^b\lambda_{jx} - \tfrac{\kappa^2}{64}\bar{\psi}_a\gamma_{bc}\psi^a\bar{\lambda}^x\gamma^{bc}\lambda_x - \tfrac{\kappa^2}{96}\bar{\psi}_a\psi_b\bar{\lambda}^x\gamma^{ab}\lambda_x \\
&+\tfrac{\kappa^2}{96}\bar{\psi}_a\gamma_b\psi_c\bar{\lambda}^x\gamma^{abc}\lambda_x - \tfrac{\kappa^2}{24}\bar{\psi}^i_a\gamma^{ab}\psi^j_b\bar{\lambda}^x_i\lambda_{jx} - \tfrac{\kappa^2}{24}\bar{\psi}^{ai}\gamma^{bc}\psi^{dj}\bar{\lambda}^x_i\gamma_{abcd}\lambda_{jx} + \tfrac{\kappa^2}{8}\bar{\psi}_a\gamma_b\psi^b\bar{\psi}^a\gamma_c\psi^c \\
&-\tfrac{\kappa^2}{16}\bar{\psi}_a\gamma_b\psi_c\bar{\psi}^a\gamma^c\psi^b - \tfrac{\kappa^2}{32}\bar{\psi}_a\gamma_b\psi_c\bar{\psi}^a\gamma^b\psi^c + \tfrac{\kappa^2}{32}\bar{\psi}_a\psi_b\bar{\psi}_c\gamma^{abcd}\psi_d - \tfrac{3}{16\kappa}\bar{\zeta}_A\gamma_{abc}\zeta^A\bar{\psi}^a\gamma^b\psi^c \\
&+\tfrac{1}{8}\bar{\psi}_a\gamma^{bc}\psi^a\bar{\zeta}_A\gamma_{bc}\zeta^A + \tfrac{1}{16}\bar{\psi}^a\psi^b\bar{\zeta}_A\gamma_{ab}\zeta^A + \tfrac{\kappa^2}{6}\sqrt{\tfrac{2}{3}}\,\mathrm{i}\,T_{xyz}\bar{\psi}_a\gamma_b\lambda^x\bar{\lambda}^y\gamma^{ab}\lambda^z \\
&+\tfrac{1}{32}\bar{\lambda}^x\gamma_{ab}\lambda_x\bar{\zeta}_A\gamma^{ab}\zeta^A + \tfrac{\kappa^2}{6}\sqrt{\tfrac{2}{3}}\,\mathrm{i}\,T_{xyz}\bar{\psi}^i_a\gamma^a\lambda^{jx}\bar{\lambda}^y_i\lambda^z_j + \tfrac{9\kappa^2}{16}\bar{\lambda}^{ix}\gamma_a\lambda^j_x\bar{\lambda}^y_i\gamma_a\lambda_{jy} \\
&+\tfrac{\kappa^2}{128}\bar{\lambda}^x\gamma_{ab}\lambda_x\bar{\lambda}^y\gamma^{ab}\lambda_y + \tfrac{\kappa^2}{6}g^{zt}T_{xyz}T_{tvw}\bar{\lambda}^{ix}\lambda^{jy}\bar{\lambda}^v_i\lambda^w_j - \tfrac{\kappa^2}{48}\bar{\lambda}^{ix}\lambda^j_x\bar{\lambda}^y_i\lambda_{jy} \\
&-\tfrac{1}{4\kappa^2}\mathcal{W}_{ABCD}\bar{\zeta}^A\zeta^B\bar{\zeta}^C\zeta^D + \tfrac{1}{32\kappa^2}\bar{\zeta}_A\gamma_{ab}\zeta^A\bar{\zeta}_B\gamma^{ab}\zeta^B\,.
\end{aligned}
\tag{7.49}
$$

The covariant derivatives are given by

$$
\begin{aligned}
\mathcal{D}_\mu\varphi^x &= \partial_\mu\varphi^x + gA^I_\mu K^x_I\,, \\
\mathcal{D}_\mu h^{\widetilde{I}} &= \partial_\mu h^{\widetilde{I}} + g t_{\widetilde{J}\widetilde{K}}{}^{\widetilde{I}}A^{\widetilde{J}}_\mu h^{\widetilde{K}} = -\sqrt{\tfrac{2}{3}}h^{\widetilde{I}}_x\mathcal{D}_\mu\varphi^x\,, \\
\mathcal{D}_\mu q^X &= \partial_\mu q^X + gA^I_\mu k^X_I\,, \\
\mathcal{D}_\mu\lambda^{xi} &= \partial_\mu\lambda^{xi} + \partial_\mu\phi^y\Gamma^x_{yz}\lambda^{zi} + \tfrac{1}{4}\omega_\mu{}^{ab}\gamma_{ab}\lambda^{xi}
\end{aligned}
\tag{7.50}
$$

$$
\begin{aligned}
& +\partial_\mu q^X \omega_{Xj}{}^i \lambda^{xj} + \kappa^2 g A_\mu^I P_{Ij}{}^i \lambda^{xj} + g A_\mu^I K_I^{x;y} \lambda_y^i \,, \\
\mathcal{D}_\mu \zeta^A &= \partial_\mu \zeta^A + \partial_\mu q^X \omega_{XB}{}^A \zeta^B + \tfrac{1}{4}\omega_\mu{}^{bc}\gamma_{bc}\zeta^A + g A_\mu^I t_{IB}{}^A \zeta^B \,, \\
\mathcal{D}_\mu \psi_{\nu i} &= (\partial_\mu + \tfrac{1}{4}\omega_\mu{}^{ab}\gamma_{ab})\psi_{\nu i} - \partial_\mu q^X \omega_{Xi}{}^j \psi_{\nu j} - \kappa^2 g A_\mu^I P_{Ii}{}^j \psi_{\nu j} \,.
\end{aligned}
$$

We chose to extract the fermionic terms from the spin connection and use $\omega_\mu{}^{ab}$ instead of $\hat{\omega}_\mu{}^{ab}$ in the covariant derivatives and the Ricci scalar, unless mentioned otherwise.

The transformation rules. The $\mathcal{N} = 2$ Poincaré supersymmetry rules that leave the above action invariant can be constructed as follows. We start from the transformation rules for the vector-tensor multiplets (6.105), for the hypermultiplets (6.112) and the two remaining transformation rules from the Weyl-multiplet (5.40) for the vielbein and gravitino. Next, the parameters corresponding to the gauge-fixed symmetries are replaced by the decomposition rules given in section 7.2.2. The remaining transformations are Poincaré supersymmetry (ϵ^i) and gauge transformations (Λ_G); they are given by:

$$
\begin{aligned}
\delta(\epsilon) e_\mu{}^a &= \tfrac{1}{2}\kappa\bar{\epsilon}\gamma^a\psi_\mu \,. \\
\delta(\epsilon)\psi_\mu^i &= \tfrac{1}{\kappa} D_\mu(\hat{\omega})\epsilon^i + \tfrac{1}{4\sqrt{6}}\,\mathrm{i}\, h_{\tilde{I}} \widehat{\mathcal{H}}^{\tilde{I}ab}(\gamma_{ab\mu} - 4 g_{\mu a}\gamma_b)\epsilon^i + \delta_Q q^X \omega_X^{ij}\psi_{\mu j} - \tfrac{1}{\kappa^2\sqrt{6}}\,\mathrm{i}\, g P^{ij}\gamma_\mu \epsilon_j \\
&\quad - \tfrac{\kappa}{6}\bar{\lambda}^{ix}\gamma_\mu \lambda_x^j \epsilon_j + \tfrac{\kappa}{12}\bar{\lambda}^{ix}\gamma^a \lambda_x^j \gamma_{\mu a}\epsilon_j - \tfrac{\kappa}{48}\bar{\lambda}^{ix}\gamma^{ab}\lambda_x^j \gamma_{\mu ab}\epsilon_j + \tfrac{\kappa}{12}\bar{\lambda}^{ix}\gamma_{\mu a}\lambda_x^j \gamma^a \epsilon_j \\
&\quad + \tfrac{1}{16\kappa}\bar{\zeta}_A \gamma^{ab}\zeta^A \gamma_{\mu ab}\epsilon^i \,, \\
\delta(\epsilon)\varphi^x &= \tfrac{\kappa}{2}\,\mathrm{i}\,\bar{\epsilon}\lambda^{\tilde{a}} f_{\tilde{a}}^x \,, \\
\delta(\epsilon) A_\mu^I &= \vartheta_\mu^I \,, \\
\delta(\epsilon)\lambda^{i\tilde{a}} &= -\tfrac{1}{2}\,\mathrm{i}\,\widehat{\not{\mathcal{D}}}\varphi^x \epsilon^i - \delta(\epsilon)\varphi^x \omega_x^{\tilde{a}\tilde{b}} f_{x\tilde{b}} + \delta(\epsilon) q^X \omega_X{}^{ij}\lambda_j^{\tilde{a}} + \tfrac{1}{4}\gamma\cdot\widehat{\mathcal{H}}^{\tilde{I}} h_{\tilde{I}}^{\tilde{a}}\epsilon^i \\
&\quad - \tfrac{1}{4\sqrt{6}} T^{\tilde{a}\tilde{b}\tilde{c}}[-3\bar{\lambda}_{\tilde{b}}^i \lambda_{\tilde{c}}^j + \bar{\lambda}_{\tilde{b}}^i \gamma_\mu \lambda_{\tilde{c}}^j \gamma^\mu + \tfrac{1}{2}\bar{\lambda}_{\tilde{b}}^i \gamma_{\mu\nu}\lambda_{\tilde{c}}^j \gamma^{\mu\nu}]\epsilon_j - \tfrac{1}{\kappa^2} g P^{\tilde{a}\,ij}\epsilon_j + \tfrac{1}{\kappa^2} g W^{\tilde{a}}\epsilon^i \,, \\
\delta(\epsilon)\widetilde{B}_{\mu\nu}^M &= 2\mathcal{D}_{[\mu}\vartheta_{\nu]}^M - \sqrt{6} g\bar{\epsilon}\gamma_{[\mu}\psi_{\nu]} h_N \Omega^{MN} - \mathrm{i}\, g\bar{\epsilon}\gamma_{\mu\nu}\lambda^x h_{xN}\Omega^{MN} \,, \\
\delta(\epsilon) q^X &= -\mathrm{i}\,\bar{\epsilon}^i \zeta^A f_{iA}^X \,, \\
\delta(\epsilon)\zeta^A &= \tfrac{1}{2}\,\mathrm{i}\,\gamma^\mu \widehat{\mathcal{D}}_\mu q^X f_X^{iA}\epsilon^i - \delta(\epsilon) q^X \omega_{XB}{}^A \zeta^B + \tfrac{1}{\kappa} g \mathcal{N}_i^A \epsilon^i \,,
\end{aligned}
\tag{7.51}
$$

with

$$
\begin{aligned}
\vartheta_\mu^{\tilde{I}} &\equiv -\tfrac{1}{2}\bar{\epsilon}\gamma_\mu \lambda^{\tilde{a}} f_{\tilde{a}}^x h_x^{\tilde{I}} - \tfrac{\sqrt{6}}{4}\,\mathrm{i}\, h^{\tilde{I}}\bar{\epsilon}\psi_\mu \,, & \widehat{\mathcal{D}}_\mu \varphi^x &= \mathcal{D}_\mu \varphi^x - \tfrac{\kappa}{2}\,\mathrm{i}\,\bar{\psi}_\mu \lambda^x \,, \\
\mathcal{D}_\mu \vartheta_\nu^{\tilde{I}} &= \partial_\mu \vartheta_\nu^{\tilde{I}} + g A_\mu^J t_{J\tilde{K}}{}^{\tilde{I}}\vartheta_\nu^{\tilde{K}} \,, & \widehat{\mathcal{D}}_\mu q^X &= \partial_\mu q^X + g A_\mu^I k_I^X + \kappa\,\mathrm{i}\,\bar{\psi}_\mu^i \zeta^B f_{iB}^X \,,
\end{aligned}
\tag{7.52}
$$

and where the (gauge) covariant derivative of the Killing spinor is given by

$$
D_\mu(\hat{\omega})\epsilon^i = \mathcal{D}_\mu(\hat{\omega})\epsilon^i - \partial_\mu q^X \omega_X^{ij}\epsilon_j - g\kappa^2 A_\mu^I P_I^{ij}\epsilon_j \,. \tag{7.53}
$$

Notice that the fermion shifts, proportional to P^{ij}, $P^{\tilde{a}ij}$, $W^{\tilde{a}}$ and $\mathcal{N}_I{}^A$, indeed appear quadratically in the scalar potential.

The transformations under the gauge group G are given by:

$$
\begin{aligned}
\delta(\Lambda_G)A_\mu^I &= \partial_\mu\Lambda_G^I + gA_\mu^J f_{JK}{}^I\Lambda_G^K\,,\\
\delta(\Lambda_G)\widetilde{B}_{\mu\nu}^M &= -g\Lambda_G^J t_{J\widetilde{K}}{}^M H_{\mu\nu}^{\widetilde{K}}\,,\\
\delta(\Lambda_G)\varphi^x &= -g\Lambda_G^I K_I^x\,,\\
\delta(\Lambda_G)q^X &= -g\Lambda_G^I k_I^X\,,\\
\delta(\Lambda_G)\zeta^A &= -\delta(\Lambda_G)q^X\omega_{XB}{}^A\zeta^B - g\Lambda_G^I t_{IB}{}^A\zeta^B\,,\\
\delta(\Lambda_G)\lambda^{i\tilde{a}} &= (-\omega_y^{\tilde{a}\tilde{b}}f_{x\tilde{b}} + \sqrt{\tfrac{2}{3}}f_w^{\tilde{a}}T_{xyz}g^{zw})\lambda^x\delta(\Lambda_G)\varphi^y + \delta(\Lambda_G)q^X\omega_X^{ij}\lambda_j^t a + \kappa^2 g\Lambda_G^I P_I^{ij}\lambda_j^{\tilde{a}}\,,\\
\delta(\Lambda_G)\psi_\mu^i &= \delta(\Lambda_G)q^X\omega_X^{ij}\psi_{\mu j} + \kappa^2 g\Lambda_G^I P_I^{ij}\psi_{\mu j}\,.
\end{aligned}
\tag{7.54}
$$

7.4 Simplified action for domain-walls

In the previous chapter we gave the full results, including the quartic fermion couplings. However, for determining the domain-wall solutions we only need the bosonic parts of the fermionic transformation rules, and the bosonic action. In this section we have collected all relevant information, needed for such an investigation.

The bosonic parts of the fermionic transformation rules immediately lead to the BPS equations:

$$
\begin{aligned}
\delta(\epsilon)\psi_\mu^i &= 0 = \tfrac{1}{\kappa}D_\mu(\omega)\epsilon^i + \tfrac{1}{4\sqrt{6}}\,\mathrm{i}\,h_{\widetilde{I}}\mathcal{H}^{\widetilde{I}ab}(\gamma_{ab\mu} - 4g_{\mu a}\gamma_b)\epsilon^i - \tfrac{1}{\kappa^2\sqrt{6}}\,\mathrm{i}\,gP^{ij}\gamma_\mu\epsilon_j\,,\\
\delta(\epsilon)\lambda^{i\tilde{a}} &= 0 = -\tfrac{1}{2}\,\mathrm{i}\,\not{\mathcal{D}}\varphi^x\epsilon^i + \tfrac{1}{4}\gamma\cdot\mathcal{H}^{\widetilde{I}}h_{\widetilde{I}}^{\tilde{a}}\epsilon^i - \tfrac{1}{\kappa^2}gP^{\tilde{a}\,ij}\epsilon_j + \tfrac{1}{\kappa^2}gW^{\tilde{a}}\epsilon^i\,,\\
\delta(\epsilon)\zeta^A &= 0 = \tfrac{1}{2}\,\mathrm{i}\,\gamma^\mu\widehat{\mathcal{D}}_\mu q^X f_X^{iA}\epsilon^i + \tfrac{1}{\kappa}g\mathcal{N}_i^A\epsilon^i\,,
\end{aligned}
\tag{7.55}
$$

with

$$
D_\mu(\omega)\epsilon^i = \mathcal{D}_\mu(\omega)\epsilon^i - \partial_\mu q^X\omega_X^{ij}\epsilon_j - g\kappa^2 A_\mu^I P_I^{ij}\epsilon_j\,. \tag{7.56}
$$

These equations can in principle be solved by choosing a specific coset manifold, i.e. specifying the constants $N_{\widetilde{I}\widetilde{J}\widetilde{K}}$ and by making the φ^x-embedding explicit. Since not every solution of the BPS equations necessarily has to satisfy the equations of motion, we will also have to give the truncated action. The bosonic equations of motion can be derived from the following truncated action:

$$
\begin{aligned}
e^{-1}\mathcal{L} = {}& \tfrac{1}{2\kappa^2}R(\omega) - \tfrac{1}{4}a_{\widetilde{I}\widetilde{J}}\mathcal{H}_{\mu\nu}^{\widetilde{I}}\mathcal{H}^{\widetilde{J}\mu\nu} - \tfrac{1}{2}g_{xy}\mathcal{D}_a\varphi^x\mathcal{D}^a\varphi^y - \tfrac{1}{2\kappa^2}h_{XY}\mathcal{D}_a q^X\mathcal{D}^a q^Y\\
&+\tfrac{1}{16g}e^{-1}\varepsilon^{\mu\nu\rho\sigma\tau}\Omega_{MN}\widetilde{B}_{\mu\nu}^M\Big(\partial_\rho\widetilde{B}_{\sigma\tau}^N + 2gt_{IJ}{}^N A_\rho^I F_{\sigma\tau}^J + g\,t_{IP}{}^N A_\rho^I\widetilde{B}_{\sigma\tau}^P\Big)\\
&+\tfrac{g^2}{\kappa^4}\Big(-2W_xW^x + 4\vec{P}\cdot\vec{P} - 2\vec{P}^x\cdot\vec{P}_x - 2\mathcal{N}_{iA}\mathcal{N}^{iA}\Big)\\
&+\tfrac{\kappa}{12}\sqrt{\tfrac{2}{3}}e^{-1}\varepsilon^{\mu\nu\lambda\rho\sigma}N_{IJK}A_\mu^I\Big[F_{\nu\lambda}^J F_{\rho\sigma}^K + f_{FG}{}^J A_\nu^F A_\lambda^G\Big(-\tfrac{1}{2}g\,F_{\rho\sigma}^K + \tfrac{1}{10}g^2 f_{HL}{}^K A_\rho^H A_\sigma^L\Big)\Big]\\
&-\tfrac{1}{8}e^{-1}\varepsilon^{\mu\nu\lambda\rho\sigma}\Omega_{MN}t_{IK}{}^M t_{FG}{}^N A_\mu^I A_\nu^F A_\lambda^G\Big(-\tfrac{1}{2}g\,F_{\rho\sigma}^K + \tfrac{1}{10}g^2 f_{HL}{}^K A_\rho^H A_\sigma^L\Big)\,,
\end{aligned}
\tag{7.57}
$$

where the relevant covariant derivatives are given by

$$\mathcal{D}_\mu \varphi^x = \partial_\mu \varphi^x + g A_\mu^I K_I^x \,, \qquad \mathcal{D}_\mu q^X = \partial_\mu q^X + g A_\mu^I k_I^X \,. \tag{7.58}$$

Conclusions

In this thesis we have used two different techniques for the construction of gauged supergravities. The first method is generalized reduction - also called Scherk-Schwarz I reduction - which exploits global symmetries of higher dimensional supergravity theories in order to introduce masses into lower dimensional supergravity theories. The global symmetries used for dimensional reduction generically appear as a gauged symmetry in the lower dimensional theory. The group-manifold equivalent of this mechanism is called Scherk-Schwarz II and deals with diffeomorphisms of the compact group manifold. In this case the masses occur as components of structure constants of a gauge group G in lower dimensions. In chapter 4 we demonstrated the Scherk-Schwarz I mechanism by reducing from eleven down to nine dimensions, by making use of several possible global scaling symmetries. Already at this level various different gaugings could be obtained, among which also non-compact gaugings. Non-compact gaugings in general are very interesting since they are believed to circumvent no-go theorems regarding the existence of de Sitter vacua and supersymmetrized brane-world scenarios. Furthermore, some of the gaugings were only defined at the level of the field equations. A better understanding of these special cases has been obtained recently in [175] in the context of eight-dimensional gauged supergravity.

The second method used in this thesis is the conformal program, which is a tool to construct matter-coupled gauged Poincaré supergravity. Motivated by recent developments like e.g. the brane-world scenario, we performed the conformal program in five dimensions. In chapter 5 the five-dimensional Poincaré algebra was extended to the full superconformal group. By gauging the superconformal group, applying the curvature constraints and introducing auxiliary matter fields, we constructed the minimal representation of the superconformal group, containing the graviton, called the Standard Weyl multiplet. In chapter 6 matter multiplets were introduced: vector-tensor multiplets and hypermultiplets. The transformation rules and corresponding actions were found in the background of the Weyl multipet fields. Finally, in chapter 7 the vector-tensor and hyper action were combined and used as starting point of the gaugefixing procedure. In this procedure we made convenient gauge choices for the symmetries that are not in the Poincaré algebra, and solved for the dependent gaugefields and auxiliary matter fields. The final result of this exercise indeed produced matter-coupled Poincaré supergravity. It furthermore provided an improved understanding of the gauge-fixing procedure, in relation with hyperkähler and quaternionic-Kähler geometry.

Note that we do not claim to have found the most general matter coupled $\mathcal{N} = 2$ Poincaré supergravity in five dimensions. First of all, in view of the applications, we chose to include only vector-tensor and hyper multiplets. Several other representations could have been included as well, like the linear and nonlinear multiplet. Secondly, it can not be excluded that more general gaugings can be found from dimensional reduction; especially if we drop the requirement of an action, several new possibilities may be possible. In all these cases, including our five-dimensional Poincaré action, it is not clear what the higher dimensional origin is, if any.

Compared to the existing formulations by e.g. Ceresole and Dall'Agata [76], or Günaydin and Zagermann [73–75], we have found a generalization by allowing off-diagonal gauge-transformations between the vector and tensor multiplets. This introduces extra terms proportional to the representation matrices ${t_{IJ}}^M$ in the action, transformation rules and most importantly in the scalar potential, that were not found in other literature.

The presence of the extra off-diagonal couplings in the action and transformation rules will probably allow for non-compact gaugings in five dimensions, leading to new classes of solutions, e.g. new domain-walls that can be used for supersymmetric brane-world models. It will be very interesting to see whether these new couplings will lead to new de Sitter vacua and improved realizations of brane-world scenarios. Hopefully, future research will teach us more.

Appendix A

Conventions

In this appendix, we will summarize our conventions. Furthermore, we will give some useful identities that have been used in the previous chapters.

A.1 Indices

Below we will summarize the different ranges and meanings of the indices used in chapters 5 and 6. First of all, the metric that we use is mostly plus: i.e. in five dimensions, we have $g_{\mu\nu} = (- + + + +)$. In chapter 5, we have used the following notations

$$\begin{array}{ccl} \mu, \nu & 0, 1, \ldots, 4 & \text{space-time}\,, \\ a, b & 0, 1, \ldots, 4 & \text{tangent space}\,, \\ \alpha, \beta & 1, \ldots, 4 & \text{spinor}\,, \\ i, j & 1, 2 & \mathrm{SU}(2)\,, \end{array} \tag{A.1}$$

In chapter 6, we have furthermore used indices labelling the components of the matter multiplets. In particular, we have used

$$\begin{array}{ccl} \widetilde{I}, \widetilde{J} & 1, 2, \ldots n+m & \text{vector-tensor multiplet}\,, \\ I, J & 1, 2, \ldots, n & \text{vector multiplet}\,, \\ M, N & 1, 2, \ldots, m & \text{tensor multiplet}\,, \\ X, Y & 1, 2, \ldots, 4r & \text{hypermultiplet target space}\,, \\ A, B & 1, 2, \ldots, 2r & \text{hypermultiplet tangent space}\,, \\ i, j & 1, 2 & \mathrm{SU}(2)\,. \end{array} \tag{A.2}$$

In chapter 7 two compensating multiplets were introduced. The X, Y and A, B indices were replaced by hatted ones to denote the increased ranges. The other indices are as above, but with $n = n_V + 1$, $m = n_T$ and $r = n_H$, where n_V, n_T and n_H respectively are the number of Poincaré vector-, tensor- and hypermultiplets. The following indices were used[1]:

$$
\begin{array}{rcll}
\widetilde{I},\widetilde{J} & 1,2,\ldots n_V+n_T+1 & \text{vector-tensor multiplet}\,, & \\
I,J & 1,2,\ldots,n_V+1 & \text{vector multiplet}\,, & \\
M,N & 1,2,\ldots,n_T & \text{tensor multiplet}\,, & \\
\tilde{x},\tilde{y} & 1,2,\ldots,n_V+n_T & \text{vector-tensor multiplet}\,, & \\
\hat{X},\hat{Y} & 1,2,\ldots,4n_H+4 & \text{hyperkähler hypermultiplet target space}\,, & \\
X,Y & 1,2,\ldots,4n_H & \text{quaternionic-Kähler hypermultiplet target space}\,, & \text{(A.3)} \\
z^\alpha & \alpha=1,2,3 & \text{SU(2) subspace of the hyperkähler hypermultiplet target space}\,, & \\
\hat{A},\hat{B} & 1,2,\ldots,2n_H+2 & \text{hyperkähler hypermultiplet tangent space}\,, & \\
A,B & 1,2,\ldots,2n_H & \text{quaternionic-Kähler hypermultiplet tangent space}\,, & \\
i,j & 1,2 & \text{SU(2)}\,, & \\
\bar{\alpha},\bar{\beta} & 1,2,3 & \text{SU(2) vector index}\,. &
\end{array}
$$

In this thesis symmetrizations are denoted with parentheses, and anti-symmetrizations with brackets around the indices. Furthermore, we (anti-)symmetrize with weight one:

$$X_{(ab)} \equiv \tfrac{1}{2}\left(X_{ab}+X_{ba}\right)\,, \quad X_{[ab]} \equiv \tfrac{1}{2}\left(X_{ab}-X_{ba}\right)\,. \tag{A.4}$$

A.2 Tensors

Our conventions for the D-dimensional Levi-Civita tensor $\varepsilon_{a_1\ldots a_D}$ are

$$\varepsilon_{01\ldots(D-1)} = -\varepsilon^{01\ldots(D-1)} = 1\,. \tag{A.5}$$

In the local case we use the Levi-Civita tensor density as a "constant tensor". It can be obtained from the Levi-Civita tensor by using vielbeins to convert the tangent space indices to space-time indices and multiplying the result with the vielbein determinant

$$\varepsilon_{\mu_1\ldots\mu_D} = e^{-1} e_{\mu_1}{}^{a_1}\cdots e_{\mu_d}{}^{a_D}\varepsilon_{a_1\ldots a_D}\,, \quad \varepsilon^{\mu_1\ldots\mu_D} = e\, e^{\mu_1}{}_{a_1}\cdots e^{\mu_D}{}_{a_D}\varepsilon^{a_1\ldots a_D}\,, \tag{A.6}$$

where we have used the Einstein summation convention in which repeated indices are summed over.

Note that raising and lowering the indices of the Levi-Civita tensor is done with the metric, which for the Levi-Civita tensor density is done by using the definition (A.6). Contractions of the Levi-Civita tensor give products of delta-functions which are normalized as

$$\varepsilon_{a_1\ldots a_p b_1\ldots b_q}\varepsilon^{a_1\ldots a_p c_1\ldots c_q} = -p!q!\delta^{[c_1}_{[b_1}\ldots\delta^{c_q]}_{b_q]}\,, \tag{A.7}$$

We have defined the dual of five-dimensional tensors as

$$\widetilde{A}^{a_1\ldots a_{5-n}} = \tfrac{1}{n!}\,\mathrm{i}\,\varepsilon_{a_1\ldots a_{5-n}b_1\ldots b_n}A^{b_n\ldots b_1}\,. \tag{A.8}$$

[1] For the hypermultiplets we now assume the presence of a metric on the scalar manifold.

Using (A.7), one finds the following identities

$$\widetilde{\widetilde{A}} = A\,, \qquad \frac{1}{n!}A^{a_1\ldots a_n}B_{a_1\ldots a_n} = \frac{1}{n!}A\cdot B = \frac{1}{(n-5)!}\widetilde{A}\cdot\widetilde{B}\,, \tag{A.9}$$

where we have introduced the generalized inner product notation $A \cdot B$ that we use throughout this thesis.

We use the same conventions for the Riemann tensor and its contractions as [176]. In particular, we define the Riemann tensor as

$$R^{\mu}{}_{\nu\lambda\rho} = \partial_\lambda \Gamma^{\mu}_{\rho\nu} - \partial_\rho \Gamma^{\mu}_{\lambda\nu} + \Gamma^{\mu}_{\sigma\lambda}\Gamma^{\sigma}_{\rho\nu} - \Gamma^{\mu}_{\sigma\rho}\Gamma^{\sigma}_{\lambda\nu}\,. \tag{A.10}$$

The Ricci tensor and Ricci scalar in this thesis are given by

$$R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu}\,, \quad R = g^{\mu\nu}R_{\mu\nu}\,. \tag{A.11}$$

With these conventions, the Einstein-Hilbert action has a positive sign.

A.3 Differential forms

At several places in this thesis, we have used differential form notation. A p-form is related to a rank-p anti-symmetric tensor according to

$$F_{(p)} = \frac{1}{p!}\mathrm{d}x^{\mu_1}\ldots\mathrm{d}x^{\mu_p}F_{\mu_1\ldots\mu_p}\,. \tag{A.12}$$

The analog of the dual of an anti-symmetric tensor (A.8), is given by the Hodge-dual: i.e a differential p-form A has a $(D-p)$-form $B = \star A$ as its dual with components

$$B_{\mu_1\ldots\mu_q} = \frac{1}{p!}e\,\varepsilon_{\mu_1\ldots\mu_q\nu_1\ldots\nu_p}A^{\nu_1\ldots\nu_p}\,, \quad q = D-p\,. \tag{A.13}$$

Note in particular the different order in which the indices in (A.13) are contracted with respect to (A.8). With this definition, we have the usual identity

$$\star\star A_{(p)} = (-)^{pq+1}A_{(p)}\,, \quad q = D-p\,. \tag{A.14}$$

A.4 Spinors in five dimensions

Our five-dimensional spinors are symplectic-Majorana spinors that transform in the $(4,2)$ of SO(5) ⊗ SU(2). The generators U_{ij} of the R-symmetry group SU(2) are defined to be anti-Hermitian and symmetric, i.e.

$$(U_i{}^j)^* = -U_j{}^i\,, \qquad U_{ij} = U_{ji}\,. \tag{A.15}$$

A symmetric traceless $U_i{}^j$ corresponds to a symmetric U^{ij} since we lower or raise SU(2) indices using the ε-symbol contracting the indices in a northwest-southeast (NW–SE) convention

$$X^i = \varepsilon^{ij}X_j\,, \qquad X_i = X^j\varepsilon_{ji}\,, \qquad \varepsilon_{12} = -\varepsilon_{21} = \varepsilon^{12} = 1\,. \tag{A.16}$$

The actual value of ε is here given as an example. It is in fact arbitrary as long as it is antisymmetric, $\varepsilon^{ij} = (\varepsilon_{ij})^*$ and $\varepsilon_{jk}\varepsilon^{ik} = \delta_j{}^i$. When the SU(2) indices on spinors are omitted, NW-SE contraction is understood

$$\bar{\lambda}\psi = \bar{\lambda}^i\psi_i\,, \tag{A.17}$$

The charge conjugation matrix C and $C\gamma_a$ are antisymmetric. The matrix C is unitary and γ_a is Hermitian apart from the timelike one, which is anti-Hermitian. The bar is the Majorana bar

$$\bar{\lambda}^i = (\lambda^i)^T C\,. \tag{A.18}$$

We define the charge conjugation operation on spinors as

$$(\lambda^i)^C \equiv \alpha^{-1}B^{-1}\varepsilon^{ij}(\lambda^j)^*\,, \qquad \bar{\lambda}^{iC} \equiv \overline{(\lambda^i)^C} = \alpha^{-1}\left(\bar{\lambda}^k\right)^* B\varepsilon^{ki}\,, \tag{A.19}$$

where $B = C\gamma_0$, and $\alpha = \pm 1$ when one uses the convention that complex conjugation does not interchange the order of spinors, or $\alpha = \pm\,\mathrm{i}$ when it does. Symplectic Majorana spinors satisfy $\lambda = \lambda^C$. Charge conjugation acts on gamma-matrices as $(\gamma_a)^C = -\gamma_a$, does not change the order of matrices, and works on matrices in SU(2) space as $M^C = \sigma_2 M^* \sigma_2$. Complex conjugation can then be replaced by charge conjugation, if for every bi-spinor one inserts a factor -1. Then, e.g. the expressions

$$\bar{\lambda}^i\gamma_\mu\lambda^j\,, \qquad \mathrm{i}\,\bar{\lambda}^i\lambda_i \tag{A.20}$$

are real for symplectic Majorana spinors. For more details, see [137].

A.5 Gamma-matrices in five dimensions

The gamma-matrices γ_a are defined as matrices that satisfy the Clifford-algebra

$$\{\gamma_a, \gamma_b\} \equiv \gamma_a\gamma_b + \gamma_b\gamma_a = 2\eta_{ab}\,. \tag{A.21}$$

Completely anti-symmetrized products of gamma-matrices are denoted in three different ways

$$\gamma_{(n)} = \gamma_{a_1\cdots a_n} = \gamma_{[a_1}\cdots\gamma_{a_n]}\,. \tag{A.22}$$

The product of all gamma-matrices is proportional to the unit matrix in odd dimensions. We use

$$\gamma^{abcde} = \mathrm{i}\,\varepsilon^{abcde}\,. \tag{A.23}$$

This implies that the dual of a $(5-n)$-antisymmetric gamma-matrix is the n-antisymmetric gamma-matrix given by

$$\gamma_{a_1\ldots a_n} = \tfrac{1}{(5-n)!}\,\mathrm{i}\,\varepsilon_{a_1\ldots a_n b_1\ldots b_{5-n}}\gamma^{b_{5-n}\ldots b_1}\,. \tag{A.24}$$

For convenience, we will give the values of gamma-contractions like

$$\gamma^{(m)}\gamma_{(n)}\gamma_{(m)} = c_{n,m}\gamma_{(n)}\,, \tag{A.25}$$

where the constants $c_{n,m}$ are given in table A.1. The constants for $n, m > 2$ can easily be obtained from (A.24) and table A.1.

Changing the order of spinors in a bilinear leads to the following signs

$$\bar{\psi}^{(1)}\gamma_{(n)}\chi^{(2)} = t_n\,\bar{\chi}^{(2)}\gamma_{(n)}\psi^{(1)} \qquad \begin{cases} t_n = +1 \text{ for } n = 0, 1 \\ t_n = -1 \text{ for } n = 2, 3 \end{cases} \tag{A.26}$$

where the labels (1) and (2) denote any SU(2) representation.

$c_{n,m}$	$m=1$	$m=2$
$n=0$	5	−20
$n=1$	−3	−4
$n=2$	1	4

Table A.1: Coefficients used in contractions of gamma-matrices.

A.6 Fierz-identities in five dimensions

The sixteen different gamma-matrices $\gamma_{(n)}$ for $n = 0,1,2$ form a complete basis for four-dimensional matrices. Similarly, the identity matrix $\mathbb{1}_2$ and the three Pauli-matrices σ^i for $i = 1,2,3$ form a basis for two-dimensional matrices. A change of basis in a product of two pseudo-Majorana spinors will give rise to so-called Fierz-rearrangement formulae, which in their simplest form are given by

$$\psi_j\bar{\lambda}^i = -\tfrac{1}{4}\bar{\lambda}^i\psi_j - \tfrac{1}{4}\bar{\lambda}^i\gamma^a\psi_j\gamma_a + \tfrac{1}{8}\bar{\lambda}^i\gamma^{ab}\psi_j\gamma_{ab}\,, \qquad \bar{\psi}^{[i}\lambda^{j]} = -\tfrac{1}{2}\bar{\psi}\lambda\varepsilon^{ij}\,. \tag{A.27}$$

Using such Fierz-rearrangements, other useful identities can be deduced for working with cubic fermion terms

$$\begin{aligned}&\lambda_j\bar{\lambda}^j\lambda^i = \gamma^a\lambda_j\bar{\lambda}^j\gamma_a\lambda^i = \tfrac{1}{8}\gamma^{ab}\lambda^i\bar{\lambda}\gamma_{ab}\lambda\,,\\ &\gamma^{cd}\gamma_{ab}\lambda^i\bar{\lambda}\gamma^{cd}\lambda = 4\lambda^i\bar{\lambda}\gamma^{ab}\lambda\,.\end{aligned} \tag{A.28}$$

When one multiplies three spinor doublets, one should be able to write the result in terms of $\binom{8}{3} = 56$ independent structures. From analyzing the representations, one can obtain that these are in the $(4,2)+(4,4)+(16,2)$ representations of $\overline{\mathrm{SO}(5)}\times\mathrm{SU}(2)$. They are

$$\begin{aligned}&\lambda_j\bar{\lambda}^j\lambda^i = \gamma^a\lambda_j\bar{\lambda}^j\gamma_a\lambda^i = \tfrac{1}{8}\gamma^{ab}\lambda^i\bar{\lambda}\gamma_{ab}\lambda\,,\\ &\lambda^{(k}\bar{\lambda}^i\lambda^{j)}\,,\\ &\lambda_j\bar{\lambda}^j\gamma_a\lambda^i\,.\end{aligned} \tag{A.29}$$

As a final Fierz-identity, we give a three-spinor identity which is needed to prove the invariance under supersymmetry of the action for a vector multiplet

$$\psi^i_{[I}\bar{\psi}_J\psi_{K]} = \gamma^a\psi^i_{[I}\bar{\psi}_J\gamma_a\psi_{K]}\,. \tag{A.30}$$

A similar identity was required to get the full hypermultiplet action from the [field]×[non-closure] method

$$\psi^i_{[\mu}\bar{\psi}_\nu\psi_{\rho]} = \gamma_a\psi^i_{[\mu}\bar{\psi}_\nu\gamma^a\psi_{\rho]}\,. \tag{A.31}$$

Appendix B

Reductions

B.1 Conventions

We use mostly plus signature $(- + \cdots +)$. All metrics are Einstein-frame metrics. Unless stated otherwise, doubly hatted fields and indices are eleven-dimensional, singly hatted fields and indices ten-dimensional while unhatted ones are nine-dimensional. Greek indices $\hat{\mu}, \hat{\nu}, \hat{\rho} \dots$ denote world coordinates and Latin indices $\hat{a}, \hat{b}, \hat{c} \dots$ represent tangent space-time. They are related by the vielbeins $\hat{e}_{\hat{\mu}}{}^{\hat{a}}$ and inverse vielbeins $\hat{e}_{\hat{a}}{}^{\hat{\mu}}$. Explicit indices x, y, z are underlined when flat and non-underlined when curved. When indices are omitted we use form notation.

B.2 Reduction of Ricci scalar

Covariant constancy of the metric translates to

$$D_\mu e_\nu{}^a = 0 = \partial_\mu e_\nu{}^a - \Gamma_{\mu\nu}{}^\rho e_\rho{}^a + \omega_\mu{}^{ab} e_{\nu b} \,. \tag{B.1}$$

Taking the antisymmetric part we obtain

$$\Omega_{\mu\nu}{}^a \equiv \omega_{[\mu}{}^{ab} e_{\nu]b} = 2\partial_\mu e_\nu{}^a \,, \qquad \Omega_{abc} = e_a{}^\mu e_b{}^\nu \Omega_{\mu\nu c} \,, \qquad \omega_{abc} = \tfrac{1}{2}(\Omega_{abc} + \Omega_{cab} - \Omega_{bca}) \,. \tag{B.2}$$

The Riemann curvature and Ricci scalar in terms of the spin connection are given by

$$R_{\mu\nu}{}^{ab} = 2D_{[\mu}\omega_{\nu]}{}^{ab} = 2\partial_{[\mu}\omega_{\nu]}^{ab} + 2\omega_{[\mu}{}^{ac}\omega_{\nu]c}{}^b \,, \qquad R = R_{\mu\nu}{}^{ab} e_a{}^\mu e_b{}^\nu \,. \tag{B.3}$$

Using the vielbein-Ansätze (3.8) the spin connections reduce as follows

$$\begin{aligned} &\hat{\omega}_{abc} = e^{-\alpha\phi}(\omega_{abc} + 2\alpha\eta_{a[b}\partial_{c]}\phi) \,, \qquad \hat{\omega}_{ab\underline{z}} = \tfrac{1}{2} e^{(\beta-2\alpha)\phi} F_{ab}(A) \,, \qquad \hat{\omega}_{a\underline{zz}} = 0 \,, \\ &\hat{\omega}_{\underline{z}bc} = -\tfrac{1}{2} e^{(\beta-2\alpha)\phi} F_{bc}(A) \,, \qquad \hat{\omega}_{\underline{z}b\underline{z}} = -\beta e^{-\alpha\phi}\partial_b\phi \,, \qquad \hat{\omega}_{\underline{zzz}} = 0 \,. \end{aligned} \tag{B.4}$$

The determinant of the metric reduces to

$$\hat{e} = e^{(\beta+D\alpha)\phi} e \,. \tag{B.5}$$

The Einstein-Hilbert action can now be written as

$$\begin{aligned}\hat{S} &= \frac{1}{2\kappa_{D+1}^2}\int \mathrm{d}^D x \mathrm{d}z\, \hat{e}\hat{R}(\hat{\omega}) \\ &= -\frac{1}{2\kappa_{D+1}^2}\int \mathrm{d}^D x \mathrm{d}z\, 2\hat{e}\left(\hat{\omega}_{[\hat{\mu}}{}^{\hat{a}\hat{c}}\hat{\omega}_{\hat{\nu}]\hat{c}}{}^{\hat{b}}\hat{e}_{\hat{a}}{}^{\hat{\mu}}\hat{e}_{\hat{b}}{}^{\hat{\nu}}\right) .\end{aligned} \tag{B.6}$$

The dω term has been partially integrated, and the boundary term is assumed to be zero. Note that in the case of non-trivial boundaries some extra requirements will have to be satisfied for these terms to vanish. Substituting the expressions for the vielbeins and spin connections, we obtain

$$\begin{aligned}S = \frac{1}{2\kappa_D^2}\int \mathrm{d}^D x \mathrm{d}z\, e^{[\beta-(D-2)\alpha]\phi} e &\Big\{2\omega_{[a}{}^{ac}\omega_{b]c}{}^{b} + 2\left[\beta+(D-2)\alpha\right]\omega_a{}^{ac}\partial_c\phi \\ &+\Big[\alpha^2(D-1)(D-2)+2\alpha\beta(D-1)\Big](\partial\phi)^2 - \tfrac{1}{4}e^{2(\beta-\alpha)\phi}F^2(A)\Big\} .\end{aligned} \tag{B.7}$$

This action can be brought into a canonical form by choosing

$$\alpha^2 = \tfrac{1}{2(D-1)(D-2)} , \qquad \beta = -(D-2)\alpha . \tag{B.8}$$

This leads to the following scalar-gravity-Maxwell action:

$$S = \frac{1}{2\kappa_D^2}\int \mathrm{d}^D x \mathrm{d}z\, e\left\{R(\omega) - \tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{4}e^{-2(D-1)\alpha\phi}F^2(A)\right\} . \tag{B.9}$$

B.3 Spinors and Γ-matrices in ten and nine dimensions

The Γ-matrices in ten ($\Gamma_{\hat{\mu}}$) and nine (γ_μ) dimensions can be chosen to satisfy

$$\Gamma_{\hat{\mu}}^\dagger = \eta_{\hat{\mu}\hat{\mu}}\Gamma_{\hat{\mu}} \qquad \text{and} \qquad \gamma_\mu^\dagger = \eta_{\mu\mu}\gamma_\mu , \tag{B.10}$$

respectively. In ten dimensions we can also choose the Γ-matrices to be real, while in nine dimensions they will be purely imaginary, which implies that

$$\Gamma_{\hat{\mu}}^T = \eta_{\hat{\mu}\hat{\mu}}\Gamma_{\hat{\mu}} \qquad \text{and} \qquad \gamma_\mu^T = -\eta_{\mu\mu}\gamma_\mu . \tag{B.11}$$

In ten dimensions the minimal spinor is a 32 component Majorana-Weyl spinor with 16 (real) degrees of freedom. With the choice

$$\Gamma_{11} \equiv -\Gamma_{\underline{0}\cdots\underline{9}} , \qquad \Gamma_{11} = \begin{pmatrix}\mathbb{1} & 0 \\ 0 & -\mathbb{1}\end{pmatrix}, \tag{B.12}$$

we can write a ten-dimensional Majorana-Weyl spinor as being composed of nine-dimensional, 16 component, Majorana-Weyl spinors according to

$$\psi_+^{MW} = \begin{pmatrix}\psi_1 \\ 0\end{pmatrix}, \qquad \psi_-^{MW} = \begin{pmatrix}0 \\ \psi_2\end{pmatrix}, \tag{B.13}$$

where ψ_i are nine-dimensional Majorana-Weyl spinors and + or − denotes the chirality of the ten-dimensional spinor. The split of an arbitrary ten-dimensional spinor into two Majorana-Weyl spinors of opposite chirality can of course be done without reference to nine dimensions (through the specific choice of Γ_{11}), but each ten-dimensional Majorana-Weyl spinor will then in general have 32 non-zero components even though it only has 16 degrees of freedom. In order to reduce to nine dimensions we use

$$\Gamma_{11} = \sigma_3 \otimes \mathbb{1}\,, \qquad \Gamma_{\underline{z}} = \sigma_1 \otimes \mathbb{1}\,, \qquad \Gamma_a = \sigma_2 \otimes \gamma_a\,, \tag{B.14}$$

where z is the reduction coordinate and the Pauli matrices are defined as

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{B.15}$$

As mentioned above the nine-dimensional γ-matrices are purely imaginary. If we work with a reduction of type IIB, where the two spinors have the same chirality, it may be convenient to introduce complex, nine-dimensional, Weyl spinors according to

$$\begin{aligned} \psi_c &= \psi_1 + i\psi_2\,, \qquad & \lambda_c &= \lambda_2 + i\lambda_1\,, \\ \epsilon_c &= \epsilon_1 + i\epsilon_2\,, \qquad & \tilde{\lambda}_c &= \tilde{\lambda}_2 + i\tilde{\lambda}_1\,, \end{aligned} \tag{B.16}$$

which in ten-dimensional notation can be written as, e.g.

$$\psi_+^W = \begin{pmatrix} \psi_1 \\ 0 \end{pmatrix} + i \begin{pmatrix} \psi_2 \\ 0 \end{pmatrix}. \tag{B.17}$$

If we instead work with a reduction of type IIA the two spinors will have opposite chirality, and can thus be composed into a ten-dimensional Majorana spinor according to

$$\psi^M = \begin{pmatrix} \psi_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \psi_2 \end{pmatrix}. \tag{B.18}$$

With the above mentioned decomposition into nine-dimensional Majorana-Weyl spinors we can write

$$\psi_\mu^M = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad \epsilon^M = \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix}, \quad \lambda^M = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}, \quad \tilde{\lambda}^M = \begin{pmatrix} \tilde{\lambda}_1 \\ \tilde{\lambda}_2 \end{pmatrix} \tag{B.19}$$

and

$$\psi_\mu^W = \begin{pmatrix} \psi_1 + i\psi_2 \\ 0 \end{pmatrix}, \quad \epsilon_\mu^W = \begin{pmatrix} \epsilon_1 + i\epsilon_2 \\ 0 \end{pmatrix}, \tag{B.20}$$

$$\lambda^W = \begin{pmatrix} 0 \\ \lambda_2 + i\lambda_1 \end{pmatrix}, \quad \tilde{\lambda}^W = \begin{pmatrix} 0 \\ \tilde{\lambda}_2 + i\tilde{\lambda}_1 \end{pmatrix}, \tag{B.21}$$

where the spinors without an M or W superscript are Majorana-Weyl spinors. Note also that it follows from the Clifford algebra and the choice of Γ_{11} that $\Gamma_{\underline{z}}$ is off-diagonal, which is crucial for this construction.

Appendix C

The geometry of scalar manifolds

In this appendix we will present the essential properties of hypercomplex manifolds, and show the relation with hyperkähler and quaternionic (Kähler) manifolds. We show how properties of the Nijenhuis tensor determine whether suitable connections for these geometries can be defined. We give the curvature relations, and finally the properties of symmetry transformations of these manifolds.

In [174] we showed that there is a map between conformal hypercomplex/hyperkähler and quaternionic(-Kähler) geometry, based on the coordinate basis chosen in section 7.1. The required geometrical properties for quaternionic manifolds were obtained by using the special coordinate basis for the identities and constraints given in chapter 6.

Hypercomplex manifolds were introduced in [177]. A very thorough paper on the subject is [178]. Examples of homogeneous hypercomplex manifolds that are not hyperkähler, can be found in [179,180], and are further discussed in appendix C of [86]. Non-compact homogeneous manifolds are dealt with in [181]. Various aspects have been treated in two workshops with mathematicians and physicists [182, 183]. To prepare this appendix, we extensively used [178]. However, in some parts we used original methods.

C.1 The family of quaternionic-like manifolds

Let V be a real vector space of dimension $4r$, whose coordinates we indicate as q^X (with $X = 1, \ldots, 4r$). We define a *hypercomplex structure* H on V to be a triple of complex structures J^α, (with $\alpha = 1, 2, 3$) which realize the algebra of quaternions,

$$J^\alpha J^\beta = -\delta^{\alpha\beta} \mathbb{1}_{4r} + \varepsilon^{\alpha\beta\gamma} J^\gamma . \qquad \text{(C.1)}$$

A *quaternionic structure* is the space of linear combinations $a_\alpha J^\alpha$ with a_α real numbers. In this case the three-dimensional space of complex structures is globally defined, but the individual complex structures do not have to be globally defined.

Let $\mathcal{M}$ be a $4r$-dimensional manifold. An *almost hypercomplex manifold* or *almost quaternionic manifold* is defined as a manifold $\mathcal{M}$ with a field of hypercomplex or quaternionic structures.

	no preserved metric	with a preserved metric
no SU(2) curvature	hypercomplex $G\ell(r,\mathbb{H})$	hyperkähler $USp(2r)$
non-zero SU(2) curvature	quaternionic $SU(2)\cdot G\ell(r,\mathbb{H})$	quaternionic-Kähler $SU(2)\cdot USp(2r)$

Table C.1: *Quaternionic-like manifolds. These are the manifolds that have a quaternionic structure satisfying (C.1) and (C.2). The holonomy group is indicated. For the right column the metric may give another real form as e.g.* $USp(2,2(r-1))$.

The 'almost' disappears under one extra condition. Different terminologies are used to express this condition. Sometimes it is said that the structure should be 1-integrable. The same condition is also expressed as the statement that the structure should be covariantly constant using some connections, and it is also sometimes expressed as the 'preservation of the structure' using that connection. The connection[1] here should be a symmetric (i.e. 'torsionless') connection $\Gamma_{(XY)}{}^Z$ and possibly an SU(2) connection $\omega_X{}^\alpha$. The condition is

$$0=\mathfrak{D}_X J^\alpha{}_Y{}^Z \equiv \partial_X J^\alpha{}_Y{}^Z - \Gamma_{XY}{}^W J^\alpha{}_W{}^Z + \Gamma_{XW}{}^Z J^\alpha{}_Y{}^W + 2\varepsilon^{\alpha\beta\gamma}\omega_X{}^\beta J^\gamma{}_Y{}^Z\,. \tag{C.2}$$

If the SU(2) connection has non-vanishing curvature, the manifold is called *quaternionic*.[2] If the condition (C.2) holds with vanishing SU(2) connection, i.e.

$$0=\mathfrak{D}_X J^\alpha{}_Y{}^Z \equiv \partial_X J^\alpha{}_Y{}^Z - \Gamma_{XY}{}^W J^\alpha{}_W{}^Z + \Gamma_{XW}{}^Z J^\alpha{}_Y{}^W\,, \tag{C.3}$$

then the manifold is *hypercomplex*. If there is a hermitian metric, i.e. a metric such that

$$J^\alpha{}_X{}^Z g_{ZY} = -J^\alpha{}_Y{}^Z g_{ZX}\,, \tag{C.4}$$

and if this metric is preserved using the connection Γ (i.e. if Γ is the Levi-Civita connection of this metric) then the hypercomplex and quaternionic manifolds are respectively promoted to hyperkähler and quaternionic-Kähler manifolds. Hence this gives rise to the scheme[3] of table C.1.

We will show in section C.4 that the spaces in the upper row have a Ricci tensor that is antisymmetric, and those in the right column have a Ricci tensor that is symmetric (and Einstein). It follows then that the hyperkähler manifolds are Ricci-flat. The restriction of the holonomy group when one goes to the right column, just follows from the fact that the presence of a metric restricts the holonomy group further to a subgroup of $O(4r)$.[4]

[1] The word 'connection' is by mathematicians mostly used as the derivative including the 'connection coefficients'. We use here 'connection' as a word denoting these coefficients, i.e. gauge fields.

[2] For $r=1$ there are subtleties in the definition, to which we will return below.

[3] The table is essentially taken over from [178], where there is also the terminology unimodular hypercomplex or unimodular quaternionic if the $G\ell(r)$ is reduced to $S\ell(r)$.

[4] The dot notation means that it is the product up to a common factor in both groups that does not contribute. In fact, one considers e.g. SU(2) and $USp(2r)$ on coset elements as working one from the left, and the other from the right. Then if both are -1, they do not contribute. Thus: $SU(2)\cdot USp(2r)=\frac{SU(2)\times USp(2r)}{\mathbb{Z}_2}$.

A theorem of Swann [168] shows that all quaternionic-Kähler manifolds have a corresponding hyperkähler manifold which admit a quaternionically extended homothety [a homothety extended to an SU(2) vector as in (6.42)] and which has three complex structures that rotate under an isometric SU(2) action. It has been shown in [164] that this can be implemented in superconformal tensor calculus to construct the actions of hypermultiplets in any quaternionic-Kähler manifold from a hyperkähler cone. Similarly, it has been proven in [184, 185] that any quaternionic manifold is related to a hypercomplex manifold.

Locally there is a vielbein f_X^{iA} (with $i = 1,2$ and $A = 1,\ldots,r$) with reality conditions as in (6.19). In supersymmetry we always start from these vielbeins and the integrability condition can be expressed as

$$\partial_X f_Y^{iA} - \Gamma_{XY}^Z f_Z^{iA} + f_Y^{jA}\omega_{Xj}{}^i + f_Y^{iB}\omega_{XB}{}^A = 0\,. \tag{C.5}$$

C.2 Conventions for curvatures and lemmas

We start with the notations for curvatures. The main conventions for target space curvature, fermion reparametrization curvature and SU(2) curvature are

$$\begin{aligned} R_{XYZ}{}^W &\equiv 2\partial_{[X}\Gamma_{Y]Z}{}^W + 2\Gamma_{V[X}{}^W\Gamma_{Y]Z}{}^V\,,\\ \mathcal{R}_{XYB}{}^A &\equiv 2\partial_{[X}\omega_{Y]B}{}^A + 2\omega_{[X|C|}{}^A\omega_{Y]B}{}^C\,,\\ \mathcal{R}_{XYi}{}^j &\equiv 2\partial_{[X}\omega_{Y]i}{}^j + 2\omega_{[X|k|}{}^j\omega_{Y]i}{}^k\,. \end{aligned} \tag{C.6}$$

The SU(2) curvature and connection $\omega_{Xi}{}^j$ are hermitian traceless,[5] and one can make the transition to triplet indices $\alpha = 1,2,3$ by using the sigma matrices

$$\begin{aligned} \mathcal{R}_{XYi}{}^j &= \mathrm{i}\,(\sigma^\alpha)_i{}^j\mathcal{R}_{XY}{}^\alpha\,,\\ \mathcal{R}_{XY}{}^\alpha &= -\tfrac{1}{2}\,\mathrm{i}\,(\sigma^\alpha)_i{}^j\mathcal{R}_{XYj}{}^i = 2\partial_{[X}\omega_{Y]}{}^\alpha + 2\varepsilon^{\alpha\beta\gamma}\omega_X{}^\beta\omega_Y{}^\gamma\,. \end{aligned} \tag{C.7}$$

This transition between doublet and triplet notation is valid for any triplet object, e.g. the complex structures. It is useful to know the translation of the inner product: $\mathcal{R}_i{}^j\mathcal{R}_j{}^i = -2\mathcal{R}^\alpha\mathcal{R}^\alpha$.

The curvatures by definition all satisfy the Bianchi identities that say that they are closed 2-forms, e.g.

$$\mathfrak{D}_{[X}R_{YZ]V}{}^W = 0\,. \tag{C.8}$$

Furthermore, due to the torsionless (symmetric) connection, also the cyclicity property holds.

$$R_{XYZ}{}^W + R_{ZXY}{}^W + R_{YZX}{}^W = 0\,. \tag{C.9}$$

The Ricci tensor is defined as

$$R_{XY} = R_{ZXY}{}^Z\,. \tag{C.10}$$

This is not necessarily symmetric. When Γ is the Levi-Civita connection of a metric, then one can raise and lower indices, $R_{WZXY} = R_{XYWZ}$ and the Ricci tensor is symmetric. Then one defines the scalar curvature as $R = g^{XY}R_{XY}$.

[5] This means symmetric if the indices are put at equal height using the raising or lowering tensor ε_{ij} (NW–SE convention).

We now present three lemmas that are useful in connecting scalar manifold indices with $G\ell(r,\mathbb{H})$ indices. These lemmas are used in section 6.3 and will simplify further derivations in this appendix.

Lemma C.2.1 *If a matrix $M_X{}^Y$ satisfies*

$$[J^\alpha, M] = 2\varepsilon^{\alpha\beta\gamma} J^\beta m^\gamma\,, \tag{C.11}$$

for some numbers m^γ, then the latter are given by

$$4\,r\,m^\alpha = \mathrm{Tr}\,(J^\alpha M)\,, \tag{C.12}$$

and the matrix can be written as

$$M = -m^\alpha J^\alpha + N\,, \qquad [N, J^\alpha] = 0\,. \tag{C.13}$$

A matrix M of this type is said to 'normalize the hypercomplex structure'.

Proof. The first statement is proven by taking the trace of (C.11) with J^δ. Inserting this value of m^α in (C.13), it is obvious that the remainder N commutes with the complex structures. ■

Lemma C.2.2 *If a matrix $M_X{}^Y$ commutes with the complex structures, then it can be written as*

$$M_X{}^Y = M_A{}^B f_X^{iA} f_{iB}^Y\,. \tag{C.14}$$

and vice-versa, any $M_A{}^B$ matrix can be transformed with (C.14) to a matrix commuting with the complex structures.

Proof. The vice-versa statement is easy. For the other direction, one replaces J^α with $J_i{}^j$ as in (6.28). Then multiply this equation with $f_{jA}^X f_Z^{kB}$ and consider the traceless part in AB. ■

Lemma C.2.3 *If a tensor $R_{[XY]Z}{}^W$ satisfies the cyclicity condition (C.9) and commutes with the complex structures,*

$$R_{XYZ}{}^V J^\alpha{}_V{}^W - J^\alpha{}_Z{}^V R_{XYV}{}^W = 0\,, \tag{C.15}$$

it can be written in terms of a tensor $W_{ABC}{}^D$ that is symmetric in its lower indices. If $R_{XYZ}{}^Z = 0$, then also W is traceless.

Proof. By the previous theorem, we can write

$$R_{XYW}{}^Z = f_W^{iA} f_{iB}^Z \mathcal{R}_{XYA}{}^B\,, \qquad \mathcal{R}_{XYA}{}^B = \tfrac{1}{2} f_{iA}^W f_Z^{iB} R_{XYW}{}^Z\,. \tag{C.16}$$

We can change all indices to tangent indices, defining

$$R_{ij,CDB}{}^A \equiv f_{Ci}^X f_{jD}^Y \mathcal{R}_{XYB}{}^A = -R_{ji,DCB}{}^A\,. \tag{C.17}$$

The cyclicity property of R can be used to obtain

$$0 = f_Z^{iA} R_{[WXY]}{}^Z = f_{[Y}^{iB} \mathcal{R}_{WX]B}{}^A\,. \tag{C.18}$$

We multiply this with $f_{iC}^X f_{Dj}^Y f_{kE}^W$, leading to

$$R_{kj,ECD}{}^A + R_{kj,CDE}{}^A + 2R_{jk,DEC}{}^A = 0\,. \tag{C.19}$$

The symmetric part in (jk) of this equation implies that $R_{(jk),ABC}{}^D = 0$ [multiply the equation by 3, and subtract both cyclicity rotated terms in (CDE)]. Thus we find

$$R_{ij,CDB}{}^A = -\tfrac{1}{2}\varepsilon_{ij} W_{CDB}{}^A\,, \tag{C.20}$$

with

$$W_{CDB}{}^A \equiv \varepsilon^{ij} f_{jC}^X f_{iD}^Y \mathcal{R}_{XYB}{}^A = \tfrac{1}{2}\varepsilon^{ij} f_{jC}^X f_{iD}^Y f_{kB}^Z f_W^{Ak} R_{XYZ}{}^W\,. \tag{C.21}$$

Now we prove that W is completely symmetric in the lower indices. The definition immediately implies symmetry in the first two. The $[jk]$ antisymmetric part of (C.19) gives

$$W_{ECD}{}^A + W_{DCE}{}^A - 2W_{EDC}{}^A = 0\,. \tag{C.22}$$

Antisymmetrizing this in two of the indices gives the desired result.

Finally, it is obvious from (C.21) that the tracelessness of R and W are equivalent. ■

The full result for such a curvature tensor is thus

$$R_{XYW}{}^Z = -\tfrac{1}{2} f_X^{Ai}\varepsilon_{ij} f_Y^{jB} f_W^{kC} f_{kD}^Z W_{ABC}{}^D\,. \tag{C.23}$$

C.3 The connections

In the definition of hypercomplex and quaternionic manifolds, the affine connection $\Gamma_{XY}{}^Z$ and an SU(2) connection $\omega_X{}^\alpha$ appear. In this subsection we will show how they can be obtained. The crucial ingredient is the Nijenhuis tensor.

C.3.1 Nijenhuis tensor

A Nijenhuis tensor $N_{XY}^{\alpha\beta Z}$ is defined for any combination of two complex structures, but we will use only the 'diagonal' Nijenhuis tensor (normalization for later convenience)

$$N_{XY}{}^Z \quad\equiv\quad \tfrac{1}{6} J^\alpha{}_X{}^W \partial_{[W} J^\alpha{}_{Y]}{}^Z - (X \leftrightarrow Y) = -N_{YX}{}^Z\,. \tag{C.24}$$

It satisfies a useful relation

$$N_{XY}{}^Z = J^\alpha{}_X{}^{X'} N_{X'Y}{}^{Z'} J^\alpha{}_{Z'}{}^Z\,, \tag{C.25}$$

from which one can deduce that it is traceless.

C.3.2 Obata connection and hypercomplex manifolds

The torsionless *Obata connection* [165] is defined as

$$\Gamma^{\rm Ob}{}_{XY}{}^Z = -\tfrac{1}{6}\left(2\partial_{(X} J^\alpha{}_{Y)}{}^W + \varepsilon^{\alpha\beta\gamma} J^\beta{}_{(X}{}^U \partial_{|U|} J^\gamma{}_{Y)}{}^W\right) J^\alpha{}_W{}^Z\,. \tag{C.26}$$

First, note that if a manifold is hypercomplex, i.e. if (C.3) is satisfied, then by inserting the expression for ∂J from that equation in the right-hand side of (C.26), one finds that the affine connection of the hypercomplex manifold should be the Obata connection, $\Gamma = \Gamma^{\rm Ob}$. One may thus answer the question whether an almost hypercomplex manifold [i.e. with three matrices satisfying (C.1)], defines a hypercomplex manifold [i.e. satisfies (C.3)]. As we now know that the affine connection in (C.3) should be (C.26), this can just be checked. For that purpose, the following equation is useful:

$$\partial_X J^\alpha{}_Y{}^Z - \left(\Gamma^{\rm Ob}{}_{XY}{}^W + N_{XY}{}^W\right) J^\alpha{}_W{}^Z + \left(\Gamma^{\rm Ob}{}_{XW}{}^Z + N_{XW}{}^Z\right) J^\alpha{}_Y{}^W = 0\,. \qquad \text{(C.27)}$$

It shows that any hypercomplex structure can be given a torsionful connection such that the complex structures are covariantly constant. The condition for a hypercomplex manifold is thus that this connection is torsionless, i.e. that the Nijenhuis tensor vanishes. In conclusion, *a hypercomplex manifold consists of the following data: a manifold $\mathcal{M}$, with a hypercomplex structure with vanishing Nijenhuis tensor*. In the main text, we only use the Obata connection, and we thus have $\Gamma = \Gamma^{\rm Ob}$.

C.3.3 Oproiu connection and quaternionic manifolds

For the quaternionic manifolds, the affine connection and SU(2) connection can not be uniquely defined. Indeed, one can easily check that (C.2) is left invariant when we change these two connections simultaneously using an arbitrary vector ξ_W as

$$\Gamma_{XY}{}^Z \to \Gamma_{XY}{}^Z + S_{XY}^{WZ}\xi_W\,, \qquad \omega_X{}^\alpha \to \omega_X{}^\alpha + J^\alpha{}_X{}^W\xi_W\,, \qquad \text{(C.28)}$$

where S is the tensor

$$S_{ZW}^{XY} \equiv 2\delta^X_{(Z}\delta^Y_{W)} - 2J^\alpha{}_Z{}^{(X}J^\alpha{}_W{}^{Y)}\,, \qquad \text{(C.29)}$$

which satisfies the relation

$$S_{ZW}^{XV}J^\alpha{}_V{}^Y - J^\alpha{}_W{}^V S_{ZV}^{XY} = 2\varepsilon^{\alpha\beta\gamma}J^\beta{}_Z{}^X J^\gamma{}_W{}^Y\,. \qquad \text{(C.30)}$$

Under this transformation, the $G\ell(r,\mathbb{H})$ connection transforms as

$$\omega_{XA}{}^B \to \omega_{XA}{}^B + \tfrac12 f_Y^{iB} f_{iA}^Z S_{XZ}^{YW}\xi_W\,. \qquad \text{(C.31)}$$

An invariant SU(2) connection is

$$\tilde\omega_X{}^\alpha = \omega_X{}^\alpha + \tfrac13 J^\alpha{}_X{}^Y J^\beta{}_Y{}^Z\omega_Z{}^\beta = \tfrac23\omega_X{}^\alpha - \tfrac13\varepsilon^{\alpha\beta\gamma}J^\beta{}_X{}^Y\omega_Y{}^\gamma\,. \qquad \text{(C.32)}$$

If we use (C.2) in the expression for the Nijenhuis tensor, (C.24), we find that quaternionic manifolds do not have a vanishing Nijenhuis tensor, but the latter should satisfy

$$N_{XY}{}^Z = -J^\alpha{}_{[X}{}^Z\tilde\omega_{Y]}{}^\alpha\,. \qquad \text{(C.33)}$$

This condition can be solved for $\tilde\omega$. We find

$$(1-2\,r)\,\tilde\omega_X{}^\alpha = N_{XY}{}^Z J^\alpha{}_Z{}^Y\,. \qquad \text{(C.34)}$$

Thus the condition for an almost quaternionic manifold to be quaternionic is that the Nijenhuis tensor satisfies

$$(1-2\,r)\,N_{XY}{}^{Z} = -J^{\alpha}{}_{[X}{}^{Z}N_{Y]V}{}^{W}J^{\alpha}{}_{W}{}^{V}\,. \tag{C.35}$$

On the other hand, one may also use (C.2) in the expression for the Obata connection (C.26). Then we find that the affine connection for the quaternionic manifolds is given by

$$\Gamma_{XY}{}^{Z} = \Gamma^{\mathrm{Ob}}{}_{XY}{}^{Z} - J^{\alpha}{}_{(X}{}^{Z}\omega_{Y)}{}^{\alpha} - \tfrac{1}{3}S^{ZU}_{XY}J^{\alpha}{}_{U}{}^{V}\omega_{V}{}^{\alpha}\,, \tag{C.36}$$

which exhibits the transformation (C.28).

One can take a gauge choice for the invariance. A convenient choice is to impose

$$J^{\alpha}{}_{Y}{}^{Z}\omega_{Z}{}^{\alpha} = 0\,. \tag{C.37}$$

With this choice $\tilde{\omega}_X{}^{\alpha} = \omega_X{}^{\alpha}$. The affine connection in (C.36) simplifies, and this expression is called the Oproiu connection [186]

$$\begin{aligned} \Gamma^{\mathrm{Op}}{}_{XY}{}^{Z} &\equiv \Gamma^{\mathrm{Ob}}{}_{XY}{}^{Z} - J^{\alpha}{}_{(X}{}^{Z}\omega_{Y)}{}^{\alpha} \\ &= \Gamma^{\mathrm{Ob}}{}_{XY}{}^{Z} + N^{Z}_{XY} - J^{\alpha}{}_{Y}{}^{Z}\omega_{X}{}^{\alpha}\,. \end{aligned} \tag{C.38}$$

The last expression shows that the Oproiu connection, which up to here was only proven to be necessary for solving (C.2), gives rise to covariantly constant complex structures under the condition (C.33). Indeed, the first two terms give a (torsionful) connection that gives rise to a covariantly constant hypercomplex structure, see (C.27), and the last term cancels the SU(2) connection. The condition (C.33) is now the condition that the connection Γ^{Op} is torsionless.

In conclusion, *a quaternionic manifold consists of the following data: a manifold $\mathcal{M}$, with a hypercomplex structure with Nijenhuis tensor satisfying (C.35).*

Levi-Civita connection and hyperkähler or quaternionic-Kähler manifolds. For hyperkähler manifolds, the Obata connection should coincide with the Levi-Civita connection of a metric. For quaternionic-Kähler manifolds, the connection that preserves the metric can be one of the equivalence class defined from the Oproiu connection by a transformation (C.28).

C.4 Curvature relations

C.4.1 Splitting according to holonomy

There are two interesting possibilities of splitting the curvature on quaternionic-like manifolds. First of all, the integrability condition of (C.5) yields that the total curvature on the manifold is the sum of the SU(2) curvature and the $\mathrm{G}\ell(r,\mathbb{H})$ curvature which shows that the (restricted) holonomy splits in these two factors:

$$\begin{aligned} R_{XYW}{}^{Z} &= R^{\mathrm{SU}(2)}{}_{XYW}{}^{Z} + R^{\mathrm{G}\ell(r,\mathbb{H})}{}_{XYW}{}^{Z} \\ &= -J^{\alpha}{}_{W}{}^{Z}\mathcal{R}_{XY}{}^{\alpha} + L_{W}{}^{Z}{}_{A}{}^{B}\mathcal{R}_{XYB}{}^{A}\,, \qquad \text{with} \qquad L_{W}{}^{Z}{}_{A}{}^{B} \equiv f^{Z}_{iA}f^{iB}_{W}\,. \end{aligned} \tag{C.39}$$

The matrices $L_A{}^B$ and J^α commute and their mutual trace vanishes

$$J^\alpha{}_X{}^Y L_Y{}^Z{}_A{}^B = L_X{}^Y{}_A{}^B J^\alpha{}_Y{}^Z\,, \qquad J^\alpha{}_Z{}^Y L_Y{}^Z{}_A{}^B = 0\,. \tag{C.40}$$

For hypercomplex (or hyperkähler) manifolds, the SU(2) curvature vanishes. Then the Riemann tensor commutes with the complex structures and using the cyclicity, one may use lemmas C.2.2 and C.2.3 to write

$$R_{XYW}{}^Z = -\tfrac{1}{2} f_X^{Ai}\varepsilon_{ij} f_Y^{jB} f_W^{kC} f_{kD}^Z W_{ABC}{}^D\,. \tag{C.41}$$

This W is symmetric in its lower indices. The Ricci tensor is then

$$R_{XY} = \tfrac{1}{2}\varepsilon_{ij} f_X^{iB} f_Y^{jC} W_{ABC}{}^A = -R_{YX}\,. \tag{C.42}$$

Thus the Ricci tensor for hypercomplex manifolds is antisymmetric. In general, the antisymmetric part can be traced back to the curvature of the U(1) part in $\mathrm{G}\ell(r,\mathbb{H}) = \mathrm{S}\ell(r,\mathbb{H})\times \mathrm{U}(1)$. Indeed, using the cyclicity condition:

$$R_{[XY]} = R_{Z[XY]}{}^Z = -\tfrac{1}{2} R_{XYZ}{}^Z = -\mathcal{R}^{\mathrm{U}(1)}_{XY}\,, \qquad \mathcal{R}^{\mathrm{U}(1)}_{XY} \equiv \mathcal{R}_{XYA}{}^A\,. \tag{C.43}$$

C.4.2 Splitting in Ricci and Weyl curvature

The separate terms in (C.39) for quaternionic manifolds do not satisfy the cyclicity condition, and thus are not bona fide curvatures. We will now discuss another splitting

$$R = R^{\mathrm{Ric}}{}_{XYW}{}^Z + R^{(\mathrm{W})}{}_{XYW}{}^Z\,. \tag{C.44}$$

Both terms will separately satisfy the cyclicity condition. The first part only depends on the Ricci tensor of the full curvature, and is called the '*Ricci part*'. The Ricci tensor of the second part will be zero, and this part will be called the '*Weyl part*' [178]. We will prove that the second part commutes with the complex structures. The lemmas of section C.2 then imply that the second part can be written in terms of a tensor $\mathcal{W}_{ABC}{}^D$, symmetric in the lower indices and traceless. This tensor appears in supersymmetric theories, which is another reason for considering this construction. The case $r = 1$ needs a separate treatment which will be discussed afterwards.

To define the splitting (C.44), we define the first term as a function of the Ricci tensor, and $R^{(\mathrm{W})}$ is just defined as the remainder. The definition of R^{Ric} again makes use of the tensor S in (C.29):

$$\begin{aligned} R^{\mathrm{Ric}}{}_{XYZ}{}^W &\equiv 2S_{Z[X}^{WV} B_{Y]V}\,,\\ B_{XY} &\equiv \frac{1}{4r}R_{(XY)} - \frac{1}{2r(r+2)}\Pi_{(XY)}{}^{ZW}R_{ZW} + \frac{1}{4(r+1)}R_{[XY]}\,. \end{aligned} \tag{C.45}$$

Here, Π projects bilinear forms onto hermitian ones, i.e.

$$\Pi_{XY}{}^{ZW} \equiv \tfrac{1}{4}\left(\delta_X{}^Z\delta_Y{}^W + J^\alpha{}_X{}^Z J^\alpha{}_Y{}^W\right). \tag{C.46}$$

The Ricci part satisfies several properties that can be checked by a straightforward calculation:

1. The Ricci tensor of R^{Ric} is just R_{XY}.

2. The cyclicity property (C.9).
3. Considered as a matrix in its last two indices, this matrix normalizes the hypercomplex structure (see lemma C.2.1).

Especially to prove the last one, the property (C.30) can be used (multiplying it with B_{UX} and antisymmetrizing in $[ZU]$). The relation is explicitly

$$J^\alpha{}_Z{}^W R^{\rm Ric}{}_{XYW}{}^V - R^{\rm Ric}{}_{XYZ}{}^W J^\alpha{}_W{}^V = 2\varepsilon^{\alpha\beta\gamma} J^\beta{}_Z{}^V \mathcal{R}^{\rm Ric}{}_{XY}{}^\gamma ,$$
$$\text{with} \qquad \mathcal{R}^{\rm Ric}{}_{XY}{}^\alpha = \frac{1}{4r} J^\alpha{}_W{}^Z R^{\rm Ric}{}_{XYZ}{}^W = 2J^\alpha{}_{[X}{}^Z B_{Y]Z} . \tag{C.47}$$

The important information is now that the full curvature also satisfies these 3 properties. The latter one is the integrability property of (C.2):

$$0 = 2\mathfrak{D}_{[X}\mathfrak{D}_{Y]} J^\alpha{}_Z{}^V = R_{XYW}{}^V J^\alpha{}_Z{}^W - R_{XYZ}{}^W J^\alpha{}_W{}^V - 2\varepsilon^{\alpha\beta\gamma}\mathcal{R}_{XY}{}^\gamma J^\beta{}_Z{}^V . \tag{C.48}$$

As in general for matrices normalizing the complex structure, we can also express $\mathcal{R}_{XY}{}^\alpha$ as

$$R_{XYZ}{}^W J^\alpha{}_W{}^Z = 4r\,\mathcal{R}_{XY}{}^\alpha . \tag{C.49}$$

This leads to properties of the Weyl part of the curvature. First of all, it implies that this part is Ricci-flat. Secondly it also satisfies the cyclicity property. Third, it also normalizes the hypercomplex structure, defining some $\mathcal{R}^{(\rm W)\alpha}_{XY}$. We will now prove that the latter is zero for $r > 1$.

The expression for this tensor satisfies a property that can be derived, starting from its definition, by first using the cyclicity of $R^{(\rm W)}$, then the equation saying that it normalizes the hypercomplex structure, and finally that it is Ricci-flat

$$\begin{aligned} r\mathcal{R}^{(\rm W)\alpha}_{XY} &= \tfrac14 J^\alpha{}_U{}^V R^{(\rm W)}{}_{XYV}{}^U = -\tfrac12 J^\alpha{}_U{}^V R^{(\rm W)}{}_{V[XY]}{}^U \\ &= -\epsilon^{\alpha\beta\gamma}\mathcal{R}^{(\rm W)\beta}_{V[X} J^\gamma{}_{Y]}{}^V . \end{aligned} \tag{C.50}$$

Multiplying with $J^\alpha{}_V{}^Y$ and antisymmetrizing leads to

$$J^\alpha{}_{[V}{}^Y \mathcal{R}^{(\rm W)\alpha}_{X]Y} = 0 . \tag{C.51}$$

Secondly, multiplying (C.50) with $J^\delta{}_Z{}^X J^\delta{}_W{}^Y$, and using (C.50) again for multiplying the complex structures at the right-hand side, leads to

$$J^\beta{}_X{}^Z J^\beta{}_Y{}^V \mathcal{R}^{(\rm W)\alpha}_{ZV} = -\mathcal{R}^{(\rm W)\alpha}_{XY} \qquad \text{or} \qquad \Pi_{XY}{}^{ZV}\mathcal{R}^{(\rm W)\alpha}_{ZV} = 0 . \tag{C.52}$$

Finally, multiplying (C.50) with $\varepsilon^{\alpha\delta\epsilon} J^\delta{}_Z{}^Y$ leads to

$$\mathcal{R}^{(\rm W)\alpha}_{XY} = 0 , \qquad \text{if} \quad r > 1 . \tag{C.53}$$

Therefore $R^{(\rm W)}{}_{XYZ}{}^V$ is a tensor that satisfies all conditions of lemma C.2.3, and we can thus write

$$R_{XYZ}{}^W = R^{\rm Ric}{}_{XYZ}{}^W - \tfrac12 f_X^{Ai}\varepsilon_{ij} f_Y^{jB} f_W^{kC} f_{kD}^Z \mathcal{W}_{ABC}{}^D . \tag{C.54}$$

For hypercomplex manifolds, we found that the full curvature can be written in terms of a tensor $W_{ABC}{}^D$, see (C.41), which is symmetric in the lower indices, but not necessarily traceless. One can straightforwardly compute the corresponding $\mathcal{W}$, and find that this is its traceless part, the trace determining the Ricci tensor:

$$\mathcal{W}_{ABC}{}^D = W_{ABC}{}^D - \frac{3}{2(r+1)}\delta^D_{(A} W_{BC)E}{}^E , \qquad R_{XY} = -\mathcal{R}_{XYA}{}^A = \tfrac12\varepsilon_{ij} f_X^{iA} f_Y^{jB} W_{ABC}{}^C . \tag{C.55}$$

C.4.3 The one-dimensional case

As

$$\mathrm{G}\ell(1,\mathbb{H}) = \mathrm{S}\ell(1,\mathbb{H})\times \mathrm{U}(1) = \mathrm{SU}(2)\times \mathrm{U}(1)\,, \tag{C.56}$$

we have now two SU(2) factors in the full holonomy group. This can be written explicitly by splitting L in (C.39) in a traceless and trace part:

$$L_X{}^Y{}_A{}^B = \tfrac{1}{2}\,\mathrm{i}\,(\sigma^\alpha)_A{}^B J^{-\alpha}{}_X{}^Y + \tfrac{1}{2}\delta_X^Y\delta_A^B\,. \tag{C.57}$$

This leads to the $r=1$ form of (C.39):

$$R_{XYW}{}^Z = -J^{+\alpha}{}_W{}^Z\mathcal{R}^{+\alpha}_{XY} - J^{-\alpha}{}_W{}^Z\mathcal{R}^{-\alpha}_{XY} + \delta_W^Z\mathcal{R}^{\mathrm{U}(1)}_{XY}\,, \tag{C.58}$$

where for emphasizing the symmetry, we indicate the original complex structures as $J^{+\alpha}{}_X{}^Y$.

We saw that for $r=1$ we could not perform all steps to get to the decomposition (C.54). However, some authors define quaternionic and quaternionic-Kähler for $r=1$ as a more restricted class of manifolds such that this decomposition is still valid [187]. For quaternionic-Kähler manifolds, the definition that is taken in general leads for $r=1$ to the manifolds with holonomy $\mathrm{SU}(2)\times\mathrm{USp}(2)$, which is just SO(4). Thus with this definition all four-dimensional Riemannian manifolds would be quaternionic-Kähler. Therefore a further restriction is imposed. This further restriction is also natural in supergravity, as it is equivalent to a constraint that follows from requiring invariance of the supergravity action.

In general, as $\mathcal{R}^{(\mathrm{W})}$ normalizes the hypercomplex structure, we can by lemma C.2.1 and lemma C.2.2 write

$$R^{(\mathrm{W})}{}_{XYZ}{}^W = -\mathcal{R}^{(\mathrm{W})\,\alpha}_{XY}J^\alpha{}_Z{}^W + \mathcal{R}^{(\mathrm{W})\,B}_{XYA}L_Z{}^W{}_A{}^B = R^{(\mathrm{W})+}{}_{XYZ}{}^W + R^{(\mathrm{W})-}{}_{XYZ}{}^W\,. \tag{C.59}$$

We impose

$$\mathcal{R}^{(\mathrm{W})\,\alpha}_{XY} = 0\,, \tag{C.60}$$

as part of the definition of quaternionic manifolds with $r=1$. This is thus the equation that is automatically valid for $r>1$. Using lemma C.2.3, this implies that (C.54) is valid for all quaternionic manifolds.

In the one-dimensional case, we can see that a possible metric is already fixed up to a multiplicative function. Indeed, the C_{AB} that is used in (6.88) can only be proportional to ε_{AB}. Therefore, it is said that there is a *conformal metric*, i.e. a metric determined up to a (local) scale function $\lambda(q)$:

$$g_{XY} \equiv \lambda(q) f_X^{iA} f_Y^{jB}\varepsilon_{ij}\varepsilon_{AB}\,. \tag{C.61}$$

One can check that this metric is hermitian for any $\lambda(q)$, i.e. $J^\alpha{}_{XY} = J^\alpha{}_X{}^Z g_{ZY}$ is antisymmetric. The remaining question is whether this metric is covariantly constant, which boils down to the covariant constancy of C_{AB}. This condition can be simplified using the Schouten identity:

$$\mathfrak{D}_X C_{AB} = \partial_X C_{AB} + 2\omega_{X[A}{}^C C_{|C|B]} = \partial_X C_{AB} + \omega_{XC}{}^C C_{AB} = \varepsilon_{AB}\left(\partial_X\lambda(q) + \omega_{XC}{}^C\lambda(q)\right). \tag{C.62}$$

We can choose a function $\lambda(q)$ such that C is covariantly constant iff $\omega_{XC}{}^C$ is a total derivative, i.e. if the U(1) curvature vanishes. Thus in the one-dimensional case hypercomplex manifolds become hyperkähler, and quaternionic manifolds become quaternionic-Kähler if and only if the U(1) factor in the curvature part $G\ell(1,\mathbb{H})$ vanishes.

C.4.4 The curvature of Quaternionic-Kähler manifolds

In quaternionic-Kähler manifolds, the affine connection is the Levi-Civita connection of a metric. Therefore, the Ricci tensor is symmetric. As we have already proven that in the hypercomplex case the symmetric part vanishes, hyperkähler manifolds have vanishing Ricci tensor. Now we will prove that the quaternionic-Kähler spaces are Einstein, and that moreover the SU(2) curvatures are proportional to the complex structures with a proportionality factor that is dependent on the scalar curvature.

We start again from the integrability property (C.48). Multiplying with $J^\delta{}_V{}^X$ gives

$$R_{YZ}\delta^{\alpha\delta} - \varepsilon^{\alpha\delta\beta} R_{XYZ}{}^W J^\beta{}_W{}^X + J^\alpha{}_Z{}^W R_{XYW}{}^V J^\delta{}_V{}^X - \\ -2\varepsilon^{\alpha\beta\delta}\mathcal{R}_{ZY}{}^\beta + 2\delta^{\alpha\delta}\mathcal{R}_{XY}{}^\beta J^\beta{}_Z{}^X - 2\mathcal{R}_{XY}{}^\delta J^\alpha{}_Z{}^X = 0\,. \tag{C.63}$$

The second and third term can be rewritten

$$\begin{aligned} R_{XYW}{}^V J^\delta{}_V{}^X &= -R_{YWX}{}^V J^\delta{}_V{}^X - R_{WXY}{}^V J^\delta{}_V{}^X \\ &= -R_{YWX}{}^V J^\delta{}_V{}^X + R_{YXW}{}^V J^\delta{}_V{}^X\,, \\ 2R_{XYW}{}^V J^\delta{}_V{}^X &= -4r\mathcal{R}_{YW}{}^\delta\,. \end{aligned} \tag{C.64}$$

In the first line, the cyclicity property of the Riemann tensor is used. Then, the symmetry in interchanging the first two and last two indices (here we use that the curvature originates from a Levi-Civita connection) and finally interchanging the indices on the last complex structure, using its antisymmetry (Hermiticity of the metric). This leads to

$$R_{YZ}\delta^{\alpha\delta} + \varepsilon^{\alpha\delta\beta}2(r-1)\mathcal{R}_{YZ}{}^\beta - 2(r-1)\mathcal{R}_{YX}{}^\delta J^\alpha{}_Z{}^X + 2\delta^{\alpha\delta}\mathcal{R}^\beta_{XY} J^\beta{}_Z{}^X = 0\,. \tag{C.65}$$

Multiplying with $\delta^{\alpha\delta}$ gives

$$R_{YZ} = -\tfrac{2}{3}(r+2)J^\beta{}_Z{}^X\mathcal{R}_{XY}{}^\beta\,. \tag{C.66}$$

On the other hand, multiplying (C.65) with $\varepsilon^{\alpha\delta\gamma}$ gives only a non-trivial result for $r \neq 1$, in which case we find

$$\text{for} \quad r > 1 \;: \qquad 2\mathcal{R}_{YZ}{}^\alpha = \varepsilon^{\alpha\beta\gamma} J^\beta{}_Y{}^X \mathcal{R}_{XZ}{}^\gamma\,. \tag{C.67}$$

We impose the same equation for $r = 1$. We will connect this equation to another requirement below.

By replacing $\varepsilon^{\alpha\beta\gamma}J^\beta{}_Y{}^X$ by $-(J^\alpha J^\gamma)_Y{}^X - \delta_Y^X\delta^{\alpha\gamma}$ we get

$$\mathcal{R}_{XY}{}^\alpha = -\tfrac{1}{3}J^\alpha{}_X{}^Z J^\beta{}_Z{}^V\mathcal{R}_{VY}{}^\beta = \frac{1}{2(r+2)}J^\alpha{}_X{}^Z R_{ZY}\,. \tag{C.68}$$

We also have

$$J^\alpha{}_X{}^Z\mathcal{R}_{ZY}{}^\beta = \varepsilon^{\alpha\beta\gamma}\mathcal{R}_{XY}{}^\gamma - \frac{1}{2(r+2)}\delta^{\alpha\beta}R_{XY}\,. \tag{C.69}$$

The final step is obtained by using (C.48) once more. Now multiply this equation with $\varepsilon^{\alpha\beta\gamma}J^{\beta YX}J^\gamma{}_V{}^U$, and use for the contraction of the Riemann curvature tensor with $J^{\beta YX}$ that we may interchange pairs of indices such that (C.49) can be used. Then everywhere $J^\alpha\mathcal{R}^\beta$ appears, for which we can use (C.69). This leads to the equation expressing that the manifold is Einstein:

$$R_{XY} = \frac{1}{4r}g_{XY}R\,. \tag{C.70}$$

With (C.68), the SU(2) curvature is proportional to the complex structure:

$$\mathcal{R}_{XY}{}^{\alpha} = \tfrac{1}{2}\nu J^{\alpha}{}_{XY}\,, \qquad \nu \equiv \frac{1}{4r(r+2)}R\,. \tag{C.71}$$

The Einstein property drastically simplifies the expression for B in (C.45) to

$$B_{XY} = \tfrac{1}{4}\nu g_{XY}\,. \tag{C.72}$$

The Ricci part of the curvature then becomes proportional to the curvature of a quaternionic projective space of the same dimension:

$$\left(R^{\mathbb{H}P^n}\right)_{XYWZ} = \tfrac{1}{2}g_{Z[X}g_{Y]W} + \tfrac{1}{2}J^{\alpha}_{XY}J^{\alpha}_{ZW} - \tfrac{1}{2}J^{\alpha}_{Z[X}J^{\alpha}_{Y]W} = \tfrac{1}{2}J^{\alpha}_{XY}J^{\alpha}_{ZW} + L_{[ZW]}{}^{AB}L_{[XY]AB}\,. \tag{C.73}$$

The full curvature decomposition is then

$$R_{XYWZ} = \nu(R^{\mathbb{H}P^n})_{XYWZ} + \tfrac{1}{2}L_{ZW}{}^{AB}\mathcal{W}_{ABCD}L_{XY}{}^{CD}\,, \tag{C.74}$$

with $\mathcal{W}_{ABCD}$ completely symmetric. The constraint appearing in supergravity fixes the value of ν to $-\kappa^2$. The quaternionic-Kähler manifolds appearing in supergravity thus have negative scalar curvature, and this implies that all such manifolds that have at least one isometry are non-compact.

Finally, we should still comment on the extra constraint (C.67) for $r = 1$. In the mathematics literature [187] the extra constraint is that the quaternionic structure annihilates the curvature tensor, which is the vanishing of

$$\begin{aligned}(J^{\alpha}\cdot R)_{XYWZ} &\equiv J^{\alpha}{}_X{}^V R_{VYWZ} + J^{\alpha}{}_Y{}^V R_{XVWZ} + J^{\alpha}{}_Z{}^V R_{XYWV} + J^{\alpha}{}_W{}^V R_{XYVZ}\\ &= \varepsilon^{\alpha\beta\gamma}\left(\mathcal{R}_{XY}{}^{\beta}J^{\gamma}_{ZW} + \mathcal{R}_{ZW}{}^{\beta}J^{\gamma}_{XY}\right),\end{aligned} \tag{C.75}$$

where the second expression is obtained using once more (C.48). We have proven that (C.67) was sufficient extra input to have $\mathcal{R}^{\alpha}_{XY}$ proportional to J^{α}_{XY} implying $J^{\alpha}\cdot R = 0$. Vice versa: multiplying (C.75) with $\varepsilon^{\alpha\delta\epsilon}J^{\epsilon}{}_{YZ}$ leads to (C.67) if $J^{\alpha}\cdot R = 0$. Thus indeed the vanishing of (C.75) is an equivalent condition that can be imposed for $r = 1$ and that is automatically satisfied for $r > 1$.

C.5 Symmetries

Symmetries of manifolds are most known as isometries for Riemannian manifolds (i.e. when there is a metric). They are transformations $\delta q^X = k^X_I(q)\Lambda^I$, where Λ^I are infinitesimal parameters. They are determined by the Killing equation[6]

$$\mathfrak{D}_{(X}k_{Y)I} = 0\,, \qquad k_{XI} \equiv g_{XY}k^Y{}_I\,. \tag{C.76}$$

[6] See also 'conformal Killing vectors' in section 5.1.

This definition can only be used when there is a metric. However, there is a weaker equation that can be used for defining symmetries also in the absence of a metric, but when parallel transport is defined. Indeed, the Killing equation implies that

$$-R_{YZX}{}^{W} k_{WI} = \mathfrak{D}_Y \mathfrak{D}_Z k_{XI} - \mathfrak{D}_Z \mathfrak{D}_Y k_{XI} = \mathfrak{D}_Y \mathfrak{D}_Z k_{XI} + \mathfrak{D}_Z \mathfrak{D}_X k_{YI} . \tag{C.77}$$

Using the cyclicity condition on the left-hand side to write

$$R_{YZX}{}^{W} = \tfrac{1}{2} \left(R_{YZX}{}^{W} - R_{ZXY}{}^{W} - R_{XYZ}{}^{W} \right) , \tag{C.78}$$

we obtain

$$\mathfrak{D}_X \mathfrak{D}_Y k_I^Z = R_{XWY}{}^{Z} k_I^W . \tag{C.79}$$

This equation does not need a metric any more. We will use it as definition of symmetries when there is no metric available. We will see that it leads to the group structure that is known from the Riemannian case.

Of course, we will require also that the symmetries respect the quaternionic structure. This is the statement that the vector k_I^X normalizes the quaternionic structure:

$$\mathcal{L}_{k_I} J^{\alpha}{}_X{}^Y \equiv k_I^Z \partial_Z J^{\alpha}{}_X{}^Y + \left(\partial_X k_I^Z \right) J^{\alpha}{}_Z{}^Y - J^{\alpha}{}_X{}^Z \left(\partial_Z k_I^Y \right) = b_I^{\alpha\beta} J^{\beta}{}_X{}^Y , \tag{C.80}$$

for some functions $b_I^{\alpha\beta}(q)$. This b_I is antisymmetric, as can be seen by multiplying the equation with $J^{\gamma}{}_Y{}^X$.

Thus we define symmetries in quaternionic-like manifolds as those $\delta q^X = k_I^X(q)\Lambda^I$, such that the vectors k_I^X satisfy (C.79) and (C.80).

We first consider (C.80). One can add an affine torsionless connection to the derivatives, because they cancel. As a total covariant derivative on J vanishes, we add in case of quaternionic manifolds the SU(2) connection to the first derivative. This addition is of the form of the right-hand side. Thus defining P_I^{γ} by $b_I^{\alpha\beta} - 2\varepsilon^{\alpha\beta\gamma}\omega_X{}^{\gamma} k_I^X = 2\varepsilon^{\alpha\beta\gamma}\nu P_I^{\gamma}$, the remaining statement is that there is a $P_I^{\alpha}(q)$ (possibly zero) such that[7]

$$J^{\alpha}{}_X{}^Z \left(\mathfrak{D}_Z k_I^Y \right) - \left(\mathfrak{D}_X k_I^Z \right) J^{\alpha}{}_Z{}^Y = -2\varepsilon^{\alpha\beta\gamma} J^{\beta}{}_X{}^Y \nu P_I^{\gamma} . \tag{C.81}$$

The equation now takes on the form of (C.11) in lemma C.2.1. Thus, using this lemma, as well as lemma C.2.2, we have

$$\mathfrak{D}_X k_I^Y = \nu J^{\alpha}{}_X{}^Y P_I^{\alpha} + L_X{}^Y{}_A{}^B t_{IB}{}^A . \tag{C.82}$$

$t_{IB}{}^A$ is the matrix that we saw in the fermion gauge transformation law (6.49). The rule (C.12) gives an expression for P_I^{α}, which is called the *moment map*:

$$4r\,\nu\, P_I^{\alpha} = -J^{\alpha}{}_X{}^Y \left(\mathfrak{D}_Y k_I^X \right) . \tag{C.83}$$

Using the second equation, (C.79) we now find

$$R_{ZWX}{}^{Y} k_I^W = \mathfrak{D}_Z \mathfrak{D}_X k_I^Y = \nu J^{\alpha}{}_X{}^Y (\mathfrak{D}_Z P_I^{\alpha}) + L_X{}^Y{}_A{}^B \left(\mathfrak{D}_Z t_{IB}{}^A \right) . \tag{C.84}$$

[7]Here we introduce in fact νP. The factor ν is included for agreement with other papers and allows a smooth limit $\nu = 0$ to the hypercomplex or hyperkähler case. In fact, we have seen in (6.55) that supersymmetry in the setting of hypercomplex manifolds demands that the right-hand side of (C.80) is zero. We will see below that this is unavoidable for hypercomplex manifolds even outside the context of supersymmetry.

Using the curvature decomposition (C.39) and projecting onto the complex structures and L, we find two equations

$$\mathcal{R}_{ZW}{}^{\alpha} k_I^W = -\nu \mathfrak{D}_Z P_I^{\alpha}\,, \qquad \mathcal{R}_{ZWB}{}^{A} k_I^W = \mathfrak{D}_Z t_{IB}{}^{A}\,. \tag{C.85}$$

The algebra that the vectors k_I^X define is

$$2k_{[I}^Y \mathfrak{D}_Y k_{J]}^X + f_{IJ}{}^K k_K^X = 0\,, \tag{C.86}$$

where $f_{IJ}{}^K$ are structure constants. Multiplying this relation with $J^{\alpha}{}_X{}^Z\mathfrak{D}_Z$, and using (C.79), and (C.83) gives

$$2J^{\alpha}{}_X{}^Z(\mathfrak{D}_Z k_{[I}^Y)(\mathfrak{D}_Y k_{J]}^X) + 2J^{\alpha}{}_X{}^Z R_{ZWY}{}^X k_{[I}^Y k_{J]}^W - 4r\nu f_{IJ}{}^K P_K^{\alpha} = 0\,. \tag{C.87}$$

The trace that appears in the first term can be evaluated by using (C.81) and once more (C.83), while in the second term we can use the cyclicity condition of the curvature and (C.49) to obtain

$$-2\nu^2 \varepsilon^{\alpha\beta\gamma} P_I^{\beta} P_J^{\gamma} + \mathcal{R}_{YW}{}^{\alpha} k_I^Y k_J^W - \nu f_{IJ}{}^K P_K^{\alpha} = 0\,. \tag{C.88}$$

We thus found that the moment maps, defined in (C.83) satisfy (C.85) and (C.88). The first of these shows that we can take $\nu = 0$ for the hypercomplex or hyperkähler manifolds. Both these two relations vanish identically in this case. However, for quaternionic-Kähler and hyperkähler manifolds, we can use (C.71), and dividing by ν leads to

$$J^{\alpha}{}_{ZW} k_I^W = -2\mathfrak{D}_Z P_I^{\alpha}\,, \tag{C.89}$$

$$-2\nu\varepsilon^{\alpha\beta\gamma} P_I^{\beta} P_J^{\gamma} + \tfrac{1}{2} J^{\alpha}{}_{YW} k_I^Y k_J^W - f_{IJ}{}^K P_K^{\alpha} = 0\,. \tag{C.90}$$

These equations are thus equivalent to the previous ones for $\nu \neq 0$ if there is a metric. This is thus the quaternionic-Kähler case, for which these relations appear already in [188]. But we did not *derive* these equations for the $\nu = 0$ (hyperkähler) case. Rather, the first one is taken as the definition of P for this case. This equation also follows from supersymmetry requirements, where the moment map P_I^{α} is an object that is needed to define the action, see (6.95). The moment map is then determined up to constants. As we saw in section 6.3, the constants are fixed when conformal symmetry is imposed. Similarly, the second equation appears in supersymmetry as a requirement, see (6.100). For a conformal invariant theory, the constants in P_I^{α} are determined and the moment map again satisfies (C.90) automatically due to a similar calculation as the one that we did above for $\nu \neq 0$. Note, however, that for the quaternionic manifolds that are not quaternionic-Kähler, we can only use (C.85) and (C.88), as (C.89) and (C.90) need a metric. For hypercomplex manifolds, on the other hand, the moment maps are not defined.

Publications

1. E.A. Bergshoeff, M. de Roo, T.C. de Wit, E. Eyras, and S. Panda, *T-duality and actions for non-BPS D-branes,* JHEP **05** (2000) 009, `hep-th/0003221`, 11pp.

2. E.A. Bergshoeff, S. Cucu, M. Derix, T.C. de Wit, R.S. Halbersma, and A. Van Proeyen, *Weyl multiplets of* $\mathcal{N} = 2$ *conformal supergravity in five dimensions,* JHEP **06** (2001) 051, `hep-th/0104113`, 39pp.

3. E.A. Bergshoeff, S. Cucu, T.C. de Wit, J. Gheerardyn, R.S. Halbersma, S. Vandoren, and A. Van Proeyen, *Superconformal N = 2, D = 5 matter with and without actions,* JHEP **10** (2002) 045, `hep-th/0205230`, 69pp.

4. E.A. Bergshoeff, T.C. de Wit, U. Gran, R. Linares, and D. Roest, *(Non-)Abelian gauged supergravities in nine dimensions,* JHEP **10** (2002) 061, `hep-th/0209205`, 36pp.

5. E.A. Bergshoeff, T.C. de Wit, U. Gran, R. Linares, and D. Roest, *(Non-)Abelian gauged supergravities in nine dimensions,* Class. Quant. Grav. 20 (2003), S425-S431, proceedings of the european workshop on the quantum structure of spacetime, Leuven, 13-19 september 2002, eds. M. Billo, M. Henneaux, A. Sevrin, S. Theisen, W. Troost, S. Vandoren and A. Van Proeyen.

6. E.A. Bergshoeff, T.C. de Wit, U. Gran, R. Linares, and D. Roest, *(Non-)Abelian gauged supergravities in nine dimensions,* proceedings of the 31st International Conference on High Energy Physics, Amsterdam, 24-31 july 2002, pp. 855-860, eds. S. Bentvelsen, P. de Jong, J. Koch and E. Laenen.

7. E.A. Bergshoeff, S. Cucu, T.C. de Wit, J. Gheerardyn, S. Vandoren, and A. Van Proeyen, $\mathcal{N} = 2$ *supergravity in five dimensions revisited,* in preparation.

8. E.A. Bergshoeff, S. Cucu, T.C. de Wit, J. Gheerardyn, S. Vandoren, and A. Van Proeyen, *The map between conformal hypercomplex/hyperkähler and quaternionic(-Kähler) geometry,* in preparation.

Bibliography

[1] http://home.uchicago.edu/~bgunn/particles.html,
http://www.sciforums.com/archive/33/2002/12/3/14248.

[2] Y. Nambu, *Quark model and the factorization of the Veneziano amplitude*, Proc. Int. Conf. on Symmetries and Quark Models (1970) proceedings of the International Conference, Wayne State University, 1969, Detroit, edited by R. Chand, Gordon and Breach, New York.

[3] T. Goto, *Relativistic Quantum Mechanics of One-Dimensional Mechanical Continuum and Subsidiary Condition of Dual Resonance Model*, Prog. Theor. Phys. **46** (1971) 1560

[4] L. Brink, P. Di Vecchia and P. S. Howe, *A locally supersymmetric and reparametrization invariant action for the spinning string*, Phys. Lett. **B65** (1976) 471–474

[5] S. Deser and B. Zumino, *A complete action for the spinning string*, Phys. Lett. **B65** (1976) 369–373

[6] A. M. Polyakov, *Quantum geometry of bosonic strings*, Phys. Lett. **B103** (1981) 207–210

[7] A. M. Polyakov, *Quantum geometry of fermionic strings*, Phys. Lett. **B103** (1981) 211–213

[8] F. Gliozzi, J. Scherk and D. I. Olive, *Supersymmetry, supergravity theories and the dual spinor model*, Nucl. Phys. **B122** (1977) 253–290

[9] N. Berkovits, *Super-Poincaré covariant quantization of the superstring*, JHEP **04** (2000) 018, hep-th/0001035

[10] N. Berkovits, *Covariant quantization of the superstring*, Int. J. Mod. Phys. **A16** (2001) 801–811, hep-th/0008145

[11] P. A. Grassi, G. Policastro, M. Porrati and P. Van Nieuwenhuizen, *Covariant quantization of superstrings without pure spinor constraints*, hep-th/0112162

[12] C. G. Callan, E. J. Martinec, M. J. Perry and D. Friedan, *Strings in background fields*, Nucl. Phys. **B262** (1985) 593

[13] D. Z. Freedman, P. van Nieuwenhuizen and S. Ferrara, *Progress towards a theory of supergravity*, Phys. Rev. **D13** (1976) 3214–3218

[14] E. Bergshoeff, C. M. Hull and T. Ortín, *Duality in the type II superstring effective action*, Nucl. Phys. **B451** (1995) 547–578, hep-th/9504081

[15] J. H. Schwarz, *Covariant field equations of chiral* $\mathcal{N} = 2\,, D = 10$ *supergravity*, Nucl. Phys. **B226** (1983) 269

[16] P. Dirac, *Quantised Singularities in the Electromagnetic Field*, Proc. Roy. Soc. **A133** (1931) 60

[17] C. Montonen and D. I. Olive, *Magnetic monopoles as gauge particles?*, Phys. Lett. **B72** (1977) 117

[18] K. Kikkawa and M. Yamasaki, *Casimir effects in superstring theories*, Phys. Lett. **B149** (1984) 357

[19] M. Dine, P. Y. Huet and N. Seiberg, *Large and small radius in string theory*, Nucl. Phys. **B322** (1989) 301

[20] J. Dai, R. G. Leigh and J. Polchinski, *New connections between string theories*, Mod. Phys. Lett. **A4** (1989) 2073–2083

[21] P. Ginsparg, *Comment on toroidal compactification of heterotic superstrings*, Phys. Rev. **D35** (1987) 648

[22] J. Polchinski and E. Witten, *Evidence for Heterotic - Type I String Duality*, Nucl. Phys. **B460** (1996) 525–540, hep-th/9510169

[23] C. M. Hull and P. K. Townsend, *Unity of superstring dualities*, Nucl. Phys. **B438** (1995) 109–137, hep-th/9410167

[24] E. Cremmer, B. Julia and J. Scherk, *Supergravity theory in 11 dimensions*, Phys. Lett. **B76** (1978) 409–412

[25] M. Huq and M. A. Namazie, *Kaluza-Klein supergravity in ten-dimensions*, Class. Quant. Grav. **2** (1985) 293

[26] P. Hořava and E. Witten, *Heterotic and type I string dynamics from eleven dimensions*, Nucl. Phys. **B460** (1996) 506–524, hep-th/9510209

[27] P. Hořava and E. Witten, *Eleven-dimensional supergravity on a manifold with boundary*, Nucl. Phys. **B475** (1996) 94–114, hep-th/9603142

[28] T. Banks, W. Fischler, S. H. Shenker and L. Susskind, *M theory as a matrix model: A conjecture*, Phys. Rev. **D55** (1997) 5112–5128, hep-th/9610043

[29] J. Polchinski, *TASI lectures on D-branes*, hep-th/9611050

[30] K. S. Stelle, *BPS branes in supergravity*, hep-th/9803116

[31] M. J. Duff, R. R. Khuri and J. X. Lu, *String solitons*, Phys. Rept. **259** (1995) 213–326, hep-th/9412184

[32] H. Lu, C. N. Pope, E. Sezgin and K. S. Stelle, *Stainless super p-branes*, Nucl. Phys. **B456** (1995) 669–698, hep-th/9508042

[33] M. J. Duff and K. S. Stelle, *Multi-membrane solutions of D = 11 supergravity*, Phys. Lett. **B253** (1991) 113–118

[34] R. Güven, *Black p-brane solutions of* $D = 11$ *supergravity theory*, Phys. Lett. **B276** (1992) 49–55

[35] J. Polchinski, *Dirichlet-Branes and Ramond-Ramond Charges*, Phys. Rev. Lett. **75** (1995) 4724–4727, hep-th/9510017

[36] R. G. Leigh, *Dirac-Born-Infeld action from Dirichlet sigma model*, Mod. Phys. Lett. **A4** (1989) 2767

[37] E. Bergshoeff, E. Eyras, R. Halbersma, C. Hull, Y. Lozano and J. P. van der Schaar, *Spacetime-filling branes and strings with sixteen supercharges*, Nucl. Phys. **B564** (2000) 29–59, hep-th/9812224

[38] E. B. Bogomolnyi, *The stability of classical solutions*, Sov. J. Nucl. Phys. **24** (1976) 499

[39] M. K. Prasad and C. M. Sommerfield, *An exact classical solution for the 't Hooft monopole and the Julia-Zee dyon*, Phys. Rev. Lett. **35** (1975) 760–762

[40] K. Behrndt, E. Bergshoeff and B. Janssen, *Intersecting D–branes in ten and six dimensions*, Phys. Rev. **D55** (1997) 3785–3792, hep-th/9604168

[41] J. R. Ellis, M. Günaydin and M. Zagermann, *Options for gauge groups in five-dimensional supergravity*, JHEP **11** (2001) 024, hep-th/0108094

[42] L. J. Romans, *Massive N=2a supergravity in ten-dimensions*, Phys. Lett. **B169** (1986) 374

[43] P. K. Townsend, *Positive energy and the scalar potential in higher dimensional (super)gravity theories*, Phys. Lett. **B148** (1984) 55

[44] R. Kallosh, A. D. Linde, S. Prokushkin and M. Shmakova, *Gauged supergravities, de Sitter space and cosmology*, Phys. Rev. **D65** (2002) 105016, hep-th/0110089

[45] S. Tsujikawa, *Introductory review of cosmic inflation*, hep-ph/0304257

[46] J. Maldacena, *The large N limit of superconformal field theories and supergravity*, Adv. Theor. Math. Phys. **2** (1998) 231–252, hep-th/9711200

[47] H. J. Boonstra, K. Skenderis and P. K. Townsend, *The domain wall/QFT correspondence*, JHEP **01** (1999) 003, hep-th/9807137

[48] K. Behrndt, E. Bergshoeff, R. Halbersma and J. P. van der Schaar, *On domain-wall/QFT dualities in various dimensions*, Class. Quant. Grav. **16** (1999) 3517–3552, hep-th/9907006

[49] M. Günaydin, L. J. Romans and N. P. Warner, *Gauged* $\mathcal{N} = 8$ *supergravity in five-dimensions*, Phys. Lett. **B154** (1985) 268

[50] M. Pernici, K. Pilch and P. van Nieuwenhuizen, *Gauged* $\mathcal{N} = 8$*,* $D = 5$ *supergravity*, Nucl. Phys. **B259** (1985) 460

[51] E. D'Hoker and D. Z. Freedman, *Supersymmetric gauge theories and the AdS/CFT correspondence*, hep-th/0201253

[52] H. Lü, C. N. Pope and P. K. Townsend, *Domain walls from Anti-de Sitter spacetime*, Phys. Lett. **B391** (1997) 39–46, hep-th/9607164

[53] K. Skenderis and P. K. Townsend, *Gravitational stability and renormalization-group flow*, Phys. Lett. **B468** (1999) 46–51, hep-th/9909070

[54] D. Z. Freedman, S. S. Gubser, K. Pilch and N. P. Warner, *Renormalization group flows from holography supersymmetry and a c-theorem*, Adv. Theor. Math. Phys. **3** (1999) 363–417, hep-th/9904017

[55] A. Ceresole, G. Dall'Agata, R. Kallosh and A. Van Proeyen, *Hypermultiplets, domain walls and supersymmetric attractors*, Phys. Rev. **D64** (2001) 104006, hep-th/0104056

[56] N. Arkani-Hamed, S. Dimopoulos and G. R. Dvali, *The hierarchy problem and new dimensions at a millimeter*, Phys. Lett. **B429** (1998) 263–272, hep-ph/9803315

[57] I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos and G. R. Dvali, *New dimensions at a millimeter to a Fermi and superstrings at a TeV*, Phys. Lett. **B436** (1998) 257–263, hep-ph/9804398

[58] V. A. Rubakov and M. E. Shaposhnikov, *Do we live inside a domain wall?*, Phys. Lett. **B125** (1983) 136–138

[59] L. Randall and R. Sundrum, *A large mass hierarchy from a small extra dimension*, Phys. Rev. Lett. **83** (1999) 3370–3373, hep-ph/9905221

[60] L. Randall and R. Sundrum, *An alternative to compactification*, Phys. Rev. Lett. **83** (1999) 4690–4693, hep-th/9906064

[61] V. A. Rubakov, *Large and infinite extra dimensions: An introduction*, Phys. Usp. **44** (2001) 871–893, hep-ph/0104152

[62] R. Dick, *Brane worlds*, Class. Quant. Grav. **18** (2001) R1–R24, hep-th/0105320

[63] S. Förste, *Strings, branes and extra dimensions*, Fortsch. Phys. **50** (2002) 221–403, hep-th/0110055

[64] E. Bergshoeff, R. Kallosh and A. Van Proeyen, *Supersymmetry in singular spaces*, JHEP **10** (2000) 033, hep-th/0007044

[65] E. Bergshoeff, R. Kallosh and A. Van Proeyen, *Supersymmetry of RS bulk and brane*, Fortsch. Phys. **49** (2001) 625–632, hep-th/0012110

[66] A. Lukas, B. A. Ovrut, K. S. Stelle and D. Waldram, *The universe as a domain wall*, Phys. Rev. **D59** (1999) 086001, hep-th/9803235

[67] A. Lukas, B. A. Ovrut, K. S. Stelle and D. Waldram, *Heterotic M-theory in five dimensions*, Nucl. Phys. **B552** (1999) 246–290, hep-th/9806051

[68] P. K. Townsend, K. Pilch and P. van Nieuwenhuizen, *Selfduality in odd dimensions*, Phys. Lett. **136B** (1984) 38

[69] E. Cremmer, *Supergravities in 5 dimensions*, Invited paper at the Nuffield Gravity Workshop, Cambridge, Eng., Jun 22 - Jul 12, 1980

[70] M. Günaydin, G. Sierra and P. K. Townsend, *The geometry of $\mathcal{N} = 2$ Maxwell-Einstein supergravity and Jordan algebras*, Nucl. Phys. **B242** (1984) 244

[71] M. Günaydin, G. Sierra and P. K. Townsend, *Gauging the $D = 5$ Maxwell-Einstein supergravity theories: more on Jordan algebras*, Nucl. Phys. **B253** (1985) 573

[72] M. Günaydin, G. Sierra and P. K. Townsend, *More on $D = 5$ Maxwell-Einstein supergravity: symmetric spaces and kinks*, Class. Quant. Grav. **3** (1986) 763

[73] M. Günaydin and M. Zagermann, *The gauging of five-dimensional, $\mathcal{N} = 2$ Maxwell-Einstein supergravity theories coupled to tensor multiplets*, Nucl. Phys. **B572** (2000) 131–150, hep-th/9912027

[74] M. Günaydin and M. Zagermann, *The vacua of 5d, $\mathcal{N} = 2$ gauged Yang-Mills/Einstein/tensor supergravity: Abelian case*, Phys. Rev. **D62** (2000) 044028, hep-th/0002228

[75] M. Günaydin and M. Zagermann, *Gauging the full R-symmetry group in five-dimensional, $\mathcal{N} = 2$ Yang-Mills/Einstein/tensor supergravity*, Phys. Rev. **D63** (2001) 064023, hep-th/0004117

[76] A. Ceresole and G. Dall'Agata, *General matter coupled $\mathcal{N} = 2$, $D = 5$ gauged supergravity*, Nucl. Phys. **B585** (2000) 143–170, hep-th/0004111

[77] R. Kallosh and A. D. Linde, *Supersymmetry and the brane world*, JHEP **02** (2000) 005, hep-th/0001071

[78] K. Behrndt and M. Cvetic, *Anti-de Sitter vacua of gauged supergravities with 8 supercharges*, Phys. Rev. **D61** (2000) 101901, hep-th/0001159

[79] A. Ceresole and G. Dall'Agata, *Brane-worlds in 5D supergravity*, Fortsch. Phys. **49** (2001) 449–454, hep-th/0101214

[80] G. W. Gibbons and N. D. Lambert, *Domain walls and solitons in odd dimensions*, Phys. Lett. **B488** (2000) 90–96, hep-th/0003197

[81] J. M. Maldacena and C. Nunez, *Supergravity description of field theories on curved manifolds and a no go theorem*, Int. J. Mod. Phys. **A16** (2001) 822–855, hep-th/0007018

[82] D. V. Alekseevsky, V. Cortes, C. Devchand and A. Van Proeyen, *Flows on quaternionic-Kaehler and very special real manifolds*, hep-th/0109094

[83] K. Behrndt and M. Cvetič, *Gauging of $\mathcal{N} = 2$ supergravity hypermultiplet and novel renormalization group flows*, Nucl. Phys. **B609** (2001) 183–192, hep-th/0101007

[84] K. Behrndt and G. Dall'Agata, *Vacua of N = 2 gauged supergravity derived from non- homogenous quaternionic spaces*, Nucl. Phys. **B627** (2002) 357–380, hep-th/0112136

[85] L. Anguelova and C. I. Lazaroiu, *Domain walls of N = 2 supergravity in five dimensions from hypermultiplet moduli spaces*, JHEP **09** (2002) 053, hep-th/0208154

[86] E. Bergshoeff, S. Cucu, T. de Wit, J. Gheerardyn, R. Halbersma, S. Vandoren and A. V. Proeyen, *Superconformal N = 2, D = 5 matter with and without actions*, JHEP **10** (2002) 045, hep-th/0205230

[87] C. Pope, *Kaluza-Klein theory*, Lecture notes, http://faculty.physics.tamu.edu/pope/ihplec.ps

[88] M. J. Duff, B. E. W. Nilsson and C. N. Pope, *Kaluza-Klein supergravity*, Phys. Rept. **130** (1986) 1–142

[89] A. Salam and J. Strathdee, *On Kaluza-Klein theory*, Annals Phys. **141** (1982) 316–352

[90] H. Lee, ed., *An introduction to Kaluza-Klein theories. Proceedings, workshop, Chalk River, Canada, august 11-16, 1983*. Singapore, Singapore: World Scientific (1984) 315p

[91] A. Einstein, *The Foundation of the General Theory of Relativity*, Annalen Phys. **49** (1916) 769–822

[92] T. Kaluza, *On the problem of unity in physics*, Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.) **K1** (1921) 966–972

[93] O. Klein, *Quantum theory and five-dimensional theory of relativity*, Z. Phys. **37** (1926) 895–906

[94] J. Scherk and J. H. Schwarz, *Spontaneous breaking of supersymmetry through dimensional reduction*, Phys. Lett. **B82** (1979) 60

[95] J. Scherk and J. H. Schwarz, *How to get masses from extra dimensions*, Nucl. Phys. **B153** (1979) 61–88

[96] C. M. Hull, *Massive string theories from M-theory and F-theory*, JHEP **11** (1998) 027, hep-th/9811021

[97] A. Dabholkar and C. Hull, *Duality twists, orbifolds, and fluxes*, hep-th/0210209

[98] B. de Wit, H. Samtleben and M. Trigiante, *On Lagrangians and gaugings of maximal supergravities*, Nucl. Phys. **B655** (2003) 93–126, hep-th/0212239

[99] N. Straumann, *On Pauli's invention of non-Abelian Kaluza-Klein theory in 1953*, gr-qc/0012054

[100] C.-N. Yang and R. L. Mills, *Conservation of isotopic spin and isotopic gauge invariance*, Phys. Rev. **96** (1954) 191–195

[101] J. Sauriau, *Relativité Multidimensionelle non Stationaire*, Les Théories Relativistes de La Gravitation (1962) 293–297, Paris, Centre National de la Recherche Scientifique

[102] J. Rayski, *Unified field theory and modern physics*, Acta Phys. Polon. **27** (1965) 89–97

[103] R. Kerner, *Generalization of the Kaluza-Klein theory for an arbitrary non-Abelian gauge group*, Ann. Inst. H. Poincaré **9** (1968) 143–152

[104] A. Trautman, *Fiber bundles associated with space-time*, Rept. Math. Phys. **1** (1970) 29–62

[105] Y. M. Cho, *Higher-dimensional unifications of gravitation and gauge theories*, J. Math. Phys. **16** (1975) 2029

[106] Y. M. Cho and P. G. O. Freund, *Non-Abelian gauge fields in Nambu-Goldstone fields*, Phys. Rev. **D12** (1975) 1711

[107] M. Cvetič, H. Lü and C. N. Pope, *Consistent Kaluza-Klein sphere reductions*, Phys. Rev. **D62** (2000) 064028, `hep-th/0003286`

[108] E. Bergshoeff, T. de Wit, U. Gran, R. Linares and D. Roest, *(Non-)Abelian gauged supergravities in nine dimensions*, JHEP **10** (2002) 061, `hep-th/0209205`

[109] E. Bergshoeff, B. Janssen and T. Ortín, *Solution-generating transformations and the string effective action*, Class. Quant. Grav. **13** (1996) 321–343, `hep-th/9506156`

[110] H. Nicolai, P. K. Townsend and P. van Nieuwenhuizen, *Comments on Eleven-dimensional Supergravity*, Nuovo Cim. Lett. **30** (1981) 315

[111] P. Meessen and T. Ortín, *An Sl(2,Z) multiplet of nine-dimensional type II supergravity theories*, Nucl. Phys. **B541** (1999) 195–245, `hep-th/9806120`

[112] E. Bergshoeff, R. Kallosh, T. Ortín, D. Roest and A. Van Proeyen, *New formulations of D = 10 supersymmetry and D8 - O8 domain walls*, Class. Quant. Grav. **18** (2001) 3359–3382, `hep-th/0103233`

[113] F. Giani and M. Pernici, *N=2 Supergravity in Ten-Dimensions*, Phys. Rev. **D30** (1984) 325–333

[114] P. S. Howe, N. D. Lambert and P. C. West, *A new massive type IIA supergravity from compactification*, Phys. Lett. **B416** (1998) 303–308, `hep-th/9707139`

[115] I. V. Lavrinenko, H. Lu and C. N. Pope, *Fibre bundles and generalised dimensional reductions*, Class. Quant. Grav. **15** (1998) 2239–2256, `hep-th/9710243`

[116] E. Bergshoeff, M. de Roo, M. B. Green, G. Papadopoulos and P. K. Townsend, *Duality of Type II 7-branes and 8-branes*, Nucl. Phys. **B470** (1996) 113–135, `hep-th/9601150`

[117] E. Bergshoeff, U. Gran and D. Roest, *Type IIB seven-brane solutions from nine-dimensional domain walls*, Class. Quant. Grav. **19** (2002) 4207–4226, `hep-th/0203202`

[118] O. DeWolfe, T. Hauer, A. Iqbal and B. Zwiebach, *Uncovering the symmetries on (p,q) 7-branes: Beyond the Kodaira classification*, Adv. Theor. Math. Phys. **3** (1999) 1785–1833, `hep-th/9812028`

[119] O. DeWolfe, T. Hauer, A. Iqbal and B. Zwiebach, *Uncovering infinite symmetries on (p,q) 7-branes: Kac-Moody algebras and beyond*, Adv. Theor. Math. Phys. **3** (1999) 1835–1891, `hep-th/9812209`

[120] J. Gheerardyn and P. Meessen, *Supersymmetry of massive D = 9 supergravity*, Phys. Lett. **B525** (2002) 322–330, `hep-th/0111130`

[121] K. Behrndt, E. Bergshoeff, D. Roest and P. Sundell, *Massive dualities in six dimensions*, Class. Quant. Grav. **19** (2002) 2171–2200, `hep-th/0112071`

[122] J. Louis and A. Micu, *Type II theories compactified on Calabi-Yau threefolds in the presence of background fluxes*, Nucl. Phys. **B635** (2002) 395–431, `hep-th/0202168`

[123] A. Salam and E. Sezgin, eds., *Supergravities in diverse dimensions. Vol. 1, 2.* Amsterdam, Netherlands: North-Holland (1989) 1499 p. Singapore, Singapore: World Scientific (1989) 1499 p

[124] M. Kaku and P. K. Townsend, *Poincaré supergravity as broken superconformal gravity*, Phys. Lett. **B76** (1978) 54

[125] P. S. Howe, *A superspace approach to extended conformal supergravity*, Phys. Lett. **B100** (1981) 389

[126] B. de Wit, M. Roček and S. Vandoren, *Hypermultiplets, hyper-Kähler cones and quaternion-Kähler geometry*, JHEP **02** (2001) 039, `hep-th/0101161`

[127] M. Kaku, P. K. Townsend and P. van Nieuwenhuizen, *Gauge theory of the conformal and superconformal group*, Phys. Lett. **B69** (1977) 304–308

[128] M. Kaku, P. K. Townsend and P. van Nieuwenhuizen, *Properties of conformal supergravity*, Phys. Rev. **D17** (1978) 3179

[129] S. W. MacDowell and F. Mansouri, *Unified geometric theory of gravity and supergravity*, Phys. Rev. Lett. **38** (1977) 739

[130] S. Ferrara, M. Kaku, P. K. Townsend and P. van Nieuwenhuizen, *Gauging the graded conformal group with unitary internal symmetries*, Nucl. Phys. **B129** (1977) 125

[131] E. Bergshoeff, S. Cucu, M. Derix, T. de Wit, R. Halbersma and A. Van Proeyen, *Weyl multiplets of $\mathcal{N}=2$ conformal supergravity in five dimensions*, JHEP **06** (2001) 051, hep-th/0104113

[132] R. S. Halbersma, *Geometry of Strings and Branes*, Ph.D. thesis, Groningen University, 2002; available from http://www.ub.rug.nl/eldoc/dis/science/r.s.halbersma/

[133] T. Fujita and K. Ohashi, *Superconformal tensor calculus in five dimensions*, Prog. Theor. Phys. **106** (2001) 221–247, hep-th/0104130

[134] A. Van Proeyen, *Superconformal tensor calculus in $\mathcal{N}=1$ and $\mathcal{N}=2$ supergravity*, To be publ. in Proc. of Karpacz Winter School, Karpacz, Poland, Feb 14-26, 1983

[135] E. A. Bergshoeff, *Conformal invariance in supergravity*, Ph.D. thesis RX-1008 (Leiden)

[136] A. Van Proeyen, *$\mathcal{N}=2$ supergravity in $D=4,5,6$ and its matter couplings*, Lecture notes, http://itf.fys.kuleuven.ac.be/ toine/lectparis.ps

[137] A. Van Proeyen, *Tools for supersymmetry*, Annals of the University of Craiova, Physics AUC **9 (part I)** (1999) 1–48, hep-th/9910030

[138] E. Sezgin and Y. Tanii, *Superconformal sigma models in higher than two dimensions*, Nucl. Phys. **B443** (1995) 70–84, hep-th/9412163

[139] B. de Wit, B. Kleijn and S. Vandoren, *Rigid $\mathcal{N}=2$ superconformal hypermultiplets*, hep-th/9808160, in *Supersymmetries and Quantum Symmetries*, proc. Int. Sem. Dubna (1997), eds. J. Wess and E.A. Ivanov, Lecture Notes in Physics, Vol. 524 (Springer, 1999), p. 37

[140] J. Michelson and A. Strominger, *The geometry of (super)conformal quantum mechanics*, Commun. Math. Phys. **213** (2000) 1–17, hep-th/9907191

[141] J. W. van Holten and A. Van Proeyen, *$\mathcal{N}=1$ supersymmetry algebras in $D=2,3,4$ mod. 8*, J. Phys. **A15** (1982) 3763

[142] R. D'Auria, S. Ferrara, M. A. Lledo and V. S. Varadarajan, *Spinor algebras*, J. Geom. Phys. **40** (2001) 101–128, hep-th/0010124

[143] E. Bergshoeff, M. de Roo and B. de Wit, *Conformal supergravity in ten dimensions*, Nucl. Phys. **B217** (1983) 489

[144] W. Nahm, *Supersymmetries and their representations*, Nucl. Phys. **B135** (1978) 149

[145] B. de Wit, J. W. van Holten and A. Van Proeyen, *Transformation rules of $\mathcal{N}=2$ supergravity multiplets*, Nucl. Phys. **B167** (1980) 186

[146] E. Bergshoeff, M. de Roo and B. de Wit, *Extended conformal supergravity*, Nucl. Phys. **B182** (1981) 173

[147] E. Bergshoeff, E. Sezgin and A. Van Proeyen, *Superconformal tensor calculus and matter couplings in six dimensions*, Nucl. Phys. **B264** (1986) 653

[148] E. Bergshoeff, E. Sezgin and A. Van Proeyen, *$(2,0)$ tensor multiplets and conformal supergravity in $D=6$*, Class. Quant. Grav. **16** (1999) 3193–3206, hep-th/9904085

[149] A. Van Proeyen, *Special geometries, from real to quaternionic*, hep-th/0110263, To be published in the proceedings of the 'DFG workshop on special geometric structures in string theory', Bonn, 8-11/9/2001

[150] N. Seiberg and E. Witten, *Electric - magnetic duality, monopole condensation, and confinement in $\mathcal{N}=2$ supersymmetric Yang–Mills theory*, Nucl. Phys. **B426** (1994) 19–52, hep-th/9407087

[151] N. Seiberg and E. Witten, *Monopoles, duality and chiral symmetry breaking in $\mathcal{N}=2$ supersymmetric QCD*, Nucl. Phys. **B431** (1994) 484–550, hep-th/9408099

[152] B. de Wit, P. G. Lauwers, R. Philippe, S. Q. Su and A. Van Proeyen, *Gauge and matter fields coupled to $\mathcal{N}=2$ supergravity*, Phys. Lett. **B134** (1984) 37

[153] B. de Wit and A. Van Proeyen, *Potentials and symmetries of general gauged $\mathcal{N}=2$ supergravity – Yang-Mills models*, Nucl. Phys. **B245** (1984) 89

[154] G. Sierra and P. K. Townsend, *An introduction to $\mathcal{N}=2$ rigid supersymmetry*, in *Supersymmetry and Supergravity 1983*, ed. B. Milewski (World Scientific, Singapore, 1983)

[155] J. Gates, S. James, *Superspace formulation of new nonlinear sigma models*, Nucl. Phys. **B238** (1984) 349

[156] N. Seiberg, *Observations on the moduli space of superconformal field theories*, Nucl. Phys. **B303** (1988) 286

[157] S. Cecotti, S. Ferrara and L. Girardello, *Geometry of type II superstrings and the moduli of superconformal field theories*, Int. J. Mod. Phys. **A4** (1989) 2475

[158] S. Ferrara and A. Strominger, *$\mathcal{N}=2$ space-time supersymmetry and Calabi–Yau moduli space*, in *Strings '89*, eds. R. Arnowitt, R. Bryan, M.J. Duff, D.V. Nanopoulos and C.N. Pope (World Scientific, Singapore, 1989), p. 245

[159] P. Candelas and X. de la Ossa, *Moduli space of Calabi–Yau manifolds*, Nucl. Phys. **B355** (1991) 455–481

[160] P. Candelas, X. C. De la Ossa, P. S. Green and L. Parkes, *An exactly soluble superconformal theory from a mirror pair of Calabi–Yau manifolds*, Phys. Lett. **B258** (1991) 118–126

[161] A. Strominger, *Special geometry*, Commun. Math. Phys. **133** (1990) 163–180

[162] T. Kugo and K. Ohashi, *Gauge and non-gauge tensor multiplets in 5D conformal supergravity*, Prog. Theor. Phys. **108** (2003) 1143–1164, `hep-th/0208082`

[163] A. Van Proeyen, *The scalars of $\mathcal{N}=2$, $D=5$ and attractor equations*, `hep-th/0105158`, in 'New developments in fundamental interaction theories', proceedings of 37th Karpacz Winter School, feb 2001; eds. J. Lukierski and J. Rembieliński, AIP proceedings 589, 2001, pp. 31–45

[164] B. de Wit, B. Kleijn and S. Vandoren, *Superconformal hypermultiplets*, Nucl. Phys. **B568** (2000) 475–502, `hep-th/9909228`

[165] M. Obata, *Affine connections on manifolds with almost complex, quaternionic or Hermitian structure*, Jap. J. Math. **26** (1956) 43–79

[166] B. de Wit, J. W. van Holten and A. Van Proeyen, *Central charges and conformal supergravity*, Phys. Lett. **B95** (1980) 51

[167] T. Kugo and K. Ohashi, *Off-shell $D=5$ supergravity coupled to matter–Yang–Mills system*, Prog. Theor. Phys. **105** (2001) 323–353, `hep-ph/0010288`

[168] A. Swann, *Hyper-Kähler and quaternionic Kähler geometry*, Math. Ann. **289** (1991) 421–450

[169] B. de Wit, P. G. Lauwers and A. Van Proeyen, *Lagrangians of N=2 Supergravity - Matter systems*, Nucl. Phys. **B255** (1985) 569

[170] R. D'Auria, S. Ferrara and P. Frè, *Special and quaternionic isometries: General couplings in $\mathcal{N}=2$ supergravity and the scalar potential*, Nucl. Phys. **B359** (1991) 705–740

[171] L. Andrianopoli, M. Bertolini, A. Ceresole, R. D'Auria, S. Ferrara, P. Fré and T. Magri, *$\mathcal{N}=2$ supergravity and $\mathcal{N}=2$ super Yang–Mills theory on general scalar manifolds: Symplectic covariance, gaugings and the momentum map*, J. Geom. Phys. **23** (1997) 111–189, `hep-th/9605032`

[172] B. de Wit, M. Roček and S. Vandoren, *Gauging isometries on hyperkähler cones and quaternion-Kähler manifolds*, Phys. Lett. **B511** (2001) 302–310, `hep-th/0104215`

[173] E. Bergshoeff, S. Cucu, T. de Wit, J. Gheerardyn, S. Vandoren and A. Van Proeyen, *$\mathcal{N}=2$ supergravity in five dimensions revisited*, in preparation

[174] E. Bergshoeff, S. Cucu, T. de Wit, J. Gheerardyn, S. Vandoren and A. Van Proeyen, *The map between conformal hypercomplex/hyperkähler and quaternionic(-Kähler) geometry*, in preparation

[175] N. Alonso Alberca *et al.*, *Domain walls of D = 8 gauged supergravities and their D = 11 origin*, `hep-th/0303113`

[176] S. Hawking and J. Ellis, *The large scale structure of spacetime*, Cambridge, UK: Univ. Pr. (1973) 391 p

[177] S. Salamon, *Differential geometry of quaternionic manifolds*, Ann. Scient. Ec. Norm. Sup., 4ème serie **19** (1986) 31–55

[178] D. V. Alekseevsky and S. Marchiafava, *Quaternionic structures on a manifold and subordinated structures*, Ann. Matem. pura appl. (IV) **171** (1996) 205–273

[179] P. Spindel, A. Sevrin, W. Troost and A. Van Proeyen, *Extended supersymmetric sigma models on group manifolds. 1. The complex structures*, Nucl. Phys. **B308** (1988) 662

[180] D. Joyce, *Compact hypercomplex and quaternionic manifolds*, J. Diff. Geom **35** (1992) 743–761

[181] M. L. Barberis and I. Dotti-Miatello, *Hypercomplex structures on a class of solvable Lie groups*, Quart. J. Math. Oxford (2) **47** (1996) 389–404

[182] G. Gentili, S. Marchiafava and M. Pontecorvo, eds., *Quaternionic structures in mathematics and physics*. World Scientific, 1996, Proceedings of workshop in Trieste, September 1994; ILAS/FM-6/1996, available on `http://www.emis.de/proceedings/QSMP94/`

[183] S. Marchiafava, P. Piccinni and M. Pontecorvo, eds., *Quaternionic structures in mathematics and physics*. World Scientific, 2001, Proceedings of workshop in Roma, September 1999; available on `http://www.univie.ac.at/EMIS/proceedings/QSMP99/`

[184] H. Pedersen, Y. a Poon and A. a Swann, *Hypercomplex structures associated to quaternionic manifolds*, Diff. Geom. Appl. **9** (1998) 273–292

[185] H. Pedersen, *Hypercomplex geometry*, in *Quaternionic structures in mathematics and physics*, World Scientific 2001, eds. S. Marchiafava, P. Piccinni and M. Pontecorvo, available on `http://www.univie.ac.at/EMIS/proceedings/QSMP99/`

[186] V. Oproiu, *Almost quaternal structures*, An. st. Univ. Iaşi **23** (1977) 287–298

[187] D. V. Alekseevsky, *Classification of quaternionic spaces with a transitive solvable group of motions*, Math. USSR Izvestija **9** (1975) 297–339

[188] K. Galicki, *A generalization of the momentum mapping construction for Quaternionic Kähler manifolds*, Commun. Math. Phys. **108** (1987) 117

Samenvatting

Sinds de geboorte van de deeltjesfysica, na de ontdekking van het elektron door Thomson in 1897, is er enorme vooruitgang geboekt in de beschrijving van waarneembare processen in de natuur. Om het gedrag van deeltjes te kunnen verklaren op (sub)atomaire schaal, werd omstreeks 1920 de quantummechanica ontwikkeld. Men realiseerde zich, afgaand op de uitkomsten van experimenten, dat alle deeltjes een fundamentele eigenschap bezitten: genaamd ‘spin’. De waarde hiervan splitst de deeltjes in twee klassen: bosonen en fermionen, elk met zeer specifieke eigenschappen. Enige tijd daarvoor, in 1905, revolutioneerde Einstein ons denken over ruimte en tijd met zijn speciale relativiteitstheorie. Deze theorie liet zien hoe de concepten ruimte en tijd zijn verweven en niet apart kunnen worden beschouwd. Een combinatie van deze twee theorieën leidde omstreeks 1970 uiteindelijk tot het Standaard Model, dat perfect in staat bleek om de wisselwerkingen te beschrijven tussen de elementaire deeltjes die de bouwstenen vormen van alle observeerbare materie in het universum. Het Standaard Model beschrijft drie soorten fundamentele interacties: de elektromagnetische, de zwakke en de sterke wisselwerking. Het begrip ijksymmetrie speelt hierbij een belangrijke rol. Door het lokaal maken van deze symmetrie, dat wil zeggen het invoeren van een coördinaatafhankelijke transformatieparameter, worden spin-1-ijkdeeltjes ingevoerd die krachten kunnen overbrengen tussen twee deeltjes. Het bekendste voorbeeld is het foton, dat het elektromagnetisch veld tussen twee geladen deeltjes veroorzaakt, waardoor deze deeltjes elkaar aantrekken of afstoten afhankelijk van hun ladingen. Op vergelijkbare wijze worden de overige fundamentele krachten ‘gedragen’ door respectievelijk de W/Z-bosonen en de gluonen. Alhoewel het Standaard Model met zeer grote precisie experimenteel bevestigd is, is er een aantal discrepanties. Zo is het Higgs-deeltje, dat nodig is om massa’s te geven aan de andere elementaire deeltjes, nog niet gevonden.[8] Een ander bezwaar is de noodzaak voor 19 ad-hocparameters — onder andere de massa’s van de elementaire deeltjes — die niet theoretisch kunnen worden bepaald, terwijl in een fundamentele theorie alles uit basisprincipes zou moeten volgen.

Een andere grootse prestatie in de 20e eeuw was Einsteins algemene relativiteitstheorie, die de vierde fundamentele kracht voor zijn rekening neemt: zwaartekracht. Deze theorie werd geconstrueerd in 1914 in een poging de grondbeginselen van de speciale relativiteit te implementeren in Newtons zwaartekrachttheorie en vergrootte ons begrip van de samenhang van ruimte en tijd. Enkele successen waren bijvoorbeeld de voorspellingen van kleine afwijkingen van planeetbanen en de buiging van licht langs massieve objecten. Van meer speculatieve aard zijn de voorspellingen van zwarte gaten en zwaartekrachtsgolven, die beide slechts indirect zijn geveri-

[8] Men hoopt dat de nieuwe LHC-versneller hier in 2006 uitsluitsel over zal kunnen geven.

kracht	ijkdeeltje	werkt op	bereik	relatieve sterkte
sterke nucleaire kracht	gluon	quarks	nucleaire afstanden	20
elektromagnetisme	foton	geladen deeltjes	oneindig	1
zwakke nucleaire kracht	W- of Z-boson	quarks en leptonen	nucleaire afstanden	10^{-7}
zwaartekracht	graviton?	massieve deeltjes	oneindig	10^{-36}

Tabel C.2: De vier fundamentele krachten. De relatieve sterktes zijn gebaseerd op twee wisselwerkende up-quarks, op een afstand van 10^{-18} m van elkaar [1].

fieerd. Tevens konden voorspellingen worden gedaan over de evolutie van ons heelal. Alhoewel de theorie de interacties tussen massieve objecten perfect kan verklaren op relatief grote lengteschaal, gaat er iets mis wanneer men probeert algemene relativiteit te beschrijven in het gebied waar de quantummechanica heerst. Aangezien de zwaartekrachtseffecten op kleine schaal verwaarloosbaar zijn ten opzichte van de andere drie fundamentele krachten, zie tabel C.2, is het niet verwonderlijk dat de algemene relativiteitstheorie slechts is getest tot een afstand van ongeveer 1 millimeter. Een poging om zwaartekracht te beschrijven met de standaard quantisatiemethoden, die ook werden gebruikt voor het Standaard Model, faalde. De theorie kampte met oneindigheden vanwege een niet-dimensieloze koppelingsconstante $\kappa = 8\pi G/c^4$, waardoor de theorie ongeschikt is om storingsrekening op toe te passen. De typische lengteschaal waar onze klassieke ideeën over zwaartekracht en de ruimte-tijd hun geldigheid verliezen wordt gegeven door de Planck-lengte:

$$\ell_P = \sqrt{\frac{hG}{c^3}} \approx 4.1 \cdot 10^{-35}\ \mathrm{m}\,, \tag{C.91}$$

waarbij h de constante van Planck is, G Newtons zwaartekrachtsconstante en c de lichtsnelheid.

Samenvattend, aan beide uiteinden van het schaalspectrum hebben we twee succesvolle theorieën die niet verenigbaar lijken te zijn. De oplossing zou gegeven moeten worden door een theorie van 'quantumzwaartekracht', die alle vier de fundamentele krachten omvat. De zoektocht naar deze geunificeerde theorie is het hoofddoel geweest van de hoge-energiefysica gedurende de laatste twintig jaar.

Een gedeeltelijk succes werd bereikt in 1976 door de ontdekking van superzwaartekracht; een uitbreiding van de algemene relativiteitstheorie die zich beter gedroeg bij hoge energieën, vanwege een gedeeltelijk tegen elkaar wegvallen van oneindigheden. Het cruciale ingrediënt hierbij was supersymmetrie, een symmetrie tussen bosonen en fermionen die voorspelt dat voor iedere boson in de natuur een corresponderend fermionisch deeltje bestaat, en visa versa. De ijktheorie van supersymmetrie wordt gegeven door superzwaartekracht. Het spin-2-ijkdeeltje dat verantwoordelijk is voor het overdragen van de zwaartekracht wordt het graviton genoemd. Zijn supersymmetrische partner is het zogenaamde gravitino. Om deze deeltjes te meten zijn echter energieën nodig die ver buiten het bereik liggen van hedendaagse (en toekomstige) versnellers.

De meest veelbelovende kandidaat tot op heden is de supersnarentheorie. Snarentheorie veronderstelt dat alle deeltjes gerepresenteerd worden door trillingstoestanden van een snaar met een typische lengte ℓ_S in de orde van de Planck lengte ℓ_P. Eén van de trillings toestanden bleek een massaloos spin-2-deeltje te beschrijven dat zich gedraagt als een graviton. Vervolgens werd gevonden dat de lage-energielimiet van supersnarentheorie wordt gegeven door superzwaartekracht. Alhoewel deze laatste niet vrij van oneindigheden was gebleken, is er een intuïtieve reden waarom supersnarentheorie dat vermoedelijk wel is. Deze oneindigheden treden namelijk meestal op in singuliere punten. Echter, een snaartje dat beweegt in de ruimte-tijd bestrijkt een tweedimensionaal oppervlak, in tegenstelling tot een lijn voor een puntdeeltje. Precies dit feit zorgt ervoor dat de interacties tussen snaren niet in één singulier punt plaatsvinden, maar verspreid zijn over een kleine ruimte. Helaas heeft ook deze theorie haar nadelen. Snarentheorie is alleen perturbatief gedefinieerd, met andere woorden: verstrooiingsamplitudes worden uitgedrukt als een oneindige reeks in machten van de snarenkoppelingsconstante g_S, die geassocieerd wordt met de 'Feynman diagrammen' van snarentheorie. De grootste tegenslag was het bestaan van maar liefst vijf supersnarentheorieën, terwijl men hoopte op één unieke theorie van quantumzwaartekracht. Enkele jaren geleden veranderde dit, door de ontdekking van dualiteiten, die verschillende energieregimes van verschillende theorieën met elkaar relateerden. Een belangrijke rol was hierbij weggelegd voor zogenaamde braanoplossingen van snarentheorie. Dit zijn solitonische membraanachtige oplossingen die kunnen worden gezien als hogerdimensionale generalisaties van snaren. De vijf op het eerste gezicht verschillende theorieën en hun braanoplossingen bleken hierdoor gerelateerd door een web van dualiteiten. Dit suggereerde echter dat de vijf supersnarentheorieën allemaal een andere limiet vormden van één fundamentele theorie, genaamd 'M-theorie'. Veel is helaas nog niet bekend over deze theorie. Echter, door de lage-energielimieten van M-theorie en de vele dualiteiten hiertussen te bestuderen, komen we hopelijk steeds een stapje dichter bij een geunificeerde theorie.

We zullen nu een korte beschrijving geven van de onderwerpen die in dit proefschrift aan bod komen. In hoofdstuk 1 beschrijven we het raamwerk van snarentheorie en superzwaartekracht, om de in dit proefschrift behandelde onderwerpen in een context te kunnen plaatsen. Hoofdstuk 2 bevat de motivatie voor het onderzoek dat in het resterend deel van deze dissertatie wordt behandeld. De hoofdmotivatie is het concept 'braanwerelden', waarbij ervan wordt uitgegaan dat ons vierdimensionaal universum kan worden gerepresenteerd als een vierdimensionale braanoplossing in vijf ruimte-tijddimensies. Dit type modellen werd gebruikt om verscheidene problemen in de kosmologie op te lossen; bijvoorbeeld het kosmologische-constanteprobleem en het hiërarchieprobleem. De branen die in dergelijke modellen worden gebruikt splitsen de ruimte-tijd in twee gebieden en worden domeinvlakken genoemd. Een supersymmetrische versie is echter niet gemakkelijk te construeren; de domeinvlakken moeten aan bepaalde voorwaarden voldoen om de juiste vacuümstructuur van de vijfdimensionale ruimte-tijd te kunnen beschrijven. Het bepalen van alle mogelijke domeinvlakkandidaten vereist een goede kennis van materiekoppelingen in de vijfdimensionale superzwaartekrachttheorie. De scalarvelden die in dergelijke theorieën voorkomen, blijken te kunnen worden opgevat als coördinaten van een manifold. De potentiële energie van deze deeltjes wordt gegeven door de scalaire potentiaal, welke een functie is van alle scalairen van de manifold. De vacuümstructuur van de vijfdimensionale ruimte-tijd wordt bepaald door de minima van de scalaire potentiaal en de geometrische eigenschappen van de scalaire manifold.

De vijfdimensionale materiegekoppelde superzwaartekrachttheorie is een speciaal geval van een geijkte superzwaartekrachttheorie, dat wil zeggen een superzwaartekrachttheorie waar één of meer globale symmetrieën lokaal zijn gemaakt. Eén manier om dergelijke theorieën te construeren, is door middel van dimensionele reductie. Men begint hiertoe met een hogerdimensionale superzwaartekrachttheorie en ‘rolt’ enkele extra dimensies op om effectief te eindigen met een superzwaartekrachttheorie in een lagere ruimte-tijddimensie. Een uitbreiding van deze methode wordt ‘algemene dimensionele reductie’ genoemd; hierbij benut men een symmetrie van een theorie om massa’s te verkrijgen in lagere dimensies. In dit geval zal de gebruikte symmetrie verschijnen als een geijkte symmetrie van de gereduceerde theorie. Wanneer deze techniek wordt toegepast op superzwaartekrachttheorie, kan men geijkte superzwaartekrachttheorieën construeren. Een algemene inleiding tot het onderwerp dimensionele reductie wordt gegeven in hoofdstuk 3, waarna deze techniek in hoofdstuk 4 zal worden toegepast op superzwaartekracht in tien en elf dimensies.

De overige hoofdstukken 5, 6 en 7 geven een tweede manier om geijkte superzwaartekracht te verkrijgen: het drie-stappen superconforme programma. We gebruikten dit programma voor de constructie van een algemenere materiegekoppelde vijfdimensionale Poincaré-superzwaartekrachttheorie dan momenteel bekend in de literatuur. De ruimte-tijdsymmetrieën van Poincaré-superzwaartekracht worden gegeven door translaties en rotaties, die deel uitmaken van de super-Poincaré-groep. Het conforme programma breidt deze groep uit tot de grootste groep van ruimte-tijdsymmetrieën, namelijk de superconforme groep. Door het invoeren van extra symmetrieën bevat de corresponderende superzwaartekrachttheorie meer structuur en is derhalve gemakkelijker te analyseren.

De eerste stap van het programma wordt behandeld in hoofdstuk 5, waar de constructie en ijking van de vijfdimensionale superconforme algebra wordt beschreven. Dit resulteert in het zogenaamd ‘Standaard Weyl-multiplet’; dit is de minimale representatie van de superconforme algebra die het graviton bevat. De superconforme achtergrondvelden worden gegeven door de velden in dit Weyl-multiplet.

De tweede stap is het onderwerp van hoofdstuk 6, waar we verscheidene materiemultipletrepresentaties van de superconforme algebra construeren, inclusief hun acties en supersymmetrietransformatieregels in de achtergrond van de Weyl-multipletvelden. Het betreft hier voornamelijk vector-tensormultipletten en hypermultipletten. Beide bevatten scalaire deeltjes die aanleiding geven tot interessante geometrie op de bijbehorende scalaire manifolds.

De laatste stap wordt gegeven in hoofdstuk 7. Hier wordt de superconforme algebra weer teruggebracht naar de super-Poincaré-algebra door het kiezen van geschikte ijkkeuzes voor de niet-Poincaré-symmetrieën. Door het kiezen van de juiste ijk vinden we vijfdimensionale materiegekoppelde Poincaré-superzwaartekracht die in tal van toepassingen kan worden gebruikt, zoals beschreven in hoofdstuk 2. Ten slotte geven we in appendices A–C onze conventies en enige extra informatie over de eigenschappen van de scalaire manifolds die gegenereerd worden door de hypermultipletkoppelingen.

Dankwoord

Allereerst wil ik van de gelegenheid gebruik maken om mijn werkgever, de Stichting voor Fundamenteel Onderzoek der Materie (FOM), te bedanken voor de financiële ondersteuning die dit proefschrift mogelijk heeft gemaakt; bovendien voor de geboden gelegenheid om conferenties en workshops bij te wonen in België, Frankrijk, Griekenland, Italië en Zweden.

Eric Bergshoeff en Mees de Roo ben ik zeer erkentelijk voor hun goede begeleiding tijdens het schrijven van dit manuscript. Hun strakke planning en nuttige feedback hebben het schrijven van dit proefschrift in goede banen geleid. I would like to thank the reading committee: Ulf Lindström, Kelly Stelle and Antoine Van Proeyen, for carefully reading this thesis.

The work published during the last four years, most of it described in this thesis, has been the result of various fruitful collaborations with the following people: Eric Bergshoeff, Sorin Cucu, Martijn Derix, Eduardo Eyras, Jos Gheerardyn, Ulf Gran, Rein Halbersma, Roman Linares, Sudhakar Panda, Diederik Roest, Mees de Roo, Stefan Vandoren and Antoine Van Proeyen.

Onze secretaresses Ynske Joustra en Iris de Roo-Kwant bedank ik voor het secuur doornemen en corrigeren van mijn inleiding en samenvatting, evenals voor de vele hulp op het administratieve vlak en de getoonde interesse.

Al mijn collega's op het centrum voor theoretische natuurkunde stonden garant voor een plezierige werkatmosfeer; vooral niet te vergeten de leuke jaarlijkse uitstapjes zoals kanoën en lasergunnen. In het bijzonder wil ik de promovendi en postdocs bedanken voor het karten en de gezellige pokeravonden.

Oud-kamergenoot Rein Halbersma ben ik dankbaar voor de zeer prettige samenwerking en de hulp bij het schrijven van mijn proefschrift. Vooral de vele (over)uren die we hebben gemaakt op kantoor en tijdens ons verblijf in Parijs staan nog vers in mijn geheugen. Nu vijf dimensies eindelijk bedwongen zijn, kijk ik al uit naar ons toekomstige projectje in zes dimensies.

Philip Jonkers bedank ik voor zijn vriendschap en de goede gesprekken die we hebben gevoerd tijdens de twee jaar dat we een kantoor deelden.

Ook denk ik met veel plezier terug aan de vele sportieve uurtjes met Wei Ji Ma en paranimf Sebastiaan Kole. Paranimf Ate Wijma zou ik willen bedanken voor zijn vriendschap en betrokkenheid.

I would like to thank the members of my Quake III clan VVF for lots of virtual bloodshed during the last two years, which helped me to get through my motivation dips; in particular I would like to express my gratitude to Peter McFerran for thoroughly checking my thesis for grammar and spelling mistakes.

Mijn moeder en mijn beide zusjes Letty en Emy ben ik erg dankbaar voor hun belangstelling en

steun. Zoals sommigen weten, is dit proefschrift tot stand gekomen in een moeilijke tijd voor mijn familie en voor mijzelf. Uiteraard had ik deze ‘mijlpaal’ het liefst bereikt in bijzijn van mijn voltallige familie, maar dit mocht helaas niet zo zijn. *Papa, ik zou je postuum willen bedanken voor al je steun. Zonder je stimulerende kijk op het leven had ik het zover niet gebracht en ik weet dan ook dat je trots op me zou zijn geweest.*